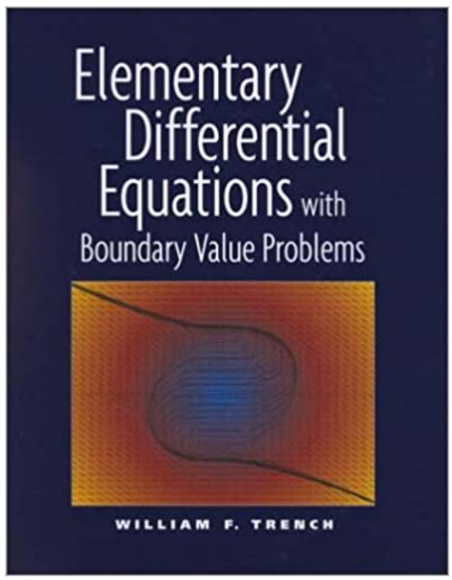
A Solution Manual For

Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001



Nasser M. Abbasi

May 15, 2024

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1.1 problem 2(a)

Internal problem ID [869]
Internal file name [OUTPUT/869_Sunday_June_05_2022_01_52_52_AM_84289303/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 2y = 0}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{2y} dy = \int dx$$
$$\frac{\ln(y)}{2} = x + c_1$$

Raising both side to exponential gives

$$\sqrt{y} = e^{x+c_1}$$

Which simplifies to

$$\sqrt{y} = c_2 e^{x}$$

Summary
The solution(s) found are the following

$$y = c_2^2 e^{2x} \tag{1}$$

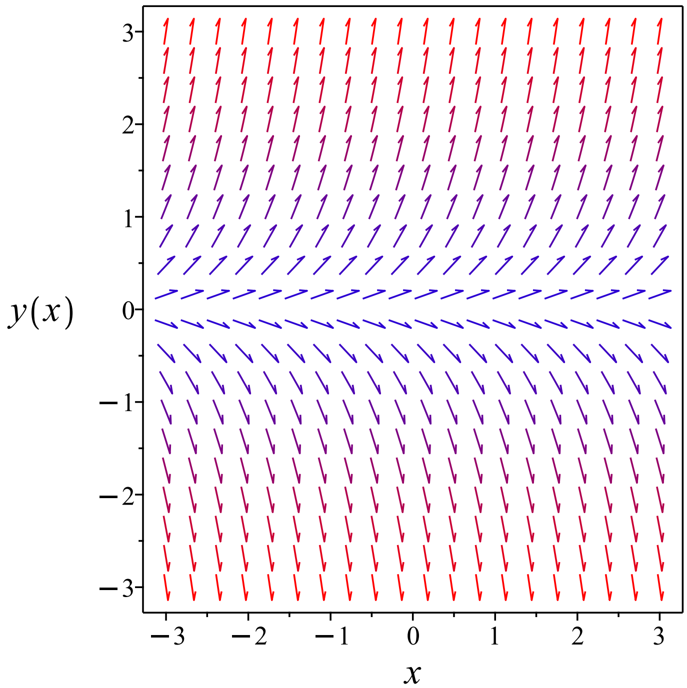


Figure 1: Slope field plot

Verification of solutions

$$y = c_2^2 e^{2x}$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' - 2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2dx + c_1$

- Evaluate integral

$$\ln(y) = 2x + c_1$$

- Solve for y

$$y = e^{2x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x) = 2*y(x),y(x), singsol=all)
```

$$y(x) = e^{2x}c_1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[y'[x]== y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

1.2 problem 2(b)

Internal problem ID [870]
Internal file name [`OUTPUT/870_Sunday_June_05_2022_01_52_53_AM_25961841/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y'x + y = x^2}$$

1.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$y' + \frac{y}{x} = x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(yx) &= (x)(x) \\
\mathrm{d}(yx) &= x^2\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
yx &= \int x^2\,\mathrm{d}x \\
yx &= \frac{x^3}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{x^2}{3} + \frac{c_1}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2}{3} + \frac{c_1}{x} \tag{1}
$$

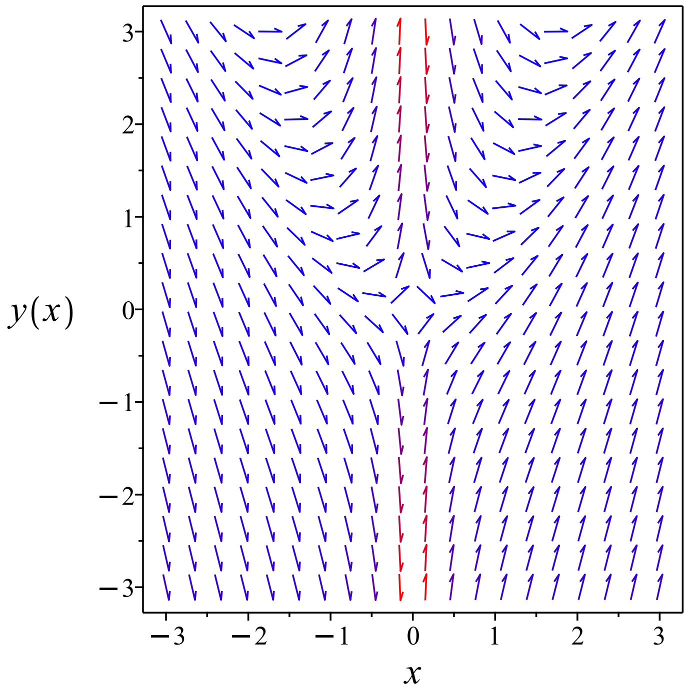


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{x^2}{3} + \frac{c_1}{x}$$

Verified OK.

1.2.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2 - y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + \left(x^2 - y\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + \left(x^2 - y\right) dx = d\left(\frac{1}{3}x^3 - yx\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{3}x^3 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^3 + 3c_1}{3x} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1}{3x} + c_1 \tag{1}$$

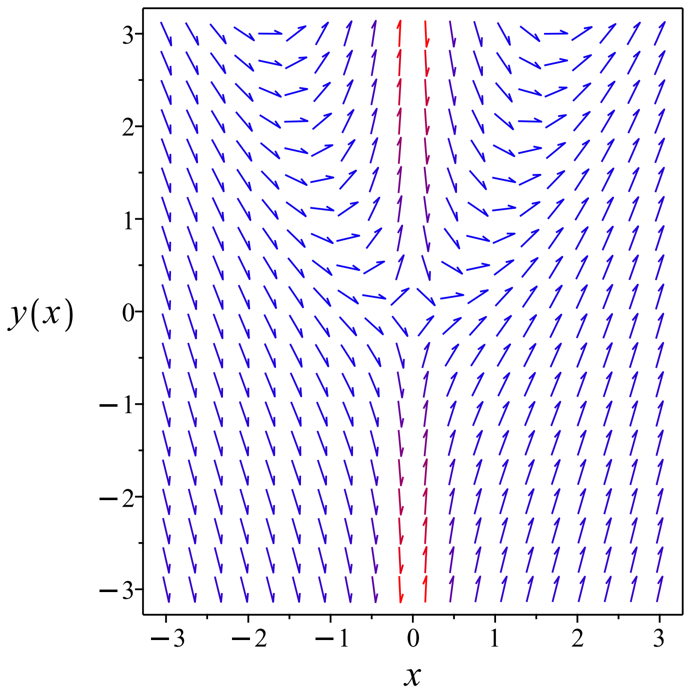


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1}{3x} + c_1$$

Verified OK.

1.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-x^2 + y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 2: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = yx
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-x^2 + y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^3}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
yx = \frac{x^3}{3} + c_1
$$

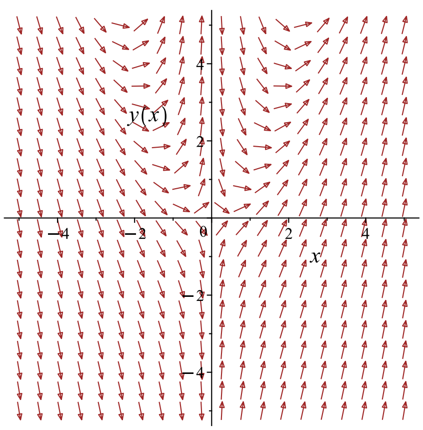
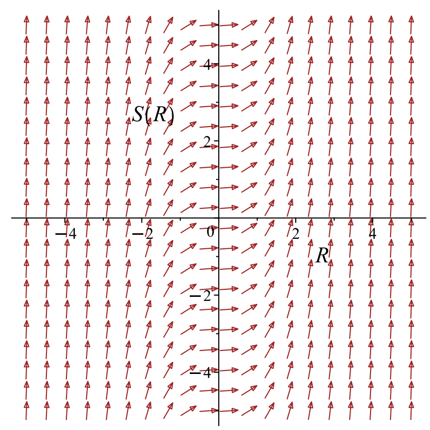
Which simplifies to

$$
yx = \frac{x^3}{3} + c_1
$$

Which gives

$$
y = \frac{x^3 + 3c_1}{3x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y}{x}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = R^2$ 

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1}{3x} \tag{1}$$

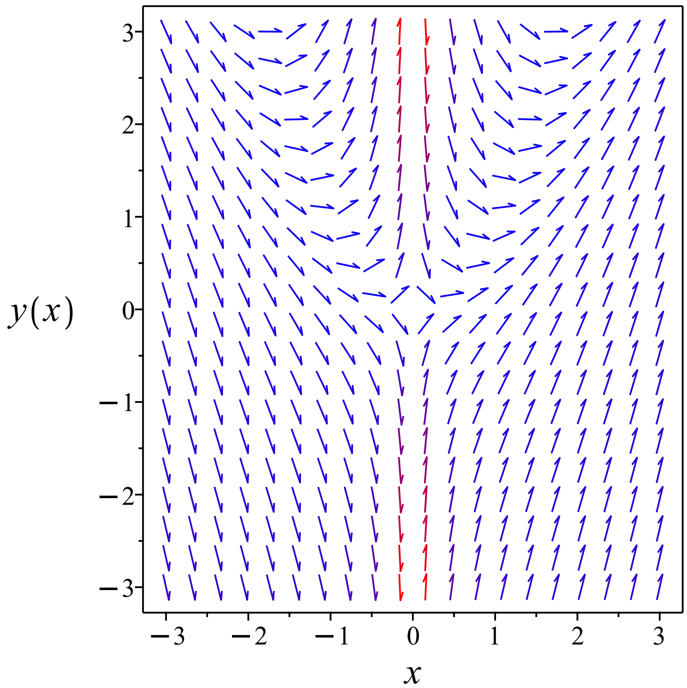


Figure 4: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1}{3x}$$

Verified OK.

1.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x)\,\mathrm{d}y &= \left(x^2 - y\right)\mathrm{d}x \\ \left(-x^2 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x^2 + y \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + y\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x^2 + y \, dx$$

$$\phi = -\frac{1}{3}x^3 + yx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + yx + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + yx$$

The solution becomes

$$y = \frac{x^3 + 3c_1}{3x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1}{3x} \tag{1}$$

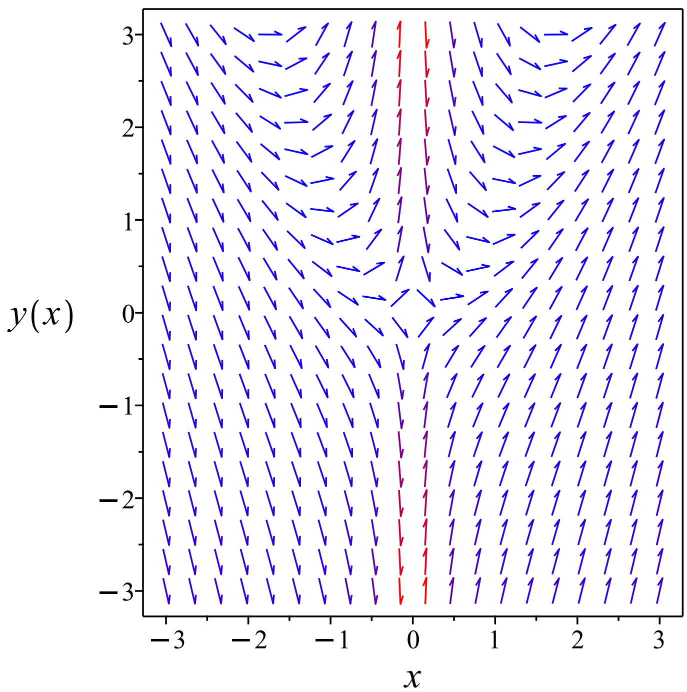


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1}{3x}$$

Verified OK.

1.2.5 Maple step by step solution

Let's solve

$y'x + y = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int x^2 dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^3}{3} + c_1}{x}$

- Simplify

$y = \frac{x^3+3c_1}{3x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x) +y(x)= x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1}{3x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y'[x] +y[x]== x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{3} + \frac{c_1}{x}$$

1.3 problem 2(c)

Internal problem ID [871]
Internal file name [`OUTPUT/871_Sunday_June_05_2022_01_52_55_AM_24961597/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 2(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yx + y' = x}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x(1-2y) \end{aligned}$$

Where $f(x) = x$ and $g(y) = 1 - 2y$. Integrating both sides gives

$$\frac{1}{1-2y}\,dy = x\,dx$$
$$\int \frac{1}{1-2y}\,dy = \int x\,dx$$

$$-\frac{\ln (1-2y)}{2} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{1-2y}} = \mathrm{e}^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{1-2y}} = c_2 \mathrm{e}^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2^2 \mathrm{e}^{x^2+2c_1} - 1\right) \mathrm{e}^{-x^2-2c_1}}{2c_2^2} \tag{1}$$

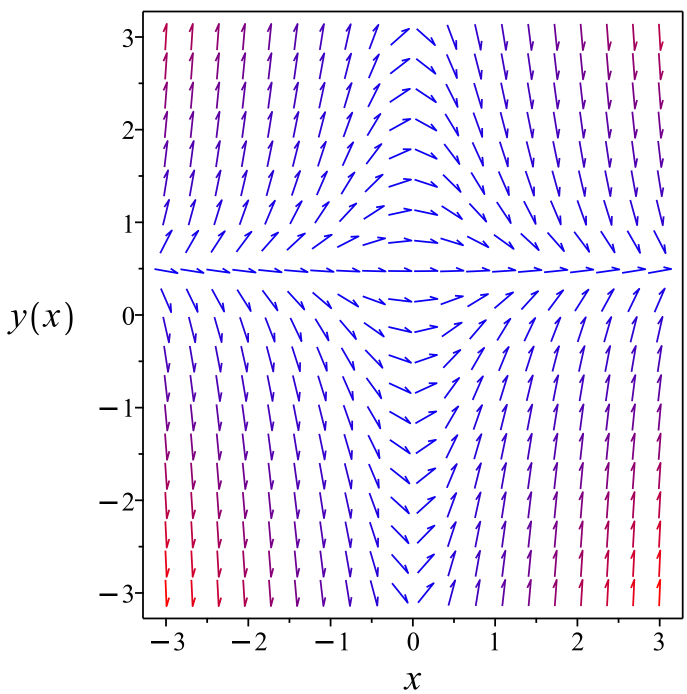


Figure 6: Slope field plot

Verification of solutions

$$y = \frac{\left(c_2^2 \mathrm{e}^{x^2+2c_1} - 1\right) \mathrm{e}^{-x^2-2c_1}}{2c_2^2}$$

Verified OK.

1.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$2yx + y' = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 2x dx} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} y\right) &= \left(\mathrm{e}^{x^2}\right)(x) \\ \mathrm{d}\left(\mathrm{e}^{x^2} y\right) &= \left(\mathrm{e}^{x^2} x\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{x^2} y &= \int \mathrm{e}^{x^2} x \,\mathrm{d}x \\ \mathrm{e}^{x^2} y &= \frac{\mathrm{e}^{x^2}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$y = \frac{\mathrm{e}^{-x^2} \mathrm{e}^{x^2}}{2} + c_1 \mathrm{e}^{-x^2}$$

which simplifies to

$$y = \frac{1}{2} + c_1 \mathrm{e}^{-x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2} + c_1 \mathrm{e}^{-x^2} \tag{1}$$

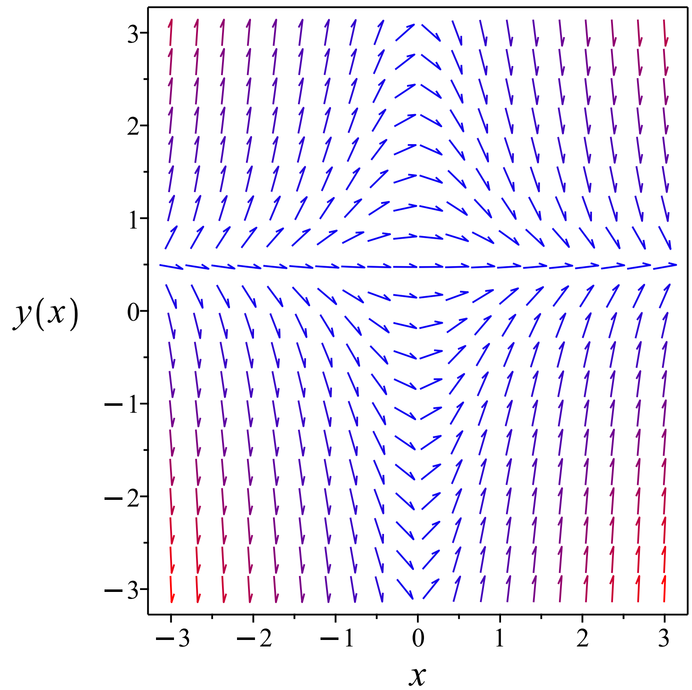


Figure 7: Slope field plot

Verification of solutions

$$y = \frac{1}{2} + c_1 \mathrm{e}^{-x^2}$$

Verified OK.

1.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 5: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\,\mathrm{e}^{x^2} y \\ S_y &= \mathrm{e}^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{x^2} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

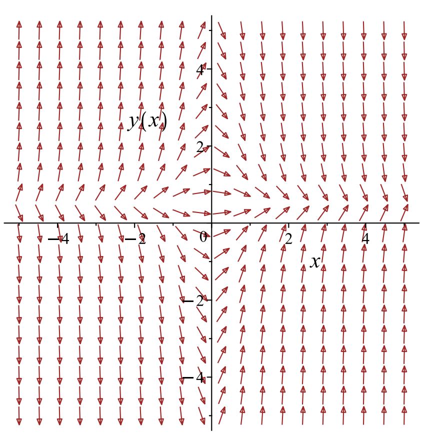
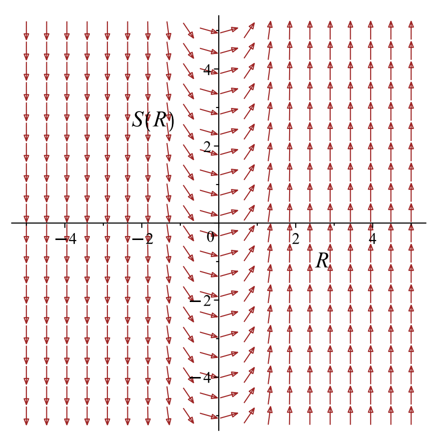
Which simplifies to

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

Which gives

$$y = \frac{\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx + x$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = e^{R^2} R$ 

Summary
The solution(s) found are the following

$$y = \frac{\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2} \tag{1}$$

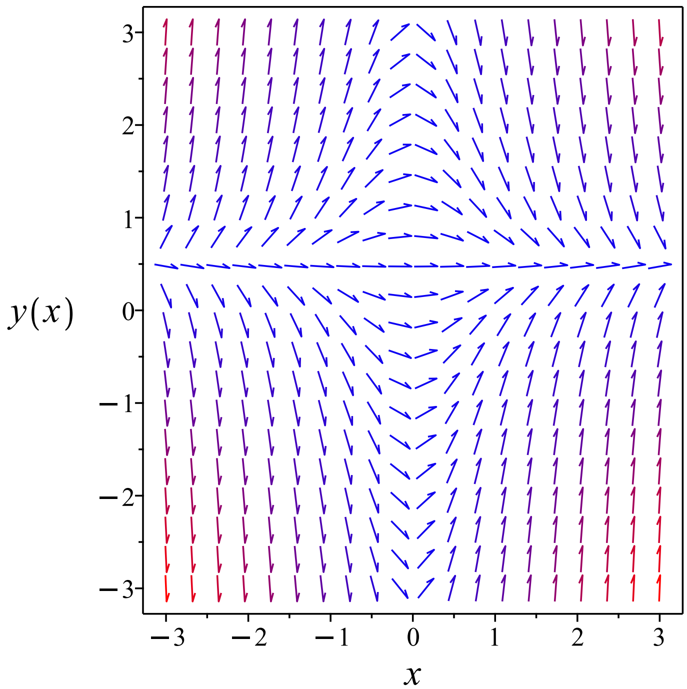


Figure 8: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

Verified OK.

1.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-2y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{1-2y}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x \\ N(x,y) &= \frac{1}{1-2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -x\, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-2y}$. Therefore equation (4) becomes

$$\frac{1}{1-2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{-1+2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{-1+2y}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(-1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-x^2-2c_1}}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-x^2-2c_1}}{2} + \frac{1}{2} \tag{1}$$

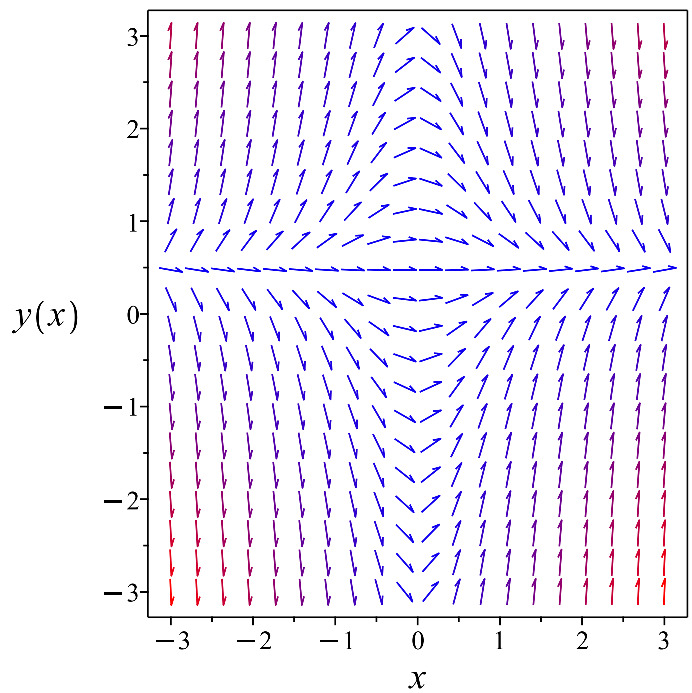


Figure 9: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2-2c_1}}{2} + \frac{1}{2}$$

Verified OK.

1.3.5 Maple step by step solution

Let's solve

$2yx + y' = x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y-1} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y-1}dx = \int -xdx + c_1$

- Evaluate integral

$$\frac{\ln(2y-1)}{2} = -\frac{x^2}{2} + c_1$$

- Solve for y

$$y = \frac{e^{-x^2+2c_1}}{2} + \frac{1}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve(diff(y(x),x) +2*x*y(x)= x,y(x), singsol=all)
```

$$y(x) = \frac{1}{2} + e^{-x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 26

```
DSolve[y'[x] +2*x*y[x]== x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} + c_1 e^{-x^2}$$
$$y(x) \to \frac{1}{2}$$

1.4 problem 2(d)

Internal problem ID [872]
Internal file name [`OUTPUT/872_Sunday_June_05_2022_01_52_56_AM_18164994/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 2(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y' + x(-1 + y^2) = 0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\left(-\frac{y^2}{2} + \frac{1}{2}\right) \end{aligned}$$

Where $f(x) = x$ and $g(y) = -\frac{y^2}{2} + \frac{1}{2}$. Integrating both sides gives

$$\frac{1}{-\frac{y^2}{2} + \frac{1}{2}}\, dy = x\, dx$$

$$\int \frac{1}{-\frac{y^2}{2}+\frac{1}{2}}dy = \int x\,dx$$

$$2\operatorname{arctanh}(y) = \frac{x^2}{2} + c_1$$

Which results in

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right) \tag{1}$$

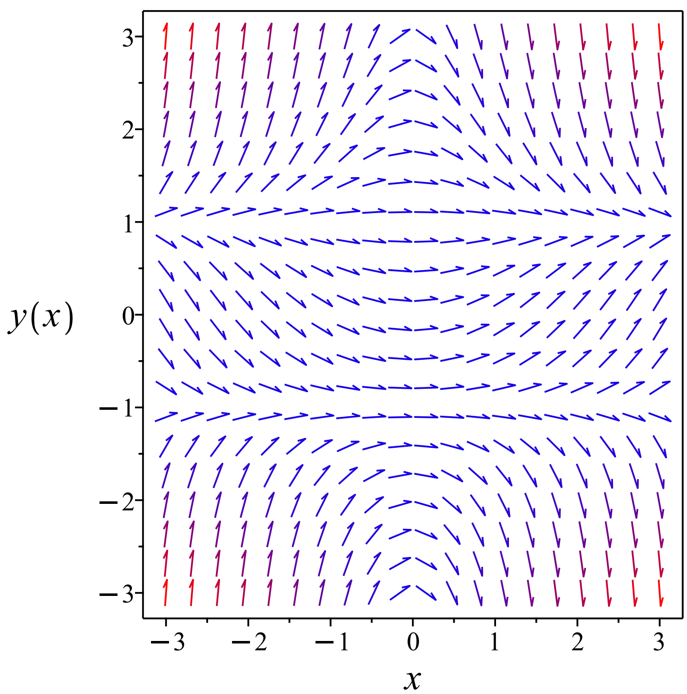


Figure 10: Slope field plot

Verification of solutions

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right)$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x(y^2-1)}{2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x(y^2-1)}{2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{y^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2 \operatorname{arctanh}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = 2 \operatorname{arctanh}(y) + c_1$$

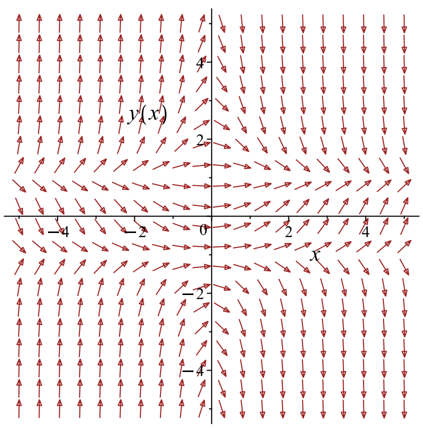
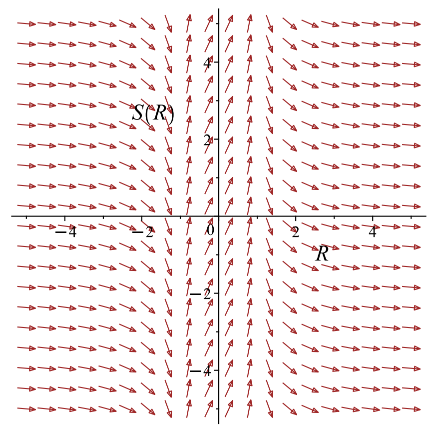
Which simplifies to

$$\frac{x^2}{2} = 2 \operatorname{arctanh}(y) + c_1$$

Which gives

$$y = -\tanh\left(-\frac{x^2}{4} + \frac{c_1}{2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^2-1)}{2}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = -\frac{2}{R^2-1}$ 

Summary

The solution(s) found are the following

$$y = -\tanh\left(-\frac{x^2}{4} + \frac{c_1}{2}\right) \tag{1}$$

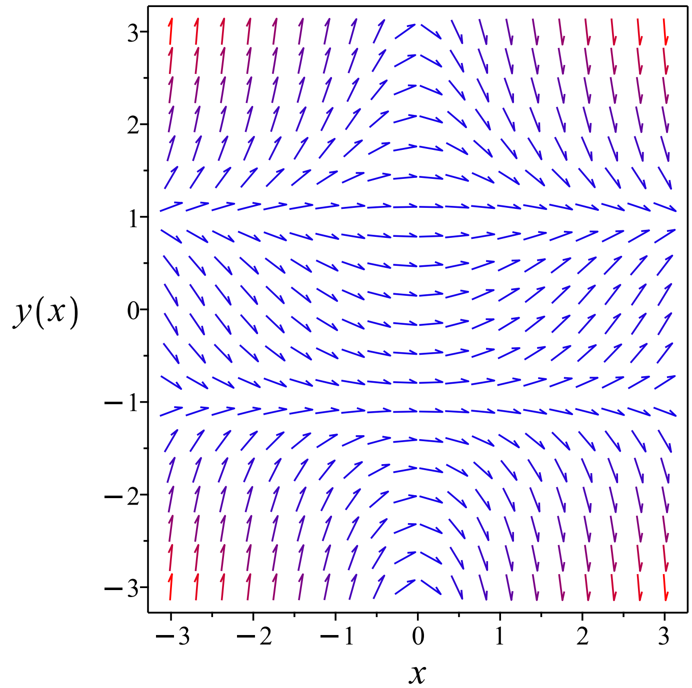


Figure 11: Slope field plot

Verification of solutions

$$y = -\tanh\left(-\frac{x^2}{4} + \frac{c_1}{2}\right)$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-\frac{y^2}{2} + \frac{1}{2}}\right) \mathrm{d}y &= (x) \, \mathrm{d}x \\
(-x) \, \mathrm{d}x + \left(\frac{1}{-\frac{y^2}{2} + \frac{1}{2}}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{-\frac{y^2}{2} + \frac{1}{2}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-\frac{y^2}{2}+\frac{1}{2}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M \,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -x \,\mathrm{d}x
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-\frac{y^2}{2}+\frac{1}{2}}$. Therefore equation (4) becomes

$$
\frac{1}{-\frac{y^2}{2}+\frac{1}{2}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2}{y^2 - 1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{2}{y^2-1}\right) \mathrm{d}y$$
$$f(y) = 2\,\operatorname{arctanh}(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + 2\,\operatorname{arctanh}(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + 2\,\operatorname{arctanh}(y)$$

The solution becomes

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right) \tag{1}$$

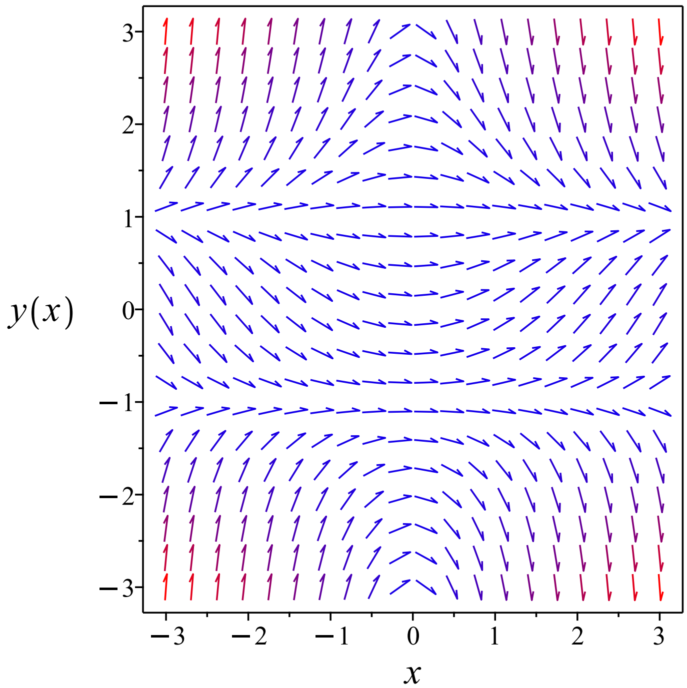


Figure 12: Slope field plot

Verification of solutions

$$y = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right)$$

Verified OK.

1.4.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x(y^2 - 1)}{2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{1}{2}x\,y^2 + \frac{1}{2}x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x}{2}$, $f_1(x) = 0$ and $f_2(x) = -\frac{x}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{xu}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{x^3}{8} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{x u''(x)}{2} + \frac{u'(x)}{2} + \frac{x^3 u(x)}{8} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sinh\left(\frac{x^2}{4}\right) + c_2 \cosh\left(\frac{x^2}{4}\right)$$

The above shows that

$$u'(x) = \frac{x\left(c_1 \cosh\left(\frac{x^2}{4}\right) + c_2 \sinh\left(\frac{x^2}{4}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \cosh\left(\frac{x^2}{4}\right) + c_2 \sinh\left(\frac{x^2}{4}\right)}{c_1 \sinh\left(\frac{x^2}{4}\right) + c_2 \cosh\left(\frac{x^2}{4}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \cosh\left(\frac{x^2}{4}\right) + \sinh\left(\frac{x^2}{4}\right)}{c_3 \sinh\left(\frac{x^2}{4}\right) + \cosh\left(\frac{x^2}{4}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 \cosh\left(\frac{x^2}{4}\right) + \sinh\left(\frac{x^2}{4}\right)}{c_3 \sinh\left(\frac{x^2}{4}\right) + \cosh\left(\frac{x^2}{4}\right)} \tag{1}$$

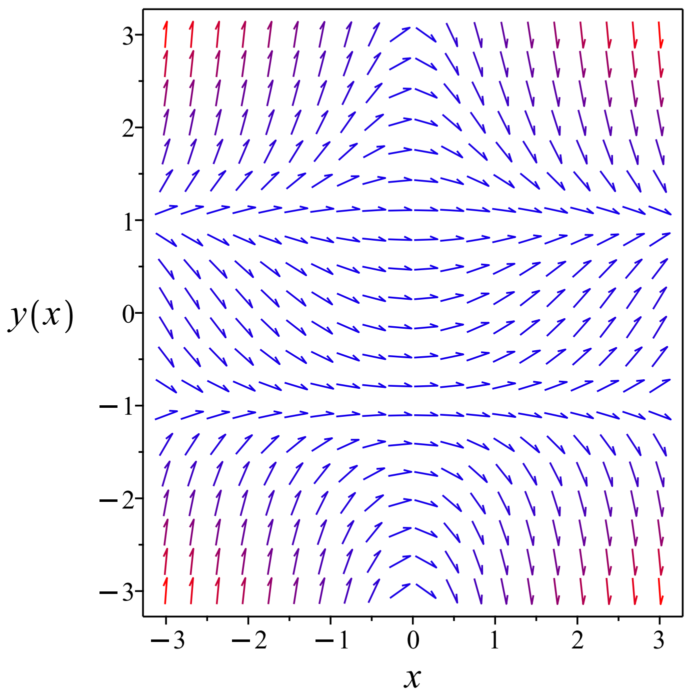


Figure 13: Slope field plot

Verification of solutions

$$y = \frac{c_3 \cosh\left(\frac{x^2}{4}\right) + \sinh\left(\frac{x^2}{4}\right)}{c_3 \sinh\left(\frac{x^2}{4}\right) + \cosh\left(\frac{x^2}{4}\right)}$$

Verified OK.

1.4.5 Maple step by step solution

Let's solve

$2y' + x(-1 + y^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1+y^2} = -\frac{x}{2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1+y^2}dx = \int -\frac{x}{2}dx + c_1$

- Evaluate integral

  $-\operatorname{arctanh}(y) = -\frac{x^2}{4} + c_1$

- Solve for y

  $y = -\tanh\left(-\frac{x^2}{4} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(2*diff(y(x),x) +x*(y(x)^2-1)= 0,y(x), singsol=all)
```

$$y(x) = \tanh\left(\frac{x^2}{4} + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 52

```
DSolve[2*y'[x] +x*(y[x]^2-1)== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{e^{\frac{x^2}{2}} - e^{2c_1}}{e^{\frac{x^2}{2}} + e^{2c_1}} \\
y(x) &\to -1 \\
y(x) &\to 1
\end{aligned}
$$

1.5 problem 2(e)

Internal problem ID [873]
Internal file name [`OUTPUT/873_Sunday_June_05_2022_01_52_57_AM_8955787/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 2(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 + y^2)\, x^2 = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y^2+1)\, x^2 \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\, dy = x^2\, dx$$

$$\int \frac{1}{y^2+1}\, dy = \int x^2\, dx$$

$$\arctan(y) = \frac{x^3}{3} + c_1$$

Which results in

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

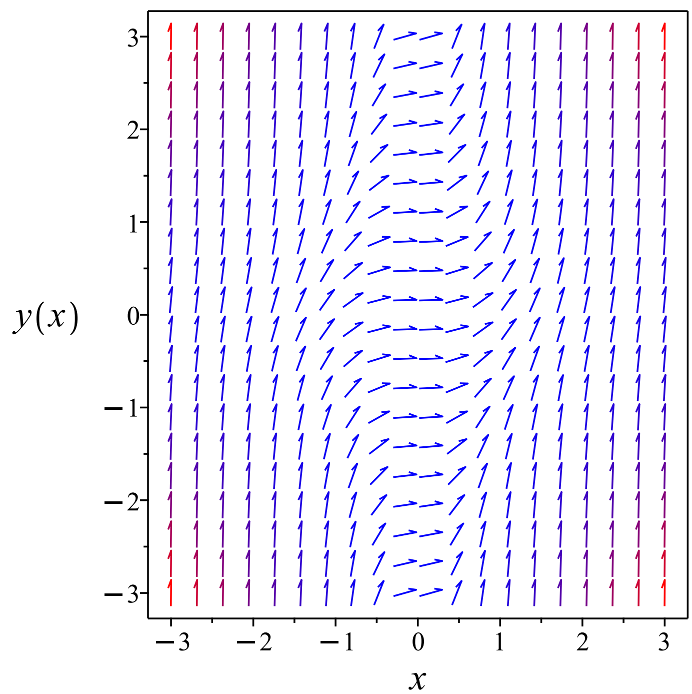


Figure 14: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= (y^2+1)\,x^2\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{x^2}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \left(y^2 + 1\right) x^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = \arctan(y) + c_1$$

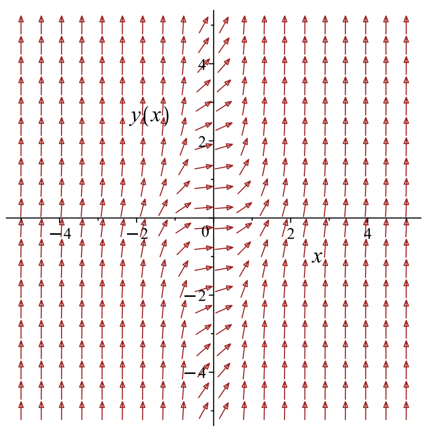
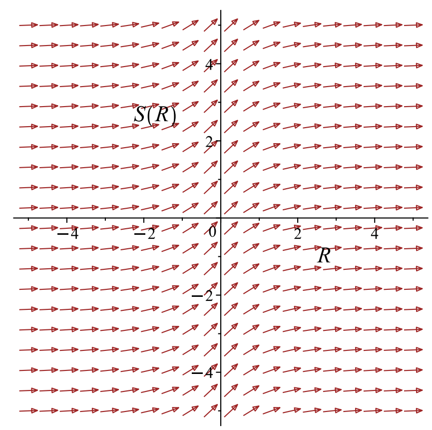
Which simplifies to

$$\frac{x^3}{3} = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (y^2 + 1)\, x^2$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right) \tag{1}$$

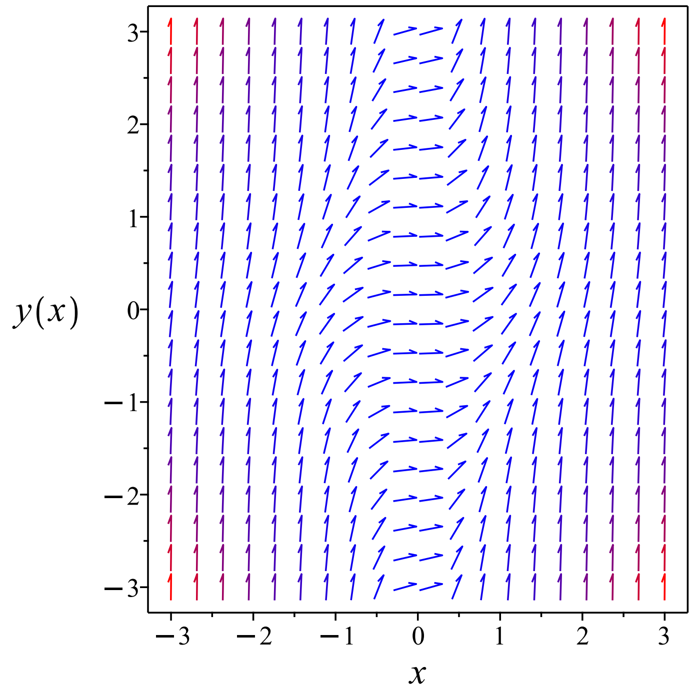


Figure 15: Slope field plot

Verification of solutions

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= \left(x^2\right)\mathrm{d}x \\ \left(-x^2\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 \\ N(x, y) &= \frac{1}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x^2\,\mathrm{d}x$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right)\mathrm{d}y$$

$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

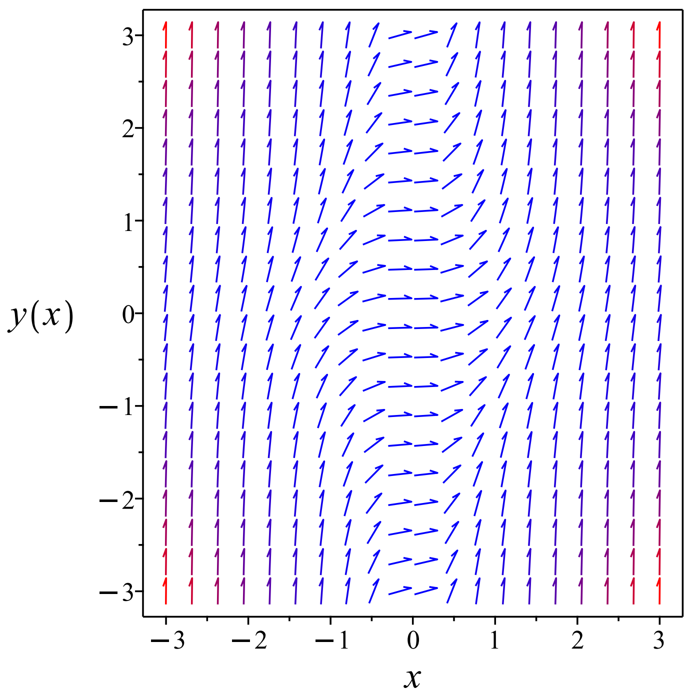


Figure 16: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

1.5.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= (y^2+1)\,x^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2y^2 + x^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{x^2u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2x \\ f_1f_2 &= 0 \\ f_2^2f_0 &= x^6 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^2u''(x) - 2xu'(x) + x^6u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x)=c_1 \sin\left(\frac{x^3}{3}\right)+c_2 \cos\left(\frac{x^3}{3}\right)
$$

The above shows that

$$
u'(x)=x^2\left(c_1 \cos\left(\frac{x^3}{3}\right)-c_2 \sin\left(\frac{x^3}{3}\right)\right)
$$

Using the above in (1) gives the solution

$$
y=-\frac{c_1 \cos\left(\frac{x^3}{3}\right)-c_2 \sin\left(\frac{x^3}{3}\right)}{c_1 \sin\left(\frac{x^3}{3}\right)+c_2 \cos\left(\frac{x^3}{3}\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$
y=\frac{-c_3 \cos\left(\frac{x^3}{3}\right)+\sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right)+\cos\left(\frac{x^3}{3}\right)}
$$

Summary

The solution(s) found are the following

$$
y=\frac{-c_3 \cos\left(\frac{x^3}{3}\right)+\sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right)+\cos\left(\frac{x^3}{3}\right)} \tag{1}
$$

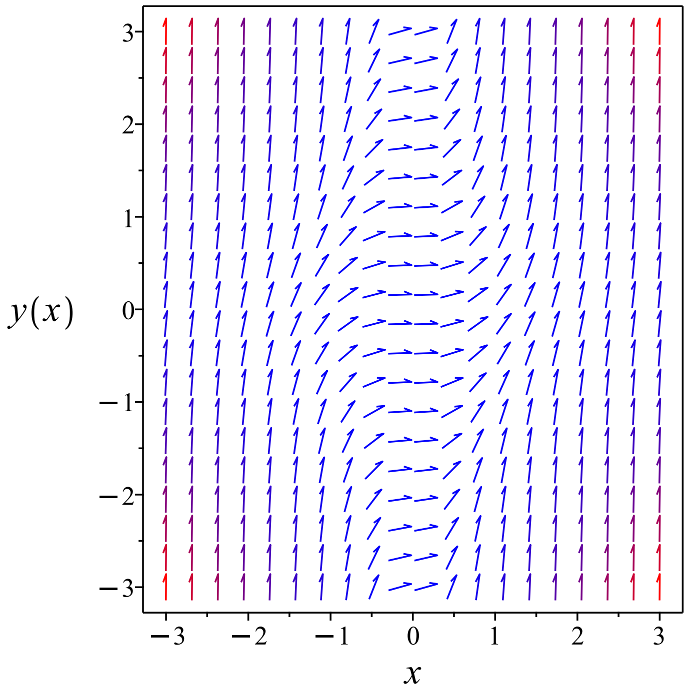


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)}$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$y' - (1 + y^2)\, x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\arctan(y) = \frac{x^3}{3} + c_1$
- Solve for y

  $y = \tan\left(\frac{x^3}{3} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = x^2*(1+y(x)^2),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{x^3}{3} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.171 (sec). Leaf size: 30

```
DSolve[y'[x] == x^2*(1+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x^3}{3} + c_1\right)$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.6 problem 3(a)

Internal problem ID [874]
Internal file name [`OUTPUT/874_Sunday_June_05_2022_01_52_58_AM_79689306/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = -x}$$

1.6.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -x \, \mathrm{d}x \\ &= -\frac{x^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{2} + c_1 \tag{1}$$

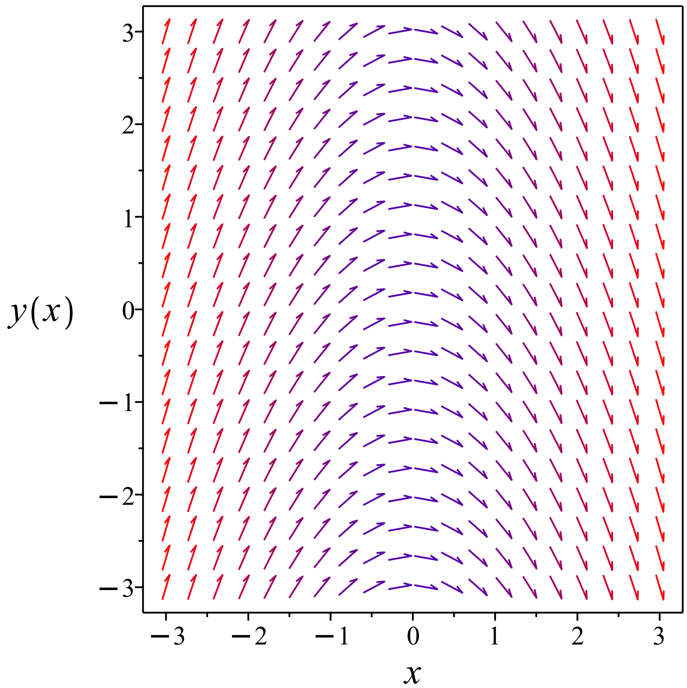


Figure 18: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{2} + c_1$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'dx = \int -xdx + c_1$

- Evaluate integral

  $y = -\frac{x^2}{2} + c_1$

- Solve for y

$$y = -\frac{x^2}{2} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) = -x,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 15

```
DSolve[y'[x] == -x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{2} + c_1$$

1.7 problem 3(b)

Internal problem ID [875]
Internal file name [`OUTPUT/875_Sunday_June_05_2022_01_52_59_AM_1317647/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = -\sin(x)\, x}$$

1.7.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -\sin(x)\, x \, \mathrm{d}x \\ &= x\cos(x) - \sin(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\cos(x) - \sin(x) + c_1 \tag{1}$$

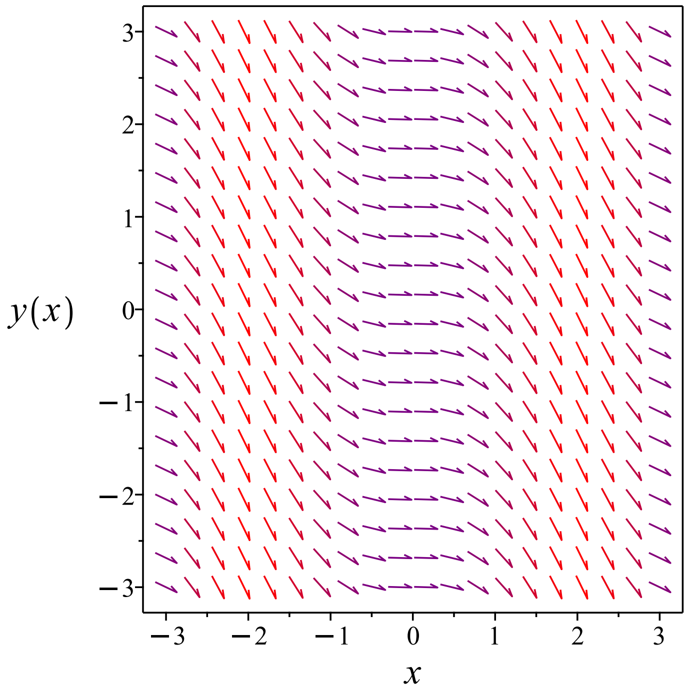


Figure 19: Slope field plot

Verification of solutions

$$y = x\cos(x) - \sin(x) + c_1$$

Verified OK.

1.7.2 Maple step by step solution

Let's solve

$y' = -\sin(x)\, x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int -\sin(x)\, x dx + c_1$

- Evaluate integral

  $y = x\cos(x) - \sin(x) + c_1$

- Solve for y

  $y = x\cos(x) - \sin(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x) = -x*sin(x),y(x), singsol=all)
```

$$y(x) = -\sin(x) + \cos(x)\, x + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 16

```
DSolve[y'[x] == -x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x) + x\cos(x) + c_1$$

1.8 problem 3(c)

Internal problem ID [876]
Internal file name [`OUTPUT/876_Sunday_June_05_2022_01_53_00_AM_12724638/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 3(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = x \ln(x)}$$

1.8.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x \ln(x)\, \mathrm{d}x \\ &= \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4} + c_1 \tag{1}$$

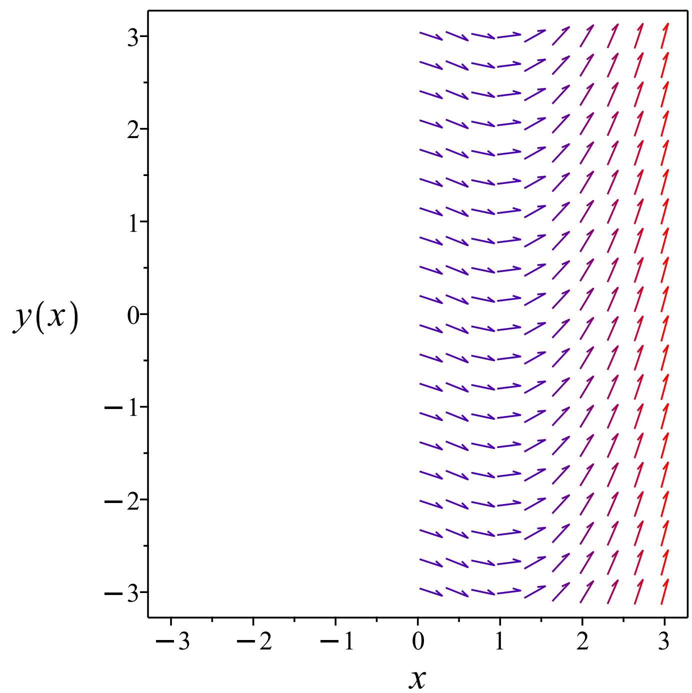


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4} + c_1$$

Verified OK.

1.8.2 Maple step by step solution

Let's solve

$y' = x \ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int x \ln(x)\, dx + c_1$

- Evaluate integral

  $y = \frac{\ln(x) x^2}{2} - \frac{x^2}{4} + c_1$

- Solve for y

$$y = \frac{\ln(x)x^2}{2} - \frac{x^2}{4} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x) = x*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4} + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 24

```
DSolve[y'[x] == x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{4} + \frac{1}{2}x^2 \log(x) + c_1$$

1.9 problem 4(a)

Internal problem ID [877]
Internal file name [`OUTPUT/877_Sunday_June_05_2022_01_53_01_AM_12765659/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 4(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = -x\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 1]$$

1.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = -x\,\mathrm{e}^{x}$$

Hence the ode is

$$y' = -x\,\mathrm{e}^{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x\,\mathrm{e}^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.9.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -x\,\mathrm{e}^x \,\mathrm{d}x \\ &= -(x-1)\,\mathrm{e}^x + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

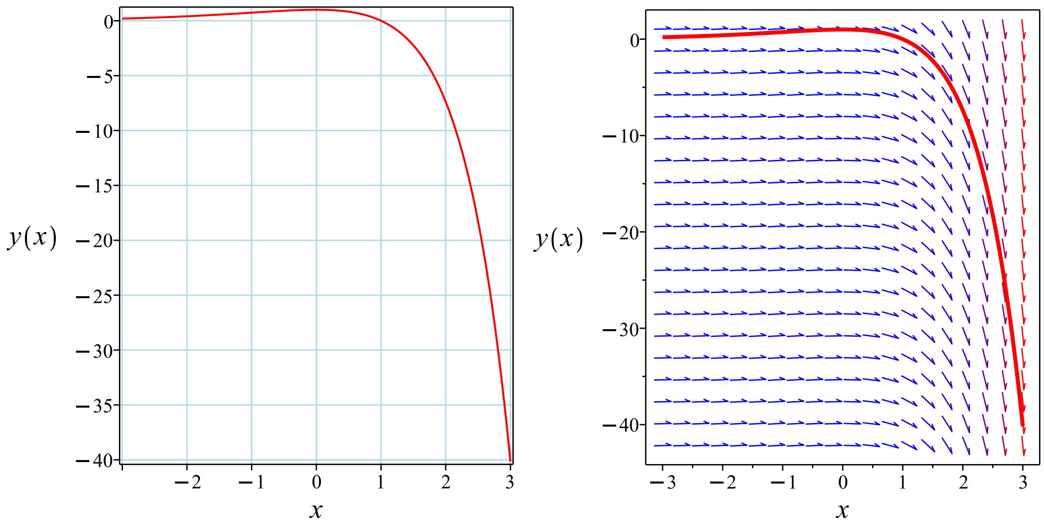
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -x\,\mathrm{e}^x + \mathrm{e}^x$$

Summary
The solution(s) found are the following

$$y = -x\,\mathrm{e}^x + \mathrm{e}^x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -x\, e^{x} + e^{x}$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$[y' = -x\, e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int -x\, e^{x} dx + c_1$

- Evaluate integral

  $y = -(x-1)\, e^{x} + c_1$

- Solve for y

  $y = -x\, e^{x} + e^{x} + c_1$

- Use initial condition $y(0) = 1$

  $1 = 1 + c_1$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = e^x(1 - x)$
- Solution to the IVP

  $y = e^x(1 - x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) = -x*exp(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -(x - 1)\, e^x$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 13

```
DSolve[{y'[x] == -x*Exp[x],y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^x(x - 1)$$

1.10 problem 4(b)

Internal problem ID [878]
Internal file name [`OUTPUT/878_Sunday_June_05_2022_01_53_02_AM_44411204/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 4(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x \sin(x^2)}$$

With initial conditions

$$\left[y\left(\frac{\sqrt{2}\,\sqrt{\pi}}{2}\right) = 1\right]$$

1.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = x \sin(x^2)$$

Hence the ode is

$$y' = x \sin(x^2)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\sqrt{2}\sqrt{\pi}}{2}$ is inside this domain. The domain of $q(x) = x \sin(x^2)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\sqrt{2}\sqrt{\pi}}{2}$ is also inside this domain. Hence solution exists and is unique.

1.10.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x \sin(x^2) \, dx \\ &= -\frac{\cos(x^2)}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\sqrt{2}\sqrt{\pi}}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

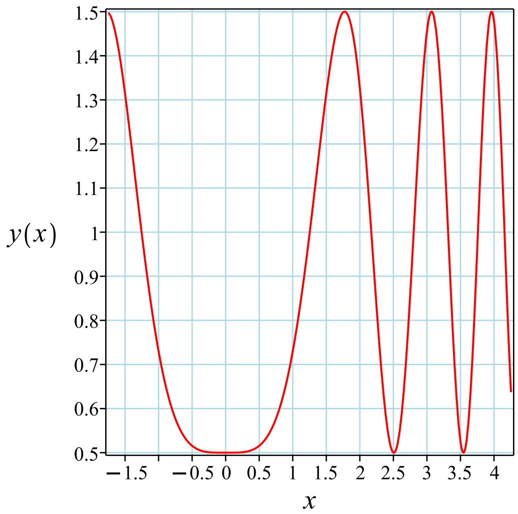
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

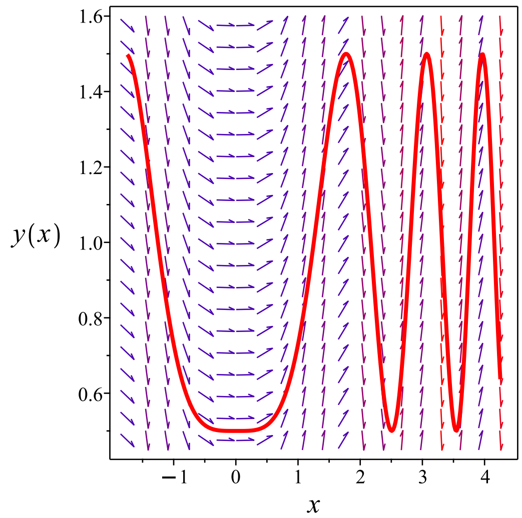
$$y = -\frac{\cos(x^2)}{2} + 1$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos(x^2)}{2} + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(x^2)}{2} + 1$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$\left[y' = x\sin(x^2), y\left(\frac{\sqrt{2}\sqrt{\pi}}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int x\sin(x^2)\, dx + c_1$

- Evaluate integral

  $y = -\frac{\cos(x^2)}{2} + c_1$

- Solve for y

  $y = -\frac{\cos(x^2)}{2} + c_1$

- Use initial condition $y\left(\frac{\sqrt{2}\sqrt{\pi}}{2}\right) = 1$

$1 = c_1$

- Solve for c_1

$c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

$y = -\frac{\cos(x^2)}{2} + 1$

- Solution to the IVP

$y = -\frac{\cos(x^2)}{2} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 12

```
dsolve([diff(y(x),x) = x*sin(x^2),y(sqrt(1/2*Pi)) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{\cos(x^2)}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 15

```
DSolve[{y'[x] == x*Sin[x^2],y[Sqrt[Pi/2]]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - \frac{\cos(x^2)}{2}$$

1.11 problem 4(c)

Internal problem ID [879]
Internal file name [`OUTPUT/879_Sunday_June_05_2022_01_53_03_AM_93000578/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 4(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \tan(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = 3\right]$$

1.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \tan(x)$$

Hence the ode is

$$y' = \tan(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z50 \vee \frac{1}{2}\pi + \pi\_Z50 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

1.11.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x)\, \mathrm{d}x \\ &= -\ln(\cos(x)) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{\ln(2)}{2} + c_1$$

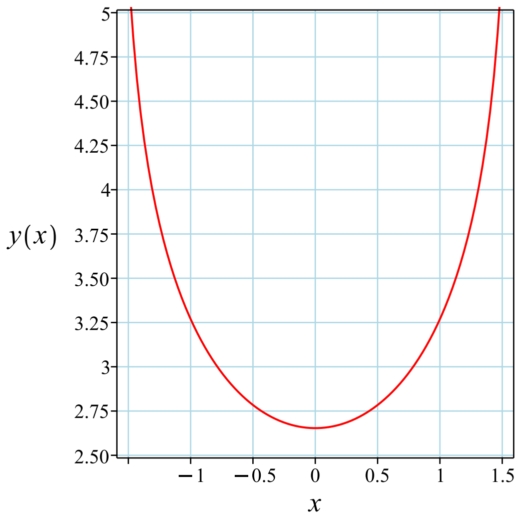
$$c_1 = -\frac{\ln(2)}{2} + 3$$

Substituting c_1 found above in the general solution gives

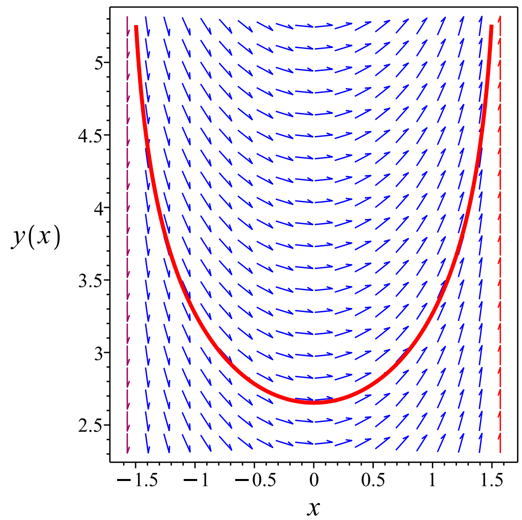
$$y = -\ln(\cos(x)) - \frac{\ln(2)}{2} + 3$$

Summary
The solution(s) found are the following

$$y = -\ln(\cos(x)) - \frac{\ln(2)}{2} + 3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\ln(\cos(x)) - \frac{\ln(2)}{2} + 3$$

Verified OK.

1.11.3 Maple step by step solution

Let's solve

$\left[y' = \tan(x), y\left(\frac{\pi}{4}\right) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \tan(x)\, dx + c_1$

- Evaluate integral

  $y = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = -\ln(\cos(x)) + c_1$

- Use initial condition $y\left(\frac{\pi}{4}\right) = 3$

$$3 = -\ln\left(\frac{\sqrt{2}}{2}\right) + c_1$$

- Solve for c_1

$$c_1 = \ln\left(\frac{\sqrt{2}}{2}\right) + 3$$

- Substitute $c_1 = \ln\left(\frac{\sqrt{2}}{2}\right) + 3$ into general solution and simplify

$$y = -\ln(\cos(x)) - \frac{\ln(2)}{2} + 3$$

- Solution to the IVP

$$y = -\ln(\cos(x)) - \frac{\ln(2)}{2} + 3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 15

```
dsolve([diff(y(x),x) = tan(x),y(1/4*Pi) = 3],y(x), singsol=all)
```

$$y(x) = -\ln(\cos(x)) + 3 - \frac{\ln(2)}{2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[{y'[x] == Tan[x],y[Pi/4]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(\cos(x)) + 3 - \frac{\log(2)}{2}$$

1.12 problem 5(a)

Internal problem ID [880]
Internal file name [`OUTPUT/880_Sunday_June_05_2022_01_53_05_AM_81107921/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 5(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \tan(x)\, y = \cos(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}\,\pi}{8}\right]$$

1.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \tan(x)$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' + \tan(x)\, y = \cos(x)$$

The domain of $p(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z50 \lor \frac{1}{2}\pi + \pi\_Z50 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

1.12.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)}\end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(\cos(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)\, y) &= (\sec(x))\,(\cos(x)) \\ \mathrm{d}(\sec(x)\, y) &= \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\sec(x)\, y &= \int \mathrm{d}x \\ \sec(x)\, y &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = x\cos(x) + c_1\cos(x)$$

which simplifies to

$$y = \cos(x)(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\sqrt{2}\,\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{2}\,\pi}{8} = \frac{\sqrt{2}\,\pi}{8} + \frac{\sqrt{2}\,c_1}{2}$$

$$c_1 = 0$$

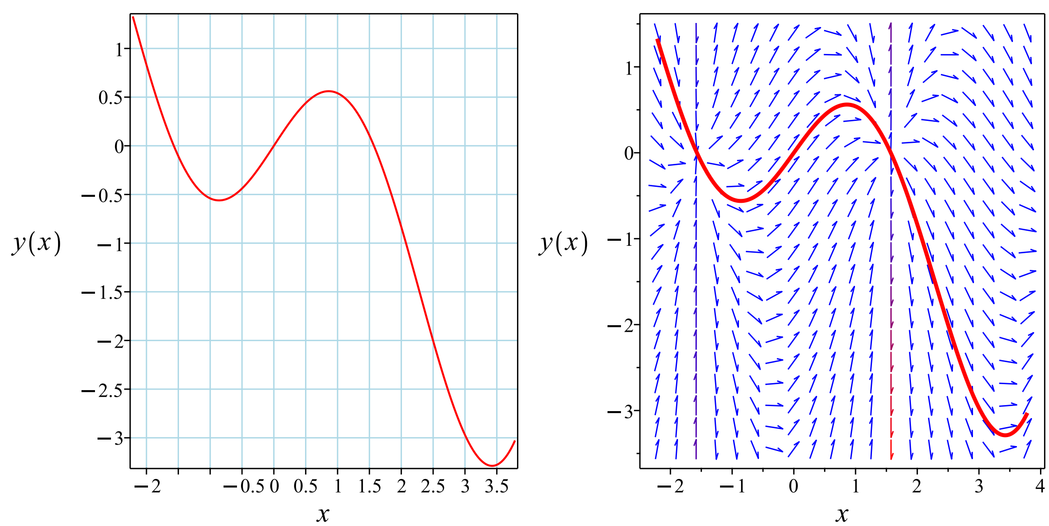
Substituting c_1 found above in the general solution gives

$$y = x\cos(x)$$

Summary

The solution(s) found are the following

$$y = x\cos(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\cos(x)$$

Verified OK.

1.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \cos(x) - \tan(x)\, y \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \cos(x)\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\cos(x)} dy\end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \cos(x) - \tan(x)\, y$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \tan(x)\sec(x)\, y\\ S_y &= \sec(x)\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec(x)\, y = x + c_1$$

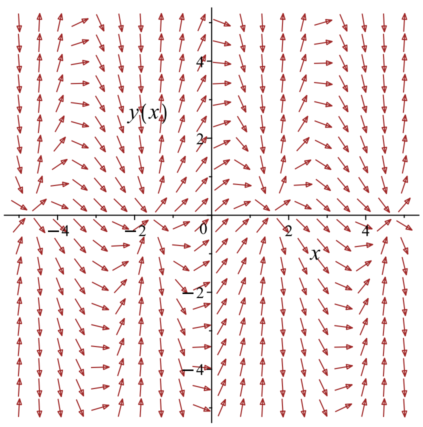
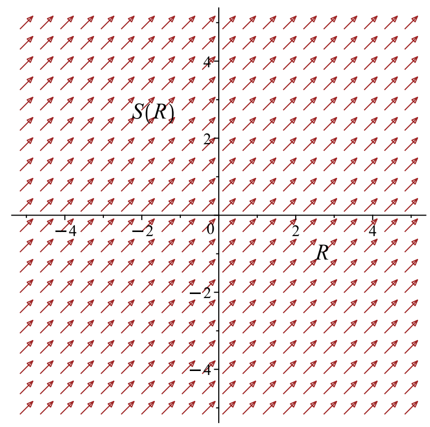
Which simplifies to

$$\sec(x)\, y = x + c_1$$

Which gives

$$y = \frac{x + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x) - \tan(x) y$ 	$R = x$ $S = \sec(x) y$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\sqrt{2}\,\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{2}\,\pi}{8} = \frac{\sqrt{2}\,\pi}{8} + \frac{\sqrt{2}\,c_1}{2}$$

$$c_1 = 0$$

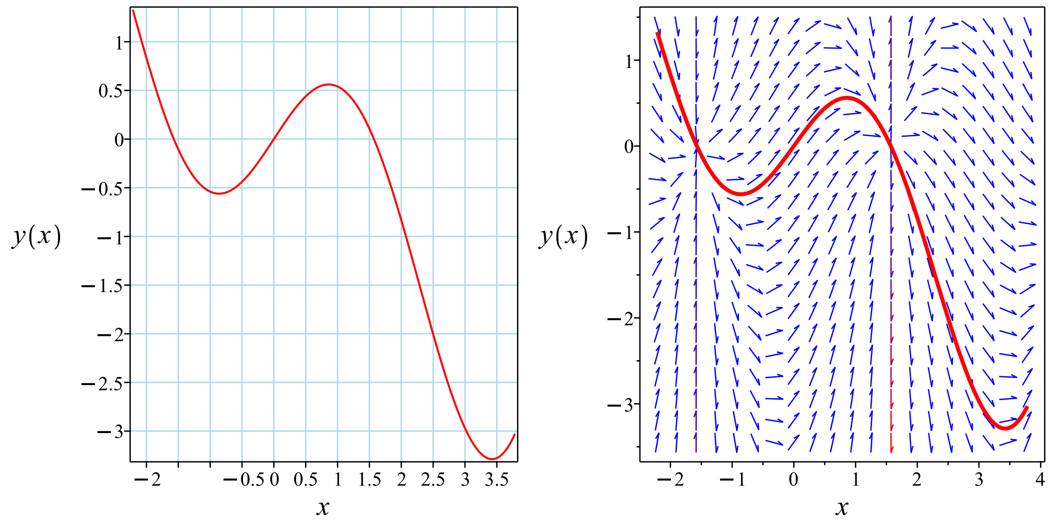
Substituting c_1 found above in the general solution gives

$$y = x\cos(x)$$

Summary

The solution(s) found are the following

$$y = x\cos(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = x\cos(x)$$

Verified OK.

1.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (\cos(x) - \tan(x)\,y)\,\mathrm{d}x \\ (-\cos(x) + \tan(x)\,y)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\cos(x) + \tan(x)\,y \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x) + \tan(x)\,y) \\ &= \tan(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\tan(x)) - (0)) \\ &= \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \tan(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(x))} \\
&= \sec(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(x)\left(-\cos(x) + \tan(x)\, y\right) \\
&= -1 + \tan(x)\sec(x)\, y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(x)\,(1) \\
&= \sec(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-1 + \tan(x)\sec(x)\, y\right) + \left(\sec(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -1 + \tan(x)\sec(x)\, y \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + \sec(x)\, y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \sec(x)\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \sec(x)\, y$$

The solution becomes

$$y = \frac{x + c_1}{\sec(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = \frac{\sqrt{2}\,\pi}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{2}\,\pi}{8} = \frac{\sqrt{2}\,\pi}{8} + \frac{\sqrt{2}\, c_1}{2}$$

$$c_1 = 0$$

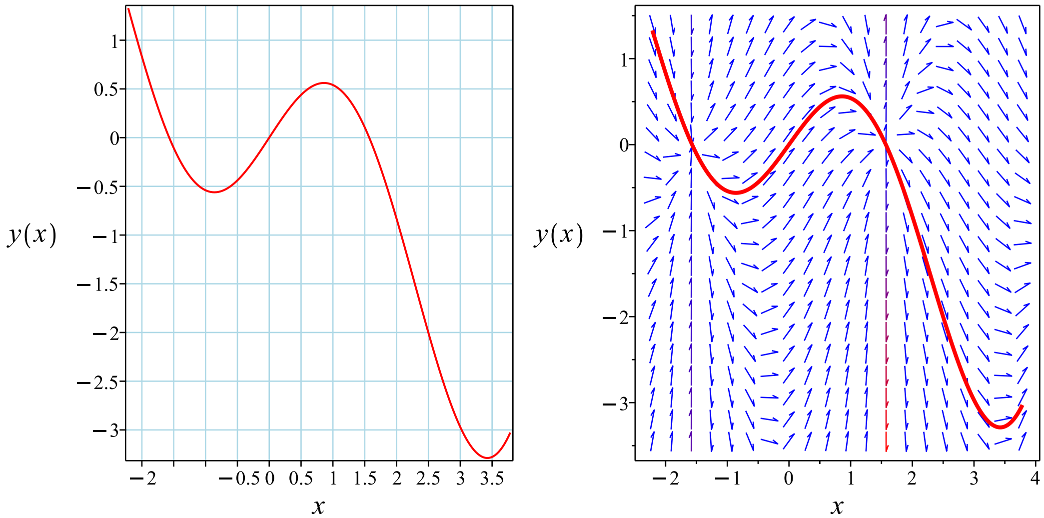
Substituting c_1 found above in the general solution gives

$$y = x\cos(x)$$

Summary

The solution(s) found are the following

$$y = x\cos(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\cos(x)$$

Verified OK.

1.12.5 Maple step by step solution

Let's solve

$\left[y' + \tan(x)\, y = \cos(x), y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}\,\pi}{8}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \cos(x) - \tan(x)\, y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \tan(x)\, y = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' + \tan(x)\, y\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$\mu(x)\left(y' + \tan(x)\, y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x)\tan(x)$

- Solve to find the integrating factor

$\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\, y = \int \mu(x)\cos(x)\, dx + c_1$

- Solve for y

$y = \frac{\int \mu(x)\cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

$y = \cos(x)\left(\int 1 dx + c_1\right)$

- Evaluate the integrals on the rhs

$y = \cos(x)(x + c_1)$

- Use initial condition $y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}\,\pi}{8}$

$\frac{\sqrt{2}\,\pi}{8} = \frac{\sqrt{2}\left(\frac{\pi}{4} + c_1\right)}{2}$

- Solve for c_1

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$y = x\cos(x)$

- Solution to the IVP

$y = x\cos(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve([diff(y(x),x) = cos(x)-y(x)*tan(x),y(1/4*Pi) = 1/4*Pi/sqrt(2)],y(x), singsol=all)
```

$$y(x) = \cos(x)\, x$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 9

```
DSolve[{y'[x] ==Cos[x]-y[x]*Tan[x],y[Pi/4]==Pi/(4*Sqrt[2])},y[x],x,IncludeSingularSolutions
```

$$y(x) \to x\cos(x)$$

1.13 problem 5(b)

Internal problem ID [881]
Internal file name [`OUTPUT/881_Sunday_June_05_2022_01_53_06_AM_62361073/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 5(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{x^2 - 2x^2y + 2}{x^3} = 0}$$

With initial conditions

$$\left[y(1) = \frac{3}{2}\right]$$

1.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{x^2 + 2}{x^3}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{x^2+2}{x^3}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{x^2+2}{x^3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.13.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2+2}{x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\, x^2\right) &= \left(x^2\right)\left(\frac{x^2+2}{x^3}\right) \\
\mathrm{d}\left(y\, x^2\right) &= \left(\frac{x^2+2}{x}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
y\, x^2 &= \int \frac{x^2+2}{x} \,\mathrm{d}x \\
y\, x^2 &= \frac{x^2}{2} + 2\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{\frac{x^2}{2} + 2\ln(x)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{1}{2} + c_1$$

$$c_1 = 1$$

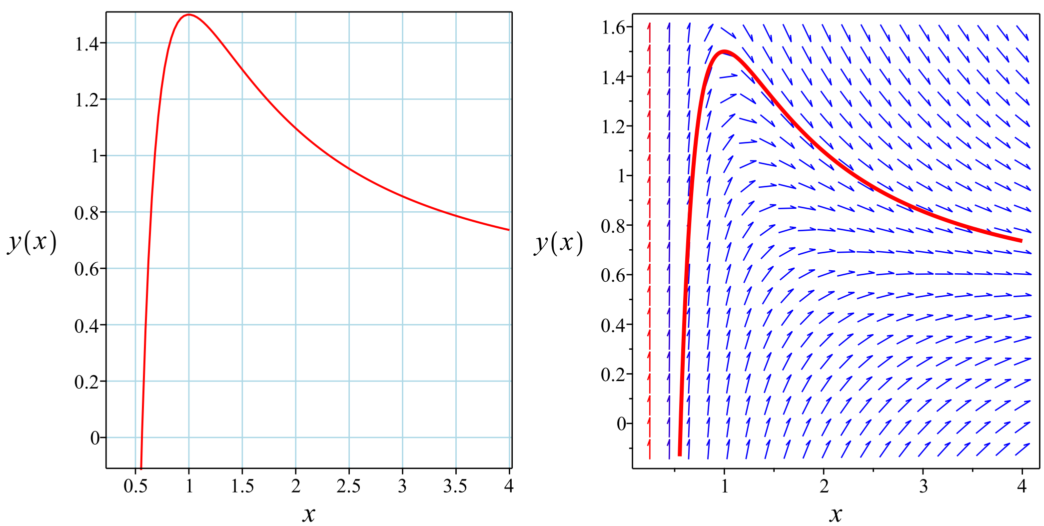
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Verified OK.

1.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2y\,x^2 - x^2 - 2}{x^3} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{x^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{x^2}} dy\end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y\,x^2 - x^2 - 2}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2yx \\
S_y &= x^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x^2+2}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R^2+2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{2} + 2\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x^2 y = \frac{x^2}{2} + 2\ln(x) + c_1
$$

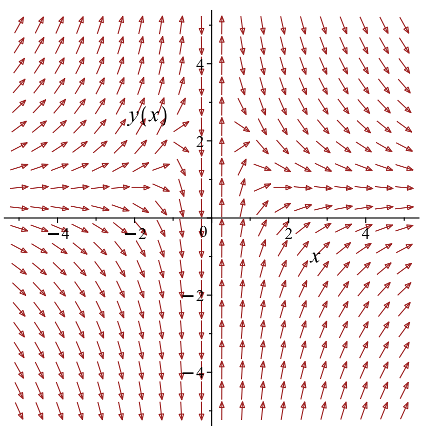
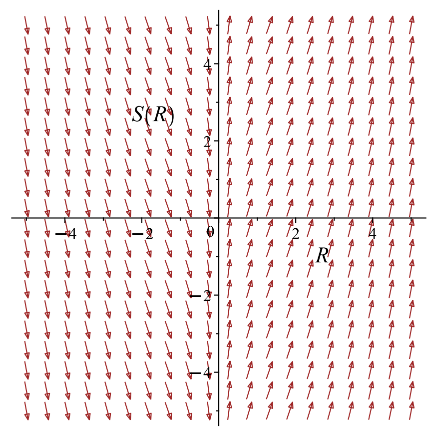
Which simplifies to

$$
x^2 y = \frac{x^2}{2} + 2\ln(x) + c_1
$$

Which gives

$$
y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2y\,x^2-x^2-2}{x^3}$

</td><td>$R = x$
$S = y\,x^2$</td><td>$\frac{dS}{dR} = \frac{R^2+2}{R}$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{1}{2} + c_1$$

$$c_1 = 1$$

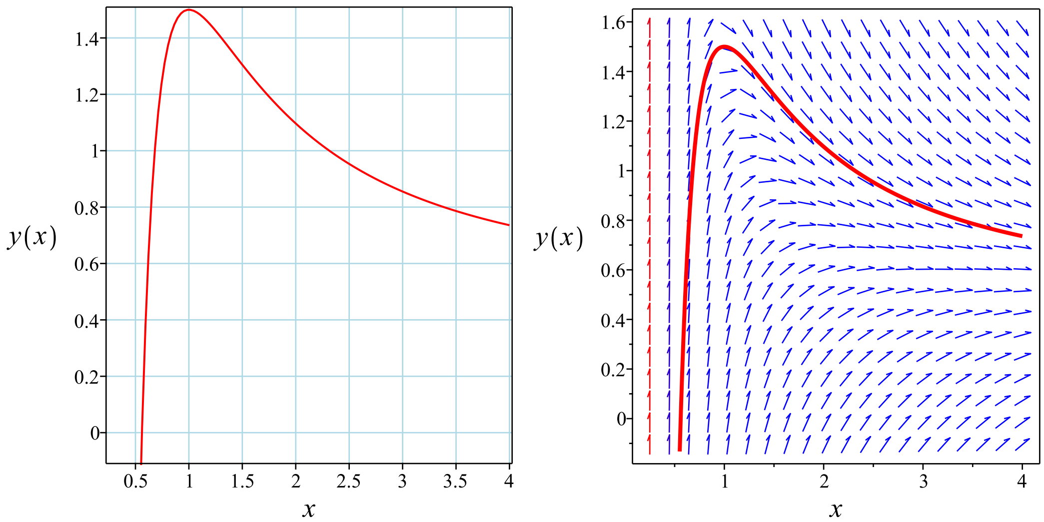
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Verified OK.

1.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{-2y\,x^2 + x^2 + 2}{x^3}\right)\mathrm{d}x$$

$$\left(-\frac{-2y\,x^2 + x^2 + 2}{x^3}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\frac{-2y\,x^2 + x^2 + 2}{x^3} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2y\,x^2 + x^2 + 2}{x^3}\right) \\ &= \frac{2}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(\frac{2}{x}\right) - (0)\right) \\ &= \frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{2}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2\ln(x)} \\ &= x^2 \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x^2\left(-\frac{-2y\,x^2 + x^2 + 2}{x^3}\right) \\ &= \frac{x^2(-1+2y) - 2}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x^2(1) \\ &= x^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^2(-1+2y) - 2}{x}\right) + (x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^2(-1+2y)-2}{x} \, \mathrm{d}x$$

$$\phi = \frac{x^2(-1+2y)}{2} - 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(-1+2y)}{2} - 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(-1+2y)}{2} - 2\ln(x)$$

The solution becomes

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{1}{2} + c_1$$

$$c_1 = 1$$

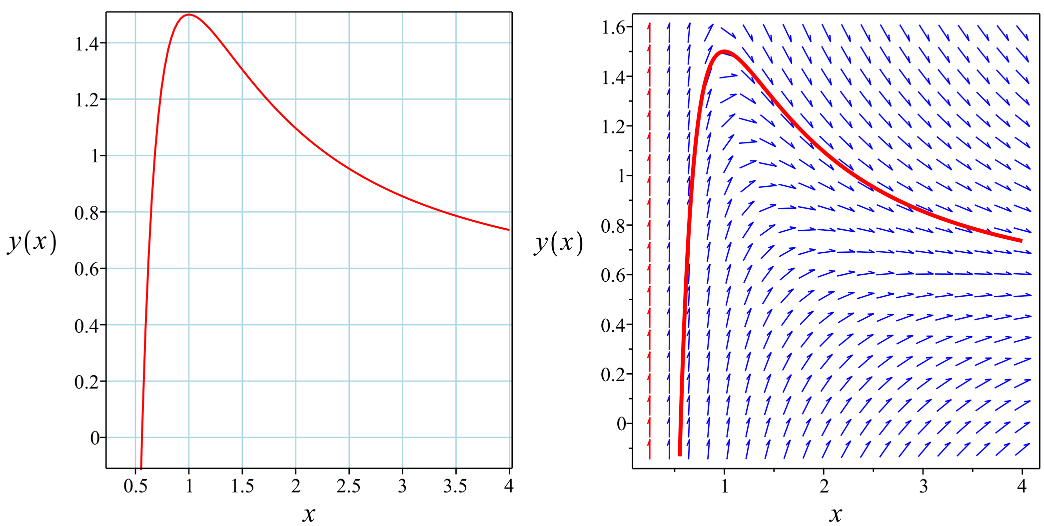
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 4\ln(x) + 2}{2x^2}$$

Verified OK.

1.13.5 Maple step by step solution

Let's solve

$\left[y' - \frac{x^2 - 2x^2 y + 2}{x^3} = 0, y(1) = \frac{3}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{x^2+2}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{x^2+2}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \frac{\mu(x)(x^2+2)}{x^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(x^2+2)}{x^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(x^2+2)}{x^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(x^2+2\right)}{x^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int \frac{x^2+2}{x} dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}$

- Use initial condition $y(1) = \frac{3}{2}$

  $\frac{3}{2} = \frac{1}{2} + c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{\frac{x^2}{2} + 2\ln(x) + 1}{x^2}$
- Solution to the IVP

  $y = \frac{\frac{x^2}{2} + 2\ln(x) + 1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x) = (x^2-2*x^2*y(x)+2)/x^3,y(1) = 3/2],y(x), singsol=all)
```

$$y(x) = \frac{\frac{x^2}{2} + 2\ln(x) + 1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 19

```
DSolve[{y'[x] ==(x^2-2*x^2*y[x]+2)/x^3,y[1]==3/2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2} + \frac{2\log(x)}{x^2} + \frac{1}{2}$$

1.14 problem 5(c)

Internal problem ID [882]
Internal file name [`OUTPUT/882_Sunday_June_05_2022_01_53_07_AM_16214606/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 5(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x(1 + y^2) = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x(y^2 + 1) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\left(y^2+1\right)\right)\\ &= 2yx\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= x\left(y^2+1\right)\end{aligned}$$

Where $f(x) = x$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2+1}\,dy &= x\,dx\\ \int \frac{1}{y^2+1}\,dy &= \int x\,dx\\ \arctan(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \tan\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan(c_1)$$

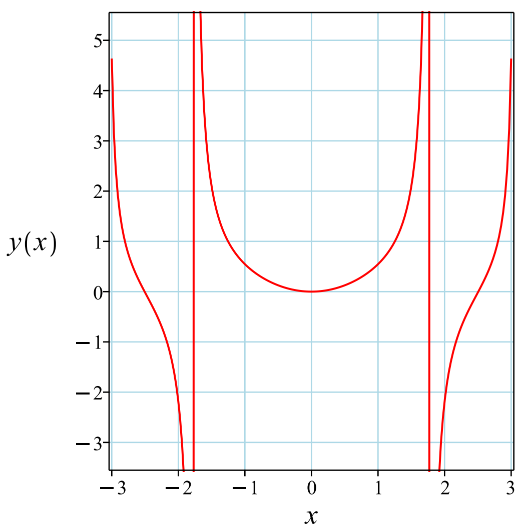
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

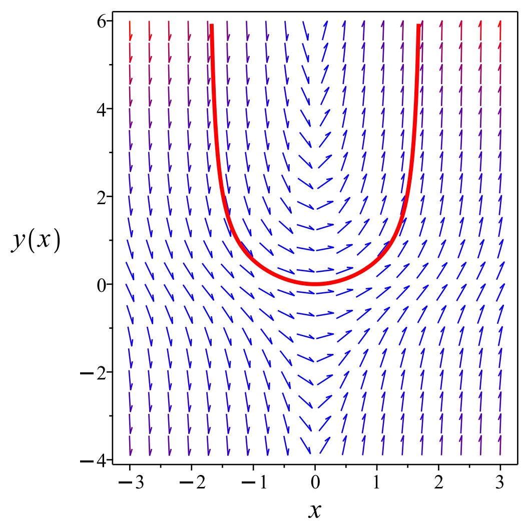
$$y = \tan\left(\frac{x^2}{2}\right)$$

Summary
The solution(s) found are the following

$$y = \tan\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^2}{2}\right)$$

Verified OK.

1.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x(y^2 + 1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\left(y^2 + 1\right)$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^2}{2} = \arctan(y) + c_1
$$

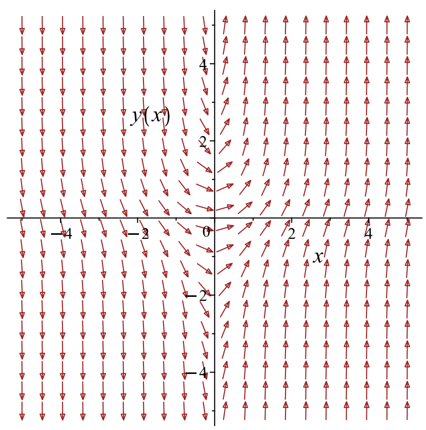
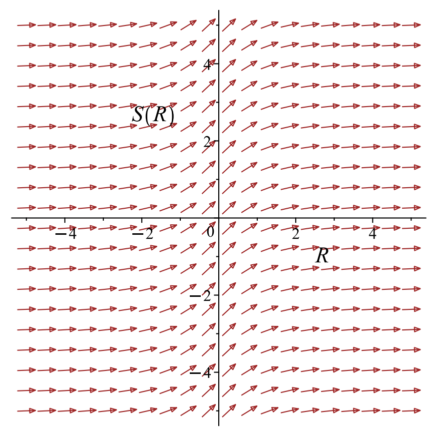
Which simplifies to

$$
\frac{x^2}{2} = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan\left(-\frac{x^2}{2} + c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x(y^2 + 1)$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan(c_1)$$

$$c_1 = 0$$

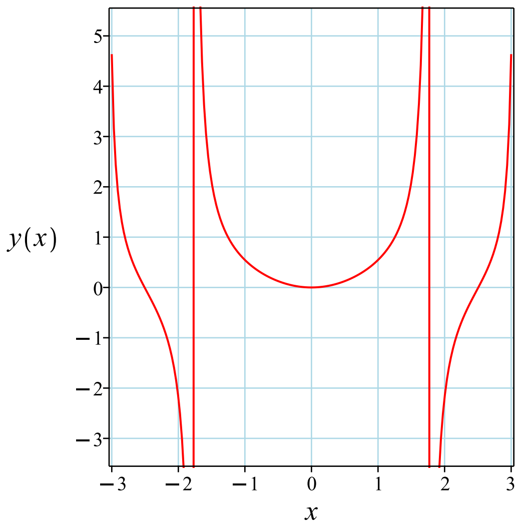
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{x^2}{2}\right)$$

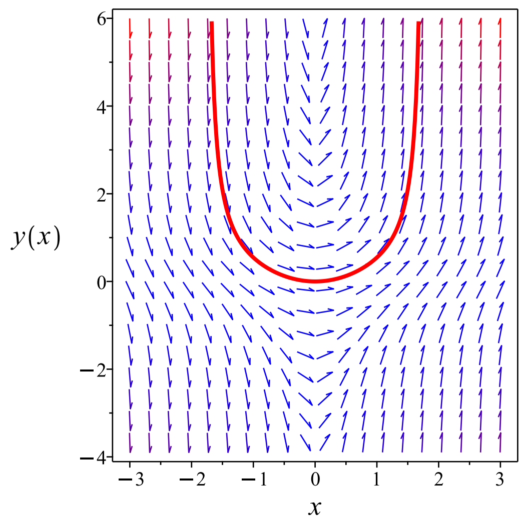
Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan\left(\frac{x^2}{2}\right)$$

Verified OK.

1.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{y^2+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2 + 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y^2 + 1}\right) \mathrm{d}y$$

$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan(c_1)$$

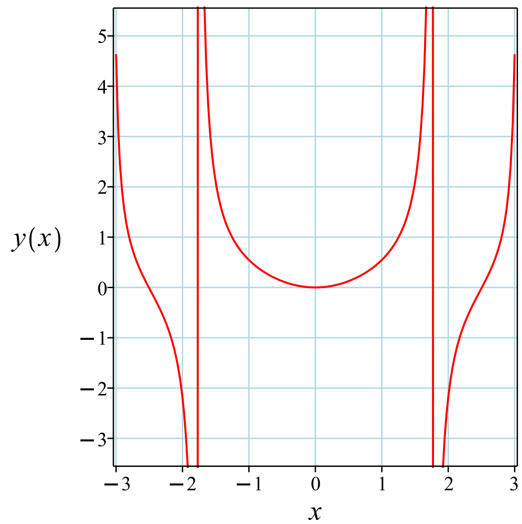
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

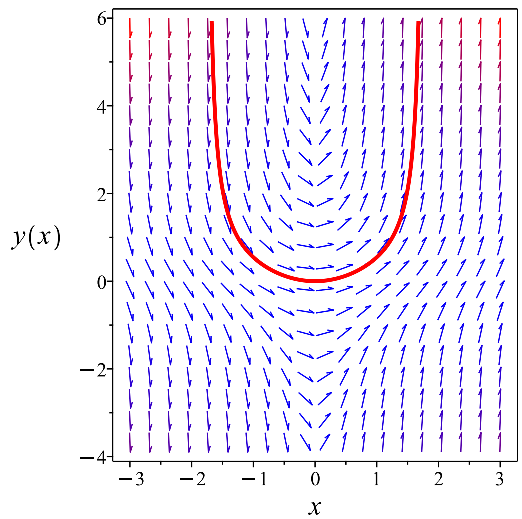
$$y = \tan\left(\frac{x^2}{2}\right)$$

Summary
The solution(s) found are the following

$$y = \tan\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan\left(\frac{x^2}{2}\right)$$

Verified OK.

1.14.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x(y^2 + 1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x\,y^2 + x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = 0$ and $f_2(x) = x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= x^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$xu''(x) - u'(x) + x^3u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)$$

The above shows that

$$u'(x) = x\left(c_1 \cos\left(\frac{x^2}{2}\right) - c_2 \sin\left(\frac{x^2}{2}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos\left(\frac{x^2}{2}\right) - c_2 \sin\left(\frac{x^2}{2}\right)}{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(\frac{x^2}{2}\right) + \sin\left(\frac{x^2}{2}\right)}{c_3 \sin\left(\frac{x^2}{2}\right) + \cos\left(\frac{x^2}{2}\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -c_3$$

$$c_3 = 0$$

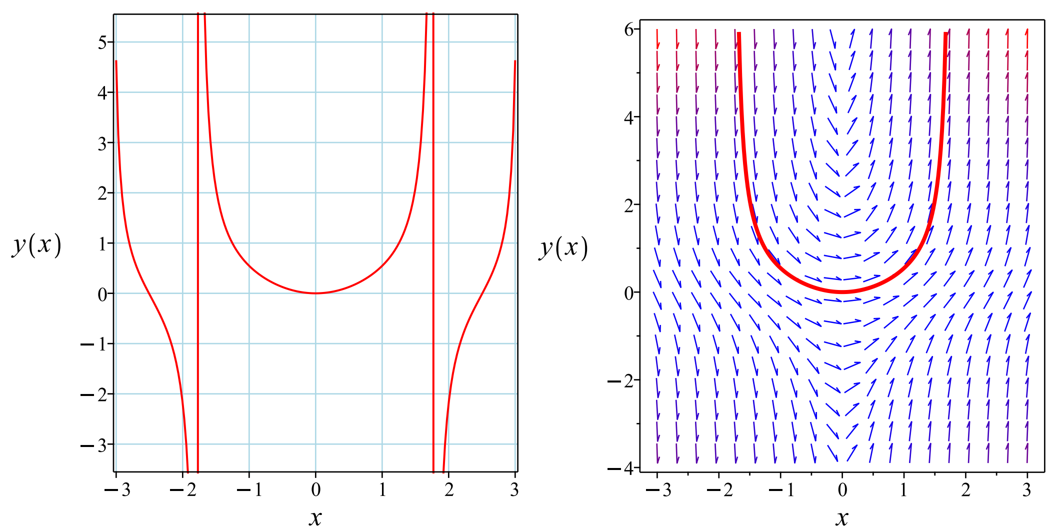
Substituting c_3 found above in the general solution gives

$$y = \frac{\sin\left(\frac{x^2}{2}\right)}{\cos\left(\frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin\left(\frac{x^2}{2}\right)}{\cos\left(\frac{x^2}{2}\right)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sin\left(\frac{x^2}{2}\right)}{\cos\left(\frac{x^2}{2}\right)}$$

Verified OK.

1.14.6 Maple step by step solution

Let's solve

$[y' - x(1 + y^2) = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int x dx + c_1$

- Evaluate integral

  $\arctan(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = \tan\left(\frac{x^2}{2} + c_1\right)$

- Use initial condition $y(0) = 0$

  $0 = \tan(c_1)$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \tan\left(\frac{x^2}{2}\right)$

- Solution to the IVP

  $y = \tan\left(\frac{x^2}{2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 10

```
dsolve([diff(y(x),x) = x*(1+y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 13

```
DSolve[{y'[x] ==x*(1+y[x]^2),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x^2}{2}\right)$$

1.15 problem 5(d)

Internal problem ID [883]
Internal file name [`OUTPUT/883_Sunday_June_05_2022_01_53_09_AM_92903899/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 5(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{y(1+y)}{x} = 0}$$

With initial conditions

$$[y(1) = -2]$$

1.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-2$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=-2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(y+1)}{x}\right) \\ &= -\frac{y+1}{x} - \frac{y}{x} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=-2$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=-2$ is inside this domain. Therefore solution exists and is unique.

1.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y(y+1)$. Integrating both sides gives

$$\frac{1}{y(y+1)}\,dy = -\frac{1}{x}\,dx$$
$$\int \frac{1}{y(y+1)}\,dy = \int -\frac{1}{x}\,dx$$
$$\ln(y) - \ln(y+1) = -\ln(x) + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y)-\ln(y+1)} = \mathrm{e}^{-\ln(x)+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = \frac{c_2}{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\frac{c_2}{-1 + c_2}$$

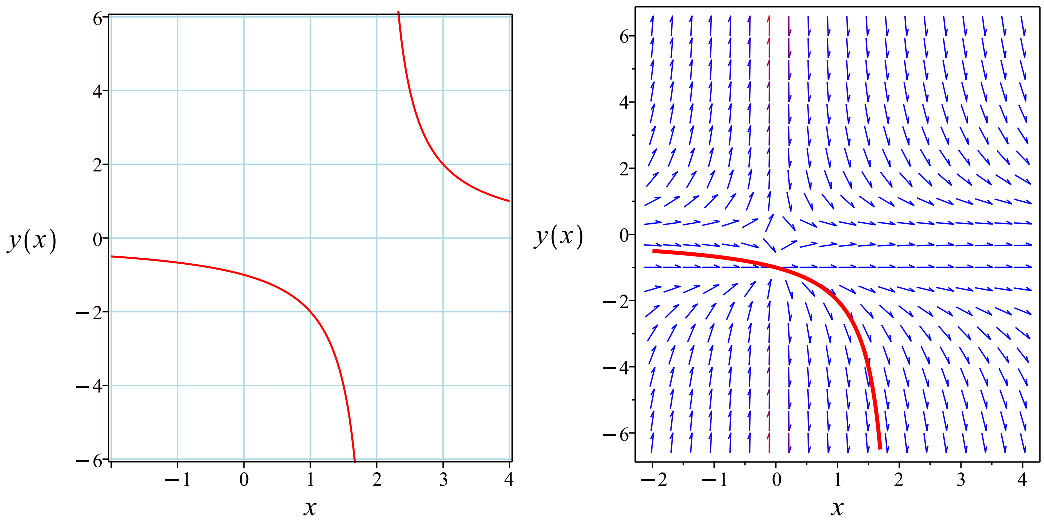
$$c_2 = 2$$

Substituting c_2 found above in the general solution gives

$$y = \frac{2}{-2 + x}$$

Summary
The solution(s) found are the following

$$y = \frac{2}{-2 + x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2}{-2+x}$$

Verified OK.

1.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y+1)}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 29: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= -x \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x} dx \end{aligned}$$

Which results in

$$S = -\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y+1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) - \ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x) = \ln(y) - \ln(1+y) + c_1$$

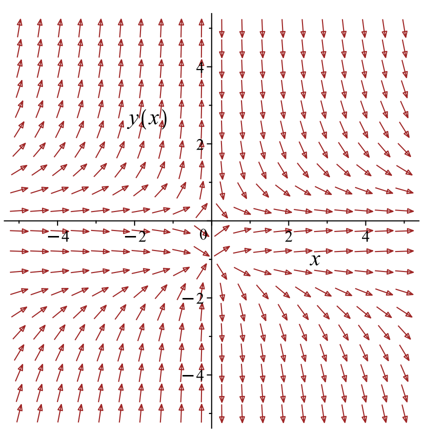
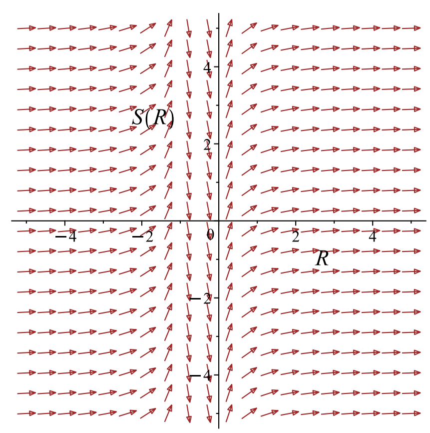
Which simplifies to

$$-\ln(x) = \ln(y) - \ln(1+y) + c_1$$

Which gives

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(y+1)}{x}$
</td><td>$R = y$
$S = -\ln(x)$</td><td>$\frac{dS}{dR} = \frac{1}{R(R+1)}$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{1}{-1 + \mathrm{e}^{c_1}}$$

$$c_1 = -\ln(2)$$

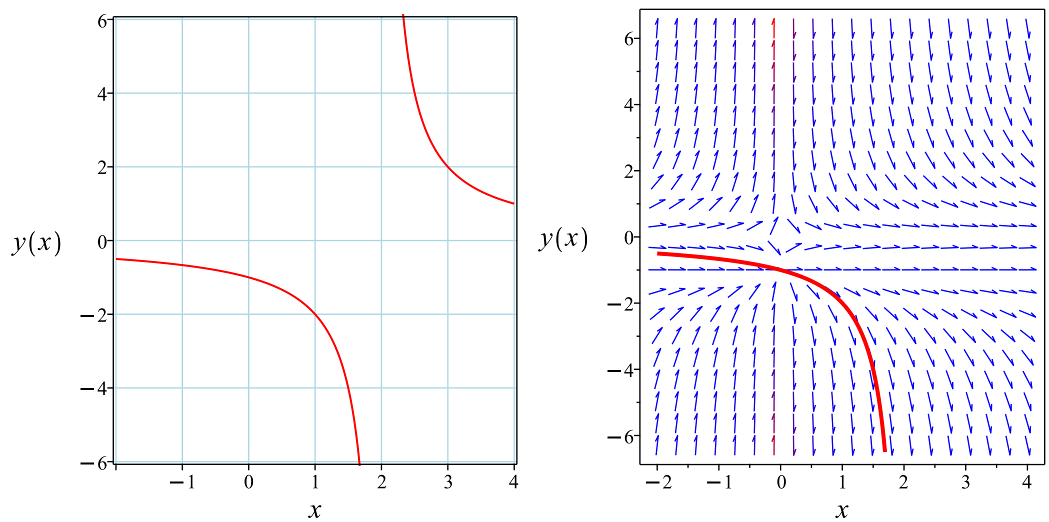
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{-2+x}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{-2+x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2}{-2+x}$$

Verified OK.

1.15.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{1}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} - \frac{1}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} - \frac{1}{x} \\ w' &= \frac{w}{x} + \frac{1}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) \\ d\left(\frac{w}{x}\right) &= \frac{1}{x^2}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x} &= \int \frac{1}{x^2}\, dx \\ \frac{w}{x} &= -\frac{1}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x - 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x - 1$$

Or

$$y = \frac{1}{c_1 x - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{1}{c_1 - 1}$$

$$c_1 = \frac{1}{2}$$

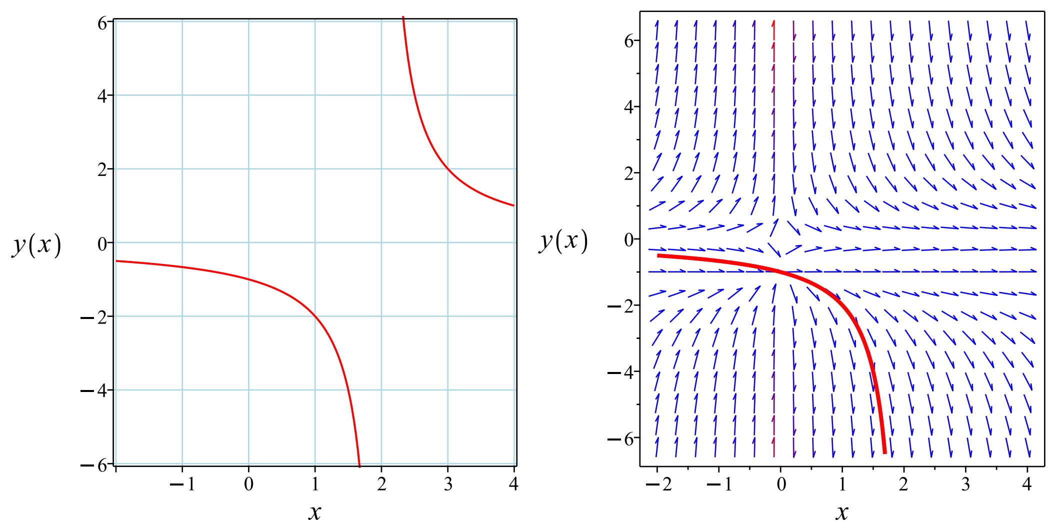
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{-2 + x}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{-2 + x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{-2 + x}$$

Verified OK.

1.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y\,(y+1)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{y\,(y+1)}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= -\frac{1}{y\,(y+1)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y\,(y+1)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y+1)}$. Therefore equation (4) becomes

$$-\frac{1}{y(y+1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y(y+1)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y(y+1)}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y) + \ln(y+1)$$

The solution becomes

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{1}{-1 + \mathrm{e}^{c_1}}$$

$$c_1 = -\ln(2)$$

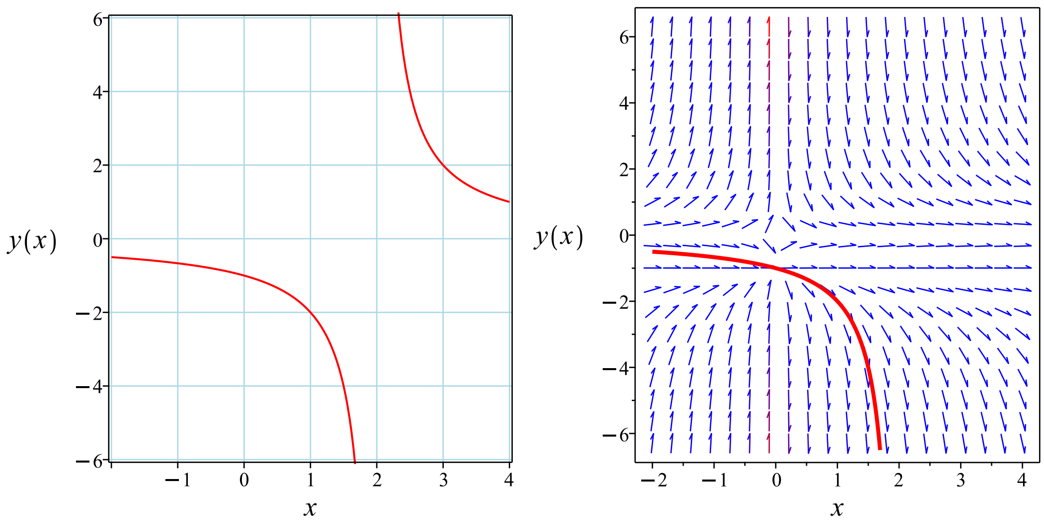
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{-2+x}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{-2+x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{-2+x}$$

Verified OK.

1.15.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} - \frac{2u'(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{x\left(c_1 + \frac{c_2}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -\frac{1}{c_3 + 1}$$

$$c_3 = -\frac{1}{2}$$

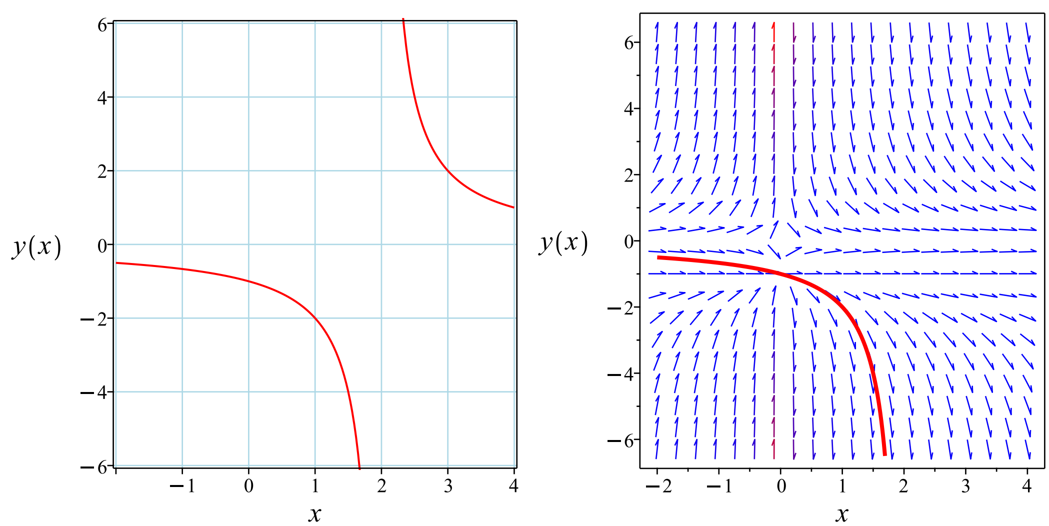
Substituting c_3 found above in the general solution gives

$$y = \frac{2}{-2 + x}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{-2 + x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2}{-2 + x}$$

Verified OK.

1.15.7 Maple step by step solution

Let's solve

$\left[y' + \frac{y(1+y)}{x} = 0, y(1) = -2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1+y)} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(1+y)} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) - \ln(1+y) = -\ln(x) + c_1$

- Solve for y

  $y = -\frac{e^{c_1}}{e^{c_1} - x}$

- Use initial condition $y(1) = -2$

  $-2 = -\frac{e^{c_1}}{-1 + e^{c_1}}$

- Solve for c_1

  $c_1 = \ln(2)$

- Substitute $c_1 = \ln(2)$ into general solution and simplify

  $y = \frac{2}{-2 + x}$

- Solution to the IVP

  $y = \frac{2}{-2 + x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) = (- y(x)*(y(x)+1))/x,y(1) = -2],y(x), singsol=all)
```

$$y(x) = \frac{2}{-2+x}$$

✓ Solution by Mathematica

Time used: 0.224 (sec). Leaf size: 12

```
DSolve[{y'[x] ==(- y[x]*(y[x]+1))/x,y[1]==-2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{x-2}$$

1.16 problem 8(a)

Internal problem ID [884]
Internal file name [`OUTPUT/884_Sunday_June_05_2022_01_53_10_AM_42109854/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 8(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - a y^{\frac{a-1}{a}} = 0}$$

1.16.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{y^{-\frac{a-1}{a}}}{a} dy = \int dx$$
$$y\, y^{-\frac{a-1}{a}} = x + c_1$$

Summary
The solution(s) found are the following

$$y\, y^{-\frac{a-1}{a}} = x + c_1 \tag{1}$$

Verification of solutions

$$y\, y^{-\frac{a-1}{a}} = x + c_1$$

Verified OK.

1.16.2 Maple step by step solution

Let's solve

$y' - a y^{\frac{a-1}{a}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{a-1}{a}}} = a$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{a-1}{a}}} dx = \int a dx + c_1$

- Evaluate integral

  $\frac{y^{-\frac{a-1}{a}+1}}{-\frac{a-1}{a}+1} = ax + c_1$

- Solve for y

  $y = \mathrm{e}^{\ln\left(\frac{ax+c_1}{a}\right)a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(diff(y(x),x) = a*y(x)^( (a-1)/a),y(x), singsol=all)
```

$$y(x) = (c_1 + x)^a$$

✓ Solution by Mathematica

Time used: 0.843 (sec). Leaf size: 28

```
DSolve[y'[x] ==a*y[x]^( (a-1)/a),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(x + \frac{c_1}{a}\right)^a$$
$$y(x) \to 0^{\frac{a}{a-1}}$$

1.17 problem 9

Internal problem ID [885]
Internal file name [`OUTPUT/885_Sunday_June_05_2022_01_53_12_AM_32529587/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - |y| = 1}$$

With initial conditions

$$[y(0) = 0]$$

1.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= |y| + 1 \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(|y| + 1) \\ &= \operatorname{abs}(1, y) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

1.17.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{|y| + 1} dy = x + c_1$$
$$\begin{cases} -\ln(-y+1) & y \leq 0 \\ \ln(y+1) & 0 < y \end{cases} = x + c_1$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= -\mathrm{e}^{-x-c_1} + 1 \\ &= \frac{-\mathrm{e}^{-x} + c_1}{c_1} \\ y_2 &= \mathrm{e}^{x+c_1} - 1 \\ &= c_1 \mathrm{e}^{x} - 1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - 1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{x} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1 - 1}{c_1}$$

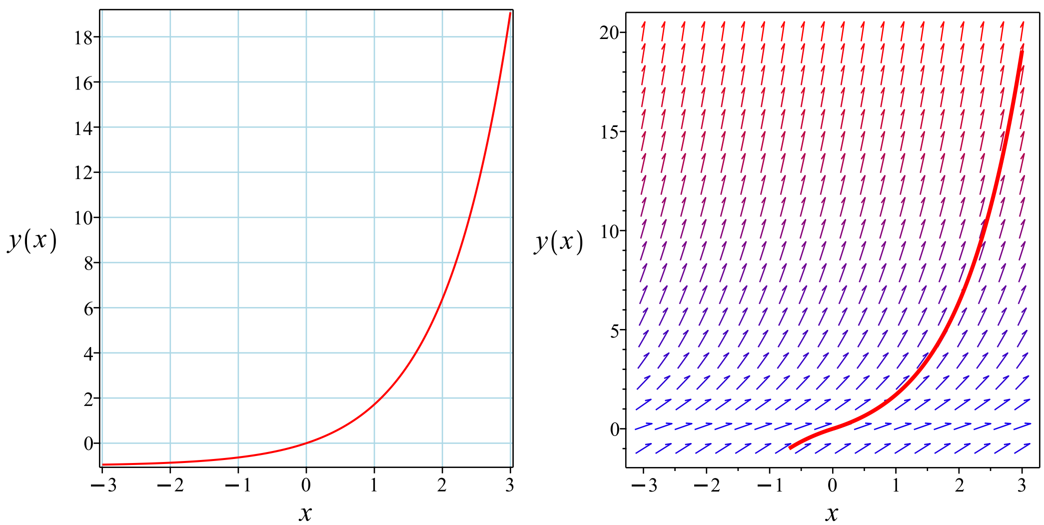
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = -e^{-x} + 1$$

Summary
The solution(s) found are the following

$$y = -e^{-x} + 1 \tag{1}$$
$$y = e^{x} - 1 \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{-x} + 1$$

Verified OK.

$$y = e^{x} - 1$$

Verified OK.

1.17.3 Maple step by step solution

Let's solve

$[y' - |y| = 1, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{|y|+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{|y|+1}dx = \int 1dx + c_1$

- Evaluate integral

  $\left\{\begin{array}{cc} -\ln(1-y) & y \leq 0 \\ \ln(1+y) & 0 < y \end{array}\right. = x + c_1$

- Solve for y

  $\{y = -e^{-x-c_1} + 1, y = e^{x+c_1} - 1\}$

- Use initial condition $y(0) = 0$

  $0 = -e^{-c_1} + 1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = -e^{-x} + 1$

- Use initial condition $y(0) = 0$

  $0 = -1 + e^{c_1}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = e^{x} - 1$

- Solutions to the IVP

  $\{y = e^{x} - 1, y = -e^{-x} + 1\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 19

```
dsolve([diff(y(x),x) = abs(y(x))+1,y(0) = 0],y(x), singsol=all)
```

$$y(x) = e^{x} - 1$$
$$y(x) = 1 - e^{-x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'[x] ==Abs[y[x]]+1,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

{}

1.18 problem 10(a)

Internal problem ID [886]
Internal file name [`OUTPUT/886_Sunday_June_05_2022_01_53_16_AM_27410281/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 1, Introduction. Section 1.2 Page 14
Problem number: 10(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y' - \frac{\sqrt{x^2+4x+4y}}{2} = -\frac{x}{2} - 1}$$

1.18.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -1 - \frac{x}{2} + \frac{\sqrt{x^2+4x+4y}}{2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\left(-1-\frac{x}{2}+\frac{\sqrt{x^2+4x+4y}}{2}\right)(b_3-a_2)-\left(-1-\frac{x}{2}+\frac{\sqrt{x^2+4x+4y}}{2}\right)^2 a_3 \\ -\left(-\frac{1}{2}+\frac{2x+4}{4\sqrt{x^2+4x+4y}}\right)(xa_2+ya_3+a_1)-\frac{xb_2+yb_3+b_1}{\sqrt{x^2+4x+4y}}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4a_1+4b_1+4x^2a_2+2xa_1+(x^2+4x+4y)^{\frac{3}{2}}a_3-2\sqrt{x^2+4x+4y}\,a_1-4\sqrt{x^2+4x+4y}\,a_2+4a_3\sqrt{x^2+}}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}&-4a_1-4b_1-4x^2a_2-2xa_1-\left(x^2+4x+4y\right)^{\frac{3}{2}}a_3\\ &+2\sqrt{x^2+4x+4y}\,a_1+4\sqrt{x^2+4x+4y}\,a_2-4a_3\sqrt{x^2+4x+4y}\\ &+4b_2\sqrt{x^2+4x+4y}-4\sqrt{x^2+4x+4y}\,b_3+2x^3a_3+12x^2a_3\\ &-8ya_2+2x^2b_3+8xb_3-12xa_2+12ya_3-4xb_2+4yb_3+6xya_3\\ &-\sqrt{x^2+4x+4y}\,x^2a_3+4\sqrt{x^2+4x+4y}\,xa_2-4\sqrt{x^2+4x+4y}\,xa_3\\ &-2\sqrt{x^2+4x+4y}\,xb_3+2\sqrt{x^2+4x+4y}\,ya_3+16xa_3=0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&-\left(x^2+4x+4y\right)^{\frac{3}{2}}a_3+2\left(x^2+4x+4y\right)xa_3-\sqrt{x^2+4x+4y}\,x^2a_3\\ &-2\left(x^2+4x+4y\right)a_2+4\left(x^2+4x+4y\right)a_3+2\left(x^2+4x+4y\right)b_3\\ &+4\sqrt{x^2+4x+4y}\,xa_2-4\sqrt{x^2+4x+4y}\,xa_3-2\sqrt{x^2+4x+4y}\,xb_3\\ &+2\sqrt{x^2+4x+4y}\,ya_3-2x^2a_2-2xya_3+2\sqrt{x^2+4x+4y}\,a_1\\ &+4\sqrt{x^2+4x+4y}\,a_2-4a_3\sqrt{x^2+4x+4y}+4b_2\sqrt{x^2+4x+4y}\\ &-4\sqrt{x^2+4x+4y}\,b_3-2xa_1-4xa_2-4xb_2-4ya_3-4yb_3-4a_1-4b_1=0\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-4a_1 - 4b_1 - 4x^2a_2 - 2xa_1 + 2\sqrt{x^2+4x+4y}\,a_1 + 4\sqrt{x^2+4x+4y}\,a_2 \\
&- 4a_3\sqrt{x^2+4x+4y} + 4b_2\sqrt{x^2+4x+4y} - 4\sqrt{x^2+4x+4y}\,b_3 + 2x^3a_3 \\
&+ 12x^2a_3 - 8ya_2 + 2x^2b_3 + 8xb_3 - 12xa_2 + 12ya_3 - 4xb_2 + 4yb_3 + 6xya_3 \\
&- 2\sqrt{x^2+4x+4y}\,x^2a_3 + 4\sqrt{x^2+4x+4y}\,xa_2 - 8\sqrt{x^2+4x+4y}\,xa_3 \\
&- 2\sqrt{x^2+4x+4y}\,xb_3 - 2\sqrt{x^2+4x+4y}\,ya_3 + 16xa_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+4x+4y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^2+4x+4y} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^3a_3 - 2v_3v_1^2a_3 - 4v_1^2a_2 + 4v_3v_1a_2 + 12v_1^2a_3 + 6v_1v_2a_3 - 8v_3v_1a_3 - 2v_3v_2a_3 \\
&+ 2v_1^2b_3 - 2v_3v_1b_3 - 2v_1a_1 + 2v_3a_1 - 12v_1a_2 - 8v_2a_2 + 4v_3a_2 + 16v_1a_3 \\
&+ 12v_2a_3 - 4a_3v_3 - 4v_1b_2 + 4b_2v_3 + 8v_1b_3 + 4v_2b_3 - 4v_3b_3 - 4a_1 - 4b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^3a_3 - 2v_3v_1^2a_3 + (-4a_2 + 12a_3 + 2b_3)\,v_1^2 + 6v_1v_2a_3 \\
&+ (4a_2 - 8a_3 - 2b_3)\,v_1v_3 + (-2a_1 - 12a_2 + 16a_3 - 4b_2 + 8b_3)\,v_1 - 2v_3v_2a_3 \\
&+ (-8a_2 + 12a_3 + 4b_3)\,v_2 + (2a_1 + 4a_2 - 4a_3 + 4b_2 - 4b_3)\,v_3 - 4a_1 - 4b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_3 &= 0\\
2a_3 &= 0\\
6a_3 &= 0\\
-4a_1 - 4b_1 &= 0\\
-8a_2 + 12a_3 + 4b_3 &= 0\\
-4a_2 + 12a_3 + 2b_3 &= 0\\
4a_2 - 8a_3 - 2b_3 &= 0\\
-2a_1 - 12a_2 + 16a_3 - 4b_2 + 8b_3 &= 0\\
2a_1 + 4a_2 - 4a_3 + 4b_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2a_2 - 2b_2\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= -2a_2 + 2b_2\\
b_2 &= b_2\\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2\\
\eta &= 2 + x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= 2 + x - \left(-1 - \frac{x}{2} + \frac{\sqrt{x^2 + 4x + 4y}}{2}\right)(-2)\\
&= \sqrt{x^2 + 4x + 4y}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2 + 4x + 4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\sqrt{x^2 + 4x + 4y}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -1 - \frac{x}{2} + \frac{\sqrt{x^2 + 4x + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2 + x}{2\sqrt{x^2 + 4x + 4y}} \\ S_y &= \frac{1}{\sqrt{x^2 + 4x + 4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2+4x+4y}}{2} = \frac{x}{2} + c_1$$

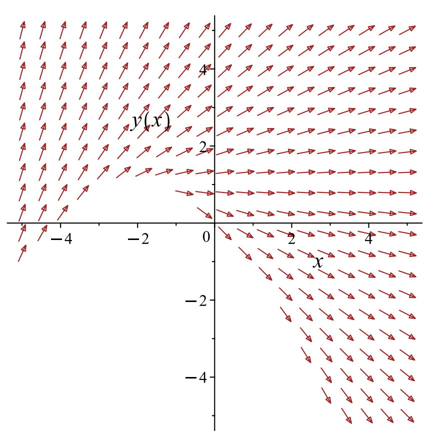
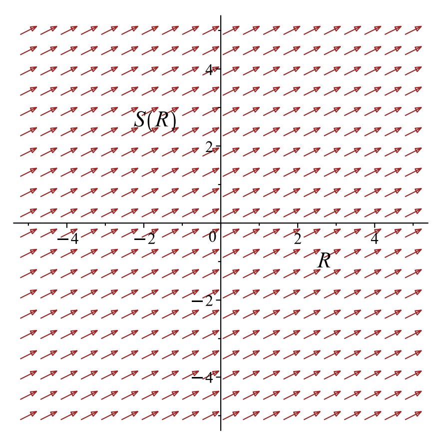
Which simplifies to

$$\frac{\sqrt{x^2+4x+4y}}{2} = \frac{x}{2} + c_1$$

Which gives

$$y = c_1^2 + c_1 x - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -1 - \frac{x}{2} + \frac{\sqrt{x^2+4x+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2+4x+4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary
The solution(s) found are the following

$$y = c_1^2 + c_1 x - x \tag{1}$$

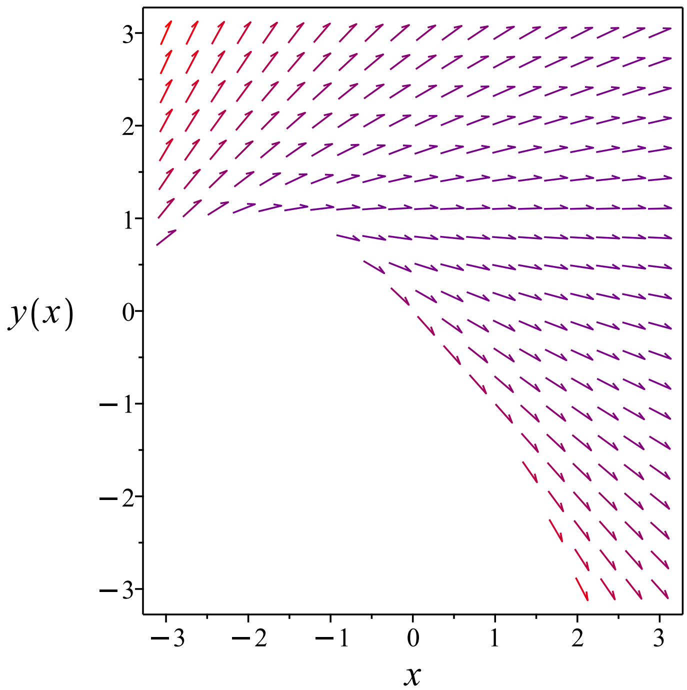


Figure 40: Slope field plot

Verification of solutions

$$y = c_1^2 + c_1 x - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1-(1/2)*x, y(x)`      *** Sublevel 2 *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x) = 1/2*(-(x+2)+sqrt(x^2+4*x+4*y(x))),y(x), singsol=all)
```

$$x-\sqrt{x^2+4x+4y(x)}-c_1=0$$

✓ Solution by Mathematica

Time used: 0.801 (sec). Leaf size: 47

```
DSolve[y'[x] ==1/2*(-(x+2)+Sqrt[x^2+4*x+4*y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{1}{4}\left(-2x+2e^{c_1}(x+1)+1+e^{2c_1}\right)$$
$$y(x)\to 1$$
$$y(x)\to \frac{1}{4}(1-2x)$$

2 Chapter 2, First order equations. Linear first order. Section 2.1 Page 41

2.1 problem 1

Internal problem ID [887]
Internal file name [`OUTPUT/887_Sunday_June_05_2022_01_53_19_AM_9951064/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + ay = 0}$$

2.1.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{ay} dy = \int dx$$
$$-\frac{\ln(y)}{a} = x + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(y)}{a}} = e^{x+c_1}$$

Which simplifies to

$$y^{-\frac{1}{a}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = (c_2 e^x)^{-a} \tag{1}$$

Verification of solutions

$$y = (c_2 e^{x})^{-a}$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$y' + ay = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -a$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -a dx + c_1$

- Evaluate integral

  $\ln(y) = -ax + c_1$

- Solve for y

  $y = e^{-ax+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) + a*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-ax}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 19

```
DSolve[y'[x] + a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ax}$$
$$y(x) \to 0$$

2.2 problem 2

Internal problem ID [888]
Internal file name [`OUTPUT/888_Sunday_June_05_2022_01_53_19_AM_56214758/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 3x^2 y = 0}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -3y\,x^2 \end{aligned}$$

Where $f(x) = -3x^2$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -3x^2\,dx \\
\int \frac{1}{y}\,dy &= \int -3x^2\,dx \\
\ln(y) &= -x^3 + c_1 \\
y &= \mathrm{e}^{-x^3+c_1} \\
&= c_1 \mathrm{e}^{-x^3}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^3} \tag{1}$$

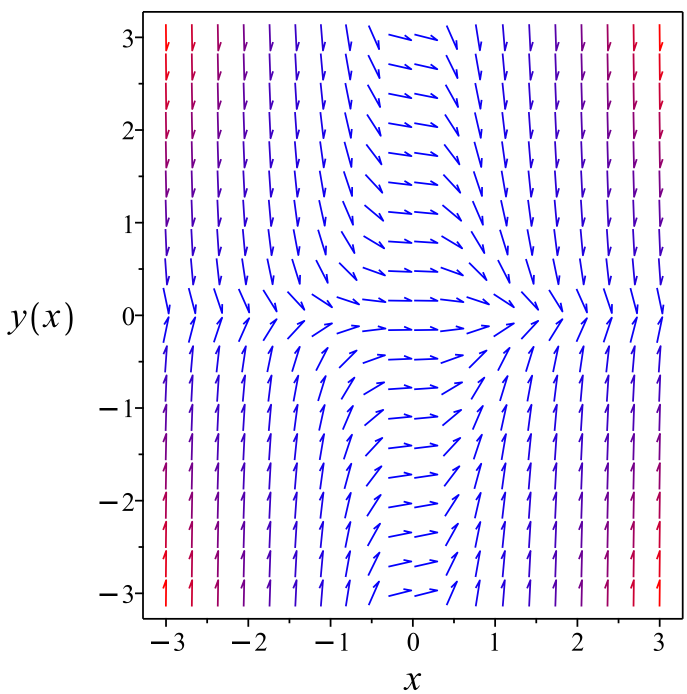


Figure 41: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^3}$$

Verified OK.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3x^2 \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + 3x^2y = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 3x^2 dx} \\ &= \mathrm{e}^{x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^3} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{x^3} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^3}$ results in

$$y = c_1 \mathrm{e}^{-x^3}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^3} \tag{1}$$

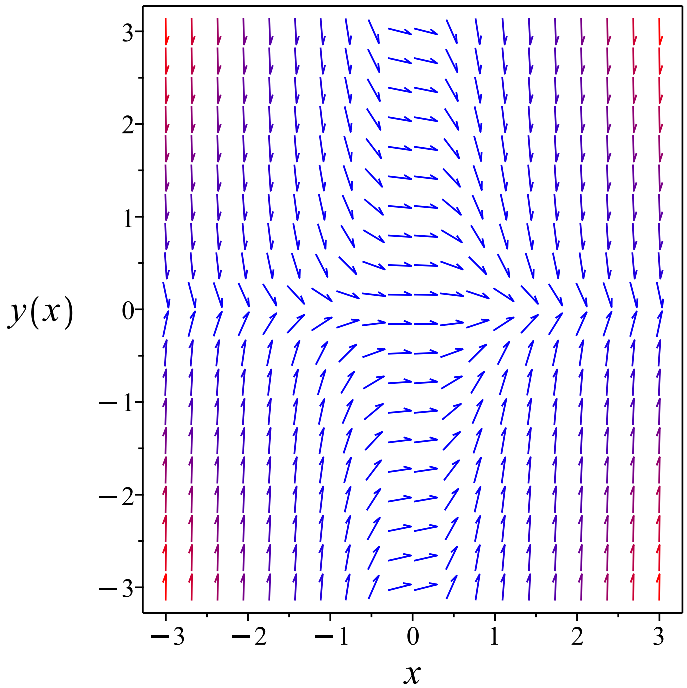


Figure 42: Slope field plot

Verification of solutions

$$y = c_1 e^{-x^3}$$

Verified OK.

2.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + 3x^3 u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(3x^3+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{3x^3+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3x^3+1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3x^3+1}{x}\, dx \\
\ln(u) &= -x^3 - \ln(x) + c_2 \\
u &= \mathrm{e}^{-x^3-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{-x^3-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{-x^3}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{-x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{-x^3} \tag{1}
$$

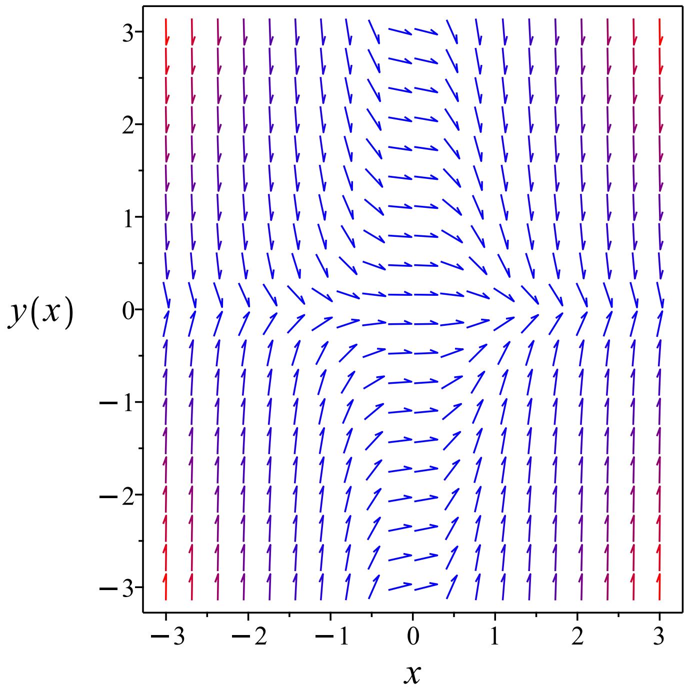


Figure 43: Slope field plot

Verification of solutions

$$y = c_2 \mathrm{e}^{-x^3}$$

Verified OK.

2.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -3y\,x^2 \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x^3}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x^3}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x^3} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -3y\, x^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3x^2 \mathrm{e}^{x^3} y \\ S_y &= \mathrm{e}^{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^3} y = c_1$$

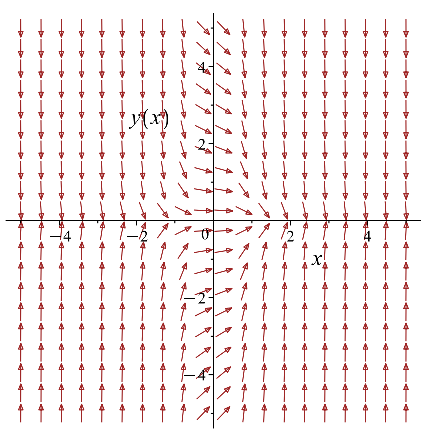
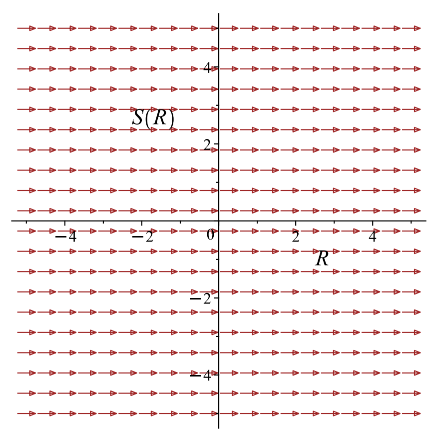
Which simplifies to

$$e^{x^3} y = c_1$$

Which gives

$$y = c_1 e^{-x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3y\,x^2$ 	$R = x$ $S = e^{x^3} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{-x^3} \tag{1}$$

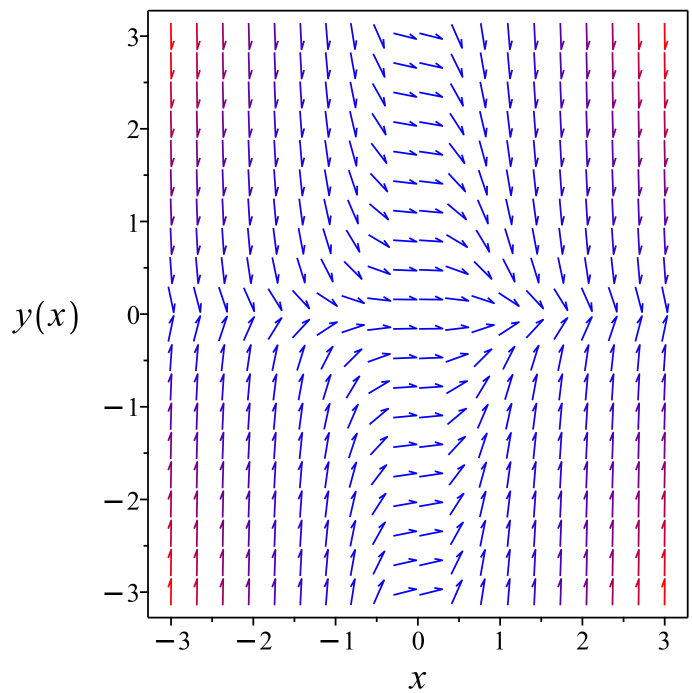


Figure 44: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x^3}$$

Verified OK.

2.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{3y}\right)\mathrm{d}y = \left(x^2\right)\mathrm{d}x$$
$$\left(-x^2\right)\mathrm{d}x + \left(-\frac{1}{3y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x^2$$
$$N(x,y) = -\frac{1}{3y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{3y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^2\,\mathrm{d}x\\ \phi &= -\frac{x^3}{3} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{3y}$. Therefore equation (4) becomes

$$-\frac{1}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{3y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{3y}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{\ln(y)}{3}$$

The solution becomes

$$y = \mathrm{e}^{-x^3-3c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x^3-3c_1} \tag{1}$$

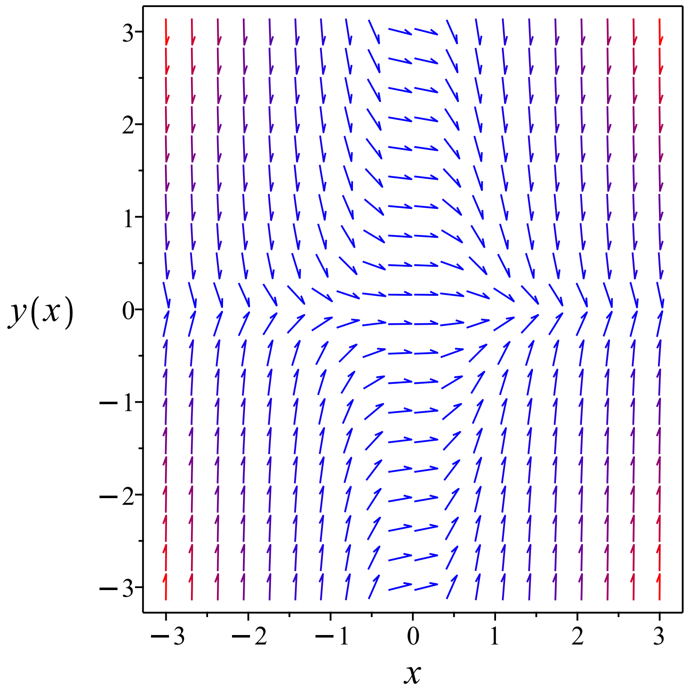


Figure 45: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x^3-3c_1}$$

Verified OK.

2.2.6 Maple step by step solution

Let's solve

$y' + 3x^2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -3x^2dx + c_1$

- Evaluate integral

$$\ln(y) = -x^3 + c_1$$

- Solve for y

$$y = e^{-x^3+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) + 3*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^3}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 20

```
DSolve[y'[x] +3*x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x^3}$$
$$y(x) \to 0$$

2.3 problem 3

Internal problem ID [889]
Internal file name [`OUTPUT/889_Sunday_June_05_2022_01_53_21_AM_54030334/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x + \ln(x)\, y = 0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\ln(x)\, y}{x} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{\ln(x)}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{\ln(x)}{x}\,dx \\
\ln(y) &= -\frac{\ln(x)^2}{2} + c_1 \\
y &= \mathrm{e}^{-\frac{\ln(x)^2}{2} + c_1} \\
&= c_1 \mathrm{e}^{-\frac{\ln(x)^2}{2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{\ln(x)^2}{2}} \tag{1}$$

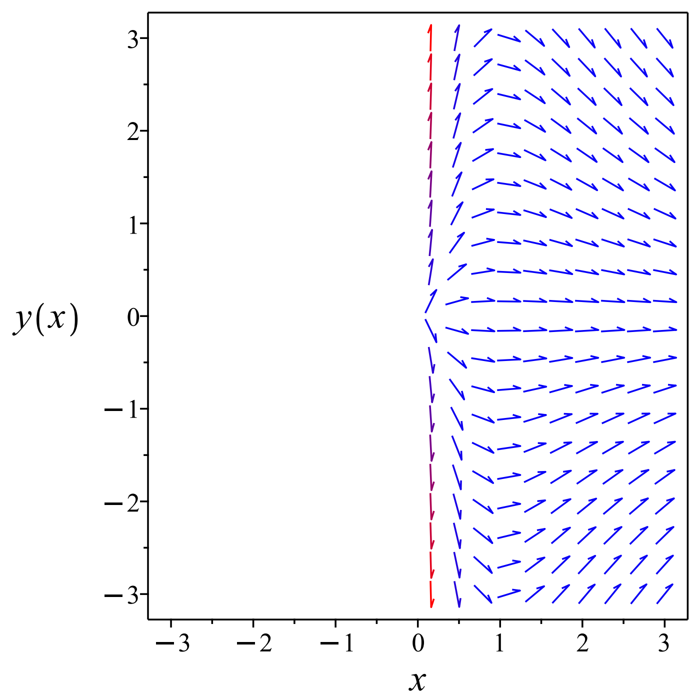


Figure 46: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{\ln(x)^2}{2}}$$

Verified OK.

2.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{\ln (x)}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{\ln (x)\, y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{\ln(x)}{x}dx} \\ &= e^{\frac{\ln(x)^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{\ln(x)^2}{2}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{\frac{\ln(x)^2}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{\ln(x)^2}{2}}$ results in

$$y = c_1 e^{-\frac{\ln(x)^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{\ln(x)^2}{2}} \tag{1}$$

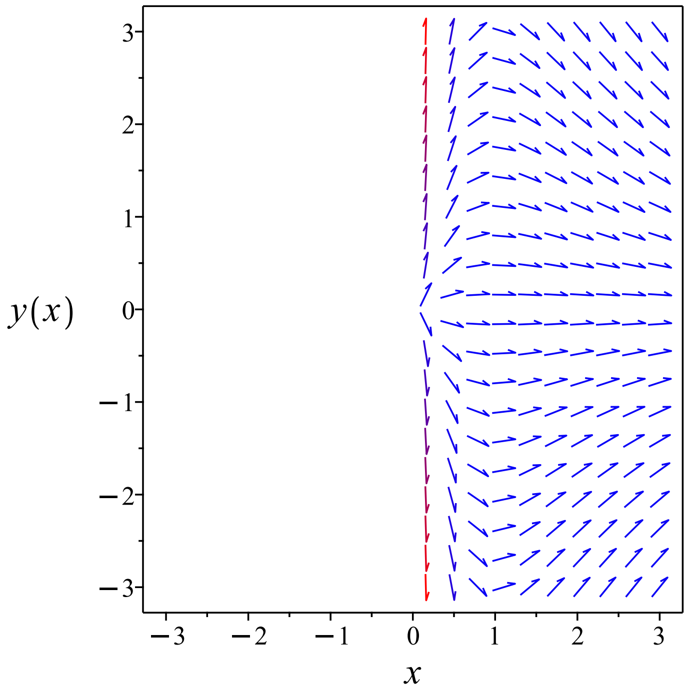


Figure 47: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{\ln(x)^2}{2}}$$

Verified OK.

2.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + \ln(x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\ln(x)+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{\ln(x)+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{\ln(x)+1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\ln(x)+1}{x}\,dx \\
\ln(u) &= -\frac{\ln(x)^2}{2} - \ln(x) + c_2 \\
u &= e^{-\frac{\ln(x)^2}{2} - \ln(x) + c_2} \\
&= c_2 e^{-\frac{\ln(x)^2}{2} - \ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 e^{-\frac{\ln(x)^2}{2}}}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 e^{-\frac{\ln(x)^2}{2}}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 e^{-\frac{\ln(x)^2}{2}} \tag{1}$$

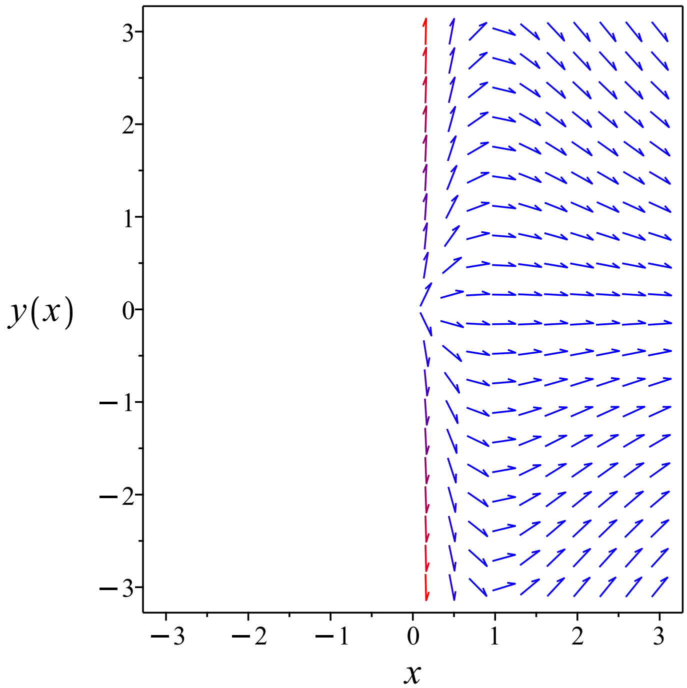


Figure 48: Slope field plot

Verification of solutions

$$y = c_2 e^{-\frac{\ln(x)^2}{2}}$$

Verified OK.

2.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\ln(x)\, y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 38: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \mathrm{e}^{-\frac{\ln(x)^2}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{\ln(x)^2}{2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{\ln(x)^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\ln(x)\, y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\ln(x)\, \mathrm{e}^{\frac{\ln(x)^2}{2}} y}{x} \\ S_y &= \mathrm{e}^{\frac{\ln(x)^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{\ln(x)^2}{2}} y = c_1$$

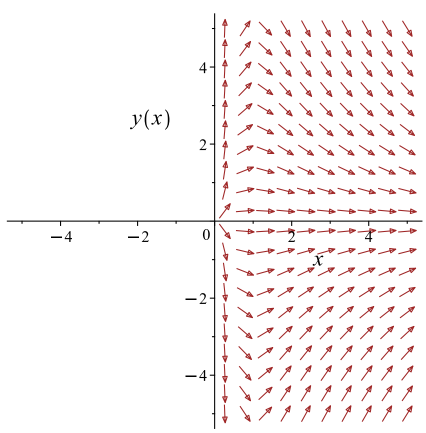
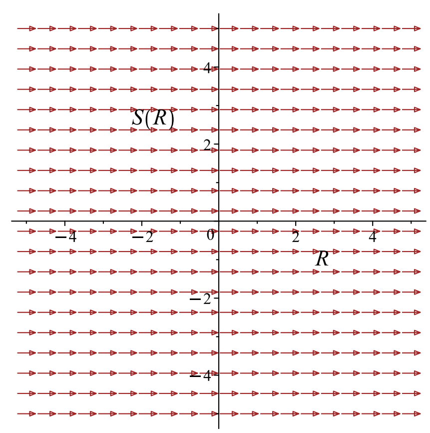
Which simplifies to

$$e^{\frac{\ln(x)^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{-\frac{\ln(x)^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{\ln(x)y}{x}$

</td><td>$R = x$
$S = e^{\frac{\ln(x)^2}{2}} y$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{\ln(x)^2}{2}} \tag{1}$$

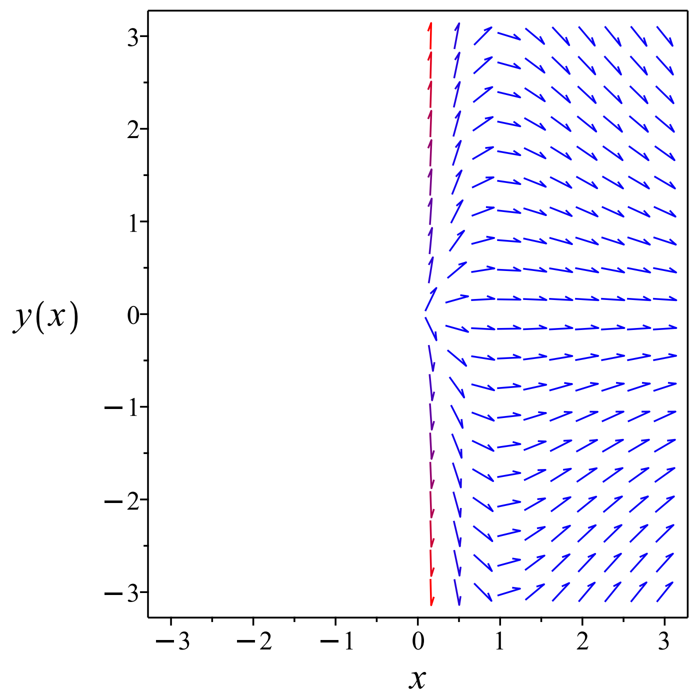


Figure 49: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{\ln(x)^2}{2}}$$

Verified OK.

2.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = \left(\frac{\ln (x)}{x}\right) \mathrm{d}x$$
$$\left(-\frac{\ln (x)}{x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\ln (x)}{x}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\ln (x)}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\ln(x)}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x)^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right)\mathrm{d}y\\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x)^2}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x)^2}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(x)^2}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\ln(x)^2}{2} - c_1} \tag{1}$$

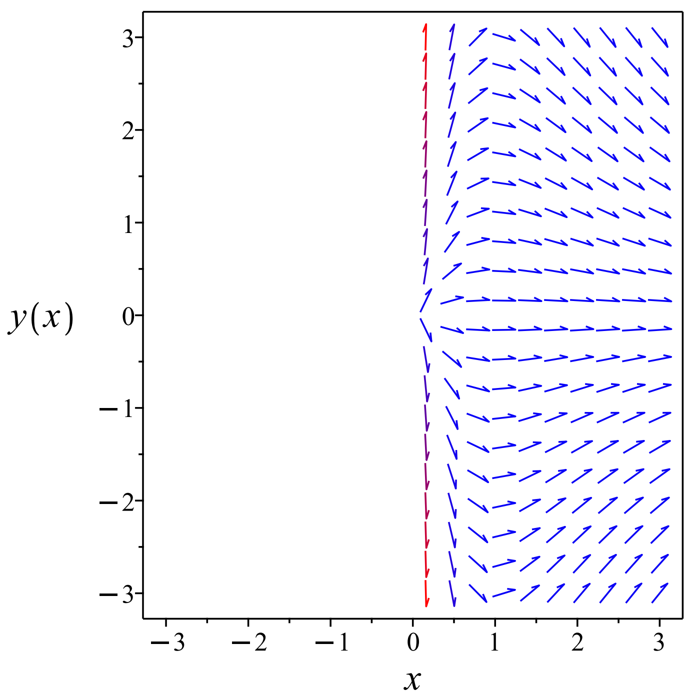


Figure 50: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x)^2}{2}-c_1}$$

Verified OK.

2.3.6 Maple step by step solution

Let's solve

$y'x + \ln(x)\,y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{\ln(x)}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{\ln(x)}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{\ln(x)^2}{2} + c_1$

- Solve for y

  $y = e^{-\frac{\ln(x)^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x) + ln(x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{\ln(x)^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 23

```
DSolve[x*y'[x] +Log[x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{1}{2}\log^2(x)}$$
$$y(x) \to 0$$

2.4 problem 4

Internal problem ID [890]
Internal file name [`OUTPUT/890_Sunday_June_05_2022_01_53_22_AM_23926869/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{3y + y'x = 0}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{3y}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{3}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{3}{x}\,dx \\
\ln(y) &= -3\ln(x) + c_1 \\
y &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

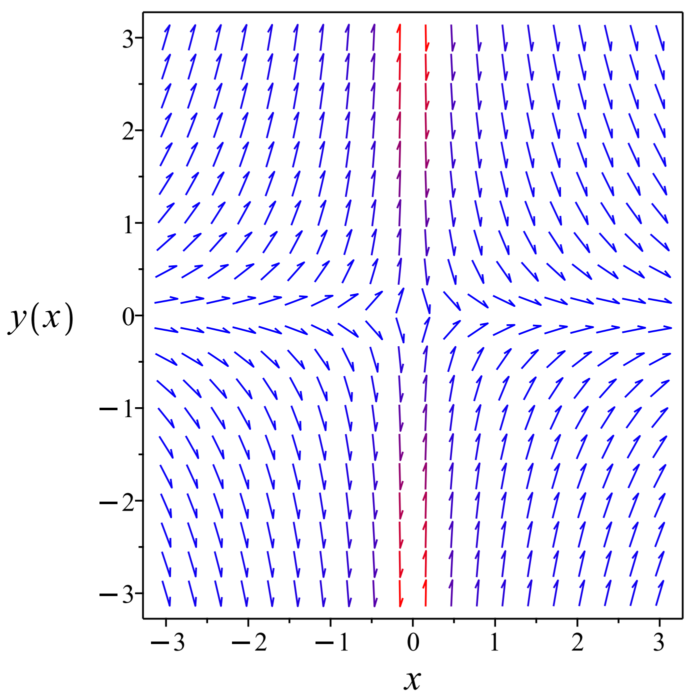


Figure 51: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.

2.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{3}{x} \\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,x^3\right) &= 0\end{aligned}$$

Integrating gives

$$y\,x^3 = c_1$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

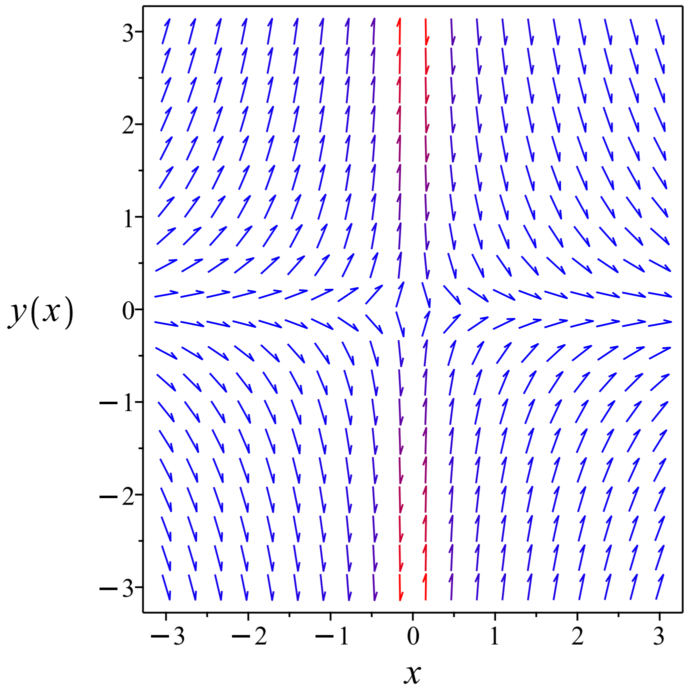


Figure 52: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.

2.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)\,x + (u'(x)\,x + u(x))\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{4}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{4}{x}\, dx \\
\ln(u) &= -4\ln(x) + c_2 \\
u &= \mathrm{e}^{-4\ln(x)+c_2} \\
&= \frac{c_2}{x^4}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x^3} \tag{1}
$$

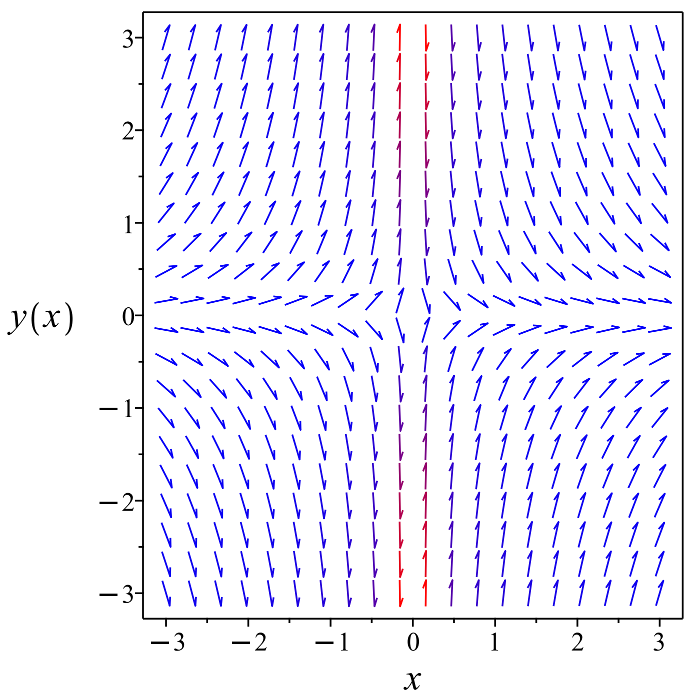


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^3}$$

Verified OK.

2.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\,x^2 \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = c_1$$

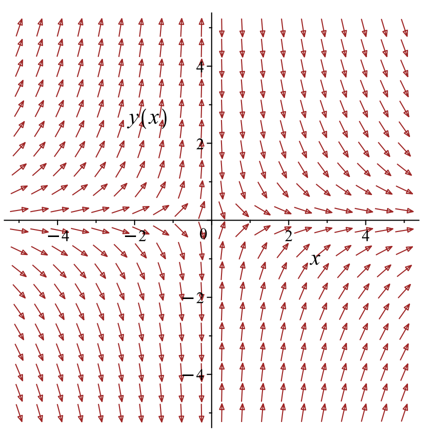
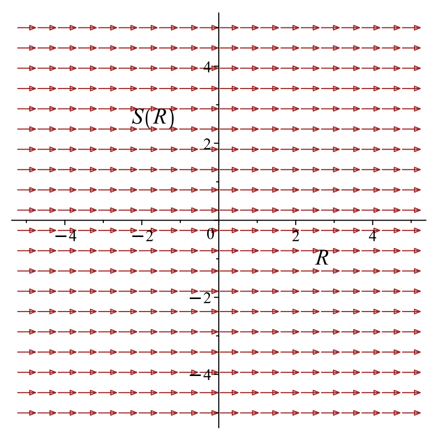
Which simplifies to

$$y x^3 = c_1$$

Which gives

$$y = \frac{c_1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y}{x}$ 	$R = x$ $S = y\, x^3$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

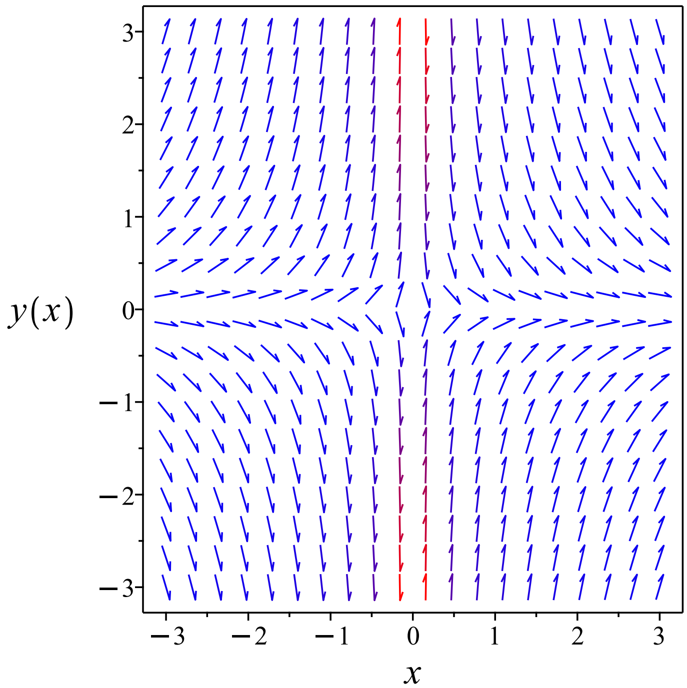


Figure 54: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.

2.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{3y}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{3y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = -\frac{1}{3y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{3y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\\ \phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{3y}$. Therefore equation (4) becomes

$$-\frac{1}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{3y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{3y}\right)\mathrm{d}y\\ f(y) &= -\frac{\ln(y)}{3} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{3}$$

The solution becomes

$$y = \frac{e^{-3c_1}}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-3c_1}}{x^3} \tag{1}$$

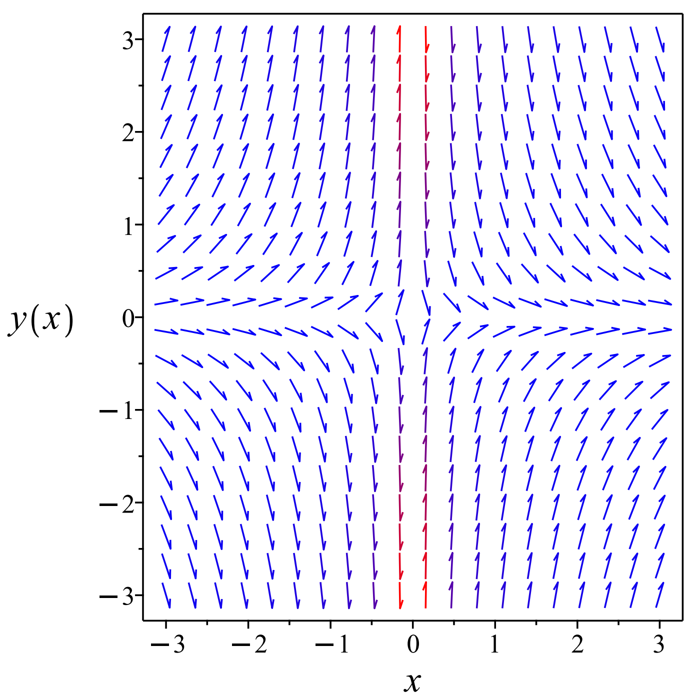


Figure 55: Slope field plot

Verification of solutions

$$y = \frac{e^{-3c_1}}{x^3}$$

Verified OK.

2.4.6 Maple step by step solution

Let's solve

$3y + y'x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{3}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{3}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = -3\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x) + 3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[x*y'[x] +3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^3}$$
$$y(x) \to 0$$

2.5 problem 5

Internal problem ID [891]
Internal file name [`OUTPUT/891_Sunday_June_05_2022_01_53_23_AM_1052067/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x^2 + y = 0}$$

2.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= -\frac{1}{x^2}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{1}{x^2}\, dx \\
\ln(y) &= \frac{1}{x} + c_1 \\
y &= e^{\frac{1}{x}+c_1} \\
&= c_1 e^{\frac{1}{x}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{1}{x}} \tag{1}$$

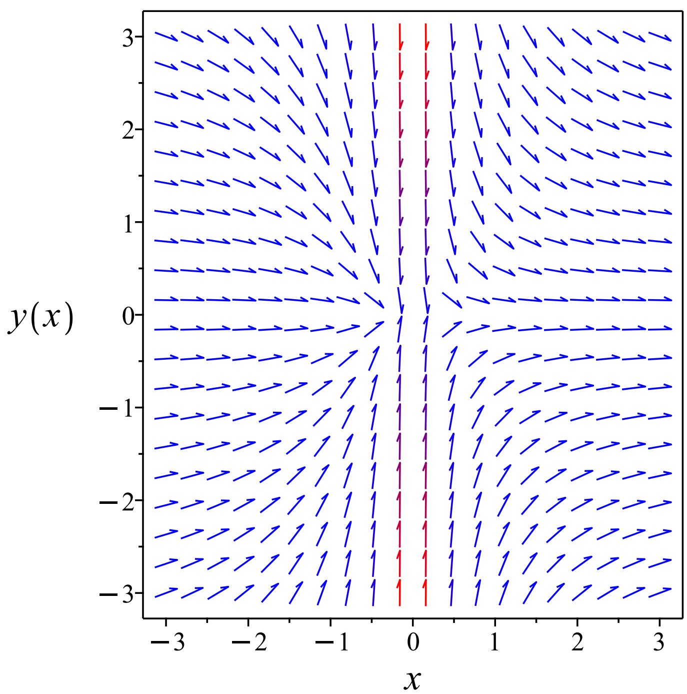


Figure 56: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{1}{x}}$$

Verified OK.

2.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x^2} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x^2} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x^2} dx} \\ &= \mathrm{e}^{-\frac{1}{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{1}{x}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{1}{x}} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{1}{x}}$ results in

$$y = c_1 \mathrm{e}^{\frac{1}{x}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{1}{x}} \tag{1}$$

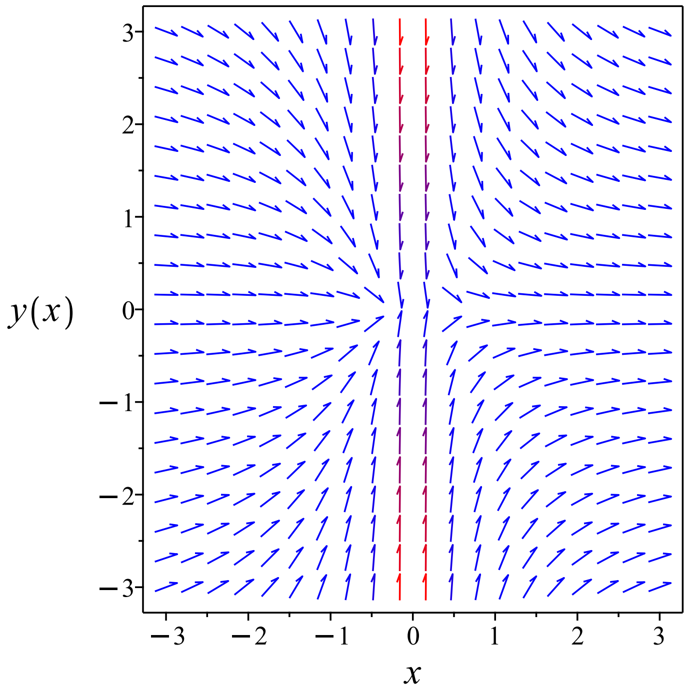


Figure 57: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{1}{x}}$$

Verified OK.

2.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 + u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x+1)}{x^2}
\end{aligned}$$

Where $f(x) = -\frac{x+1}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{x+1}{x^2}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x+1}{x^2}\,dx \\
\ln(u) &= -\ln(x) + \frac{1}{x} + c_2 \\
u &= \mathrm{e}^{-\ln(x)+\frac{1}{x}+c_2} \\
&= c_2 \mathrm{e}^{-\ln(x)+\frac{1}{x}}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{1}{x}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= c_2 \mathrm{e}^{\frac{1}{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{1}{x}} \tag{1}
$$

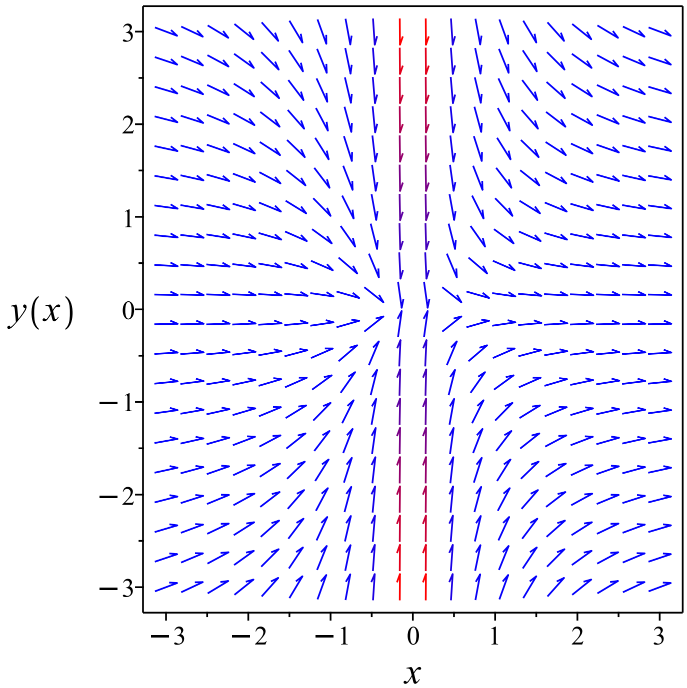


Figure 58: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{1}{x}}$$

Verified OK.

2.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = e^{\frac{1}{x}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{x}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{1}{x}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{-\frac{1}{x}} y}{x^2} \\ S_y &= e^{-\frac{1}{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{1}{x}} y = c_1$$

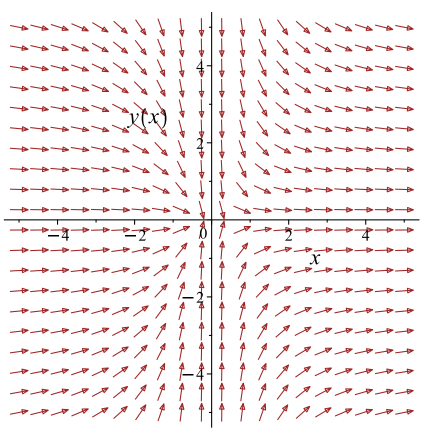
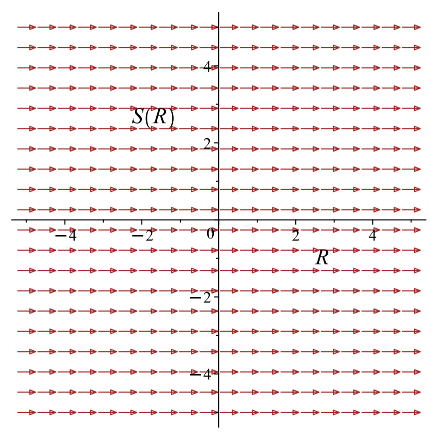
Which simplifies to

$$e^{-\frac{1}{x}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x^2}$ 	$R = x$ $S = e^{-\frac{1}{x}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{1}{x}} \tag{1}$$

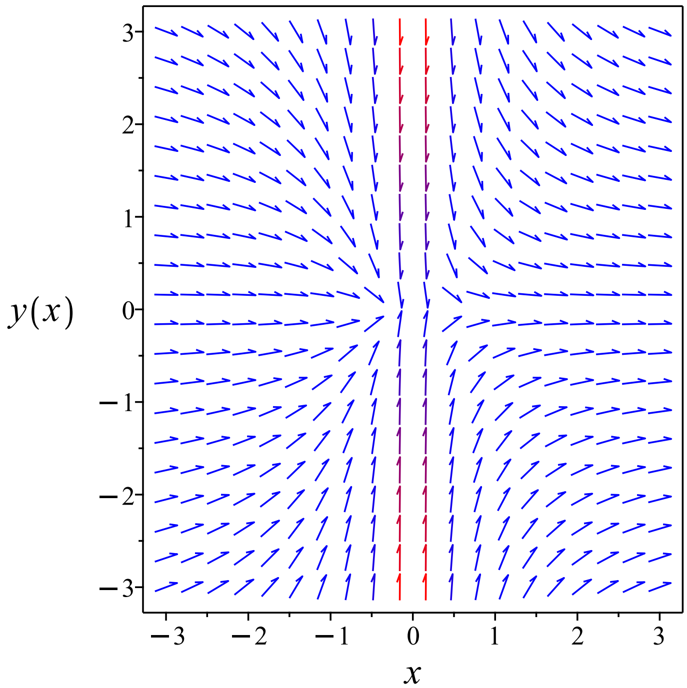


Figure 59: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{1}{x}}$$

Verified OK.

2.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x^2}\right) \mathrm{d}x \\
\left(-\frac{1}{x^2}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x^2} \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^2}\,\mathrm{d}x\\ \phi &= \frac{1}{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{c_1 x - 1}{x}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{c_1 x - 1}{x}} \tag{1}$$

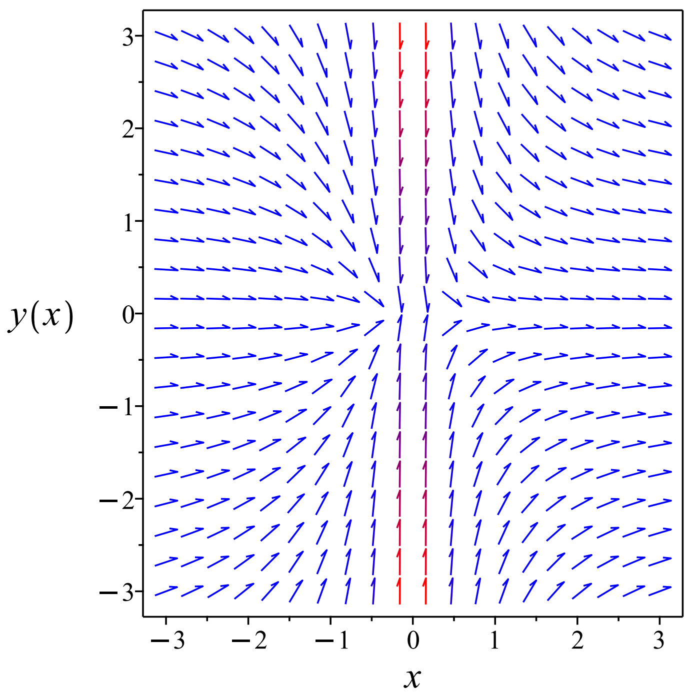


Figure 60: Slope field plot

Verification of solutions

$$y = e^{-\frac{c_1 x - 1}{x}}$$

Verified OK.

2.5.6 Maple step by step solution

Let's solve

$y'x^2 + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1}{x^2} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{1}{x} + c_1$

- Solve for y

  $y = e^{\frac{c_1 x + 1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve(x^2*diff(y(x),x) + y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{1}{x}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[x^2*y'[x] +y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1}{x}}$$
$$y(x) \to 0$$

2.6 problem 6

Internal problem ID [892]
Internal file name [`OUTPUT/892_Sunday_June_05_2022_01_53_24_AM_72879044/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{(x+1)\,y}{x} = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-x-1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{(-x-1)\,y}{x} = 0$$

The domain of $p(x) = -\frac{-x-1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

2.6.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{(x+1)\,y}{x} \end{aligned}$$

Where $f(x) = -\frac{x+1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= -\frac{x+1}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int -\frac{x+1}{x}\,dx \\ \ln(y) &= -x - \ln(x) + c_1 \\ y &= \mathrm{e}^{-x-\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-x-\ln(x)} \end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1 \mathrm{e}^{-x}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{-1} c_1$$

$$c_1 = e$$

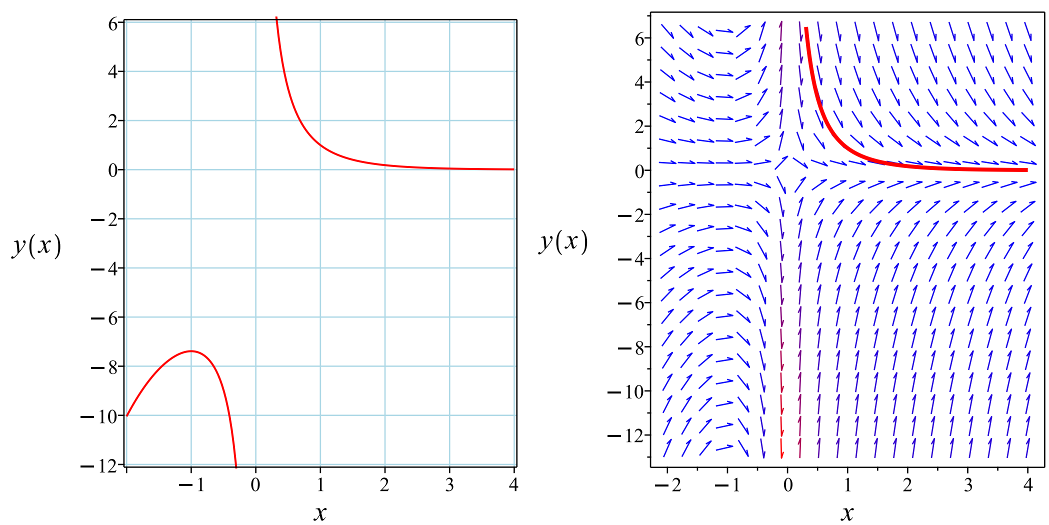
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{1-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{1-x}}{x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{1-x}}{x}$$

Verified OK.

2.6.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-x-1}{x}dx} \\ &= \mathrm{e}^{x+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x\,\mathrm{e}^{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(x\,\mathrm{e}^{x} y) &= 0\end{aligned}$$

Integrating gives

$$x\,\mathrm{e}^{x} y = c_1$$

Dividing both sides by the integrating factor $\mu = x\,\mathrm{e}^{x}$ results in

$$y = \frac{c_1 \mathrm{e}^{-x}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1} c_1$$

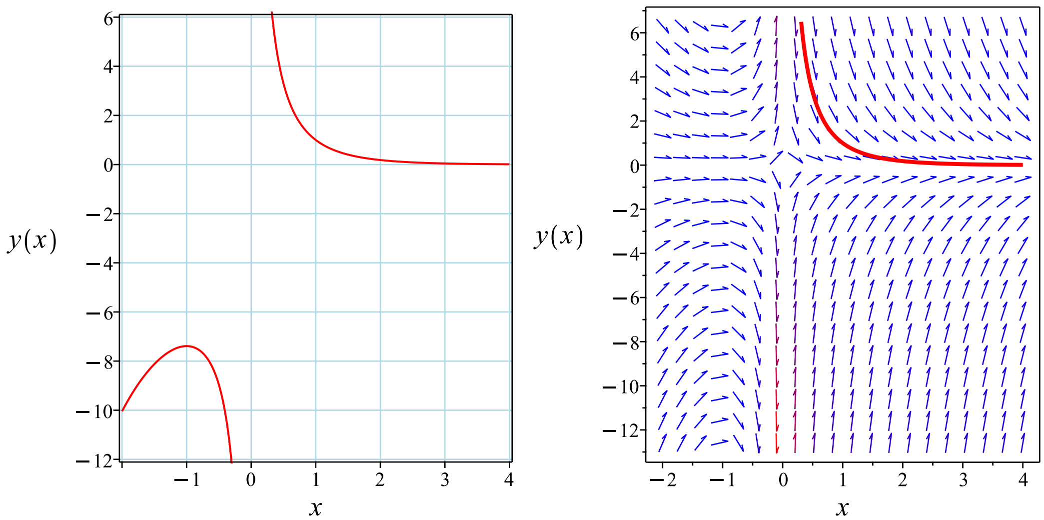
$$c_1 = \mathrm{e}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{1-x}}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{1-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{1-x}}{x}$$

Verified OK.

2.6.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + (x + 1)\,u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(2 + x)}{x} \end{aligned}$$

Where $f(x) = -\frac{2+x}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2+x}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2+x}{x}\, dx \\
\ln(u) &= -x - 2\ln(x) + c_2 \\
u &= \mathrm{e}^{-x-2\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{-x-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 \mathrm{e}^{-x}}{x^2}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2 \mathrm{e}^{-x}}{x}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2 \mathrm{e}^{-1}$$

$$c_2 = \mathrm{e}$$

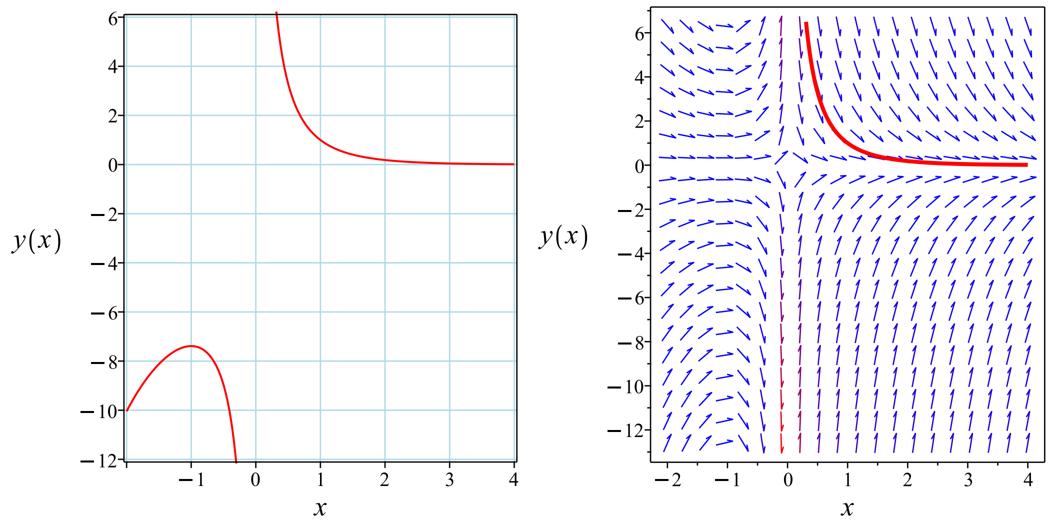
Substituting c_2 found above in the general solution gives

$$y = \frac{\mathrm{e}^{1-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{1-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{1-x}}{x}$$

Verified OK.

2.6.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(x+1)\,y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x-\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,\mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(x+1)\,y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y(x+1) \\ S_y &= x\,\mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} x = c_1$$

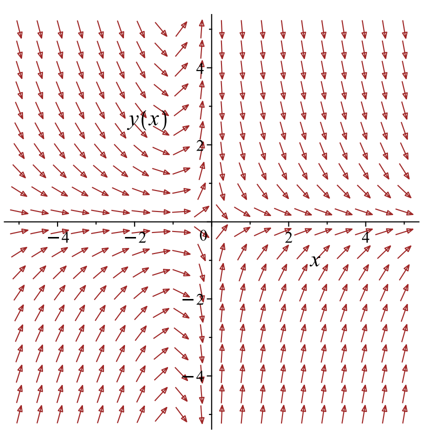
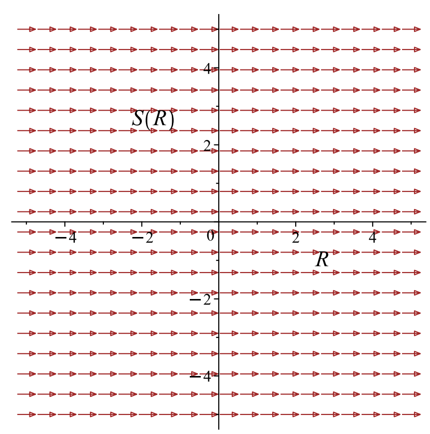
Which simplifies to

$$y\,\mathrm{e}^{x} x = c_1$$

Which gives

$$y = \frac{c_1 \mathrm{e}^{-x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(x+1)y}{x}$ 	$R = x$ $S = x\,\mathrm{e}^{x} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1} c_1$$

$$c_1 = e$$

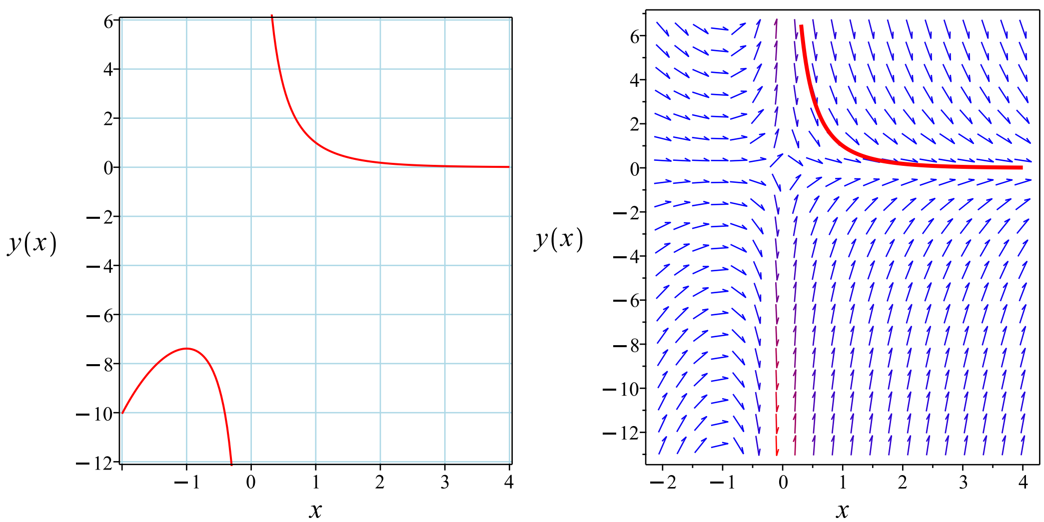
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{1-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{1-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{1-x}}{x}$$

Verified OK.

2.6.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{x+1}{x}\right)\mathrm{d}x\\ \left(-\frac{x+1}{x}\right)\mathrm{d}x+\left(-\frac{1}{y}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x+1}{x}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{x+1}{x} \,\mathrm{d}x$$
$$\phi = -x - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{\mathrm{e}^{-x-c_1}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1-c_1}$$

$$c_1 = -1$$

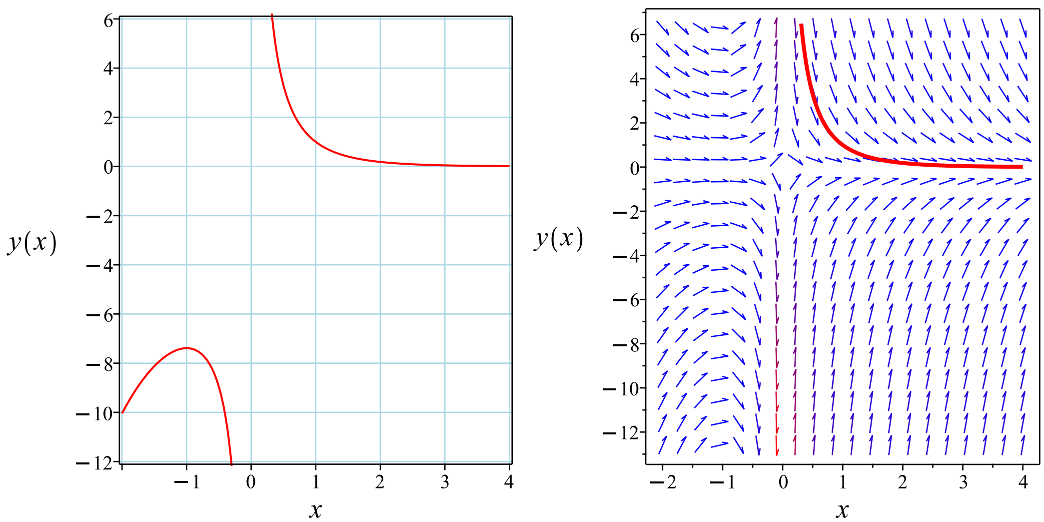
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{1-x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{1-x}}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{1-x}}{x}$$

Verified OK.

2.6.7 Maple step by step solution

Let's solve

$\left[y' + \frac{(x+1)y}{x} = 0, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x+1}{x}$

- Integrate both sides with respect to x

$\int \frac{y'}{y} dx = \int -\frac{x+1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -x - \ln(x) + c_1$
- Solve for y

  $y = \frac{e^{-x+c_1}}{x}$
- Use initial condition $y(1) = 1$

  $1 = e^{c_1 - 1}$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{e^{1-x}}{x}$
- Solution to the IVP

  $y = \frac{e^{1-x}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x) + ((1+x)/x)*y(x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^{1-x}}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 16

```
DSolve[{y'[x] +((1+x)/x)*y[x]==0,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{1-x}}{x}$$

2.7 problem 7

Internal problem ID [893]
Internal file name [`OUTPUT/893_Sunday_June_05_2022_01_53_25_AM_41652620/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x + \left(1 + \frac{1}{\ln(x)}\right) y = 0}$$

With initial conditions

$$[y(\mathrm{e}) = 1]$$

2.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{-\ln(x) - 1}{x \ln(x)}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' - \frac{(-\ln(x) - 1)\, y}{x \ln(x)} = 0$$

The domain of $p(x) = -\frac{-\ln(x)-1}{x\ln(x)}$ is

$$\{0 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = \mathrm{e}$ is inside this domain. Hence solution exists and is unique.

2.7.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{y(\ln(x) + 1)}{x \ln(x)}\end{aligned}$$

Where $f(x) = -\frac{\ln(x)+1}{x\ln(x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\, dy &= -\frac{\ln(x) + 1}{x \ln(x)}\, dx\\ \int \frac{1}{y}\, dy &= \int -\frac{\ln(x) + 1}{x \ln(x)}\, dx\\ \ln(y) &= -\ln(x) - \ln(\ln(x)) + c_1\\ y &= \mathrm{e}^{-\ln(x)-\ln(\ln(x))+c_1}\\ &= c_1 \mathrm{e}^{-\ln(x)-\ln(\ln(x))}\end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1}{x \ln(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \mathrm{e}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1} c_1$$

$$c_1 = \mathrm{e}$$

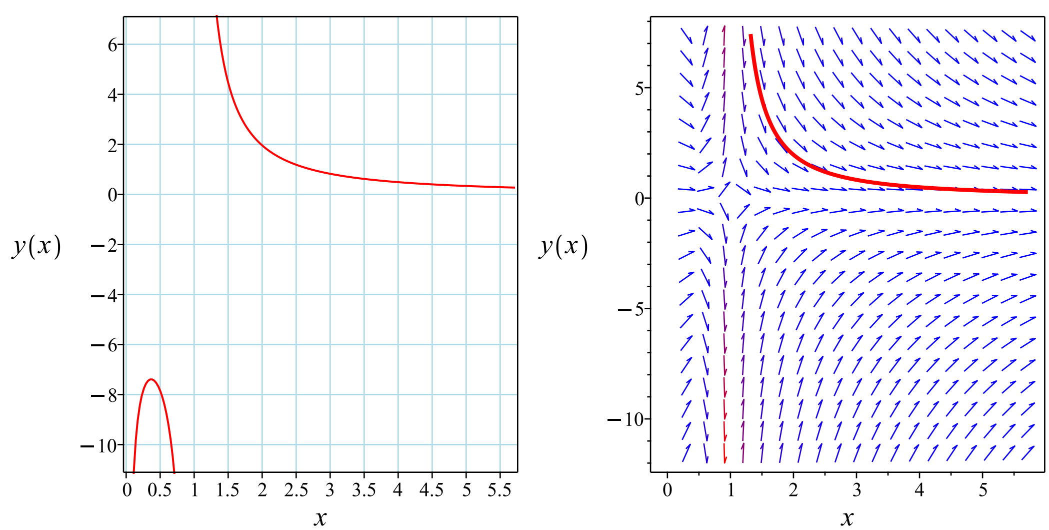
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}}{x \ln(x)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\mathrm{e}}{x \ln(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\mathrm{e}}{x \ln(x)}$$

Verified OK.

2.7.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{-\ln(x)-1}{x\ln(x)}dx} \\ &= e^{\ln(x)+\ln(\ln(x))}\end{aligned}$$

Which simplifies to

$$\mu = x\ln(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}(x\ln(x)\,y) &= 0\end{aligned}$$

Integrating gives

$$x\ln(x)\,y = c_1$$

Dividing both sides by the integrating factor $\mu = x\ln(x)$ results in

$$y = \frac{c_1}{x\ln(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = e$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{-1}c_1$$

$$c_1 = e$$

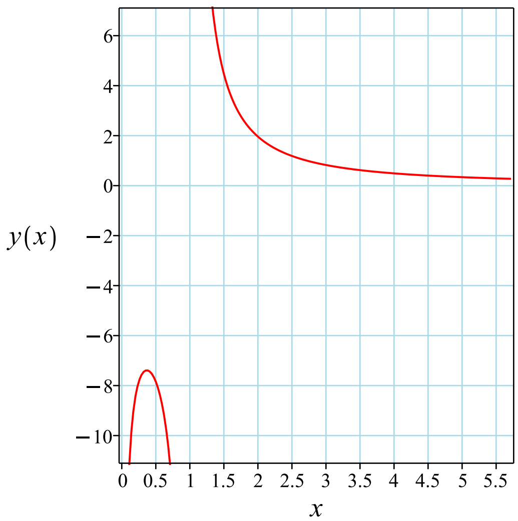
Substituting c_1 found above in the general solution gives

$$y = \frac{e}{x\ln(x)}$$

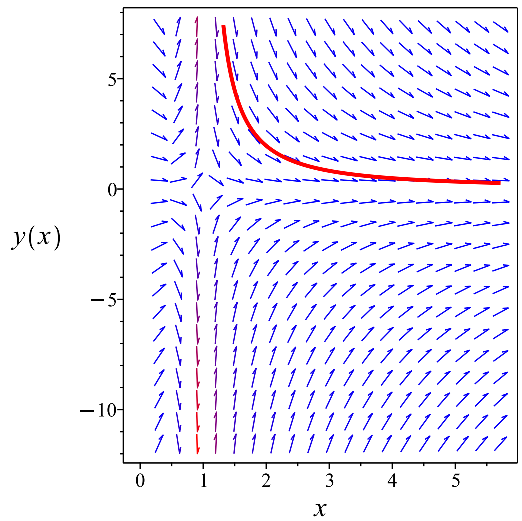
Summary

The solution(s) found are the following

$$y = \frac{e}{x\ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\mathrm{e}}{x \ln(x)}$$

Verified OK.

2.7.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + \left(1 + \frac{1}{\ln(x)}\right) u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(2\ln(x)+1)}{x\ln(x)} \end{aligned}$$

Where $f(x) = -\frac{2\ln(x)+1}{x\ln(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2\ln(x)+1}{x\ln(x)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2\ln(x)+1}{x\ln(x)}\,dx \\
\ln(u) &= -\ln(\ln(x)) - 2\ln(x) + c_2 \\
u &= \mathrm{e}^{-\ln(\ln(x))-2\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{-\ln(\ln(x))-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x^2\ln(x)}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2}{x\ln(x)}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = \mathrm{e}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2 \mathrm{e}^{-1}$$

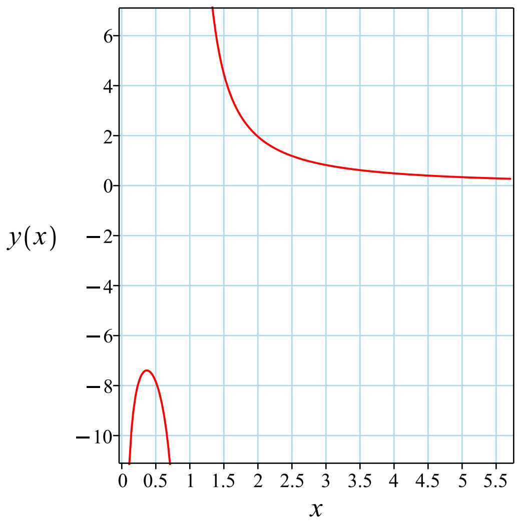
$$c_2 = \mathrm{e}$$

Substituting c_2 found above in the general solution gives

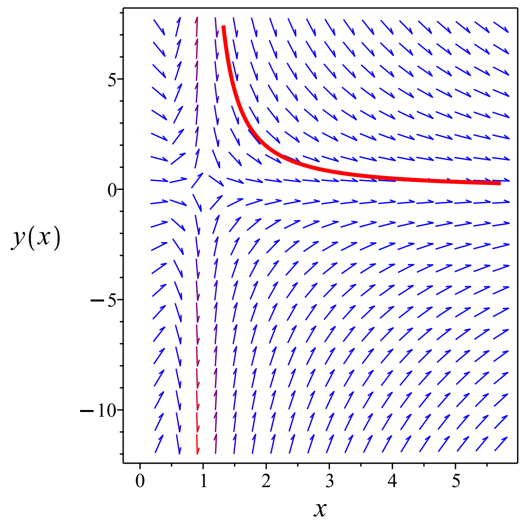
$$y = \frac{\mathrm{e}}{x\ln(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{e}}{x\ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\mathrm{e}}{x \ln(x)}$$

Verified OK.

2.7.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(\ln(x) + 1)}{x \ln(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\ln(x) - \ln(\ln(x))} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\ln(x)-\ln(\ln(x))}} dy \end{aligned}$$

Which results in

$$S = x \ln(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(\ln(x)+1)}{x\ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y(\ln(x)+1) \\ S_y &= x\ln(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x \ln(x) y = c_1$$

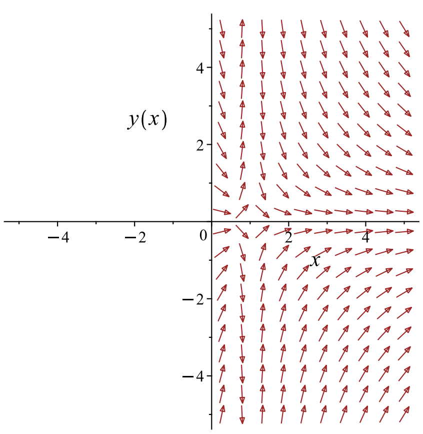
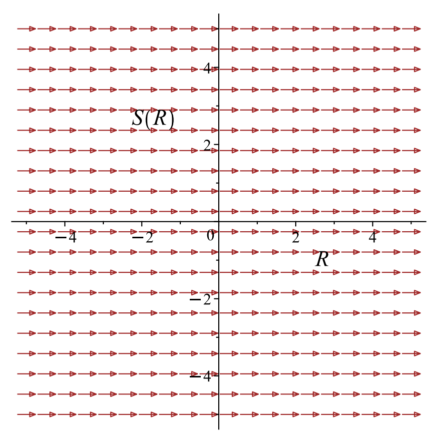
Which simplifies to

$$x \ln(x) y = c_1$$

Which gives

$$y = \frac{c_1}{x \ln(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(\ln(x)+1)}{x \ln(x)}$ 	$R = x$ $S = x \ln(x) y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \mathrm{e}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1} c_1$$

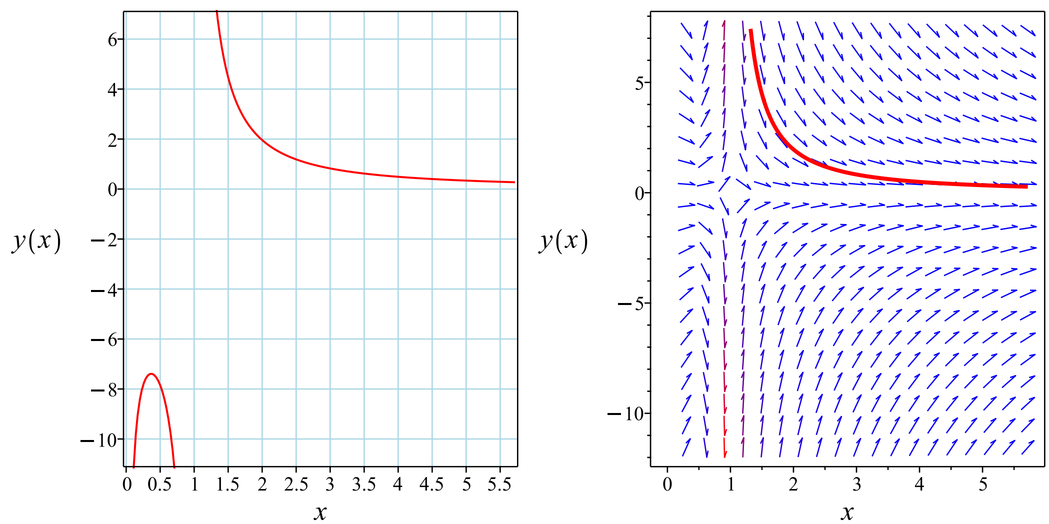
$$c_1 = e$$

Substituting c_1 found above in the general solution gives

$$y = \frac{e}{x \ln(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{e}{x \ln(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e}{x \ln(x)}$$

Verified OK.

2.7.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{\ln(x)+1}{x\ln(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\ln(x)+1}{x\ln(x)}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{\ln(x)+1}{x\ln(x)}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\ln(x)+1}{x\ln(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\ln(x)+1}{x\ln(x)}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) - \ln(\ln(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(\ln(x)) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(\ln(x)) - \ln(y)$$

The solution becomes

$$y = \frac{\mathrm{e}^{-c_1}}{\ln(x)\,x}$$

Initial conditions are used to solve for c_1. Substituting $x = \mathrm{e}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{-1-c_1}$$

$$c_1 = -1$$

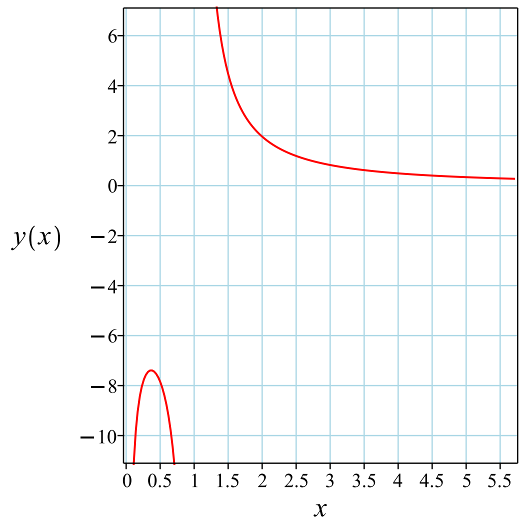
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}}{x\ln(x)}$$

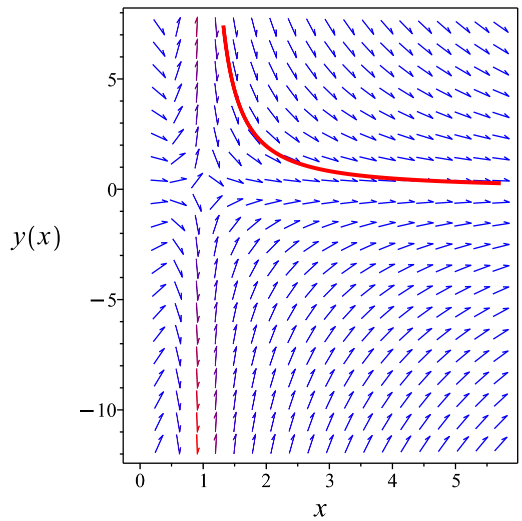
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}}{x\ln(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e}{x \ln(x)}$$

Verified OK.

2.7.7 Maple step by step solution

Let's solve

$\left[y'x + \left(1 + \frac{1}{\ln(x)}\right) y = 0, y(e) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1+\frac{1}{\ln(x)}}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1+\frac{1}{\ln(x)}}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) - \ln(\ln(x)) + c_1$

- Solve for y

$y = \frac{e^{c_1}}{\ln(x) x}$

- Use initial condition $y(e) = 1$

  $1 = \frac{e^{c_1}}{e}$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{e}{x \ln(x)}$

- Solution to the IVP

  $y = \frac{e}{x \ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x) + (1+1/ln(x))*y(x)=0,y(exp(1)) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e}{x \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 18

```
DSolve[{y'[x] +(1+1/Log[x])*y[x]==0,y[Exp[1]]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\text{LogIntegral}(x)+\text{LogIntegral}(e)-x+e}$$

2.8 problem 8

Internal problem ID [894]
Internal file name [`OUTPUT/894_Sunday_June_05_2022_01_53_26_AM_75667082/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x + (1 + x\cot(x))\, y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 2\right]$$

2.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{(1 + x\cot(x))\, y}{x} \end{aligned}$$

Where $f(x) = -\frac{1+x\cot(x)}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{1 + x\cot\left(x\right)}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{1 + x\cot\left(x\right)}{x}\,dx \\
\ln\left(y\right) &= -\ln\left(\sin\left(x\right)\right) - \ln\left(x\right) + c_1 \\
y &= \mathrm{e}^{-\ln(\sin(x))-\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-\ln(\sin(x))-\ln(x)}
\end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1}{\sin\left(x\right)x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2c_1}{\pi}$$

$$c_1 = \pi$$

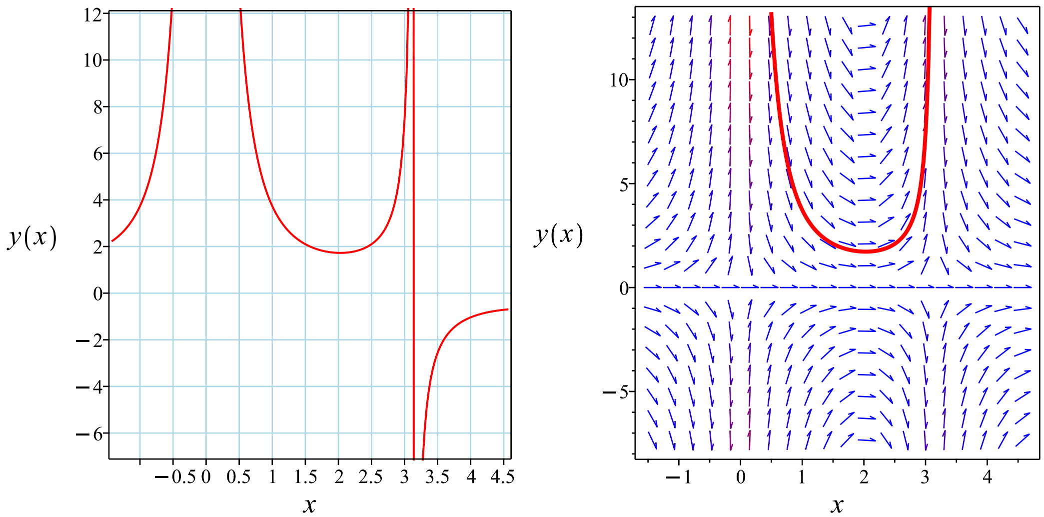
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{\sin\left(x\right)x}$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{\sin\left(x\right)x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\pi}{\sin(x)\, x}$$

Verified OK.

2.8.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-1-x\cot(x)}{x}dx}$$
$$= \mathrm{e}^{\ln(\sin(x))+\ln(x)}$$

Which simplifies to

$$\mu = \sin(x)\, x$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)\, xy) = 0$$

Integrating gives

$$\sin(x)\, xy = c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)\, x$ results in

$$y = \frac{c_1 \csc(x)}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2c_1}{\pi}$$

$$c_1 = \pi$$

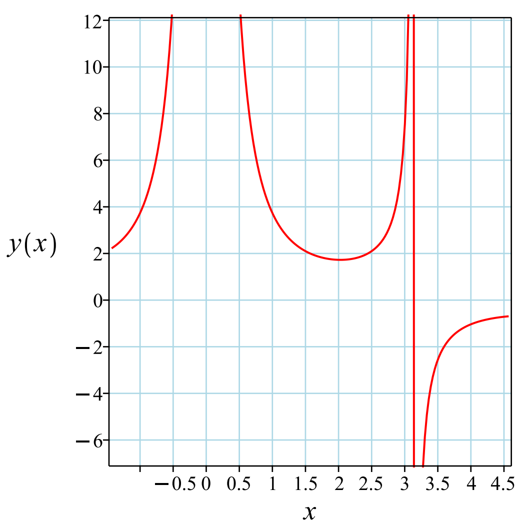
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{\sin(x)\, x}$$

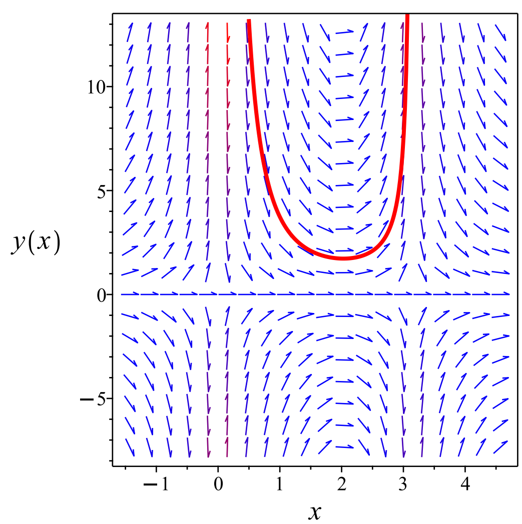
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\pi}{\sin(x)\, x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\pi}{\sin(x)\, x}$$

Verified OK.

2.8.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + (1 + x\cot(x))\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x\cot(x)+2)}{x}
\end{aligned}$$

Where $f(x) = -\frac{x\cot(x)+2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x\cot(x)+2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x\cot(x)+2}{x}\,dx \\
\ln(u) &= -\ln(\sin(x)) - 2\ln(x) + c_2 \\
u &= \mathrm{e}^{-\ln(\sin(x))-2\ln(x)+c_2} \\
&= c_2\mathrm{e}^{-\ln(\sin(x))-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{\sin(x)\,x^2}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2}{x\sin(x)}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2c_2}{\pi}$$

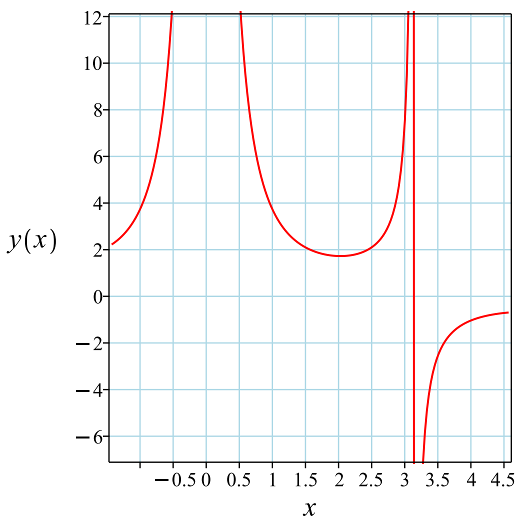
$$c_2 = \pi$$

Substituting c_2 found above in the general solution gives

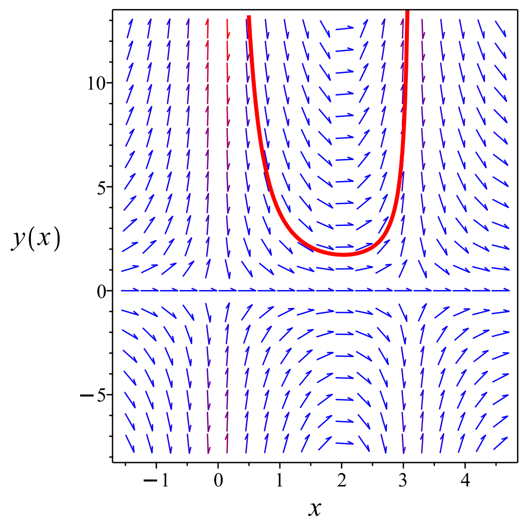
$$y = \frac{\pi}{\sin(x)\,x}$$

Summary
The solution(s) found are the following

$$y = \frac{\pi}{\sin(x)\, x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{\sin(x)\, x}$$

Verified OK.

2.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(1 + x\cot(x))\, y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\ln(\sin(x)) - \ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\ln(\sin(x))-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(1 + x\cot(x))\, y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y(x\cos(x) + \sin(x)) \\ S_y &= \sin(x)\, x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)\, xy = c_1$$

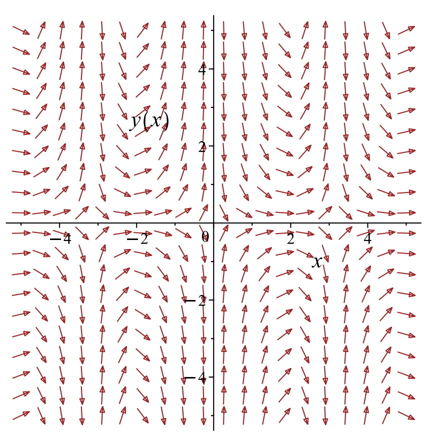
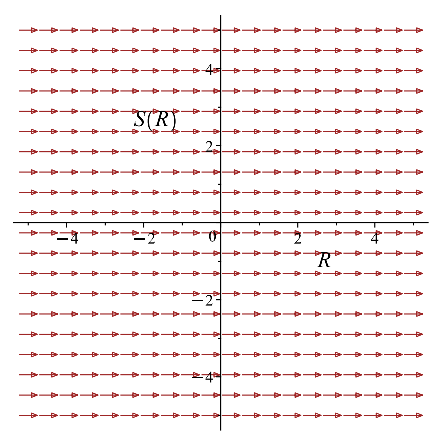
Which simplifies to

$$\sin(x)\, xy = c_1$$

Which gives

$$y = \frac{c_1}{\sin(x)\, x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(1+x\cot(x))y}{x}$ 	$R = x$ $S = \sin(x)\, xy$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2c_1}{\pi}$$

$$c_1 = \pi$$

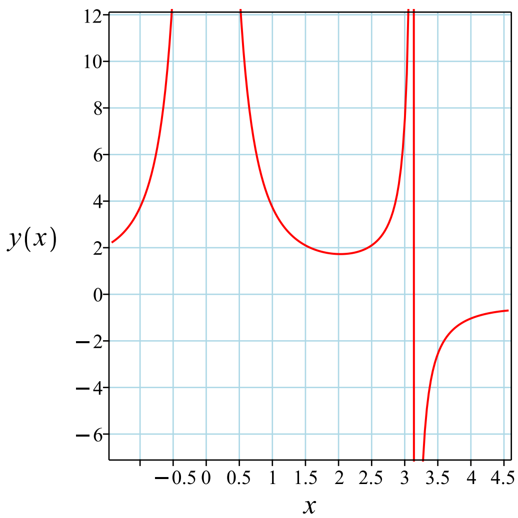
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{\sin(x) x}$$

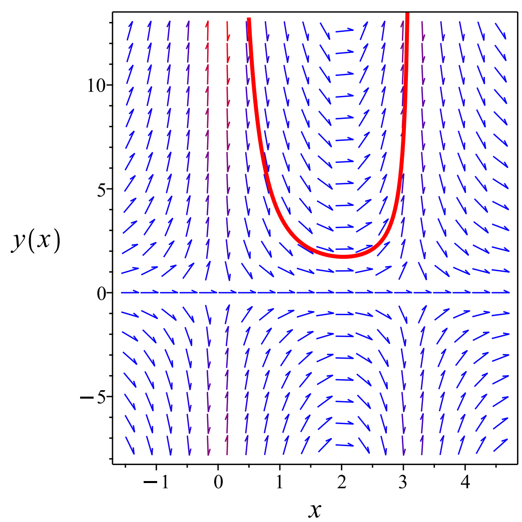
Summary

The solution(s) found are the following

$$y = \frac{\pi}{\sin(x) x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{\sin(x) x}$$

Verified OK.

2.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1 + x\cot(x)}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1 + x\cot(x)}{x}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1 + x\cot(x)}{x}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1+x\cot(x)}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1+x\cot(x)}{x} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\sin(x)) - \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) - \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) - \ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{\mathrm{e}^{-c_1}}{\sin(x)\,x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2\,\mathrm{e}^{-c_1}}{\pi}$$

$$c_1 = -\ln(\pi)$$

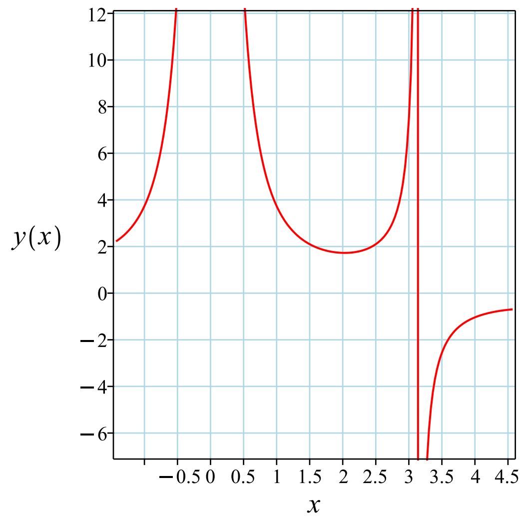
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{\sin(x)\,x}$$

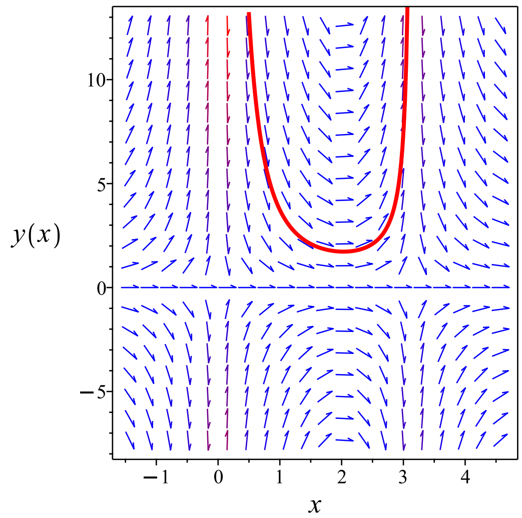
Summary

The solution(s) found are the following

$$y = \frac{\pi}{\sin(x)\,x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{\sin(x)\, x}$$

Verified OK.

2.8.6 Maple step by step solution

Let's solve

$\left[y'x + (1 + x\cot(x))\, y = 0, y\left(\frac{\pi}{2}\right) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1+x\cot(x)}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1+x\cot(x)}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(\sin(x)) - \ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x\sin(x)}$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 2$

  $2 = \frac{2\,e^{c_1}}{\pi}$

- Solve for c_1

  $c_1 = \ln\left(\pi\right)$

- Substitute $c_1 = \ln\left(\pi\right)$ into general solution and simplify

  $y = \frac{\csc(x)\pi}{x}$

- Solution to the IVP

  $y = \frac{\csc(x)\pi}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([x*diff(y(x),x) + (1+x*cot(x))*y(x)=0,y(1/2*Pi) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\csc\left(x\right)\pi}{x}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 66

```
DSolve[{y'[x] +(1+x*Cot[x])*y[x]==0,y[Pi/2]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 2^{1+\frac{\pi}{2}}\left(1-e^{2ix}\right)^{-x}\exp\left(-\frac{1}{12}i\left(-6\,\text{PolyLog}\left(2,e^{2ix}\right)-6x(x+2i)+\pi^2+6i\pi\right)\right)$$

2.9 problem 9

Internal problem ID [895]
Internal file name [`OUTPUT/895_Sunday_June_05_2022_01_53_28_AM_47903859/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2xy}{x^2 + 1} = 0}$$

With initial conditions

$$[y(0) = 2]$$

2.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x}{x^2+1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{2xy}{x^2+1} = 0$$

The domain of $p(x) = -\frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

2.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2xy}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{2x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{2x}{x^2+1}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{2x}{x^2+1}\,dx \\ \ln(y) &= \ln\left(x^2+1\right) + c_1 \\ y &= \mathrm{e}^{\ln(x^2+1)+c_1} \\ &= c_1\left(x^2+1\right) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

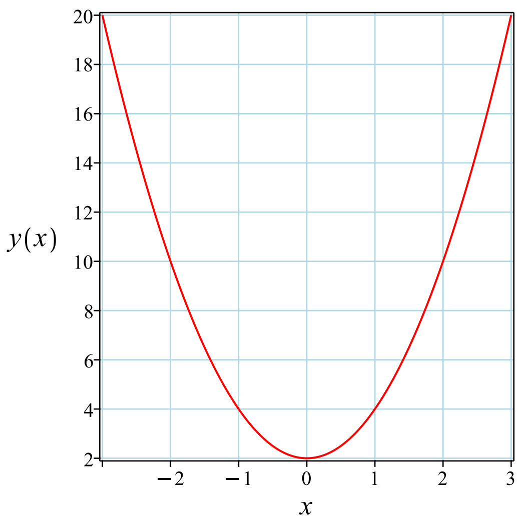
Substituting c_1 found above in the general solution gives

$$y = 2x^2 + 2$$

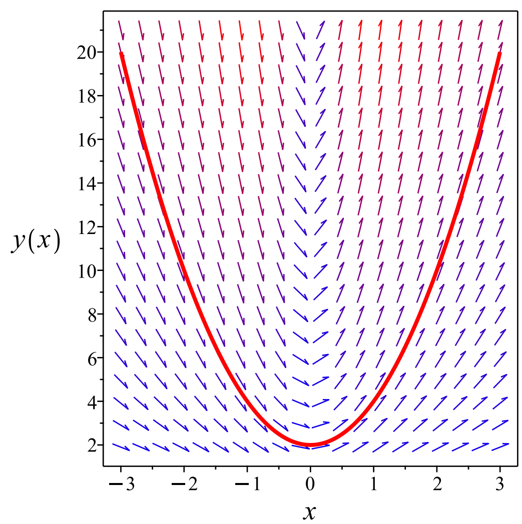
Summary

The solution(s) found are the following

$$y = 2x^2 + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2x^2 + 2$$

Verified OK.

2.9.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2x}{x^2+1}dx} \\ &= \frac{1}{x^2+1}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2+1}\right) = 0$$

Integrating gives

$$\frac{y}{x^2+1} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+1}$ results in

$$y = c_1(x^2+1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

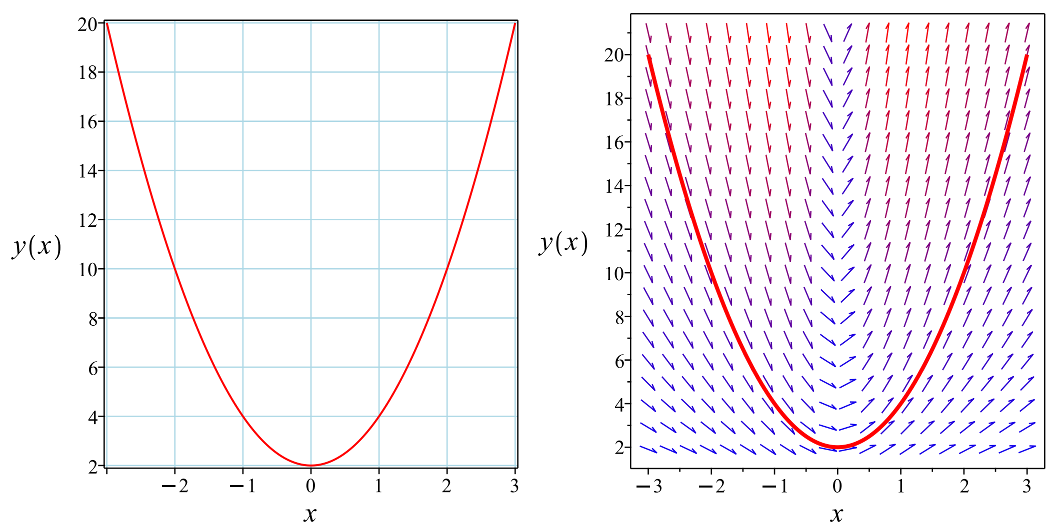
Substituting c_1 found above in the general solution gives

$$y = 2x^2 + 2$$

Summary

The solution(s) found are the following

$$y = 2x^2 + 2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2x^2 + 2$$

Verified OK.

2.9.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2x^2 u(x)}{x^2 + 1} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u(x^2 - 1)}{x\,(x^2 + 1)} \end{aligned}$$

Where $f(x) = \frac{x^2-1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{x^2 - 1}{x\,(x^2 + 1)}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{x^2 - 1}{x\,(x^2 + 1)}\, dx \\ \ln(u) &= -\ln(x) + \ln\left(x^2 + 1\right) + c_2 \\ u &= \mathrm{e}^{-\ln(x) + \ln(x^2+1) + c_2} \\ &= c_2 \mathrm{e}^{-\ln(x) + \ln(x^2+1)} \end{aligned}$$

Which simplifies to

$$u(x) = c_2 \left(x + \frac{1}{x}\right)$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x c_2 \left(x + \frac{1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2$$

$$c_2 = 2$$

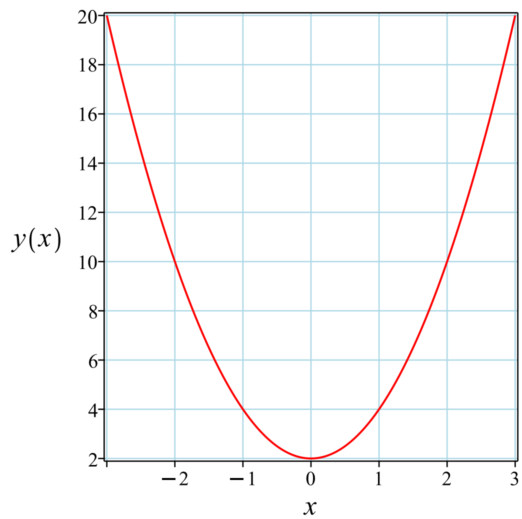
Substituting c_2 found above in the general solution gives

$$y = 2x^2 + 2$$

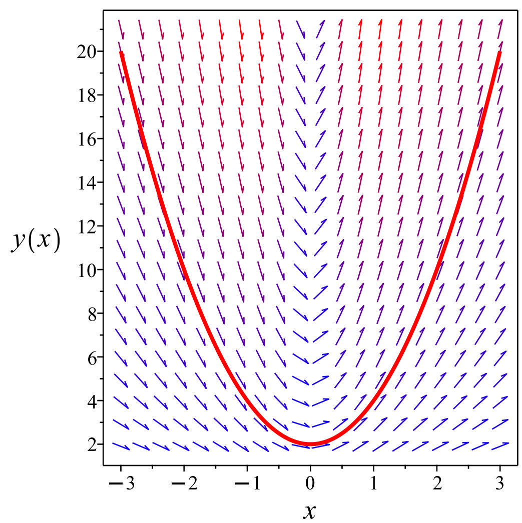
Summary

The solution(s) found are the following

$$y = 2x^2 + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2x^2 + 2$$

Verified OK.

2.9.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2xy}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2 + 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2xy}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2+1)^2} \\ S_y &= \frac{1}{x^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2+1} = c_1$$

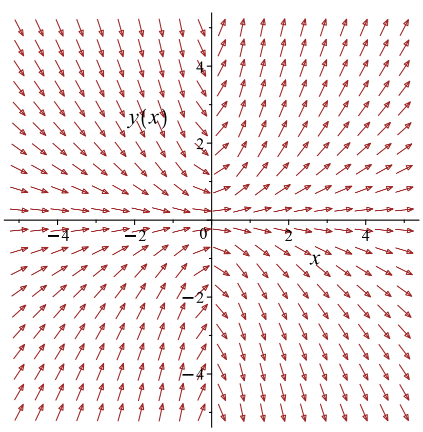
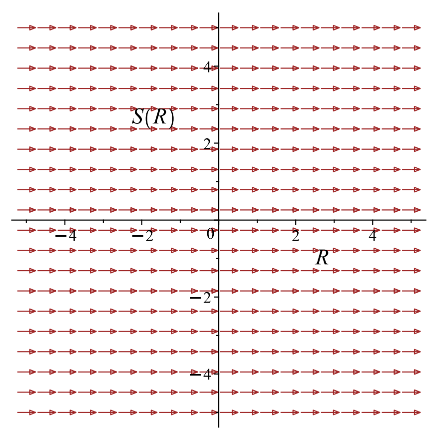
Which simplifies to

$$\frac{y}{x^2+1} = c_1$$

Which gives

$$y = c_1(x^2+1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2xy}{x^2+1}$ 	$R = x$ $S = \frac{y}{x^2+1}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

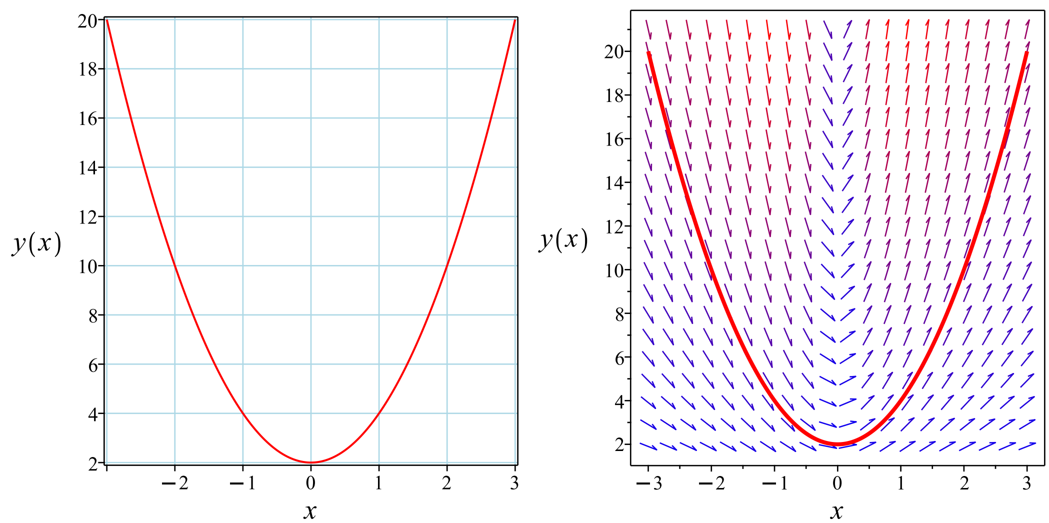
Substituting c_1 found above in the general solution gives

$$y = 2x^2 + 2$$

Summary

The solution(s) found are the following

$$y = 2x^2 + 2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2x^2 + 2$$

Verified OK.

2.9.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{2 y}\right) \mathrm{d} y &= \left(\frac{x}{x^2+1}\right) \mathrm{d} x \\
\left(-\frac{x}{x^2+1}\right) \mathrm{d} x+\left(\frac{1}{2 y}\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{x}{x^2+1} \\
N(x, y) &= \frac{1}{2 y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x\\ \phi &= -\frac{\ln\left(x^2+1\right)}{2} + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{2c_1}\left(x^2+1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{2c_1}$$

$$c_1 = \frac{\ln(2)}{2}$$

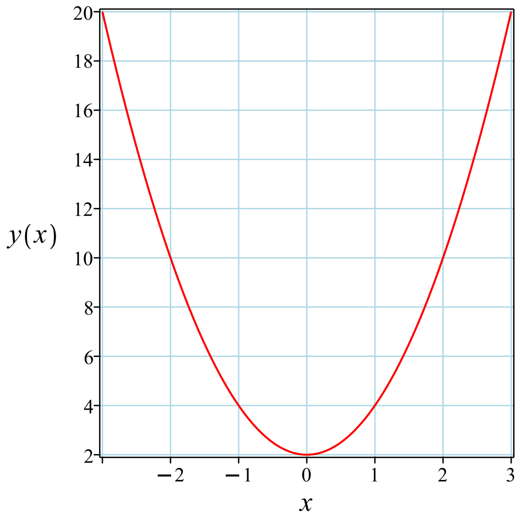
Substituting c_1 found above in the general solution gives

$$y = 2x^2 + 2$$

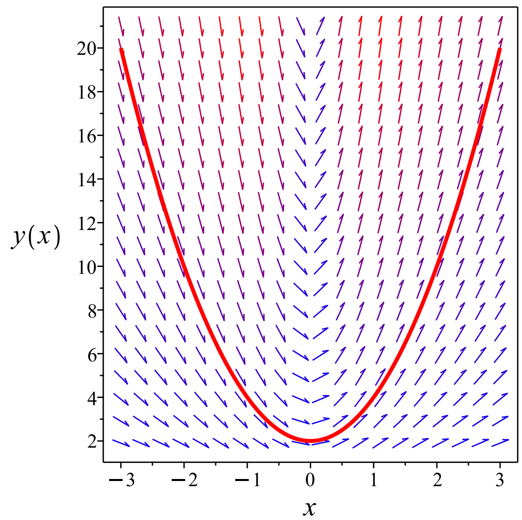
Summary

The solution(s) found are the following

$$y = 2x^2 + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2x^2 + 2$$

Verified OK.

2.9.7 Maple step by step solution

Let's solve

$\left[y' - \frac{2xy}{x^2+1} = 0, y(0) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{2x}{x^2+1} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x^2+1) + c_1$

- Solve for y

  $y = e^{c_1}(x^2+1)$

- Use initial condition $y(0) = 2$

  $2 = e^{c_1}$

- Solve for c_1

  $c_1 = \ln(2)$

- Substitute $c_1 = \ln(2)$ into general solution and simplify

  $y = 2x^2 + 2$

- Solution to the IVP

  $y = 2x^2 + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) - (2*x)/(1+x^2)*y(x)=0,y(0) = 2],y(x), singsol=all)
```

$$y(x) = 2x^2 + 2$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 12

```
DSolve[{y'[x] -(2*x)/(1+x^2)*y[x]==0,y[0]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2(x^2 + 1)$$

2.10 problem 10

Internal problem ID [896]

Internal file name [`OUTPUT/896_Sunday_June_05_2022_01_53_29_AM_56359127/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001

Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41

Problem number: 10.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{ky}{x} = 0}$$

With initial conditions

$$[y(1) = 3]$$

2.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{k}{x}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{ky}{x} = 0$$

The domain of $p(x) = \frac{k}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

2.10.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{ky}{x}\end{aligned}$$

Where $f(x) = -\frac{k}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\,dy &= -\frac{k}{x}\,dx\\ \int \frac{1}{y}\,dy &= \int -\frac{k}{x}\,dx\\ \ln(y) &= -k\ln(x) + c_1\\ y &= \mathrm{e}^{-k\ln(x)+c_1}\\ &= c_1\mathrm{e}^{-k\ln(x)}\end{aligned}$$

Which can be simplified to become

$$y = c_1 x^{-k}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = 3x^{-k}$$

Summary
The solution(s) found are the following

$$y = 3x^{-k} \tag{1}$$

Verification of solutions

$$y = 3x^{-k}$$

Verified OK.

2.10.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{k}{x} dx} \\ &= e^{k \ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = x^k$$

The ode becomes

$$\begin{aligned} \frac{d}{dx} \mu y &= 0 \\ \frac{d}{dx}\left(x^k y\right) &= 0 \end{aligned}$$

Integrating gives

$$x^k y = c_1$$

Dividing both sides by the integrating factor $\mu = x^k$ results in

$$y = c_1 x^{-k}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = 3x^{-k}$$

Summary
The solution(s) found are the following

$$y = 3x^{-k} \tag{1}$$

Verification of solutions

$$y = 3x^{-k}$$

Verified OK.

2.10.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) + ku(x) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{(-1-k)\, u}{x}
\end{aligned}$$

Where $f(x) = \frac{-1-k}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{-1-k}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{-1-k}{x}\, dx \\
\ln(u) &= (-1-k)\ln(x) + c_2 \\
u &= \mathrm{e}^{(-1-k)\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{(-1-k)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 x^{-k}}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x^{-k} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2$$

$$c_2 = 3$$

Substituting c_2 found above in the general solution gives

$$y = 3x^{-k}$$

Summary

The solution(s) found are the following

$$y = 3x^{-k} \tag{1}$$

Verification of solutions

$$y = 3x^{-k}$$

Verified OK.

2.10.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{ky}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-k \ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-k \ln(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{k \ln(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{ky}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= ky\, x^{k-1} \\ S_y &= x^k \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^k y = c_1$$

Which simplifies to

$$x^k y = c_1$$

Which gives

$$y = c_1 x^{-k}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = 3x^{-k}$$

Summary
The solution(s) found are the following

$$y = 3x^{-k} \tag{1}$$

Verification of solutions

$$y = 3x^{-k}$$

Verified OK.

2.10.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{ky}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(-\frac{1}{ky}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x}\\
N(x,y) &= -\frac{1}{ky}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{ky}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{ky}$. Therefore equation (4) becomes

$$-\frac{1}{ky} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{ky}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(-\frac{1}{ky}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{k} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{k} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{k}$$

The solution becomes

$$y = \mathrm{e}^{-k\ln(x) - c_1 k}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \mathrm{e}^{-c_1 k}$$

$$c_1 = -\frac{\ln(3)}{k}$$

Substituting c_1 found above in the general solution gives

$$y = 3x^{-k}$$

Summary
The solution(s) found are the following

$$y = 3x^{-k} \tag{1}$$

Verification of solutions

$$y = 3x^{-k}$$

Verified OK.

2.10.7 Maple step by step solution

Let's solve

$\left[y' + \frac{ky}{x} = 0, y(1) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{k}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{k}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -k\ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{-k\ln(x)+c_1}$

- Use initial condition $y(1) = 3$

  $3 = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = \ln(3)$

- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = 3x^{-k}$

- Solution to the IVP

$$y = 3x^{-k}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) +k/x*y(x)=0,y(1) = 3],y(x), singsol=all)
```

$$y(x) = 3x^{-k}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 12

```
DSolve[{y'[x] +k/x*y[x]==0,y[1]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^{-k}$$

2.11 problem 11

Internal problem ID [897]
Internal file name [`OUTPUT/897_Sunday_June_05_2022_01_53_30_AM_19393587/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \tan(kx)\, y = 0}$$

With initial conditions

$$[y(0) = 2]$$

2.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(kx) \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \tan(kx)\,y = 0$$

The domain of $p(x) = \tan(kx)$ is

$$\left\{x < \frac{\pi(1 + 2\_Z51)}{2k} \vee \frac{\pi(1 + 2\_Z51)}{2k} < x\right\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.11.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\tan(kx)\,y \end{aligned}$$

Where $f(x) = -\tan(kx)$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= -\tan(kx)\,dx \\ \int \frac{1}{y}\,dy &= \int -\tan(kx)\,dx \\ \ln(y) &= \frac{\ln(\cos(kx))}{k} + c_1 \\ y &= \mathrm{e}^{\frac{\ln(\cos(kx))}{k} + c_1} \\ &= c_1 \mathrm{e}^{\frac{\ln(\cos(kx))}{k}} \end{aligned}$$

Which can be simplified to become

$$y = c_1 \cos(kx)^{\frac{1}{k}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Summary

The solution(s) found are the following

$$y = 2\cos(kx)^{\frac{1}{k}} \tag{1}$$

Verification of solutions

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Verified OK.

2.11.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \tan(kx)dx} \\ &= e^{-\frac{\ln(\cos(kx))}{k}}\end{aligned}$$

Which simplifies to

$$\mu = \cos(kx)^{-\frac{1}{k}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\cos(kx)^{-\frac{1}{k}} y\right) &= 0\end{aligned}$$

Integrating gives

$$\cos(kx)^{-\frac{1}{k}} y = c_1$$

Dividing both sides by the integrating factor $\mu = \cos(kx)^{-\frac{1}{k}}$ results in

$$y = c_1 \cos(kx)^{\frac{1}{k}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Summary
The solution(s) found are the following

$$y = 2\cos(kx)^{\frac{1}{k}} \tag{1}$$

Verification of solutions

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Verified OK.

2.11.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + \tan(kx)\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\tan(kx)\,x + 1)}{x} \end{aligned}$$

Where $f(x) = -\frac{\tan(kx)x+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{\tan(kx)\,x+1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\tan(kx)\,x+1}{x}\,dx \\
\ln(u) &= \frac{\ln(\cos(kx))}{k} - \ln(kx) + c_2 \\
u &= \mathrm{e}^{\frac{\ln(\cos(kx))}{k} - \ln(kx) + c_2} \\
&= c_2 \mathrm{e}^{\frac{\ln(\cos(kx))}{k} - \ln(kx)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 \cos(kx)^{\frac{1}{k}}}{kx}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2 \cos(kx)^{\frac{1}{k}}}{k}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_2}{k}$$

$$c_2 = 2k$$

Substituting c_2 found above in the general solution gives

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Summary

The solution(s) found are the following

$$y = 2\cos(kx)^{\frac{1}{k}} \tag{1}$$

Verification of solutions

$$y = 2\cos(kx)^{\frac{1}{k}}$$

Verified OK.

2.11.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\tan(kx)\,y \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\frac{\ln\left(1+\tan(kx)^2\right)}{2k}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{\ln\left(1+\tan(kx)^2\right)}{2k}}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{\ln\left(\sqrt{1+\tan(kx)^2}\right)}{k}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\tan(kx)\, y$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \sec(kx)^{\frac{1}{k}} \tan(kx) \\
S_y &= \sec(kx)^{\frac{1}{k}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y \sec(kx)^{\frac{1}{k}} = c_1
$$

Which simplifies to

$$
y \sec(kx)^{\frac{1}{k}} = c_1
$$

Which gives

$$
y = c_1 \sec(kx)^{-\frac{1}{k}}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = c_1
$$

$$
c_1 = 2
$$

Substituting c_1 found above in the general solution gives

$$y = 2\left(\frac{1}{\cos(kx)}\right)^{-\frac{1}{k}}$$

Summary

The solution(s) found are the following

$$y = 2\left(\frac{1}{\cos(kx)}\right)^{-\frac{1}{k}} \tag{1}$$

Verification of solutions

$$y = 2\left(\frac{1}{\cos(kx)}\right)^{-\frac{1}{k}}$$

Verified OK.

2.11.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y=(\tan(kx))\,\mathrm{d}x$$
$$(-\tan(kx))\,\mathrm{d}x+\left(-\frac{1}{y}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=-\tan(kx)$$
$$N(x,y)=-\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(-\tan(kx))$$
$$=0$$

And

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{1}{y}\right)$$
$$=0$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\tan(kx) \,\mathrm{d}x$$
$$\phi = -\frac{\ln\left(\sec(kx)^2\right)}{2k} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln\left(\sec(kx)^2\right)}{2k} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln\left(\sec(kx)^2\right)}{2k} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{2c_1k+\ln\left(\frac{1}{\cos(kx)^2}\right)}{2k}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{-c_1}$$

$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

$$y = 2\left(\frac{2}{1+\cos(2kx)}\right)^{-\frac{1}{2k}}$$

Summary

The solution(s) found are the following

$$y = 2\left(\frac{2}{1+\cos(2kx)}\right)^{-\frac{1}{2k}} \tag{1}$$

Verification of solutions

$$y = 2\left(\frac{2}{1+\cos(2kx)}\right)^{-\frac{1}{2k}}$$

Verified OK.

2.11.7 Maple step by step solution

Let's solve

$[y' + \tan(kx)\,y = 0, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\tan(kx)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\tan(kx)\,dx + c_1$

- Evaluate integral

$$\ln(y) = -\frac{\ln\left(1+\tan(kx)^2\right)}{2k} + c_1$$

- Solve for y

$$y = e^{-\frac{-2c_1k+\ln\left(\frac{1}{\cos(kx)^2}\right)}{2k}}$$

- Use initial condition $y(0) = 2$

$$2 = e^{c_1}$$

- Solve for c_1

$$c_1 = \ln(2)$$

- Substitute $c_1 = \ln(2)$ into general solution and simplify

$$y = 2\left(\sec(kx)^2\right)^{-\frac{1}{2k}}$$

- Solution to the IVP

$$y = 2\left(\sec(kx)^2\right)^{-\frac{1}{2k}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 18

```
dsolve([diff(y(x),x) +tan(k*x)*y(x)=0,y(0) = 2],y(x), singsol=all)
```

$$y(x) = 2\left(\sec(kx)^2\right)^{-\frac{1}{2k}}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 15

```
DSolve[{y'[x] +Tan[k*x]*y[x]==0,y[0]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sqrt[k]{\cos(kx)}$$

2.12 problem 12

Internal problem ID [898]
Internal file name [`OUTPUT/898_Sunday_June_05_2022_01_53_32_AM_13100521/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{3y + y' = 1}$$

2.12.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-3y+1} dy = \int dx$$
$$-\frac{\ln(-3y+1)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = c_2 \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-3x}}{3c_2^3} + \frac{1}{3} \tag{1}$$

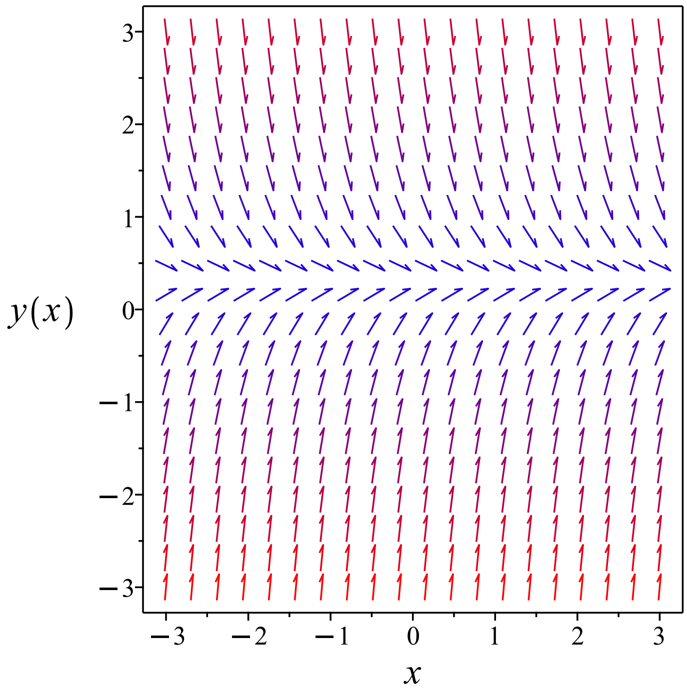


Figure 81: Slope field plot

Verification of solutions

$$y = -\frac{e^{-3x}}{3c_2^3} + \frac{1}{3}$$

Verified OK.

2.12.2 Maple step by step solution

Let's solve

$3y + y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{-3y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-3y+1}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{\ln(-3y+1)}{3} = x + c_1$

- Solve for y

  $y = -\frac{e^{-3x-3c_1}}{3} + \frac{1}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) +3*y(x)=1,y(x), singsol=all)
```

$$y(x) = \frac{1}{3} + c_1 e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[y'[x] +3*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} + c_1 e^{-3x}$$
$$y(x) \to \frac{1}{3}$$

2.13 problem 13

Internal problem ID [899]
Internal file name [`OUTPUT/899_Sunday_June_05_2022_01_53_33_AM_13407723/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \left(\frac{1}{x} - 1\right) y = -\frac{2}{x}}$$

2.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = -\frac{2}{x}$$

Hence the ode is

$$y' - \frac{(x-1)\, y}{x} = -\frac{2}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x-1}{x}dx}$$
$$= e^{-x+\ln(x)}$$

Which simplifies to

$$\mu = x\,e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{2}{x}\right)$$
$$\frac{d}{dx}\left(e^{-x}xy\right) = \left(x\,e^{-x}\right)\left(-\frac{2}{x}\right)$$
$$d\left(e^{-x}xy\right) = \left(-2\,e^{-x}\right)\,dx$$

Integrating gives

$$e^{-x}xy = \int -2\,e^{-x}\,dx$$
$$e^{-x}xy = 2\,e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = x\,e^{-x}$ results in

$$y = \frac{2\,e^{x}e^{-x}}{x} + \frac{c_1e^{x}}{x}$$

which simplifies to

$$y = \frac{c_1e^{x} + 2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1e^{x} + 2}{x} \tag{1}$$

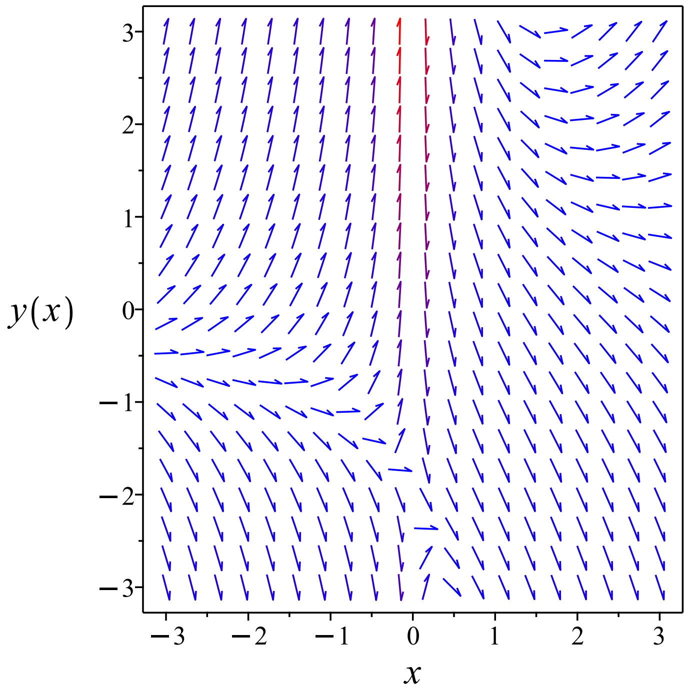


Figure 82: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^x + 2}{x}$$

Verified OK.

2.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{yx - y - 2}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 66: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = e^{x-\ln(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-x} x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{yx - y - 2}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{-x} y(1 - x) \\ S_y &= x\,\mathrm{e}^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{-x} x = 2\,\mathrm{e}^{-x} + c_1$$

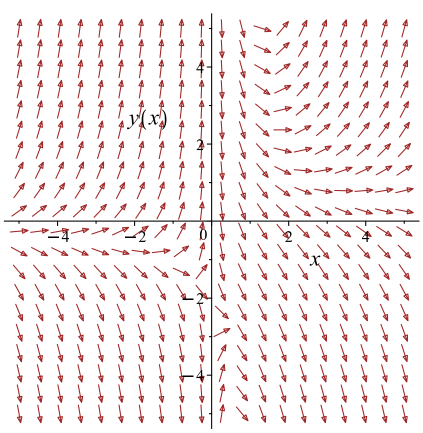
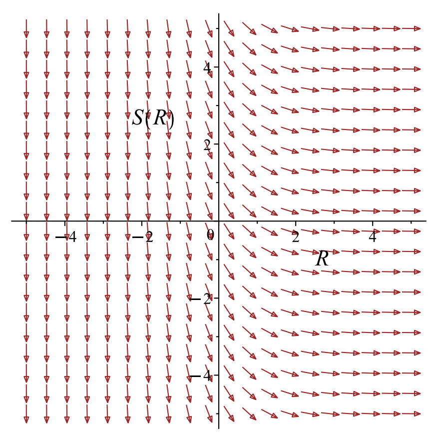
Which simplifies to

$$y\,\mathrm{e}^{-x} x = 2\,\mathrm{e}^{-x} + c_1$$

Which gives

$$y = \frac{(2\,\mathrm{e}^{-x} + c_1)\,\mathrm{e}^{x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{yx-y-2}{x}$ 	$R = x$ $S = \mathrm{e}^{-x} x y$	$\frac{dS}{dR} = -2\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(2\,\mathrm{e}^{-x} + c_1)\,\mathrm{e}^{x}}{x} \tag{1}$$

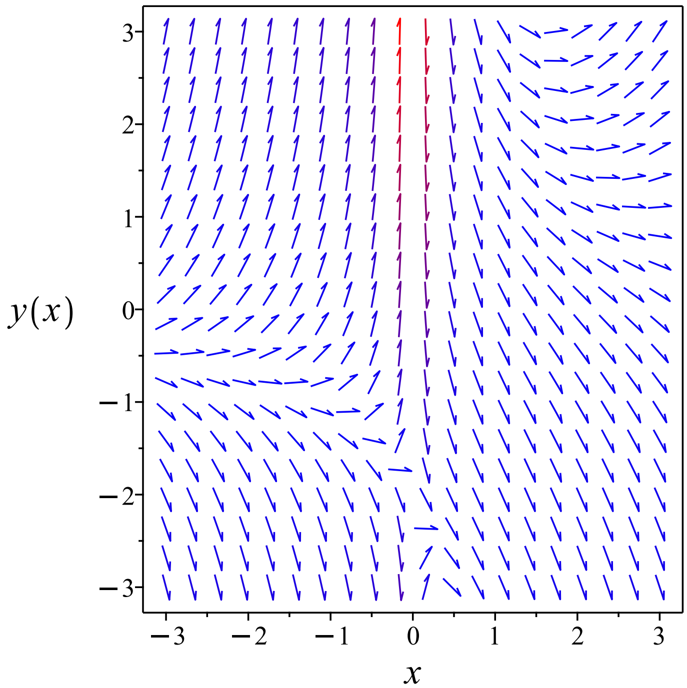


Figure 83: Slope field plot

Verification of solutions

$$y = \frac{(2\,\mathrm{e}^{-x} + c_1)\,\mathrm{e}^{x}}{x}$$

Verified OK.

2.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\left(\frac{1}{x} - 1\right) y - \frac{2}{x}\right) \mathrm{d}x$$

$$\left(\left(\frac{1}{x} - 1\right) y + \frac{2}{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \left(\frac{1}{x} - 1\right) y + \frac{2}{x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\left(\frac{1}{x} - 1\right) y + \frac{2}{x}\right)\\ &= \frac{1}{x} - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{1}{x} - 1\right) - (0)\right)\\ &= \frac{1}{x} - 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{1}{x}-1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x+\ln(x)}\\ &= x\,\mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\,\mathrm{e}^{-x}\left(\left(\frac{1}{x} - 1\right)y + \frac{2}{x}\right)\\ &= -((x-1)\,y - 2)\,\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x\,\mathrm{e}^{-x}(1)\\ &= x\,\mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-((x-1)\,y-2)\,\mathrm{e}^{-x}\right) + \left(x\,\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -((x-1)\,y-2)\,\mathrm{e}^{-x}\,\mathrm{d}x$$
$$\phi = \mathrm{e}^{-x}(yx-2) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$x\,\mathrm{e}^{-x} = x\,\mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x}(yx-2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-x}(yx - 2)$$

The solution becomes

$$y = \frac{(2\,e^{-x} + c_1)\,e^{x}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\,e^{-x} + c_1)\,e^{x}}{x} \tag{1}$$

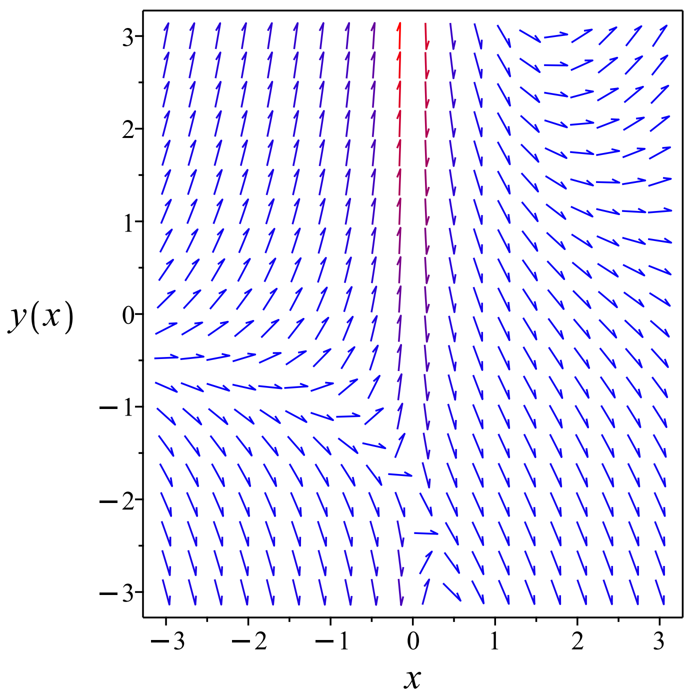


Figure 84: Slope field plot

Verification of solutions

$$y = \frac{(2\,e^{-x} + c_1)\,e^{x}}{x}$$

Verified OK.

2.13.4 Maple step by step solution

Let's solve

$y' + \left(\frac{1}{x} - 1\right) y = -\frac{2}{x}$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = \frac{(x-1)y}{x} - \frac{2}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(x-1)y}{x} = -\frac{2}{x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{(x-1)y}{x}\right) = -\frac{2\mu(x)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{(x-1)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)(x-1)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x\,\mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{2\mu(x)}{x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{2\mu(x)}{x} dx + c_1$
- Solve for y

  $y = \frac{\int -\frac{2\mu(x)}{x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x\mathrm{e}^{-x}$

  $y = \frac{\int -2\,\mathrm{e}^{-x} dx + c_1}{x\,\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{2\,\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x} x}$
- Simplify

$y = \frac{c_1 e^x + 2}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x) +(1/x-1)*y(x)=-2/x,y(x), singsol=all)
```

$$y(x) = \frac{e^x c_1 + 2}{x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[y'[x] +(1/x-1)*y[x]==-2/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2 + c_1 e^x}{x}$$

2.14 problem 14

Internal problem ID [900]
Internal file name [`OUTPUT/900_Sunday_June_05_2022_01_53_34_AM_57306810/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2yx + y' = x\,\mathrm{e}^{-x^2}}$$

2.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= x\,\mathrm{e}^{-x^2} \end{aligned}$$

Hence the ode is

$$2yx + y' = x\,\mathrm{e}^{-x^2}$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x\,e^{-x^2}\right)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)\left(x\,e^{-x^2}\right)$$
$$d\left(e^{x^2} y\right) = x\,dx$$

Integrating gives

$$e^{x^2} y = \int x\,dx$$
$$e^{x^2} y = \frac{x^2}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = \frac{x^2 e^{-x^2}}{2} + c_1 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2}\left(\frac{x^2}{2} + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{-x^2}\left(\frac{x^2}{2} + c_1\right) \tag{1}$$

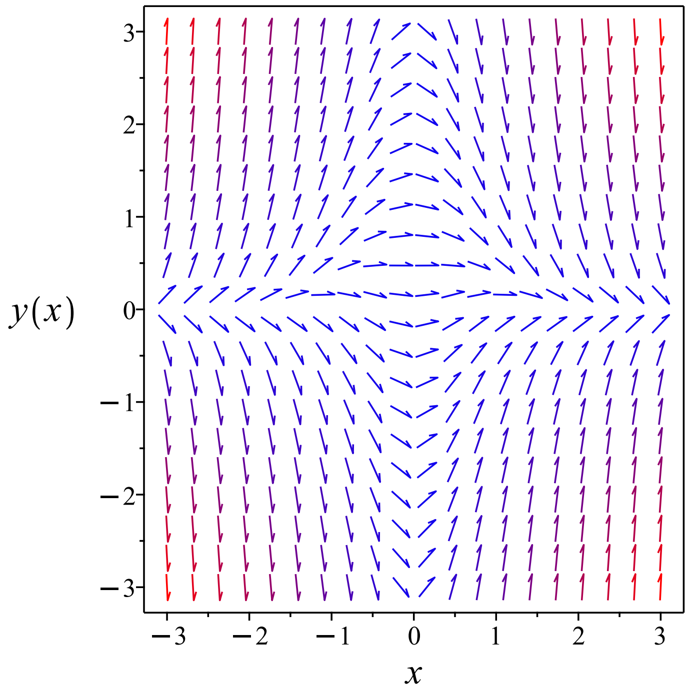


Figure 85: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-x^2}\left(\frac{x^2}{2} + c_1\right)$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx + x\,\mathrm{e}^{-x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 69: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx + x\,e^{-x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\,e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = \frac{x^2}{2} + c_1$$

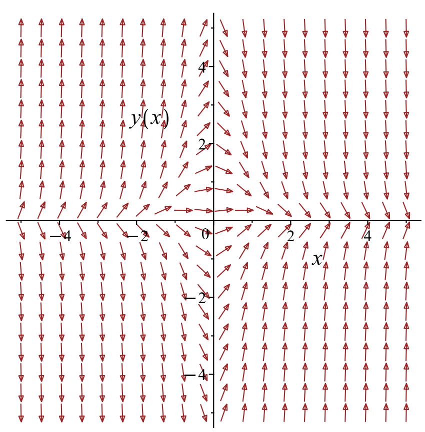
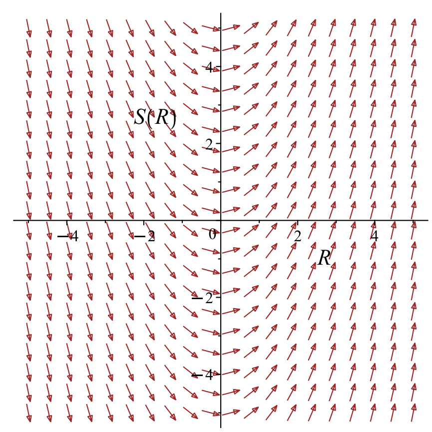
Which simplifies to

$$e^{x^2} y = \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx + x\,e^{-x^2}$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2} \tag{1}$$

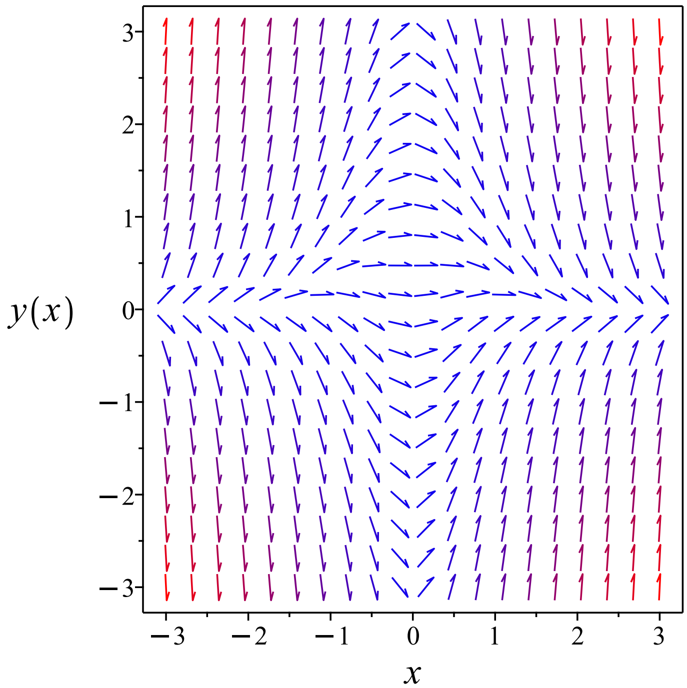


Figure 86: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2}$$

Verified OK.

2.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2yx + x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x$$
$$\left(2yx - x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2yx - x\,\mathrm{e}^{-x^2}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2yx - x\,\mathrm{e}^{-x^2}\right)$$
$$= 2x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2x) - (0)) \\ &= 2x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int 2x \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x^2} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x^2}\left(2yx - x\,\mathrm{e}^{-x^2}\right) \\ &= 2x\,\mathrm{e}^{x^2} y - x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x^2}(1) \\ &= \mathrm{e}^{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(2x\,\mathrm{e}^{x^2} y - x\right) + \left(\mathrm{e}^{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x \, \mathrm{e}^{x^2} y - x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + \mathrm{e}^{x^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{x^2} = \mathrm{e}^{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \mathrm{e}^{x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \mathrm{e}^{x^2} y$$

The solution becomes

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2} \tag{1}$$

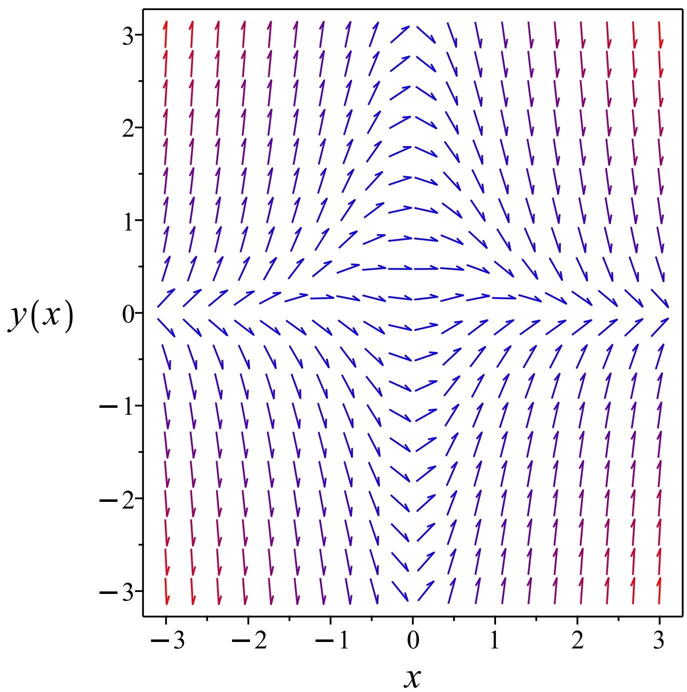


Figure 87: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}(x^2 + 2c_1)}{2}$$

Verified OK.

2.14.4 Maple step by step solution

Let's solve

$2yx + y' = x\,\mathrm{e}^{-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yx + x\,\mathrm{e}^{-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $2yx + y' = x\,\mathrm{e}^{-x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(2yx + y') = \mu(x)\,x\,\mathrm{e}^{-x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(2yx + y') = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x\,\mathrm{e}^{-x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x\,\mathrm{e}^{-x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x\,\mathrm{e}^{-x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x^2}$

  $y = \frac{\int x\,\mathrm{e}^{-x^2} \mathrm{e}^{x^2} dx + c_1}{\mathrm{e}^{x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^2}{2} + c_1}{\mathrm{e}^{x^2}}$

- Simplify

$$y = \frac{e^{-x^2}(x^2+2c_1)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x) +2*x*y(x)=x*exp(-x^2),y(x), singsol=all)
```

$$y(x) = \frac{(x^2+2c_1)\,e^{-x^2}}{2}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 24

```
DSolve[y'[x] +2*x*y[x]==x*Exp[-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x^2}(x^2+2c_1)$$

2.15 problem 15

Internal problem ID [901]
Internal file name [`OUTPUT/901_Sunday_June_05_2022_01_53_35_AM_86813562/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{2xy}{x^2+1} = \frac{e^{-x^2}}{x^2+1}}$$

2.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2+1}$$
$$q(x) = \frac{e^{-x^2}}{x^2+1}$$

Hence the ode is

$$y' + \frac{2xy}{x^2+1} = \frac{e^{-x^2}}{x^2+1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2x}{x^2+1}dx}$$
$$= x^2 + 1$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{e^{-x^2}}{x^2+1}\right)$$
$$\frac{d}{dx}\left(\left(x^2+1\right)y\right) = \left(x^2+1\right)\left(\frac{e^{-x^2}}{x^2+1}\right)$$
$$d\left(\left(x^2+1\right)y\right) = e^{-x^2}\, dx$$

Integrating gives

$$\left(x^2+1\right)y = \int e^{-x^2}\, dx$$
$$\left(x^2+1\right)y = \frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = \frac{\sqrt{\pi}\,\operatorname{erf}(x)}{2x^2+2} + \frac{c_1}{x^2+1}$$

which simplifies to

$$y = \frac{\sqrt{\pi}\,\operatorname{erf}(x) + 2c_1}{2x^2+2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{\pi}\,\operatorname{erf}(x) + 2c_1}{2x^2+2} \tag{1}$$

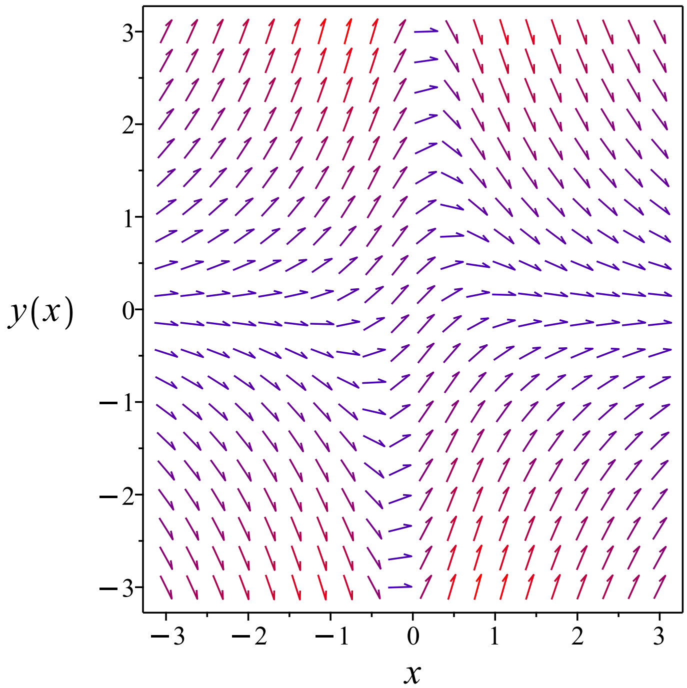


Figure 88: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\pi}\, \mathrm{erf}(x) + 2c_1}{2x^2 + 2}$$

Verified OK.

2.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2yx + \mathrm{e}^{-x^2}}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2 + 1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2+1}} dy \end{aligned}$$

Which results in

$$S = (x^2 + 1) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-2yx + e^{-x^2}}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 + 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\sqrt{\pi}\,\mathrm{erf}\,(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y(x^2+1) = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x)}{2} + c_1$$

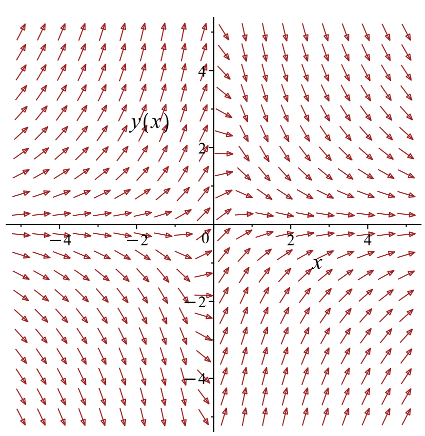
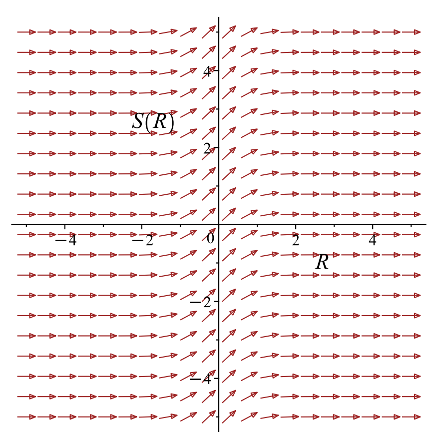
Which simplifies to

$$y(x^2+1) = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x)}{2} + c_1$$

Which gives

$$y = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x) + 2c_1}{2x^2+2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2yx+\mathrm{e}^{-x^2}}{x^2+1}$ 	$R = x$ $S = (x^2+1)\,y$	$\frac{dS}{dR} = \mathrm{e}^{-R^2}$ 

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{\pi}\, \operatorname{erf}(x) + 2c_1}{2x^2 + 2} \tag{1}$$

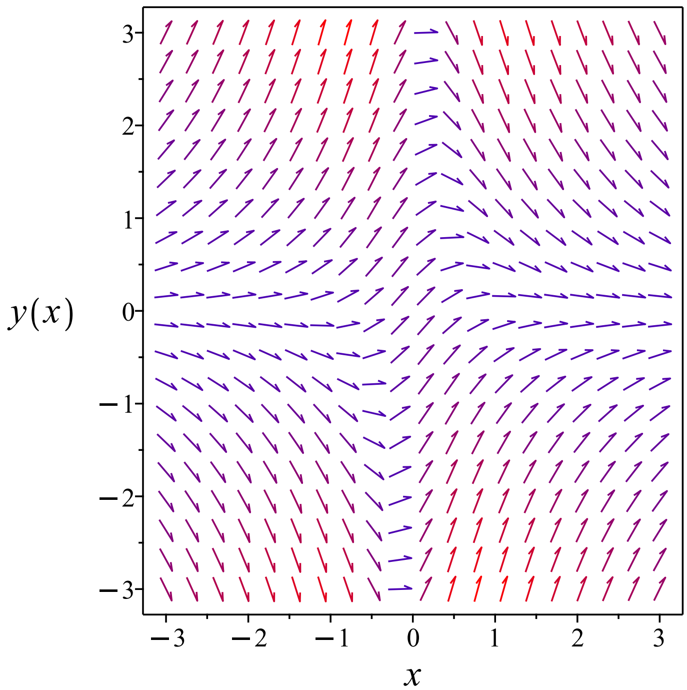


Figure 89: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\pi}\, \operatorname{erf}(x) + 2c_1}{2x^2 + 2}$$

Verified OK.

2.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2+1)\,\mathrm{d}y &= \left(-2yx + \mathrm{e}^{-x^2}\right)\mathrm{d}x\\ \left(2yx - \mathrm{e}^{-x^2}\right)\mathrm{d}x + (x^2+1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2yx - \mathrm{e}^{-x^2}\\ N(x,y) &= x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2yx - \mathrm{e}^{-x^2}\right)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+1)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 2yx - \mathrm{e}^{-x^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = y\,x^2 - \frac{\sqrt{\pi}\,\mathrm{erf}\,(x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 1$. Therefore equation (4) becomes

$$x^2 + 1 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (1)\,\mathrm{d}y\\ f(y) &= y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 - \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right)}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 - \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right)}{2} + y$$

The solution becomes

$$y = \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right) + 2c_1}{2x^2 + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{\pi}\ \mathrm{erf}\left(x\right) + 2c_1}{2x^2 + 2} \tag{1}$$

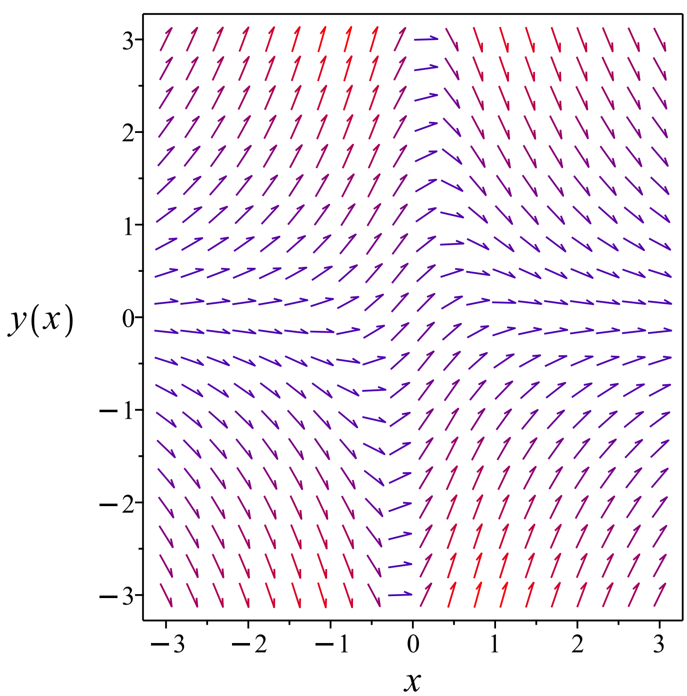


Figure 90: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\pi}\, \operatorname{erf}(x) + 2c_1}{2x^2 + 2}$$

Verified OK.

2.15.4 Maple step by step solution

Let's solve

$y' + \frac{2xy}{x^2+1} = \frac{e^{-x^2}}{x^2+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2xy}{x^2+1} + \frac{e^{-x^2}}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2xy}{x^2+1} = \frac{e^{-x^2}}{x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = \frac{\mu(x)e^{-x^2}}{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2xy}{x^2+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x) = x^2 + 1$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)e^{-x^2}}{x^2+1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)e^{-x^2}}{x^2+1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)e^{-x^2}}{x^2+1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2 + 1$

  $y = \frac{\int e^{-x^2}dx + c_1}{x^2+1}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\sqrt{\pi}\,\mathrm{erf}(x)}{2} + c_1}{x^2+1}$

- Simplify

  $y = \frac{\sqrt{\pi}\,\mathrm{erf}(x) + 2c_1}{2x^2+2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x) +(2*x)/(1+x^2)*y(x)=exp(-x^2)/(1+x^2),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\pi}\,\mathrm{erf}\,(x) + 2c_1}{2x^2+2}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 28

```
DSolve[y'[x] +(2*x)/(1+x^2)*y[x]==Exp[-x^2]/(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\pi}\mathrm{erf}(x) + 2c_1}{2x^2+2}$$

2.16 problem 16

Internal problem ID [902]
Internal file name [`OUTPUT/902_Sunday_June_05_2022_01_53_36_AM_97350989/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{x} = \frac{7}{x^2} + 3}$$

2.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{3x^2 + 7}{x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{3x^2 + 7}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x}dx}$$
$$= x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{3x^2+7}{x^2}\right)$$
$$\frac{d}{dx}(yx) = (x)\left(\frac{3x^2+7}{x^2}\right)$$
$$d(yx) = \left(\frac{3x^2+7}{x}\right)dx$$

Integrating gives

$$yx = \int \frac{3x^2+7}{x}\,dx$$
$$yx = \frac{3x^2}{2} + 7\ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{3x^2}{2} + 7\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$y = \frac{\frac{3x^2}{2} + 7\ln(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{3x^2}{2} + 7\ln(x) + c_1}{x} \tag{1}$$

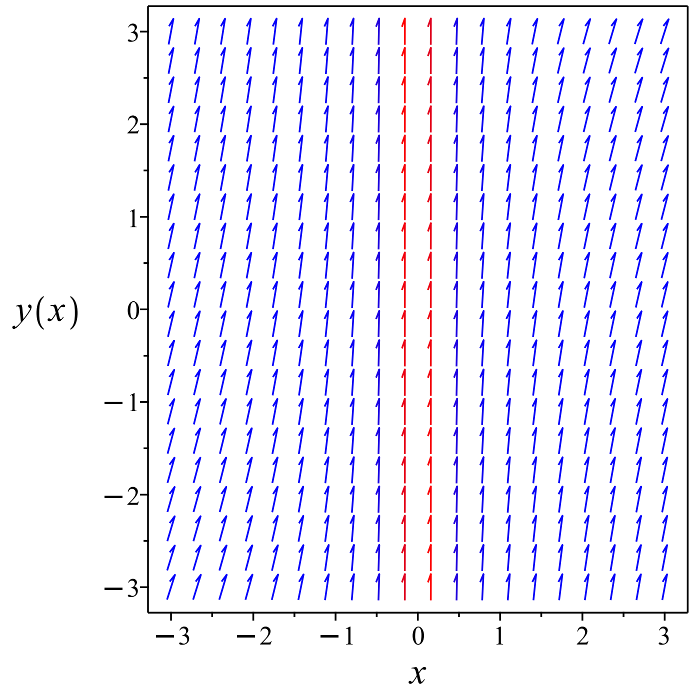


Figure 91: Slope field plot

Verification of solutions

$$y = \frac{\frac{3x^2}{2} + 7\ln(x) + c_1}{x}$$

Verified OK.

2.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-3x^2 + yx - 7}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 75: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-3x^2 + yx - 7}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3x^2 + 7}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3R^2 + 7}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3R^2}{2} + 7\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \frac{3x^2}{2} + 7\ln(x) + c_1$$

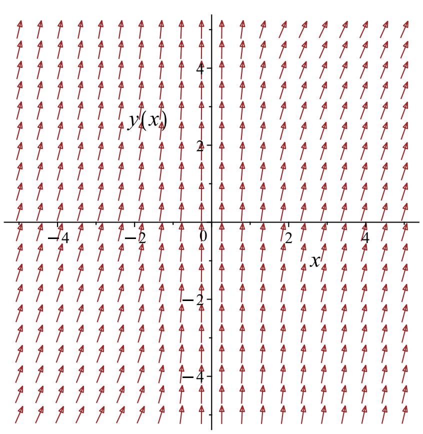
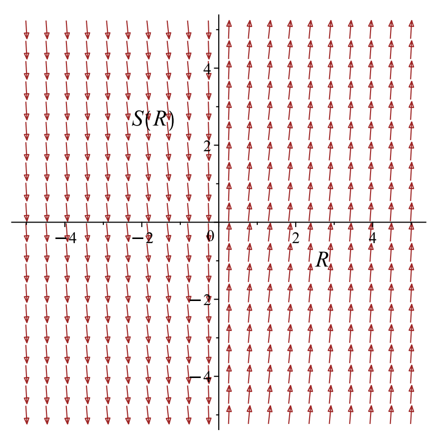
Which simplifies to

$$yx = \frac{3x^2}{2} + 7\ln(x) + c_1$$

Which gives

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x^2+yx-7}{x^2}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = \frac{3R^2+7}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x} \tag{1}$$

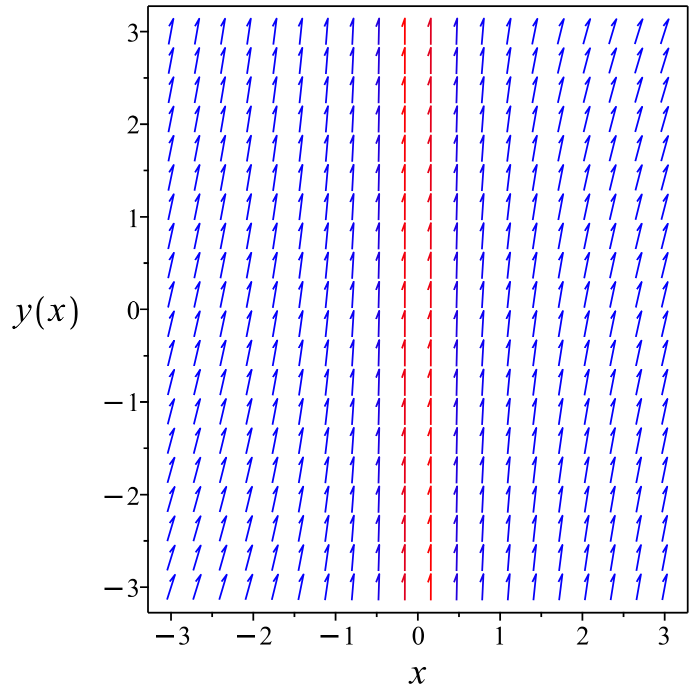


Figure 92: Slope field plot

Verification of solutions

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x}$$

Verified OK.

2.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{x} + \frac{7}{x^2} + 3\right) \mathrm{d}x$$

$$\left(-3 + \frac{y}{x} - \frac{7}{x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -3 + \frac{y}{x} - \frac{7}{x^2} \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3 + \frac{y}{x} - \frac{7}{x^2}\right) \\ &= \frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{1}{x}\right) - (0)\right)\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\, \mathrm{d}x}\\ &= e^{\int \frac{1}{x}\, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(-3 + \frac{y}{x} - \frac{7}{x^2}\right)\\ &= \frac{-3x^2 + yx - 7}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x(1)\\ &= x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-3x^2 + yx - 7}{x}\right) + (x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-3x^2 + yx - 7}{x}\,\mathrm{d}x$$
$$\phi = -\frac{3x^2}{2} + yx - 7\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3x^2}{2} + yx - 7\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3x^2}{2} + yx - 7\ln(x)$$

The solution becomes

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x} \tag{1}$$

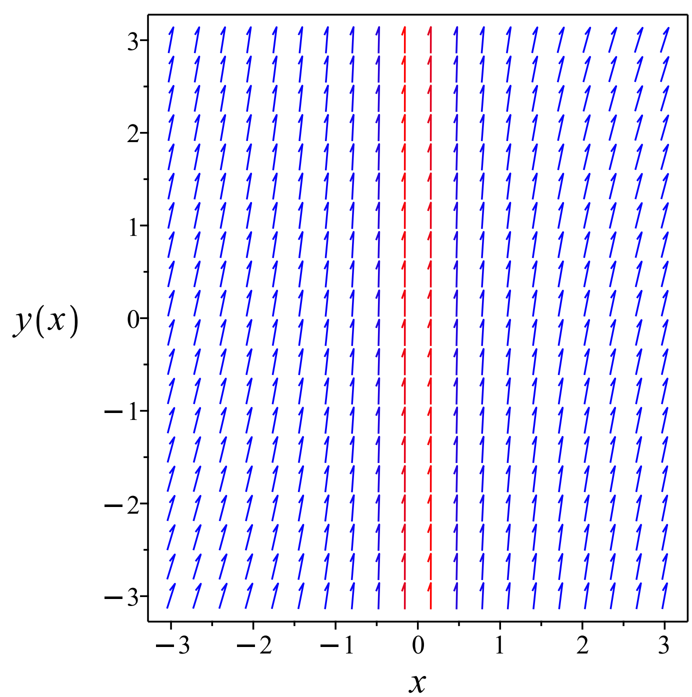


Figure 93: Slope field plot

Verification of solutions

$$y = \frac{3x^2 + 14\ln(x) + 2c_1}{2x}$$

Verified OK.

2.16.4 Maple step by step solution

Let's solve

$y' + \frac{y}{x} = \frac{7}{x^2} + 3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \frac{3x^2+7}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{3x^2+7}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \frac{\mu(x)(3x^2+7)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(3x^2+7)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(3x^2+7)}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(3x^2+7\right)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

$$y = \frac{\int \frac{3x^2+7}{x}dx + c_1}{x}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{3x^2}{2} + 7\ln(x) + c_1}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x) +1/x*y(x)=7/x^2+3,y(x), singsol=all)
```

$$y(x) = \frac{\frac{3x^2}{2} + 7\ln(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[y'[x] +1/x*y[x]==7/x^2+3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x}{2} + \frac{7\log(x)}{x} + \frac{c_1}{x}$$

2.17 problem 17

Internal problem ID [903]
Internal file name [`OUTPUT/903_Sunday_June_05_2022_01_53_37_AM_47617246/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{4y}{x-1} = \frac{1}{(x-1)^5} + \frac{\sin(x)}{(x-1)^4}}$$

2.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{x-1} \\ q(x) &= \frac{1 + \sin(x)(x-1)}{(x-1)^5} \end{aligned}$$

Hence the ode is

$$y' + \frac{4y}{x-1} = \frac{1 + \sin(x)(x-1)}{(x-1)^5}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{x-1}dx}$$
$$= (x-1)^4$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1+\sin(x)(x-1)}{(x-1)^5}\right)$$
$$\frac{d}{dx}\left(y(x-1)^4\right) = \left((x-1)^4\right)\left(\frac{1+\sin(x)(x-1)}{(x-1)^5}\right)$$
$$d\left(y(x-1)^4\right) = \left(\frac{1+\sin(x)(x-1)}{x-1}\right) dx$$

Integrating gives

$$y(x-1)^4 = \int \frac{1+\sin(x)(x-1)}{x-1} dx$$
$$y(x-1)^4 = -\cos(x) + \ln(x-1) + c_1$$

Dividing both sides by the integrating factor $\mu = (x-1)^4$ results in

$$y = \frac{-\cos(x) + \ln(x-1)}{(x-1)^4} + \frac{c_1}{(x-1)^4}$$

which simplifies to

$$y = \frac{-\cos(x) + \ln(x-1) + c_1}{(x-1)^4}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x) + \ln(x-1) + c_1}{(x-1)^4} \tag{1}$$

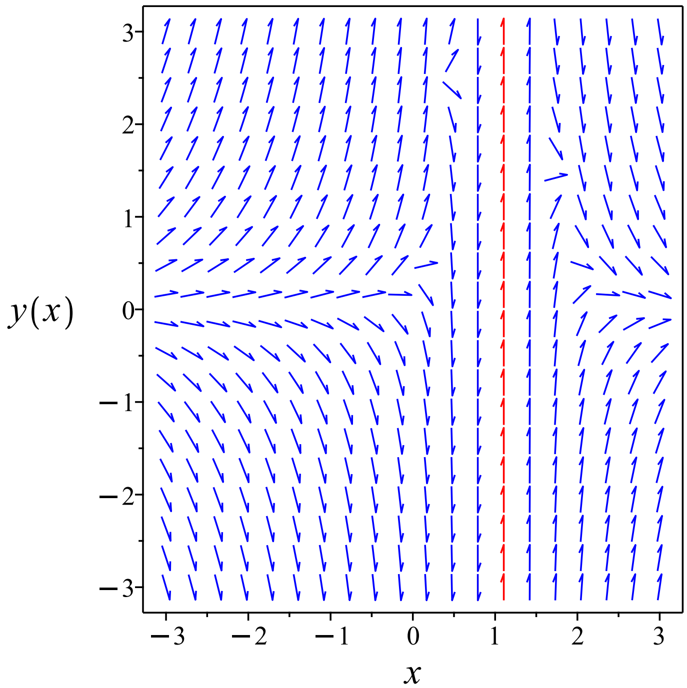


Figure 94: Slope field plot

Verification of solutions

$$y = \frac{-\cos(x) + \ln(x-1) + c_1}{(x-1)^4}$$

Verified OK.

2.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-4x^4y + 16y\,x^3 - 24y\,x^2 + \sin(x)\,x + 16yx - \sin(x) - 4y + 1}{(x-1)^5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 78: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{(x-1)^4}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x-1)^4}} dy \end{aligned}$$

Which results in

$$S = y(x-1)^4$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-4x^4 y + 16y\, x^3 - 24y\, x^2 + \sin(x)\, x + 16yx - \sin(x) - 4y + 1}{(x-1)^5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4y(x-1)^3 \\ S_y &= (x-1)^4 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1 + \sin(x)(x-1)}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1 + \sin(R)(R-1)}{R-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R) + \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-1)^4 y = -\cos(x) + \ln(x-1) + c_1$$

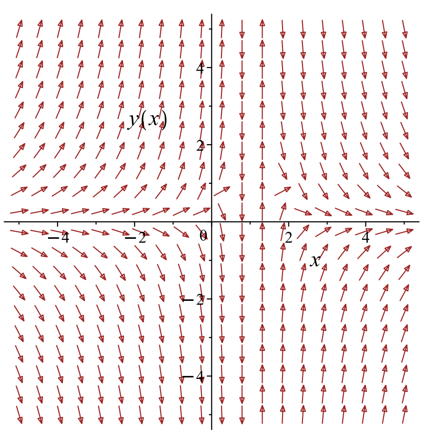
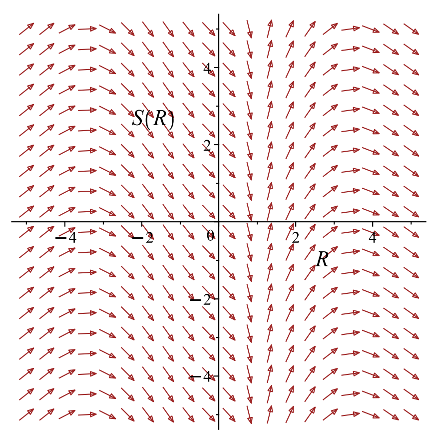
Which simplifies to

$$(x-1)^4 y = -\cos(x) + \ln(x-1) + c_1$$

Which gives

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{(x-1)^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-4x^4y+16y\,x^3-24y\,x^2+\sin(x)x+16yx-\sin(x)-4y+1}{(x-1)^5}$ 	$R = x$ $S = y(x-1)^4$	$\frac{dS}{dR} = \frac{1+\sin(R)(R-1)}{R-1}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{(x-1)^4} \tag{1}$$

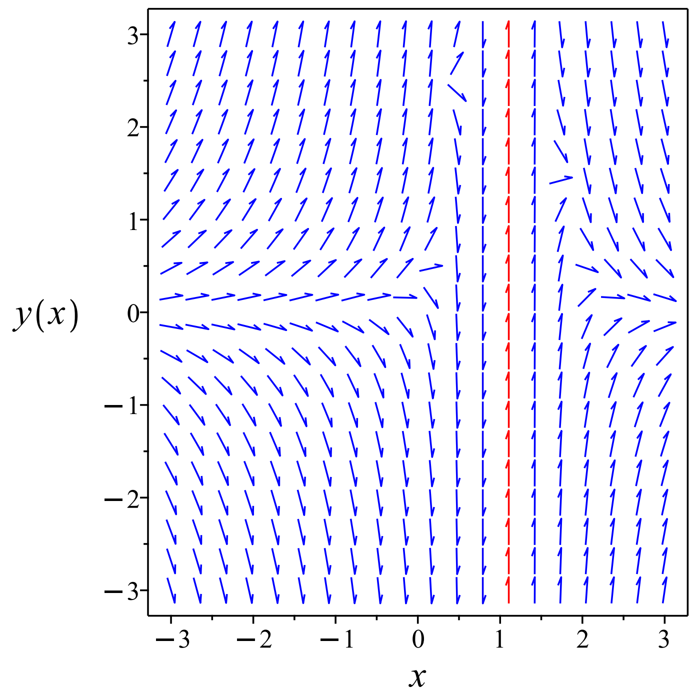


Figure 95: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{(x-1)^4}$$

Verified OK.

2.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-\frac{4y}{x-1}+\frac{1}{(x-1)^5}+\frac{\sin(x)}{(x-1)^4}\right)\mathrm{d}x$$

$$\left(\frac{4y}{x-1}-\frac{1}{(x-1)^5}-\frac{\sin(x)}{(x-1)^4}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\frac{4y}{x-1}-\frac{1}{(x-1)^5}-\frac{\sin(x)}{(x-1)^4}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{4y}{x-1} - \frac{1}{(x-1)^5} - \frac{\sin(x)}{(x-1)^4}\right) \\
&= \frac{4}{x-1}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{4}{x-1}\right) - (0)\right) \\
&= \frac{4}{x-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{4}{x-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{4\ln(x-1)} \\
&= (x-1)^4
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= (x-1)^4\left(\frac{4y}{x-1} - \frac{1}{(x-1)^5} - \frac{\sin(x)}{(x-1)^4}\right) \\
&= \left(\frac{4y}{x-1} - \frac{1}{(x-1)^5} - \frac{\sin(x)}{(x-1)^4}\right)(x-1)^4
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= (x-1)^4 (1) \\
&= (x-1)^4
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(\frac{4y}{x-1} - \frac{1}{(x-1)^5} - \frac{\sin(x)}{(x-1)^4}\right)(x-1)^4\right) + \left((x-1)^4\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \left(\frac{4y}{x-1} - \frac{1}{(x-1)^5} - \frac{\sin(x)}{(x-1)^4}\right)(x-1)^4 \mathrm{d}x \\
\phi &= x^4 y + 6y\,x^2 - 4yx - 4y\,x^3 + \cos(x) - \ln(x-1) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= x^4 - 4x^3 + 6x^2 - 4x + f'(y) \qquad (4) \\
&= x(-2+x)\left(x^2 - 2x + 2\right) + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x-1)^4$. Therefore equation (4) becomes

$$
(x-1)^4 = x(-2+x)\left(x^2 - 2x + 2\right) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^4y + 6y\,x^2 - 4yx - 4y\,x^3 + \cos(x) - \ln(x-1) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^4y + 6y\,x^2 - 4yx - 4y\,x^3 + \cos(x) - \ln(x-1) + y$$

The solution becomes

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{x^4 - 4x^3 + 6x^2 - 4x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{x^4 - 4x^3 + 6x^2 - 4x + 1} \tag{1}$$

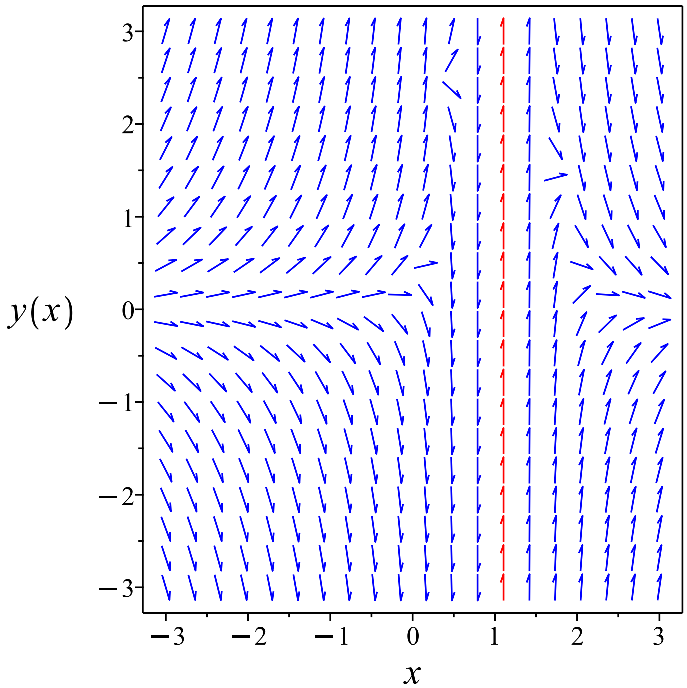


Figure 96: Slope field plot

Verification of solutions

$$y = -\frac{\cos (x) - \ln (x-1) - c_1}{x^4 - 4x^3 + 6x^2 - 4x + 1}$$

Verified OK.

2.17.4 Maple step by step solution

Let's solve

$y' + \frac{4y}{x-1} = \frac{1}{(x-1)^5} + \frac{\sin(x)}{(x-1)^4}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4y}{x-1} + \frac{\sin(x)x-\sin(x)+1}{(x-1)^5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4y}{x-1} = \frac{\sin(x)x-\sin(x)+1}{(x-1)^5}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{4y}{x-1}\right)=\frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^5}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{4y}{x-1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{4\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x)=(x-1)^4$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^5}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^5}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^5}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=(x-1)^4$

  $y=\frac{\int\frac{\sin(x)x-\sin(x)+1}{x-1}dx+c_1}{(x-1)^4}$

- Evaluate the integrals on the rhs

  $y=\frac{-\cos(x)+\ln(x-1)+c_1}{(x-1)^4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x) +4/(x-1)*y(x)=1/(x-1)^5+sin(x)/(x-1)^4,y(x), singsol=all)
```

$$y(x) = \frac{-\cos(x) + \ln(x-1) + c_1}{(x-1)^4}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 22

```
DSolve[y'[x] +4/(x-1)*y[x]==1/(x-1)^5+Sin[x]/(x-1)^4,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\log(x-1) - \cos(x) + c_1}{(x-1)^4}$$

2.18 problem 18

Internal problem ID [904]
Internal file name [`OUTPUT/904_Sunday_June_05_2022_01_53_39_AM_69619743/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + y(2x^2 + 1) = x^3 \mathrm{e}^{-x^2}}$$

2.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x^2 - 1}{x}$$
$$q(x) = x^2 \mathrm{e}^{-x^2}$$

Hence the ode is

$$y' - \frac{(-2x^2 - 1)\, y}{x} = x^2 \mathrm{e}^{-x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-2x^2-1}{x}dx} \\
&= \mathrm{e}^{x^2+\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{x^2} x
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(x^2 \mathrm{e}^{-x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\mathrm{e}^{x^2} y\right) &= \left(\mathrm{e}^{x^2} x\right) \left(x^2 \mathrm{e}^{-x^2}\right) \\
\mathrm{d}\left(x\,\mathrm{e}^{x^2} y\right) &= x^3 \,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x\,\mathrm{e}^{x^2} y &= \int x^3 \,\mathrm{d}x \\
x\,\mathrm{e}^{x^2} y &= \frac{x^4}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2} x$ results in

$$
y = \frac{x^3 \mathrm{e}^{-x^2}}{4} + \frac{c_1 \mathrm{e}^{-x^2}}{x}
$$

which simplifies to

$$
y = \frac{\mathrm{e}^{-x^2} (x^4 + 4c_1)}{4x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{-x^2} (x^4 + 4c_1)}{4x} \tag{1}
$$

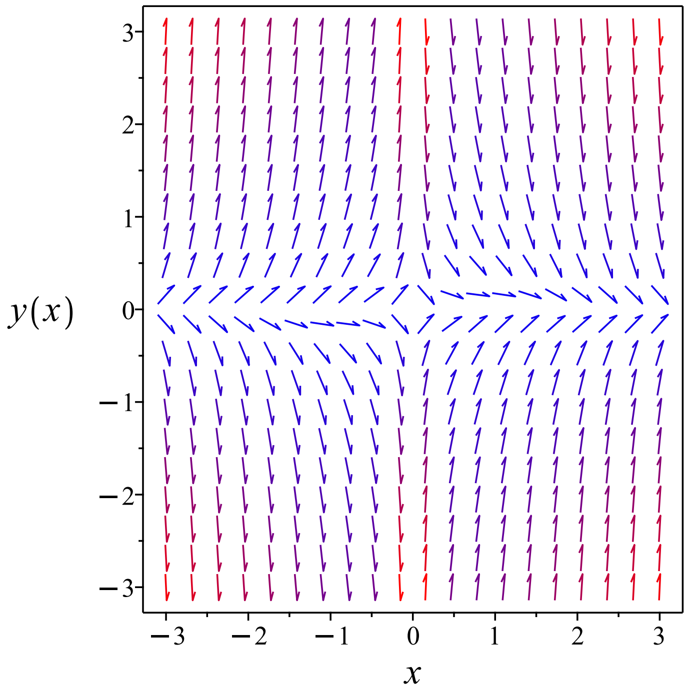


Figure 97: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}(x^4 + 4c_1)}{4x}$$

Verified OK.

2.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 e^{-x^2} - 2y\,x^2 - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 81: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2 - \ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2 - \ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 e^{-x^2} - 2y\,x^2 - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x^2} y\left(2x^2 + 1\right) \\ S_y &= e^{x^2} x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x\,\mathrm{e}^{x^2} y = \frac{x^4}{4} + c_1$$

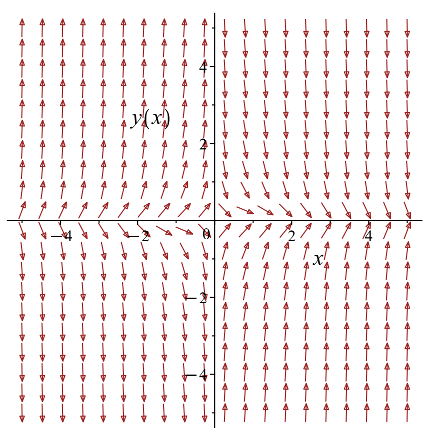
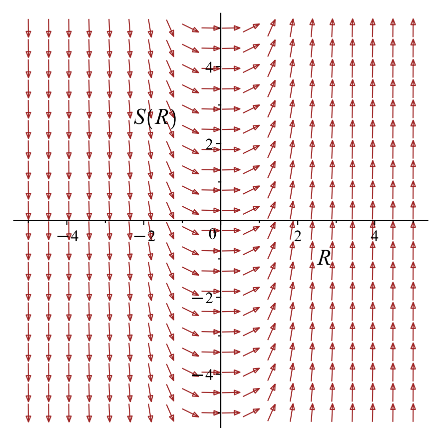
Which simplifies to

$$x\,\mathrm{e}^{x^2} y = \frac{x^4}{4} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-x^2}(x^4 + 4c_1)}{4x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3\mathrm{e}^{-x^2} - 2y\,x^2 - y}{x}$ 	$R = x$ $S = x\,\mathrm{e}^{x^2} y$	$\frac{dS}{dR} = R^3$ 

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-x^2}(x^4 + 4c_1)}{4x} \tag{1}$$

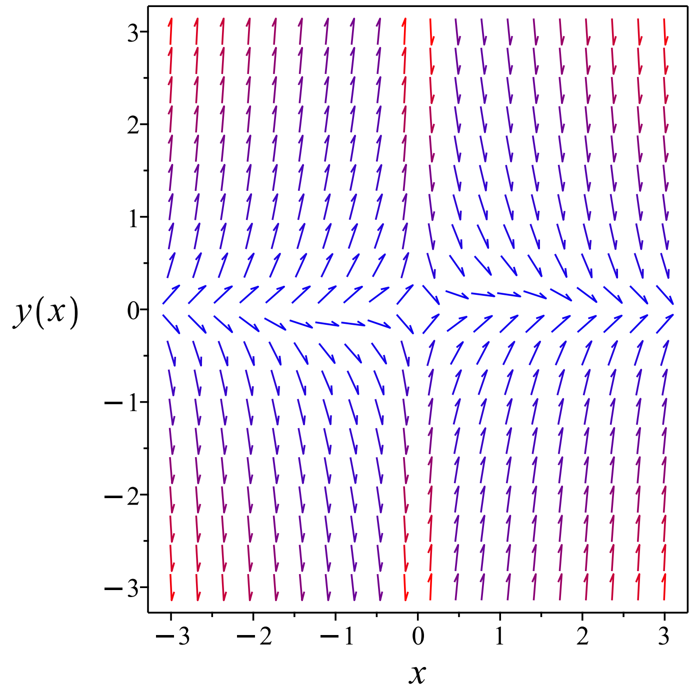


Figure 98: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}(x^4 + 4c_1)}{4x}$$

Verified OK.

2.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-y(2x^2+1) + x^3 \mathrm{e}^{-x^2}\right) \mathrm{d}x$$
$$\left(y(2x^2+1) - x^3 \mathrm{e}^{-x^2}\right) \mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y(2x^2+1) - x^3 \mathrm{e}^{-x^2}$$
$$N(x,y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y(2x^2+1) - x^3 \mathrm{e}^{-x^2}\right) \\ &= 2x^2 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}\left(\left(2x^2+1\right) - (1)\right) \\ &= 2x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2x \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x^2} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x^2}\left(y\left(2x^2+1\right) - x^3\mathrm{e}^{-x^2}\right) \\ &= \mathrm{e}^{x^2} y\left(2x^2+1\right) - x^3 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x^2}(x) \\ &= \mathrm{e}^{x^2} x \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{x^2} y\left(2x^2+1\right) - x^3\right) + \left(\mathrm{e}^{x^2} x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{align*}
\frac{\partial \phi}{\partial x} &= \overline{M} \tag{1}\\
\frac{\partial \phi}{\partial y} &= \overline{N} \tag{2}
\end{align*}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align*}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \mathrm{e}^{x^2} y(2x^2 + 1) - x^3 \, \mathrm{d}x \\
\phi &= x\,\mathrm{e}^{x^2} y - \frac{x^4}{4} + f(y) \tag{3}
\end{align*}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} x$. Therefore equation (4) becomes

$$
\mathrm{e}^{x^2} x = \mathrm{e}^{x^2} x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = x\,\mathrm{e}^{x^2} y - \frac{x^4}{4} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = x\,\mathrm{e}^{x^2} y - \frac{x^4}{4}
$$

The solution becomes

$$y = \frac{e^{-x^2}(x^4 + 4c_1)}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x^2}(x^4 + 4c_1)}{4x} \tag{1}$$

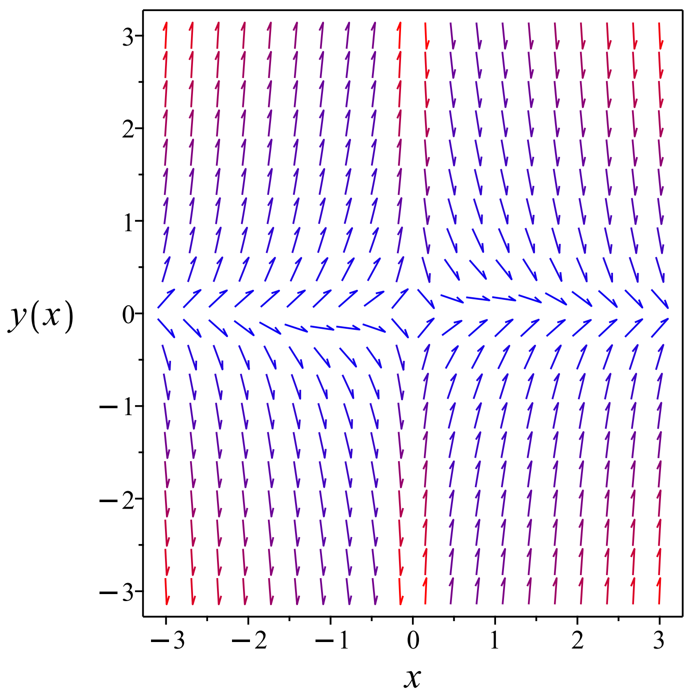


Figure 99: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}(x^4 + 4c_1)}{4x}$$

Verified OK.

2.18.4 Maple step by step solution

Let's solve

$y'x + y(2x^2+1) = x^3 e^{-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(2x^2+1)y}{x} + x^2 e^{-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(2x^2+1)y}{x} = x^2 e^{-x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(2x^2+1)y}{x}\right) = \mu(x)\, x^2 e^{-x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{(2x^2+1)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(2x^2+1)}{x}$

- Solve to find the integrating factor

  $\mu(x) = e^{x^2} x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 e^{-x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 e^{-x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 e^{-x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{x^2} x$

  $y = \frac{\int x^3 e^{-x^2} e^{x^2} dx + c_1}{e^{x^2} x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^4}{4} + c_1}{e^{x^2} x}$

- Simplify

  $$y = \frac{e^{-x^2}(x^4+4c_1)}{4x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x) +(1+2*x^2)*y(x)=x^3*exp(-x^2),y(x), singsol=all)
```

$$y(x) = \frac{(x^4+4c_1)\,e^{-x^2}}{4x}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 27

```
DSolve[x*y'[x] +(1+2*x^2)*y[x]==x^3*Exp[-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x^2}(x^4+4c_1)}{4x}$$

2.19 problem 19

Internal problem ID [905]
Internal file name [`OUTPUT/905_Sunday_June_05_2022_01_53_40_AM_11648253/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + y'x = \frac{2}{x^2} + 1}$$

2.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{x^2 + 2}{x^3}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{x^2 + 2}{x^3}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2+2}{x^3}\right) \\
\frac{d}{dx}\left(y\,x^2\right) &= \left(x^2\right)\left(\frac{x^2+2}{x^3}\right) \\
d\left(y\,x^2\right) &= \left(\frac{x^2+2}{x}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^2 &= \int \frac{x^2+2}{x}\, dx \\
y\,x^2 &= \frac{x^2}{2} + 2\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{\frac{x^2}{2} + 2\ln(x)}{x^2} + \frac{c_1}{x^2}
$$

which simplifies to

$$
y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2} \tag{1}
$$

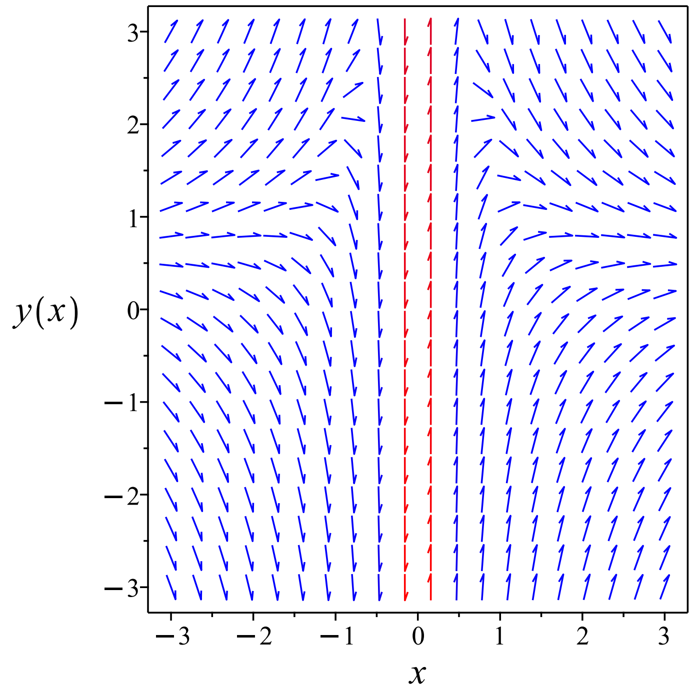


Figure 100: Slope field plot

Verification of solutions

$$y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}$$

Verified OK.

2.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y\,x^2 - x^2 - 2}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y\,x^2 - x^2 - 2}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2 + 2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2 + 2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = \frac{x^2}{2} + 2\ln(x) + c_1$$

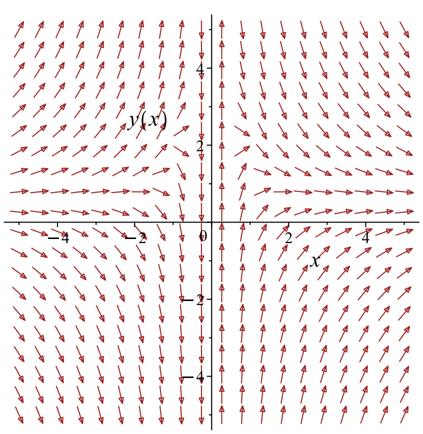
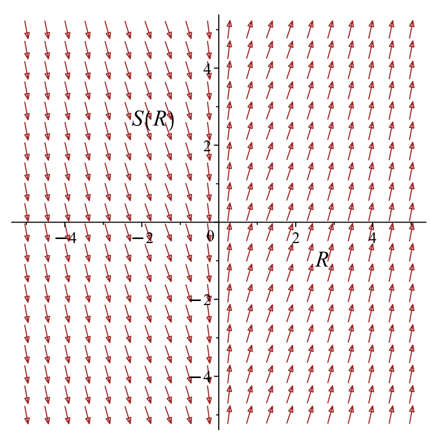
Which simplifies to

$$x^2 y = \frac{x^2}{2} + 2\ln(x) + c_1$$

Which gives

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y\,x^2 - x^2 - 2}{x^3}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = \frac{R^2 + 2}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2} \tag{1}$$

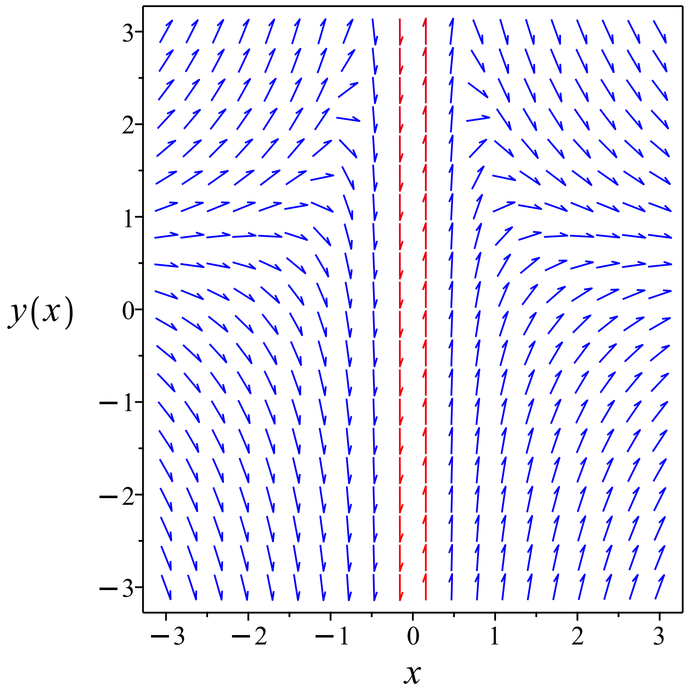


Figure 101: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}$$

Verified OK.

2.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \, dy &= \left(-2y + \frac{2}{x^2} + 1\right) dx \\
\left(2y - 1 - \frac{2}{x^2}\right) dx + (x) \, dy &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - 1 - \frac{2}{x^2} \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - 1 - \frac{2}{x^2}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2) - (1)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(2y - 1 - \frac{2}{x^2}\right) \\
&= \left(2y - 1 - \frac{2}{x^2}\right)x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\left(2y - 1 - \frac{2}{x^2}\right)x\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \left(2y - 1 - \frac{2}{x^2}\right)x\,\mathrm{d}x$$
$$\phi = \frac{x^2(-1 + 2y)}{2} - 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2(-1+2y)}{2} - 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2(-1+2y)}{2} - 2\ln(x)$$

The solution becomes

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2} \tag{1}$$

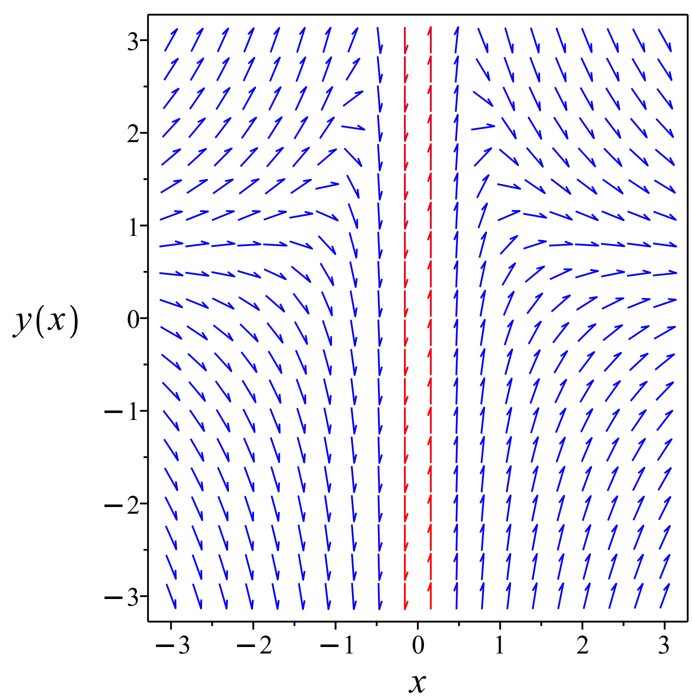


Figure 102: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x^2}$$

Verified OK.

2.19.4 Maple step by step solution

Let's solve

$2y + y'x = \frac{2}{x^2} + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{x^2+2}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{x^2+2}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)(x^2+2)}{x^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)(x^2+2)}{x^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)(x^2+2)}{x^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(x^2+2\right)}{x^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

$$y = \frac{\int \frac{x^2+2}{x}dx + c_1}{x^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x) +2*y(x)=2/x^2+1,y(x), singsol=all)
```

$$y(x) = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 22

```
DSolve[x*y'[x] +2*y[x]==2/x^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\log(x)}{x^2} + \frac{c_1}{x^2} + \frac{1}{2}$$

2.20 problem 20

Internal problem ID [906]
Internal file name [`OUTPUT/906_Sunday_June_05_2022_01_53_41_AM_75690544/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \tan(x)\, y = \cos(x)}$$

2.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \tan(x)$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' + \tan(x)\, y = \cos(x)$$

The integrating factor μ is

$$\mu = e^{\int \tan(x)dx}$$
$$= \frac{1}{\cos(x)}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\cos(x))$$
$$\frac{d}{dx}(\sec(x)\, y) = (\sec(x))(\cos(x))$$
$$d(\sec(x)\, y) = dx$$

Integrating gives

$$\sec(x)\, y = \int dx$$
$$\sec(x)\, y = x + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = x\cos(x) + c_1 \cos(x)$$

which simplifies to

$$y = \cos(x)(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \cos(x)(x + c_1) \tag{1}$$

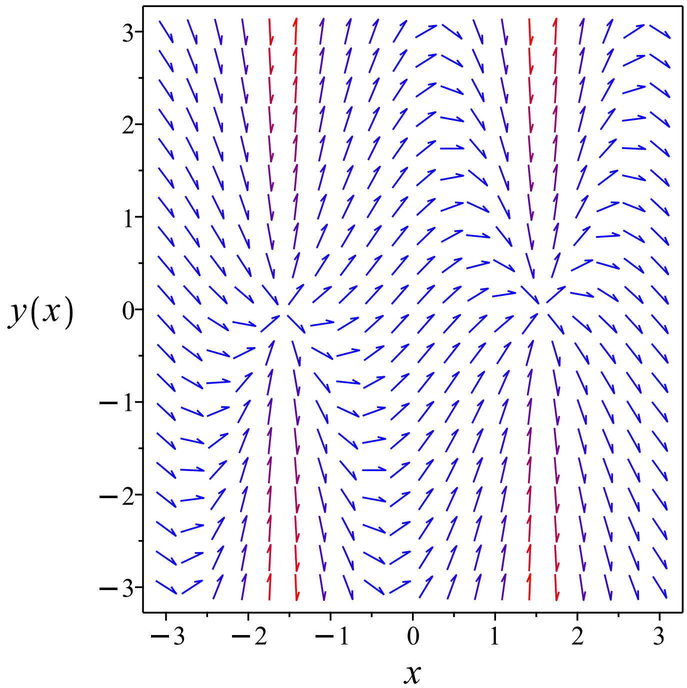


Figure 103: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(x)(x + c_1)$$

Verified OK.

2.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \cos(x) - \tan(x)\, y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \cos(x)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(x) - \tan(x) y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \tan(x) \sec(x) y \\ S_y &= \sec(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec(x)\, y = x + c_1$$

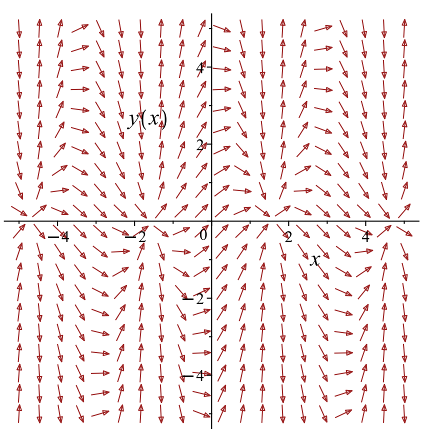
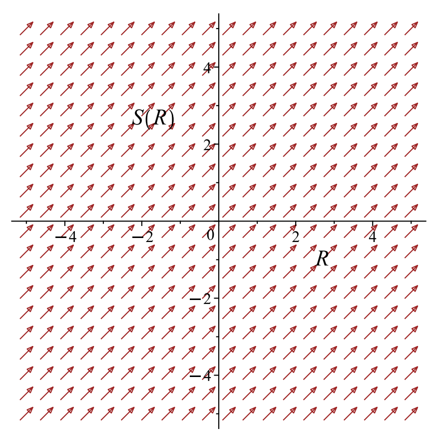
Which simplifies to

$$\sec(x)\, y = x + c_1$$

Which gives

$$y = \frac{x + c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \cos(x) - \tan(x)\, y$

</td><td>$R = x$
$S = \sec(x)\, y$</td><td>$\frac{dS}{dR} = 1$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{x + c_1}{\sec(x)} \tag{1}$$

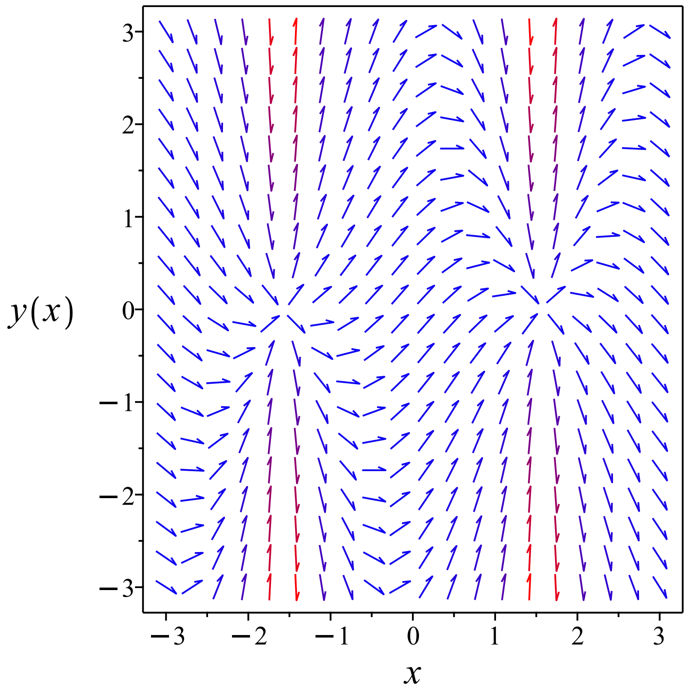


Figure 104: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\sec(x)}$$

Verified OK.

2.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (\cos(x) - \tan(x)\,y)\,\mathrm{d}x \\ (\tan(x)\,y - \cos(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \tan(x)\,y - \cos(x) \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\tan(x)\,y - \cos(x)) \\ &= \tan(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\tan(x)) - (0)) \\ &= \tan(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \tan(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(x))} \\ &= \sec(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sec(x)(\tan(x) y - \cos(x)) \\ &= -1 + \tan(x) \sec(x) y \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sec(x)(1) \\ &= \sec(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-1 + \tan(x)\sec(x) y) + (\sec(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -1 + \tan(x) \sec(x) y \mathrm{d}x$$
$$\phi = -x + \sec(x) y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(x)$. Therefore equation (4) becomes

$$\sec(x) = \sec(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \sec(x) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \sec(x) y$$

The solution becomes

$$y = \frac{x + c_1}{\sec(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{\sec(x)} \tag{1}$$

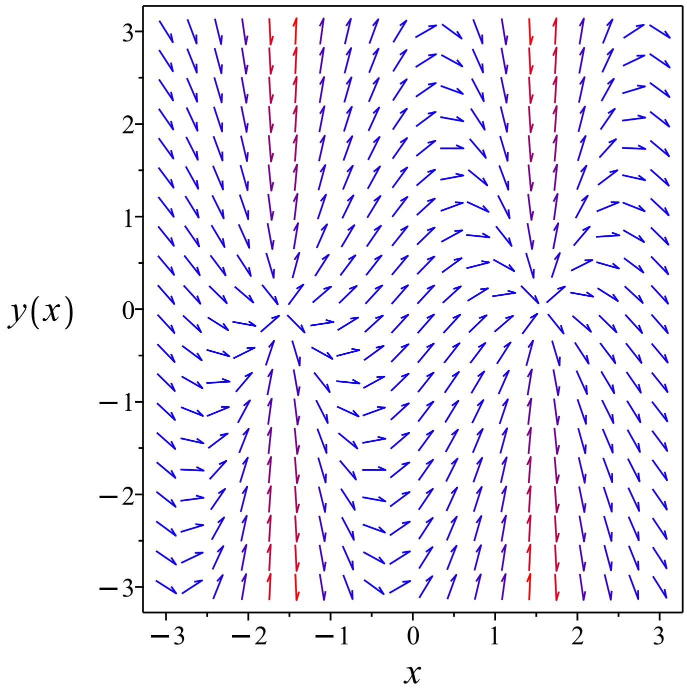


Figure 105: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\sec(x)}$$

Verified OK.

2.20.4 Maple step by step solution

Let's solve

$y' + \tan(x)\, y = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = \cos(x) - \tan(x)\, y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \tan(x)\, y = \cos(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \tan(x)\, y\right) = \mu(x)\cos(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + \tan(x)\, y\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\tan(x)$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\cos(x)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\cos(x)\, dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\cos(x)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

$$y = \cos(x)\left(\int 1dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \cos(x)\,(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve(diff(y(x),x) +tan(x)*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = (c_1 + x) \cos(x)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 12

```
DSolve[y'[x] +Tan[x]*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_1) \cos(x)$$

2.21 problem 21

Internal problem ID [907]
Internal file name [`OUTPUT/907_Sunday_June_05_2022_01_53_42_AM_41989435/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x+1)\,y'+2y=\frac{\sin(x)}{x+1}}$$

2.21.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{2}{x+1}$$
$$q(x)=\frac{\sin(x)}{(x+1)^2}$$

Hence the ode is

$$y'+\frac{2y}{x+1}=\frac{\sin(x)}{(x+1)^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x+1}dx}$$
$$= (x+1)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(x)}{(x+1)^2}\right)$$
$$\frac{d}{dx}\left((x+1)^2 y\right) = \left((x+1)^2\right)\left(\frac{\sin(x)}{(x+1)^2}\right)$$
$$d\left((x+1)^2 y\right) = \sin(x)\ dx$$

Integrating gives

$$(x+1)^2 y = \int \sin(x)\ dx$$
$$(x+1)^2 y = -\cos(x) + c_1$$

Dividing both sides by the integrating factor $\mu = (x+1)^2$ results in

$$y = -\frac{\cos(x)}{(x+1)^2} + \frac{c_1}{(x+1)^2}$$

which simplifies to

$$y = \frac{-\cos(x) + c_1}{(x+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x) + c_1}{(x+1)^2} \tag{1}$$

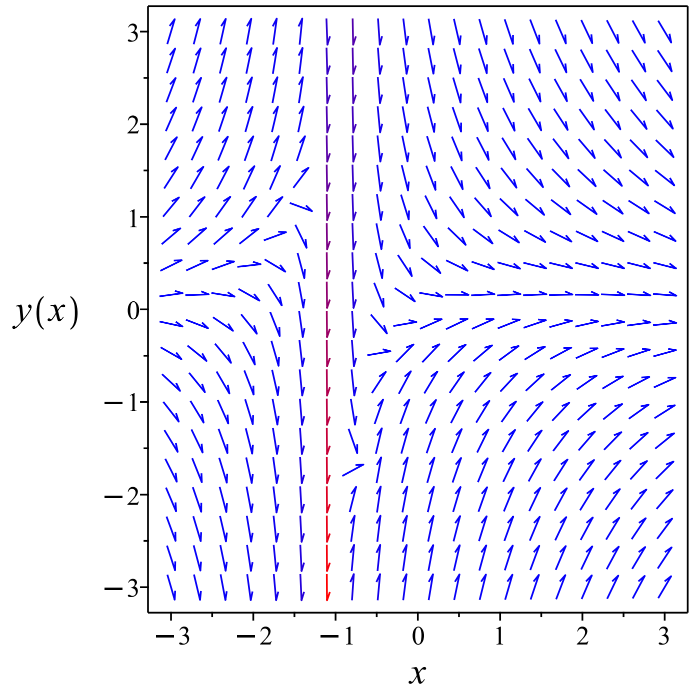


Figure 106: Slope field plot

Verification of solutions

$$y = \frac{-\cos(x) + c_1}{(x+1)^2}$$

Verified OK.

2.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2yx + \sin(x) - 2y}{(x+1)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 90: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{(x+1)^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x+1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2yx + \sin(x) - 2y}{(x+1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2(x+1)\, y \\ S_y &= (x+1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x+1)^2 y = -\cos(x) + c_1$$

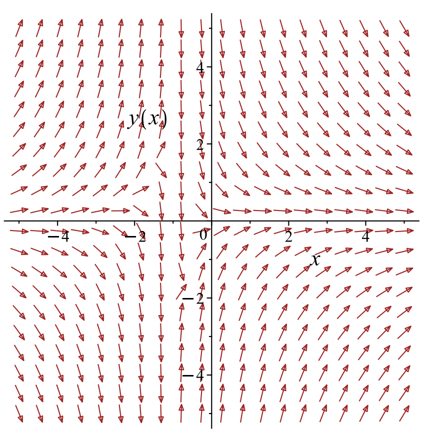
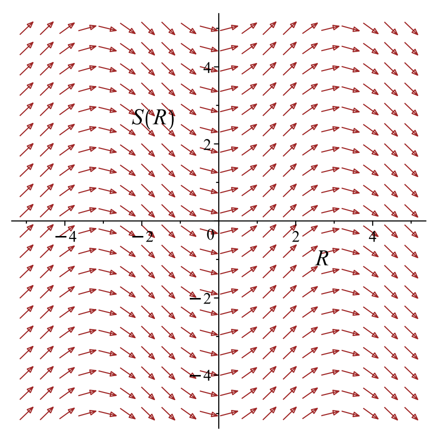
Which simplifies to

$$(x+1)^2 y = -\cos(x) + c_1$$

Which gives

$$y = -\frac{\cos(x) - c_1}{(x+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2yx+\sin(x)-2y}{(x+1)^2}$ 	$R = x$ $S = (x+1)^2 y$	$\frac{dS}{dR} = \sin(R)$ 

Summary
The solution(s) found are the following

$$y = -\frac{\cos(x) - c_1}{(x+1)^2} \tag{1}$$

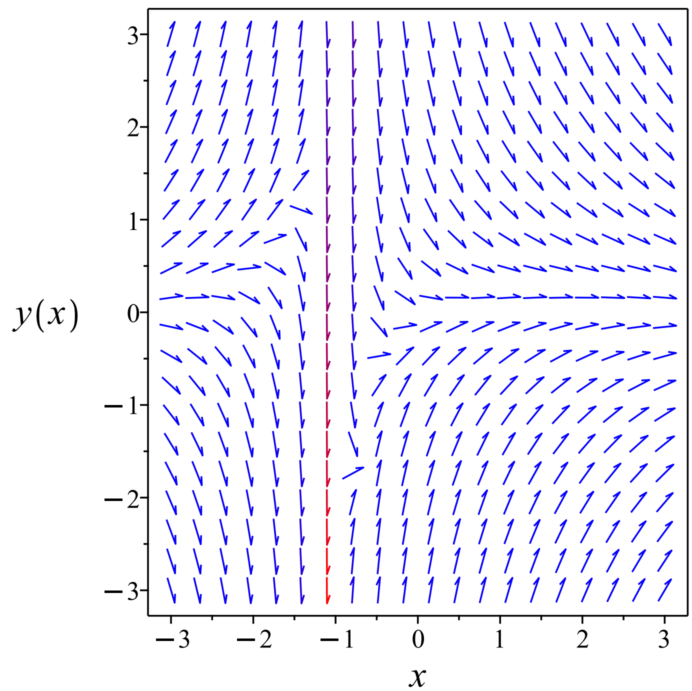


Figure 107: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - c_1}{(x+1)^2}$$

Verified OK.

2.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left((x+1)^2\right)\mathrm{d}y = (-2yx + \sin(x) - 2y)\,\mathrm{d}x$$
$$(2yx - \sin(x) + 2y)\,\mathrm{d}x + \left((x+1)^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2yx - \sin(x) + 2y$$
$$N(x, y) = (x+1)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx - \sin(x) + 2y) \\
&= 2 + 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left((x+1)^2\right) \\
&= 2 + 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 2yx - \sin(x) + 2y \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = y\,x^2 + 2yx + \cos(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + 2x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x+1)^2$. Therefore equation (4) becomes

$$
(x+1)^2 = x^2 + 2x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 + 2yx + \cos(x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + 2yx + \cos(x) + y$$

The solution becomes

$$y = -\frac{\cos(x) - c_1}{x^2 + 2x + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\cos(x) - c_1}{x^2 + 2x + 1} \tag{1}$$

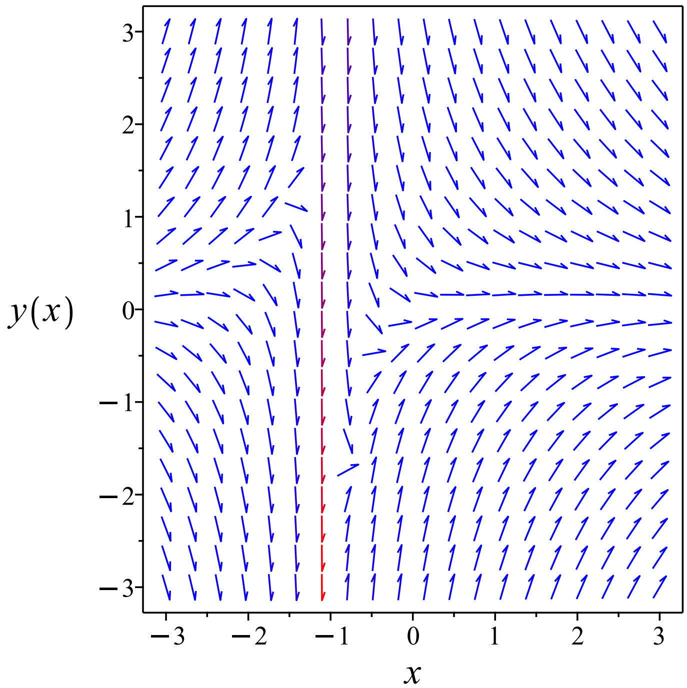


Figure 108: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x) - c_1}{x^2 + 2x + 1}$$

Verified OK.

2.21.4 Maple step by step solution

Let's solve

$(x+1)\, y' + 2y = \frac{\sin(x)}{x+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x+1} + \frac{\sin(x)}{(x+1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x+1} = \frac{\sin(x)}{(x+1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{2y}{x+1}\right)=\frac{\mu(x)\sin(x)}{(x+1)^2}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{2y}{x+1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{2\mu(x)}{x+1}$$

- Solve to find the integrating factor

$$\mu(x)=(x+1)^2$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\sin(x)}{(x+1)^2}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{\mu(x)\sin(x)}{(x+1)^2}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{\mu(x)\sin(x)}{(x+1)^2}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=(x+1)^2$

$$y=\frac{\int\sin(x)dx+c_1}{(x+1)^2}$$

- Evaluate the integrals on the rhs

$$y=\frac{-\cos(x)+c_1}{(x+1)^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+x)*diff(y(x),x) +2*y(x)=sin(x)/(1+x),y(x), singsol=all)
```

$$y(x) = \frac{-\cos(x) + c_1}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 18

```
DSolve[(1+x)*y'[x] +2*y[x]==Sin[x]/(1+x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\cos(x) + c_1}{(x+1)^2}$$

2.22 problem 22

Internal problem ID [908]
Internal file name [`OUTPUT/908_Sunday_June_05_2022_01_53_44_AM_46748277/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(-2+x)(x-1)y'-(4x-3)y=(-2+x)^3}$$

2.22.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{4x-3}{(-2+x)(x-1)}$$
$$q(x)=\frac{(-2+x)^2}{x-1}$$

Hence the ode is

$$y'-\frac{(4x-3)y}{(-2+x)(x-1)}=\frac{(-2+x)^2}{x-1}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{4x-3}{(-2+x)(x-1)}dx}$$
$$= \mathrm{e}^{\ln(x-1)-5\ln(-2+x)}$$

Which simplifies to

$$\mu = \frac{x-1}{(-2+x)^5}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{(-2+x)^2}{x-1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x-1)\,y}{(-2+x)^5}\right) = \left(\frac{x-1}{(-2+x)^5}\right)\left(\frac{(-2+x)^2}{x-1}\right)$$
$$\mathrm{d}\left(\frac{(x-1)\,y}{(-2+x)^5}\right) = \frac{1}{(-2+x)^3}\,\mathrm{d}x$$

Integrating gives

$$\frac{(x-1)\,y}{(-2+x)^5} = \int \frac{1}{(-2+x)^3}\,\mathrm{d}x$$
$$\frac{(x-1)\,y}{(-2+x)^5} = -\frac{1}{2\,(-2+x)^2} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{x-1}{(-2+x)^5}$ results in

$$y = -\frac{(-2+x)^3}{2\,(x-1)} + \frac{c_1(-2+x)^5}{x-1}$$

which simplifies to

$$y = \frac{2\left(-\frac{1}{2} + c_1(-2+x)^2\right)(-2+x)^3}{2x-2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{2\left(-\frac{1}{2} + c_1(-2+x)^2\right)(-2+x)^3}{2x-2} \tag{1}$$

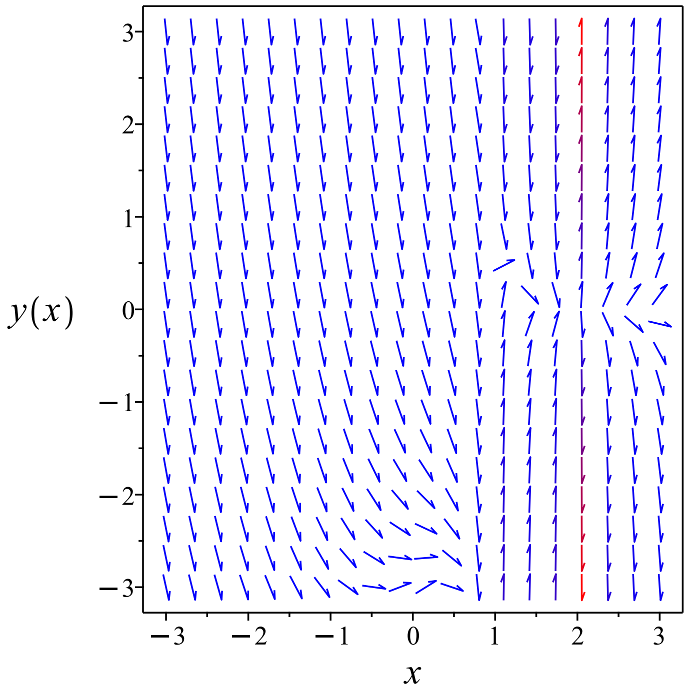


Figure 109: Slope field plot

Verification of solutions

$$y = \frac{2\left(-\frac{1}{2} + c_1(-2+x)^2\right)(-2+x)^3}{2x-2}$$

Verified OK.

2.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 - 6x^2 + 4yx + 12x - 3y - 8}{(-2+x)(x-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-\ln(x-1)+5\ln(-2+x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\ln(x-1)+5\ln(-2+x)}} dy \end{aligned}$$

Which results in

$$S = \frac{(x-1)\,y}{(-2+x)^5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 - 6x^2 + 4yx + 12x - 3y - 8}{(-2+x)(x-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(-4x+3)}{(-2+x)^6} \\ S_y &= \frac{x-1}{(-2+x)^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(-2+x)^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(-2+R)^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2\left(-2+R\right)^{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x-1)\, y}{\left(-2+x\right)^{5}} = -\frac{1}{2\left(-2+x\right)^{2}} + c_1$$

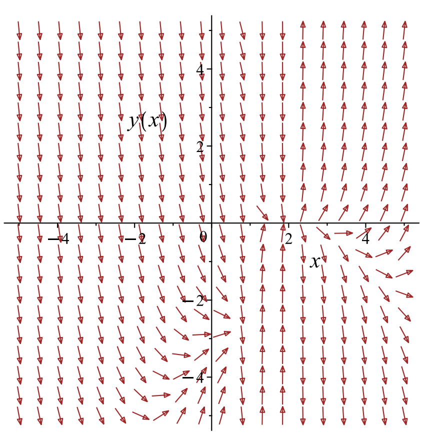
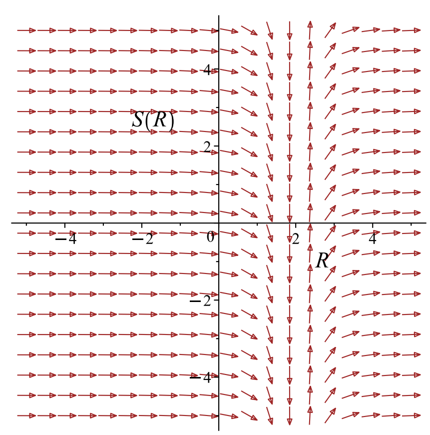
Which simplifies to

$$\frac{(x-1)\, y}{\left(-2+x\right)^{5}} = -\frac{1}{2\left(-2+x\right)^{2}} + c_1$$

Which gives

$$y = \frac{\left(-2+x\right)^{3}\left(2c_1x^{2} - 8c_1x + 8c_1 - 1\right)}{2x-2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3-6x^2+4yx+12x-3y-8}{(-2+x)(x-1)}$ 	$R = x$ $S = \frac{(x-1)\, y}{\left(-2+x\right)^{5}}$	$\frac{dS}{dR} = \frac{1}{(-2+R)^3}$ 

Summary
The solution(s) found are the following

$$y = \frac{(-2+x)^3 (2c_1x^2 - 8c_1x + 8c_1 - 1)}{2x - 2} \tag{1}$$

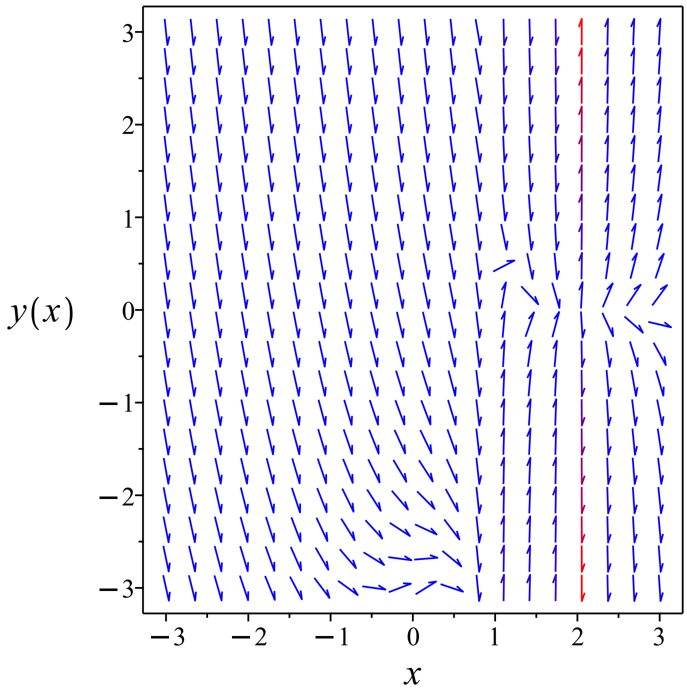


Figure 110: Slope field plot

Verification of solutions

$$y = \frac{(-2+x)^3 (2c_1x^2 - 8c_1x + 8c_1 - 1)}{2x - 2}$$

Verified OK.

2.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}((-2+x)(x-1))\,\mathrm{d}y&=\left((4x-3)\,y+(-2+x)^3\right)\mathrm{d}x\\ \left(-(4x-3)\,y-(-2+x)^3\right)\mathrm{d}x+((-2+x)(x-1))\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-(4x-3)\,y-(-2+x)^3\\ N(x,y)&=(-2+x)(x-1)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-(4x-3)\,y-(-2+x)^3\right)\\ &= -4x+3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}((-2+x)\,(x-1))\\ &= -3+2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{(-2+x)\,(x-1)}((-4x+3)-(-3+2x))\\ &= -\frac{6}{-2+x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{6}{-2+x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-6\ln(-2+x)}\\ &= \frac{1}{(-2+x)^6}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(-2+x)^6}\left(-(4x-3)\,y-(-2+x)^3\right)\\ &= \frac{-(4x-3)\,y-(-2+x)^3}{(-2+x)^6}\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(-2+x)^6}((-2+x)(x-1)) \\
&= \frac{x-1}{(-2+x)^5}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-(4x-3)\,y-(-2+x)^3}{(-2+x)^6}\right) + \left(\frac{x-1}{(-2+x)^5}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-(4x-3)\,y-(-2+x)^3}{(-2+x)^6}\,\mathrm{d}x
\end{aligned}$$

$$\phi = \frac{y}{(-2+x)^5} + \frac{y}{(-2+x)^4} + \frac{1}{2\,(-2+x)^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{(-2+x)^5} + \frac{1}{(-2+x)^4} + f'(y) \tag{4}$$

$$= \frac{x-1}{(-2+x)^5} + f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{x-1}{(-2+x)^5}$. Therefore equation (4) becomes

$$\frac{x-1}{(-2+x)^5}=\frac{x-1}{(-2+x)^5}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{y}{(-2+x)^5}+\frac{y}{(-2+x)^4}+\frac{1}{2(-2+x)^2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\frac{y}{(-2+x)^5}+\frac{y}{(-2+x)^4}+\frac{1}{2(-2+x)^2}$$

The solution becomes

$$y=\frac{(-2+x)^3(2c_1x^2-8c_1x+8c_1-1)}{2x-2}$$

<u>Summary</u>
The solution(s) found are the following

$$y=\frac{(-2+x)^3(2c_1x^2-8c_1x+8c_1-1)}{2x-2} \tag{1}$$

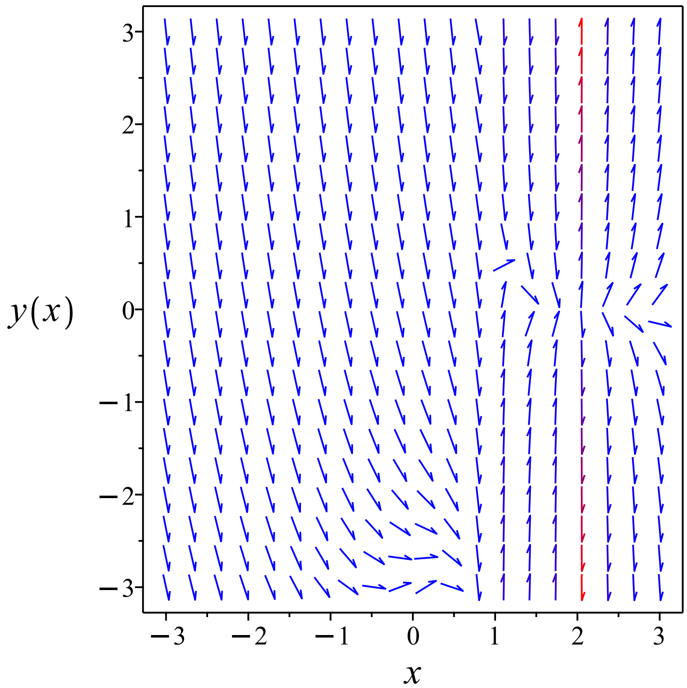


Figure 111: Slope field plot

Verification of solutions

$$y = \frac{(-2+x)^3 (2c_1x^2 - 8c_1x + 8c_1 - 1)}{2x - 2}$$

Verified OK.

2.22.4 Maple step by step solution

Let's solve

$(-2+x)(x-1)y' - (4x-3)y = (-2+x)^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{(4x-3)y}{(-2+x)(x-1)} + \frac{(-2+x)^2}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(4x-3)y}{(-2+x)(x-1)} = \frac{(-2+x)^2}{x-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' - \frac{(4x-3)y}{(-2+x)(x-1)}\right) = \frac{\mu(x)(-2+x)^2}{x-1}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' - \frac{(4x-3)y}{(-2+x)(x-1)}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)(4x-3)}{(-2+x)(x-1)}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{x-1}{(-2+x)^5}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{\mu(x)(-2+x)^2}{x-1}dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)(-2+x)^2}{x-1}dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)(-2+x)^2}{x-1}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{x-1}{(-2+x)^5}$

$$y = \frac{(-2+x)^5\left(\int \frac{1}{(-2+x)^3}dx + c_1\right)}{x-1}$$

- Evaluate the integrals on the rhs

$$y = \frac{(-2+x)^5\left(-\frac{1}{2(-2+x)^2} + c_1\right)}{x-1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve((x-2)*(x-1)*diff(y(x),x) -(4*x-3)*y(x)=(x-2)^3,y(x), singsol=all)
```

$$y(x) = \frac{2\left(-\frac{1}{2} + (-2+x)^2 c_1\right)(-2+x)^3}{2x-2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 30

```
DSolve[(x-2)*(x-1)*y'[x] -(4*x-3)*y[x]==(x-2)^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x-2)^3 \left(-1+2c_1(x-2)^2\right)}{2(x-1)}$$

2.23 problem 23

Internal problem ID [909]
Internal file name [`OUTPUT/909_Sunday_June_05_2022_01_53_45_AM_64182429/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2\cos(x)\sin(x)\,y = \mathrm{e}^{-\sin(x)^2}}$$

2.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \sin(2x)$$
$$q(x) = \mathrm{e}^{-\sin(x)^2}$$

Hence the ode is

$$y' + y\sin(2x) = \mathrm{e}^{-\sin(x)^2}$$

The integrating factor μ is

$$\mu = e^{\int \sin(2x)dx}$$
$$= e^{-\frac{\cos(2x)}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{-\sin(x)^2}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{\cos(2x)}{2}} y\right) = \left(e^{-\frac{\cos(2x)}{2}}\right)\left(e^{-\sin(x)^2}\right)$$
$$d\left(e^{-\frac{\cos(2x)}{2}} y\right) = e^{-\sin(x)^2-\frac{\cos(2x)}{2}}\,dx$$

Integrating gives

$$e^{-\frac{\cos(2x)}{2}} y = \int e^{-\sin(x)^2-\frac{\cos(2x)}{2}}\,dx$$
$$e^{-\frac{\cos(2x)}{2}} y = x\,e^{-\frac{1}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{\cos(2x)}{2}}$ results in

$$y = e^{\frac{\cos(2x)}{2}} x\,e^{-\frac{1}{2}} + c_1 e^{\frac{\cos(2x)}{2}}$$

which simplifies to

$$y = e^{\frac{\cos(2x)}{2}}\left(x\,e^{-\frac{1}{2}} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\cos(2x)}{2}}\left(x\,e^{-\frac{1}{2}} + c_1\right) \tag{1}$$

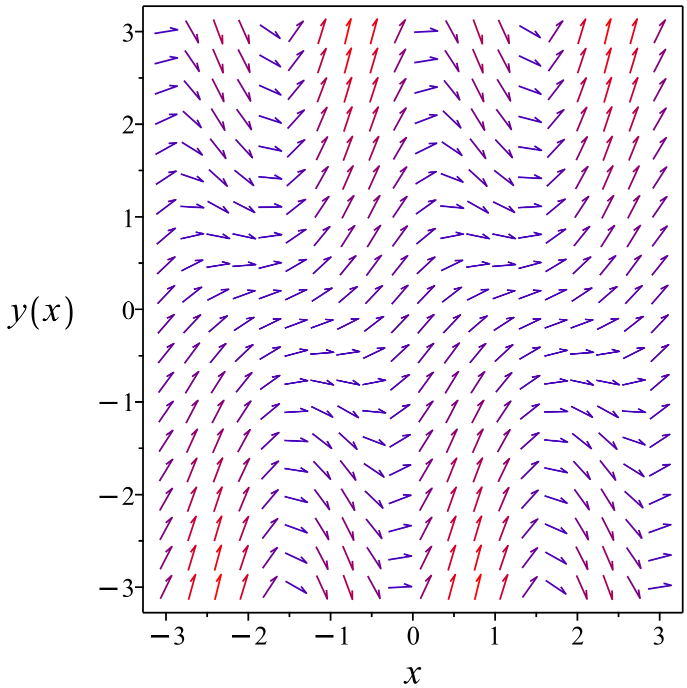


Figure 112: Slope field plot

Verification of solutions

$$y = e^{\frac{\cos(2x)}{2}} \left(x\, e^{-\frac{1}{2}} + c_1\right)$$

Verified OK.

2.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2\cos(x)\sin(x)\, y + e^{-\sin(x)^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\cos(2x)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\cos(2x)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{\cos(2x)}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2 \cos(x) \sin(x) y + e^{-\sin(x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sin(2x) e^{-\frac{\cos(2x)}{2}} y \\ S_y &= e^{-\frac{\cos(2x)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-\sin(x)^2 - \frac{\cos(2x)}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-\sin(R)^2 - \frac{\cos(2R)}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{-\frac{1}{2}} R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{\cos(2x)}{2}} y = x\,\mathrm{e}^{-\frac{1}{2}} + c_1$$

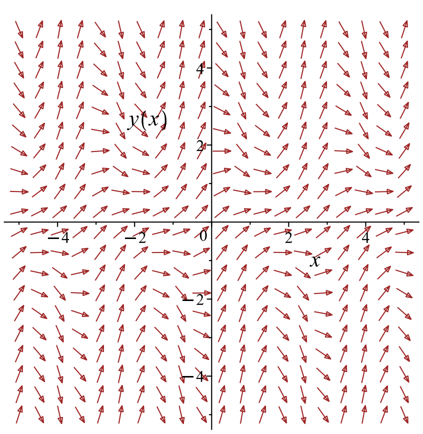
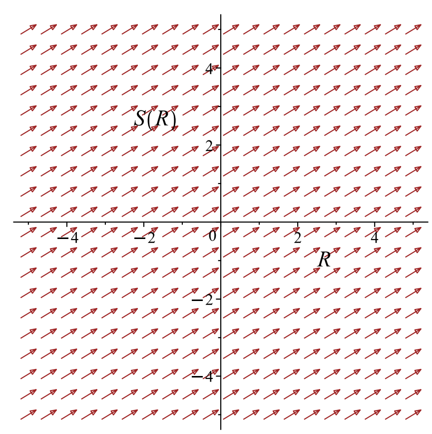
Which simplifies to

$$\mathrm{e}^{-\frac{\cos(2x)}{2}} y = x\,\mathrm{e}^{-\frac{1}{2}} + c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\cos(2x)}{2}} \left(x\,\mathrm{e}^{-\frac{1}{2}} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2\cos(x)\sin(x)\,y + \mathrm{e}^{-\sin(x)^2}$ 	$R = x$ $S = \mathrm{e}^{-\frac{\cos(2x)}{2}} y$	$\frac{dS}{dR} = \mathrm{e}^{-\sin(R)^2 - \frac{\cos(2R)}{2}}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\cos(2x)}{2}} \left(x\,\mathrm{e}^{-\frac{1}{2}} + c_1\right) \tag{1}$$

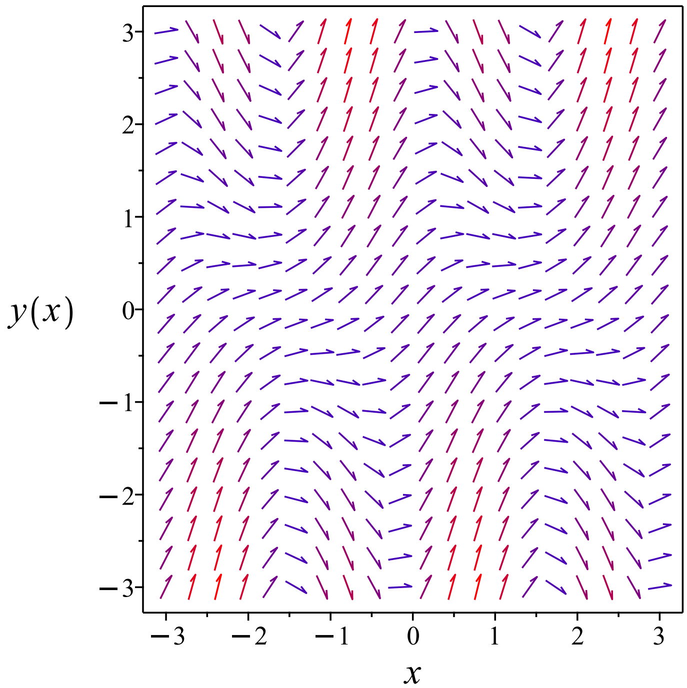


Figure 113: Slope field plot

Verification of solutions

$$y = e^{\frac{\cos(2x)}{2}} \left(x\, e^{-\frac{1}{2}} + c_1\right)$$

Verified OK.

2.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2\cos(x)\sin(x)\,y + \mathrm{e}^{-\sin(x)^2}\right)\mathrm{d}x$$
$$\left(2\cos(x)\sin(x)\,y - \mathrm{e}^{-\sin(x)^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2\cos(x)\sin(x)\,y - \mathrm{e}^{-\sin(x)^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2\cos(x)\sin(x)\,y - \mathrm{e}^{-\sin(x)^2}\right)$$
$$= \sin(2x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2\cos(x)\sin(x)) - (0)) \\ &= \sin(2x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \sin(2x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{\cos(2x)}{2}} \\ &= \mathrm{e}^{-\frac{\cos(2x)}{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-\frac{\cos(2x)}{2}}\left(2\cos(x)\sin(x)\,y - \mathrm{e}^{-\sin(x)^2}\right) \\ &= \left(y\sin(2x) - \mathrm{e}^{-\sin(x)^2}\right)\mathrm{e}^{-\frac{\cos(2x)}{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{\cos(2x)}{2}}(1) \\ &= \mathrm{e}^{-\frac{\cos(2x)}{2}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(y\sin(2x) - \mathrm{e}^{-\sin(x)^2}\right)\mathrm{e}^{-\frac{\cos(2x)}{2}}\right) + \left(\mathrm{e}^{-\frac{\cos(2x)}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\left(y\sin(2x)-\mathrm{e}^{-\sin(x)^2}\right)\mathrm{e}^{-\frac{\cos(2x)}{2}}\,\mathrm{d}x$$
$$\phi=\mathrm{e}^{-\frac{\cos(2x)}{2}}y-x\,\mathrm{e}^{-\frac{1}{2}}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\mathrm{e}^{-\frac{\cos(2x)}{2}}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\mathrm{e}^{-\frac{\cos(2x)}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{\cos(2x)}{2}}=\mathrm{e}^{-\frac{\cos(2x)}{2}}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\mathrm{e}^{-\frac{\cos(2x)}{2}}y-x\,\mathrm{e}^{-\frac{1}{2}}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\mathrm{e}^{-\frac{\cos(2x)}{2}}y-x\,\mathrm{e}^{-\frac{1}{2}}$$

The solution becomes

$$y = e^{\frac{\cos(2x)}{2}} \left(x\, e^{-\frac{1}{2}} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\cos(2x)}{2}} \left(x\, e^{-\frac{1}{2}} + c_1\right) \tag{1}$$

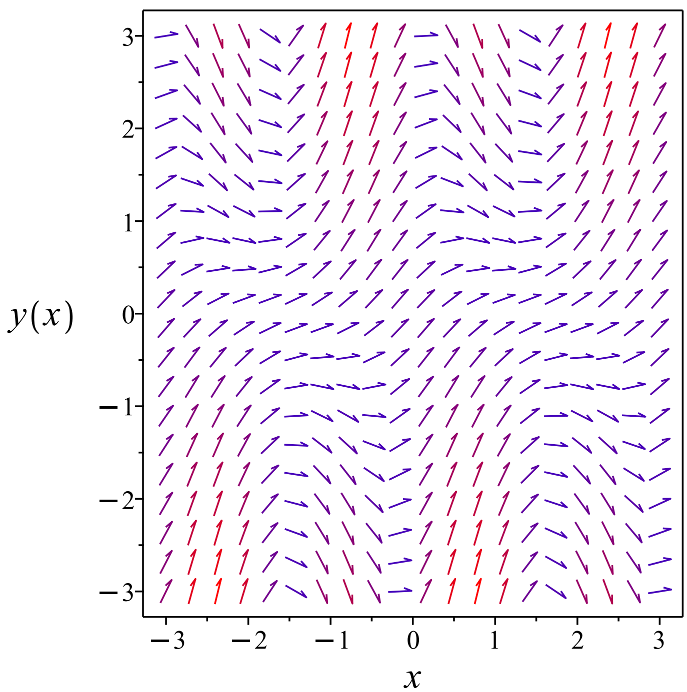


Figure 114: Slope field plot

Verification of solutions

$$y = e^{\frac{\cos(2x)}{2}} \left(x\, e^{-\frac{1}{2}} + c_1\right)$$

Verified OK.

2.23.4 Maple step by step solution

Let's solve

$y' + 2\cos(x)\sin(x)\, y = \mathrm{e}^{-\sin(x)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2\cos(x)\sin(x)\, y + \mathrm{e}^{-\sin(x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2\cos(x)\sin(x)\, y = \mathrm{e}^{-\sin(x)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + 2\cos(x)\sin(x)\, y\right) = \mu(x)\,\mathrm{e}^{-\sin(x)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + 2\cos(x)\sin(x)\, y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)\cos(x)\sin(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\sin(x)^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,\mathrm{e}^{-\sin(x)^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,\mathrm{e}^{-\sin(x)^2} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{-\sin(x)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)^2}$

  $y = \frac{\int \mathrm{e}^{\sin(x)^2}\mathrm{e}^{-\sin(x)^2} dx + c_1}{\mathrm{e}^{\sin(x)^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\mathrm{e}^{\sin(x)^2}}$

- Simplify

$$y = e^{-\sin(x)^2}(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x) +(2*sin(x)*cos(x))*y(x)=exp(-sin(x)^2),y(x), singsol=all)
```

$$y(x) = (c_1 + x)\, e^{-\frac{1}{2}+\frac{\cos(2x)}{2}}$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 24

```
DSolve[y'[x] +(2*Sin[x]*Cos[x])*y[x]==Exp[-Sin[x]^2],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \left(x + \sqrt{e}c_1\right) e^{-\sin^2(x)}$$

2.24 problem 24

Internal problem ID [910]
Internal file name [`OUTPUT/910_Sunday_June_05_2022_01_53_46_AM_72494338/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x^2 + 3yx = e^x}$$

2.24.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{e^x}{x^2}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{e^x}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{e^x}{x^2}\right) \\ \frac{d}{dx}\left(y\,x^3\right) &= \left(x^3\right)\left(\frac{e^x}{x^2}\right) \\ d\left(y\,x^3\right) &= \left(x\,e^x\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^3 &= \int x\,e^x\,dx \\ y\,x^3 &= (x-1)\,e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{(x-1)\,e^x}{x^3} + \frac{c_1}{x^3}$$

which simplifies to

$$y = \frac{(x-1)\,e^x + c_1}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{(x-1)\,e^x + c_1}{x^3} \tag{1}$$

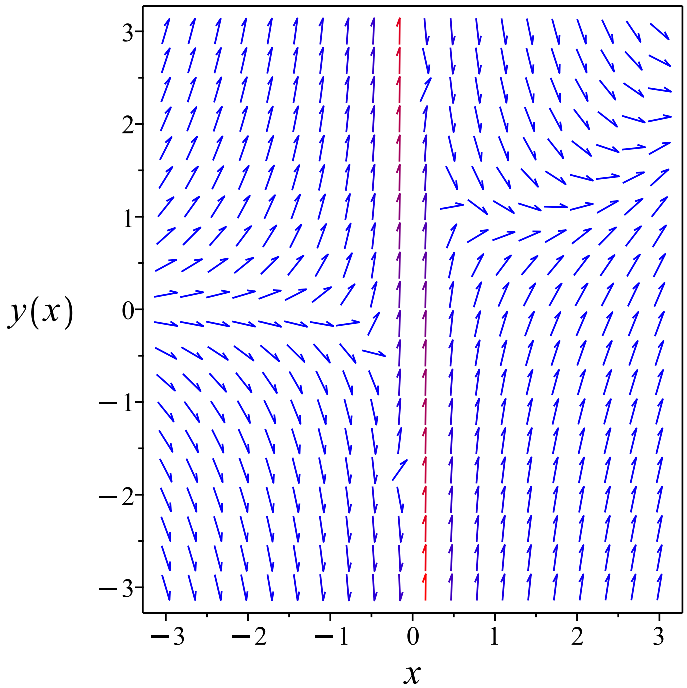


Figure 115: Slope field plot

Verification of solutions

$$y = \frac{(x-1)\,\mathrm{e}^{x} + c_1}{x^3}$$

Verified OK.

2.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-3yx + \mathrm{e}^{x}}{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 99: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^3}} dy
\end{aligned}$$

Which results in

$$S = y\, x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3yx + \mathrm{e}^x}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3y\, x^2 \\
S_y &= x^3
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, \mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, \mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R-1)\,e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = (x-1)\,e^{x} + c_1$$

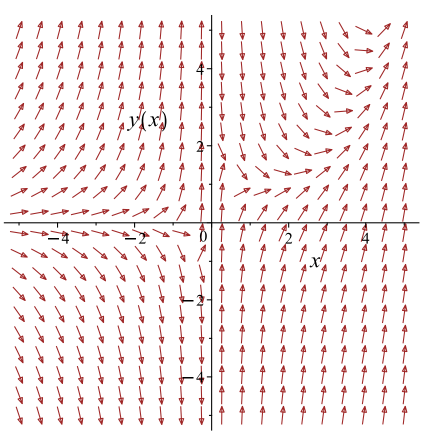
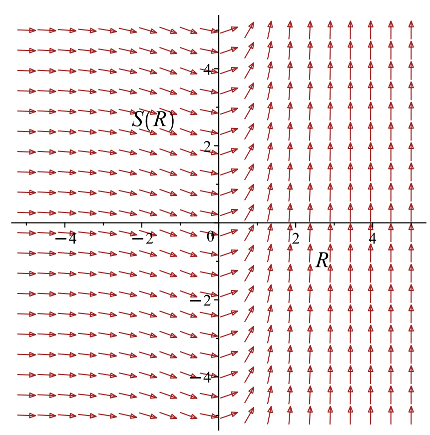
Which simplifies to

$$y x^3 = (x-1)\,e^{x} + c_1$$

Which gives

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3yx+e^{x}}{x^2}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = R\,e^{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3} \tag{1}$$

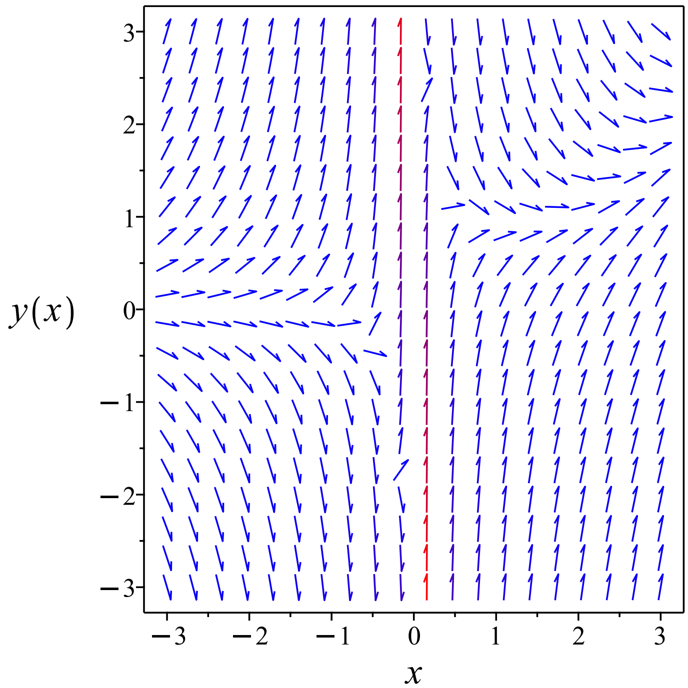


Figure 116: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3}$$

Verified OK.

2.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x^2)\,\mathrm{d}y &= (-3yx + \mathrm{e}^x)\,\mathrm{d}x \\
(3yx - \mathrm{e}^x)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3yx - \mathrm{e}^x \\
N(x, y) &= x^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3yx - \mathrm{e}^x) \\
&= 3x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2}((3x) - (2x)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x(3yx - \mathrm{e}^{x}) \\ &= 3y\,x^2 - x\,\mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x\left(x^2\right) \\ &= x^3 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(3y\,x^2 - x\,\mathrm{e}^{x}\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3y\, x^2 - x\, \mathrm{e}^x \,\mathrm{d}x$$
$$\phi = -x\, \mathrm{e}^x + \mathrm{e}^x + y\, x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\, \mathrm{e}^x + \mathrm{e}^x + y\, x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\, \mathrm{e}^x + \mathrm{e}^x + y\, x^3$$

The solution becomes

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3} \tag{1}$$

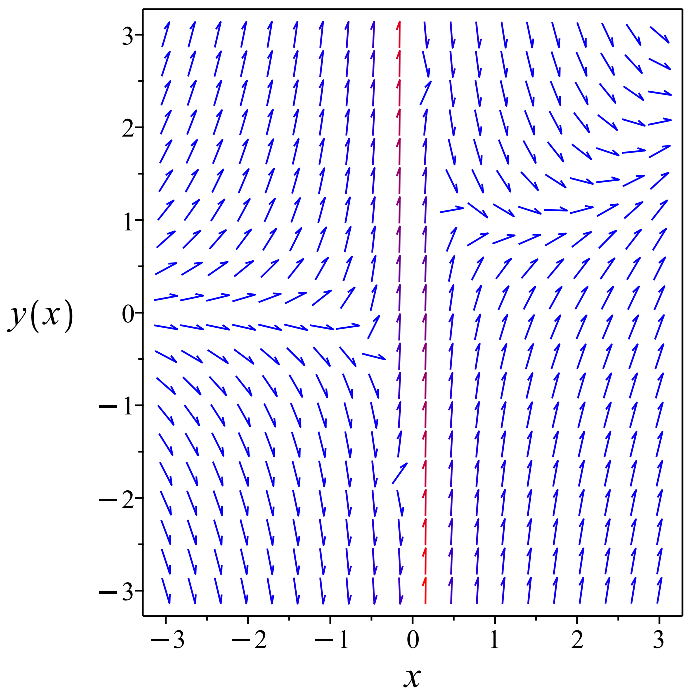


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{x} - e^{x} + c_1}{x^3}$$

Verified OK.

2.24.4 Maple step by step solution

Let's solve

$y'x^2 + 3yx = e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + \frac{e^x}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = \frac{e^x}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \frac{\mu(x)e^x}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)e^x}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)e^x}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)e^x}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

  $y = \frac{\int x\,e^x dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

  $y = \frac{(x-1)e^x + c_1}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x) +3*x*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(x-1)\,e^{x} + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 19

```
DSolve[x^2*y'[x] +3*x*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{x}(x-1) + c_1}{x^3}$$

2.25 problem 25

Internal problem ID [911]
Internal file name [`OUTPUT/911_Sunday_June_05_2022_01_53_48_AM_5786991/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 7y = e^{3x}}$$

With initial conditions

$$[y(0) = 0]$$

2.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 7$$
$$q(x) = e^{3x}$$

Hence the ode is

$$y' + 7y = e^{3x}$$

The domain of $p(x) = 7$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.25.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 7dx} \\ &= e^{7x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{3x}\right) \\ \frac{d}{dx}\left(e^{7x}y\right) &= \left(e^{7x}\right)\left(e^{3x}\right) \\ d\left(e^{7x}y\right) &= e^{10x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{7x}y &= \int e^{10x}\, dx \\ e^{7x}y &= \frac{e^{10x}}{10} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{7x}$ results in

$$y = \frac{e^{-7x}e^{10x}}{10} + c_1 e^{-7x}$$

which simplifies to

$$y = \frac{\left(e^{10x} + 10c_1\right)e^{-7x}}{10}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{10} + c_1$$

$$c_1 = -\frac{1}{10}$$

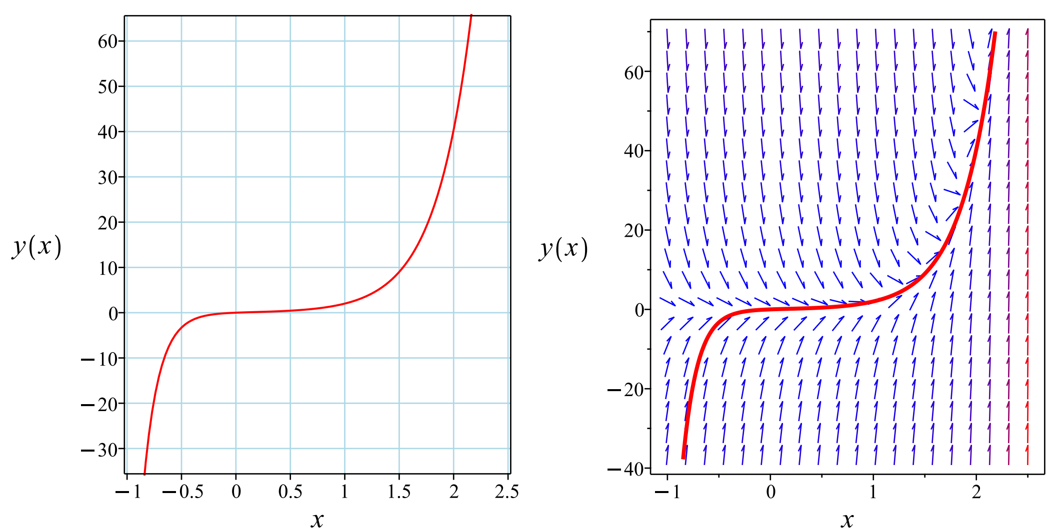
Substituting c_1 found above in the general solution gives

$$y = \frac{(e^{10x} - 1)\, e^{-7x}}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{10x} - 1)\, e^{-7x}}{10} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{(e^{10x} - 1)\, e^{-7x}}{10}$$

Verified OK.

2.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -7y + \mathrm{e}^{3x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 102: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= e^{-7x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{e^{-7x}} dy\end{aligned}$$

Which results in

$$S = e^{7x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -7y + e^{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= 7\,e^{7x} y\\ S_y &= e^{7x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{10x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{10R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{10R}}{10} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{7x} y = \frac{e^{10x}}{10} + c_1$$

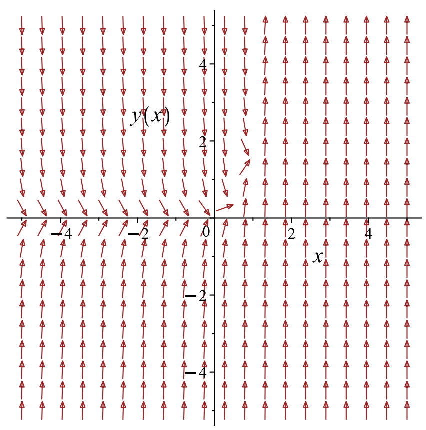
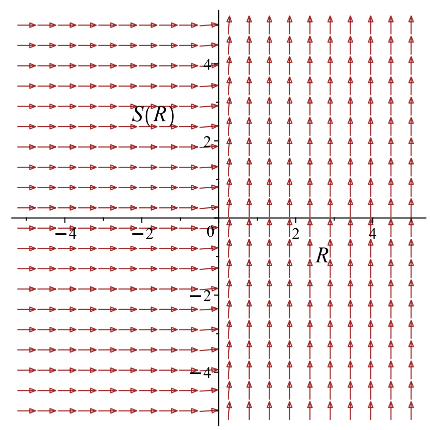
Which simplifies to

$$e^{7x} y = \frac{e^{10x}}{10} + c_1$$

Which gives

$$y = \frac{(e^{10x} + 10c_1)\, e^{-7x}}{10}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -7y + e^{3x}$ 	$R = x$ $S = e^{7x} y$	$\frac{dS}{dR} = e^{10R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{10} + c_1$$

$$c_1 = -\frac{1}{10}$$

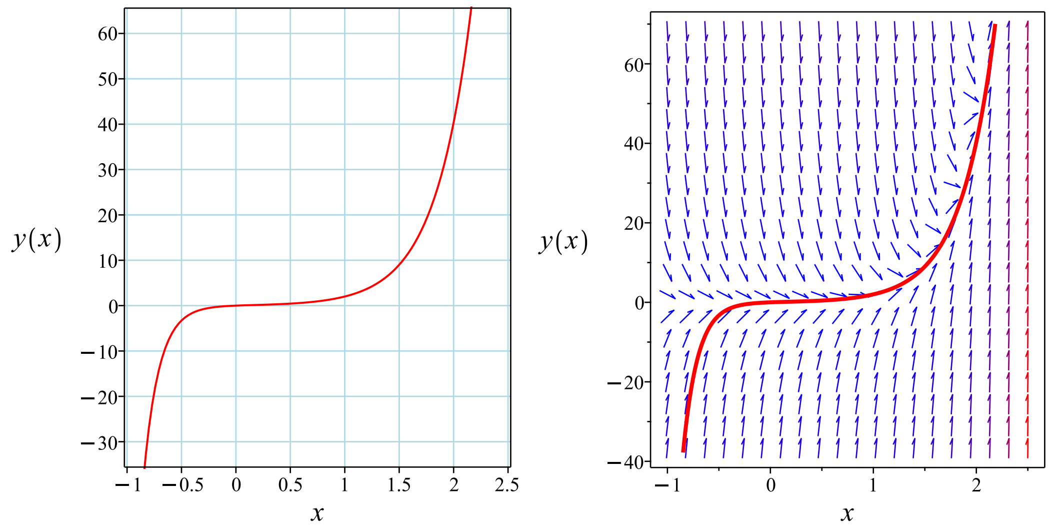
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10}$$

Verified OK.

2.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-7y + \mathrm{e}^{3x}\right)\mathrm{d}x \\ \left(7y - \mathrm{e}^{3x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 7y - \mathrm{e}^{3x} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(7y - \mathrm{e}^{3x}\right) \\ &= 7\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((7) - (0)) \\ &= 7 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 7 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{7x} \\ &= \mathrm{e}^{7x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{7x}(7y - \mathrm{e}^{3x}) \\ &= (7y - \mathrm{e}^{3x})\,\mathrm{e}^{7x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{7x}(1) \\ &= \mathrm{e}^{7x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((7y - \mathrm{e}^{3x})\,\mathrm{e}^{7x}\right) + \left(\mathrm{e}^{7x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int (7y - \mathrm{e}^{3x})\,\mathrm{e}^{7x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\mathrm{e}^{10x}}{10} + \mathrm{e}^{7x}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{7x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{7x}$. Therefore equation (4) becomes

$$\mathrm{e}^{7x} = \mathrm{e}^{7x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{10x}}{10} + \mathrm{e}^{7x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{10x}}{10} + \mathrm{e}^{7x} y$$

The solution becomes

$$y = \frac{(\mathrm{e}^{10x} + 10c_1)\,\mathrm{e}^{-7x}}{10}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{10} + c_1$$

$$c_1 = -\frac{1}{10}$$

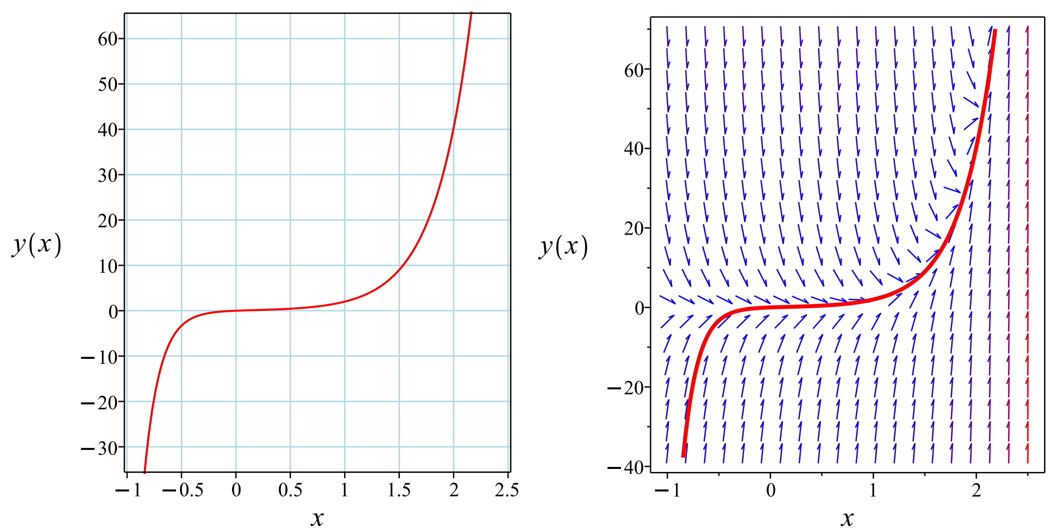
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{3x}}{10} - \frac{e^{-7x}}{10}$$

Verified OK.

2.25.5 Maple step by step solution

Let's solve

$[y' + 7y = e^{3x}, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -7y + \mathrm{e}^{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 7y = \mathrm{e}^{3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + 7y\right) = \mu(x)\,\mathrm{e}^{3x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + 7y\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 7\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{7x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{3x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{3x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{3x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{7x}$

  $y = \frac{\int \mathrm{e}^{3x}\mathrm{e}^{7x} dx + c_1}{\mathrm{e}^{7x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\mathrm{e}^{10x}}{10} + c_1}{\mathrm{e}^{7x}}$

- Simplify

  $y = \frac{(\mathrm{e}^{10x} + 10c_1)\,\mathrm{e}^{-7x}}{10}$

- Use initial condition $y(0) = 0$

  $0 = \frac{1}{10} + c_1$

- Solve for c_1

  $c_1 = -\frac{1}{10}$

- Substitute $c_1 = -\frac{1}{10}$ into general solution and simplify

  $y = \frac{(\mathrm{e}^{10x} - 1)\,\mathrm{e}^{-7x}}{10}$

- Solution to the IVP

  $y = \frac{(e^{10x}-1)e^{-7x}}{10}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(x),x) +7*y(x)=exp(3*x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{(e^{10x}-1)\,e^{-7x}}{10}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 21

```
DSolve[{y'[x] +7*y[x]==Exp[3*x],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}e^{-7x}\left(e^{10x}-1\right)$$

2.26 problem 26

Internal problem ID [912]
Internal file name [`OUTPUT/912_Sunday_June_05_2022_01_53_49_AM_49208102/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(x^2+1\right) y'+4yx=\frac{2}{x^2+1}}$$

With initial conditions

$$[y(0)=1]$$

2.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{4x}{x^2+1}$$
$$q(x)=\frac{2}{\left(x^2+1\right)^2}$$

Hence the ode is

$$y' + \frac{4xy}{x^2+1} = \frac{2}{(x^2+1)^2}$$

The domain of $p(x) = \frac{4x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{(x^2+1)^2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.26.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{4x}{x^2+1}dx} \\ &= (x^2+1)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2}{(x^2+1)^2}\right) \\ \frac{d}{dx}\left((x^2+1)^2 y\right) &= \left((x^2+1)^2\right)\left(\frac{2}{(x^2+1)^2}\right) \\ d\left((x^2+1)^2 y\right) &= 2\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(x^2+1)^2 y &= \int 2\,dx \\ (x^2+1)^2 y &= 2x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x^2+1)^2$ results in

$$y = \frac{2x}{(x^2+1)^2} + \frac{c_1}{(x^2+1)^2}$$

which simplifies to

$$y = \frac{2x + c_1}{(x^2+1)^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

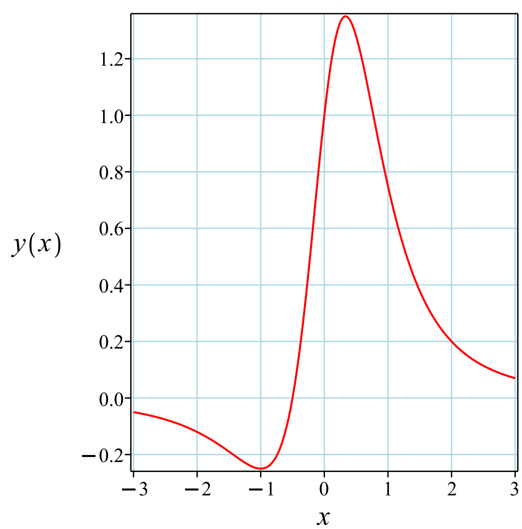
Substituting c_1 found above in the general solution gives

$$y = \frac{1 + 2x}{(x^2+1)^2}$$

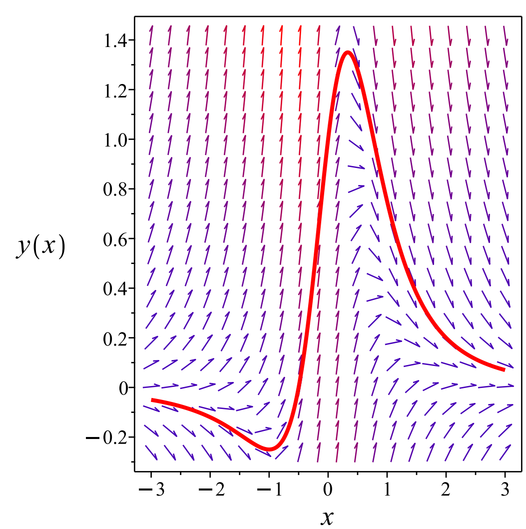
Summary

The solution(s) found are the following

$$y = \frac{1 + 2x}{(x^2+1)^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1 + 2x}{(x^2+1)^2}$$

Verified OK.

2.26.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-4yx + \frac{2}{x^2+1}}{x^2+1} \tag{1}$$

Which becomes

$$0 = \left(-x^4 - 2x^2 - 1\right) dy + \left(-4y\, x^3 - 4yx + 2\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^4 - 2x^2 - 1\right) dy + \left(-4y\, x^3 - 4yx + 2\right) dx = d\left(-x^4 y - 2y\, x^2 + 2x - y\right)$$

Hence (2) becomes

$$0 = d\left(-x^4 y - 2y\, x^2 + 2x - y\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{2x + c_1}{x^4 + 2x^2 + 1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 2c_1$$

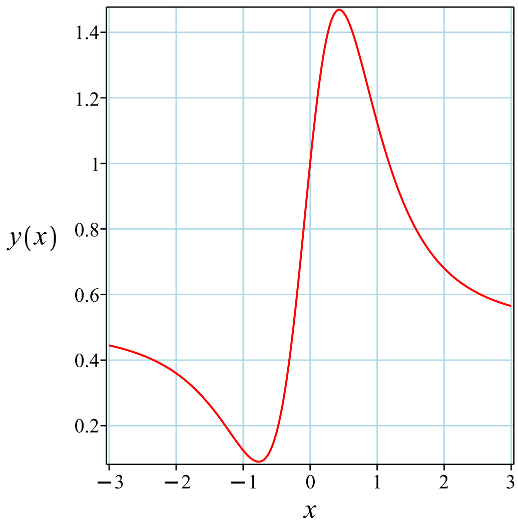
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

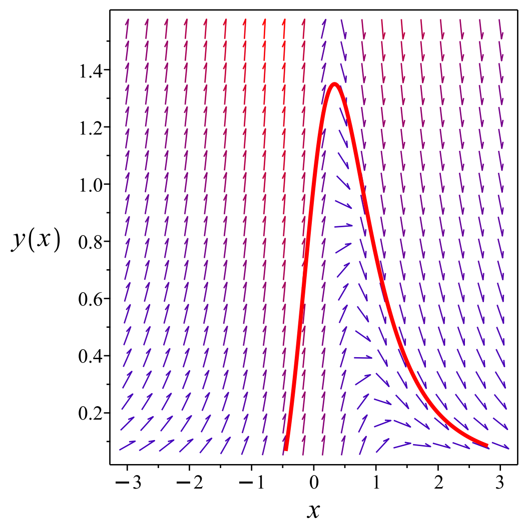
$$y = \frac{x^4 + 2x^2 + 4x + 2}{2x^4 + 4x^2 + 2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^4 + 2x^2 + 4x + 2}{2x^4 + 4x^2 + 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^4 + 2x^2 + 4x + 2}{2x^4 + 4x^2 + 2}$$

Verified OK.

2.26.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2(2y\,x^3 + 2yx - 1)}{(x^2 + 1)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 105: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{(x^2+1)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x^2+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x^2 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2(2y\, x^3 + 2yx - 1)}{(x^2 + 1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4yx(x^2 + 1) \\ S_y &= (x^2 + 1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\left(x^2+1\right)^2 y = 2x + c_1$$

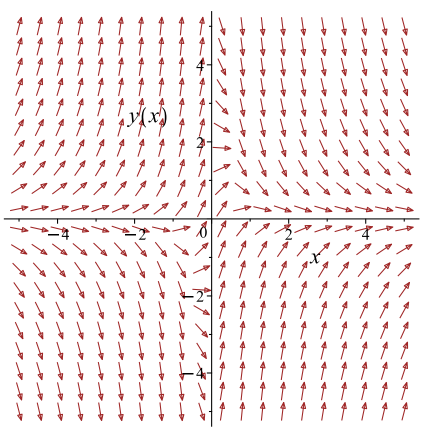
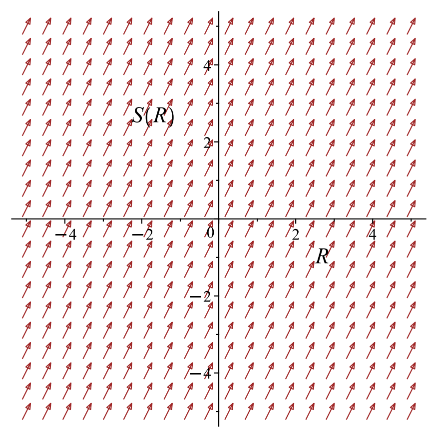
Which simplifies to

$$\left(x^2+1\right)^2 y = 2x + c_1$$

Which gives

$$y = \frac{2x + c_1}{\left(x^2+1\right)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(2y\,x^3+2yx-1)}{(x^2+1)^2}$ 	$R = x$ $S = \left(x^2+1\right)^2 y$	$\frac{dS}{dR} = 2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

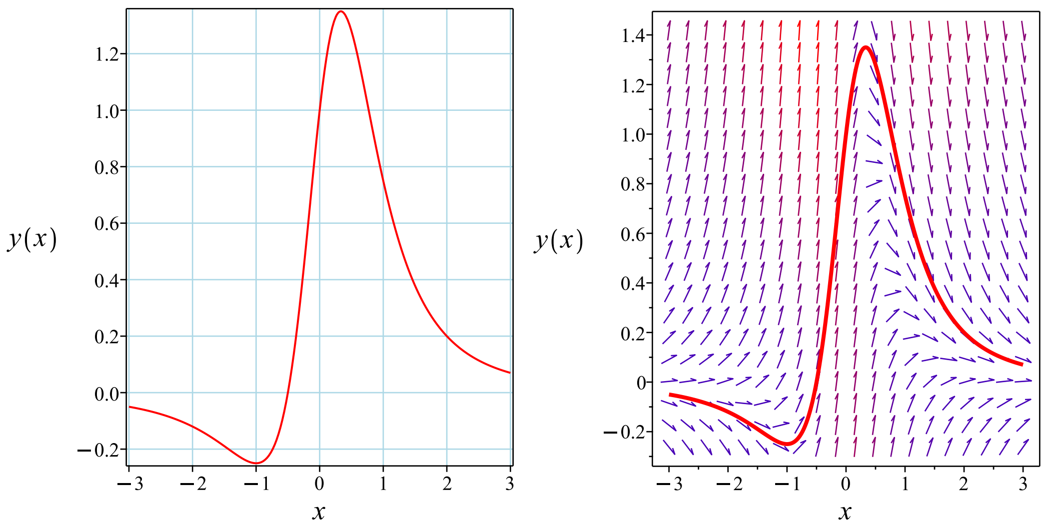
Substituting c_1 found above in the general solution gives

$$y = \frac{1 + 2x}{(x^2 + 1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{1 + 2x}{(x^2 + 1)^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1 + 2x}{(x^2 + 1)^2}$$

Verified OK.

2.26.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\left(x^2+1\right)^2\right) \mathrm{d}y = \left(-4y\, x^3 - 4yx + 2\right) \mathrm{d}x$$
$$\left(4y\, x^3 + 4yx - 2\right) \mathrm{d}x + \left(\left(x^2+1\right)^2\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 4y\, x^3 + 4yx - 2$$
$$N(x, y) = \left(x^2+1\right)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y\,x^3 + 4yx - 2\right) \\ &= 4x^3 + 4x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(x^2+1\right)^2\right) \\ &= 4x\left(x^2+1\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 4y\,x^3 + 4yx - 2\,\mathrm{d}x\end{aligned}$$

$$\phi = x^4 y + 2y\,x^2 - 2x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^4 + 2x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(x^2+1\right)^2$. Therefore equation (4) becomes

$$\left(x^2+1\right)^2 = x^4 + 2x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (1) \,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^4 y + 2y\,x^2 - 2x + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^4 y + 2y\,x^2 - 2x + y$$

The solution becomes

$$y = \frac{2x + c_1}{x^4 + 2x^2 + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

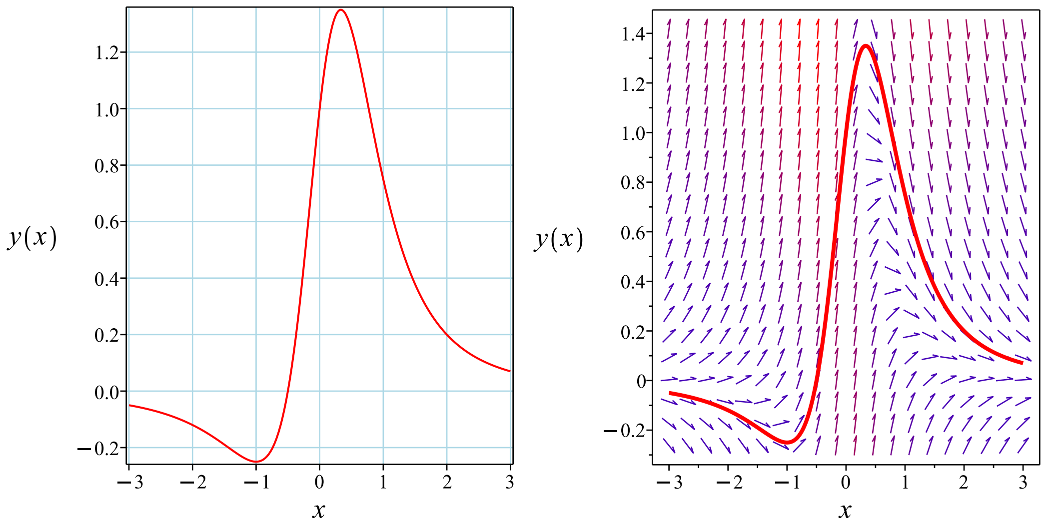
Substituting c_1 found above in the general solution gives

$$y = \frac{1 + 2x}{x^4 + 2x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1 + 2x}{x^4 + 2x^2 + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1 + 2x}{x^4 + 2x^2 + 1}$$

Verified OK.

2.26.6 Maple step by step solution

Let's solve

$\left[(x^2+1)\,y' + 4yx = \frac{2}{x^2+1}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4xy}{x^2+1} + \frac{2}{(x^2+1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4xy}{x^2+1} = \frac{2}{(x^2+1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{4xy}{x^2+1}\right) = \frac{2\mu(x)}{(x^2+1)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y' + \frac{4xy}{x^2+1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = \frac{4\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

$\mu(x) = \left(x^2+1\right)^2$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{2\mu(x)}{\left(x^2+1\right)^2} dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int \frac{2\mu(x)}{\left(x^2+1\right)^2} dx + c_1$

- Solve for y

$y = \frac{\int \frac{2\mu(x)}{\left(x^2+1\right)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \left(x^2+1\right)^2$

$y = \frac{\int 2dx + c_1}{\left(x^2+1\right)^2}$

- Evaluate the integrals on the rhs

$y = \frac{2x+c_1}{\left(x^2+1\right)^2}$

- Use initial condition $y(0) = 1$

$1 = c_1$

- Solve for c_1

$c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

$y = \frac{1+2x}{\left(x^2+1\right)^2}$

- Solution to the IVP

$y = \frac{1+2x}{\left(x^2+1\right)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([(1+x^2)*diff(y(x),x)+4*x*y(x)=2/(1+x^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1+2x}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 18

```
DSolve[{(1+x^2)*y'[x]+4*x*y[x]==2/(1+x^2),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x+1}{(x^2+1)^2}$$

2.27 problem 27

Internal problem ID [913]
Internal file name [`OUTPUT/913_Sunday_June_05_2022_01_53_50_AM_83433225/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{3y + y'x = \frac{2}{x\left(x^2+1\right)}}$$

With initial conditions

$$[y(-1) = 0]$$

2.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{2}{x^2\left(x^2+1\right)}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{2}{x^2(x^2+1)}$$

The domain of $p(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{2}{x^2(x^2+1)}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

2.27.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2}{x^2(x^2+1)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,x^3\right) &= \left(x^3\right)\left(\frac{2}{x^2(x^2+1)}\right) \\ \mathrm{d}\left(y\,x^3\right) &= \left(\frac{2x}{x^2+1}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^3 &= \int \frac{2x}{x^2+1}\,\mathrm{d}x \\ y\,x^3 &= \ln\left(x^2+1\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{\ln(x^2+1)}{x^3} + \frac{c_1}{x^3}$$

which simplifies to

$$y = \frac{\ln(x^2+1)+c_1}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(2) - c_1$$

$$c_1 = -\ln(2)$$

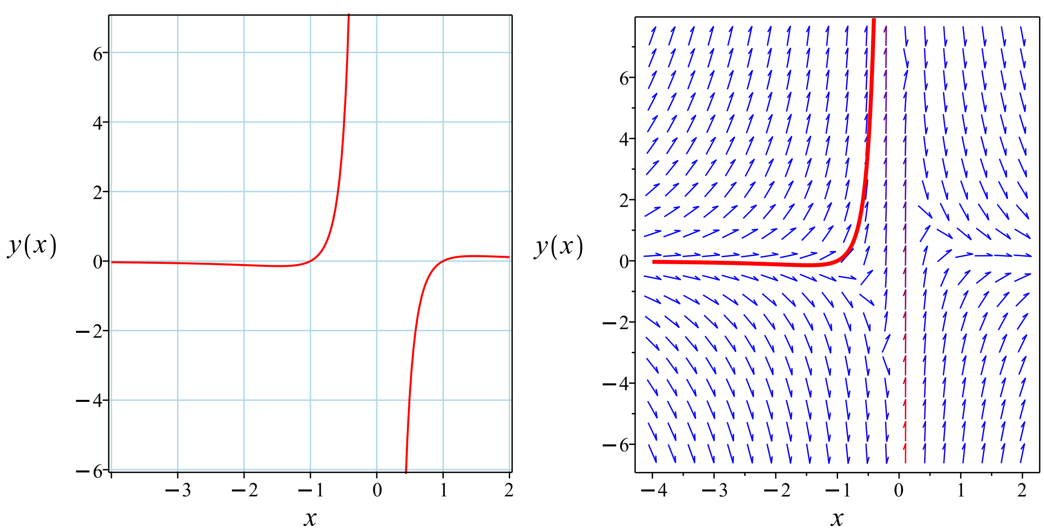
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$$

Verified OK.

2.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y\,x^3+3yx-2}{x^2\,(x^2+1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 108: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{x^3}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{x^3}} dy\end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y\,x^3 + 3yx - 2}{x^2\,(x^2+1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3y\,x^2 \\
S_y &= x^3
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x}{x^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R^2+1\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = \ln\left(x^2+1\right) + c_1$$

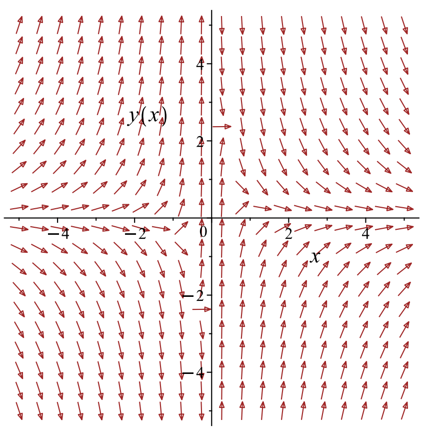
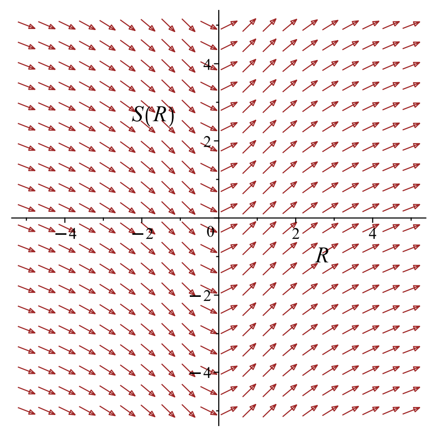
Which simplifies to

$$y x^3 = \ln\left(x^2+1\right) + c_1$$

Which gives

$$y = \frac{\ln\left(x^2+1\right) + c_1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y\,x^3+3yx-2}{x^2(x^2+1)}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = \frac{2R}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(2) - c_1$$

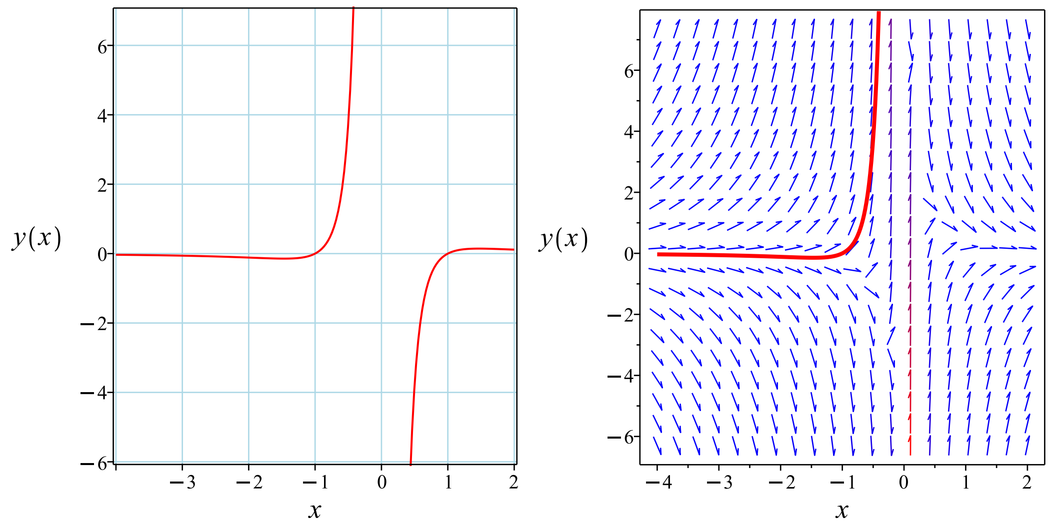
$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x^2+1) - \ln(2)}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x^2+1) - \ln(2)}{x^3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x^2+1) - \ln(2)}{x^3}$$

Verified OK.

2.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-3y + \frac{2}{x\,(x^2+1)}\right)\mathrm{d}x$$

$$\left(3y - \frac{2}{x\,(x^2+1)}\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 3y - \frac{2}{x\,(x^2+1)}$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - \frac{2}{x\,(x^2+1)}\right)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((3) - (1)) \\
&= \frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x)} \\
&= x^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^2\left(3y - \frac{2}{x\left(x^2+1\right)}\right) \\
&= \frac{3x^4y + 3y\,x^2 - 2x}{x^2+1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(x) \\
&= x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x^4y + 3y\,x^2 - 2x}{x^2+1}\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{3x^4y + 3y\,x^2 - 2x}{x^2+1} \mathrm{d}x$$
$$\phi = y\,x^3 - \ln\left(x^2+1\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^3 - \ln\left(x^2+1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^3 - \ln\left(x^2+1\right)$$

The solution becomes

$$y = \frac{\ln(x^2+1)+c_1}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x=-1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(2) - c_1$$

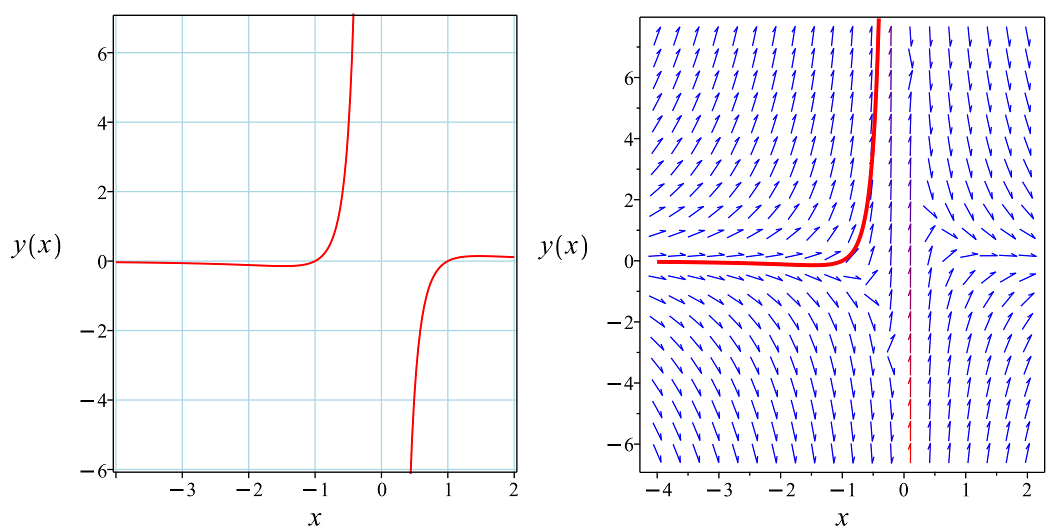
$$c_1 = -\ln(2)$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$$

Verified OK.

2.27.5 Maple step by step solution

Let's solve

$\left[3y + y'x = \frac{2}{x(x^2+1)}, y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + \frac{2}{x^2(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = \frac{2}{x^2(x^2+1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \frac{2\mu(x)}{x^2(x^2+1)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{2\mu(x)}{x^2(x^2+1)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{2\mu(x)}{x^2(x^2+1)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)}{x^2\left(x^2+1\right)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

  $y = \frac{\int \frac{2x}{x^2+1} dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

  $y = \frac{\ln(x^2+1) + c_1}{x^3}$

- Use initial condition $y(-1) = 0$

  $0 = -\ln(2) - c_1$
- Solve for c_1

  $c_1 = -\ln(2)$
- Substitute $c_1 = -\ln(2)$ into general solution and simplify

  $y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$
- Solution to the IVP

  $y = \frac{\ln(x^2+1)-\ln(2)}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([x*diff(y(x),x)+3*y(x)=2/(x*(1+x^2)),y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\ln(x^2+1) - \ln(2)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 19

```
DSolve[{x*y'[x]+3*y[x]==2/(x*(1+x^2)),y[-1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log\left(\frac{1}{2}(x^2+1)\right)}{x^3}$$

2.28 problem 28

Internal problem ID [914]
Internal file name [`OUTPUT/914_Sunday_June_05_2022_01_53_52_AM_53037594/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = \cos(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

2.28.1 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cot(x)dx}\\ &= \sin(x)\end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\cos\left(x\right)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(x\right) y\right) &= \left(\sin\left(x\right)\right)\left(\cos\left(x\right)\right) \\
\mathrm{d}\left(\sin\left(x\right) y\right) &= \left(\frac{\sin\left(2x\right)}{2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin\left(x\right) y &= \int \frac{\sin\left(2x\right)}{2} \,\mathrm{d}x \\
\sin\left(x\right) y &= -\frac{\cos\left(2x\right)}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin\left(x\right)$ results in

$$
y = -\frac{\csc\left(x\right)\cos\left(2x\right)}{4} + \csc\left(x\right) c_1
$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{1}{4} + c_1
$$

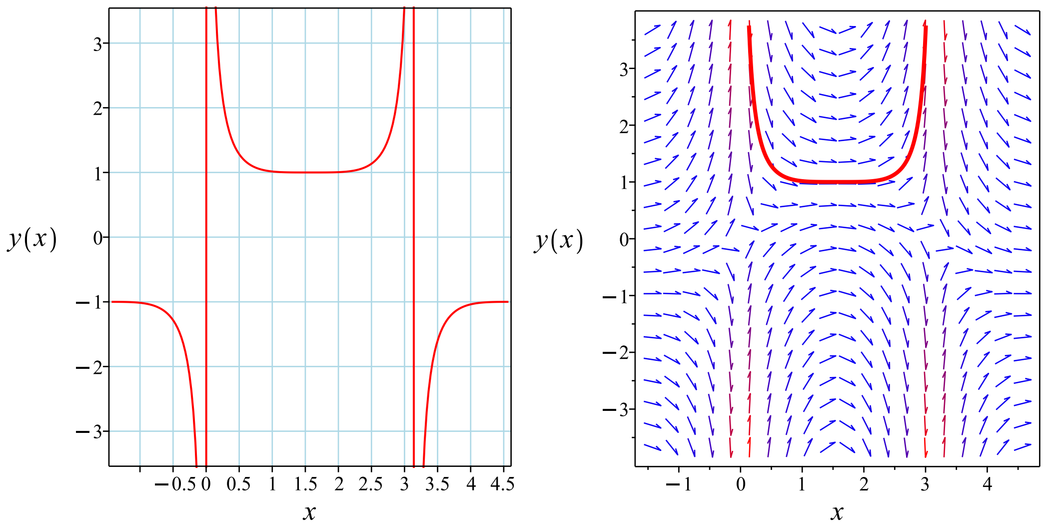
$$
c_1 = \frac{3}{4}
$$

Substituting c_1 found above in the general solution gives

$$
y = -\frac{\csc\left(x\right)\cos\left(x\right)^2}{2} + \csc\left(x\right)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{\csc\left(x\right)\cos\left(x\right)^2}{2} + \csc\left(x\right) \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x)$$

Verified OK.

2.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cot(x) + \cos(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 111: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cot(x) + \cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

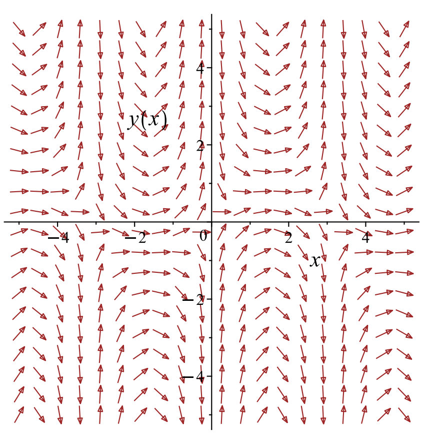
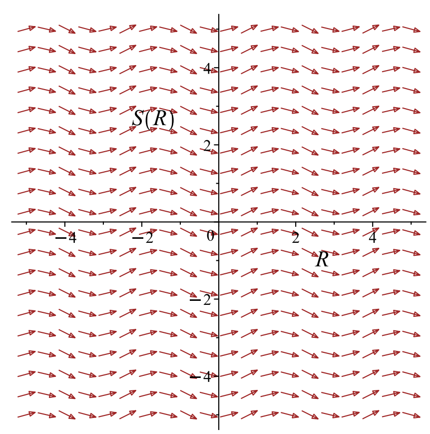
Which simplifies to

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

Which gives

$$y = -\frac{\cos(2x) - 4c_1}{4\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + \cos(x)$ 	$R = x$ $S = \sin(x)\, y$	$\frac{dS}{dR} = \frac{\sin(2R)}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{4} + c_1$$

$$c_1 = \frac{3}{4}$$

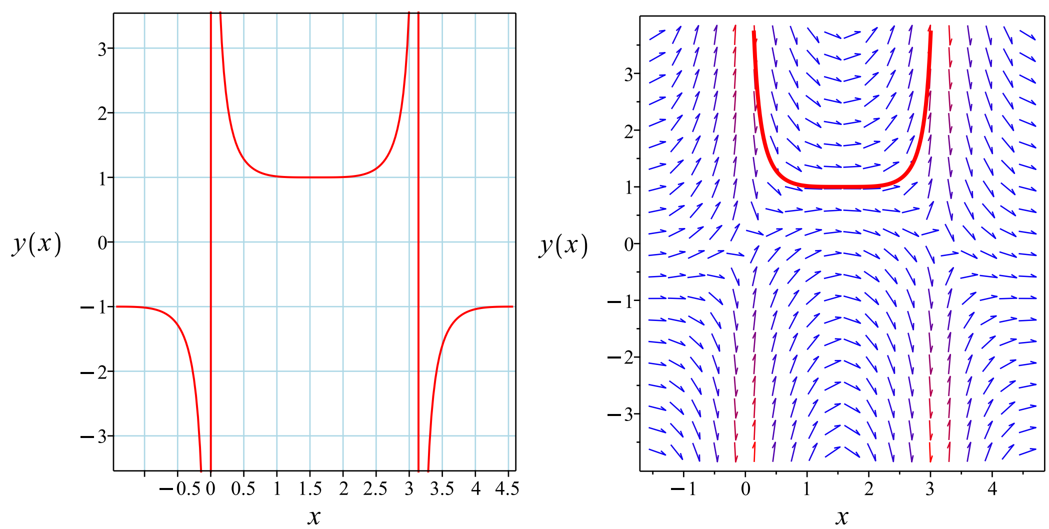
Substituting c_1 found above in the general solution gives

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x)$$

Verified OK.

2.28.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (-y\cot(x) + \cos(x))\,\mathrm{d}x\\ (y\cot(x) - \cos(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\cot(x) - \cos(x)\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cot(x) - \cos(x)) \\ &= \cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \cot(x) \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sin(x))} \\ &= \sin(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sin(x)(y\cot(x) - \cos(x)) \\ &= \cos(x)(-\sin(x) + y)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)(1) \\
&= \sin(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\cos(x)(-\sin(x) + y)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \cos(x)(-\sin(x) + y)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\sin(x)(\sin(x) - 2y)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$
\sin(x) = \sin(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(x)(\sin(x) - 2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(x)(\sin(x) - 2y)}{2}$$

The solution becomes

$$y = \frac{\sin(x)^2 + 2c_1}{2\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

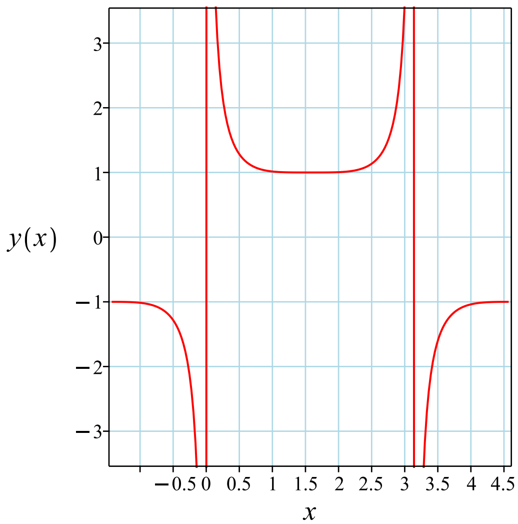
Substituting c_1 found above in the general solution gives

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x)$$

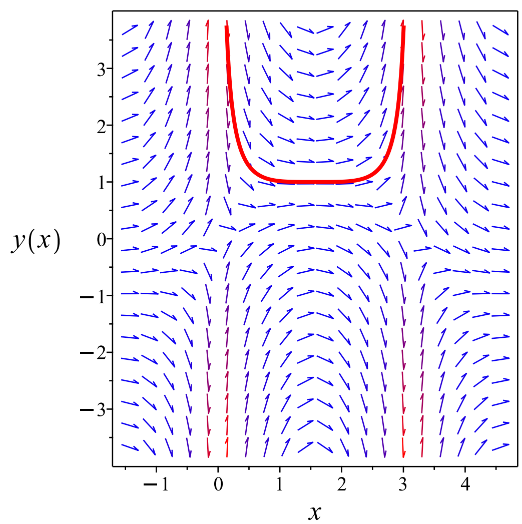
Summary

The solution(s) found are the following

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\csc(x)\cos(x)^2}{2} + \csc(x)$$

Verified OK.

2.28.4 Maple step by step solution

Let's solve

$\left[y' + y\cot(x) = \cos(x), y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cot(x) + \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cot(x) = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\cot(x)\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + y\cot(x)\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x) \cot(x)$
- Solve to find the integrating factor

  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \cos(x)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) \cos(x) dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int \cos(x) \sin(x) dx + c_1}{\sin(x)}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{\sin(x)^2}{2} + c_1}{\sin(x)}$
- Simplify

  $y = \frac{\sin(x)}{2} + \csc(x)\, c_1$
- Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

  $1 = \frac{1}{2} + c_1$
- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{\sin(x)}{2} + \frac{\csc(x)}{2}$
- Solution to the IVP

  $y = \frac{\sin(x)}{2} + \frac{\csc(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)+cot(x)*y(x)=cos(x),y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{\cos(x)\cot(x)}{2} + \csc(x)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 16

```
DSolve[{y'[x]+Cot[x]*y[x]==Cos[x],y[Pi/2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \csc(x) - \frac{1}{2}\cos(x)\cot(x)$$

2.29 problem 29

Internal problem ID [915]
Internal file name [`OUTPUT/915_Sunday_June_05_2022_01_53_54_AM_14525349/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{x} = \frac{2}{x^2} + 1}$$

With initial conditions

$$[y(-1) = 0]$$

2.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 + 2}{x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{x^2+2}{x^2}$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{x^2+2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

2.29.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^2+2}{x^2}\right) \\ \frac{d}{dx}(yx) &= (x)\left(\frac{x^2+2}{x^2}\right) \\ d(yx) &= \left(\frac{x^2+2}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}yx &= \int \frac{x^2+2}{x}\, dx \\ yx &= \frac{x^2}{2} + 2\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{x^2}{2} + 2\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2i\pi - c_1 - \frac{1}{2}$$

$$c_1 = -2i\pi - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Verified OK.

2.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2 + yx - 2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 114: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2 + yx - 2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2 + 2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2 + 2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \frac{x^2}{2} + 2\ln(x) + c_1$$

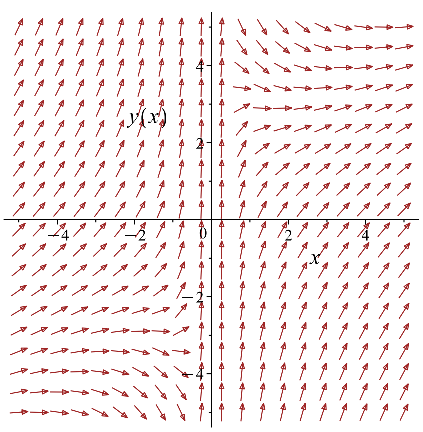
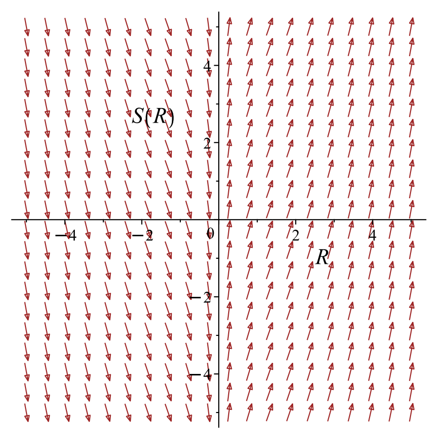
Which simplifies to

$$yx = \frac{x^2}{2} + 2\ln(x) + c_1$$

Which gives

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+yx-2}{x^2}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = \frac{R^2+2}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2i\pi - c_1 - \frac{1}{2}$$

$$c_1 = -2i\pi - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Verified OK.

2.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{x} + \frac{2}{x^2} + 1\right)\mathrm{d}x$$

$$\left(-1 + \frac{y}{x} - \frac{2}{x^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -1 + \frac{y}{x} - \frac{2}{x^2} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-1 + \frac{y}{x} - \frac{2}{x^2}\right) \\ &= \frac{1}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(\frac{1}{x}\right) - (0)\right) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(-1 + \frac{y}{x} - \frac{2}{x^2}\right) \\ &= \frac{-x^2 + yx - 2}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(1) \\ &= x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^2 + yx - 2}{x}\right) + (x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^2 + yx - 2}{x} \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + yx - 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + yx - 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + yx - 2\ln(x)$$

The solution becomes

$$y = \frac{x^2 + 4\ln(x) + 2c_1}{2x}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2i\pi - c_1 - \frac{1}{2}$$

$$c_1 = -2i\pi - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

Verified OK.

2.29.5 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{x} = \frac{2}{x^2} + 1, y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \frac{x^2+2}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{x^2+2}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \frac{\mu(x)(x^2+2)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$\mu(x) \left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(x^2+2)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(x^2+2)}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(x^2+2\right)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \frac{x^2+2}{x} dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^2}{2} + 2\ln(x) + c_1}{x}$

- Use initial condition $y(-1) = 0$

  $0 = -2\,\mathrm{I}\pi - c_1 - \frac{1}{2}$

- Solve for c_1

  $c_1 = -2\,\mathrm{I}\pi - \frac{1}{2}$

- Substitute $c_1 = -2\,\mathrm{I}\pi - \frac{1}{2}$ into general solution and simplify

  $y = \frac{-4\,\mathrm{I}\pi + x^2 + 4\ln(x) - 1}{2x}$

- Solution to the IVP

  $y = \frac{-4\,\mathrm{I}\pi + x^2 + 4\ln(x) - 1}{2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x)+y(x)/x=2/x^2+1,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{-4i\pi + x^2 + 4\ln(x) - 1}{2x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 26

```
DSolve[{y'[x]+y[x]/x==2/x^2+1,y[-1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 + 4\log(x) - 4i\pi - 1}{2x}$$

2.30 problem 30

Internal problem ID [916]
Internal file name [`OUTPUT/916_Sunday_June_05_2022_01_53_55_AM_42718370/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x-1)\,y'+3y = \frac{1}{(x-1)^{3}}+\frac{\sin (x)}{(x-1)^{2}}}$$

With initial conditions

$$[y(0) = 1]$$

2.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x-1}$$
$$q(x) = \frac{1+\sin (x)\,(x-1)}{(x-1)^{4}}$$

Hence the ode is

$$y' + \frac{3y}{x-1} = \frac{1+\sin(x)(x-1)}{(x-1)^4}$$

The domain of $p(x) = \frac{3}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1+\sin(x)(x-1)}{(x-1)^4}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.30.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{3}{x-1} dx} \\
&= (x-1)^3
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1+\sin(x)(x-1)}{(x-1)^4}\right) \\
\frac{d}{dx}\left(y(x-1)^3\right) &= \left((x-1)^3\right)\left(\frac{1+\sin(x)(x-1)}{(x-1)^4}\right) \\
d\left(y(x-1)^3\right) &= \left(\frac{1+\sin(x)(x-1)}{x-1}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
y(x-1)^3 &= \int \frac{1+\sin(x)(x-1)}{x-1} dx \\
y(x-1)^3 &= -\cos(x) + \ln(x-1) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x-1)^3$ results in

$$y = \frac{-\cos(x) + \ln(x-1)}{(x-1)^3} + \frac{c_1}{(x-1)^3}$$

which simplifies to

$$y = \frac{-\cos(x) + \ln(x-1) + c_1}{(x-1)^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1 + 1$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Summary
The solution(s) found are the following

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1} \tag{1}$$

Verification of solutions

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Verified OK.

2.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-3y\,x^3 + 9y\,x^2 + \sin(x)\,x - 9yx - \sin(x) + 3y + 1}{(x-1)^4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{(x-1)^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x-1)^3}} dy \end{aligned}$$

Which results in

$$S = y(x-1)^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3y\,x^3 + 9y\,x^2 + \sin(x)\,x - 9yx - \sin(x) + 3y + 1}{(x-1)^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y(x-1)^2 \\ S_y &= (x-1)^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1 + \sin(x)(x-1)}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1 + \sin(R)(R-1)}{R-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\cos(R) + \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-1)^3 y = -\cos(x) + \ln(x-1) + c_1$$

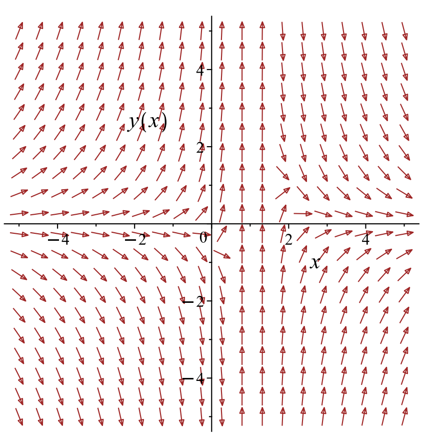
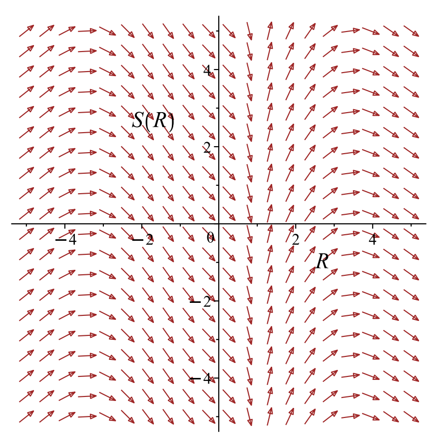
Which simplifies to

$$(x-1)^3 y = -\cos(x) + \ln(x-1) + c_1$$

Which gives

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{(x-1)^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3y\,x^3+9y\,x^2+\sin(x)x-9yx-\sin(x)+3y+1}{(x-1)^4}$ 	$R = x$ $S = y(x-1)^3$	$\frac{dS}{dR} = \frac{1+\sin(R)(R-1)}{R-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1 + 1$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1} \tag{1}$$

Verification of solutions

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Verified OK.

2.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x-1)\,\mathrm{d}y = \left(-3y + \frac{1}{(x-1)^3} + \frac{\sin(x)}{(x-1)^2}\right)\mathrm{d}x$$

$$\left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right)\mathrm{d}x + (x-1)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\\ N(x, y) &= x - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-1)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x-1}((3) - (1)) \\ &= \frac{2}{x-1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{2}{x-1} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2\ln(x-1)} \\ &= (x-1)^2 \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= (x-1)^2 \left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right) \\ &= \left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right)(x-1)^2 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= (x-1)^2 (x-1) \\ &= (x-1)^3 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right)(x-1)^2\right) + \left((x-1)^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \left(3y - \frac{1}{(x-1)^3} - \frac{\sin(x)}{(x-1)^2}\right)(x-1)^2 \mathrm{d}x$$

$$\phi = 3yx - 3y\,x^2 + y\,x^3 + \cos(x) - \ln(x-1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 - 3x^2 + 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x-1)^3$. Therefore equation (4) becomes

$$(x-1)^3 = x^3 - 3x^2 + 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (-1) \mathrm{d}y$$

$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 3yx - 3y\,x^2 + y\,x^3 + \cos(x) - \ln(x-1) - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3yx - 3y\,x^2 + y\,x^3 + \cos(x) - \ln(x-1) - y$$

The solution becomes

$$y = -\frac{\cos(x) - \ln(x-1) - c_1}{x^3 - 3x^2 + 3x - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1 + 1$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1} \tag{1}$$

Verification of solutions

$$y = \frac{-\cos(x) + \ln(x-1) - i\pi}{x^3 - 3x^2 + 3x - 1}$$

Verified OK.

2.30.5 Maple step by step solution

Let's solve

$\left[(x-1)\,y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin(x)}{(x-1)^2}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\frac{3y}{x-1} + \frac{\sin(x)x-\sin(x)+1}{(x-1)^4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x-1} = \frac{\sin(x)x-\sin(x)+1}{(x-1)^4}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x-1}\right) = \frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x-1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x) = (x-1)^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^4} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^4} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(\sin(x)x-\sin(x)+1)}{(x-1)^4} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x-1)^3$

  $y = \frac{\int \frac{\sin(x)x-\sin(x)+1}{x-1} dx + c_1}{(x-1)^3}$

- Evaluate the integrals on the rhs

  $y = \frac{-\cos(x)+\ln(x-1)+c_1}{(x-1)^3}$

- Use initial condition $y(0) = 1$

  $1 = 1 - \mathrm{I}\pi - c_1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi$

- Substitute $c_1 = -\mathrm{I}\pi$ into general solution and simplify

  $y = \frac{-\cos(x)+\ln(x-1)-\mathrm{I}\pi}{(x-1)^3}$

- Solution to the IVP

  $y = \frac{-\cos(x)+\ln(x-1)-\mathrm{I}\pi}{(x-1)^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([(x-1)*diff(y(x),x)+3*y(x)=1/(x-1)^3+sin(x)/(x-1)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-\cos(x)+\ln(x-1)-i\pi}{(x-1)^3}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 25

```
DSolve[{(x-1)*y'[x]+3*y[x]==1/(x-1)^3+Sin[x]/(x-1)^2,y[0]==1},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{\log(x-1)-\cos(x)-i\pi}{(x-1)^3}$$

2.31 problem 31

Internal problem ID [917]
Internal file name [`OUTPUT/917_Sunday_June_05_2022_01_53_57_AM_26854052/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{2y + y'x = 8x^2}$$

With initial conditions

$$[y(1) = 3]$$

2.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 8x$$

Hence the ode is

$$y' + \frac{2y}{x} = 8x$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 8x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.31.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(8x) \\ \frac{d}{dx}(y\,x^2) &= (x^2)(8x) \\ d(y\,x^2) &= (8x^3)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int 8x^3\, dx \\ y\,x^2 &= 2x^4 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = 2x^2 + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 + 2$$

$$c_1 = 1$$

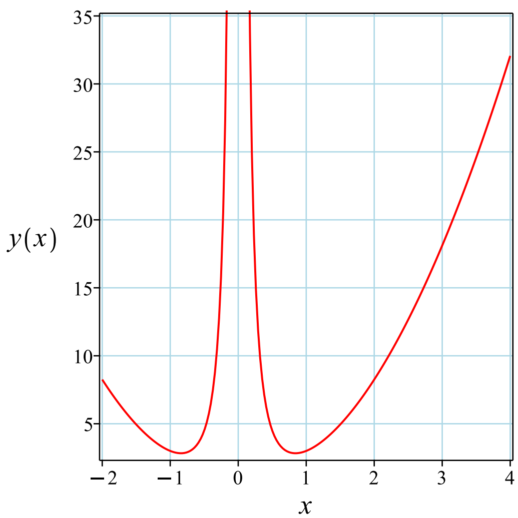
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^4 + 1}{x^2}$$

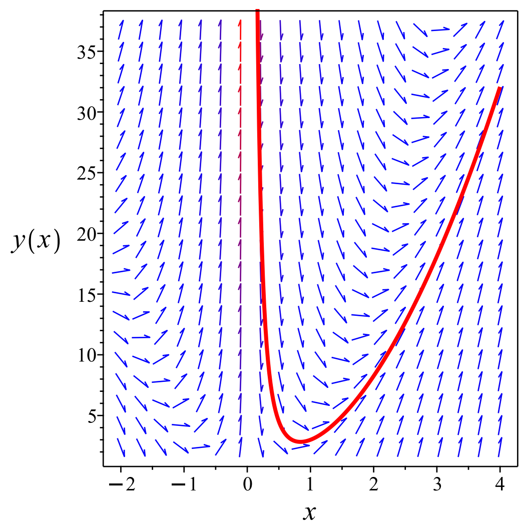
Summary

The solution(s) found are the following

$$y = \frac{2x^4 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^4 + 1}{x^2}$$

Verified OK.

2.31.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2(-4x^2+y)}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^2}} dy
\end{aligned}
$$

Which results in

$$
S = y\,x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2(-4x^2 + y)}{x}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2yx \\
S_y &= x^2
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 8x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 8R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R^4 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2y = 2x^4 + c_1$$

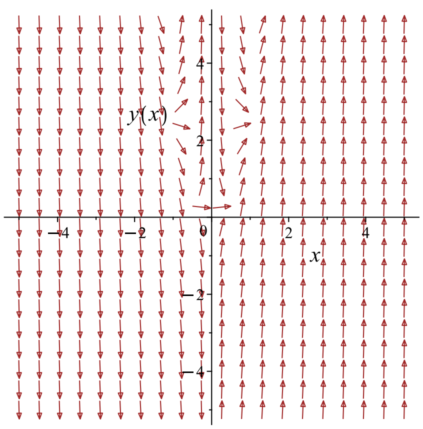
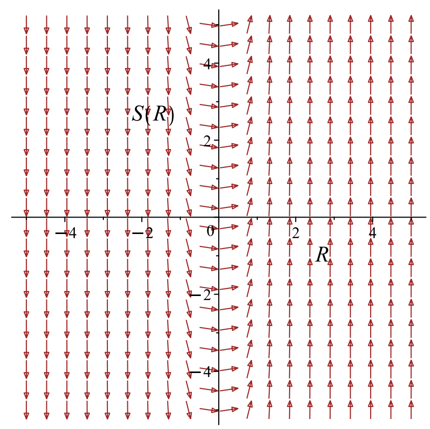
Which simplifies to

$$x^2y = 2x^4 + c_1$$

Which gives

$$y = \frac{2x^4 + c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(-4x^2+y)}{x}$ 	$R = x$ $S = y\, x^2$	$\frac{dS}{dR} = 8R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 + 2$$

$$c_1 = 1$$

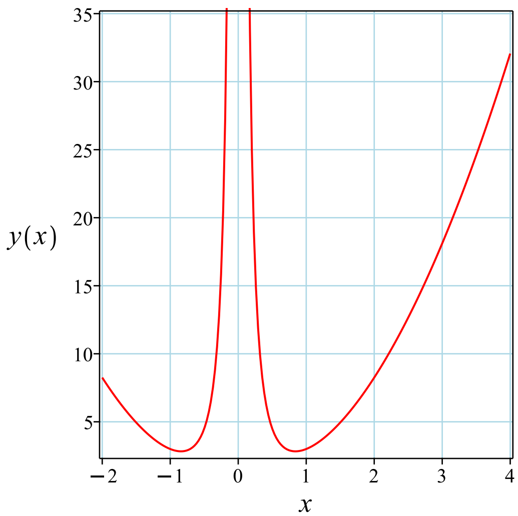
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^4 + 1}{x^2}$$

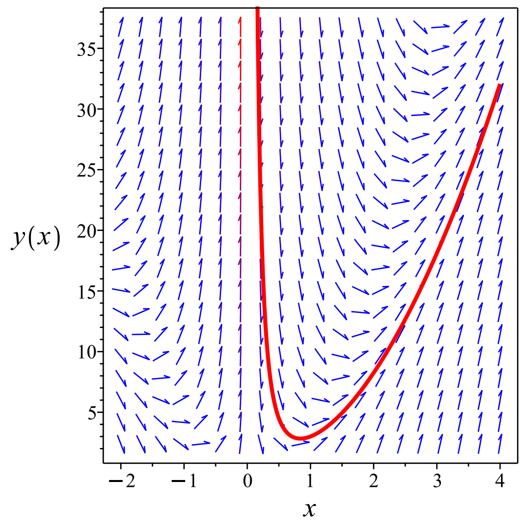
Summary

The solution(s) found are the following

$$y = \frac{2x^4 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2x^4 + 1}{x^2}$$

Verified OK.

2.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x)\,\mathrm{d}y &= \left(8x^2 - 2y\right)\mathrm{d}x \\
\left(-8x^2 + 2y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -8x^2 + 2y \\
N(x, y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-8x^2 + 2y\right) \\
&= 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2) - (1)) \\
&= \frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\, \mathrm{d}x}\\ &= e^{\int \frac{1}{x}\, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(-8x^2 + 2y\right)\\ &= -8x^3 + 2yx\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x(x)\\ &= x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-8x^3 + 2yx\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \overline{M}\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -8x^3 + 2yx\, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\left(4x^2 - y\right)^2}{8} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 - \frac{y}{4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 - \frac{y}{4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{4}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{4}\right)\mathrm{d}y$$

$$f(y) = \frac{y^2}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(4x^2 - y)^2}{8} + \frac{y^2}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(4x^2 - y)^2}{8} + \frac{y^2}{8}$$

The solution becomes

$$y = \frac{2x^4 + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 + 2$$

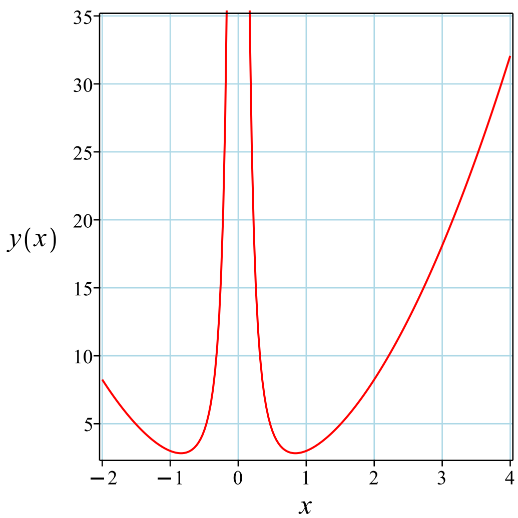
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

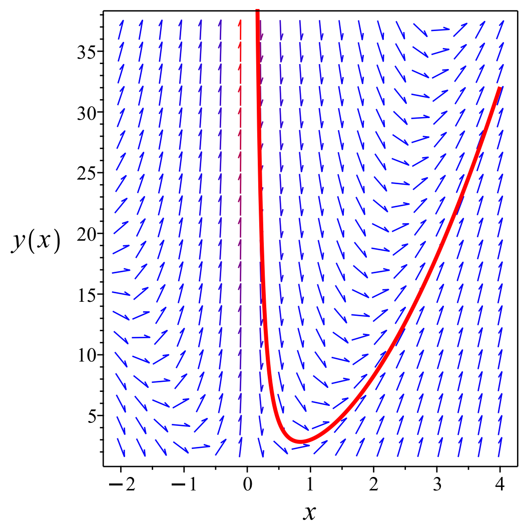
$$y = \frac{2x^4 + 1}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^4 + 1}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^4 + 1}{x^2}$$

Verified OK.

2.31.5 Maple step by step solution

Let's solve

$[2y + y'x = 8x^2, y(1) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + 8x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 8x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = 8\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 8\mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 8\mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int 8\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int 8x^3 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{2x^4 + c_1}{x^2}$

- Use initial condition $y(1) = 3$

  $3 = c_1 + 2$

- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{2x^4+1}{x^2}$
- Solution to the IVP

  $y = \frac{2x^4+1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([x*diff(y(x),x)+2*y(x)=8*x^2,y(1) = 3],y(x), singsol=all)
```

$$y(x) = \frac{2x^4+1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 14

```
DSolve[{x*y'[x]+2*y[x]==8*x^2,y[1]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x^2 + \frac{1}{x^2}$$

2.32 problem 32

Internal problem ID [918]
Internal file name [`OUTPUT/918_Sunday_June_05_2022_01_53_58_AM_39126823/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y'x - 2y = -x^2}$$

With initial conditions

$$[y(1) = 1]$$

2.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = -x$$

Hence the ode is

$$y' - \frac{2y}{x} = -x$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.32.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-x) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)(-x) \\ d\left(\frac{y}{x^2}\right) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int -\frac{1}{x}\,dx \\ \frac{y}{x^2} &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = -\ln(x)\,x^2 + c_1 x^2$$

which simplifies to

$$y = x^2(-\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

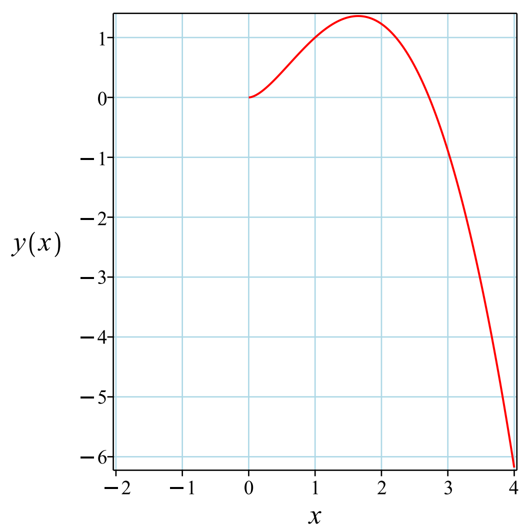
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

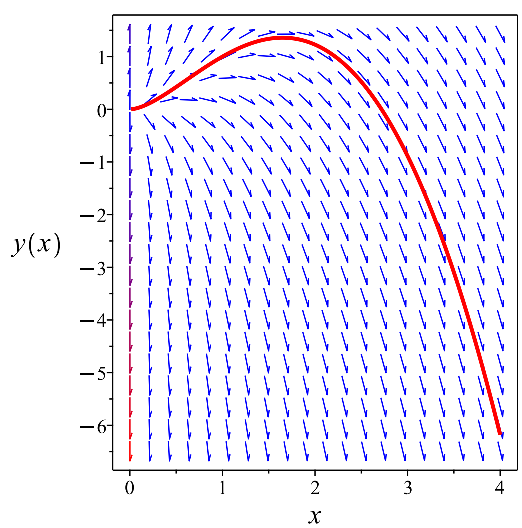
$$y = -\ln(x) x^2 + x^2$$

Summary
The solution(s) found are the following

$$y = -\ln(x) x^2 + x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\ln(x) x^2 + x^2$$

Verified OK.

2.32.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^2 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 123: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x^2 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{x^3} \\
S_y &= \frac{1}{x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^2} = -\ln(x) + c_1
$$

Which simplifies to

$$
\frac{y}{x^2} = -\ln(x) + c_1
$$

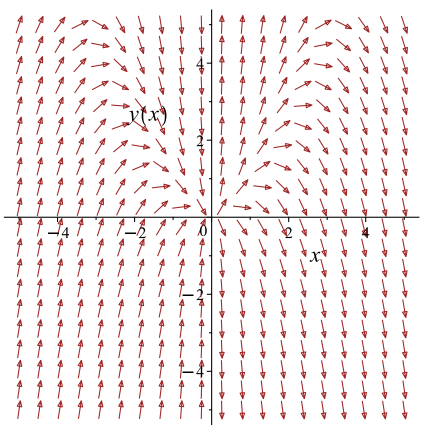
Which gives

$$
y = -x^2(\ln(x) - c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-x^2+2y}{x}$



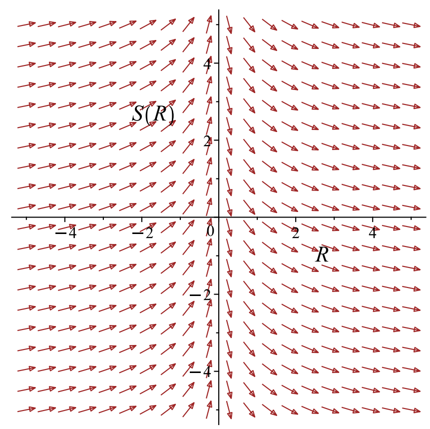
</td><td>

$R = x$

$S = \frac{y}{x^2}$

</td><td>

$\frac{dS}{dR} = -\frac{1}{R}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

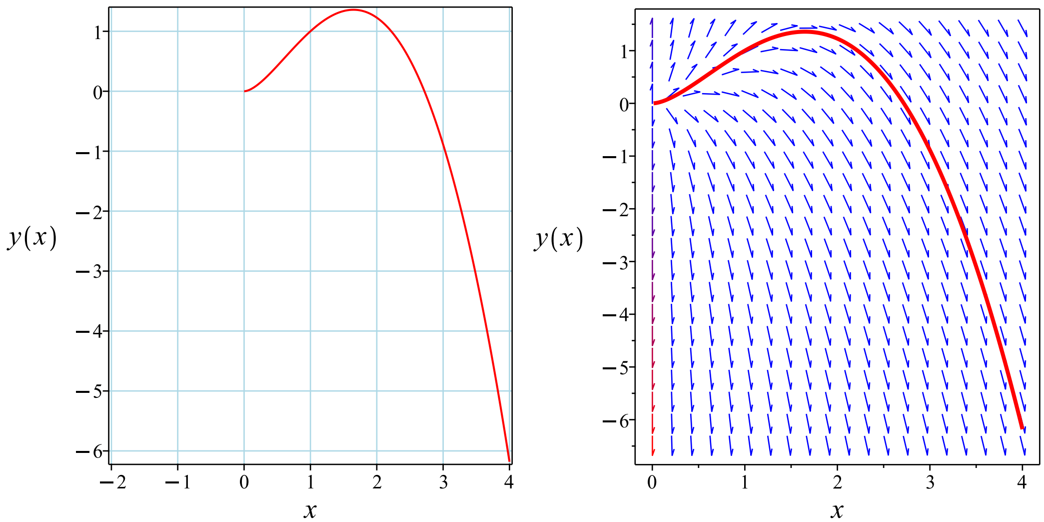
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = -\ln(x)\, x^2 + x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\ln(x)\, x^2 + x^2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\ln(x)\, x^2 + x^2$$

Verified OK.

2.32.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-x^2 + 2y\right)\mathrm{d}x\\ \left(x^2 - 2y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 - 2y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 - 2y\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-2) - (1))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^3}(x^2 - 2y) \\ &= \frac{x^2 - 2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^2 - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^2 - 2y}{x^3} \, \mathrm{d}x$$
$$\phi = \ln(x) + \frac{y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + \frac{y}{x^2}$$

The solution becomes

$$y = -x^2(\ln(x) - c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

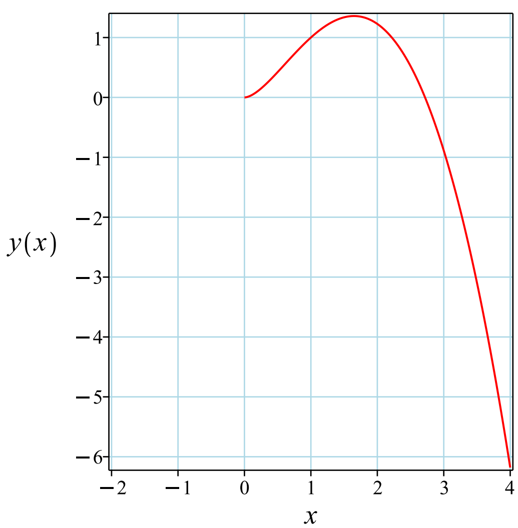
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

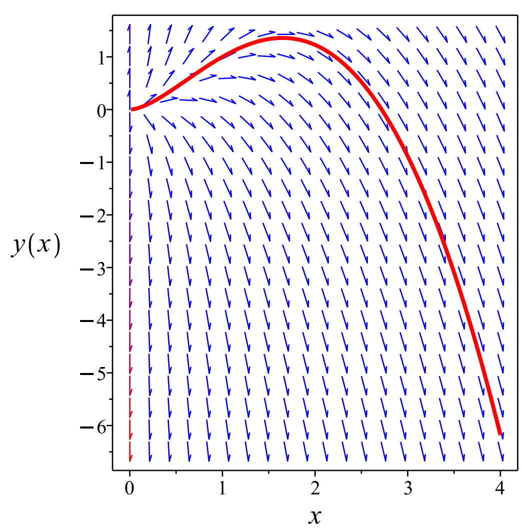
$$y = -\ln(x)\,x^2 + x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\ln(x)\,x^2 + x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\ln(x)\,x^2 + x^2$$

Verified OK.

2.32.5 Maple step by step solution

Let's solve

$[y'x - 2y = -x^2, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} - x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = -x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = -\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,x dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int -\frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(-\ln(x) + c_1)$

- Use initial condition $y(1) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = (-\ln(x) + 1)\, x^2$
- Solution to the IVP

  $y = (-\ln(x) + 1)\, x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x)-2*y(x)=-x^2,y(1) = 1],y(x), singsol=all)
```

$$y(x) = (1 - \ln(x))\, x^2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 14

```
DSolve[{x*y'[x]-2*y[x]==-x^2,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2(\log(x) - 1)$$

2.33 problem 33

Internal problem ID [919]
Internal file name [`OUTPUT/919_Sunday_June_05_2022_01_53_59_AM_52983431/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2yx + y' = x}$$

With initial conditions

$$[y(0) = 3]$$

2.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = x$$

Hence the ode is

$$2yx + y' = x$$

The domain of $p(x) = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.33.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(1 - 2y) \end{aligned}$$

Where $f(x) = x$ and $g(y) = 1 - 2y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1-2y}\, dy &= x\, dx \\ \int \frac{1}{1-2y}\, dy &= \int x\, dx \\ -\frac{\ln(1-2y)}{2} &= \frac{x^2}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{1-2y}} = \mathrm{e}^{\frac{x^2}{2} + c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{1-2y}} = c_2 \mathrm{e}^{\frac{x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{e^{2c_1} e^{-2c_1} c_2^2 - e^{-2c_1}}{2c_2^2}$$

$$c_1 = -\frac{\ln(-5c_2^2)}{2}$$

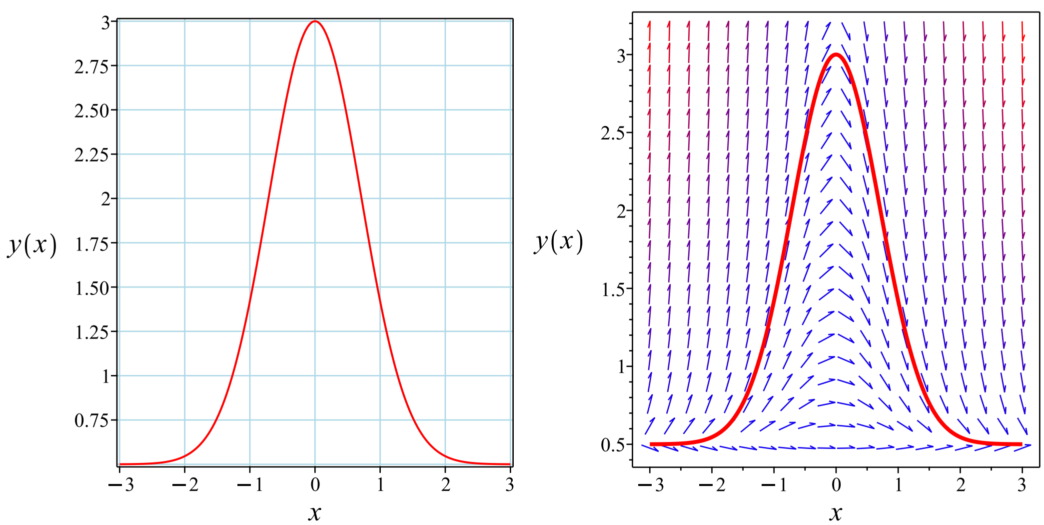
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} + \frac{5\,e^{-x^2}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} + \frac{5\,e^{-x^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} + \frac{5\,e^{-x^2}}{2}$$

Verified OK.

2.33.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int 2x dx} \\
&= e^{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x) \\
\frac{d}{dx}\left(e^{x^2} y\right) &= \left(e^{x^2}\right)(x) \\
d\left(e^{x^2} y\right) &= \left(e^{x^2} x\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{x^2} y &= \int e^{x^2} x \, dx \\
e^{x^2} y &= \frac{e^{x^2}}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = \frac{e^{-x^2} e^{x^2}}{2} + c_1 e^{-x^2}$$

which simplifies to

$$y = \frac{1}{2} + c_1 e^{-x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{1}{2} + c_1$$

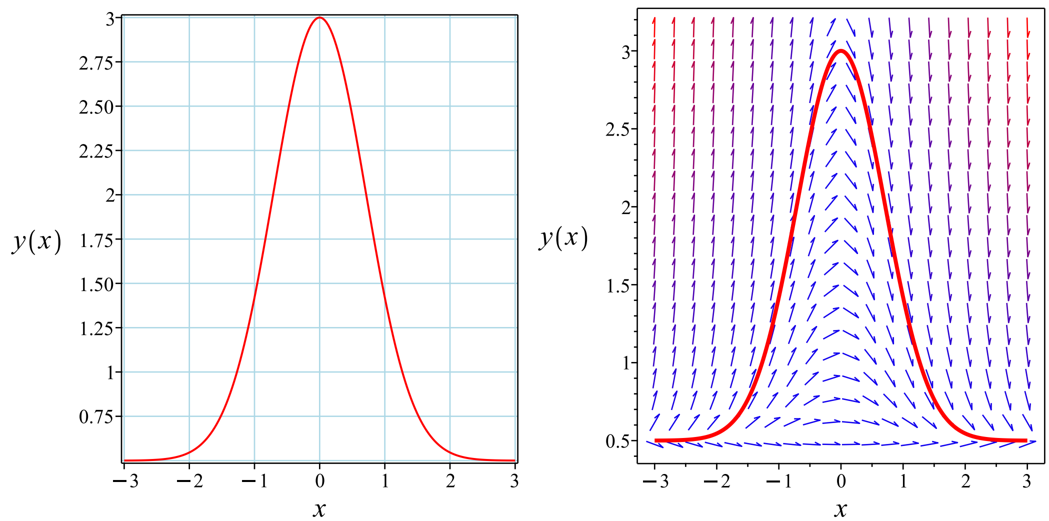
$$c_1 = \frac{5}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} + \frac{5 e^{-x^2}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2}$$

Verified OK.

2.33.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 126: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\, e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x^2} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

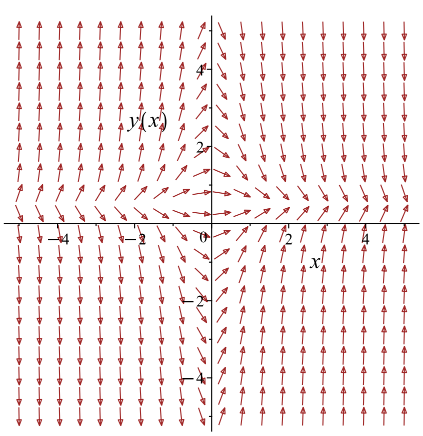
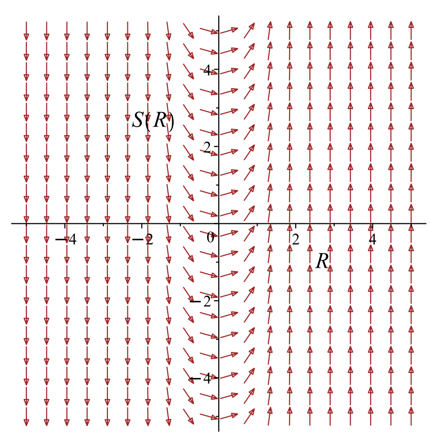
Which simplifies to

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

Which gives

$$y = \frac{\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx + x$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = e^{R^2} R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{1}{2} + c_1$$

$$c_1 = \frac{5}{2}$$

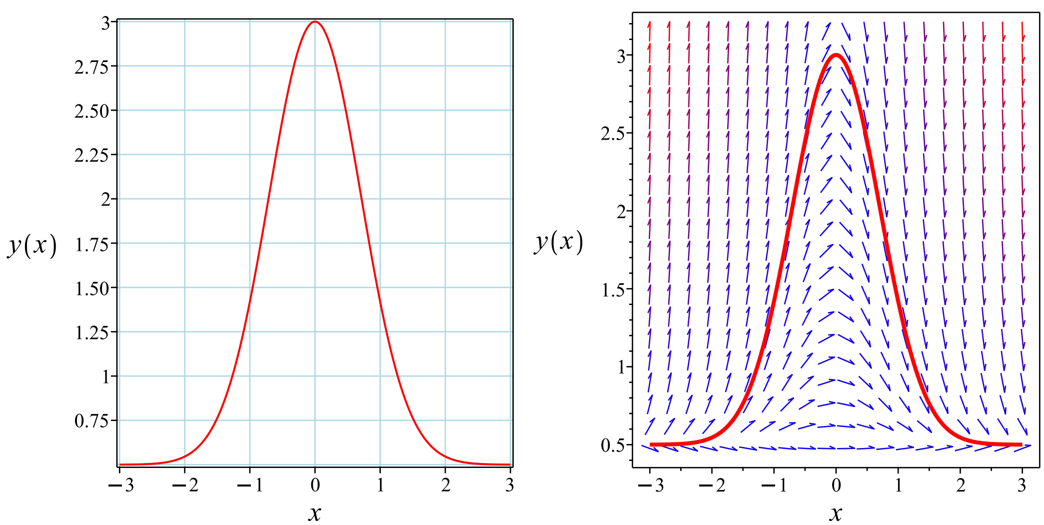
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2}$$

Verified OK.

2.33.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-2y}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{1-2y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{1-2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{1-2y}$. Therefore equation (4) becomes

$$\frac{1}{1-2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{-1+2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{-1+2y}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(-1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-x^2-2c_1}}{2} + \frac{1}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \frac{\mathrm{e}^{-2c_1}}{2} + \frac{1}{2}$$

$$c_1 = -\frac{\ln(5)}{2}$$

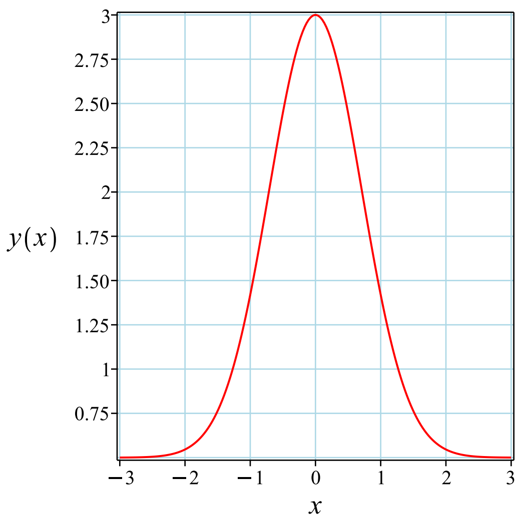
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2}$$

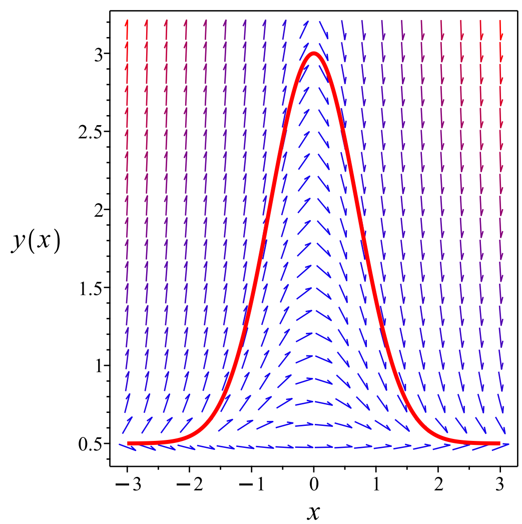
Summary

The solution(s) found are the following

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} + \frac{5\,\mathrm{e}^{-x^2}}{2}$$

Verified OK.

2.33.6 Maple step by step solution

Let's solve

$[2yx + y' = x, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{2y-1} = -x$$

- Integrate both sides with respect to x

$$\int \frac{y'}{2y-1}dx = \int -xdx + c_1$$

- Evaluate integral

$$\frac{\ln(2y-1)}{2} = -\frac{x^2}{2} + c_1$$

- Solve for y

$$y = \frac{e^{-x^2+2c_1}}{2} + \frac{1}{2}$$

- Use initial condition $y(0) = 3$

$$3 = \frac{e^{2c_1}}{2} + \frac{1}{2}$$

- Solve for c_1

$$c_1 = \frac{\ln(5)}{2}$$

- Substitute $c_1 = \frac{\ln(5)}{2}$ into general solution and simplify

$$y = \frac{1}{2} + \frac{5\,e^{-x^2}}{2}$$

- Solution to the IVP

$$y = \frac{1}{2} + \frac{5\,e^{-x^2}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+2*x*y(x)=x,y(0) = 3],y(x), singsol=all)
```

$$y(x) = \frac{1}{2} + \frac{5\,e^{-x^2}}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[{y'[x]+2*x*y[x]==x,y[0]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5e^{-x^2}}{2} + \frac{1}{2}$$

2.34 problem 34

Internal problem ID [920]
Internal file name [`OUTPUT/920_Sunday_June_05_2022_01_54_01_AM_22703504/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x-1)\,y'+3y=\frac{1+(x-1)\sec(x)^2}{(x-1)^3}}$$

With initial conditions

$$[y(0)=-1]$$

2.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{3}{x-1}$$
$$q(x)=\frac{-\tan(x)^2+\sec(x)^2 x}{(x-1)^4}$$

Hence the ode is

$$y' + \frac{3y}{x-1} = \frac{-\tan(x)^2 + \sec(x)^2 x}{(x-1)^4}$$

The domain of $p(x) = \frac{3}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{-\tan(x)^2+\sec(x)^2 x}{(x-1)^4}$ is

$$\left\{-\infty \leq x < 1, 1 < x < \frac{1}{2}\pi + \pi\_Z50, \frac{1}{2}\pi + \pi\_Z50 < x \leq \infty\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.34.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x-1} dx} \\ &= (x-1)^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-\tan(x)^2 + \sec(x)^2 x}{(x-1)^4}\right) \\ \frac{d}{dx}\left(y(x-1)^3\right) &= \left((x-1)^3\right)\left(\frac{-\tan(x)^2 + \sec(x)^2 x}{(x-1)^4}\right) \\ d\left(y(x-1)^3\right) &= \left(\frac{-\tan(x)^2 + \sec(x)^2 x}{x-1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y(x-1)^3 &= \int \frac{-\tan(x)^2 + \sec(x)^2 x}{x-1} dx \\ y(x-1)^3 &= \ln(x-1) + \tan(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x-1)^3$ results in

$$y = \frac{\ln(x-1) + \tan(x)}{(x-1)^3} + \frac{c_1}{(x-1)^3}$$

which simplifies to

$$y = \frac{\ln (x-1)+\tan (x)+c_1}{(x-1)^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln (x-1)+\tan (x)-i\pi+1}{x^3-3x^2+3x-1}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln (x-1)+\tan (x)-i\pi+1}{x^3-3x^2+3x-1} \tag{1}$$

Verification of solutions

$$y = \frac{\ln (x-1)+\tan (x)-i\pi+1}{x^3-3x^2+3x-1}$$

Verified OK.

2.34.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-3y\,x^3+\sec (x)^2 x+9y\,x^2-\sec (x)^2-9yx+3y+1}{(x-1)^4}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 129: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{(x-1)^3} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x-1)^3}} dy \end{aligned}$$

Which results in

$$S = y(x-1)^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3y\, x^3 + \sec(x)^2 x + 9y\, x^2 - \sec(x)^2 - 9yx + 3y + 1}{(x-1)^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y(x-1)^2 \\ S_y &= (x-1)^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-\tan(x)^2 + \sec(x)^2 x}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-\tan(R)^2 + \sec(R)^2 R}{R-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R-1) + \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-1)^3 y = \ln(x-1) + \tan(x) + c_1$$

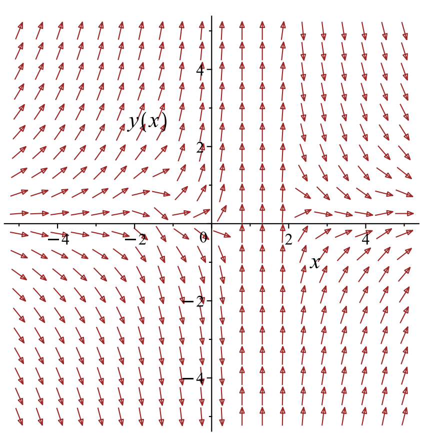
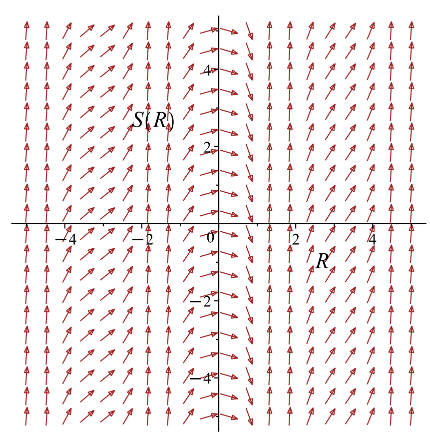
Which simplifies to

$$(x-1)^3 y = \ln(x-1) + \tan(x) + c_1$$

Which gives

$$y = \frac{\ln(x-1) + \tan(x) + c_1}{(x-1)^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3y\,x^3+\sec(x)^2 x+9y\,x^2-\sec(x)^2-9yx+3y+1}{(x-1)^4}$ 	$R = x$ $S = y(x-1)^3$	$\frac{dS}{dR} = \frac{-\tan(R)^2+\sec(R)^2 R}{R-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3 - 3x^2 + 3x - 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3 - 3x^2 + 3x - 1} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3 - 3x^2 + 3x - 1}$$

Verified OK.

2.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x-1)\,\mathrm{d}y = \left(-3y + \frac{1 + (x-1)\sec(x)^2}{(x-1)^3}\right)\mathrm{d}x$$

$$\left(3y - \frac{1 + (x-1)\sec(x)^2}{(x-1)^3}\right)\mathrm{d}x + (x-1)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 3y - \frac{1 + (x-1)\sec(x)^2}{(x-1)^3} \\ N(x, y) &= x - 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - \frac{1 + (x-1)\sec(x)^2}{(x-1)^3}\right) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-1) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x-1}((3) - (1)) \\
&= \frac{2}{x-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{x-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(x-1)} \\
&= (x-1)^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= (x-1)^2 \left(3y - \frac{1 + (x-1)\sec(x)^2}{(x-1)^3}\right) \\
&= \left(3y + \frac{-1 + (1-x)\sec(x)^2}{(x-1)^3}\right)(x-1)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= (x-1)^2 (x-1) \\
&= (x-1)^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(3y + \frac{-1 + (1-x)\sec(x)^2}{(x-1)^3}\right)(x-1)^2\right) + \left((x-1)^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \left(3y + \frac{-1 + (1-x)\sec(x)^2}{(x-1)^3}\right)(x-1)^2 \,\mathrm{d}x$$
$$\phi = \frac{-\ln(x-1)\cos(x) + y(x^3 - 3x^2 + 3x + 9)\cos(x) - \sin(x)}{\cos(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 - 3x^2 + 3x + 9 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x-1)^3$. Therefore equation (4) becomes

$$(x-1)^3 = x^3 - 3x^2 + 3x + 9 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -10$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-10) \,\mathrm{d}y$$
$$f(y) = -10y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\ln(x-1)\cos(x) + y(x^3 - 3x^2 + 3x + 9)\cos(x) - \sin(x)}{\cos(x)} - 10y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\ln(x-1)\cos(x) + y(x^3-3x^2+3x+9)\cos(x) - \sin(x)}{\cos(x)} - 10y$$

The solution becomes

$$y = \frac{\ln(x-1)\cos(x) + c_1\cos(x) + \sin(x)}{\cos(x)(x^3-3x^2+3x-1)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -i\pi - c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3-3x^2+3x-1}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3-3x^2+3x-1} \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x-1) + \tan(x) - i\pi + 1}{x^3-3x^2+3x-1}$$

Verified OK.

2.34.5 Maple step by step solution

Let's solve

$\left[(x-1)y' + 3y = \frac{1+(x-1)\sec(x)^2}{(x-1)^3}, y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

  $y' = -\frac{3y}{x-1}+\frac{\sec(x)^2 x-\sec(x)^2+1}{(x-1)^4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{3y}{x-1} = \frac{\sec(x)^2 x-\sec(x)^2+1}{(x-1)^4}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{3y}{x-1}\right) = \frac{\mu(x)\left(\sec(x)^2 x-\sec(x)^2+1\right)}{(x-1)^4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'+\frac{3y}{x-1}\right) = \mu'(x)\, y+\mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x) = (x-1)^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\left(\sec(x)^2 x-\sec(x)^2+1\right)}{(x-1)^4} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\left(\sec(x)^2 x-\sec(x)^2+1\right)}{(x-1)^4} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(\sec(x)^2 x-\sec(x)^2+1\right)}{(x-1)^4} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x-1)^3$

  $y = \frac{\int \frac{\sec(x)^2 x-\sec(x)^2+1}{x-1} dx + c_1}{(x-1)^3}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{2\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)^2-1}+\ln(x-1)+c_1}{(x-1)^3}$

- Simplify

  $y = \frac{\ln(x-1)\cos(x)+c_1\cos(x)+\sin(x)}{(x-1)^3\cos(x)}$

- Use initial condition $y(0) = -1$

  $-1 = -\mathrm{I}\pi - c_1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi + 1$

- Substitute $c_1 = -\mathrm{I}\pi + 1$ into general solution and simplify

  $y = \frac{\ln(x-1)+\tan(x)-\mathrm{I}\pi+1}{(x-1)^3}$

- Solution to the IVP

  $y = \frac{\ln(x-1)+\tan(x)-\mathrm{I}\pi+1}{(x-1)^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve([(x-1)*diff(y(x),x)+3*y(x)= (1+(x-1)*sec(x)^2)/(x-1)^3,y(0) = -1],y(x), singsol=all)
```

$$y(x) = \frac{\ln(x-1) + \tan(x) + 1 - i\pi}{(x-1)^3}$$

✓ Solution by Mathematica

Time used: 0.171 (sec). Leaf size: 21

```
DSolve[{(x-1)*y'[x]+3*y[x]==(1+(x-1)*Sec[x]^2)/(x-1)^3,y[0]==-1},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{\log(1-x) + \tan(x) + 1}{(x-1)^3}$$

2.35 problem 35

Internal problem ID [921]
Internal file name [`OUTPUT/921_Sunday_June_05_2022_01_54_03_AM_82947288/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(2+x)\,y' + 4y = \frac{2x^2+1}{x\,(2+x)^3}}$$

With initial conditions

$$[y(-1) = 2]$$

2.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{4}{2+x}$$
$$q(x) = \frac{2x^2+1}{(2+x)^4\,x}$$

Hence the ode is

$$y' + \frac{4y}{2+x} = \frac{2x^2+1}{(2+x)^4 x}$$

The domain of $p(x) = \frac{4}{2+x}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{2x^2+1}{(2+x)^4 x}$ is

$$\{-\infty \leq x < -2, -2 < x < 0, 0 < x \leq \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

2.35.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{4}{2+x} dx} \\
&= (2+x)^4
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x^2+1}{(2+x)^4 x}\right) \\
\frac{d}{dx}\left((2+x)^4 y\right) &= \left((2+x)^4\right)\left(\frac{2x^2+1}{(2+x)^4 x}\right) \\
d\left((2+x)^4 y\right) &= \left(\frac{2x^2+1}{x}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(2+x)^4 y &= \int \frac{2x^2+1}{x} dx \\
(2+x)^4 y &= x^2 + \ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (2+x)^4$ results in

$$y = \frac{x^2 + \ln(x)}{(2+x)^4} + \frac{c_1}{(2+x)^4}$$

which simplifies to

$$y = \frac{x^2 + \ln(x) + c_1}{(2 + x)^4}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = i\pi + c_1 + 1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Verified OK.

2.35.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4x^4y + 24y\,x^3 + 48y\,x^2 - 2x^2 + 32yx - 1}{(2 + x)^4\,x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 132: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{(2+x)^4}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(2+x)^4}} dy \end{aligned}$$

Which results in

$$S = (2+x)^4 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4x^4y + 24y\,x^3 + 48y\,x^2 - 2x^2 + 32yx - 1}{(2+x)^4\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4(2+x)^3\,y \\ S_y &= (2+x)^4 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x^2+1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R^2+1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(2+x)^4 y = x^2 + \ln(x) + c_1$$

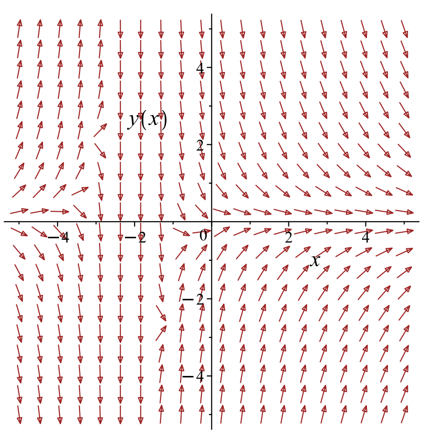
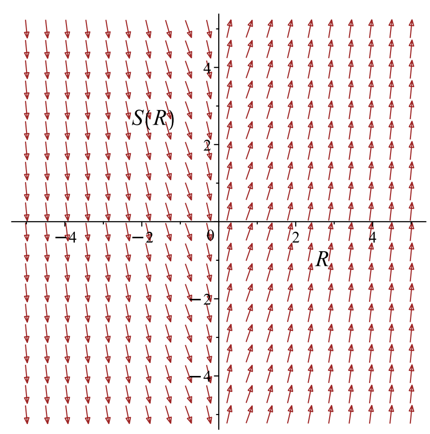
Which simplifies to

$$(2+x)^4 y = x^2 + \ln(x) + c_1$$

Which gives

$$y = \frac{x^2 + \ln(x) + c_1}{(2+x)^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x^4y+24y\,x^3+48y\,x^2-2x^2+32yx-1}{(2+x)^4 x}$ 	$R = x$ $S = (2+x)^4 y$	$\frac{dS}{dR} = \frac{2R^2+1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = i\pi + c_1 + 1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Verified OK.

2.35.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(2+x)\,\mathrm{d}y=\left(-4y+\frac{2x^2+1}{x\left(2+x\right)^3}\right)\mathrm{d}x$$

$$\left(4y-\frac{2x^2+1}{x\left(2+x\right)^3}\right)\mathrm{d}x+(2+x)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=4y-\frac{2x^2+1}{x\left(2+x\right)^3}\\N(x,y)&=2+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(4y-\frac{2x^2+1}{x\left(2+x\right)^3}\right)\\&=4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2+x)\\&=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{2+x}((4)-(1))\\&=\frac{3}{2+x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{3}{2+x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(2+x)}\\ &= (2+x)^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= (2+x)^3\left(4y-\frac{2x^2+1}{x\left(2+x\right)^3}\right)\\ &= \left(4y-\frac{2x^2+1}{x\left(2+x\right)^3}\right)(2+x)^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= (2+x)^3\,(2+x)\\ &= (2+x)^4\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(4y-\frac{2x^2+1}{x\left(2+x\right)^3}\right)(2+x)^3\right)+\left((2+x)^4\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \left(4y - \frac{2x^2+1}{x(2+x)^3}\right)(2+x)^3 \mathrm{d}x$$

$$\phi = -\ln(x) + x^4 y + 8y\,x^3 + (24y-1)\,x^2 + 32yx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x^4 + 8x^3 + 24x^2 + 32x + f'(y) \\ &= x(x+4)\left(x^2+4x+8\right) + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (2+x)^4$. Therefore equation (4) becomes

$$(2+x)^4 = x(x+4)\left(x^2+4x+8\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 16$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (16) \mathrm{d}y$$

$$f(y) = 16y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + x^4 y + 8y\,x^3 + (24y-1)\,x^2 + 32yx + 16y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + x^4 y + 8y\,x^3 + (24y-1)\,x^2 + 32yx + 16y$$

The solution becomes

$$y = \frac{x^2 + \ln(x) + c_1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = i\pi + c_1 + 1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 + \ln(x) - i\pi + 1}{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Verified OK.

2.35.5 Maple step by step solution

Let's solve

$\left[(2+x)\, y' + 4y = \frac{2x^2+1}{x(2+x)^3}, y(-1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4y}{2+x} + \frac{2x^2+1}{(2+x)^4 x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4y}{2+x} = \frac{2x^2+1}{(2+x)^4 x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{4y}{2+x}\right)=\frac{\mu(x)(2x^2+1)}{(2+x)^4x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{4y}{2+x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{4\mu(x)}{2+x}$

- Solve to find the integrating factor

  $\mu(x)=(2+x)^4$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)(2x^2+1)}{(2+x)^4x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)(2x^2+1)}{(2+x)^4x}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)\left(2x^2+1\right)}{(2+x)^4x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=(2+x)^4$

  $y=\frac{\int\frac{2x^2+1}{x}dx+c_1}{(2+x)^4}$

- Evaluate the integrals on the rhs

  $y=\frac{x^2+\ln(x)+c_1}{(2+x)^4}$

- Use initial condition $y(-1)=2$

  $2=\mathrm{I}\pi+c_1+1$

- Solve for c_1

  $c_1=-\mathrm{I}\pi+1$

- Substitute $c_1=-\mathrm{I}\pi+1$ into general solution and simplify

  $y=\frac{x^2+\ln(x)-\mathrm{I}\pi+1}{(2+x)^4}$

- Solution to the IVP

  $y=\frac{x^2+\ln(x)-\mathrm{I}\pi+1}{(2+x)^4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve([(x+2)*diff(y(x),x)+4*y(x)= (1+2*x^2)/(x*(x+2)^3),y(-1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{x^2 + \ln(x) + 1 - i\pi}{(2+x)^4}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 23

```
DSolve[{(x+2)*y'[x]+4*y[x]== (1+2*x^2)/(x*(x+2)^3),y[-1]==2},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{x^2 + \log(x) - i\pi + 1}{(x+2)^4}$$

2.36 problem 36

Internal problem ID [922]
Internal file name [`OUTPUT/922_Sunday_June_05_2022_01_54_04_AM_39182112/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2-1)\,y'-2yx = x(x^2-1)}$$

With initial conditions

$$[y(0) = 4]$$

2.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x}{x^2-1}$$
$$q(x) = x$$

Hence the ode is

$$y' - \frac{2xy}{x^2-1} = x$$

The domain of $p(x) = -\frac{2x}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.36.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2x}{x^2-1}dx} \\ &= \mathrm{e}^{-\ln(x-1)-\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{x^2-1}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2-1}\right) &= \left(\frac{1}{x^2-1}\right)(x) \\ \mathrm{d}\left(\frac{y}{x^2-1}\right) &= \left(\frac{x}{x^2-1}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2-1} &= \int \frac{x}{x^2-1}\,\mathrm{d}x \\ \frac{y}{x^2-1} &= \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2-1}$ results in

$$y = (x^2-1)\left(\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}\right) + c_1(x^2-1)$$

which simplifies to

$$y = \frac{(x^2-1)(\ln(x-1)+\ln(x+1)+2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\frac{i\pi}{2} - c_1$$

$$c_1 = -\frac{i\pi}{2} - 4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln(x-1)\, x^2}{2} + \frac{\ln(x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + 4$$

Summary

The solution(s) found are the following

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln(x-1)\, x^2}{2} + \frac{\ln(x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + 4 \tag{1}$$

Verification of solutions

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln(x-1)\, x^2}{2} + \frac{\ln(x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + 4$$

Verified OK.

2.36.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x(x^2+2y-1)}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 135: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\ln(x-1) + \ln(x+1)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(x-1)+\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(x-1)(x+1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(x^2 + 2y - 1)}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2-1)^2} \\ S_y &= \frac{1}{x^2-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{x^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2-1} = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1$$

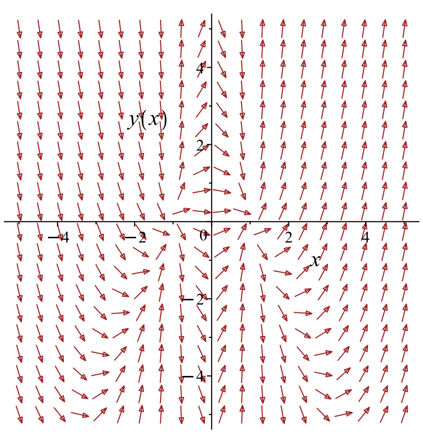
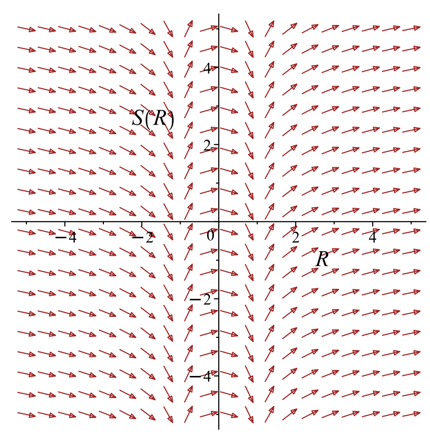
Which simplifies to

$$\frac{y}{x^2-1} = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1$$

Which gives

$$y = \frac{(x^2-1)(\ln(x-1) + \ln(x+1) + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(x^2+2y-1)}{x^2-1}$ 	$R = x$ $S = \frac{y}{x^2-1}$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\frac{i\pi}{2} - c_1$$

$$c_1 = -\frac{i\pi}{2} - 4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln\left(x-1\right) x^2}{2} + \frac{\ln\left(x+1\right) x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(x+1\right)}{2} + 4$$

Summary

The solution(s) found are the following

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln\left(x-1\right) x^2}{2} + \frac{\ln\left(x+1\right) x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(x+1\right)}{2} + 4 \tag{14}$$

Verification of solutions

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln\left(x-1\right) x^2}{2} + \frac{\ln\left(x+1\right) x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln\left(x-1\right)}{2} - \frac{\ln\left(x+1\right)}{2} + 4$$

Verified OK.

2.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2-1)\,\mathrm{d}y &= \left(2yx + x(x^2-1)\right)\mathrm{d}x\\ \left(-2yx - x(x^2-1)\right)\mathrm{d}x + (x^2-1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2yx - x(x^2-1)\\ N(x,y) &= x^2-1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2yx - x(x^2-1)\right)\\ &= -2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2-1)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2-1}((-2x) - (2x))\\ &= -\frac{4x}{x^2-1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4x}{x^2-1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x-1)-2\ln(x+1)}\\ &= \frac{1}{\left(x-1\right)^{2}\left(x+1\right)^{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\left(x-1\right)^{2}\left(x+1\right)^{2}}\left(-2yx-x\left(x^{2}-1\right)\right)\\ &= -\frac{x(x^{2}+2y-1)}{\left(x-1\right)^{2}\left(x+1\right)^{2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\left(x-1\right)^{2}\left(x+1\right)^{2}}\left(x^{2}-1\right)\\ &= \frac{1}{x^{2}-1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{x(x^{2}+2y-1)}{\left(x-1\right)^{2}\left(x+1\right)^{2}}\right)+\left(\frac{1}{x^{2}-1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{x(x^2+2y-1)}{(x-1)^2 (x+1)^2} \mathrm{d}x$$

$$\phi = -\frac{\ln(x+1)}{2} - \frac{y}{2+2x} - \frac{\ln(x-1)}{2} + \frac{y}{2x-2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\frac{1}{2+2x} + \frac{1}{2x-2} + f'(y) \\ &= \frac{1}{x^2-1} + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2-1}$. Therefore equation (4) becomes

$$\frac{1}{x^2-1} = \frac{1}{x^2-1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x+1)}{2} - \frac{y}{2+2x} - \frac{\ln(x-1)}{2} + \frac{y}{2x-2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x+1)}{2} - \frac{y}{2+2x} - \frac{\ln(x-1)}{2} + \frac{y}{2x-2}$$

The solution becomes

$$y = \frac{(\ln (x-1) + \ln (x+1) + 2c_1)(x-1)(x+1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\frac{i\pi}{2} - c_1$$

$$c_1 = -\frac{i\pi}{2} - 4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln (x-1)\, x^2}{2} + \frac{\ln (x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} + 4$$

Summary

The solution(s) found are the following

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln (x-1)\, x^2}{2} + \frac{\ln (x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} + 4 \quad (1)$$

Verification of solutions

$$y = -\frac{i\pi\, x^2}{2} + \frac{\ln (x-1)\, x^2}{2} + \frac{\ln (x+1)\, x^2}{2} + \frac{i\pi}{2} - 4x^2 - \frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} + 4$$

Verified OK.

2.36.5 Maple step by step solution

Let's solve

$[(x^2-1)\, y' - 2yx = x(x^2-1)\,, y(0) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2xy}{x^2-1} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2xy}{x^2-1} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{2xy}{x^2-1}\right)=\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{2xy}{x^2-1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{2\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,xdx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,xdx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{(x-1)(x+1)}$

  $y=(x-1)\,(x+1)\left(\int\frac{x}{(x-1)(x+1)}dx+c_1\right)$

- Evaluate the integrals on the rhs

  $y=(x-1)\,(x+1)\left(\frac{\ln((x-1)(x+1))}{2}+c_1\right)$

- Simplify

  $y=\frac{(\ln(x^2-1)+2c_1)\,(x^2-1)}{2}$

- Use initial condition $y(0)=4$

  $4=-\frac{\mathrm{I}\pi}{2}-c_1$

- Solve for c_1

  $c_1=-\frac{\mathrm{I}\pi}{2}-4$

- Substitute $c_1=-\frac{\mathrm{I}\pi}{2}-4$ into general solution and simplify

  $y=-\frac{(\mathrm{I}\pi-\ln(x^2-1)+8)\,(x^2-1)}{2}$

- Solution to the IVP

  $y=-\frac{(\mathrm{I}\pi-\ln(x^2-1)+8)\,(x^2-1)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 29

```
dsolve([(x^2-1)*diff(y(x),x)-2*x*y(x)= x*(x^2-1),y(0) = 4],y(x), singsol=all)
```

$$y(x) = -\frac{(i\pi - \ln(x+1) - \ln(x-1) + 8)(x^2-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 27

```
DSolve[{(x^2-1)*y'[x]-2*x*y[x]== x*(x^2-1),y[0]==4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x^2-1)\left(\log(x^2-1) - i\pi - 8\right)$$

2.37 problem 44

Internal problem ID [923]
Internal file name [`OUTPUT/923_Sunday_June_05_2022_01_54_06_AM_51720512/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x - 2y = -1}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

2.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -\frac{1}{x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = -\frac{1}{x}$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.37.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-1+2y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = -1 + 2y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-1+2y}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{-1+2y}\,dy &= \int \frac{1}{x}\,dx \\ \frac{\ln(-1+2y)}{2} &= \ln(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{-1+2y} = \mathrm{e}^{\ln(x)+c_1}$$

Which simplifies to

$$\sqrt{-1+2y} = c_2 x$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2}$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{2c_1} c_2^2 x^2}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{e}^{2c_1} c_2^2 x^2}{2} + \frac{1}{2}$$

Verified OK.

2.37.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{1}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-\frac{1}{x}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(-\frac{1}{x^3}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = \int -\frac{1}{x^3}\,\mathrm{d}x$$
$$\frac{y}{x^2} = \frac{1}{2x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = \frac{1}{2} + c_1 x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2}$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = \frac{1}{2} + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{2} + c_1 x^2$$

Verified OK.

2.37.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 - 1}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = \frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{2Y}{X}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= 2u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) X - u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= \frac{u}{X}
\end{aligned}
$$

Where $f(X) = \frac{1}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{X}\,dX \\
\int \frac{1}{u}\,du &= \int \frac{1}{X}\,dX \\
\ln(u) &= \ln(X) + c_2 \\
u &= e^{\ln(X)+c_2} \\
&= c_2 X
\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X^2 c_2$$

Using the solution for $Y(X)$

$$Y(X) = X^2 c_2$$

And replacing back terms in the above solution using

$$\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}$$

Or

$$\begin{aligned}
Y &= y + \frac{1}{2} \\
X &= x
\end{aligned}$$

Then the solution in y becomes

$$y - \frac{1}{2} = c_2 x^2$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_2. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y - \frac{1}{2} = c_2 x^2 \tag{1}$$

Verification of solutions

$$y - \frac{1}{2} = c_2 x^2$$

Verified OK.

2.37.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-1+2y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = x^2 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-1 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{x^3} \\
S_y &= \frac{1}{x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x^3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R^3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{1}{2R^2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^2} = \frac{1}{2x^2} + c_1
$$

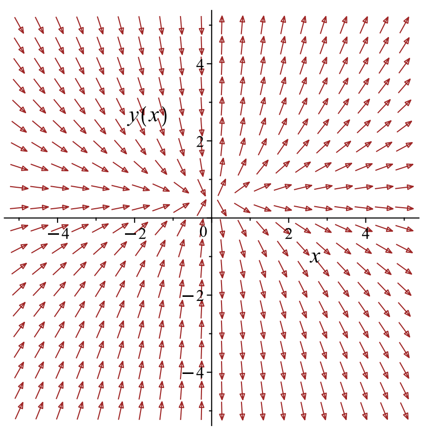
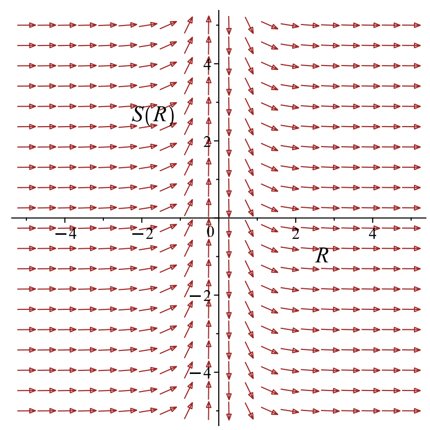
Which simplifies to

$$
\frac{y}{x^2} = \frac{1}{2x^2} + c_1
$$

Which gives

$$
y = \frac{1}{2} + c_1 x^2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-1+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = -\frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2}$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = \frac{1}{2} + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{2} + c_1 x^2$$

Verified OK.

2.37.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-1+2y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{-1+2y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{1}{-1+2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-1+2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-1+2y}$. Therefore equation (4) becomes

$$\frac{1}{-1+2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{-1+2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{-1+2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(-1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{\ln(-1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{\ln(-1+2y)}{2}$$

The solution becomes

$$y = \frac{x^2 \mathrm{e}^{2c_1}}{2} + \frac{1}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2}$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = \frac{x^2 e^{2c_1}}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 e^{2c_1}}{2} + \frac{1}{2}$$

Verified OK.

2.37.7 Maple step by step solution

Let's solve

$\left[y'x - 2y = -1, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y-1} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y-1} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\frac{\ln(2y-1)}{2} = \ln(x) + c_1$

- Solve for y

  $y = \frac{x^2 e^{2c_1}}{2} + \frac{1}{2}$

- Use initial condition $y(0) = \frac{1}{2}$

  $\frac{1}{2} = \frac{1}{2}$

- Solve for c_1

  $c_1 = c_1$

- Substitute $c_1 = c_1$ into general solution and simplify

  $y = \frac{x^2 e^{2c_1}}{2} + \frac{1}{2}$

- Solution to the IVP

  $y = \frac{x^2 e^{2c_1}}{2} + \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([x*diff(y(x),x)-2*y(x)= -1,y(0) = 1/2],y(x), singsol=all)
```

$$y(x) = \frac{1}{2} + c_1 x^2$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 8

```
DSolve[{x*y'[x]-2*y[x]== -1,y[0]==1/2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}$$

2.38 problem 48(a)

Internal problem ID [924]
Internal file name [`OUTPUT/924_Sunday_June_05_2022_01_54_07_AM_72624317/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 48(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\sec (y)^{2} y'-3 \tan (y)=-1}$$

2.38.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} \int \frac{\sec (y)^{2}}{3 \tan (y)-1}dy &= \int dx \\ \frac{\ln (3 \tan (y)-1)}{3} &= x +c_{1} \end{aligned}$$

Raising both side to exponential gives

$$(3 \tan (y)-1)^{\frac{1}{3}} = {\mathrm e}^{x +c_{1}}$$

Which simplifies to

$$(3 \tan (y)-1)^{\frac{1}{3}} = c_{2} {\mathrm e}^{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \arctan \left(\frac{c_{2}^{3} {\mathrm e}^{3 x}}{3}+\frac{1}{3}\right) \tag{1}$$

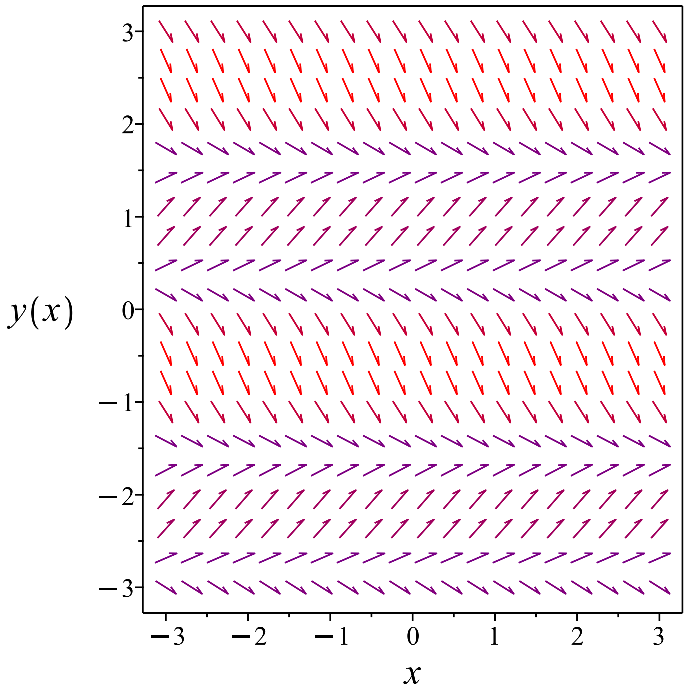


Figure 141: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{c_2^3 e^{3x}}{3} + \frac{1}{3}\right)$$

Verified OK.

2.38.2 Maple step by step solution

Let's solve

$\sec(y)^2 y' - 3\tan(y) = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sec(y)^2}{3\tan(y)-1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y' \sec(y)^2}{3\tan(y)-1} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{\ln(3\tan(y)-1)}{3} = x + c_1$
- Solve for y

  $y = \arctan\left(\frac{e^{3x+3c_1}}{3} + \frac{1}{3}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(sec(y(x))^2*diff(y(x),x)-3*tan(y(x))= -1,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{c_1 e^{3x}}{3} + \frac{1}{3}\right)$$

✓ Solution by Mathematica

Time used: 60.217 (sec). Leaf size: 177

```
DSolve[Sec[y[x]]^2*y'[x]-3*Tan[y[x]]== -1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(-\frac{3e^{6c_1}}{\sqrt{e^{6x} - 2e^{3x+6c_1} + 10e^{12c_1}}}\right)$$
$$y(x) \to \arccos\left(-\frac{3e^{6c_1}}{\sqrt{e^{6x} - 2e^{3x+6c_1} + 10e^{12c_1}}}\right)$$
$$y(x) \to -\arccos\left(\frac{3e^{6c_1}}{\sqrt{e^{6x} - 2e^{3x+6c_1} + 10e^{12c_1}}}\right)$$
$$y(x) \to \arccos\left(\frac{3e^{6c_1}}{\sqrt{e^{6x} - 2e^{3x+6c_1} + 10e^{12c_1}}}\right)$$

2.39 problem 48(b)

Internal problem ID [925]
Internal file name [OUTPUT/925_Sunday_June_05_2022_01_54_09_AM_99212434/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 48(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{e^{y^2}\left(2yy'+\frac{2}{x}\right)=\frac{1}{x^2}}$$

2.39.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(2\,\mathrm{e}^{y^2}y\,x^2\right)\mathrm{d}y=\left(-2\,\mathrm{e}^{y^2}x+1\right)\mathrm{d}x$$
$$\left(2\,\mathrm{e}^{y^2}x-1\right)\mathrm{d}x+\left(2\,\mathrm{e}^{y^2}y\,x^2\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=2\,\mathrm{e}^{y^2}x-1$$
$$N(x,y)=2\,\mathrm{e}^{y^2}y\,x^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(2\,\mathrm{e}^{y^2}x-1\right)$$
$$=4\,\mathrm{e}^{y^2}yx$$

And

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(2\,\mathrm{e}^{y^2}y\,x^2\right)$$
$$=4\,\mathrm{e}^{y^2}yx$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2\,\mathrm{e}^{y^2} x - 1 \,\mathrm{d}x$$
$$\phi = x\left(\mathrm{e}^{y^2} x - 1\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{y^2} y\, x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\,\mathrm{e}^{y^2} y\, x^2$. Therefore equation (4) becomes

$$2\,\mathrm{e}^{y^2} y\, x^2 = 2\,\mathrm{e}^{y^2} y\, x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x\left(\mathrm{e}^{y^2} x - 1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\left(\mathrm{e}^{y^2} x - 1\right)$$

Summary

The solution(s) found are the following

$$x\left(\mathrm{e}^{y^2} x - 1\right) = c_1 \tag{1}$$

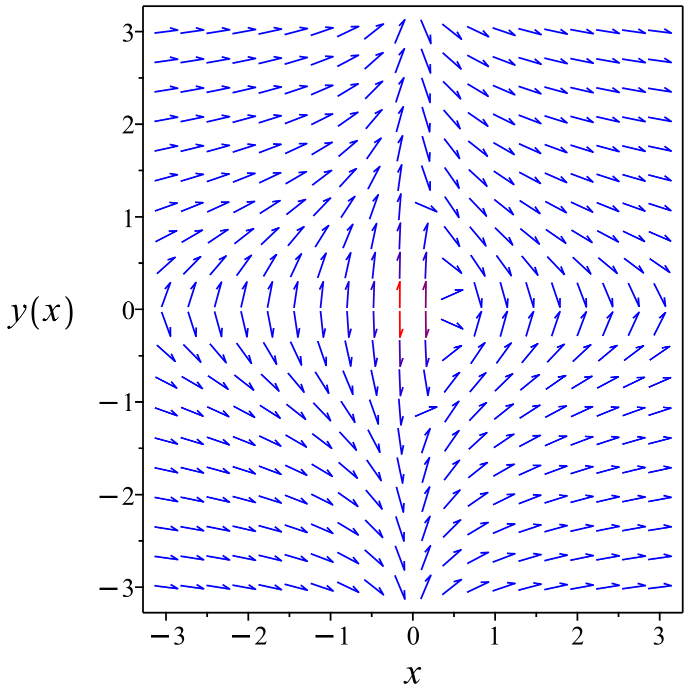


Figure 142: Slope field plot

Verification of solutions

$$x\left(e^{y^2}x-1\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(exp(y(x)^2)*(2*y(x)*diff(y(x),x)+2/x)= 1/x^2,y(x), singsol=all)
```

$$y(x) = \sqrt{\ln\left(\frac{-c_1 + x}{x^2}\right)}$$
$$y(x) = -\sqrt{\ln\left(\frac{-c_1 + x}{x^2}\right)}$$

✓ Solution by Mathematica

Time used: 7.286 (sec). Leaf size: 37

```
DSolve[Exp[y[x]^2]*(2*y[x]*y'[x]+2/x)== 1/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\log\left(\frac{x + c_1}{x^2}\right)}$$
$$y(x) \to \sqrt{\log\left(\frac{x + c_1}{x^2}\right)}$$

2.40 problem 48(c)

Internal problem ID [926]
Internal file name [OUTPUT/926_Sunday_June_05_2022_01_54_11_AM_71946659/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 48(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{2\ln(y) = -\frac{xy'}{y} + 4x^2}$$

2.40.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2y(-2x^2 + \ln(y))}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7 + y\,x^2a_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^3b_7 + y\,x^2b_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{gathered}
3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\
- \frac{2y(-2x^2 + \ln(y))\left(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + \right.}{x} \\
- \frac{4y^2(-2x^2 + \ln(y))^2 (x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{x^2} \\
- \left(8y + \frac{2y(-2x^2 + \ln(y))}{x^2}\right)\left(x^3a_7 + y\,x^2a_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4\right. \\
\left. + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) - \left(-\frac{2(-2x^2 + \ln(y))}{x} - \frac{2}{x}\right)\left(x^3b_7\right. \\
\left. + y\,x^2b_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{gathered} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4\ln(y)\,x^3ya_7 - 2\ln(y)\,x^2y^2a_8 + 2\ln(y)\,x^2y^2b_9 + 4\ln(y)\,x\,y^3b_{10} - 16\ln(y)\,x^4y^2a_8 - 32\ln(y)\,x^3y^3a_9 - }{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{gathered}
4\ln(y)\,x^3ya_7 + 2\ln(y)\,x^2y^2a_8 - 2\ln(y)\,x^2y^2b_9 - 4\ln(y)\,x\,y^3b_{10} \\
+ 16\ln(y)\,x^4y^2a_8 + 32\ln(y)\,x^3y^3a_9 + 48\ln(y)\,x^2y^4a_{10} \\
- 4\ln(y)^2\,x^2y^2a_8 - 8\ln(y)^2\,x\,y^3a_9 + 16\ln(y)\,x^3y^2a_5 \\
+ 32\ln(y)\,x^2y^3a_6 - 4\ln(y)^2\,x\,y^2a_5 + 2\ln(y)\,x^2ya_4 \\
- 2\ln(y)\,x\,y^2b_6 + 16\ln(y)\,x^2y^2a_3 + 3yb_5x^2 - 16x^5y^2a_5 \\
- 32x^4y^3a_6 - 12x^4ya_4 - 8x^3y^2a_5 + 4x^3y^2b_6 - 4x^2y^3a_6 \\
+ 2x\,y^2b_6 + 4x^3b_4 - 4x^5b_4 - 16x^4y^2a_3 - 8x^3ya_2 - 4x^2y^2a_3 \\
- 4x^2ya_1 + 2xyb_3 - 4\ln(y)^2\,y^2a_3 + 2\ln(y)\,x^2b_2 - 2\ln(y)\,y^2a_3 \\
+ 2\ln(y)\,xb_1 - 2\ln(y)\,ya_1 + 4x^3yb_8 + 3y^2b_9x^2 - 16x^5ya_7 \\
- 12x^4y^2a_8 + 4x^4y^2b_9 - 8x^3y^3a_9 + 8x^3y^3b_{10} - 16x^6y^2a_8 \\
- 32x^5y^3a_9 - 48x^4y^4a_{10} - 4x^2y^4a_{10} + 2x\,y^3b_{10} - 12\ln(y)^2\,y^4a_{10} \\
- 2\ln(y)\,y^4a_{10} + 2\ln(y)\,x^4b_7 + 5x^4b_7 - 4x^6b_7 + 3b_2x^2 - 4x^4b_2 \\
- 4x^3b_1 + 2xb_1 - 8\ln(y)^2\,y^3a_6 + 2\ln(y)\,x^3b_4 - 2\ln(y)\,y^3a_6 = 0
\end{gathered} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(y) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&4v_3v_1^3v_2a_7 + 2v_3v_1^2v_2^2a_8 - 2v_3v_1^2v_2^2b_9 - 4v_3v_1v_2^3b_{10} + 16v_3v_1^4v_2^2a_8 \\
&+ 32v_3v_1^3v_2^3a_9 + 48v_3v_1^2v_2^4a_{10} - 4v_3^2v_1^2v_2^2a_8 - 8v_3^2v_1v_2^3a_9 + 16v_3v_1^3v_2^2a_5 \\
&+ 32v_3v_1^2v_2^3a_6 - 4v_3^2v_1v_2^2a_5 + 2v_3v_1^2v_2a_4 - 2v_3v_1v_2^2b_6 + 16v_3v_1^2v_2^2a_3 \\
&+ 4v_1^3b_4 - 4v_1^5b_4 + 5v_1^4b_7 - 4v_1^6b_7 + 3b_2v_1^2 - 4v_1^4b_2 - 4v_1^3b_1 + 2v_1b_1 \\
&+ 3v_2b_5v_1^2 - 16v_1^5v_2^2a_5 - 32v_1^4v_2^3a_6 - 12v_1^4v_2a_4 - 8v_1^3v_2^2a_5 + 4v_1^3v_2^2b_6 \\
&- 4v_1^2v_2^3a_6 + 2v_1v_2^2b_6 - 16v_1^4v_2^2a_3 - 8v_1^3v_2a_2 - 4v_1^2v_2^2a_3 - 4v_1^2v_2a_1 \\
&+ 2v_1v_2b_3 - 4v_3^2v_2^2a_3 + 2v_3v_1^2b_2 - 2v_3v_2^2a_3 + 2v_3v_1b_1 - 2v_3v_2a_1 \\
&+ 4v_1^3v_2b_8 + 3v_2^2b_9v_1^2 - 16v_1^5v_2a_7 - 12v_1^4v_2^2a_8 + 4v_1^4v_2^2b_9 - 8v_1^3v_2^3a_9 \\
&+ 8v_1^3v_2^3b_{10} - 16v_1^6v_2^2a_8 - 32v_1^5v_2^3a_9 - 48v_1^4v_2^4a_{10} - 4v_1^2v_2^4a_{10} + 2v_1v_2^3b_{10} \\
&- 12v_3^2v_2^4a_{10} - 2v_3v_2^4a_{10} + 2v_3v_1^4b_7 - 8v_3^2v_2^3a_6 + 2v_3v_1^3b_4 - 2v_3v_2^3a_6 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&4v_3v_1^3v_2a_7 + (-4a_1 + 3b_5)\, v_1^2v_2 + (-8a_5 + 4b_6)\, v_2^2v_1^3 \\
&+ (-16a_3 - 12a_8 + 4b_9)\, v_2^2v_1^4 + (-8a_2 + 4b_8)\, v_2v_1^3 + (-4a_3 + 3b_9)\, v_2^2v_1^2 \\
&+ (-8a_9 + 8b_{10})\, v_2^3v_1^3 - 4v_3v_1v_2^3b_{10} + 16v_3v_1^4v_2^2a_8 + 32v_3v_1^3v_2^3a_9 \\
&+ 48v_3v_1^2v_2^4a_{10} - 4v_3^2v_1^2v_2^2a_8 - 8v_3^2v_1v_2^3a_9 + 16v_3v_1^3v_2^2a_5 + 32v_3v_1^2v_2^3a_6 \\
&- 4v_3^2v_1v_2^2a_5 + 2v_3v_1^2v_2a_4 - 2v_3v_1v_2^2b_6 - 4v_1^5b_4 - 4v_1^6b_7 + 3b_2v_1^2 + 2v_1b_1 \\
&+ (2a_8 - 2b_9 + 16a_3)\, v_2^2v_1^2v_3 + (-4b_1 + 4b_4)\, v_1^3 + (5b_7 - 4b_2)\, v_1^4 \\
&- 16v_1^5v_2^2a_5 - 32v_1^4v_2^3a_6 - 12v_1^4v_2a_4 - 4v_1^2v_2^3a_6 + 2v_1v_2^2b_6 + 2v_1v_2b_3 \\
&- 4v_3^2v_2^2a_3 + 2v_3v_1^2b_2 - 2v_3v_2^2a_3 + 2v_3v_1b_1 - 2v_3v_2a_1 - 16v_1^5v_2a_7 \\
&- 16v_1^6v_2^2a_8 - 32v_1^5v_2^3a_9 - 48v_1^4v_2^4a_{10} - 4v_1^2v_2^4a_{10} + 2v_1v_2^3b_{10} \\
&- 12v_3^2v_2^4a_{10} - 2v_3v_2^4a_{10} + 2v_3v_1^4b_7 - 8v_3^2v_2^3a_6 + 2v_3v_1^3b_4 - 2v_3v_2^3a_6 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-12a_4 &= 0 \\
2a_4 &= 0 \\
-16a_5 &= 0 \\
-4a_5 &= 0 \\
16a_5 &= 0 \\
-32a_6 &= 0 \\
-8a_6 &= 0 \\
-4a_6 &= 0 \\
-2a_6 &= 0 \\
32a_6 &= 0 \\
-16a_7 &= 0 \\
4a_7 &= 0 \\
-16a_8 &= 0 \\
-4a_8 &= 0 \\
16a_8 &= 0 \\
-32a_9 &= 0 \\
-8a_9 &= 0 \\
32a_9 &= 0 \\
-48a_{10} &= 0 \\
-12a_{10} &= 0 \\
-4a_{10} &= 0 \\
-2a_{10} &= 0 \\
48a_{10} &= 0 \\
2b_1 &= 0 \\
2b_2 &= 0 \\
3b_2 &= 0 \\
2b_3 &= 0 \\
-4b_4 &= 0 \\
2b_4 &= 0 \\
-2b_6 &= 0 \\
2b_6 &= 0 \\
-4b_7 &= 0 \\
2b_7 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_8}{2} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= b_8 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{2} \\
\eta &= y\,x^2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y\,x^2 - \left(-\frac{2y(-2x^2 + \ln(y))}{x}\right)\left(\frac{x}{2}\right) \\
&= -y\,x^2 + y\ln(y) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-y\,x^2 + y \ln(y)} dy \end{aligned}$$

Which results in

$$S = \ln\left(-x^2 + \ln(y)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y(-2x^2 + \ln(y))}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x}{-x^2 + \ln(y)} \\ S_y &= \frac{1}{y\left(-x^2 + \ln(y)\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(-x^2 + \ln(y)\right) = -2\ln(x) + c_1$$

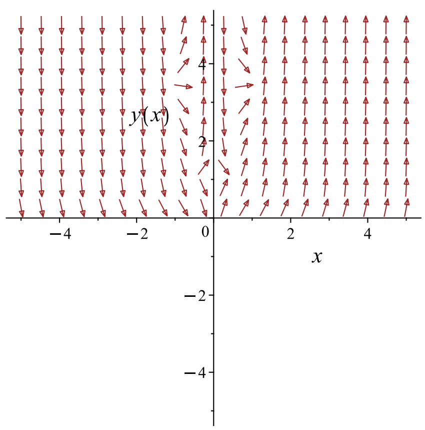
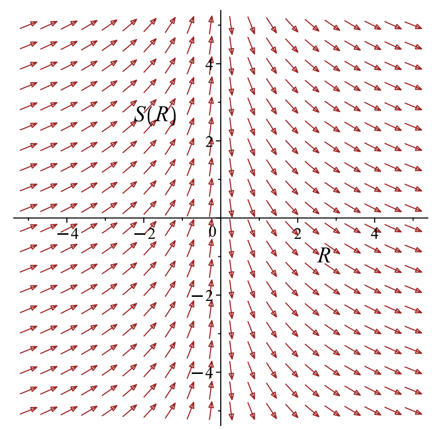
Which simplifies to

$$\ln\left(-x^2 + \ln(y)\right) = -2\ln(x) + c_1$$

Which gives

$$y = \mathrm{e}^{\frac{x^4 + \mathrm{e}^{c_1}}{x^2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(-2x^2+\ln(y))}{x}$ 	$R = x$ $S = \ln\left(-x^2 + \ln\left(y\right)\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{x^4+e^{c_1}}{x^2}} \tag{1}$$

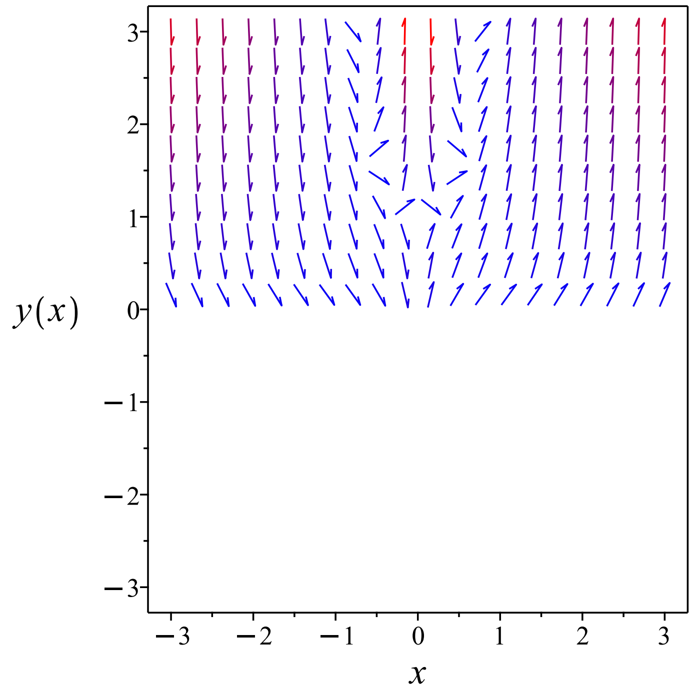


Figure 143: Slope field plot

Verification of solutions

$$y = e^{\frac{x^4 + e^{c_1}}{x^2}}$$

Verified OK.

2.40.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{x}{y}\right) \mathrm{d}y &= \left(-2\ln(y) + 4x^2\right) \mathrm{d}x \\
\left(-4x^2 + 2\ln(y)\right) \mathrm{d}x + \left(\frac{x}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -4x^2 + 2\ln(y) \\
N(x, y) &= \frac{x}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-4x^2 + 2\ln(y)\right) \\
&= \frac{2}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y}\right) \\
&= \frac{1}{y}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{y}{x}\left(\left(\frac{2}{y}\right) - \left(\frac{1}{y}\right)\right) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{1}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(-4x^2 + 2\ln(y)\right) \\
&= -4x^3 + 2x\ln(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(\frac{x}{y}\right) \\
&= \frac{x^2}{y}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-4x^3 + 2x\ln(y)\right) + \left(\frac{x^2}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -4x^3 + 2x\ln(y)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\left(-2x^2 + \ln(y)\right)^2}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = -\frac{-2x^2 + \ln(y)}{2y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{x^2}{y}$. Therefore equation (4) becomes

$$
\frac{x^2}{y} = \frac{2x^2 - \ln(y)}{2y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\ln(y)}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\ln(y)}{2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(-2x^2 + \ln(y))^2}{4} + \frac{\ln(y)^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(-2x^2 + \ln(y))^2}{4} + \frac{\ln(y)^2}{4}$$

The solution becomes

$$y = \mathrm{e}^{\frac{x^4+c_1}{x^2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x^4+c_1}{x^2}} \tag{1}$$

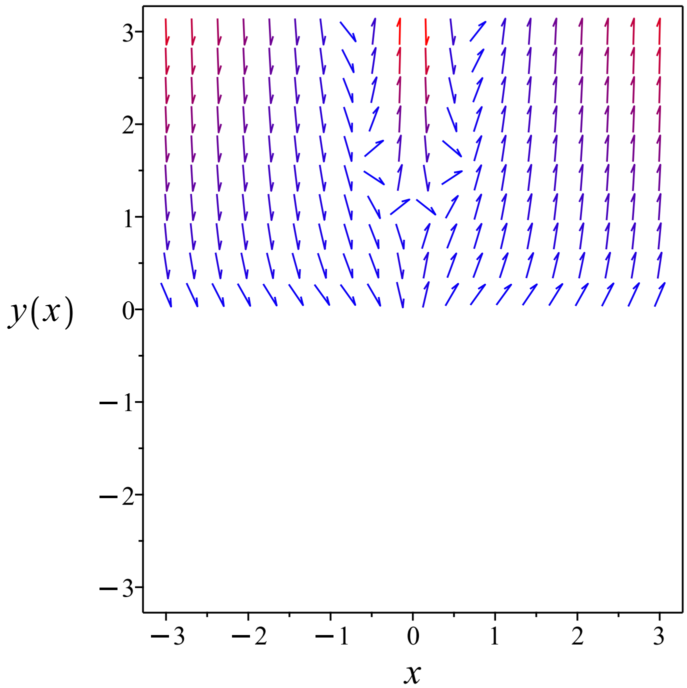


Figure 144: Slope field plot

Verification of solutions

$$y = e^{\frac{x^4+c_1}{x^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)/y(x)+2*ln(y(x))= 4*x^2,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{x^4 - c_1}{x^2}}$$

✓ Solution by Mathematica

Time used: 0.248 (sec). Leaf size: 17

```
DSolve[x*y'[x]/y[x]+2*Log[y[x]]== 4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2 + \frac{c_1}{x^2}}$$

2.41 problem 48(d)

Internal problem ID [927]
Internal file name [`OUTPUT/927_Sunday_June_05_2022_01_54_13_AM_92514859/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Linear first order. Section 2.1 Page 41
Problem number: 48(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, _Riccati]
```

$$\boxed{\frac{y'}{(1+y)^2}-\frac{1}{x(1+y)}=-\frac{3}{x^2}}$$

2.41.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{(1+Y(X)+y_0)(3Y(X)+3y_0+3-X-x_0)}{(X+x_0)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=0$$
$$y_0=-1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=-\frac{-Y(X)X+3Y(X)^2}{X^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{Y(-X+3Y)}{X^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y(X-3Y)$ and $N = X^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -3u^2 + u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= -\frac{3u(X)^2}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) + \frac{3u(X)^2}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{3u^2}{X} \end{aligned}$$

Where $f(X) = -\frac{3}{X}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^2}\,du &= -\frac{3}{X}\,dX \\ \int \frac{1}{u^2}\,du &= \int -\frac{3}{X}\,dX \\ -\frac{1}{u} &= -3\ln(X) + c_2 \end{aligned}$$

The solution is

$$-\frac{1}{u(X)} + 3\ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$-\frac{X}{Y(X)} + 3\ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$-\frac{X}{Y(X)} + 3\ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y - 1 \\ X &= x \end{aligned}$$

Then the solution in y becomes

$$-\frac{x}{1+y} + 3\ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{1+y} + 3\ln(x) - c_2 = 0 \tag{1}$$

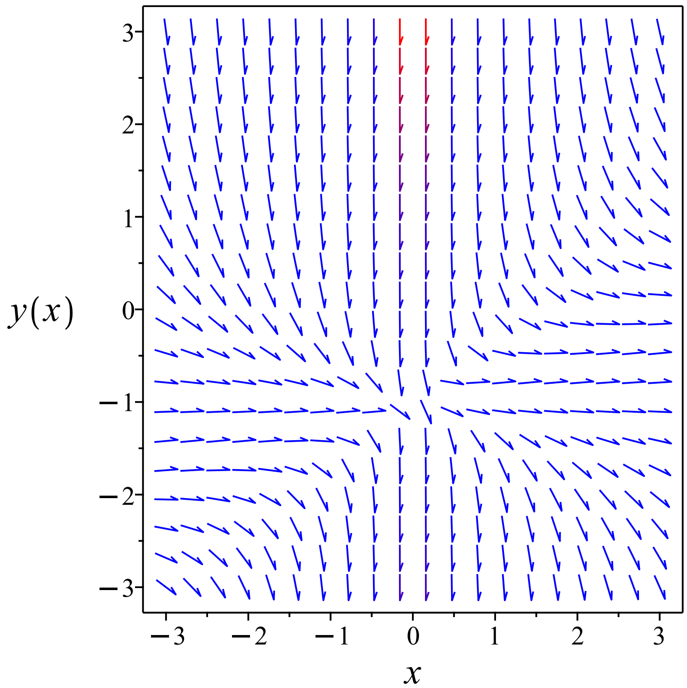


Figure 145: Slope field plot

<u>Verification of solutions</u>

$$-\frac{x}{1+y}+3\ln\left(x\right)-c_2=0$$

Verified OK.

2.41.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{-yx+3y^2-x+6y+3}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 - \frac{(-yx + 3y^2 - x + 6y + 3)(b_3 - a_2)}{x^2} - \frac{(-yx + 3y^2 - x + 6y + 3)^2 a_3}{x^4} \\ &- \left(-\frac{-y-1}{x^2} + \frac{-2yx + 6y^2 - 2x + 12y + 6}{x^3}\right)(xa_2 + ya_3 + a_1) \\ &+ \frac{(6y - x + 6)(xb_2 + yb_3 + b_1)}{x^2} = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^3yb_2 - 3x^2y^2a_2 + 3x^2y^2b_3 - 9y^4a_3 - x^3b_1 + 6x^3b_2 + x^3b_3 + x^2ya_1 - 6x^2ya_2 - x^2ya_3 + 6x^2yb_1 - 6x\,y^2a_1}{} \\ = 0$$

Setting the numerator to zero gives

$$\begin{aligned}&6x^3yb_2 - 3x^2y^2a_2 + 3x^2y^2b_3 - 9y^4a_3 - x^3b_1 + 6x^3b_2 + x^3b_3 + x^2ya_1 - 6x^2ya_2 \\ &- x^2ya_3 + 6x^2yb_1 - 6x\,y^2a_1 + 6x\,y^2a_3 - 36y^3a_3 + x^2a_1 - 3x^2a_2 - x^2a_3 + 6x^2b_1 \\ &- 3x^2b_3 - 12xya_1 + 12xya_3 - 54y^2a_3 - 6xa_1 + 6xa_3 - 36ya_3 - 9a_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-3a_2v_1^2v_2^2 - 9a_3v_2^4 + 6b_2v_1^3v_2 + 3b_3v_1^2v_2^2 + a_1v_1^2v_2 - 6a_1v_1v_2^2 - 6a_2v_1^2v_2 - a_3v_1^2v_2 \\ &+ 6a_3v_1v_2^2 - 36a_3v_2^3 - b_1v_1^3 + 6b_1v_1^2v_2 + 6b_2v_1^3 + b_3v_1^3 + a_1v_1^2 - 12a_1v_1v_2 - 3a_2v_1^2 \\ &- a_3v_1^2 + 12a_3v_1v_2 - 54a_3v_2^2 + 6b_1v_1^2 - 3b_3v_1^2 - 6a_1v_1 + 6a_3v_1 - 36a_3v_2 - 9a_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &6b_2v_1^3v_2 + (-b_1 + 6b_2 + b_3)\, v_1^3 + (-3a_2 + 3b_3)\, v_1^2v_2^2 + (a_1 - 6a_2 - a_3 + 6b_1)\, v_1^2v_2 \\ &+ (a_1 - 3a_2 - a_3 + 6b_1 - 3b_3)\, v_1^2 + (-6a_1 + 6a_3)\, v_1v_2^2 + (-12a_1 + 12a_3)\, v_1v_2 \\ &+ (-6a_1 + 6a_3)\, v_1 - 9a_3v_2^4 - 36a_3v_2^3 - 54a_3v_2^2 - 36a_3v_2 - 9a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -54a_3 &= 0 \\ -36a_3 &= 0 \\ -9a_3 &= 0 \\ 6b_2 &= 0 \\ -12a_1 + 12a_3 &= 0 \\ -6a_1 + 6a_3 &= 0 \\ -3a_2 + 3b_3 &= 0 \\ -b_1 + 6b_2 + b_3 &= 0 \\ a_1 - 6a_2 - a_3 + 6b_1 &= 0 \\ a_1 - 3a_2 - a_3 + 6b_1 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y + 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y + 1 - \left(-\frac{-yx + 3y^2 - x + 6y + 3}{x^2}\right)(x) \\
&= \frac{3y^2 + 6y + 3}{x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3y^2+6y+3}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x}{3\,(y + 1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-yx + 3y^2 - x + 6y + 3}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{3y+3} \\
S_y &= \frac{x}{3(y+1)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x}{3y+3} = -\ln(x) + c_1
$$

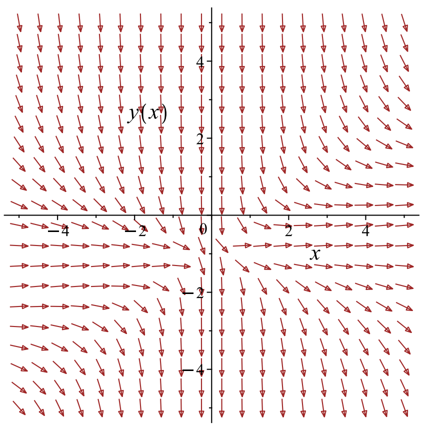
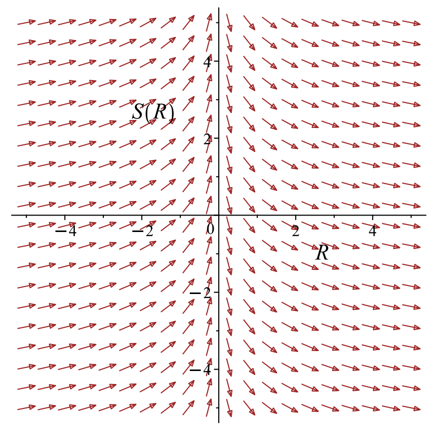
Which simplifies to

$$
-\frac{x}{3y+3} = -\ln(x) + c_1
$$

Which gives

$$
y = -\frac{3\ln(x) - 3c_1 - x}{3(\ln(x) - c_1)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-yx+3y^2-x+6y+3}{x^2}$ 	$R = x$ $S = -\frac{x}{3y+3}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{3\ln(x) - 3c_1 - x}{3(\ln(x) - c_1)} \tag{1}$$

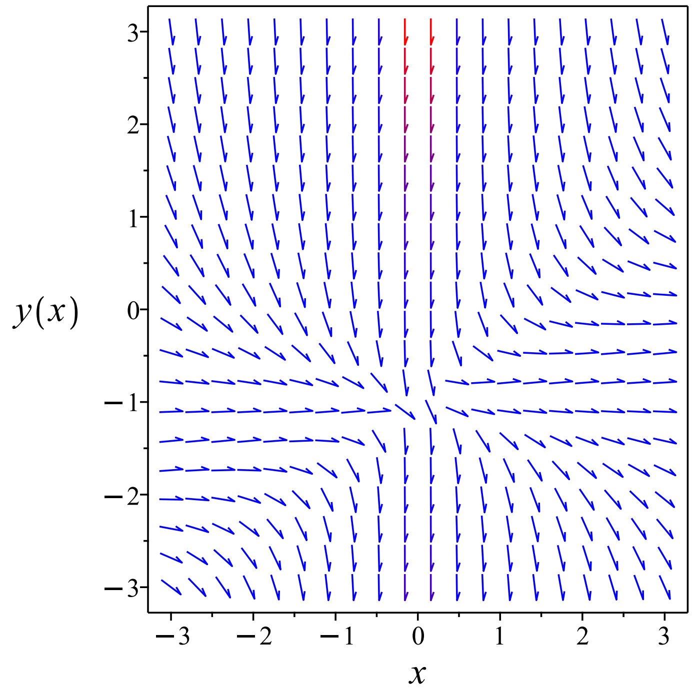


Figure 146: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{3\ln(x) - 3c_1 - x}{3(\ln(x) - c_1)}$$

Verified OK.

2.41.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{(y+1)^2}\right)\mathrm{d}y = \left(\frac{1}{x(y+1)} - \frac{3}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x(y+1)} + \frac{3}{x^2}\right)\mathrm{d}x + \left(\frac{1}{(y+1)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x(y+1)} + \frac{3}{x^2}$$
$$N(x,y) = \frac{1}{(y+1)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x(y+1)} + \frac{3}{x^2}\right)$$
$$= \frac{1}{x(y+1)^2}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{(y+1)^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= (y+1)^2\left(\left(\frac{1}{x(y+1)^2}\right) - (0)\right)\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int \frac{1}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(-\frac{1}{x(y+1)} + \frac{3}{x^2}\right)\\ &= \frac{3y - x + 3}{x(y+1)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x\left(\frac{1}{(y+1)^2}\right)\\ &= \frac{x}{(y+1)^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3y - x + 3}{x\left(y + 1\right)}\right) + \left(\frac{x}{\left(y + 1\right)^{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{3y - x + 3}{x\left(y + 1\right)}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{-x + \ln\left(x\right)\left(3y + 3\right)}{y + 1} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{-x + \ln\left(x\right)\left(3y + 3\right)}{\left(y + 1\right)^{2}} + \frac{3\ln\left(x\right)}{y + 1} + f'(y) \tag{4}
$$

$$
= \frac{x}{\left(y + 1\right)^{2}} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{\left(y+1\right)^{2}}$. Therefore equation (4) becomes

$$
\frac{x}{\left(y + 1\right)^{2}} = \frac{x}{\left(y + 1\right)^{2}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x + \ln(x)(3y + 3)}{y + 1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x + \ln(x)(3y + 3)}{y + 1}$$

The solution becomes

$$y = -\frac{3\ln(x) - c_1 - x}{3\ln(x) - c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{3\ln(x) - c_1 - x}{3\ln(x) - c_1} \tag{1}$$

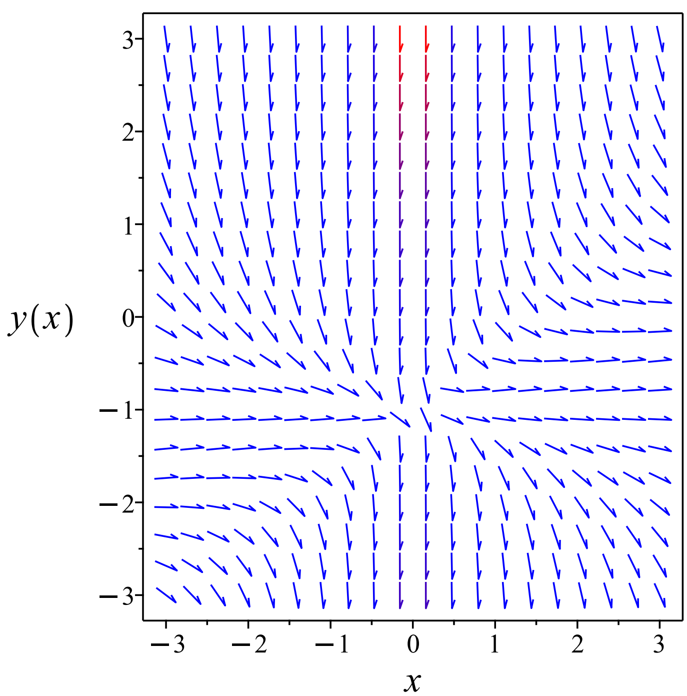


Figure 147: Slope field plot

Verification of solutions

$$y = -\frac{3 \ln (x) - c_1 - x}{3 \ln (x) - c_1}$$

Verified OK.

2.41.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-yx + 3y^2 - x + 6y + 3}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{3y^2}{x^2} + \frac{1}{x} - \frac{6y}{x^2} - \frac{3}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{3-x}{x^2}$, $f_1(x) = -\frac{-x+6}{x^2}$ and $f_2(x) = -\frac{3}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{3u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{6}{x^3} \\ f_1 f_2 &= \frac{-3x + 18}{x^4} \\ f_2^2 f_0 &= -\frac{9(3-x)}{x^6} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{3u''(x)}{x^2} - \left(\frac{6}{x^3} + \frac{-3x + 18}{x^4}\right) u'(x) - \frac{9(3-x)\, u(x)}{x^6} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{3}{x}}(c_1 + c_2 \ln(x))$$

The above shows that

$$u'(x) = \frac{e^{\frac{3}{x}}(-3c_2 \ln(x) + c_2 x - 3c_1)}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{-3c_2 \ln(x) + c_2 x - 3c_1}{3c_1 + 3c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-3\ln(x) + x - 3c_3}{3c_3 + 3\ln(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{-3\ln(x) + x - 3c_3}{3c_3 + 3\ln(x)} \tag{1}$$

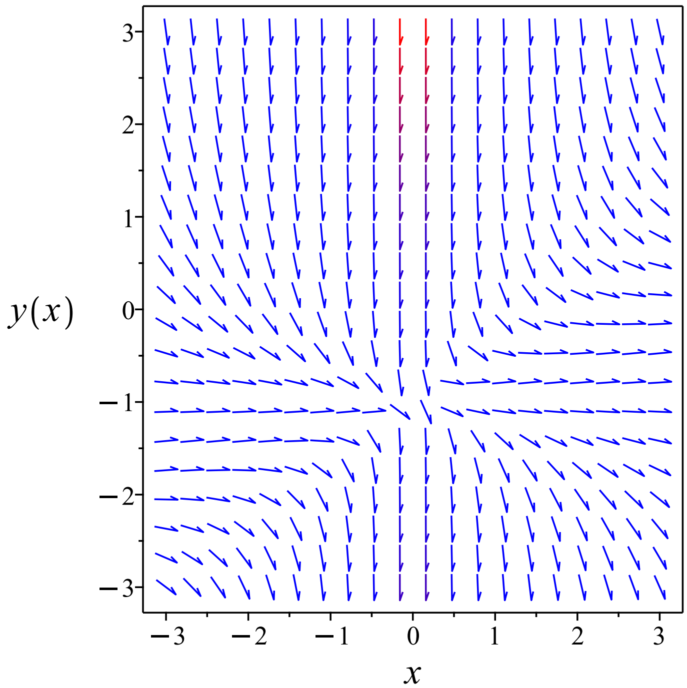


Figure 148: Slope field plot

Verification of solutions

$$y = \frac{-3\ln(x) + x - 3c_3}{3c_3 + 3\ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)/(1+y(x))^2-1/(x*(1+y(x)))= -3/x^2,y(x), singsol=all)
```

$$y(x) = -1 + \frac{x}{3\ln(x) + 3c_1}$$

✓ Solution by Mathematica

Time used: 0.252 (sec). Leaf size: 31

```
DSolve[y'[x]/(1+y[x])^2-1/(x*(1+y[x]))== -3/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x - 3\log(x) - 3c_1}{3(\log(x) + c_1)}$$
$$y(x) \to -1$$

3 Chapter 2, First order equations. separable equations. Section 2.2 Page 52

3.1 problem 1

Internal problem ID [928]
Internal file name [`OUTPUT/928_Sunday_June_05_2022_01_54_15_AM_81198624/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{3x^2 + 2x + 1}{y - 2} = 0}$$

3.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{3x^2 + 2x + 1}{y - 2} \end{aligned}$$

Where $f(x) = 3x^2 + 2x + 1$ and $g(y) = \frac{1}{y-2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-2}} \, dy = 3x^2 + 2x + 1 \, dx$$

$$\int \frac{1}{\frac{1}{y-2}}dy = \int 3x^2 + 2x + 1\,dx$$
$$\frac{1}{2}y^2 - 2y = x^3 + x^2 + c_1 + x$$

Which results in

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$
$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$

Summary

The solution(s) found are the following

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} \tag{1}$$
$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} \tag{2}$$

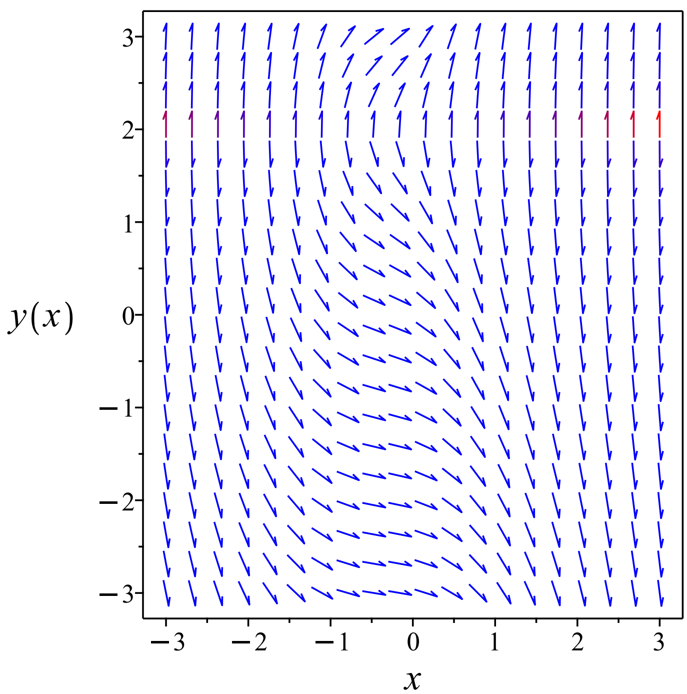


Figure 149: Slope field plot

Verification of solutions

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$

Verified OK.

$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$

Verified OK.

3.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{3x^2 + 2x + 1}{y - 2} \tag{1}$$

Which becomes

$$(y - 2)\, dy = \left(3x^2 + 2x + 1\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(3x^2 + 2x + 1\right) dx = d\left(x^3 + x^2 + x\right)$$

Hence (2) becomes

$$(y - 2)\, dy = d\left(x^3 + x^2 + x\right)$$

Integrating both sides gives gives these solutions

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1$$
$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1$$

Summary

The solution(s) found are the following

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1 \tag{1}$$
$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1 \tag{2}$$

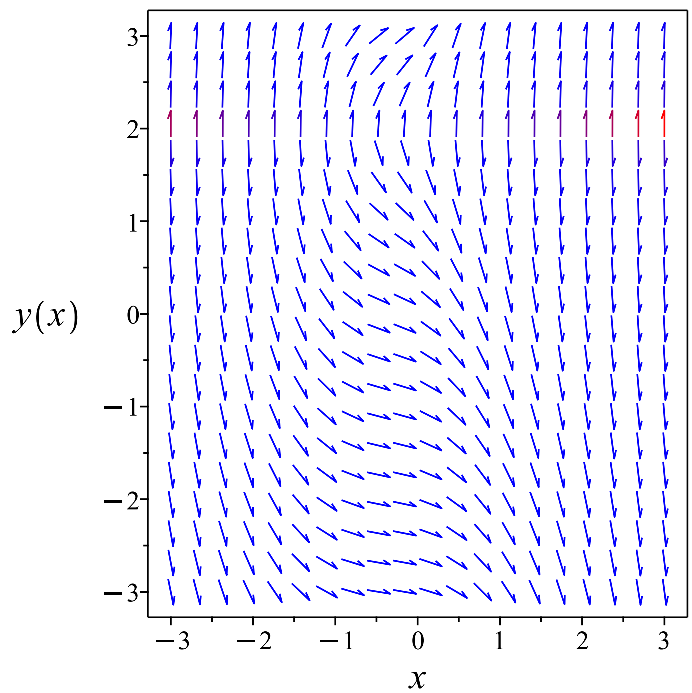


Figure 150: Slope field plot

<u>Verification of solutions</u>

$$y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1$$

Verified OK.

$$y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4} + c_1$$

Verified OK.

3.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^2 + 2x + 1}{y - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 142: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{3x^2 + 2x + 1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x^2+2x+1}} dx \end{aligned}$$

Which results in

$$S = x^3 + x^2 + x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3x^2 + 2x + 1}{y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x^2 + 2x + 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - 2R + c_1 \tag{4}$$

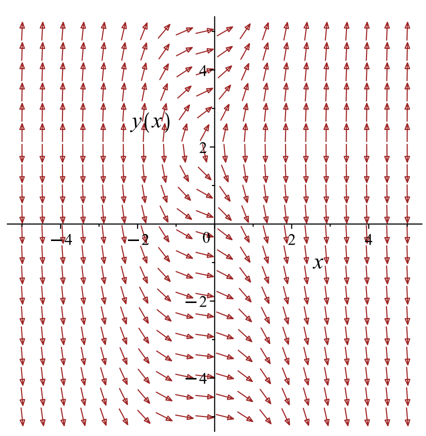
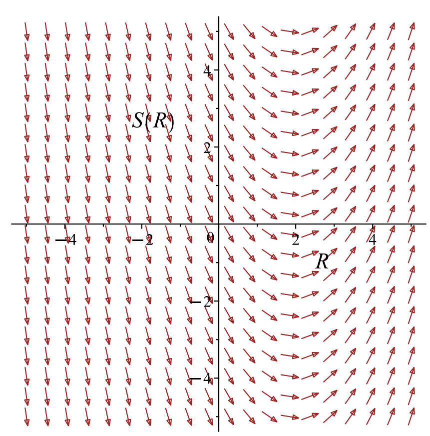
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^3 + x^2 + x = \frac{y^2}{2} - 2y + c_1$$

Which simplifies to

$$x^3 + x^2 + x = \frac{y^2}{2} - 2y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2+2x+1}{y-2}$ 	$R = y$ $S = x^3 + x^2 + x$	$\frac{dS}{dR} = R - 2$ 

Summary

The solution(s) found are the following

$$x^3 + x^2 + x = \frac{y^2}{2} - 2y + c_1 \tag{1}$$

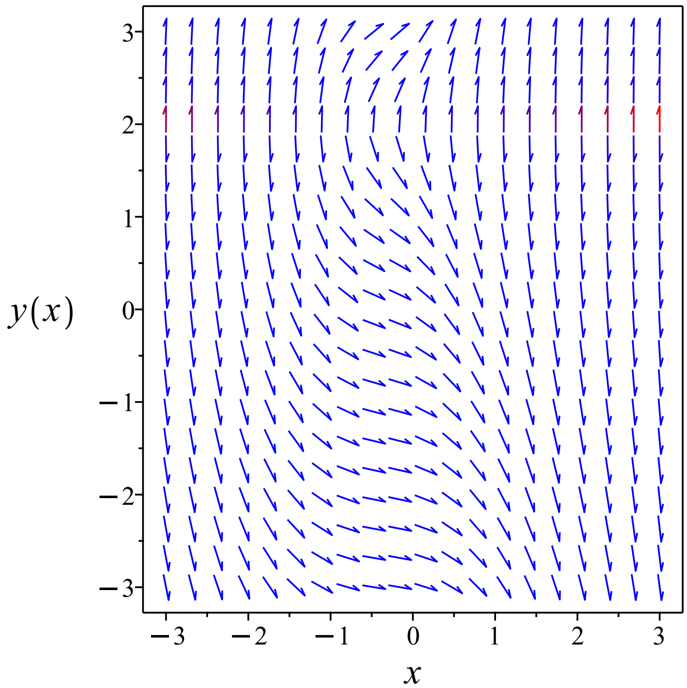


Figure 151: Slope field plot

Verification of solutions

$$x^3 + x^2 + x = \frac{y^2}{2} - 2y + c_1$$

Verified OK.

3.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(y - 2) \, \mathrm{d}y &= \left(3x^2 + 2x + 1\right) \mathrm{d}x \\
\left(-3x^2 - 2x - 1\right) \mathrm{d}x + (y - 2) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -3x^2 - 2x - 1 \\
N(x, y) &= y - 2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2 - 2x - 1\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - 2) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -3x^2-2x-1\,\mathrm{d}x$$
$$\phi=-x^3-x^2-x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y-2$. Therefore equation (4) becomes

$$y-2=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=y-2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (y-2)\,\mathrm{d}y$$
$$f(y)=\frac{1}{2}y^2-2y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-x^3-x^2-x+\frac{1}{2}y^2-2y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 - x^2 - x + \frac{1}{2}y^2 - 2y$$

Summary

The solution(s) found are the following

$$-x^3 + \frac{y^2}{2} - x^2 - 2y - x = c_1 \tag{1}$$

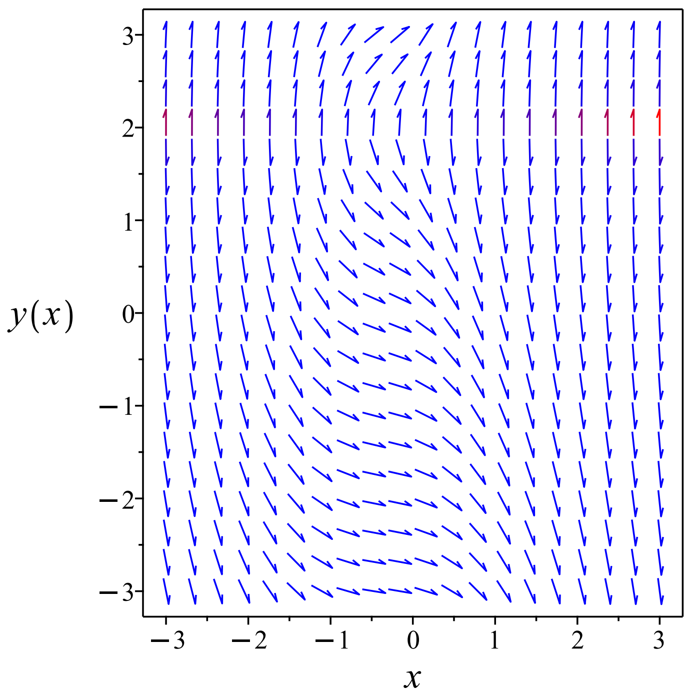


Figure 152: Slope field plot

Verification of solutions

$$-x^3 + \frac{y^2}{2} - x^2 - 2y - x = c_1$$

Verified OK.

3.1.5 Maple step by step solution

Let's solve

$y' - \frac{3x^2+2x+1}{y-2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(y-2)\, y' = 3x^2 + 2x + 1$

- Integrate both sides with respect to x

  $\int (y-2)\, y' dx = \int (3x^2 + 2x + 1)\, dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} - 2y = x^3 + x^2 + c_1 + x$

- Solve for y

  $\{y = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}, y = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)= (3*x^2+2*x+1)/(y(x)-2),y(x), singsol=all)
```

$$y(x) = 2 - \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$
$$y(x) = 2 + \sqrt{2x^3 + 2x^2 + 2c_1 + 2x + 4}$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 56

```
DSolve[y'[x]== (3*x^2+2*x+1)/(y[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 - \sqrt{2}\sqrt{x^3 + x^2 + x + 2 + c_1}$$
$$y(x) \to 2 + \sqrt{2}\sqrt{x^3 + x^2 + x + 2 + c_1}$$

3.2 problem 2

Internal problem ID [929]
Internal file name [`OUTPUT/929_Sunday_June_05_2022_01_54_16_AM_63498818/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin(y)\sin(x)+\cos(y)\,y'=0}$$

3.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\sin(x)\tan(y) \end{aligned}$$

Where $f(x)=-\sin(x)$ and $g(y)=\tan(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(y)}\,dy &= -\sin(x)\,dx\\ \int\frac{1}{\tan(y)}\,dy &= \int -\sin(x)\,dx\\ \ln(\sin(y)) &= \cos(x)+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(y) = e^{\cos(x)+c_1}$$

Which simplifies to

$$\sin(y) = c_2 e^{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(c_2 e^{\cos(x)+c_1}\right) \tag{1}$$

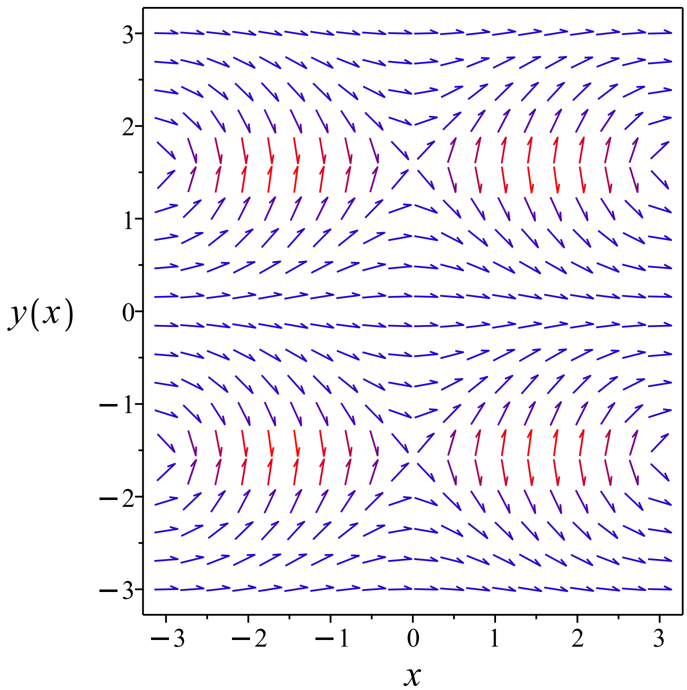


Figure 153: Slope field plot

Verification of solutions

$$y = \arcsin\left(c_2 e^{\cos(x)+c_1}\right)$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(y)\sin(x)}{\cos(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 145: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{1}{\sin(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \cos(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(y)\sin(x)}{\cos(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\sin(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cos(x) = \ln(\sin(y)) + c_1$$

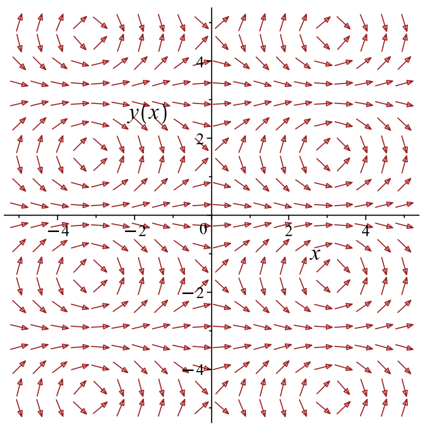
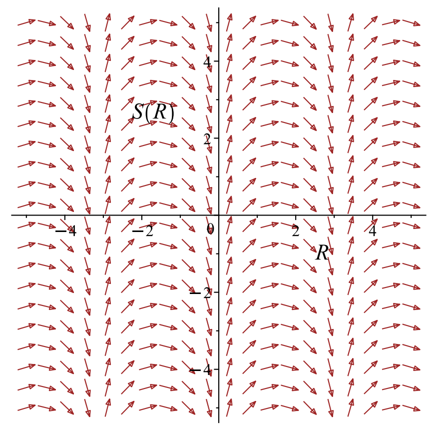
Which simplifies to

$$\cos(x) = \ln(\sin(y)) + c_1$$

Which gives

$$y = \arcsin\left(e^{\cos(x)-c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(y)\sin(x)}{\cos(y)}$ 	$R = y$ $S = \cos(x)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(e^{\cos(x)-c_1}\right) \tag{1}$$

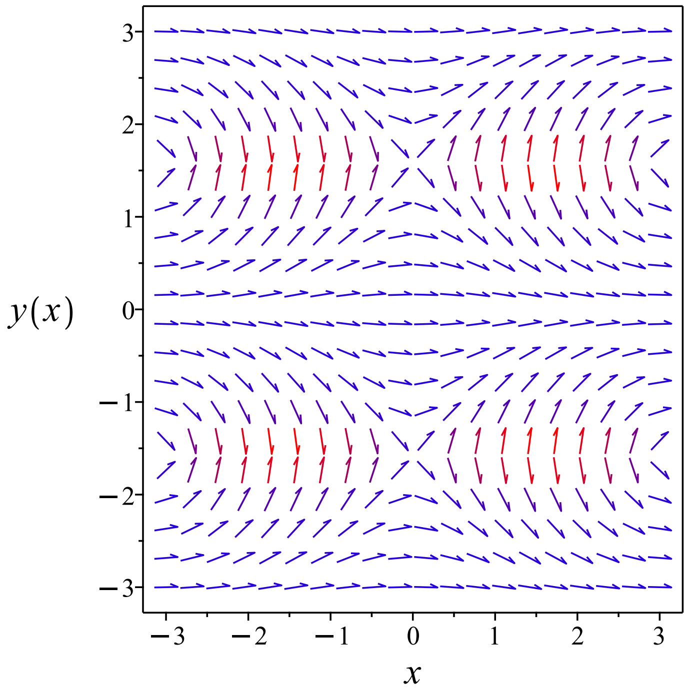


Figure 154: Slope field plot

Verification of solutions

$$y = \arcsin\left(e^{\cos(x)-c_1}\right)$$

Verified OK.

3.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\cos(y)}{\sin(y)}\right) dy &= (\sin(x)) \, dx \\ (-\sin(x)) \, dx + \left(-\frac{\cos(y)}{\sin(y)}\right) dy &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sin(x) \\ N(x, y) &= -\frac{\cos(y)}{\sin(y)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(y)}{\sin(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\sin(x) \, \mathrm{d}x \\
\phi &= \cos(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos(y)}{\sin(y)}$. Therefore equation (4) becomes

$$
-\frac{\cos(y)}{\sin(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\cos(y)}{\sin(y)} \\
&= -\cot(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (-\cot(y))\,\mathrm{d}y$$
$$f(y) = -\ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) - \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) - \ln(\sin(y))$$

Summary
The solution(s) found are the following

$$\cos(x) - \ln(\sin(y)) = c_1 \tag{1}$$

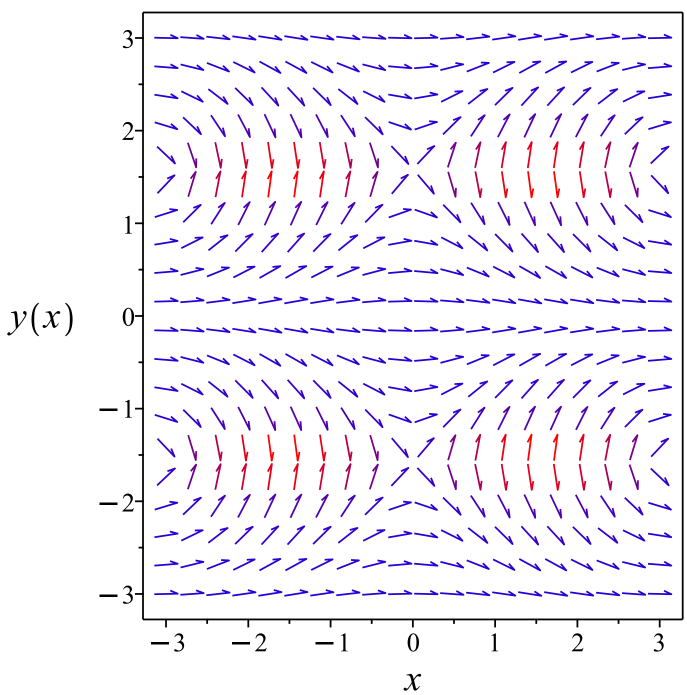


Figure 155: Slope field plot

Verification of solutions

$$\cos(x) - \ln(\sin(y)) = c_1$$

Verified OK.

3.2.4 Maple step by step solution

Let's solve

$\sin(y)\sin(x) + \cos(y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{\cos(y)y'}{\sin(y)} = -\sin(x)$

- Integrate both sides with respect to x

  $\int \frac{\cos(y)y'}{\sin(y)}dx = \int -\sin(x)\,dx + c_1$

- Evaluate integral

  $\ln(\sin(y)) = \cos(x) + c_1$

- Solve for y

  $y = \arcsin\left(\mathrm{e}^{\cos(x)+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 12

```
dsolve(sin(x)*sin(y(x))+cos(y(x))*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{e^{\cos(x)}}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 29.953 (sec). Leaf size: 22

```
DSolve[Sin[x]*Sin[y[x]]+Cos[y[x]]*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(e^{\cos(x)+\frac{c_1}{2}}\right)$$
$$y(x) \to 0$$

3.3 problem 3

Internal problem ID [930]
Internal file name [`OUTPUT/930_Sunday_June_05_2022_01_54_18_AM_67213606/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x + y^2 + y = 0}$$

3.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y(y+1)$. Integrating both sides gives

$$\frac{1}{y\,(y+1)}\,dy = -\frac{1}{x}\,dx$$

$$\int \frac{1}{y(y+1)}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln(y) - \ln(y+1) = -\ln(x) + c_1$$

Raising both side to exponential gives

$$e^{\ln(y)-\ln(y+1)} = e^{-\ln(x)+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = \frac{c_2}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_2}{-x + c_2} \tag{1}$$

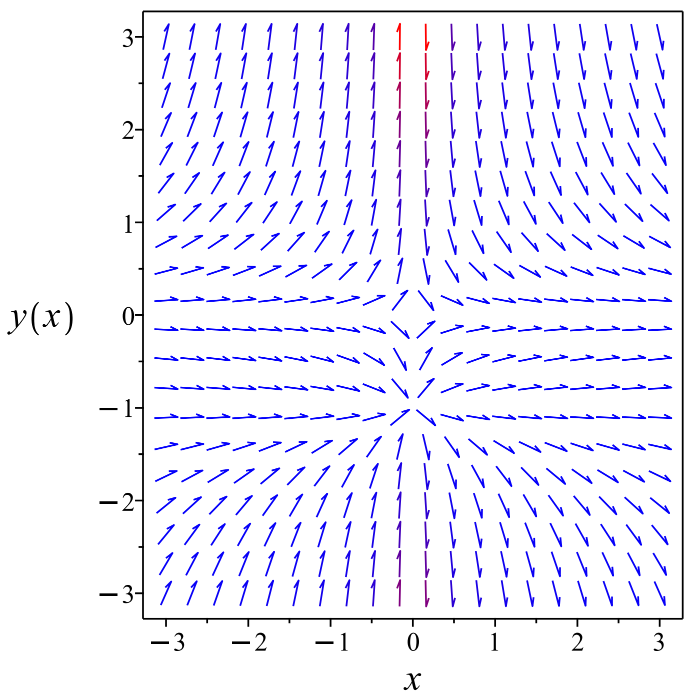


Figure 156: Slope field plot

Verification of solutions

$$y = -\frac{c_2}{-x + c_2}$$

Verified OK.

3.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(y+1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 148: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -x \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-x} dx
\end{aligned}
$$

Which results in

$$
S = -\ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(y+1)}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\left(y+1\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\left(R+1\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R\right) - \ln\left(R+1\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(x\right) = \ln\left(y\right) - \ln\left(1+y\right) + c_1$$

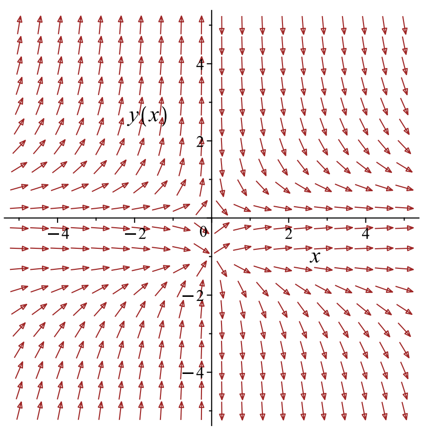
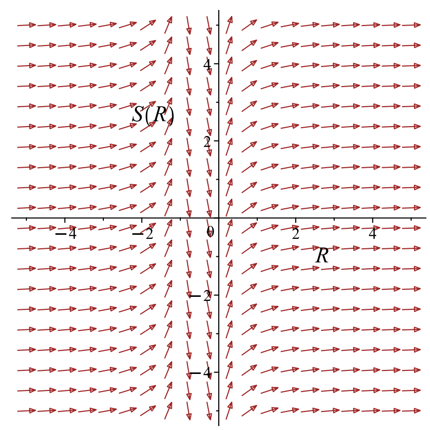
Which simplifies to

$$-\ln\left(x\right) = \ln\left(y\right) - \ln\left(1+y\right) + c_1$$

Which gives

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y+1)}{x}$ 	$R = y$ $S = -\ln(x)$	$\frac{dS}{dR} = \frac{1}{R(R+1)}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}} \tag{1}$$

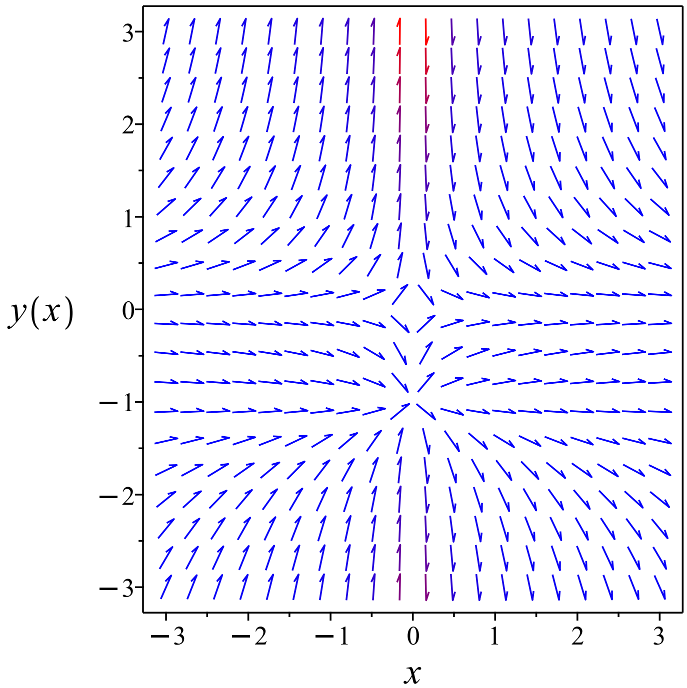


Figure 157: Slope field plot

Verification of solutions

$$y = \frac{1}{-1 + x\,e^{c_1}}$$

Verified OK.

3.3.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -\frac{1}{x} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = -\frac{1}{yx} - \frac{1}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -\frac{w(x)}{x} - \frac{1}{x} \\
w' &= \frac{w}{x} + \frac{1}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) \\ d\left(\frac{w}{x}\right) &= \frac{1}{x^2}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{1}{x^2}\, dx \\ \frac{w}{x} &= -\frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x - 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x - 1$$

Or

$$y = \frac{1}{c_1 x - 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1}{c_1 x - 1} \tag{1}$$

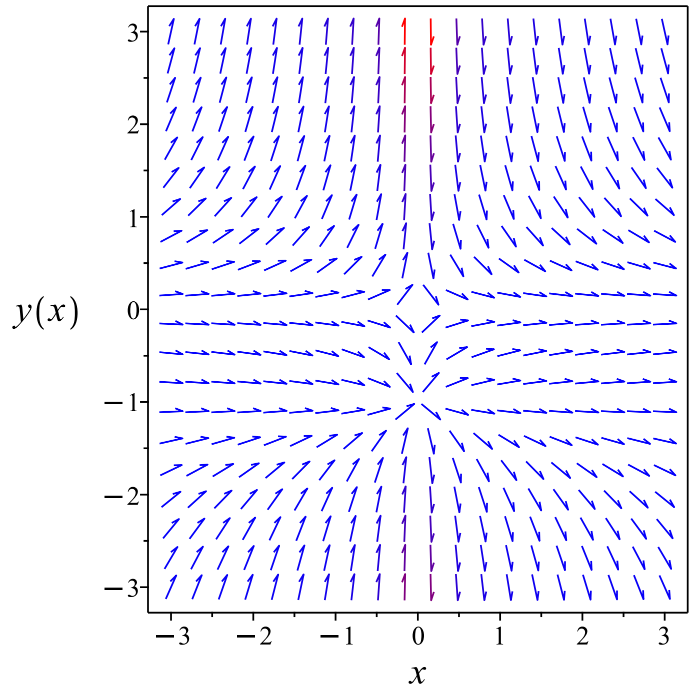


Figure 158: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x - 1}$$

Verified OK.

3.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y(y+1)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{y(y+1)}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x}\\ N(x, y) &= -\frac{1}{y(y+1)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y\,(y+1)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y+1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\,(y+1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\,(y+1)}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y\,(y+1)}\right)\mathrm{d}y\\ f(y) &= -\ln(y) + \ln(y+1) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y) + \ln(y+1)$$

The solution becomes

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}} \tag{1}$$

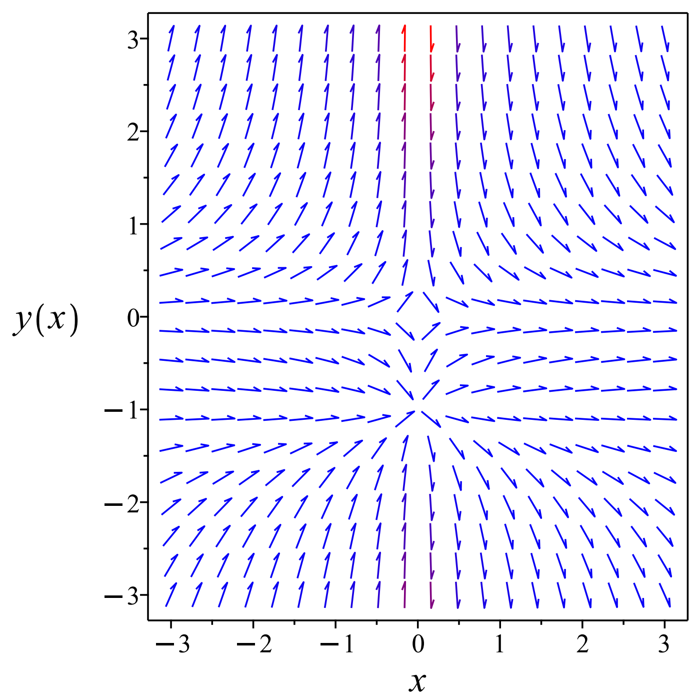


Figure 159: Slope field plot

Verification of solutions

$$y = \frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

Verified OK.

3.3.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y+1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} - \frac{2u'(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{x\left(c_1 + \frac{c_2}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{c_3 x + 1} \tag{1}$$

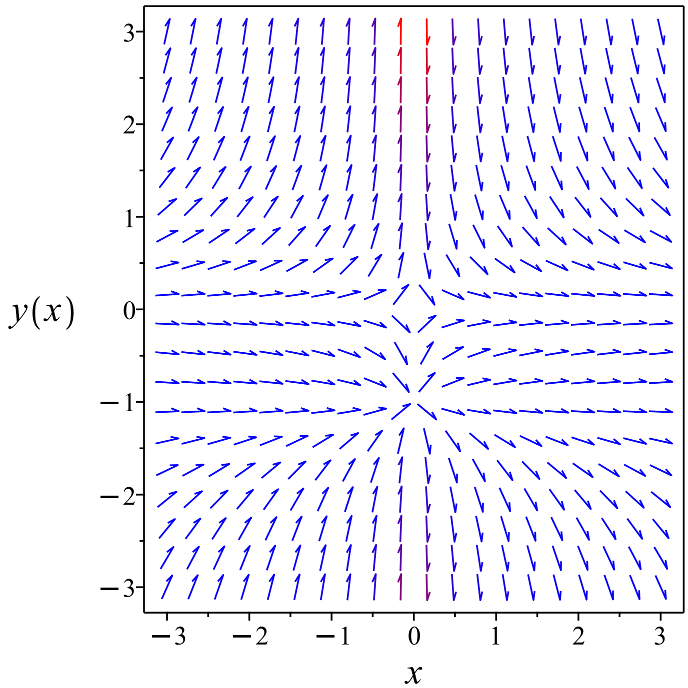


Figure 160: Slope field plot

Verification of solutions

$$y = -\frac{1}{c_3 x + 1}$$

Verified OK.

3.3.6 Maple step by step solution

Let's solve

$y'x + y^2 + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+y}dx = \int -\frac{1}{x}dx + c_1$

- Evaluate integral

$$\ln(y) - \ln(1+y) = -\ln(x) + c_1$$

- Solve for y

$$y = -\frac{e^{c_1}}{e^{c_1} - x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)+y(x)^2+y(x)= 0,y(x), singsol=all)
```

$$y(x) = \frac{1}{c_1 x - 1}$$

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 31

```
DSolve[x*y'[x]+y[x]^2+y[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{c_1}}{-x + e^{c_1}}$$
$$y(x) \to -1$$
$$y(x) \to 0$$

3.4 problem 5

Internal problem ID [931]
Internal file name [`OUTPUT/931_Sunday_June_05_2022_01_54_19_AM_99537949/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\left(3y^{3}+3y\cos\left(y\right)+1\right)y'+\frac{\left(1+2x\right)y}{x^{2}+1}=0}$$

3.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{\left(1+2x\right)y}{\left(x^{2}+1\right)\left(3y^{3}+3y\cos\left(y\right)+1\right)}
\end{aligned}$$

Where $f(x)=-\frac{1+2x}{x^{2}+1}$ and $g(y)=\frac{y}{3y^{3}+3y\cos(y)+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{3y^{3}+3y\cos(y)+1}}\,dy=-\frac{1+2x}{x^{2}+1}\,dx$$

$$\int \frac{1}{\frac{y}{3y^3+3y\cos(y)+1}}dy = \int -\frac{1+2x}{x^2+1}dx$$

$$y^3 + 3\sin(y) + \ln(y) = -\ln(x^2+1) - \arctan(x) + c_1$$

Which results in

$$y = \operatorname{RootOf}\left(-\_Z^3 - \ln(x^2+1) - \arctan(x) - 3\sin(\_Z) - \ln(\_Z) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(-\_Z^3 - \ln(x^2+1) - \arctan(x) - 3\sin(\_Z) - \ln(\_Z) + c_1\right) \tag{1}$$

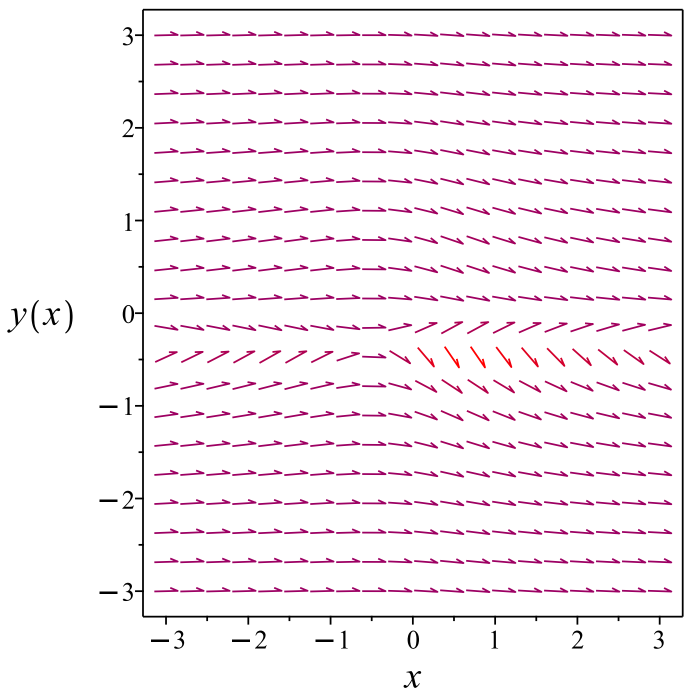


Figure 161: Slope field plot

Verification of solutions

$$y = \operatorname{RootOf}\left(-\_Z^3 - \ln(x^2+1) - \arctan(x) - 3\sin(\_Z) - \ln(\_Z) + c_1\right)$$

Verified OK.

3.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(1+2x)}{3y^3x^2+3y^3+3y\cos(y)\,x^2+3y\cos(y)+x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{x^2+1}{1+2x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^2+1}{1+2x}} dx \end{aligned}$$

Which results in

$$S = -\ln\left(x^2+1\right) - \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(1+2x)}{3y^3x^2 + 3y^3 + 3y\cos(y)x^2 + 3y\cos(y) + x^2 + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-1 - 2x}{x^2 + 1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{3y^3 + 3y\cos(y) + 1}{y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{3R^3 + 3R\cos(R) + 1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R^3 + 3\sin(R) + \ln(R) + c_1 \tag{4}
$$

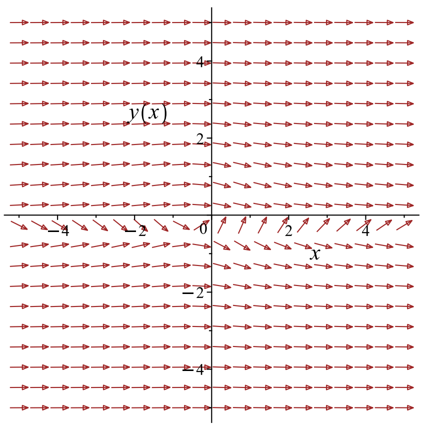
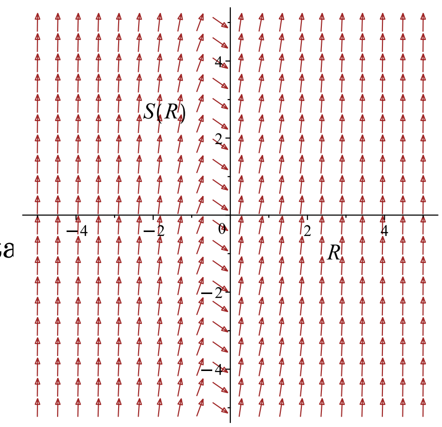
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln\left(x^2 + 1\right) - \arctan(x) = y^3 + 3\sin(y) + \ln(y) + c_1
$$

Which simplifies to

$$
-\ln\left(x^2 + 1\right) - \arctan(x) = y^3 + 3\sin(y) + \ln(y) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(1+2x)}{3y^3x^2+3y^3+3y\cos(y)x^2+3y\cos(y)+x^2+1}$ 	$R = y$ $S = -\ln\left(x^2+1\right) - \text{arcta}$	$\frac{dS}{dR} = \frac{3R^3+3R\cos(R)+1}{R}$ 

Summary
The solution(s) found are the following

$$-\ln\left(x^2+1\right) - \arctan(x) = y^3 + 3\sin(y) + \ln(y) + c_1 \tag{1}$$

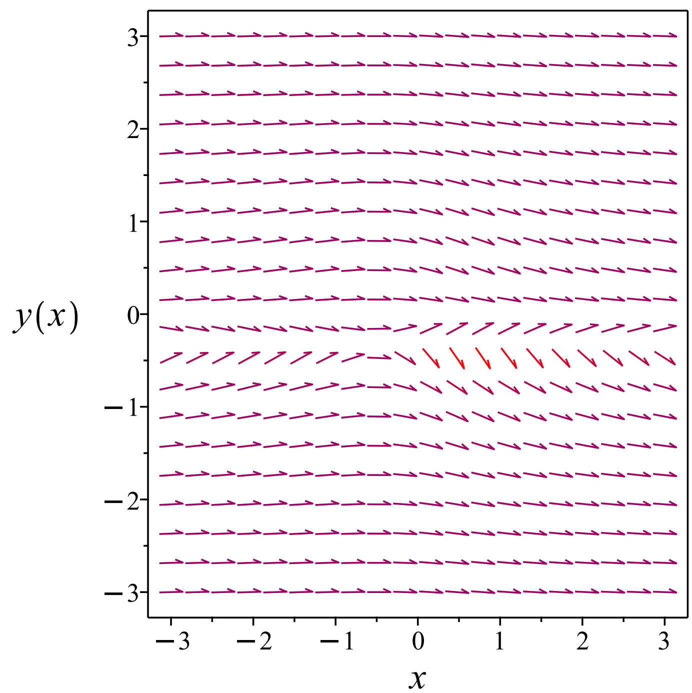


Figure 162: Slope field plot

Verification of solutions

$$-\ln\left(x^2+1\right)-\arctan\left(x\right)=y^3+3\sin\left(y\right)+\ln\left(y\right)+c_1$$

Verified OK.

3.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{3y^3 + 3y\cos(y) + 1}{y}\right)\mathrm{d}y = \left(\frac{1 + 2x}{x^2 + 1}\right)\mathrm{d}x$$
$$\left(-\frac{1 + 2x}{x^2 + 1}\right)\mathrm{d}x + \left(-\frac{3y^3 + 3y\cos(y) + 1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1 + 2x}{x^2 + 1}$$
$$N(x, y) = -\frac{3y^3 + 3y\cos(y) + 1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1 + 2x}{x^2 + 1}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{3y^3+3y\cos(y)+1}{y}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1+2x}{x^2+1}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln\left(x^2+1\right) - \arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3y^3+3y\cos(y)+1}{y}$. Therefore equation (4) becomes

$$-\frac{3y^3+3y\cos(y)+1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\frac{3y^3+3y\cos(y)+1}{y} \\ &= \frac{-3y^3-3y\cos(y)-1}{y}\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-3y^3 - 3y\cos(y) - 1}{y}\right) \mathrm{d}y$$
$$f(y) = -y^3 - 3\sin(y) - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x^2+1) - \arctan(x) - y^3 - 3\sin(y) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x^2+1) - \arctan(x) - y^3 - 3\sin(y) - \ln(y)$$

Summary

The solution(s) found are the following

$$-y^3 - 3\sin(y) - \ln(x^2+1) - \arctan(x) - \ln(y) = c_1 \tag{1}$$

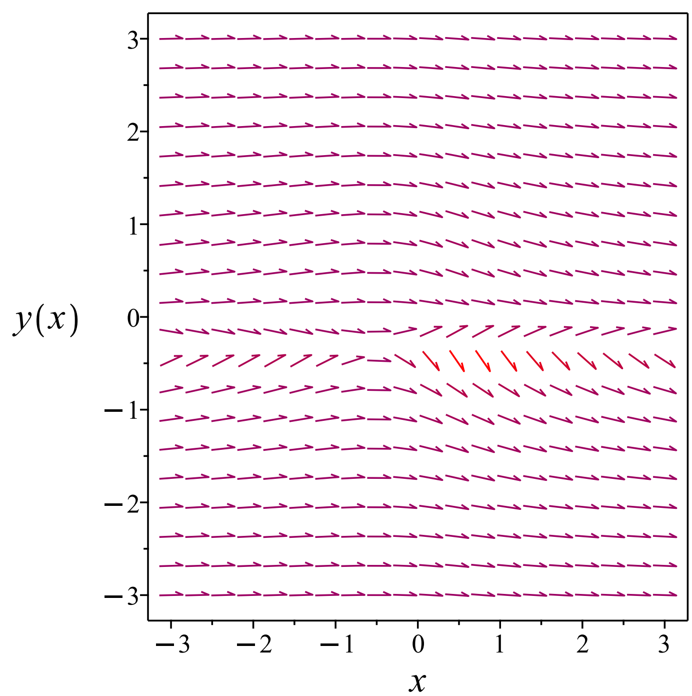


Figure 163: Slope field plot

Verification of solutions

$$-y^3 - 3\sin(y) - \ln\left(x^2+1\right) - \arctan(x) - \ln(y) = c_1$$

Verified OK.

3.4.4 Maple step by step solution

Let's solve

$\left(3y^3+3y\cos(y)+1\right)y' + \frac{(1+2x)y}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\left(3y^3+3y\cos(y)+1\right)}{y} = -\frac{1+2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'\left(3y^3+3y\cos(y)+1\right)}{y}dx = \int -\frac{1+2x}{x^2+1}dx + c_1$

- Evaluate integral

  $y^3 + 3\sin(y) + \ln(y) = -\ln(x^2+1) - \arctan(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve((3*y(x)^3+3*y(x)*cos(y(x))+1)*diff(y(x),x)+((2*x+1)*y(x))/(1+x^2)= 0,y(x), singsol=al
```

$$\ln\left(x^2+1\right) + \arctan(x) + y(x)^3 + 3\sin(y(x)) + \ln(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.379 (sec). Leaf size: 40

```
DSolve[(3*y[x]^3+3*y[x]*Cos[y[x]]+1)*y'[x]+((2*x+1)*y[x])/(1+x^2)== 0,y[x],x,IncludeSingular
```

$$y(x) \to \text{InverseFunction}\left[\#1^3 + \log(\#1) + 3\sin(\#1)\&\right]\left[-\arctan(x) - \log\left(x^2+1\right) + c_1\right]$$
$$y(x) \to 0$$

3.5 problem 6

Internal problem ID [932]
Internal file name [`OUTPUT/932_Sunday_June_05_2022_01_54_21_AM_32716405/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x^2y - \left(-1+y^2\right)^{\frac{3}{2}} = 0}$$

3.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(y^2-1)^{\frac{3}{2}}}{x^2y} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = \frac{(y^2-1)^{\frac{3}{2}}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{(y^2-1)^{\frac{3}{2}}}{y}}\,dy = \frac{1}{x^2}\,dx$$

$$\int \frac{1}{\frac{(y^2-1)^{\frac{3}{2}}}{y}}\,dy = \int \frac{1}{x^2}\,dx$$

$$-\frac{(y-1)(y+1)}{(y^2-1)^{\frac{3}{2}}} = -\frac{1}{x} + c_1$$

The solution is

$$-\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + \frac{1}{x} - c_1 = 0$$

Summary

The solution(s) found are the following

$$-\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + \frac{1}{x} - c_1 = 0 \tag{1}$$

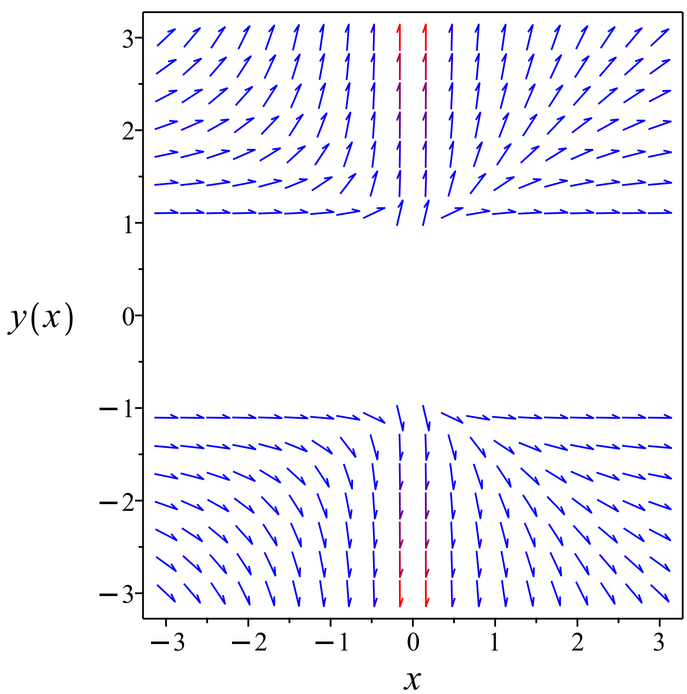


Figure 164: Slope field plot

Verification of solutions

$$-\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + \frac{1}{x} - c_1 = 0$$

Verified OK.

3.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{(y^2-1)^{\frac{3}{2}}}{x^2 y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 154: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{(y^2 - 1)^{\frac{3}{2}}}{x^2 y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y}{(y^2-1)^{\frac{3}{2}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{(R^2-1)^{\frac{3}{2}}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{(R-1)(R+1)}{(R^2-1)^{\frac{3}{2}}} + c_1 \tag{4}
$$

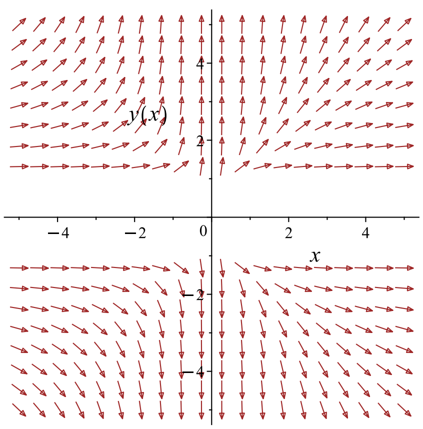
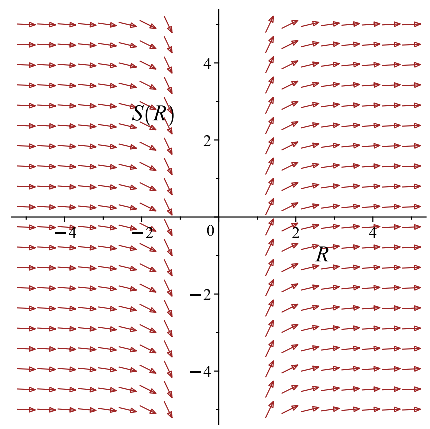
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{1}{x} = -\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + c_1
$$

Which simplifies to

$$
-\frac{1}{x} = -\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(y^2-1)^{\frac{3}{2}}}{x^2 y}$ 	$R = y$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \frac{R}{(R^2-1)^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$-\frac{1}{x} = -\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + c_1 \tag{1}$$

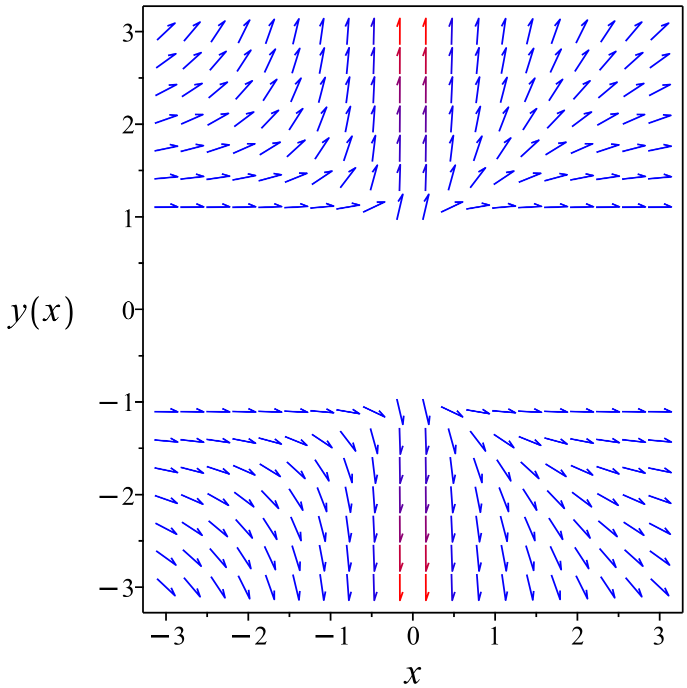


Figure 165: Slope field plot

Verification of solutions

$$-\frac{1}{x} = -\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + c_1$$

Verified OK.

3.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y}{(y^2-1)^{\frac{3}{2}}}\right)\mathrm{d}y = \left(\frac{1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2}\right)\mathrm{d}x + \left(\frac{y}{(y^2-1)^{\frac{3}{2}}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2}$$
$$N(x, y) = \frac{y}{(y^2-1)^{\frac{3}{2}}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{(y^2-1)^{\frac{3}{2}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{(y^2-1)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$
\frac{y}{(y^2-1)^{\frac{3}{2}}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{(y^2-1)^{\frac{3}{2}}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{y}{(y^2-1)^{\frac{3}{2}}}\right) \mathrm{d}y$$

$$f(y) = -\frac{(y-1)(y+1)}{(y^2-1)^{\frac{3}{2}}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{(y-1)(y+1)}{(y^2-1)^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{(y-1)(y+1)}{(y^2-1)^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$-\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}} + \frac{1}{x} = c_1 \tag{1}$$

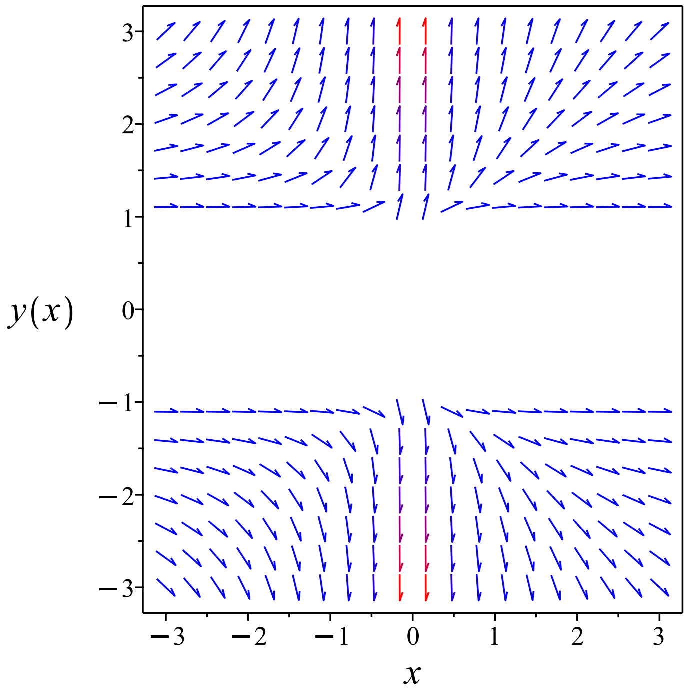


Figure 166: Slope field plot

Verification of solutions

$$-\frac{(y-1)(1+y)}{(-1+y^2)^{\frac{3}{2}}}+\frac{1}{x}=c_1$$

Verified OK.

3.5.4 Maple step by step solution

Let's solve

$y'x^2y-(-1+y^2)^{\frac{3}{2}}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{(-1+y^2)^{\frac{3}{2}}}=\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{(-1+y^2)^{\frac{3}{2}}}dx=\int \frac{1}{x^2}dx+c_1$

- Evaluate integral

  $-\frac{1}{\sqrt{-1+y^2}} = -\frac{1}{x} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{c_1^2 x^2 - 2c_1 x + x^2 + 1}}{c_1 x - 1}, y = -\frac{\sqrt{c_1^2 x^2 - 2c_1 x + x^2 + 1}}{c_1 x - 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*y(x)*diff(y(x),x)= (y(x)^2-1)^(3/2),y(x), singsol=all)
```

$$-\frac{1}{x} + \frac{(y(x)-1)(y(x)+1)}{(y(x)^2-1)^{\frac{3}{2}}} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.707 (sec). Leaf size: 111

```
DSolve[x^2*y[x]*y'[x]== (y[x]^2-1)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{(1+c_1{}^2)\,x^2 - 2c_1x + 1}}{1 - c_1x}$$
$$y(x) \to \frac{\sqrt{(1+c_1{}^2)\,x^2 - 2c_1x + 1}}{-1 + c_1x}$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to -\frac{\sqrt{x^2}}{x}$$
$$y(x) \to \frac{\sqrt{x^2}}{x}$$

3.6 problem 7

Internal problem ID [933]
Internal file name [`OUTPUT/933_Sunday_June_05_2022_01_54_23_AM_59717124/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (1 + y^2)\, x^2 = 0}$$

3.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y^2 + 1)\, x^2 \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\, dy = x^2\, dx$$

$$\int \frac{1}{y^2+1}\, dy = \int x^2\, dx$$

$$\arctan(y) = \frac{x^3}{3} + c_1$$

Which results in

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

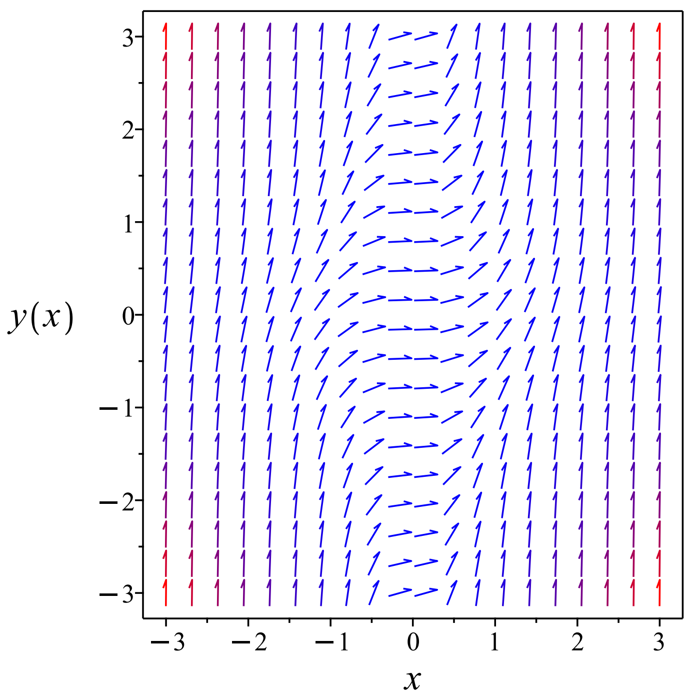


Figure 167: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

3.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (y^2 + 1)\,x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 157: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{1}{x^2}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{\frac{1}{x^2}} dx\end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \left(y^2+1\right) x^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2 + 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^3}{3} = \arctan(y) + c_1
$$

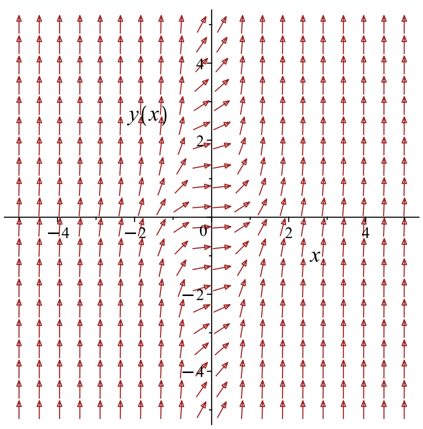
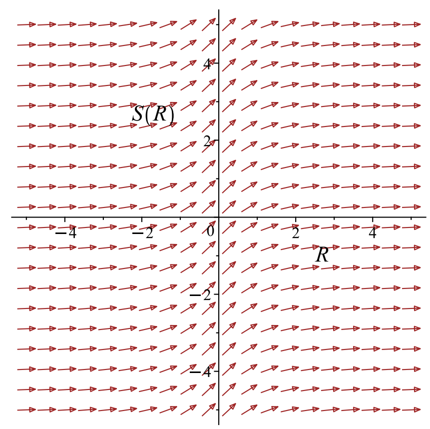
Which simplifies to

$$
\frac{x^3}{3} = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan\left(-\frac{x^3}{3} + c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (y^2+1)\,x^2$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right) \tag{1}$$

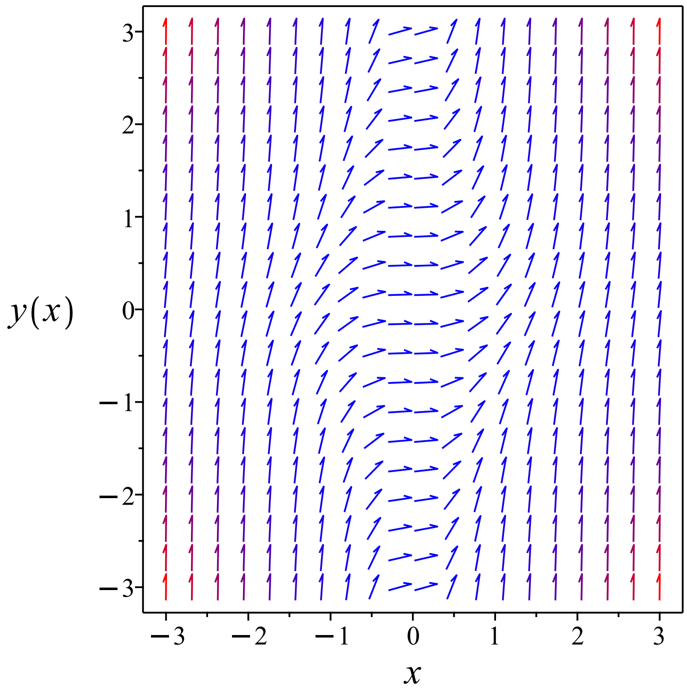


Figure 168: Slope field plot

Verification of solutions

$$y = -\tan\left(-\frac{x^3}{3} + c_1\right)$$

Verified OK.

3.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y^2+1}\right) \mathrm{d}y &= \left(x^2\right) \mathrm{d}x \\
\left(-x^2\right) \mathrm{d}x + \left(\frac{1}{y^2+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= \frac{1}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x^2 \,\mathrm{d}x\end{aligned}$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y\\ f(y) &= \arctan(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{x^3}{3} + c_1\right) \tag{1}$$

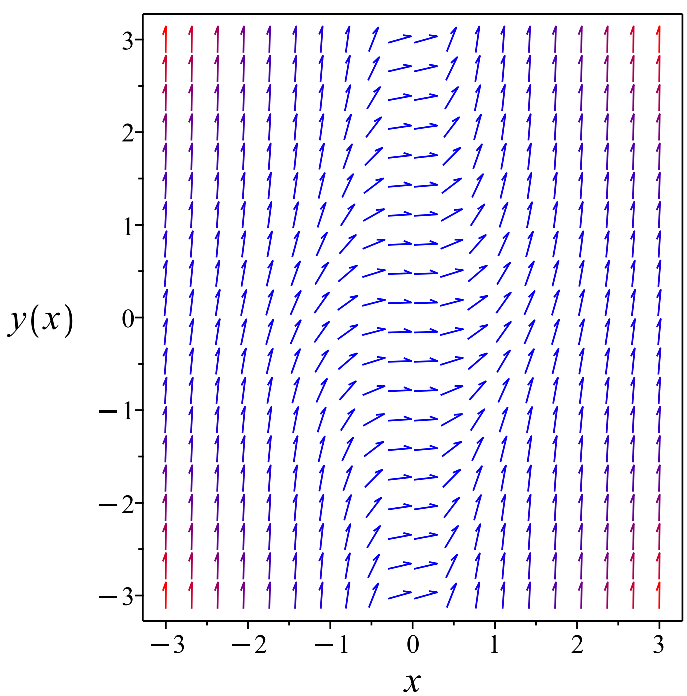


Figure 169: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{x^3}{3} + c_1\right)$$

Verified OK.

3.6.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (y^2 + 1)\, x^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2y^2 + x^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2x \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^6 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^2 u''(x) - 2xu'(x) + x^6 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^3}{3}\right) + c_2 \cos\left(\frac{x^3}{3}\right)$$

The above shows that

$$u'(x) = x^2\left(c_1 \cos\left(\frac{x^3}{3}\right) - c_2 \sin\left(\frac{x^3}{3}\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos\left(\frac{x^3}{3}\right) - c_2 \sin\left(\frac{x^3}{3}\right)}{c_1 \sin\left(\frac{x^3}{3}\right) + c_2 \cos\left(\frac{x^3}{3}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)} \tag{1}$$

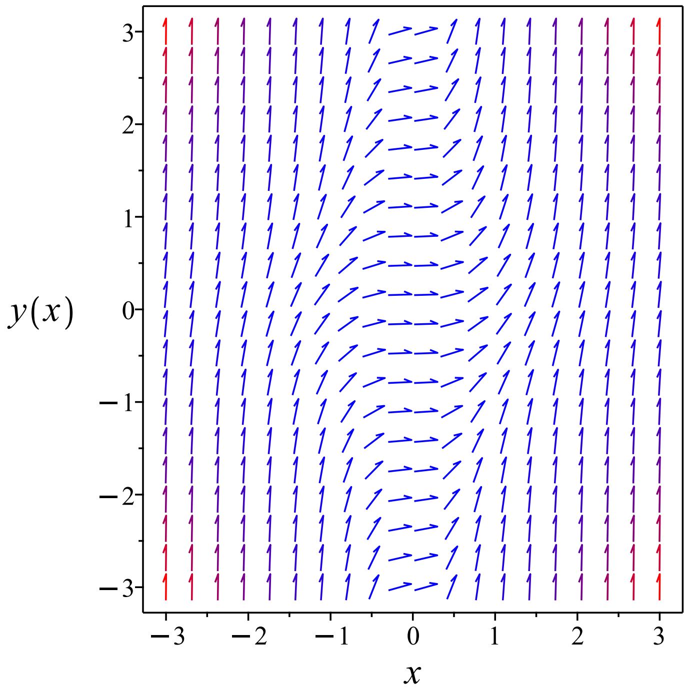


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \cos\left(\frac{x^3}{3}\right) + \sin\left(\frac{x^3}{3}\right)}{c_3 \sin\left(\frac{x^3}{3}\right) + \cos\left(\frac{x^3}{3}\right)}$$

Verified OK.

3.6.5 Maple step by step solution

Let's solve

$y' - (1 + y^2)\, x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\arctan(y) = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \tan\left(\frac{x^3}{3} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)= x^2*(1+y(x)^2),y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{x^3}{3} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 30

```
DSolve[y'[x]== x^2*(1+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x^3}{3} + c_1\right)$$
$$y(x) \to -i$$
$$y(x) \to i$$

3.7 problem 8

Internal problem ID [934]
Internal file name [`OUTPUT/934_Sunday_June_05_2022_01_54_24_AM_4654346/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'+yx=0}$$

3.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy}{x^2+1} \end{aligned}$$

Where $f(x) = -\frac{x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{x}{x^2+1}\,dx$$
$$\int \frac{1}{y}\,dy = \int -\frac{x}{x^2+1}\,dx$$
$$\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$$
$$y = e^{-\frac{\ln(x^2+1)}{2}+c_1}$$
$$= \frac{c_1}{\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

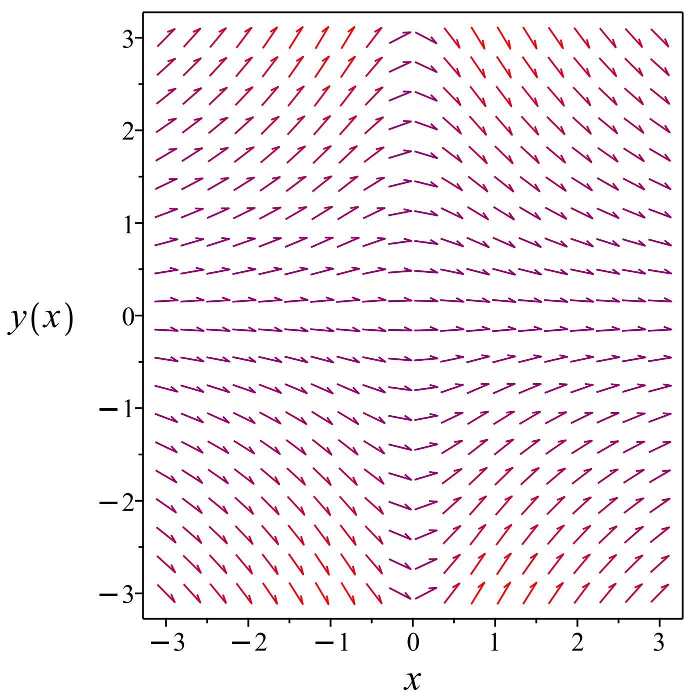


Figure 171: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

3.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{x}{x^2+1}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{xy}{x^2+1} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{x}{x^2+1}dx}\\ &= \sqrt{x^2+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^2+1}\,y\right) &= 0\end{aligned}$$

Integrating gives

$$\sqrt{x^2+1}\,y = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

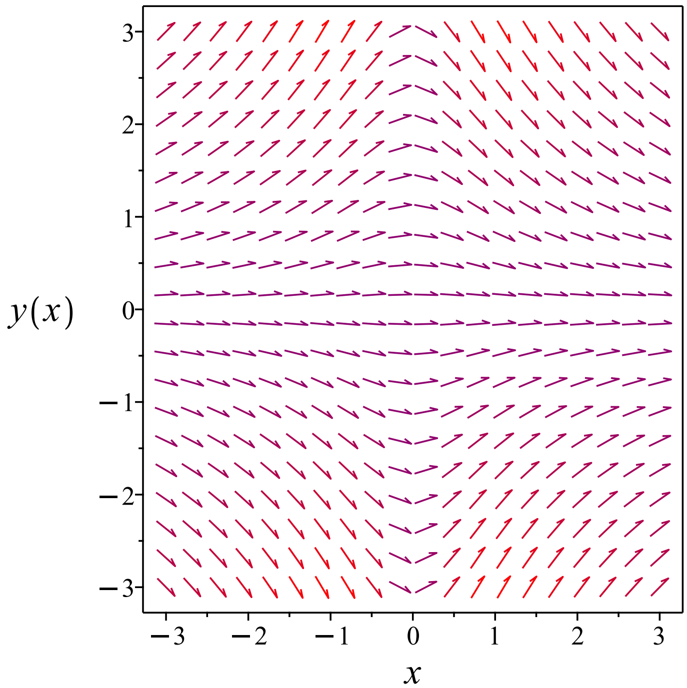


Figure 172: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

3.7.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(x^2+1\right)\left(u'(x)\,x+u(x)\right)+u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(2x^2+1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2x^2+1}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2x^2+1}{x\,(x^2+1)}\,dx \\
\ln(u) &= -\ln(x) - \frac{\ln(x^2+1)}{2} + c_2 \\
u &= e^{-\ln(x)-\frac{\ln\left(x^2+1\right)}{2}+c_2} \\
&= c_2 e^{-\ln(x)-\frac{\ln\left(x^2+1\right)}{2}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x\sqrt{x^2+1}}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{\sqrt{x^2+1}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{\sqrt{x^2+1}} \tag{1}$$

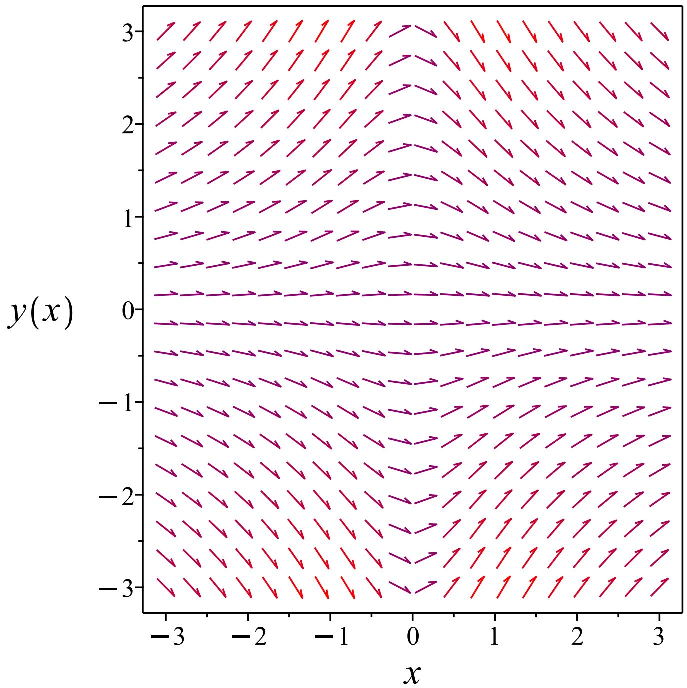


Figure 173: Slope field plot

Verification of solutions

$$y = \frac{c_2}{\sqrt{x^2+1}}$$

Verified OK.

3.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy}{x^2+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 160: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy\end{aligned}$$

Which results in

$$S = \sqrt{x^2 + 1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2 + 1}} \\ S_y &= \sqrt{x^2 + 1}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1}\, y = c_1$$

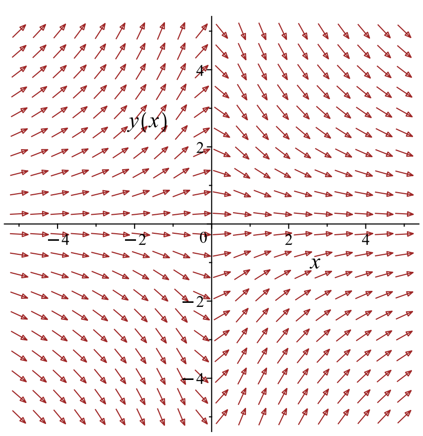
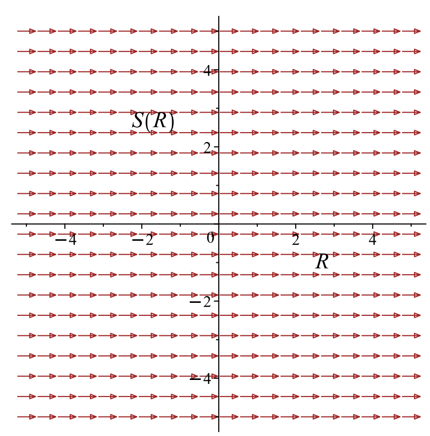
Which simplifies to

$$\sqrt{x^2+1}\, y = c_1$$

Which gives

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy}{x^2+1}$ 	$R = x$ $S = \sqrt{x^2+1}\, y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

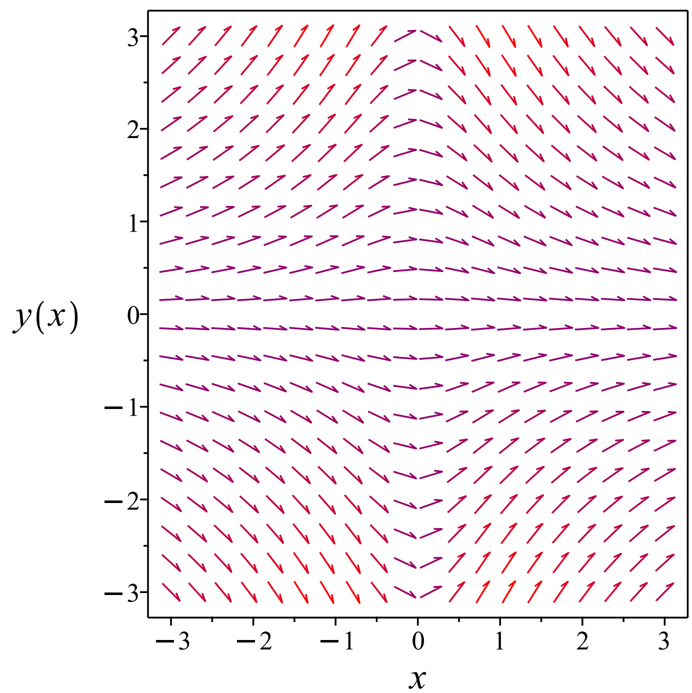


Figure 174: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

3.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x^2+1}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{x^2+1} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1} \tag{1}$$

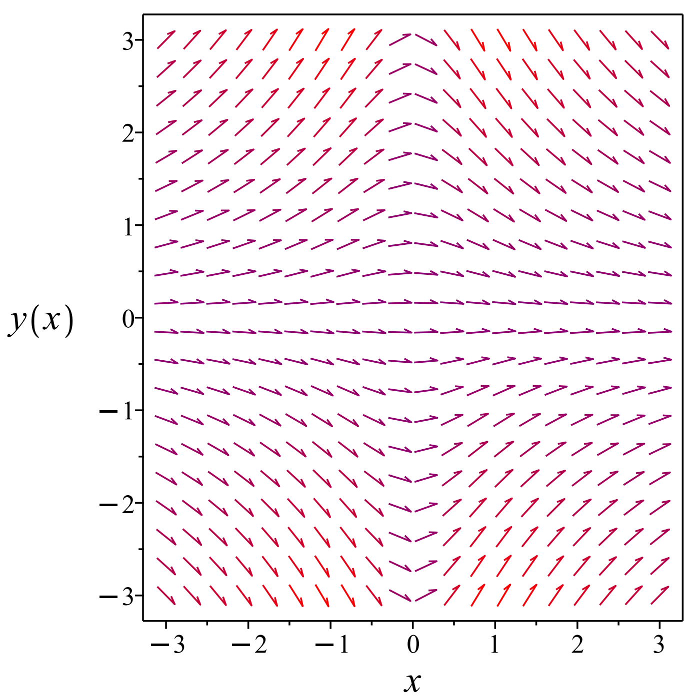


Figure 175: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1}$$

Verified OK.

3.7.6 Maple step by step solution

Let's solve

$(x^2+1)\, y' + yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{x}{x^2+1} dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{\ln(x^2+1)}{2} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)*(1+x^2)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 22

```
DSolve[y'[x]*(1+x^2)+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x^2+1}}$$
$$y(x) \to 0$$

3.8 problem 9

Internal problem ID [935]
Internal file name [`OUTPUT/935_Sunday_June_05_2022_01_54_25_AM_80057781/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (x-1)(y-1)(y-2) = 0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (x-1)(y-1)(y-2) \end{aligned}$$

Where $f(x) = x - 1$ and $g(y) = (y-1)(y-2)$. Integrating both sides gives

$$\frac{1}{(y-1)(y-2)}\,dy = x - 1\,dx$$

$$\int \frac{1}{(y-1)(y-2)}\,dy = \int x - 1\,dx$$

$$-\ln(y-1)+\ln(y-2)=\frac{1}{2}x^2-x+c_1$$

Raising both side to exponential gives

$$e^{-\ln(y-1)+\ln(y-2)}=e^{\frac{1}{2}x^2-x+c_1}$$

Which simplifies to

$$\frac{y-2}{y-1}=c_2e^{\frac{1}{2}x^2-x}$$

Summary

The solution(s) found are the following

$$y=\frac{c_2e^{\frac{1}{2}x^2-x}-2}{-1+c_2e^{\frac{1}{2}x^2-x}} \tag{1}$$

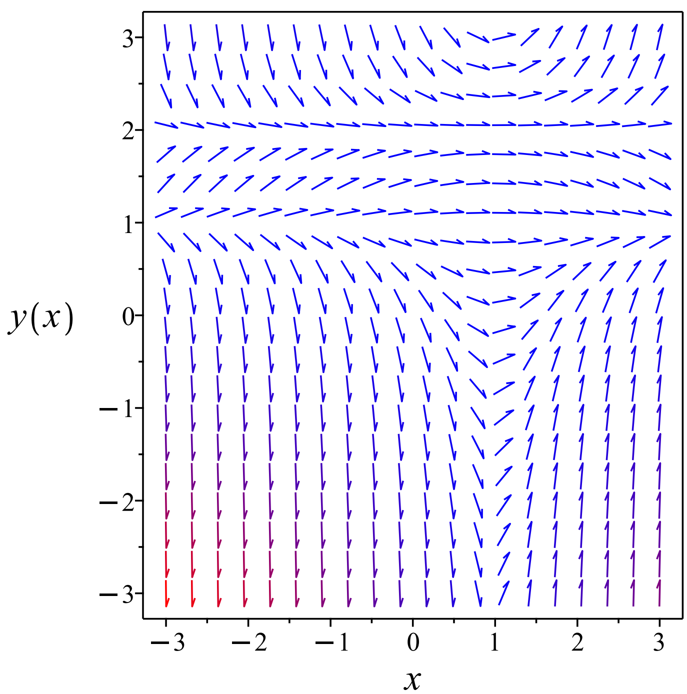


Figure 176: Slope field plot

Verification of solutions

$$y=\frac{c_2e^{\frac{1}{2}x^2-x}-2}{-1+c_2e^{\frac{1}{2}x^2-x}}$$

Verified OK.

3.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= (x-1)(y-1)(y-2) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 163: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x-1} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x-1}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{2}x^2 - x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (x-1)(y-1)(y-2)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x-1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y-1)(y-2)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)(R-2)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R-1) + \ln(R-2) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 - x = -\ln(y-1) + \ln(y-2) + c_1$$

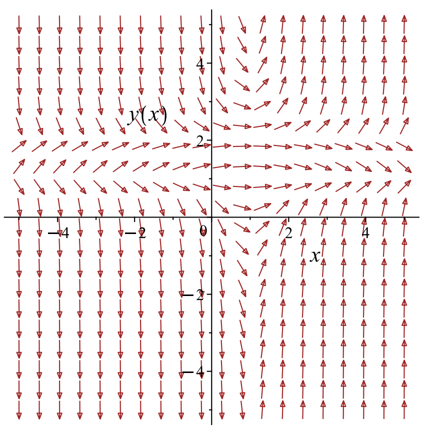
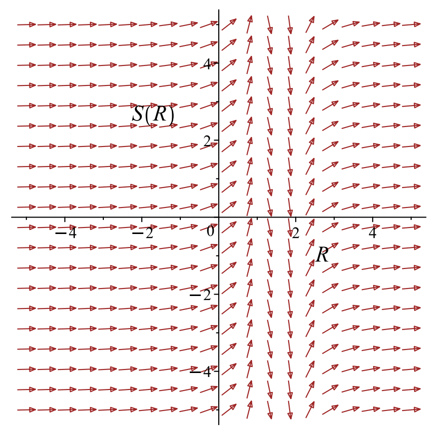
Which simplifies to

$$\frac{1}{2}x^2 - x = -\ln(y-1) + \ln(y-2) + c_1$$

Which gives

$$y = \frac{2\,\mathrm{e}^{-\frac{1}{2}x^2 + c_1 + x} - 1}{-1 + \mathrm{e}^{-\frac{1}{2}x^2 + c_1 + x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (x-1)(y-1)(y-2)$ 	$R = y$ $S = \frac{1}{2}x^2 - x$	$\frac{dS}{dR} = \frac{1}{(R-1)(R-2)}$ 

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{-\frac{1}{2}x^2 + c_1 + x} - 1}{-1 + \mathrm{e}^{-\frac{1}{2}x^2 + c_1 + x}} \tag{1}$$

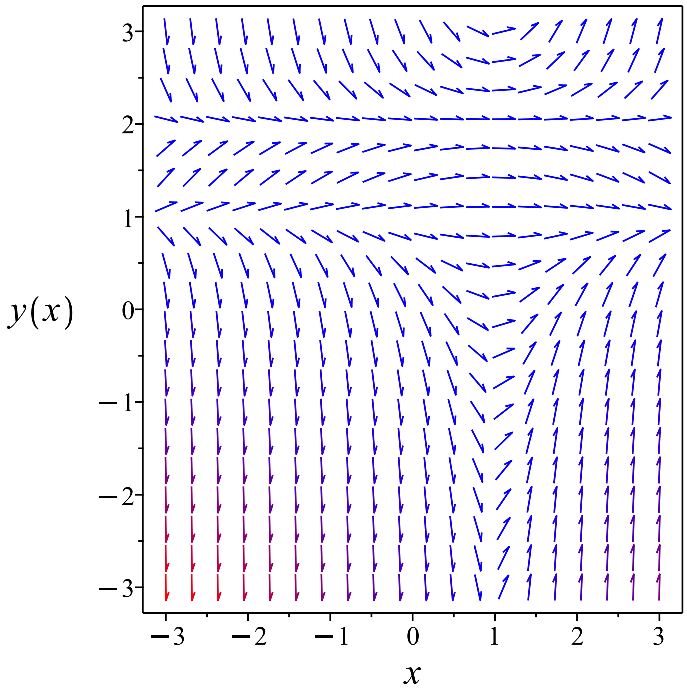


Figure 177: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2\,\mathrm{e}^{-\frac{1}{2}x^2+c_1+x} - 1}{-1+\mathrm{e}^{-\frac{1}{2}x^2+c_1+x}}$$

Verified OK.

3.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{(y-1)(y-2)}\right)\mathrm{d}y &= (x-1)\,\mathrm{d}x \\
(1-x)\,\mathrm{d}x + \left(\frac{1}{(y-1)(y-2)}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 1 - x \\
N(x, y) &= \frac{1}{(y-1)(y-2)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 - x) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{(y-1)(y-2)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 1 - x \, \mathrm{d}x
\end{aligned}$$

$$\phi = x - \frac{1}{2}x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(y-1)(y-2)}$. Therefore equation (4) becomes

$$\frac{1}{(y-1)(y-2)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{(y-1)(y-2)}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{(y-1)(y-2)}\right) \mathrm{d}y \\
f(y) &= -\ln(y-1) + \ln(y-2) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{x^2}{2} - \ln(y-1) + \ln(y-2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{x^2}{2} - \ln(y-1) + \ln(y-2)$$

The solution becomes

$$y = \frac{-2 + e^{\frac{1}{2}x^2 - x + c_1}}{e^{\frac{1}{2}x^2 - x + c_1} - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-2 + e^{\frac{1}{2}x^2 - x + c_1}}{e^{\frac{1}{2}x^2 - x + c_1} - 1} \tag{1}$$

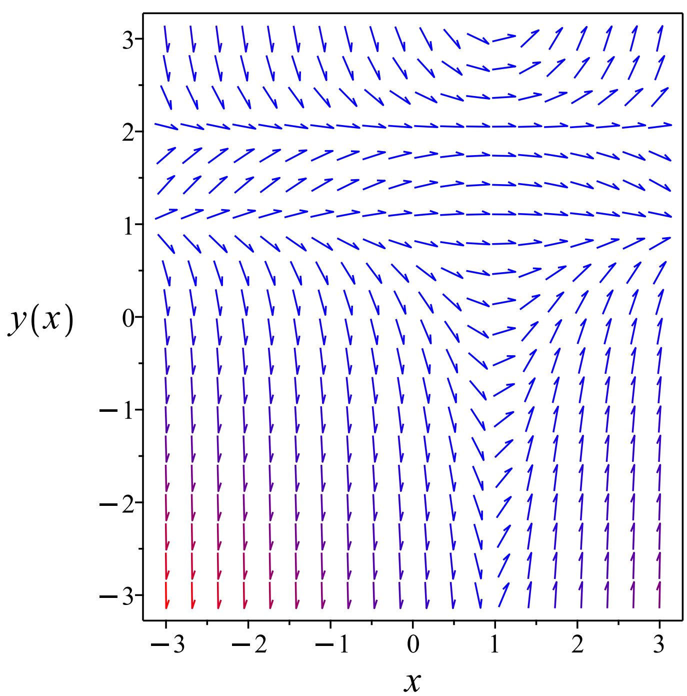


Figure 178: Slope field plot

Verification of solutions

$$y = \frac{-2 + \mathrm{e}^{\frac{1}{2}x^2 - x + c_1}}{\mathrm{e}^{\frac{1}{2}x^2 - x + c_1} - 1}$$

Verified OK.

3.8.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (x-1)(y-1)(y-2) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x\,y^2 - 3yx - y^2 + 2x + 3y - 2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2x - 2$, $f_1(x) = 3 - 3x$ and $f_2(x) = x - 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(x-1)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= (3-3x)(x-1) \\ f_2^2 f_0 &= (x-1)^2 (2x-2) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(x-1)\,u''(x) - (1 + (3-3x)(x-1))\,u'(x) + (x-1)^2 (2x-2)\,u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{-x(-2+x)} + c_2 \mathrm{e}^{-\frac{x(-2+x)}{2}}$$

The above shows that

$$u'(x) = -\left(2c_1 \mathrm{e}^{-x(-2+x)} + c_2 \mathrm{e}^{-\frac{x(-2+x)}{2}}\right)(x-1)$$

Using the above in (1) gives the solution

$$y = \frac{2c_1 \mathrm{e}^{-x(-2+x)} + c_2 \mathrm{e}^{-\frac{x(-2+x)}{2}}}{c_1 \mathrm{e}^{-x(-2+x)} + c_2 \mathrm{e}^{-\frac{x(-2+x)}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2c_3 \mathrm{e}^{-\frac{x(-2+x)}{2}} + 1}{c_3 \mathrm{e}^{-\frac{x(-2+x)}{2}} + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{2c_3 \mathrm{e}^{-\frac{x(-2+x)}{2}} + 1}{c_3 \mathrm{e}^{-\frac{x(-2+x)}{2}} + 1} \tag{1}$$

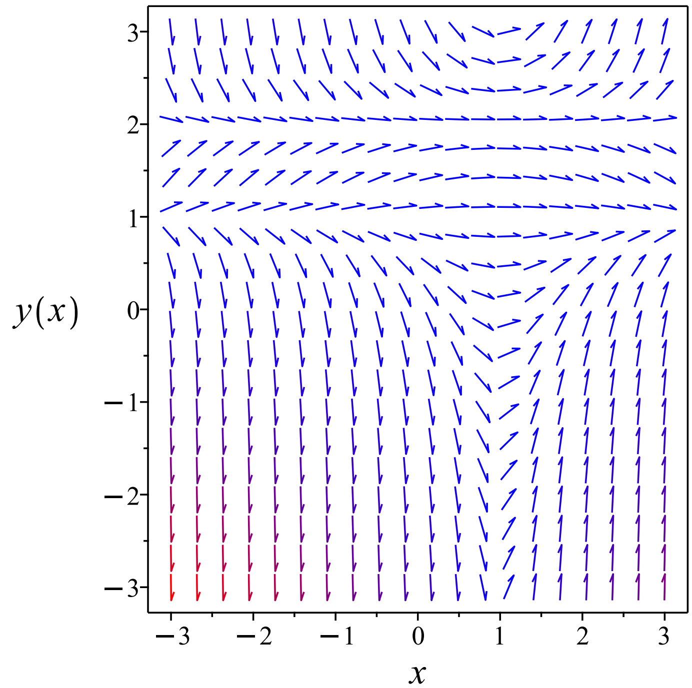


Figure 179: Slope field plot

Verification of solutions

$$y = \frac{2c_3 e^{-\frac{x(-2+x)}{2}} + 1}{c_3 e^{-\frac{x(-2+x)}{2}} + 1}$$

Verified OK.

3.8.5 Maple step by step solution

Let's solve

$y' - (x-1)(y-1)(y-2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(y-1)(y-2)} = x - 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{(y-1)(y-2)} dx = \int (x-1)\, dx + c_1$

- Evaluate integral

  $-\ln(y-1)+\ln(y-2)=\frac{1}{2}x^2-x+c_1$

- Solve for y

  $y=\frac{-2+e^{\frac{1}{2}x^2-x+c_1}}{e^{\frac{1}{2}x^2-x+c_1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)=(x-1)*(y(x)-1)*(y(x)-2),y(x), singsol=all)
```

$$y(x)=\frac{-2+c_1e^{\frac{x(-2+x)}{2}}}{c_1e^{\frac{x(-2+x)}{2}}-1}$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 56

```
DSolve[y'[x]==(x-1)*(y[x]-1)*(y[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{2e^x-e^{\frac{x^2}{2}+c_1}}{e^x-e^{\frac{x^2}{2}+c_1}}$$
$$y(x)\to 1$$
$$y(x)\to 2$$

3.9 problem 10

Internal problem ID [936]
Internal file name [`OUTPUT/936_Sunday_June_05_2022_01_54_27_AM_35833158/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(y-1)^2 y' = 2x+3}$$

3.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2x+3}{y^2-2y+1} \end{aligned}$$

Where $f(x) = 2x+3$ and $g(y) = \frac{1}{y^2-2y+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2-2y+1}}\,dy = 2x+3\,dx$$

$$\int \frac{1}{\frac{1}{y^2-2y+1}}\, dy = \int 2x+3\, dx$$

$$\frac{(y-1)^3}{3} = x^2+c_1+3x$$

Which results in

$$y = \left(3x^2+3c_1+9x\right)^{\frac{1}{3}}+1$$

$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1$$

$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1$$

Summary

The solution(s) found are the following

$$y = \left(3x^2+3c_1+9x\right)^{\frac{1}{3}}+1 \tag{1}$$

$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1 \tag{2}$$

$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1 \tag{3}$$

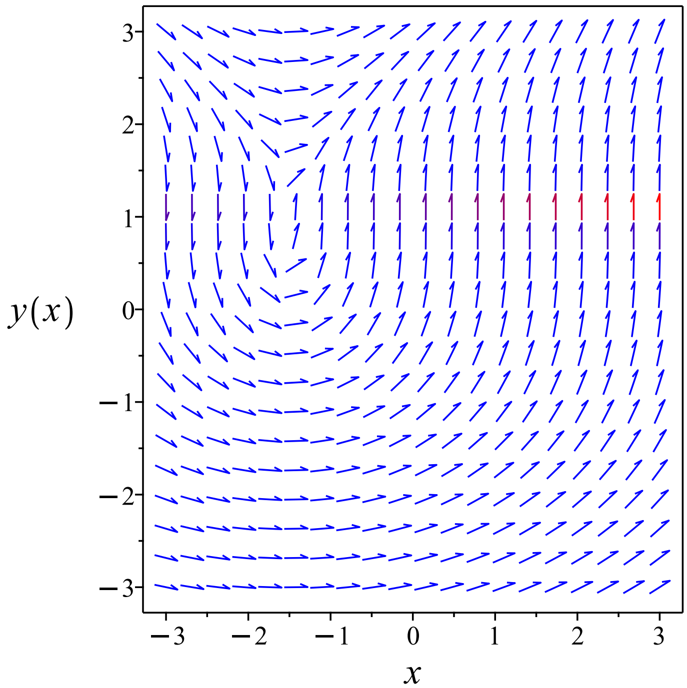


Figure 180: Slope field plot

Verification of solutions

$$y = \left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}} + 1$$

Verified OK.

$$y = -\frac{\left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}}}{2} + 1$$

Verified OK.

$$y = -\frac{\left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}}}{2} + 1$$

Verified OK.

3.9.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x+3}{(y-1)^2} \tag{1}$$

Which becomes

$$\left(y^2-2y+1\right)dy = (2x+3)\,dx \tag{2}$$

But the RHS is complete differential because

$$(2x+3)\,dx = d\left(x^2+3x\right)$$

Hence (2) becomes

$$\left(y^2-2y+1\right)dy = d\left(x^2+3x\right)$$

Integrating both sides gives gives these solutions

$$y = \left(3x^2+3c_1+9x\right)^{\frac{1}{3}}+1+c_1$$
$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1+c_1$$
$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1+c_1$$

Summary
The solution(s) found are the following

$$y = \left(3x^2+3c_1+9x\right)^{\frac{1}{3}}+1+c_1 \tag{1}$$
$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1+c_1 \tag{2}$$
$$y = -\frac{\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\left(3x^2+3c_1+9x\right)^{\frac{1}{3}}}{2}+1+c_1 \tag{3}$$

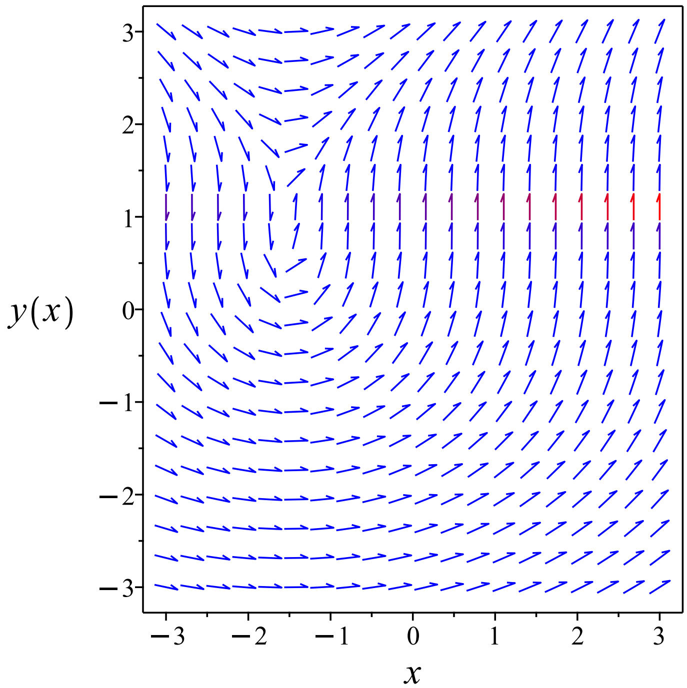


Figure 181: Slope field plot

Verification of solutions

$$y = \left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}} + 1 + c_1$$

Verified OK.

$$y = -\frac{(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + 1 + c_1$$

Verified OK.

$$y = -\frac{(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + 1 + c_1$$

Verified OK.

3.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x+3}{y^2-2y+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 166: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= \frac{1}{2x+3} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{2x+3}} dx
\end{aligned}$$

Which results in

$$S = x^2 + 3x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x+3}{y^2 - 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 2x + 3 \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (y-1)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R-1)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(R-1)^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 + 3x = \frac{(y-1)^3}{3} + c_1$$

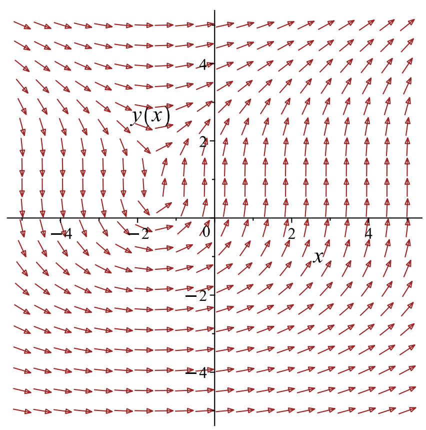
Which simplifies to

$$x^2 + 3x = \frac{(y-1)^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2x+3}{y^2-2y+1}$



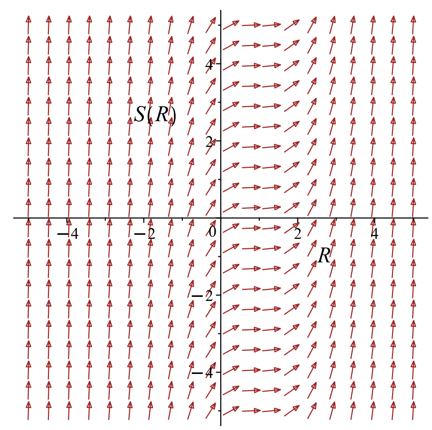
</td><td>

$R = y$

$S = x^2 + 3x$

</td><td>

$\frac{dS}{dR} = (R-1)^2$



</td></tr>
</table>

<u>Summary</u>

The solution(s) found are the following

$$x^2 + 3x = \frac{(y-1)^3}{3} + c_1 \tag{1}$$

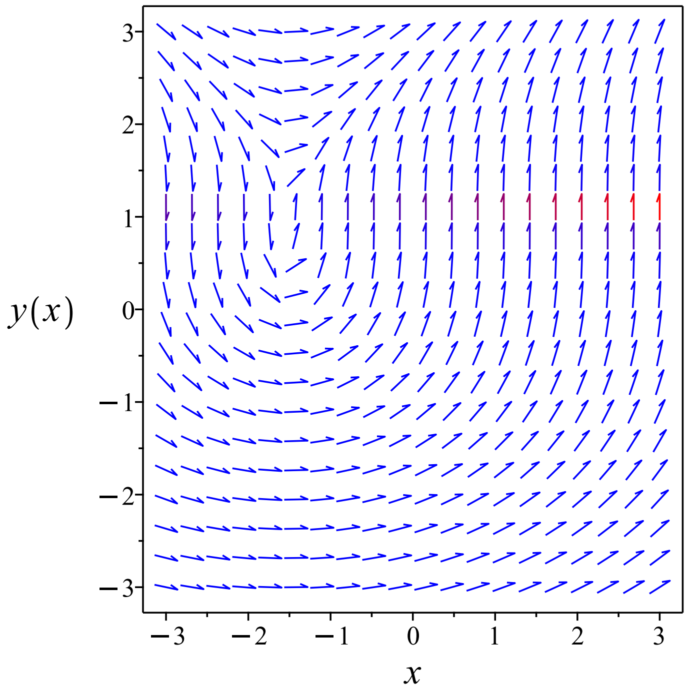


Figure 182: Slope field plot

Verification of solutions

$$x^2 + 3x = \frac{(y-1)^3}{3} + c_1$$

Verified OK.

3.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y^2 - 2y + 1\right) \mathrm{d}y &= (2x + 3)\,\mathrm{d}x \\
(-3 - 2x)\,\mathrm{d}x + \left(y^2 - 2y + 1\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -3 - 2x \\
N(x, y) &= y^2 - 2y + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3 - 2x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 - 2y + 1\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -3-2x\,\mathrm{d}x\\ \phi &= -x^2-3x+f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y^2-2y+1$. Therefore equation (4) becomes

$$y^2-2y+1=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= y^2-2y+1\\ &= (y-1)^2\end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left((y-1)^2\right)\mathrm{d}y\\ f(y) &= \frac{(y-1)^3}{3}+c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - 3x + \frac{(y-1)^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - 3x + \frac{(y-1)^3}{3}$$

Summary

The solution(s) found are the following

$$-x^2 - 3x + \frac{(y-1)^3}{3} = c_1 \tag{1}$$

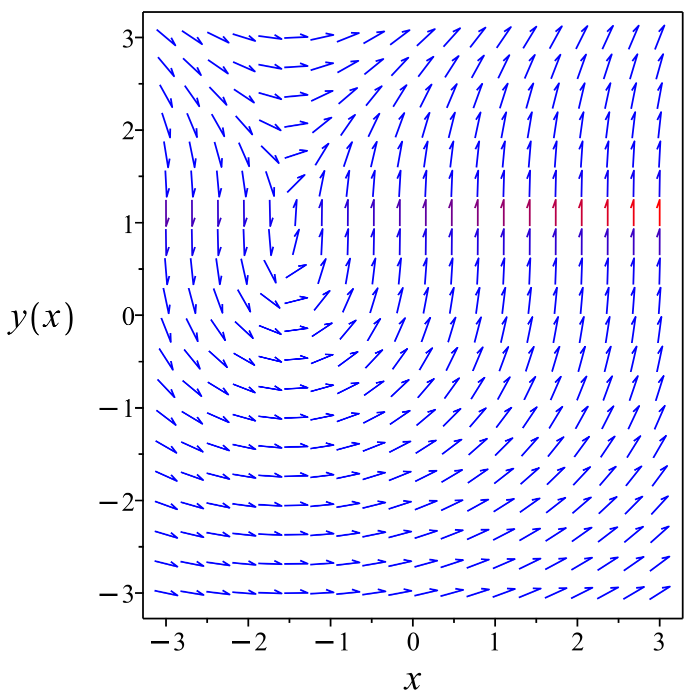


Figure 183: Slope field plot

Verification of solutions

$$-x^2 - 3x + \frac{(y-1)^3}{3} = c_1$$

Verified OK.

3.9.5 Maple step by step solution

Let's solve

$(y-1)^2 y' = 2x+3$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (y-1)^2 y' dx = \int (2x+3)\, dx + c_1$

- Evaluate integral

  $\frac{(y-1)^3}{3} = x^2 + c_1 + 3x$

- Solve for y

  $y = (3x^2 + 3c_1 + 9x)^{\frac{1}{3}} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 102

```
dsolve((y(x)-1)^2*diff(y(x),x)=2*x+3,y(x), singsol=all)
```

$$y(x) = \left(3x^2 + 3c_1 + 9x\right)^{\frac{1}{3}} + 1$$

$$y(x) = -\frac{(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + 1$$

$$y(x) = -\frac{(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(3x^2 + 3c_1 + 9x)^{\frac{1}{3}}}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.482 (sec). Leaf size: 103

```
DSolve[(y[x]-1)^2*y'[x]==2*x+3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + \sqrt[3]{3x^2 + 9x - 1 + 3c_1}$$
$$y(x) \to 1 + \frac{1}{2}i\left(\sqrt{3} + i\right)\sqrt[3]{3x^2 + 9x - 1 + 3c_1}$$
$$y(x) \to 1 - \frac{1}{2}\left(1 + i\sqrt{3}\right)\sqrt[3]{3x^2 + 9x - 1 + 3c_1}$$

3.10 problem 11

Internal problem ID [937]
Internal file name [`OUTPUT/937_Sunday_June_05_2022_01_54_28_AM_23520406/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x^2 + 3x + 2}{y - 2} = 0}$$

With initial conditions

$$[y(1) = 4]$$

3.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + 3x + 2}{y - 2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 2 \lor 2 < y\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2 + 3x + 2}{y - 2}\right) \\ &= -\frac{x^2 + 3x + 2}{(y - 2)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 2 \lor 2 < y\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

3.10.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x^2 + 3x + 2}{y - 2} \end{aligned}$$

Where $f(x) = x^2 + 3x + 2$ and $g(y) = \frac{1}{y-2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y-2}} dy = x^2 + 3x + 2 \, dx$$

$$\int \frac{1}{\frac{1}{y-2}} dy = \int x^2 + 3x + 2 \, dx$$

$$\frac{1}{2}y^2 - 2y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + c_1$$

Which results in

$$y = 2 + \frac{\sqrt{6x^3 + 27x^2 + 18c_1 + 36x + 36}}{3}$$

$$y = 2 - \frac{\sqrt{6x^3 + 27x^2 + 18c_1 + 36x + 36}}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 2 - \frac{\sqrt{105 + 18c_1}}{3}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 2 + \frac{\sqrt{105 + 18c_1}}{3}$$

$$c_1 = -\frac{23}{6}$$

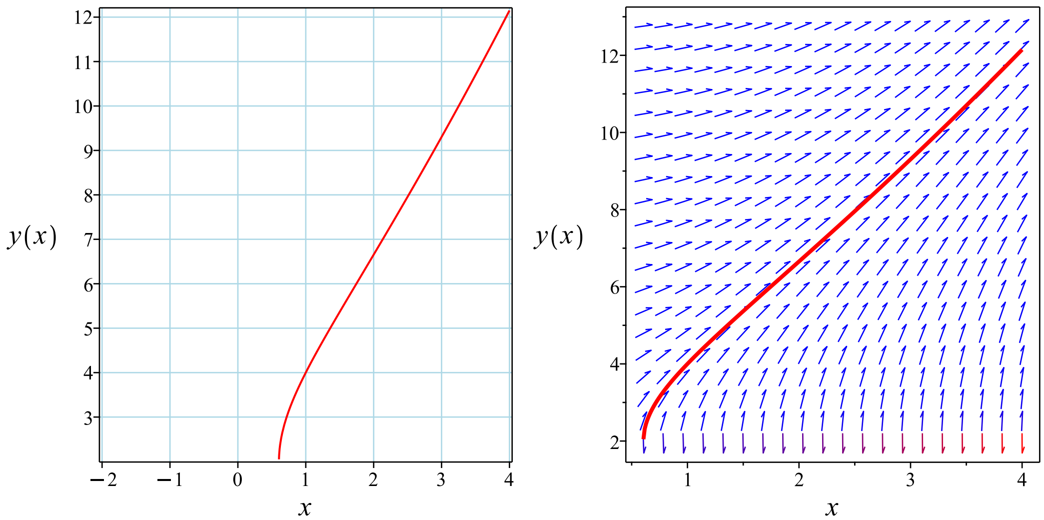
Substituting c_1 found above in the general solution gives

$$y = 2 + \frac{\sqrt{6x^3 + 27x^2 + 36x - 33}}{3}$$

Summary

The solution(s) found are the following

$$y = 2 + \frac{\sqrt{6x^3 + 27x^2 + 36x - 33}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2 + \frac{\sqrt{6x^3 + 27x^2 + 36x - 33}}{3}$$

Verified OK.

3.10.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x^2 + 3x + 2}{y - 2} \tag{1}$$

Which becomes

$$(y - 2)\, dy = (x^2 + 3x + 2)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x^2 + 3x + 2)\, dx = d\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)$$

Hence (2) becomes

$$(y - 2)\, dy = d\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= 2+\frac{\sqrt{6x^{3}+27x^{2}+18c_{1}+36x+36}}{3}+c_{1} \\
y &= 2-\frac{\sqrt{6x^{3}+27x^{2}+18c_{1}+36x+36}}{3}+c_{1}
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$
4 = 2-\frac{\sqrt{105+18c_{1}}}{3}+c_{1}
$$

$$
c_{1} = 3+\frac{5\sqrt{6}}{3}
$$

Substituting c_1 found above in the general solution gives

$$
y = 5-\frac{\sqrt{6x^{3}+27x^{2}+90+30\sqrt{6}+36x}}{3}+\frac{5\sqrt{6}}{3}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$
4 = 2+\frac{\sqrt{105+18c_{1}}}{3}+c_{1}
$$

$$
c_{1} = 3-\frac{5\sqrt{6}}{3}
$$

Substituting c_1 found above in the general solution gives

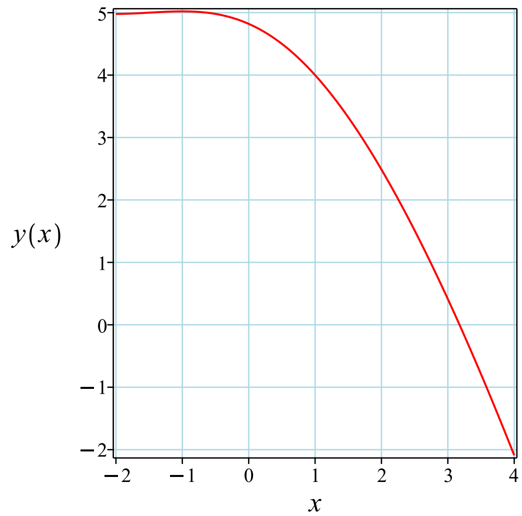
$$
y = 5+\frac{\sqrt{6x^{3}+27x^{2}+90-30\sqrt{6}+36x}}{3}-\frac{5\sqrt{6}}{3}
$$

Summary

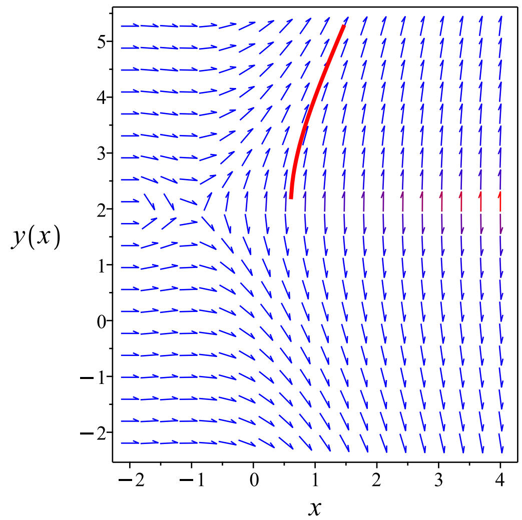
The solution(s) found are the following

$$
y = 5+\frac{\sqrt{6x^{3}+27x^{2}+90-30\sqrt{6}+36x}}{3}-\frac{5\sqrt{6}}{3} \tag{1}
$$

$$
y = 5-\frac{\sqrt{6x^{3}+27x^{2}+90+30\sqrt{6}+36x}}{3}+\frac{5\sqrt{6}}{3} \tag{2}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 5 + \frac{\sqrt{6x^3 + 27x^2 + 90 - 30\sqrt{6} + 36x}}{3} - \frac{5\sqrt{6}}{3}$$

Verified OK.

$$y = 5 - \frac{\sqrt{6x^3 + 27x^2 + 90 + 30\sqrt{6} + 36x}}{3} + \frac{5\sqrt{6}}{3}$$

Verified OK.

3.10.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + 3x + 2}{y - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 169: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x^2 + 3x + 2}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2+3x+2}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + 3x + 2}{y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^2 + 3x + 2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y - 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R - 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{2}R^2 - 2R + c_1 \tag{4}$$

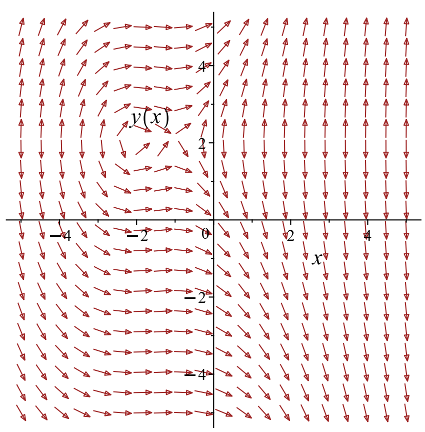
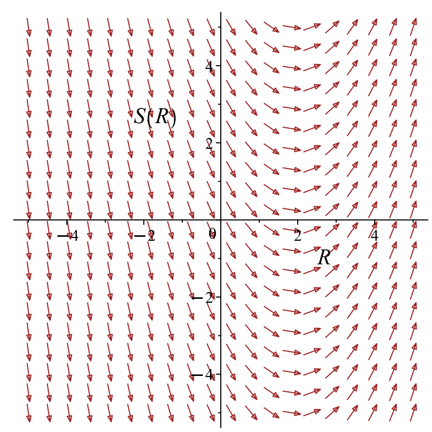
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x = \frac{y^2}{2} - 2y + c_1$$

Which simplifies to

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x = \frac{y^2}{2} - 2y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^2+3x+2}{y-2}$

</td><td>$R = y$
$S = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x$</td><td>$\frac{dS}{dR} = R - 2$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{23}{6} = c_1$$

$$c_1 = \frac{23}{6}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x = \frac{1}{2}y^2 - 2y + \frac{23}{6}$$

Summary

The solution(s) found are the following

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x = \frac{y^2}{2} - 2y + \frac{23}{6} \tag{1}$$

Verification of solutions

$$\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x = \frac{y^2}{2} - 2y + \frac{23}{6}$$

Verified OK.

3.10.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-2)\,\mathrm{d}y &= \left(x^2+3x+2\right)\mathrm{d}x\\ \left(-x^2-3x-2\right)\mathrm{d}x + (y-2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2-3x-2\\ N(x,y) &= y-2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2-3x-2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x^2 - 3x - 2 \mathrm{d}x \\
\phi &= -\frac{x\left(x^2 + \frac{9}{2}x + 6\right)}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - 2$. Therefore equation (4) becomes

$$
y - 2 = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y - 2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (y - 2) \mathrm{d}y \\
f(y) &= \frac{1}{2}y^2 - 2y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{x\left(x^2 + \frac{9}{2}x + 6\right)}{3} + \frac{y^2}{2} - 2y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{x\left(x^2 + \frac{9}{2}x + 6\right)}{3} + \frac{y^2}{2} - 2y
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{23}{6} = c_1$$

$$c_1 = -\frac{23}{6}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x\left(x^2 + \frac{9}{2}x + 6\right)}{3} + \frac{y^2}{2} - 2y = -\frac{23}{6}$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} - \frac{3x^2}{2} + \frac{y^2}{2} - 2x - 2y = -\frac{23}{6} \tag{1}$$

Verification of solutions

$$-\frac{x^3}{3} - \frac{3x^2}{2} + \frac{y^2}{2} - 2x - 2y = -\frac{23}{6}$$

Verified OK.

3.10.6 Maple step by step solution

Let's solve

$\left[y' - \frac{x^2+3x+2}{y-2} = 0, y(1) = 4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $(y-2)\, y' = x^2 + 3x + 2$

- Integrate both sides with respect to x

  $\int (y-2)\, y' dx = \int (x^2 + 3x + 2)\, dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} - 2y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + c_1$

- Solve for y

$$\left\{y=2-\frac{\sqrt{6x^3+27x^2+18c_1+36x+36}}{3}, y=2+\frac{\sqrt{6x^3+27x^2+18c_1+36x+36}}{3}\right\}$$

- Use initial condition $y(1)=4$

  $4=2-\frac{\sqrt{105+18c_1}}{3}$

- Solution does not satisfy initial condition
- Use initial condition $y(1)=4$

  $4=2+\frac{\sqrt{105+18c_1}}{3}$

- Solve for c_1

  $c_1=-\frac{23}{6}$

- Substitute $c_1=-\frac{23}{6}$ into general solution and simplify

  $y=2+\frac{\sqrt{6x^3+27x^2+36x-33}}{3}$

- Solution to the IVP

  $y=2+\frac{\sqrt{6x^3+27x^2+36x-33}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 25

```
dsolve([diff(y(x),x)=(x^2+3*x+2)/(y(x)-2),y(1) = 4],y(x), singsol=all)
```

$$y(x)=2+\frac{\sqrt{6x^3+27x^2+36x-33}}{3}$$

✓ Solution by Mathematica

Time used: 0.167 (sec). Leaf size: 30

```
DSolve[{y'[x]==(x^2+3*x+2)/(y[x]-2),y[1]==4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{\frac{2x^3}{3} + 3x^2 + 4x - \frac{11}{3}} + 2$$

3.11 problem 12

Internal problem ID [938]
Internal file name [`OUTPUT/938_Sunday_June_05_2022_01_54_29_AM_12806827/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + x(y^2 + y) = 0}$$

With initial conditions

$$[y(2) = 1]$$

3.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -yx(y+1) \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x=2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-yx(y+1)) \\ &= -x(y+1) - yx \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.11.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -yx(y+1) \end{aligned}$$

Where $f(x) = -x$ and $g(y) = y(y+1)$. Integrating both sides gives

$$\frac{1}{y(y+1)}\,dy = -x\,dx$$
$$\int \frac{1}{y(y+1)}\,dy = \int -x\,dx$$
$$\ln(y) - \ln(y+1) = -\frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$e^{\ln(y)-\ln(y+1)} = e^{-\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = c_2 e^{-\frac{x^2}{2}}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{c_2 e^{-2}}{c_2 e^{-2} - 1}$$

$$c_2 = \frac{e^2}{2}$$

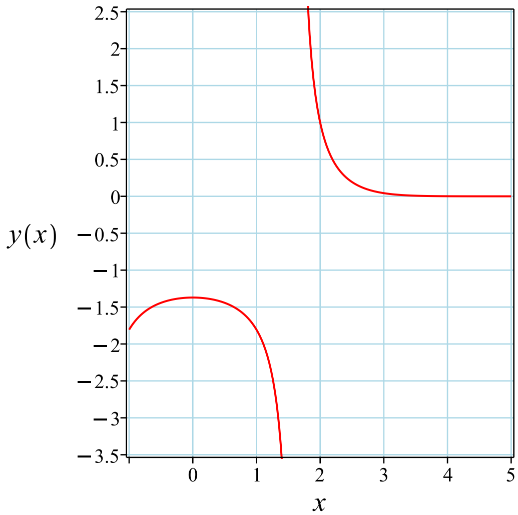
Substituting c_2 found above in the general solution gives

$$y = -\frac{e^{-\frac{(-2+x)(2+x)}{2}}}{e^{-\frac{(-2+x)(2+x)}{2}} - 2}$$

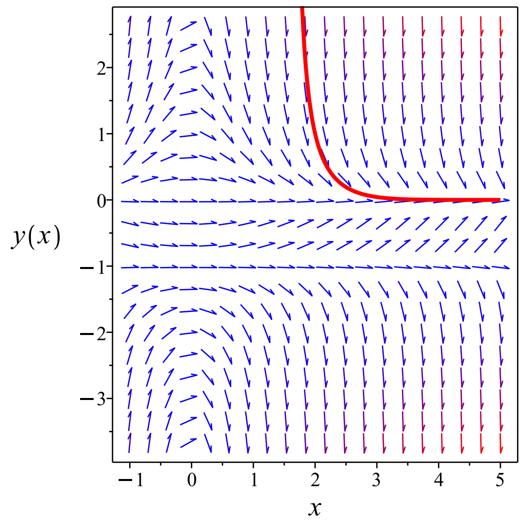
Summary

The solution(s) found are the following

$$y = -\frac{e^{-\frac{(-2+x)(2+x)}{2}}}{e^{-\frac{(-2+x)(2+x)}{2}} - 2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{(-2+x)(2+x)}{2}}}{e^{-\frac{(-2+x)(2+x)}{2}} - 2}$$

Verified OK.

3.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -yx(y+1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 172: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= -\frac{1}{x} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -yx(y+1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) - \ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \ln(y) - \ln(1+y) + c_1$$

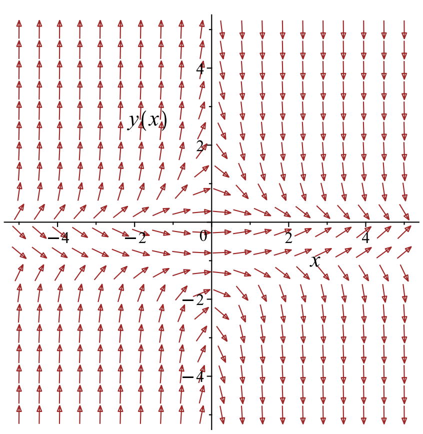
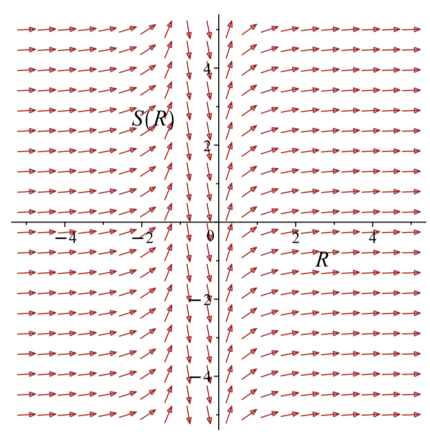
Which simplifies to

$$-\frac{x^2}{2} = \ln(y) - \ln(1+y) + c_1$$

Which gives

$$y = \frac{1}{\mathrm{e}^{\frac{x^2}{2}+c_1} - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -yx(y+1)$  	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R(R+1)}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{e^{c_1+2} - 1}$$

$$c_1 = \ln(2) - 2$$

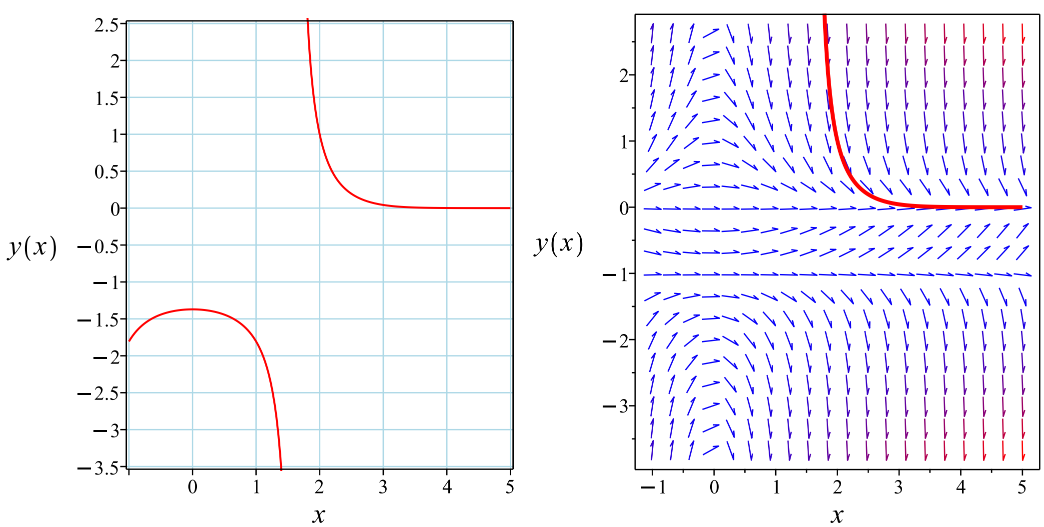
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\,e^{\frac{(-2+x)(2+x)}{2}} - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2\,e^{\frac{(-2+x)(2+x)}{2}} - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\,e^{\frac{(-2+x)(2+x)}{2}} - 1}$$

Verified OK.

3.11.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -yx(y+1)
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -xy - xy^2 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -x \\
f_1(x) &= -x \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = -\frac{x}{y} - x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -w(x)\,x - x \\
w' &= xw + x \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -x \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - w(x)\,x = x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -x dx} \\
&= e^{-\frac{x^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\,(x) \\
\frac{d}{dx}\left(e^{-\frac{x^2}{2}} w\right) &= \left(e^{-\frac{x^2}{2}}\right)(x) \\
d\left(e^{-\frac{x^2}{2}} w\right) &= \left(x\,e^{-\frac{x^2}{2}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{x^2}{2}} w &= \int x\,e^{-\frac{x^2}{2}}\, dx \\
e^{-\frac{x^2}{2}} w &= -e^{-\frac{x^2}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$
w(x) = -e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} + c_1 e^{\frac{x^2}{2}}
$$

which simplifies to

$$w(x) = -1 + c_1 e^{\frac{x^2}{2}}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -1 + c_1 e^{\frac{x^2}{2}}$$

Or

$$y = \frac{1}{-1 + c_1 e^{\frac{x^2}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{-1 + e^2 c_1}$$

$$c_1 = 2\,e^{-2}$$

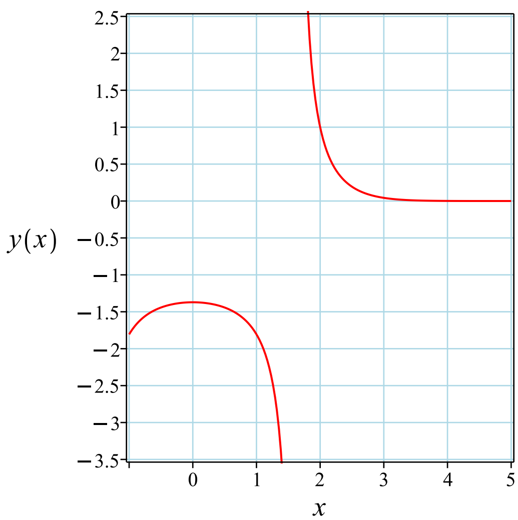
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\,e^{\frac{(-2+x)(2+x)}{2}} - 1}$$

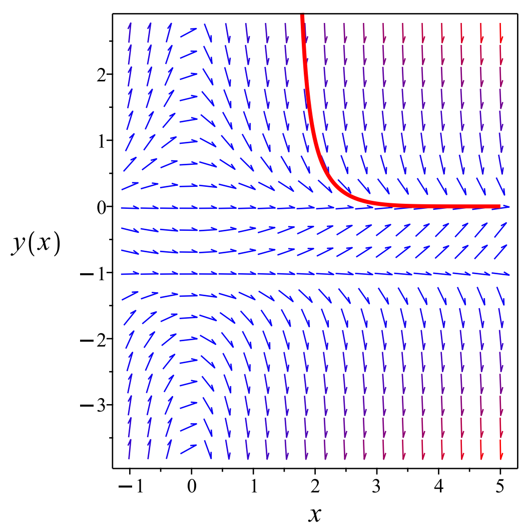
Summary

The solution(s) found are the following

$$y = \frac{1}{2\,e^{\frac{(-2+x)(2+x)}{2}} - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\,\mathrm{e}^{\frac{(-2+x)(2+x)}{2}}-1}$$

Verified OK.

3.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y\left(y+1\right)}\right) \mathrm{d}y &= (x) \mathrm{d}x \\
(-x) \mathrm{d}x + \left(-\frac{1}{y\left(y+1\right)}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= -\frac{1}{y\left(y+1\right)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y\left(y+1\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \qquad (1)
$$

$$
\frac{\partial \phi}{\partial y} = N \qquad (2)
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \qquad (3)
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y+1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\,(y+1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\,(y+1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y\,(y+1)}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \ln(y) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \ln(y) + \ln(y+1)$$

The solution becomes

$$y = \frac{1}{\mathrm{e}^{\frac{x^2}{2}+c_1} - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\mathrm{e}^{c_1+2} - 1}$$

$$c_1 = \ln(2) - 2$$

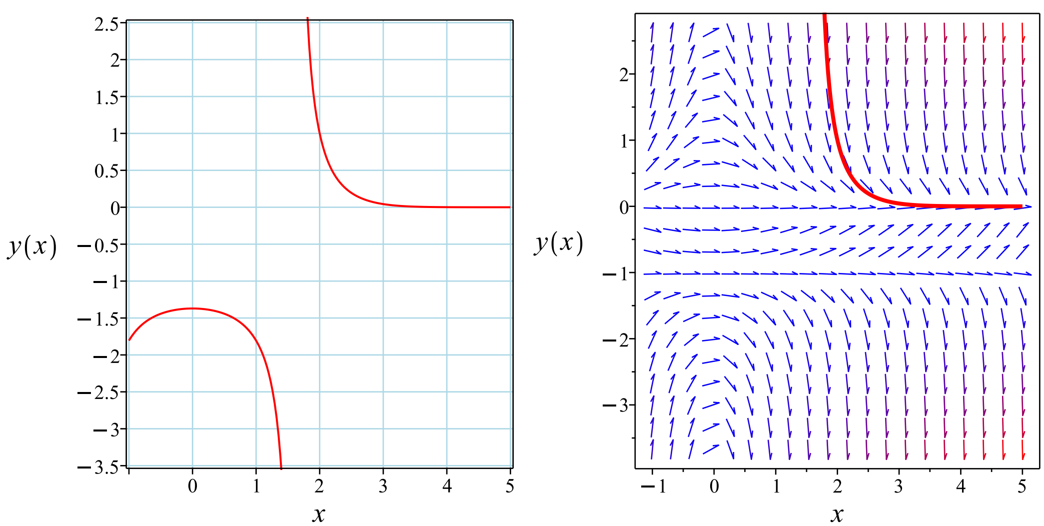
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\,\mathrm{e}^{\frac{(-2+x)(2+x)}{2}} - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2\,\mathrm{e}^{\frac{(-2+x)(2+x)}{2}} - 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\,\mathrm{e}^{\frac{(-2+x)(2+x)}{2}} - 1}$$

Verified OK.

3.11.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -yx(y+1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x\,y^2 - yx$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -x$ and $f_2(x) = -x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -1 \\ f_1 f_2 &= x^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-xu''(x) - \left(x^2 - 1\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \mathrm{e}^{-\frac{x^2}{2}}$$

The above shows that

$$u'(x) = -c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 e^{-\frac{x^2}{2}}}{c_1 + c_2 e^{-\frac{x^2}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{e^{-\frac{x^2}{2}}}{c_3 + e^{-\frac{x^2}{2}}}$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{e^{-2}}{e^{-2} + c_3}$$

$$c_3 = -2\,e^{-2}$$

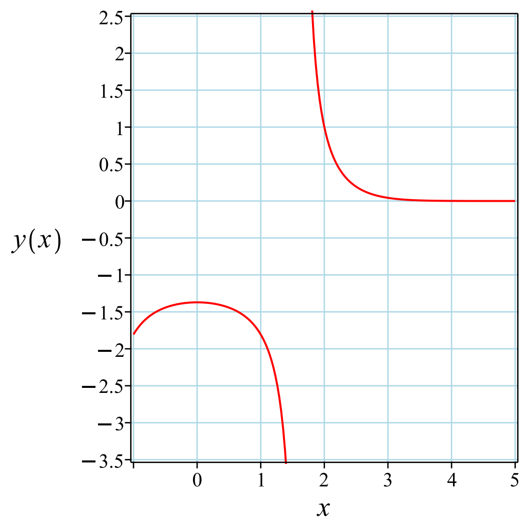
Substituting c_3 found above in the general solution gives

$$y = \frac{e^{-\frac{x^2}{2}}}{2\,e^{-2} - e^{-\frac{x^2}{2}}}$$

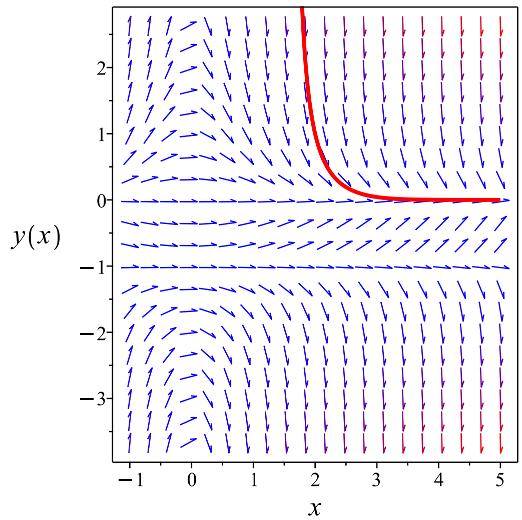
Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{x^2}{2}}}{2\,e^{-2} - e^{-\frac{x^2}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-\frac{x^2}{2}}}{2\,e^{-2} - e^{-\frac{x^2}{2}}}$$

Verified OK.

3.11.7 Maple step by step solution

Let's solve

$[y' + x(y^2 + y) = 0, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+y} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+y} dx = \int -x dx + c_1$

- Evaluate integral

  $\ln(y) - \ln(1+y) = -\frac{x^2}{2} + c_1$

- Solve for y

$$y = -\frac{e^{-\frac{x^2}{2}+c_1}}{-1+e^{-\frac{x^2}{2}+c_1}}$$

- Use initial condition $y(2) = 1$

$$1 = -\frac{e^{c_1-2}}{-1+e^{c_1-2}}$$

- Solve for c_1

$$c_1 = 2 - \ln(2)$$

- Substitute $c_1 = 2 - \ln(2)$ into general solution and simplify

$$y = -\frac{e^{-\frac{(-2+x)(2+x)}{2}}}{e^{-\frac{(-2+x)(2+x)}{2}}-2}$$

- Solution to the IVP

$$y = -\frac{e^{-\frac{(-2+x)(2+x)}{2}}}{e^{-\frac{(-2+x)(2+x)}{2}}-2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)+x*(y(x)^2+y(x))=0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{-1+2\,e^{\frac{(2+x)(-2+x)}{2}}}$$

✓ Solution by Mathematica

Time used: 0.241 (sec). Leaf size: 27

```
DSolve[{y'[x]+x*(y[x]^2+y[x])==0,y[2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^2}{e^2 - 2e^{\frac{x^2}{2}}}$$

3.12 problem 13

Internal problem ID [939]
Internal file name [`OUTPUT/939_Sunday_June_05_2022_01_54_31_AM_4937947/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(3y^2 + 4y)\, y' = -2x - \cos(x)}$$

With initial conditions

$$[y(0) = 1]$$

3.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{2x + \cos(x)}{y\,(3y+4)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{-\infty \leq y < 0, 0 < y < -\frac{4}{3}, -\frac{4}{3} < y \leq \infty\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + \cos(x)}{y(3y+4)}\right) \\ &= \frac{2x + \cos(x)}{y^2(3y+4)} + \frac{6x + 3\cos(x)}{y(3y+4)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{-\infty \leq y < 0, 0 < y < -\frac{4}{3}, -\frac{4}{3} < y \leq \infty\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-2x - \cos(x)}{y(3y+4)}\end{aligned}$$

Where $f(x) = -2x - \cos(x)$ and $g(y) = \frac{1}{y(3y+4)}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y(3y+4)}}\, dy &= -2x - \cos(x)\, dx \\ \int \frac{1}{\frac{1}{y(3y+4)}}\, dy &= \int -2x - \cos(x)\, dx \\ y^3 + 2y^2 &= -x^2 - \sin(x) + c_1\end{aligned}$$

Which results in

$$y = \frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x^2 \cdots}\right.}{6} + \frac{8}{3\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1 \cdots}\right.} - \frac{2}{3}$$

$$y = -\frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x \cdots}\right.}{12} - \frac{4}{3\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c \cdots}\right.} - \frac{2}{3} + \frac{i\sqrt{3}\left(\frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x^2+81\sin(x)^2-162\sin(x)c_1+81c_1^2}\right)^{\frac{1}{3}}}{6} - \cdots\right.}{2}$$

$$y = -\frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x \cdots}\right.}{12} - \frac{4}{3\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c \cdots}\right.} - \frac{2}{3} - \frac{i\sqrt{3}\left(\frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x^2+81\sin(x)^2-162\sin(x)c_1+81c_1^2}\right)^{\frac{1}{3}}}{6} - \cdots\right.}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-i\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{2}{3}}\sqrt{3}+16i\sqrt{3}-\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{2}{3}}-8}{12\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{2}{3}}\sqrt{3}-16i\sqrt{3}-\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{2}{3}}-8(}{12\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{2}{3}}-4\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{1}{3}}+16}{6\left(-64+108c_1+12\sqrt{3}\,\sqrt{27c_1^2-32c_1}\right)^{\frac{1}{3}}}$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

3.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2x+\cos(x)}{y(3y+4)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 175: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{-2x - \cos(x)} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{-2x-\cos(x)}} dx \end{aligned}$$

Which results in

$$S = -x^2 - \sin(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + \cos(x)}{y(3y + 4)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -2x - \cos(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3y^2 + 4y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2 + 4R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^3 + 2R^2 + c_1 \tag{4}$$

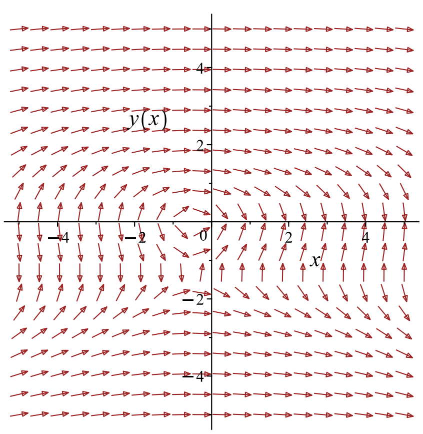
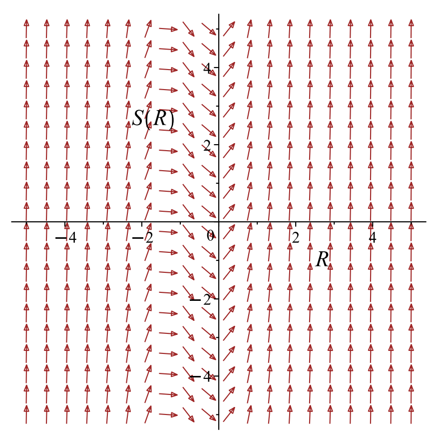
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x^2 - \sin(x) = y^3 + 2y^2 + c_1$$

Which simplifies to

$$-x^2 - \sin(x) = y^3 + 2y^2 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+\cos(x)}{y(3y+4)}$ 	$R = y$ $S = -x^2 - \sin(x)$	$\frac{dS}{dR} = 3R^2 + 4R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 3 + c_1$$

$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

$$-x^2 - \sin(x) = y^3 + 2y^2 - 3$$

Summary
The solution(s) found are the following

$$-x^2 - \sin(x) = y^3 + 2y^2 - 3 \tag{1}$$

Verification of solutions

$$-x^2 - \sin(x) = y^3 + 2y^2 - 3$$

Verified OK.

3.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y(3y+4))\,\mathrm{d}y &= (2x+\cos(x))\,\mathrm{d}x\\ (-2x-\cos(x))\,\mathrm{d}x+(-y(3y+4))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x-\cos(x)\\ N(x,y) &= -y(3y+4)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x-\cos(x))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y(3y+4))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -2x-\cos(x)\,\mathrm{d}x\\ \phi &= -x^2-\sin(x)+f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y(3y+4)$. Therefore equation (4) becomes

$$-y(3y+4) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y(3y+4)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-3y^2 - 4y\right)\mathrm{d}y$$
$$f(y) = -y^3 - 2y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - \sin(x) - y^3 - 2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - \sin(x) - y^3 - 2y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_1$$

$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

$$-x^2 - \sin(x) - y^3 - 2y^2 = -3$$

Summary
The solution(s) found are the following

$$-x^2 - \sin(x) - y^3 - 2y^2 = -3 \tag{1}$$

Verification of solutions

$$-x^2 - \sin(x) - y^3 - 2y^2 = -3$$

Verified OK.

3.12.5 Maple step by step solution

Let's solve

$[(3y^2 + 4y)\, y' = -2x - \cos(x)\,, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (3y^2 + 4y)\, y' dx = \int (-2x - \cos(x))\, dx + c_1$

- Evaluate integral

  $y^3 + 2y^2 = -x^2 - \sin(x) + c_1$

- Solve for y

  $y = \frac{\left(-64-108x^2-108\sin(x)+108c_1+12\sqrt{96x^2+96\sin(x)-96c_1+81x^4+162\sin(x)x^2-162c_1x^2+81\sin(x)^2-162\sin(x)c_1+81c_1^2}\right)^{\frac{1}{3}}}{6}$

- Use initial condition $y(0) = 1$

  $1 = \frac{\left(-64+108c_1+12\sqrt{81c_1^2-96c_1}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(-64+108c_1+12\sqrt{81c_1^2-96c_1}\right)^{\frac{1}{3}}} - \frac{2}{3}$

- Solve for c_1

  $c_1 = RootOf\left(-\left(-64 + 108\_Z + 12\sqrt{3}\,\sqrt{27\_Z^2 - 32\_Z}\right)^{\frac{2}{3}} + 10\left(-64 + 108\_Z + 12\sqrt{3}\,\sqrt{27\_Z^2\right.\right.$

- Substitute $c_1 = RootOf\left(-\left(-64 + 108\_Z + 12\sqrt{3}\,\sqrt{27\_Z^2 - 32\_Z}\right)^{\frac{2}{3}} + 10\left(-64 + 108\_Z + 12\right.\right.$

$$y = -\frac{2\left(-\frac{\left(-64-108x^{2}-108\sin(x)+108\,RootOf\left(-\left(-64+108\_Z+12\sqrt{3}\,\sqrt{\_Z\left(27\_Z-32\right)}\right)^{\frac{2}{3}}+10\left(-64+108\_Z+12\sqrt{3}\,\sqrt{\_Z\left(27\_Z-32\right.}\right.\right.\right.}{}\right.}{}$$

- Solution to the IVP

$$y = -\frac{2\left(-\frac{\left(-64-108x^{2}-108\sin(x)+108\,RootOf\left(-\left(-64+108\_Z+12\sqrt{3}\,\sqrt{\_Z\left(27\_Z-32\right)}\right)^{\frac{2}{3}}+10\left(-64+108\_Z+12\sqrt{3}\,\sqrt{\_Z\left(27\_Z-32\right.}\right.\right.\right.}{}\right.}{}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 102

```
dsolve([(3*y(x)^2+4*y(x))*diff(y(x),x)+2*x+cos(x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(260-108x^2-108\sin(x)+12\sqrt{441-390x^2-390\sin(x)+81x^4+162\sin(x)x^2+81\sin(x)^2}\right)^{\frac{1}{3}}}{6} + \frac{8}{3\left(260-108x^2-108\sin(x)+12\sqrt{441-390x^2-390\sin(x)+81x^4+162\sin(x)x^2+81\sin(x)^2}\right)^{\frac{1}{3}}} - \frac{2}{3}$$

✓ Solution by Mathematica

Time used: 2.549 (sec). Leaf size: 127

```
DSolve[{(3*y[x]^2+4*y[x])*y'[x]+2*x+Cos[x]==0,y[0]==1},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{2^{2/3}\left(-27x^2+\sqrt{(27x^2+27\sin(x)-65)^2-256}-27\sin(x)+65\right)^{2/3} - 4\sqrt[3]{-27x^2+\sqrt{(27x^2+27\sin(}}{6\sqrt[3]{-27x^2+\sqrt{(27x^2+27\sin(x)-65)^2-256}-27\sin(x)+65}}$$

3.13 problem 14

Internal problem ID [940]
Internal file name [`OUTPUT/940_Sunday_June_05_2022_01_54_33_AM_69820171/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "abelFirstKind", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{(1+y)(y-1)(y-2)}{x+1} = 0}$$

With initial conditions

$$[y(1) = 0]$$

3.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{(y+1)(y-1)(y-2)}{x+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x<-1\lor -1<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{(y+1)\,(y-1)\,(y-2)}{x+1}\right)\\&=-\frac{(y-1)\,(y-2)}{x+1}-\frac{(y+1)\,(y-2)}{x+1}-\frac{(y+1)\,(y-1)}{x+1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$\{x<-1\lor -1<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=0$ is inside this domain. Therefore solution exists and is unique.

3.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y'&=F(x,y)\\&=f(x)g(y)\\&=-\frac{y^3-2y^2-y+2}{x+1}\end{aligned}$$

Where $f(x)=-\frac{1}{x+1}$ and $g(y)=y^3-2y^2-y+2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^3-2y^2-y+2}\,dy&=-\frac{1}{x+1}\,dx\\\int\frac{1}{y^3-2y^2-y+2}\,dy&=\int-\frac{1}{x+1}\,dx\\\frac{\ln(y+1)}{6}-\frac{\ln(y-1)}{2}+\frac{\ln(y-2)}{3}&=-\ln(x+1)+c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\frac{\ln(y+1)}{6}-\frac{\ln(y-1)}{2}+\frac{\ln(y-2)}{3}} = e^{-\ln(x+1)+c_1}$$

Which simplifies to

$$\frac{(y+1)^{\frac{1}{6}}(y-2)^{\frac{1}{3}}}{\sqrt{y-1}} = \frac{c_2}{x+1}$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(\left(c_2^6 - x^6 - 6x^5 - 15x^4 - 20x^3 - 15x^2 - 6x - 1\right)\_Z^{18} + \left(-6c_2^6 + 6x^6 + 36x^5 + 90x^4 + 120x^3 + \right.\right. \tag{1}$$
$$-1$$

Verification of solutions

$$y = \text{RootOf}\left(\left(c_2^6 - x^6 - 6x^5 - 15x^4 - 20x^3 - 15x^2 - 6x - 1\right)\_Z^{18} + \left(-6c_2^6 + 6x^6 + 36x^5 + 90x^4 + 120x^3 + \right.\right.$$
$$-1$$

Verified OK.

3.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{(y+1)(y-1)(y-2)}{x+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 178: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x - 1 \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x-1} dx \end{aligned}$$

Which results in

$$S = -\ln(-x-1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(y+1)(y-1)(y-2)}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{-x-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y-2)(y^2-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-2)(R^2-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R+1)}{6} - \frac{\ln(R-1)}{2} + \frac{\ln(R-2)}{3} + c_1 \tag{4}$$

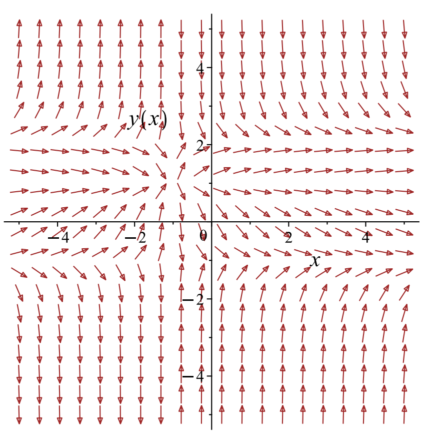
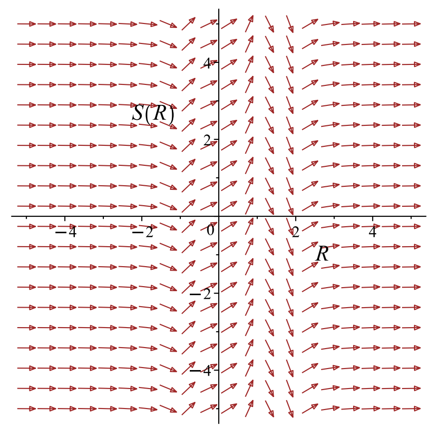
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-x-1) = \frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} + c_1$$

Which simplifies to

$$-\ln(-x-1) = \frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(y+1)(y-1)(y-2)}{x+1}$ 	$R = y$ $S = -\ln(-x-1)$	$\frac{dS}{dR} = \frac{1}{(R-2)(R^2-1)}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) - i\pi = -\frac{i\pi}{6} + \frac{\ln(2)}{3} + c_1$$

$$c_1 = -\frac{5i\pi}{6} - \frac{4\ln(2)}{3}$$

Substituting c_1 found above in the general solution gives

$$-\ln(-x-1) = \frac{\ln(y+1)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} - \frac{5i\pi}{6} - \frac{4\ln(2)}{3}$$

Summary

The solution(s) found are the following

$$-\ln(-x-1) = \frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} - \frac{5i\pi}{6} - \frac{4\ln(2)}{3} \tag{1}$$

Verification of solutions

$$-\ln(-x-1) = \frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} - \frac{5i\pi}{6} - \frac{4\ln(2)}{3}$$

Verified OK.

3.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{(y+1)\,(y-1)\,(y-2)}\right)\mathrm{d}y = \left(\frac{1}{x+1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x+1}\right)\mathrm{d}x + \left(-\frac{1}{(y+1)\,(y-1)\,(y-2)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x+1}$$
$$N(x,y) = -\frac{1}{(y+1)\,(y-1)\,(y-2)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{(y+1)\,(y-1)\,(y-2)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$
$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x+1}\,\mathrm{d}x$$
$$\phi = -\ln(x+1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{1}{(y+1)(y-1)(y-2)}$. Therefore equation (4) becomes

$$-\frac{1}{(y+1)(y-1)(y-2)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{(y+1)(y-1)(y-2)}$$
$$= -\frac{1}{(y-2)(y^2-1)}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{(y-2)(y^2-1)}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y+1)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) - \frac{\ln(y+1)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) - \frac{\ln(y+1)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{4\ln(2)}{3} + \frac{i\pi}{6} = c_1$$

$$c_1 = -\frac{4\ln(2)}{3} + \frac{i\pi}{6}$$

Substituting c_1 found above in the general solution gives

$$-\ln(x+1) - \frac{\ln(y+1)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3} = -\frac{4\ln(2)}{3} + \frac{i\pi}{6}$$

Summary

The solution(s) found are the following

$$-\ln(x+1) - \frac{\ln(1+y)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3} = -\frac{4\ln(2)}{3} + \frac{i\pi}{6} \tag{1}$$

Verification of solutions

$$-\ln(x+1) - \frac{\ln(1+y)}{6} + \frac{\ln(y-1)}{2} - \frac{\ln(y-2)}{3} = -\frac{4\ln(2)}{3} + \frac{i\pi}{6}$$

Verified OK.

3.13.5 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -\frac{y^3}{x+1} + \frac{2y^2}{x+1} + \frac{y}{x+1} - \frac{2}{x+1} \tag{1}$$

Therefore

$$
\begin{aligned}
f_0(x) &= -\frac{2}{x+1} \\
f_1(x) &= \frac{1}{x+1} \\
f_2(x) &= \frac{2}{x+1} \\
f_3(x) &= -\frac{1}{x+1}
\end{aligned}
$$

Since $f_2(x) = \frac{2}{x+1}$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$
\begin{aligned}
y &= u(x) - \left(\frac{\frac{2}{x+1}}{-\frac{3}{x+1}}\right) \\
&= u(x) + \frac{2}{3}
\end{aligned}
$$

The above transformation applied to (1) gives a new ODE as

$$
u'(x) = \frac{7u(x)}{3\,(x+1)} - \frac{20}{27\,(x+1)} - \frac{u(x)^3}{x+1} \tag{2}
$$

The above ODE (2) can now be solved as separable.

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\frac{7}{3}u - \frac{20}{27} - u^3}{x+1}
\end{aligned}
$$

Where $f(x) = \frac{1}{x+1}$ and $g(u) = \frac{7}{3}u - \frac{20}{27} - u^3$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{7}{3}u - \frac{20}{27} - u^3}\,du &= \frac{1}{x+1}\,dx \\
\int \frac{1}{\frac{7}{3}u - \frac{20}{27} - u^3}\,du &= \int \frac{1}{x+1}\,dx \\
\frac{\ln(3u-1)}{2} - \frac{\ln(3u+5)}{6} - \frac{\ln(3u-4)}{3} &= \ln(x+1) + c_3
\end{aligned}
$$

Raising both side to exponential gives

$$e^{\frac{\ln(3u-1)}{2}-\frac{\ln(3u+5)}{6}-\frac{\ln(3u-4)}{3}} = e^{\ln(x+1)+c_3}$$

Which simplifies to

$$\frac{\sqrt{3u-1}}{(3u+5)^{\frac{1}{6}}(3u-4)^{\frac{1}{3}}} = c_4(x+1)$$

Now we transform the solution $u(x) = \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5}{3}$ $\frac{4}{3}$ to y using $y = u(x) - \frac{f_2}{3f_3}$, which gives

$$y = \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5+\right.}{3}$$

$$= \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5+\right.}{3}$$

$$= \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5+\right.}{3}$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5+1\right.}{3} + \frac{10}{3} \tag{1}$$

Verification of solutions

$$y = \frac{\operatorname{RootOf}\left(-27+(c_4^6x^6+6c_4^6x^5+15c_4^6x^4+20c_4^6x^3+15c_4^6x^2+6c_4^6x+c_4^6-1)\_Z^9+(9c_4^6x^6+54c_4^6x^5+1\right.}{3} + \frac{10}{3}$$

Warning, solution could not be verified

3.13.6 Maple step by step solution

Let's solve

$\left[y' + \frac{(1+y)(y-1)(y-2)}{x+1} = 0, y(1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1+y)(y-1)(y-2)} = -\frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y)(y-1)(y-2)}dx = \int -\frac{1}{x+1}dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} = -\ln(x+1) + c_1$

- Use initial condition $y(1) = 0$

  $-\frac{I\pi}{6} + \frac{\ln(2)}{3} = -\ln(2) + c_1$

- Solve for c_1

  $c_1 = \frac{4\ln(2)}{3} - \frac{I\pi}{6}$

- Substitute $c_1 = \frac{4\ln(2)}{3} - \frac{I\pi}{6}$ into general solution and simplify

  $\frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} = -\ln(x+1) + \frac{4\ln(2)}{3} - \frac{I\pi}{6}$

- Solution to the IVP

  $\frac{\ln(1+y)}{6} - \frac{\ln(y-1)}{2} + \frac{\ln(y-2)}{3} = -\ln(x+1) + \frac{4\ln(2)}{3} - \frac{I\pi}{6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 6.86 (sec). Leaf size: 633

```
dsolve([diff(y(x),x)+((y(x)+1)*(y(x)-1)*(y(x)-2))/(x+1)=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{8\,\mathrm{RootOf}\left(512\_Z^6 + \left(96\,2^{\frac{1}{3}}x^2 + 192x2^{\frac{1}{3}} + 96\,2^{\frac{1}{3}}\right)\_Z^4 - x^6 - 6x^5 - 15x^4 - 20x^3 - 15x^2 - 6x - 257\right)^2}{8\,\mathrm{RootOf}\left(512\_Z^6 + \left(96\,2^{\frac{1}{3}}x^2 + 192x2^{\frac{1}{3}} + 96\,2^{\frac{1}{3}}\right)\_Z^4 - x^6 - 6x^5 - 15x^4 - 20x^3 - 15x^2 - 6x -}$$

$$y(x) = \frac{16\,\mathrm{RootOf}\left(512\_Z^6 + \left(48ix^2\sqrt{3}\,2^{\frac{1}{3}} + 96ix\sqrt{3}\,2^{\frac{1}{3}} + 48i\sqrt{3}\,2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}x^2 - 96x2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}\right)\_Z^4 - x^6 -}{16\,\mathrm{RootOf}\left(512\_Z^6 + \left(48ix^2\sqrt{3}\,2^{\frac{1}{3}} + 96ix\sqrt{3}\,2^{\frac{1}{3}} + 48i\sqrt{3}\,2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}x^2 - 96x2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}\right)}$$

$$y(x) = \frac{16\,\mathrm{RootOf}\left(512\_Z^6 + \left(-48ix^2\sqrt{3}\,2^{\frac{1}{3}} - 96ix\sqrt{3}\,2^{\frac{1}{3}} - 48i\sqrt{3}\,2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}x^2 - 96x2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}\right)\_Z^4 - x^6}{16\,\mathrm{RootOf}\left(512\_Z^6 + \left(-48ix^2\sqrt{3}\,2^{\frac{1}{3}} - 96ix\sqrt{3}\,2^{\frac{1}{3}} - 48i\sqrt{3}\,2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}x^2 - 96x2^{\frac{1}{3}} - 48\,2^{\frac{1}{3}}\right.}$$

✓ Solution by Mathematica

Time used: 60.912 (sec). Leaf size: 1618

```
DSolve[{y'[x]+((y[x]+1)*(y[x]-1)*(y[x]-2))/(x+1)==0,y[1]==0},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\left(-1 - i\sqrt{3}\right)x^{12} - 12i\left(-i + \sqrt{3}\right)x^{11} - 66i\left(-i + \sqrt{3}\right)x^{10} - 220i\left(-i + \sqrt{3}\right)x^9 - 495i\left(-i + \sqrt{3}\right)x^8 - 7}{}$$

3.14 problem 15

Internal problem ID [941]
Internal file name [`OUTPUT/941_Sunday_June_05_2022_01_54_42_AM_8081077/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 2x(1 + y) = 0}$$

With initial conditions

$$[y(0) = 2]$$

3.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = -2x$$

Hence the ode is

$$2yx + y' = -2x$$

The domain of $p(x) = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(-2y - 2) \end{aligned}$$

Where $f(x) = x$ and $g(y) = -2y - 2$. Integrating both sides gives

$$\frac{1}{-2y - 2} \, dy = x \, dx$$

$$\int \frac{1}{-2y - 2} \, dy = \int x \, dx$$

$$-\frac{\ln(y + 1)}{2} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{y + 1}} = e^{\frac{x^2}{2} + c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{y + 1}} = c_2 e^{\frac{x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{-e^{-2c_1} e^{2c_1} c_2^2 + e^{-2c_1}}{c_2^2}$$

$$c_1 = -\frac{\ln(3c_2^2)}{2}$$

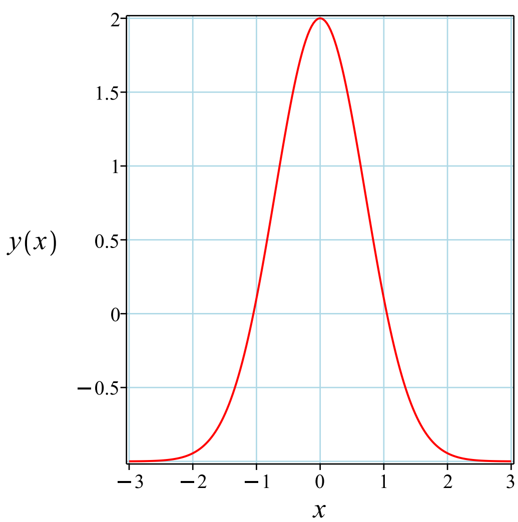
Substituting c_1 found above in the general solution gives

$$y = -1 + 3\,e^{-x^2}$$

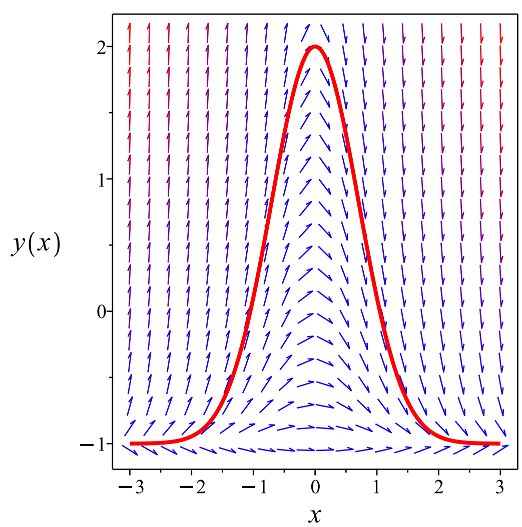
<u>Summary</u>

The solution(s) found are the following

$$y = -1 + 3\,e^{-x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -1 + 3\,e^{-x^2}$$

Verified OK.

3.14.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int 2x dx} \\ = e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-2x) \\ \frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)(-2x) \\ d\left(e^{x^2} y\right) = \left(-2\,e^{x^2} x\right) dx$$

Integrating gives

$$e^{x^2} y = \int -2\,e^{x^2} x \, dx \\ e^{x^2} y = -e^{x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = -e^{-x^2} e^{x^2} + c_1 e^{-x^2}$$

which simplifies to

$$y = -1 + c_1 e^{-x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 - 1$$

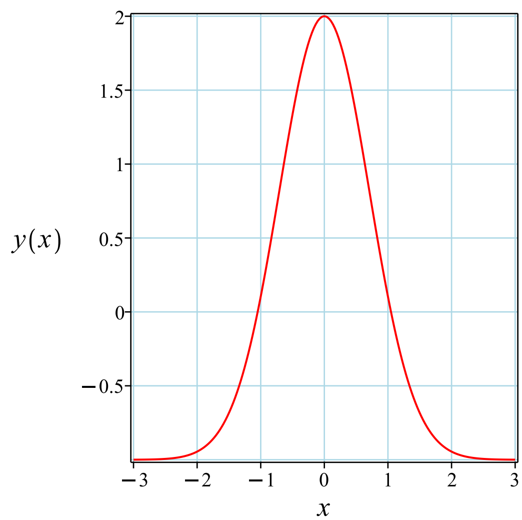
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

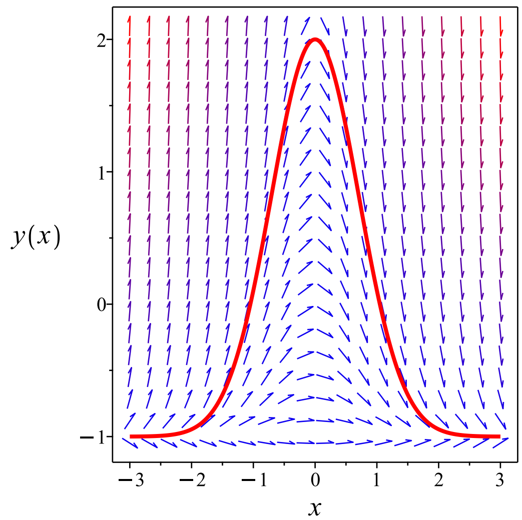
$$y = -1 + 3\,e^{-x^2}$$

Summary
The solution(s) found are the following

$$y = -1 + 3\,e^{-x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -1 + 3\,\mathrm{e}^{-x^2}$$

Verified OK.

3.14.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2x(y+1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 181: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2x(y+1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\,\mathrm{e}^{x^2} y \\ S_y &= \mathrm{e}^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\,\mathrm{e}^{x^2} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\,\mathrm{e}^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = -e^{x^2} + c_1$$

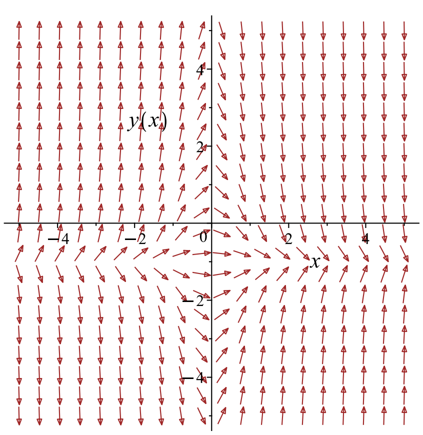
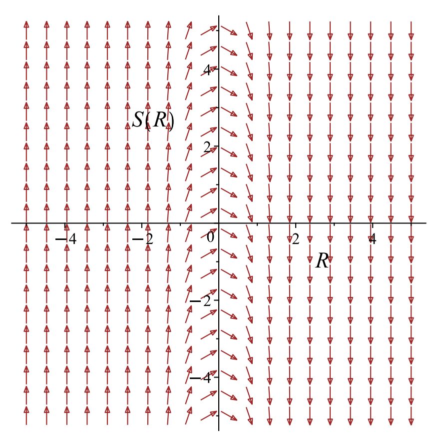
Which simplifies to

$$e^{x^2} y = -e^{x^2} + c_1$$

Which gives

$$y = -\left(e^{x^2} - c_1\right) e^{-x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2x(y+1)$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = -2 e^{R^2} R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 - 1$$

$$c_1 = 3$$

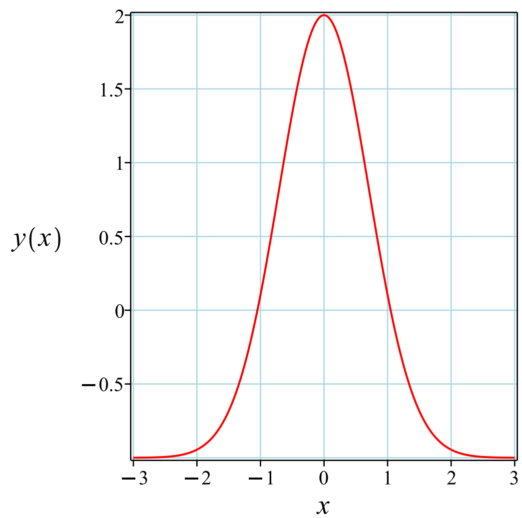
Substituting c_1 found above in the general solution gives

$$y = -1 + 3\,e^{-x^2}$$

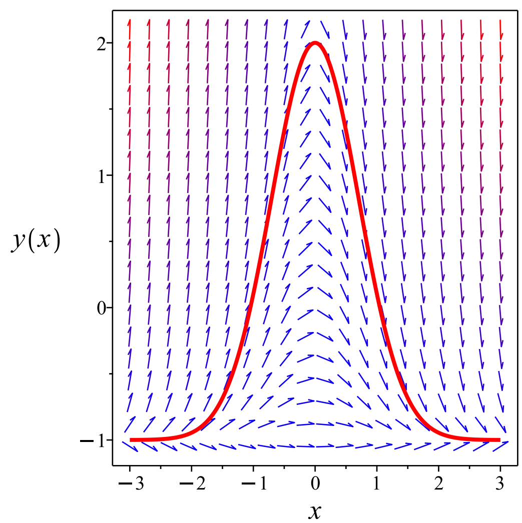
Summary

The solution(s) found are the following

$$y = -1 + 3\,e^{-x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + 3\,e^{-x^2}$$

Verified OK.

3.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-2y-2}\right) \mathrm{d}y &= (x)\, \mathrm{d}x \\ (-x)\, \mathrm{d}x + \left(\frac{1}{-2y-2}\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= \frac{1}{-2y-2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-2y-2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-2y-2}$. Therefore equation (4) becomes

$$
\frac{1}{-2y-2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2(y+1)}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y+2}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y+1)}{2}$$

The solution becomes

$$y = \mathrm{e}^{-x^2-2c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{-2c_1} - 1$$

$$c_1 = -\frac{\ln(3)}{2}$$

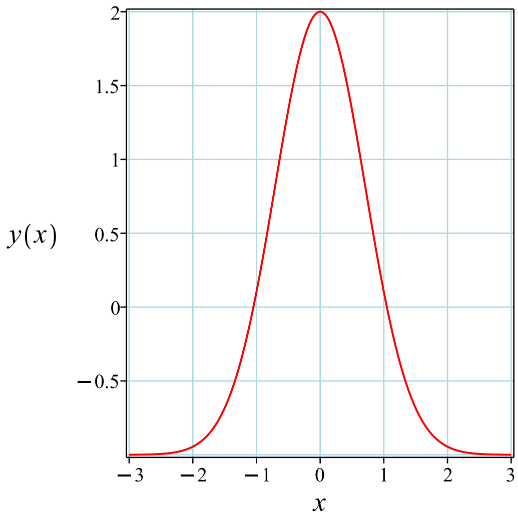
Substituting c_1 found above in the general solution gives

$$y = -1 + 3\,\mathrm{e}^{-x^2}$$

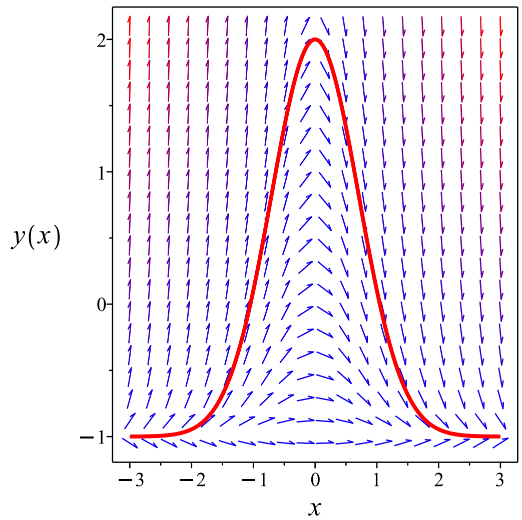
Summary

The solution(s) found are the following

$$y = -1 + 3\,\mathrm{e}^{-x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + 3\,\mathrm{e}^{-x^2}$$

Verified OK.

3.14.6 Maple step by step solution

Let's solve

$[y' + 2x(1+y) = 0, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int -2x dx + c_1$

- Evaluate integral

  $\ln(1+y) = -x^2 + c_1$

- Solve for y

  $y = \mathrm{e}^{-x^2+c_1} - 1$

- Use initial condition $y(0) = 2$

  $2 = -1 + e^{c_1}$

- Solve for c_1

  $c_1 = \ln(3)$

- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = -1 + 3\,e^{-x^2}$

- Solution to the IVP

  $y = -1 + 3\,e^{-x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+2*x*(y(x)+1)=0,y(0) = 2],y(x), singsol=all)
```

$$y(x) = -1 + 3\,e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 16

```
DSolve[{y'[x]+2*x*(y[x]+1)==0,y[0]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3e^{-x^2} - 1$$

3.15 problem 16

Internal problem ID [942]
Internal file name [`OUTPUT/942_Sunday_June_05_2022_01_54_44_AM_26322134/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2xy(1 + y^2) = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 2xy(y^2 + 1) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2xy\left(y^2+1\right)\right)\\ &= 2\left(y^2+1\right)x + 4x\,y^2\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= 2xy\left(y^2+1\right)\end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y(y^2+1)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y\left(y^2+1\right)}\,dy &= 2x\,dx\\ \int \frac{1}{y\left(y^2+1\right)}\,dy &= \int 2x\,dx\\ \ln\left(y\right) - \frac{\ln\left(y^2+1\right)}{2} &= x^2 + c_1\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y)-\frac{\ln\left(y^2+1\right)}{2}} = \mathrm{e}^{x^2+c_1}$$

Which simplifies to

$$\frac{y}{\sqrt{y^2+1}} = c_2 e^{x^2}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{-\frac{1}{c_2^2-1}}\, c_2$$

$$c_2 = \frac{\sqrt{2}}{2}$$

Substituting c_2 found above in the general solution gives

$$y = \lim_{c_2 \to \frac{\sqrt{2}}{2}} c_2 e^{x^2} \sqrt{-\frac{1}{c_2^2 e^{2x^2}-1}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \lim_{c_2 \to \frac{\sqrt{2}}{2}} c_2 e^{x^2} \sqrt{-\frac{1}{c_2^2 e^{2x^2}-1}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \lim_{c_2 \to \frac{\sqrt{2}}{2}} c_2 e^{x^2} \sqrt{-\frac{1}{c_2^2 e^{2x^2}-1}}$$

Verified OK.

3.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2xy(y^2+1)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 184: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{2x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2x}} dx \end{aligned}$$

Which results in

$$S = x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2xy\left(y^2 + 1\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\,(y^2 + 1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\,(R^2 + 1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) - \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

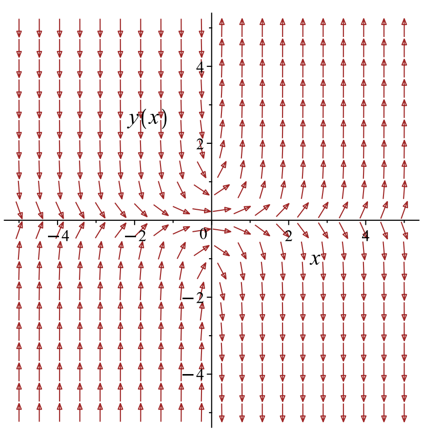
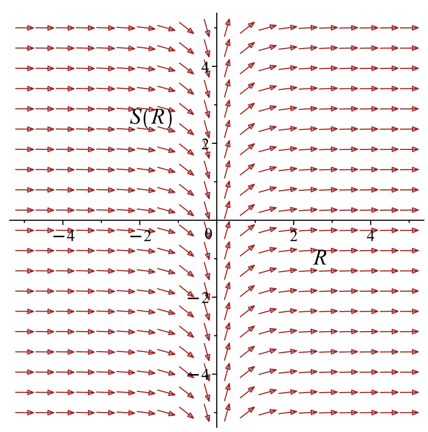
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$x^2 = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2xy(y^2+1)$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = \frac{1}{R(R^2+1)}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{\ln(2)}{2} + c_1$$

$$c_1 = \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$x^2 = \ln(y) - \frac{\ln(y^2+1)}{2} + \frac{\ln(2)}{2}$$

Summary
The solution(s) found are the following

$$x^2 = \ln(y) - \frac{\ln(1+y^2)}{2} + \frac{\ln(2)}{2} \tag{1}$$

Verification of solutions

$$x^2 = \ln(y) - \frac{\ln(1+y^2)}{2} + \frac{\ln(2)}{2}$$

Verified OK.

3.15.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 2xy(y^2+1) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 2xy + 2xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 2x \\ f_1(x) &= 2x \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{2x}{y^2} + 2x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= 2w(x)\, x + 2x \\ w' &= -4xw - 4x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4x \\ q(x) &= -4x \end{aligned}$$

Hence the ode is

$$w'(x) + 4w(x)\, x = -4x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 4x dx} \\ &= e^{2x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\,(-4x) \\ \frac{d}{dx}\left(e^{2x^2} w\right) &= \left(e^{2x^2}\right)(-4x) \\ d\left(e^{2x^2} w\right) &= \left(-4x\, e^{2x^2}\right)\, dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{2x^2} w &= \int -4x\, e^{2x^2}\, dx \\
e^{2x^2} w &= -e^{2x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2x^2}$ results in

$$
w(x) = -e^{-2x^2} e^{2x^2} + c_1 e^{-2x^2}
$$

which simplifies to

$$
w(x) = -1 + c_1 e^{-2x^2}
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = -1 + c_1 e^{-2x^2}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = c_1 - 1
$$

$$
c_1 = 2
$$

Substituting c_1 found above in the general solution gives

$$
\frac{1}{y^2} = -1 + 2\, e^{-2x^2}
$$

The above simplifies to

$$
-2\, e^{-2x^2} y^2 + y^2 + 1 = 0
$$

Summary

The solution(s) found are the following

$$
-2\, e^{-2x^2} y^2 + y^2 + 1 = 0 \tag{1}
$$

Verification of solutions

$$
-2\, e^{-2x^2} y^2 + y^2 + 1 = 0
$$

Verified OK.

3.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y\,(y^2+1)}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{2y\,(y^2+1)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{2y\left(y^2+1\right)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y\left(y^2+1\right)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y(y^2+1)}$. Therefore equation (4) becomes

$$\frac{1}{2y\left(y^2+1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y\left(y^2+1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y\left(y^2+1\right)}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y)}{2} - \frac{\ln\left(y^2+1\right)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y)}{2} - \frac{\ln\left(y^2+1\right)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y)}{2} - \frac{\ln\left(y^2+1\right)}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(2)}{4} = c_1$$

$$c_1 = -\frac{\ln(2)}{4}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} + \frac{\ln(y)}{2} - \frac{\ln\left(y^2+1\right)}{4} = -\frac{\ln(2)}{4}$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2}+\frac{\ln(y)}{2}-\frac{\ln(1+y^2)}{4}=-\frac{\ln(2)}{4} \tag{1}$$

Verification of solutions

$$-\frac{x^2}{2}+\frac{\ln(y)}{2}-\frac{\ln(1+y^2)}{4}=-\frac{\ln(2)}{4}$$

Verified OK.

3.15.6 Maple step by step solution

Let's solve

$[y'-2xy(1+y^2)=0, y(0)=1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1+y^2)}=2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(1+y^2)}dx=\int 2xdx+c_1$

- Evaluate integral

  $\ln(y)-\frac{\ln(1+y^2)}{2}=x^2+c_1$

- Solve for y

  $\left\{y=\frac{\sqrt{-\left(\mathrm{e}^{2x^2+2c_1}-1\right)\mathrm{e}^{2x^2+2c_1}}}{\mathrm{e}^{2x^2+2c_1}-1}, y=-\frac{\sqrt{-\left(\mathrm{e}^{2x^2+2c_1}-1\right)\mathrm{e}^{2x^2+2c_1}}}{\mathrm{e}^{2x^2+2c_1}-1}\right\}$

- Use initial condition $y(0)=1$

  $1=\frac{\sqrt{-(\mathrm{e}^{2c_1}-1)\mathrm{e}^{2c_1}}}{\mathrm{e}^{2c_1}-1}$

- Solution does not satisfy initial condition
- Use initial condition $y(0)=1$

  $1=-\frac{\sqrt{-(\mathrm{e}^{2c_1}-1)\mathrm{e}^{2c_1}}}{\mathrm{e}^{2c_1}-1}$

- Solve for c_1

$$c_1 = -\frac{\ln(2)}{2}$$

- Substitute $c_1 = -\frac{\ln(2)}{2}$ into general solution and simplify

$$y = -\frac{\sqrt{-\left(\left(e^{x^2}\right)^2-2\right)\left(e^{x^2}\right)^2}}{e^{2x^2}-2}$$

- Solution to the IVP

$$y = -\frac{\sqrt{-\left(\left(e^{x^2}\right)^2-2\right)\left(e^{x^2}\right)^2}}{e^{2x^2}-2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 16

```
dsolve([diff(y(x),x)=2*x*y(x)*(1+y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{2\,e^{-2x^2}-1}}$$

✓ Solution by Mathematica

Time used: 60.086 (sec). Leaf size: 27

```
DSolve[{y'[x]==2*x*y[x]*(1+y[x]^2),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{ie^{x^2}}{\sqrt{e^{2x^2}-2}}$$

3.16 problem 17

Internal problem ID [943]
Internal file name [`OUTPUT/943_Sunday_June_05_2022_01_54_46_AM_76131040/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x^2+2)\,y' - 4x(y^2+2y+1) = 0}$$

3.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(4y^2+8y+4)}{x^2+2} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+2}$ and $g(y) = 4y^2+8y+4$. Integrating both sides gives

$$\frac{1}{4y^2+8y+4}\,dy = \frac{x}{x^2+2}\,dx$$

$$\int \frac{1}{4y^2+8y+4}\,dy = \int \frac{x}{x^2+2}\,dx$$
$$-\frac{1}{4\,(y+1)} = \frac{\ln\left(x^2+2\right)}{2}+c_1$$

Which results in

$$y = -\frac{2\ln\left(x^2+2\right)+4c_1+1}{2\left(\ln\left(x^2+2\right)+2c_1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\ln\left(x^2+2\right)+4c_1+1}{2\left(\ln\left(x^2+2\right)+2c_1\right)} \tag{1}$$

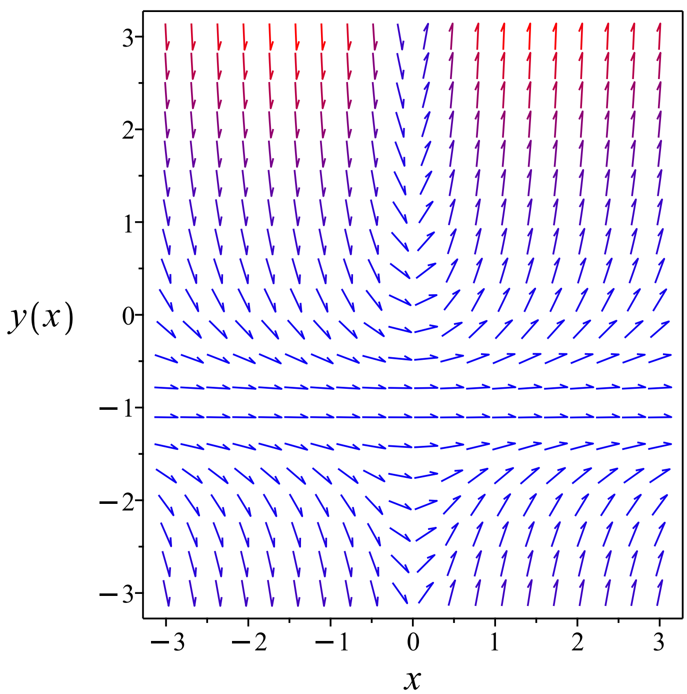


Figure 195: Slope field plot

Verification of solutions

$$y = -\frac{2\ln\left(x^2+2\right)+4c_1+1}{2\left(\ln\left(x^2+2\right)+2c_1\right)}$$

Verified OK.

3.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{4x(y^2+2y+1)}{x^2+2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 187: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x^2+2}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2+2}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(x^2+2)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{4x(y^2+2y+1)}{x^2+2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{x^2+2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{4\left(y+1\right)^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{4\left(R+1\right)^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{4\left(R+1\right)} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(x^2+2\right)}{2} = -\frac{1}{4\left(1+y\right)} + c_1
$$

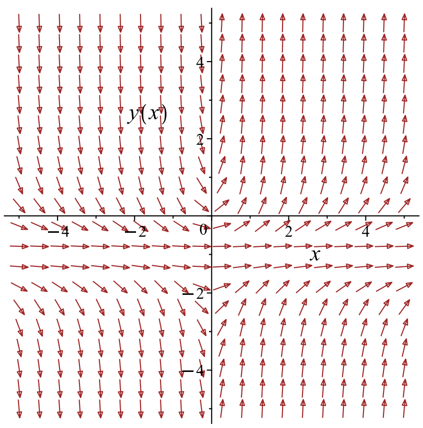
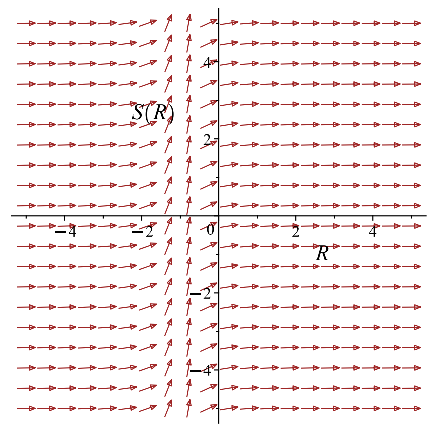
Which simplifies to

$$
\frac{\ln\left(x^2+2\right)}{2} = -\frac{1}{4\left(1+y\right)} + c_1
$$

Which gives

$$
y = -\frac{2\ln\left(x^2+2\right) - 4c_1 + 1}{2\left(\ln\left(x^2+2\right) - 2c_1\right)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4x(y^2+2y+1)}{x^2+2}$ 	$R = y$ $S = \frac{\ln(x^2+2)}{2}$	$\frac{dS}{dR} = \frac{1}{4(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2\ln(x^2+2) - 4c_1 + 1}{2(\ln(x^2+2) - 2c_1)} \tag{1}$$

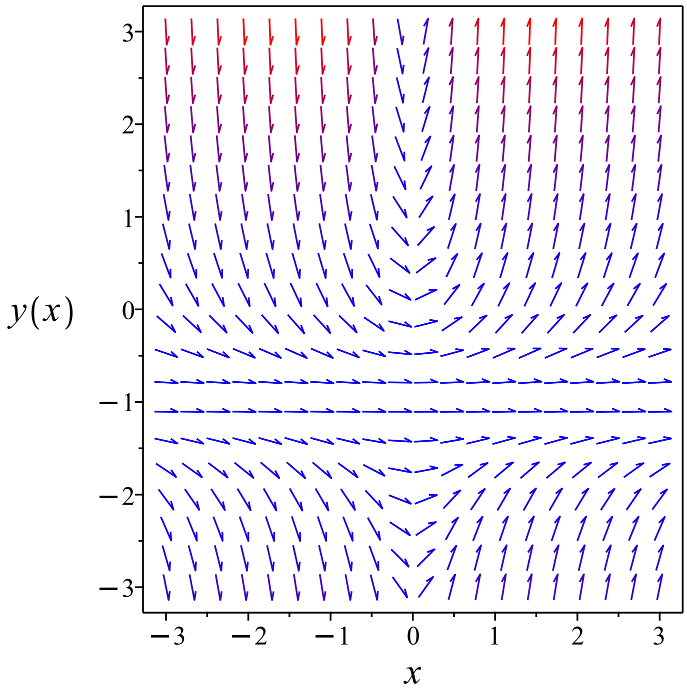


Figure 196: Slope field plot

Verification of solutions

$$y = -\frac{2\ln\left(x^2+2\right)-4c_1+1}{2\left(\ln\left(x^2+2\right)-2c_1\right)}$$

Verified OK.

3.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{4y^2 + 8y + 4}\right) \mathrm{d}y = \left(\frac{x}{x^2 + 2}\right) \mathrm{d}x$$
$$\left(-\frac{x}{x^2 + 2}\right) \mathrm{d}x + \left(\frac{1}{4y^2 + 8y + 4}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2 + 2}$$
$$N(x, y) = \frac{1}{4y^2 + 8y + 4}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2 + 2}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{4y^2+8y+4}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^2+2)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{4y^2+8y+4}$. Therefore equation (4) becomes

$$
\frac{1}{4y^2+8y+4} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{4y^2+8y+4}\\
&= \frac{1}{4(y+1)^2}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{4(y+1)^2}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{4(y+1)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+2)}{2} - \frac{1}{4(y+1)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+2)}{2} - \frac{1}{4(y+1)}$$

The solution becomes

$$y = -\frac{2\ln(x^2+2) + 4c_1 + 1}{2(\ln(x^2+2) + 2c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\ln(x^2+2) + 4c_1 + 1}{2(\ln(x^2+2) + 2c_1)} \tag{1}$$

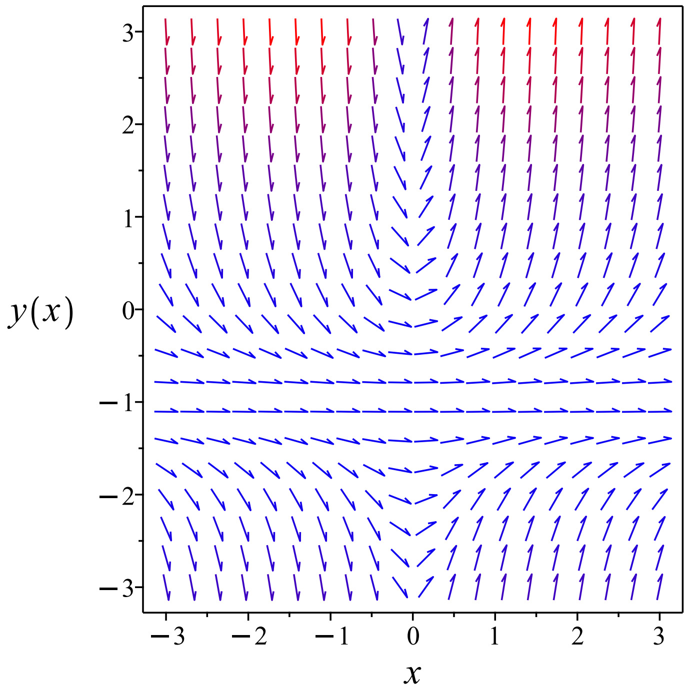


Figure 197: Slope field plot

Verification of solutions

$$y = -\frac{2\ln(x^2+2)+4c_1+1}{2(\ln(x^2+2)+2c_1)}$$

Verified OK.

3.16.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{4x(y^2+2y+1)}{x^2+2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{4x\,y^2}{x^2+2} + \frac{8xy}{x^2+2} + \frac{4x}{x^2+2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{4x}{x^2+2}$, $f_1(x) = \frac{8x}{x^2+2}$ and $f_2(x) = \frac{4x}{x^2+2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{4xu}{x^2+2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{4}{x^2+2} - \frac{8x^2}{(x^2+2)^2} \\ f_1 f_2 &= \frac{32x^2}{(x^2+2)^2} \\ f_2^2 f_0 &= \frac{64x^3}{(x^2+2)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{4xu''(x)}{x^2+2} - \left(\frac{4}{x^2+2} + \frac{24x^2}{(x^2+2)^2}\right) u'(x) + \frac{64x^3 u(x)}{(x^2+2)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (x^2+2)^2 \left(\ln(x^2+2)\, c_1 + c_2\right)$$

The above shows that

$$u'(x) = 4x\left(\ln(x^2+2)\, c_1 + \frac{c_1}{2} + c_2\right)(x^2+2)$$

Using the above in (1) gives the solution

$$y = -\frac{\ln(x^2+2)\, c_1 + \frac{c_1}{2} + c_2}{\ln(x^2+2)\, c_1 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2 \ln \left(x^2+2\right) c_3 - c_3 - 2}{2 \ln \left(x^2+2\right) c_3 + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{-2 \ln \left(x^2+2\right) c_3 - c_3 - 2}{2 \ln \left(x^2+2\right) c_3 + 2} \tag{1}$$

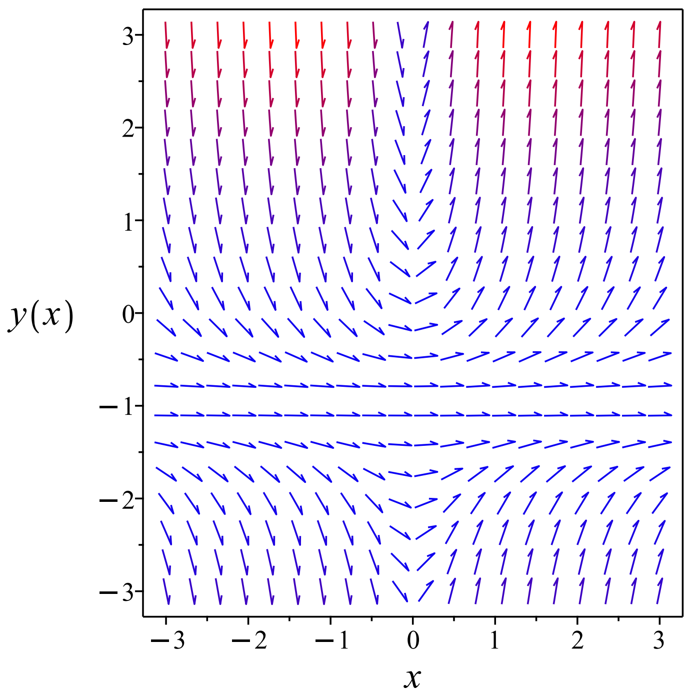


Figure 198: Slope field plot

Verification of solutions

$$y = \frac{-2 \ln \left(x^2+2\right) c_3 - c_3 - 2}{2 \ln \left(x^2+2\right) c_3 + 2}$$

Verified OK.

3.16.5 Maple step by step solution

Let's solve

$(x^2+2)\,y'-4x(y^2+2y+1)=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+2y+1}=\frac{4x}{x^2+2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+2y+1}dx=\int \frac{4x}{x^2+2}dx+c_1$

- Evaluate integral

  $-\frac{1}{1+y}=2\ln(x^2+2)+c_1$

- Solve for y

  $y=-\frac{2\ln(x^2+2)+c_1+1}{2\ln(x^2+2)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)*(x^2+2)=4*x*(y(x)^2+2*y(x)+1),y(x), singsol=all)
```

$$y(x)=\frac{-2\ln(x^2+2)-4c_1-1}{2\ln(x^2+2)+4c_1}$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 37

```
DSolve[y'[x]*(x^2+2)==4*x*(y[x]^2+2*y[x]+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2\log(x^2+2)+1+c_1}{2\log(x^2+2)+c_1}$$
$$y(x) \to -1$$

3.17 problem 18

Internal problem ID [944]
Internal file name [`OUTPUT/944_Sunday_June_05_2022_01_54_47_AM_93818467/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "abelFirstKind", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 2x(y^3 - 3y + 2) = 0}$$

With initial conditions

$$[y(0) = 3]$$

3.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -2x(y^3 - 3y + 2) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-2x\left(y^3 - 3y + 2\right)\right) \\ &= -2x\left(3y^2 - 3\right) \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

3.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\left(-2y^3 + 6y - 4\right) \end{aligned}$$

Where $f(x) = x$ and $g(y) = -2y^3 + 6y - 4$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2y^3 + 6y - 4}\,dy &= x\,dx \\ \int \frac{1}{-2y^3 + 6y - 4}\,dy &= \int x\,dx \\ \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18} &= \frac{x^2}{2} + c_1 \end{aligned}$$

Which results in

$$y = \mathrm{e}^{\mathrm{RootOf}\left(9x^2 \mathrm{e}^{-}{}^{Z} - \ln\left(\mathrm{e}^{-}{}^{Z} - 3\right)\mathrm{e}^{-}{}^{Z} + 18c_1 \mathrm{e}^{-}{}^{Z} + \_Z \mathrm{e}^{-}{}^{Z} - 27x^2 + 3\ln\left(\mathrm{e}^{-}{}^{Z} - 3\right) - 54c_1 - 3\_Z - 3\right)} - 2$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$y = e^{\text{RootOf}\left(9x^2 e^{\_Z} - \ln\left(e^{\_Z}-3\right) e^{\_Z} + 18c_1 e^{\_Z} + \_Z e^{\_Z} - 27x^2 + 3\ln\left(e^{\_Z}-3\right) - 54c_1 - 3\_Z - 3\right)} - 2 \tag{1}$$

Verification of solutions

$$y = e^{\text{RootOf}\left(9x^2 e^{\_Z} - \ln\left(e^{\_Z}-3\right) e^{\_Z} + 18c_1 e^{\_Z} + \_Z e^{\_Z} - 27x^2 + 3\ln\left(e^{\_Z}-3\right) - 54c_1 - 3\_Z - 3\right)} - 2$$

Verified OK.

3.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2x\left(y^3 - 3y + 2\right) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 190: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2x(y^3 - 3y + 2)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2y^3 - 6y + 4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R^3 - 6R + 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{6R-6} + \frac{\ln(R-1)}{18} - \frac{\ln(R+2)}{18} + c_1 \tag{4}$$

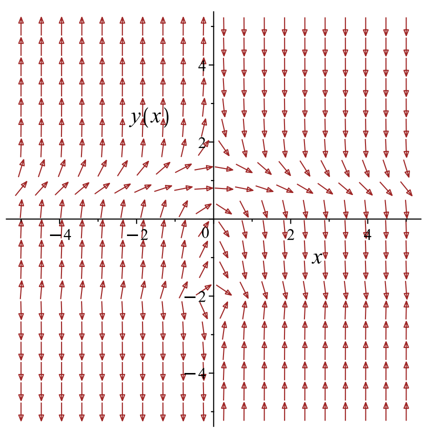
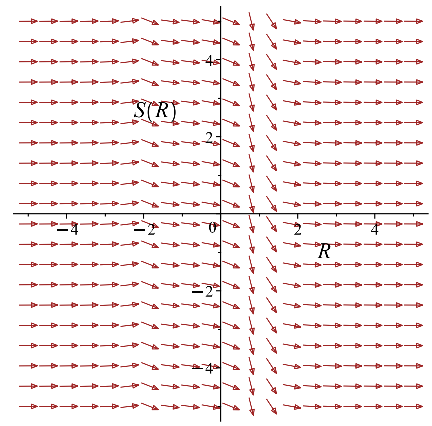
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2x(y^3 - 3y + 2)$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = -\frac{1}{2R^3-6R+4}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{18} - \frac{\ln(5)}{18} + c_1 + \frac{1}{12}$$

$$c_1 = -\frac{1}{12} - \frac{\ln(2)}{18} + \frac{\ln(5)}{18}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{2} = \frac{-2\ln(2)\,y + 2\ln(5)\,y + 2\ln(y-1)\,y - 2\ln(2+y)\,y + 2\ln(2) - 2\ln(5) - 2\ln(y-1) + 2\ln(2+}{36y - 36}$$

The above simplifies to

$$18y\,x^2 + 2\ln(2)\,y - 2\ln(5)\,y - 2\ln(y-1)\,y + 2\ln(2+y)\,y - 18x^2 - 2\ln(2) + 2\ln(5) + 2\ln(y-1) - 2$$

Summary

The solution(s) found are the following

$$\begin{aligned}&(-2+2y)\ln(2+y) + (-2y+2)\ln(y-1) + (-2+2y)\ln(2)\\&+(-2y+2)\ln(5) + 18x^2y - 18x^2 + 3y - 9 = 0\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}&(-2+2y)\ln(2+y) + (-2y+2)\ln(y-1) + (-2+2y)\ln(2)\\&+(-2y+2)\ln(5) + 18x^2y - 18x^2 + 3y - 9 = 0\end{aligned}$$

Verified OK.

3.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-2y^3+6y-4}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{-2y^3+6y-4}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{-2y^3+6y-4}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-2y^3+6y-4}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-2y^3+6y-4}$. Therefore equation (4) becomes

$$\frac{1}{-2y^3 + 6y - 4} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{2\,(y^3 - 3y + 2)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{2y^3 - 6y + 4}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{6y - 6} + \frac{\ln (y - 1)}{18} - \frac{\ln (2 + y)}{18} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{12} + \frac{\ln(2)}{18} - \frac{\ln(5)}{18} = c_1$$

$$c_1 = \frac{1}{12} + \frac{\ln(2)}{18} - \frac{\ln(5)}{18}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} + \frac{1}{6y-6} + \frac{\ln(y-1)}{18} - \frac{\ln(2+y)}{18} = \frac{1}{12} + \frac{\ln(2)}{18} - \frac{\ln(5)}{18}$$

The above simplifies to

$$-18y\,x^2 - 2\ln(2)\,y + 2\ln(5)\,y + 2\ln(y-1)\,y - 2\ln(2+y)\,y + 18x^2 + 2\ln(2) - 2\ln(5) - 2\ln(y-1) +$$

Summary

The solution(s) found are the following

$$(-2y+2)\ln(2+y) + (-2+2y)\ln(y-1) + (-2y+2)\ln(2) + (-2+2y)\ln(5) - 18x^2y + 18x^2 - 3y + 9 = 0 \tag{1}$$

Verification of solutions

$$(-2y+2)\ln(2+y) + (-2+2y)\ln(y-1) + (-2y+2)\ln(2) + (-2+2y)\ln(5) - 18x^2y + 18x^2 - 3y + 9 = 0$$

Verified OK.

3.17.5 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -2xy^3 + 6yx - 4x \tag{1}$$

Therefore

$$\begin{aligned} f_0(x) &= -4x \\ f_1(x) &= 6x \\ f_2(x) &= 0 \\ f_3(x) &= -2x \end{aligned}$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$\frac{27}{4}$$

Since the Abel invariant does not depend on x then this ode can be solved directly.

3.17.6 Maple step by step solution

Let's solve

$[y' + 2x(y^3 - 3y + 2) = 0, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3-3y+2} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3-3y+2}dx = \int -2xdx + c_1$

- Evaluate integral

  $-\frac{1}{3(y-1)}-\frac{\ln(y-1)}{9}+\frac{\ln(2+y)}{9}=-x^2+c_1$

- Use initial condition $y(0)=3$

  $-\frac{1}{6}-\frac{\ln(2)}{9}+\frac{\ln(5)}{9}=c_1$

- Solve for c_1

  $c_1=-\frac{1}{6}-\frac{\ln(2)}{9}+\frac{\ln(5)}{9}$

- Substitute $c_1=-\frac{1}{6}-\frac{\ln(2)}{9}+\frac{\ln(5)}{9}$ into general solution and simplify

  $-\frac{1}{3y-3}-\frac{\ln(y-1)}{9}+\frac{\ln(2+y)}{9}=-x^2-\frac{1}{6}-\frac{\ln(2)}{9}+\frac{\ln(5)}{9}$

- Solution to the IVP

  $-\frac{1}{3y-3}-\frac{\ln(y-1)}{9}+\frac{\ln(2+y)}{9}=-x^2-\frac{1}{6}-\frac{\ln(2)}{9}+\frac{\ln(5)}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 70

```
dsolve([diff(y(x),x)=-2*x*(y(x)^3-3*y(x)+2),y(0) = 3],y(x), singsol=all)
```

$$y(x) = e^{\mathrm{RootOf}\left(18x^2 e^{\_Z} - 2\ln\left(e^{\_Z}-3\right)e^{\_Z} + 2\,e^{\_Z}\ln(2) - 2\,e^{\_Z}\ln(5) + 2\_Z e^{\_Z} - 54x^2 + 3\,e^{\_Z} + 6\ln\left(e^{\_Z}-3\right) - 6\ln(2) + 6\ln(5) - 6\_Z - 15\right)} - 2$$

✓ Solution by Mathematica

Time used: 1.071 (sec). Leaf size: 49

```
DSolve[{y'[x]==-2*x*(y[x]^3-3*y[x]+2),y[0]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{9}\left(-\frac{3}{\#1-1} - \log(\#1-1) + \log(\#1+2)\right)\&\right]\left[-x^2 - \frac{1}{6} + \frac{1}{9}\log\left(\frac{5}{2}\right)\right]$$

3.18 problem 19

Internal problem ID [945]
Internal file name [`OUTPUT/945_Sunday_June_05_2022_01_54_51_AM_48473993/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2x}{1 + 2y} = 0}$$

With initial conditions

$$[y(2) = 0]$$

3.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= \frac{2x}{1+2y}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x=2$ is

$$
\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}
$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{1+2y}\right) \\
&= -\frac{4x}{(1+2y)^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=2$ is

$$
\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{2x}{1+2y}
\end{aligned}$$

Where $f(x) = 2x$ and $g(y) = \frac{1}{1+2y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{1+2y}}\,dy &= 2x\,dx \\
\int \frac{1}{\frac{1}{1+2y}}\,dy &= \int 2x\,dx \\
y^2 + y &= x^2 + c_1
\end{aligned}$$

Which results in

$$\begin{aligned}
y &= -\frac{1}{2} + \frac{\sqrt{4x^2 + 4c_1 + 1}}{2} \\
y &= -\frac{1}{2} - \frac{\sqrt{4x^2 + 4c_1 + 1}}{2}
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} - \frac{\sqrt{17 + 4c_1}}{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + \frac{\sqrt{17 + 4c_1}}{2}$$

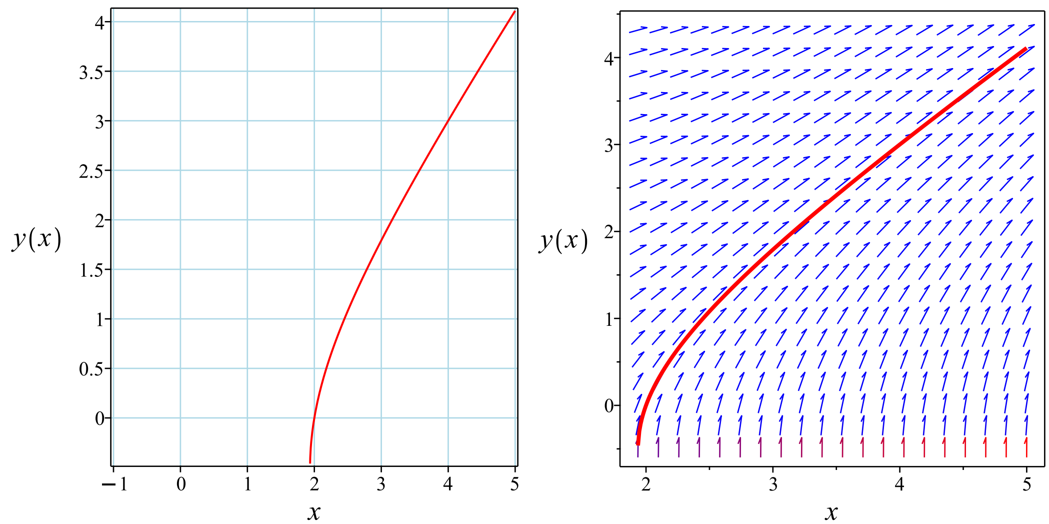
$$c_1 = -4$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}$$

Verified OK.

3.18.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x}{1 + 2y} \tag{1}$$

Which becomes

$$(1 + 2y)\, dy = (2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(2x)\, dx = d(x^2)$$

Hence (2) becomes

$$(1+2y)\,dy = d\left(x^2\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{1}{2}+\frac{\sqrt{4x^2+4c_1+1}}{2}+c_1$$
$$y = -\frac{1}{2}-\frac{\sqrt{4x^2+4c_1+1}}{2}+c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2}-\frac{\sqrt{17+4c_1}}{2}+c_1$$

$$c_1 = \sqrt{5}+1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2}-\frac{\sqrt{4x^2+4\sqrt{5}+5}}{2}+\sqrt{5}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2}+\frac{\sqrt{17+4c_1}}{2}+c_1$$

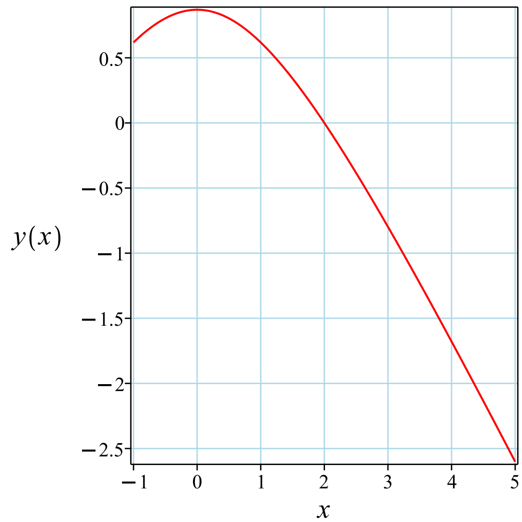
$$c_1 = -\sqrt{5}+1$$

Substituting c_1 found above in the general solution gives

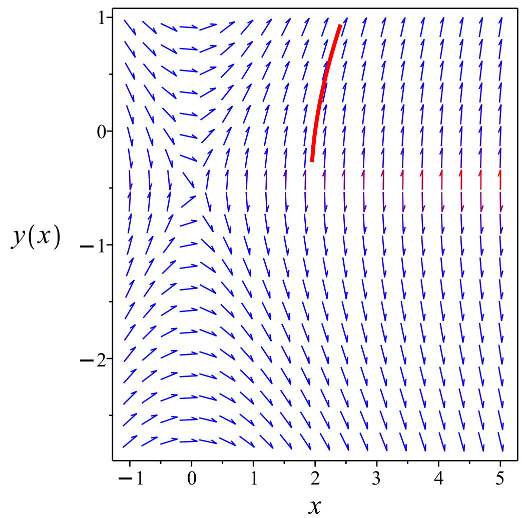
$$y = \frac{1}{2}+\frac{\sqrt{4x^2-4\sqrt{5}+5}}{2}-\sqrt{5}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}+\frac{\sqrt{4x^2-4\sqrt{5}+5}}{2}-\sqrt{5} \tag{1}$$
$$y = \frac{1}{2}-\frac{\sqrt{4x^2+4\sqrt{5}+5}}{2}+\sqrt{5} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} + \frac{\sqrt{4x^2 - 4\sqrt{5} + 5}}{2} - \sqrt{5}$$

Verified OK.

$$y = \frac{1}{2} - \frac{\sqrt{4x^2 + 4\sqrt{5} + 5}}{2} + \sqrt{5}$$

Verified OK.

3.18.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2X + 2x_0}{1 + 2Y(X) + 2y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{1}{u} \\ \frac{du}{dX} &= \frac{\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - 1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{X}\,dX \\
\int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{X}\,dX \\
\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\ln(X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(X) + 2c_2 \\
\ln(u-1) + \ln(u+1) &= (2)(-\ln(X) + 2c_2) \\
&= -2\ln(X) + 4c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(X)+2c_2}
$$

Which simplifies to

$$
\begin{aligned}
u^2 - 1 &= \frac{2c_2}{X^2} \\
&= \frac{c_3}{X^2}
\end{aligned}
$$

The solution is

$$
u(X)^2 - 1 = \frac{c_3}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{Y(X)^2}{X^2} - 1 = \frac{c_3}{X^2}
$$

Which simplifies to

$$
-(X - Y(X))(X + Y(X)) = c_3
$$

Using the solution for $Y(X)$

$$
-(X - Y(X))(X + Y(X)) = c_3
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - \frac{1}{2} \\
X &= x
\end{aligned}
$$

Then the solution in y becomes

$$
-\left(x - \frac{1}{2} - y\right)\left(x + \frac{1}{2} + y\right) = c_3
$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
-\frac{15}{4} = c_3
$$

$$
c_3 = -\frac{15}{4}
$$

Substituting c_3 found above in the general solution gives

$$
-\left(x - \frac{1}{2} - y\right)\left(x + \frac{1}{2} + y\right) = -\frac{15}{4}
$$

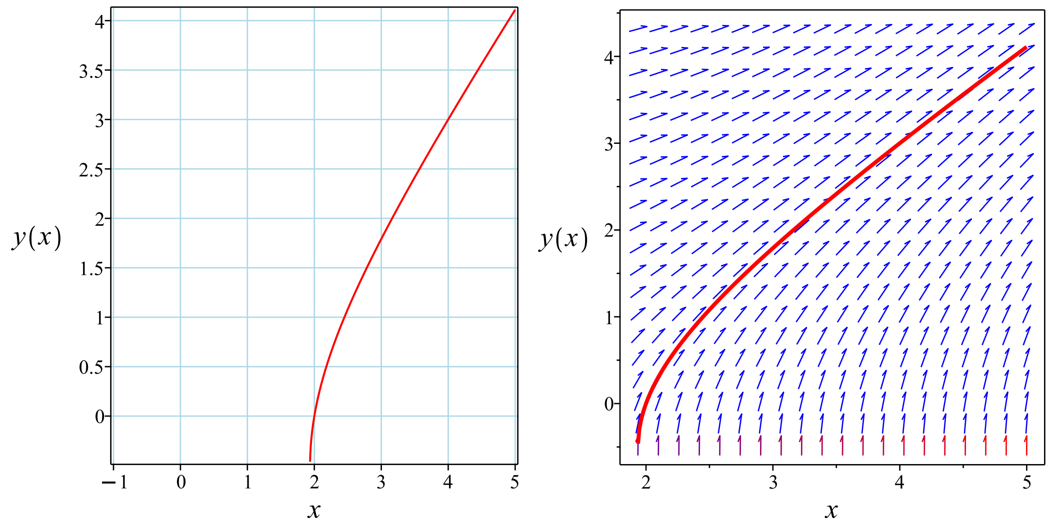
Solving for y from the above gives

$$
y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2} \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{2} + \frac{\sqrt{4x^2 - 15}}{2}$$

Verified OK.

3.18.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2x}{1+2y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 193: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{1}{2x}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2x}} dx \end{aligned}$$

Which results in

$$S = x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x}{1+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + 2y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + R + c_1 \tag{4}$$

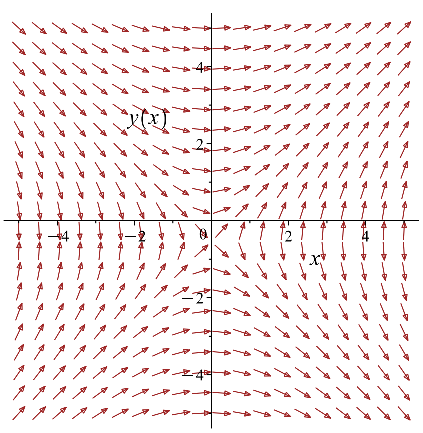
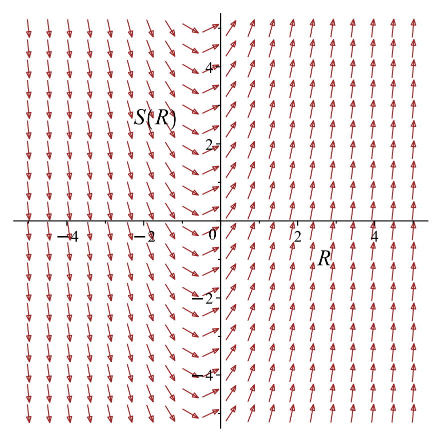
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = y^2 + c_1 + y$$

Which simplifies to

$$x^2 = y^2 + c_1 + y$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x}{1+2y}$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = 1 + 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$4 = c_1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$x^2 = y^2 + y + 4$$

Summary
The solution(s) found are the following

$$x^2 = y^2 + y + 4 \tag{1}$$

Verification of solutions

$$x^2 = y^2 + y + 4$$

Verified OK.

3.18.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2}+y\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{2}+y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{2}+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}+y\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2} + y$. Therefore equation (4) becomes

$$\frac{1}{2} + y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2} + y$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{2} + y\right) \mathrm{d}y$$
$$f(y) = \frac{1}{2}y + \frac{1}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^2 + \frac{1}{2}y + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^2 + \frac{1}{2}y + \frac{1}{2}y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{2}x^2 + \frac{1}{2}y + \frac{1}{2}y^2 = -2$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y}{2} + \frac{y^2}{2} = -2 \tag{1}$$

Verification of solutions

$$-\frac{x^2}{2} + \frac{y}{2} + \frac{y^2}{2} = -2$$

Verified OK.

3.18.7 Maple step by step solution

Let's solve
$\left[y' - \frac{2x}{1+2y} = 0, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $y'(1 + 2y) = 2x$
- Integrate both sides with respect to x
  $\int y'(1 + 2y)\,dx = \int 2x dx + c_1$
- Evaluate integral
  $y^2 + y = x^2 + c_1$
- Solve for y

$$\left\{y = -\frac{1}{2} - \frac{\sqrt{4x^2+4c_1+1}}{2}, y = -\frac{1}{2} + \frac{\sqrt{4x^2+4c_1+1}}{2}\right\}$$

- Use initial condition $y(2) = 0$

  $0 = -\frac{1}{2} - \frac{\sqrt{17+4c_1}}{2}$

- Solution does not satisfy initial condition
- Use initial condition $y(2) = 0$

  $0 = -\frac{1}{2} + \frac{\sqrt{17+4c_1}}{2}$

- Solve for c_1

  $c_1 = -4$

- Substitute $c_1 = -4$ into general solution and simplify

  $y = -\frac{1}{2} + \frac{\sqrt{4x^2-15}}{2}$

- Solution to the IVP

  $y = -\frac{1}{2} + \frac{\sqrt{4x^2-15}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=2*x/(1+2*y(x)),y(2) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + \frac{\sqrt{4x^2-15}}{2}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 22

```
DSolve[{y'[x]==2*x/(1+2*y[x]),y[2]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 - 15} - 1\right)$$

3.19 problem 20

Internal problem ID [946]
Internal file name [`OUTPUT/946_Sunday_June_05_2022_01_54_52_AM_16422908/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 2y + y^2 = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -y^2 + 2y \end{aligned}$$

The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2 + 2y) \\ &= -2y + 2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.19.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + 2y} dy = \int dx$$
$$\frac{\ln(y)}{2} - \frac{\ln(y-2)}{2} = x + c_1$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(y) - \ln(y-2)) = x + c_1$$
$$\ln(y) - \ln(y-2) = (2)(x + c_1)$$
$$= 2x + 2c_1$$

Raising both side to exponential gives

$$e^{\ln(y)-\ln(y-2)} = 2c_1 e^{2x}$$

Which simplifies to

$$\frac{y}{y-2} = c_2 e^{2x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{2c_2}{c_2 - 1}$$

$$c_2 = -1$$

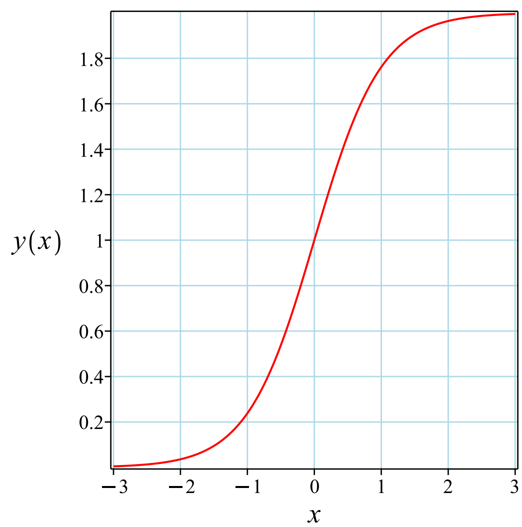
Substituting c_2 found above in the general solution gives

$$y = \frac{2\,e^{2x}}{e^{2x} + 1}$$

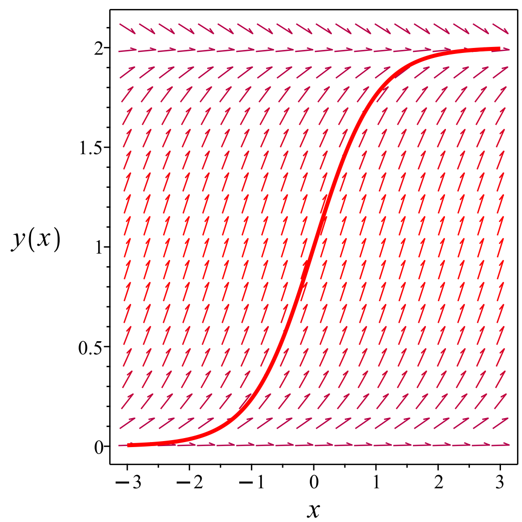
Summary

The solution(s) found are the following

$$y = \frac{2\,e^{2x}}{e^{2x}+1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\,e^{2x}}{e^{2x}+1}$$

Verified OK.

3.19.3 Maple step by step solution

Let's solve

$[y' - 2y + y^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y-y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y-y^2}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(y)}{2} - \frac{\ln(y-2)}{2} = x + c_1$
- Solve for y

  $y = \frac{2\,e^{2x+2c_1}}{e^{2x+2c_1}-1}$
- Use initial condition $y(0) = 1$

  $1 = \frac{2\,e^{2c_1}}{e^{2c_1}-1}$
- Solve for c_1

  $c_1 = \frac{I}{2}\pi$
- Substitute $c_1 = \frac{I}{2}\pi$ into general solution and simplify

  $y = \frac{2\,e^{2x}}{e^{2x}+1}$
- Solution to the IVP

  $y = \frac{2\,e^{2x}}{e^{2x}+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=2*y(x)-y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{2}{1 + e^{-2x}}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 21

```
DSolve[{y'[x]==2*y[x]-y[x]^2,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^{2x}}{e^{2x} + 1}$$

3.20 problem 21

Internal problem ID [947]
Internal file name [`OUTPUT/947_Sunday_June_05_2022_01_54_54_AM_33089805/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{yy' = -x}$$

With initial conditions

$$[y(3) = -4]$$

3.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $f(x,y)$ when $x = 3$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{y}\right) \\ &= \frac{x}{y^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=-4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 3$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -4$ is inside this domain. Therefore solution exists and is unique.

3.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{-x^2 + 2c_1}$$

$$y = -\sqrt{-x^2 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = -\sqrt{-9 + 2c_1}$$

$$c_1 = \frac{25}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-x^2 + 25}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

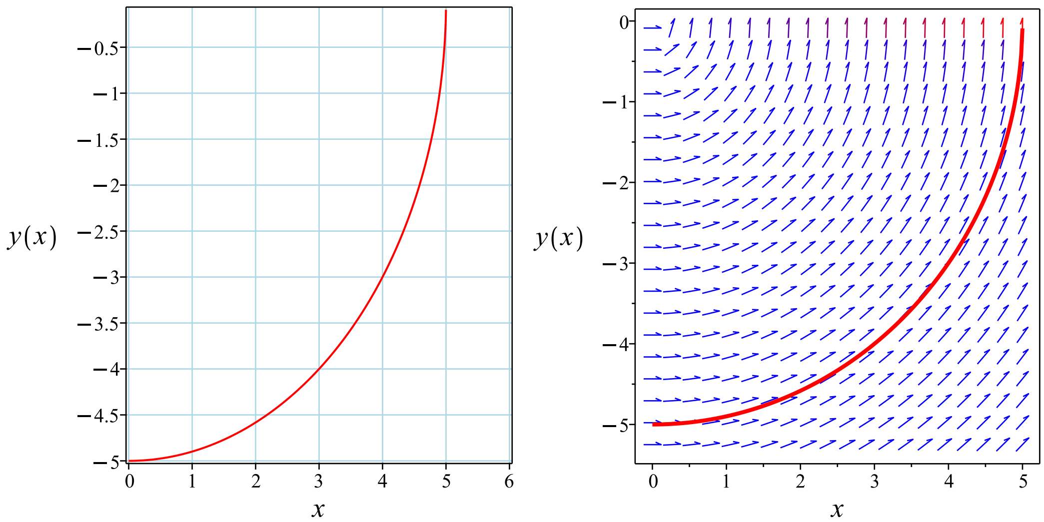
$$-4 = \sqrt{-9 + 2c_1}$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\sqrt{-x^2 + 25}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sqrt{-x^2+25}$$

Verified OK.

3.20.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(u'(x)\,x+u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2+1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln(u^2+1)}{2} &= -\ln(x)+c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2+1} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{25}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x=3$ and $y=-4$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{5}{3} = \frac{5c_3\sqrt{\frac{1}{c_3^2}}}{3}$$

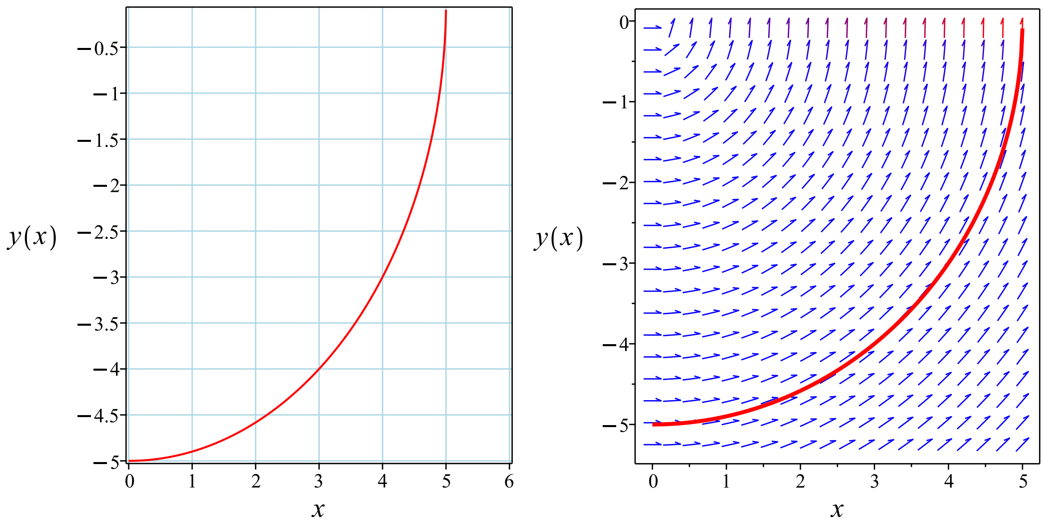
This solution is valid for any c_3. Hence there are infinite number of solutions.

Solving for y from the above gives

$$y = -\sqrt{-x^2+25}$$

Summary
The solution(s) found are the following

$$y = -\sqrt{-x^2+25} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sqrt{-x^2 + 25}$$

Verified OK. `{positive}`

3.20.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy = (-x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dx = d\left(-\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(-\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$
$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = -\sqrt{-9 + 2c_1} + c_1$$

$$c_1 = -3 - 4i$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-x^2 - 8i - 6} - 3 - 4i$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = \sqrt{-9 + 2c_1} + c_1$$

Warning: Unable to solve for constant of integration.

Summary

The solution(s) found are the following

$$y = -\sqrt{-x^2 - 8i - 6} -$$

Verification of solutions

$$y = -\sqrt{-x^2 - 8i - 6} - 3 - 4i$$

Verified OK. {positive}

3.20.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 197: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

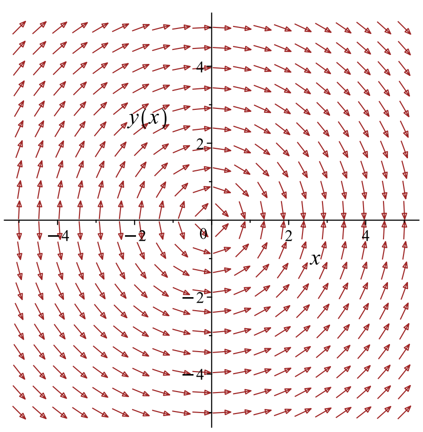
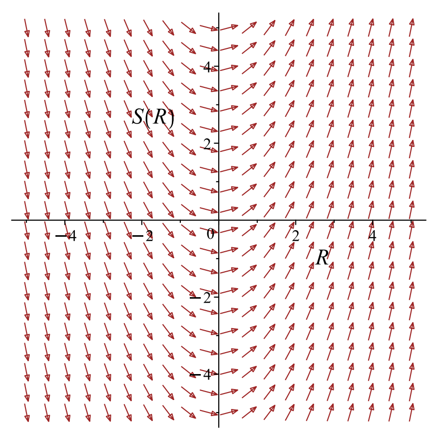
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{9}{2} = 8 + c_1$$

$$c_1 = -\frac{25}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} = \frac{y^2}{2} - \frac{25}{2}$$

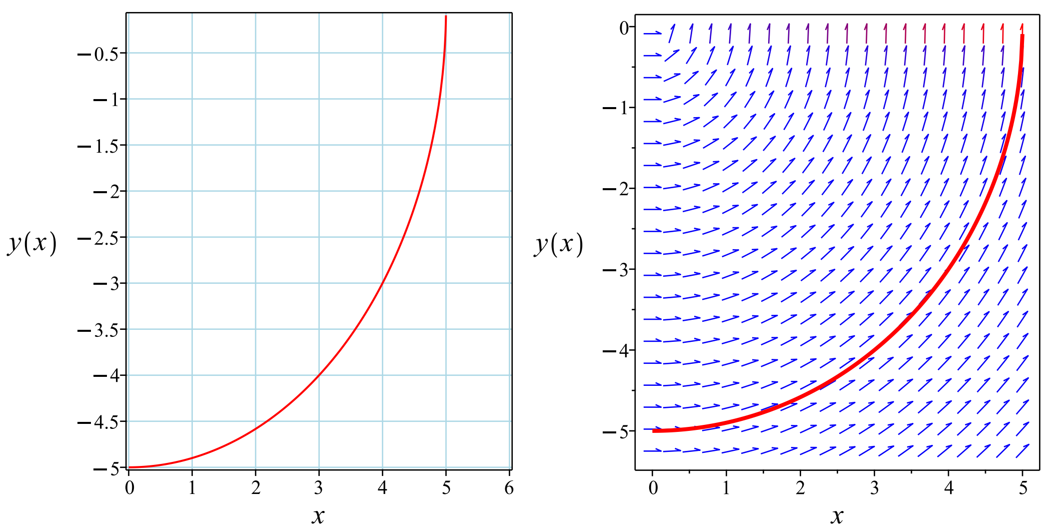
Solving for y from the above gives

$$y = -\sqrt{-x^2 + 25}$$

Summary

The solution(s) found are the following

$$y = -\sqrt{-x^2 + 25} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\sqrt{-x^2 + 25}$$

Verified OK. `{positive}`

3.20.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+(-y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=-y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-y)\,\mathrm{d}y$$

$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{25}{2} = c_1$$

$$c_1 = -\frac{25}{2}$$

Substituting c_1 found above in the general solution gives

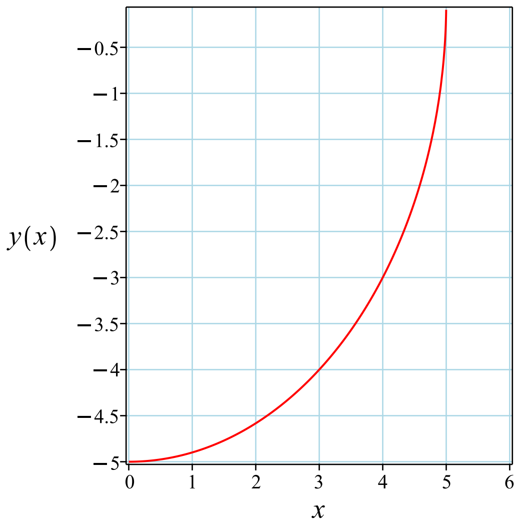
$$-\frac{x^2}{2} - \frac{y^2}{2} = -\frac{25}{2}$$

Solving for y from the above gives

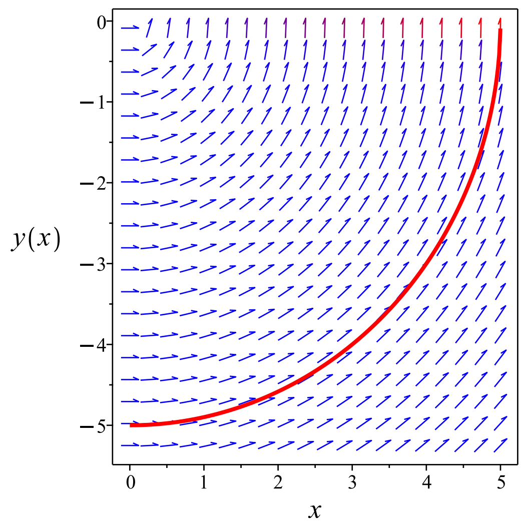
$$y = -\sqrt{-x^2 + 25}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\sqrt{-x^2 + 25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sqrt{-x^2 + 25}$$

Verified OK. `{positive}`

3.20.7 Maple step by step solution

Let's solve

$[yy' = -x, y(3) = -4]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\{y = \sqrt{-x^2 + 2c_1}, y = -\sqrt{-x^2 + 2c_1}\}$

- Use initial condition $y(3) = -4$

  $-4 = \sqrt{-9 + 2c_1}$

- Solution does not satisfy initial condition
- Use initial condition $y(3) = -4$

  $-4 = -\sqrt{-9 + 2c_1}$
- Solve for c_1

  $c_1 = \frac{25}{2}$
- Substitute $c_1 = \frac{25}{2}$ into general solution and simplify

  $y = -\sqrt{-x^2 + 25}$
- Solution to the IVP

  $y = -\sqrt{-x^2 + 25}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 15

```
dsolve([x+y(x)*diff(y(x),x)=0,y(3) = -4],y(x), singsol=all)
```

$$y(x) = -\sqrt{-x^2 + 25}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 18

```
DSolve[{x+y[x]*y'[x]==0,y[3]==-4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{25 - x^2}$$

3.21 problem 22

Internal problem ID [948]
Internal file name [`OUTPUT/948_Sunday_June_05_2022_01_54_55_AM_27304392/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "abelFirstKind", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + x^2(1+y)\left(y-2\right)^2 = 0}$$

3.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -x^2(y+1)\left(y-2\right)^2 \end{aligned}$$

Where $f(x) = -x^2$ and $g(y) = (y+1)\left(y-2\right)^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{(y+1)\left(y-2\right)^2}\,dy &= -x^2\,dx \\ \int \frac{1}{(y+1)\left(y-2\right)^2}\,dy &= \int -x^2\,dx \end{aligned}$$

$$\frac{\ln(y+1)}{9} - \frac{1}{3(y-2)} - \frac{\ln(y-2)}{9} = -\frac{x^3}{3} + c_1$$

Which results in

$$y = e^{\operatorname{RootOf}\left(-3x^3 e^{\_Z} - \ln\left(e^{\_Z}+3\right)e^{\_Z} + 9c_1 e^{\_Z} + \_Z e^{\_Z} + 3\right)} + 2$$

Summary

The solution(s) found are the following

$$y = e^{\operatorname{RootOf}\left(-3x^3 e^{\_Z} - \ln\left(e^{\_Z}+3\right)e^{\_Z} + 9c_1 e^{\_Z} + \_Z e^{\_Z} + 3\right)} + 2 \tag{1}$$

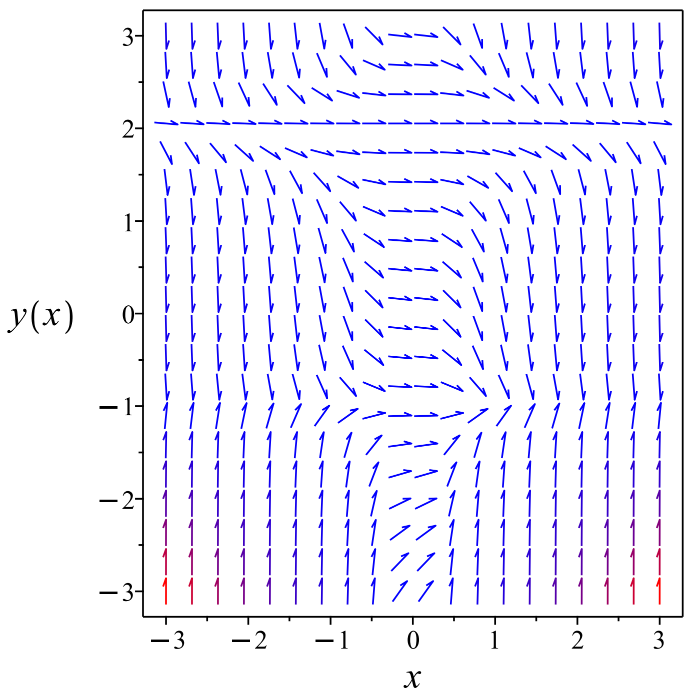


Figure 207: Slope field plot

Verification of solutions

$$y = e^{\operatorname{RootOf}\left(-3x^3 e^{\_Z} - \ln\left(e^{\_Z}+3\right)e^{\_Z} + 9c_1 e^{\_Z} + \_Z e^{\_Z} + 3\right)} + 2$$

Verified OK.

3.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -x^2(y+1)(y-2)^2 \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x^2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{x^2}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^3}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -x^2 (y + 1) (y - 2)^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y+1)(y-2)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R+1)(R-2)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R+1)}{9} - \frac{1}{3(R-2)} - \frac{\ln(R-2)}{9} + c_1 \tag{4}$$

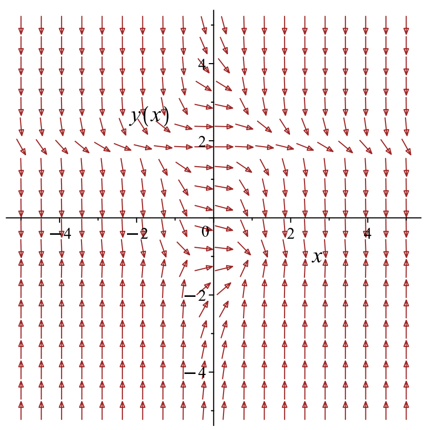
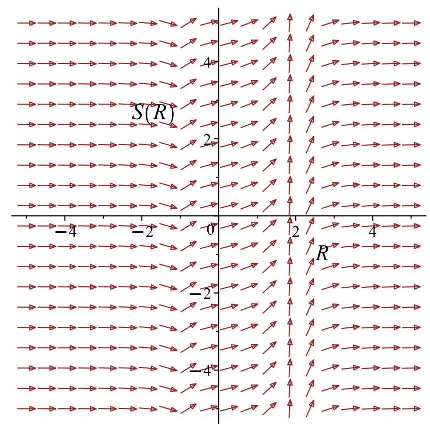
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{3} = \frac{\ln(1+y)}{9} - \frac{1}{3(y-2)} - \frac{\ln(y-2)}{9} + c_1$$

Which simplifies to

$$-\frac{x^3}{3} = \frac{\ln(1+y)}{9} - \frac{1}{3(y-2)} - \frac{\ln(y-2)}{9} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x^2(y+1)(y-2)^2$  	$R = y$ $S = -\frac{x^3}{3}$	$\frac{dS}{dR} = \frac{1}{(R+1)(R-2)^2}$ 

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} = \frac{\ln(1+y)}{9} - \frac{1}{3(y-2)} - \frac{\ln(y-2)}{9} + c_1 \tag{1}$$

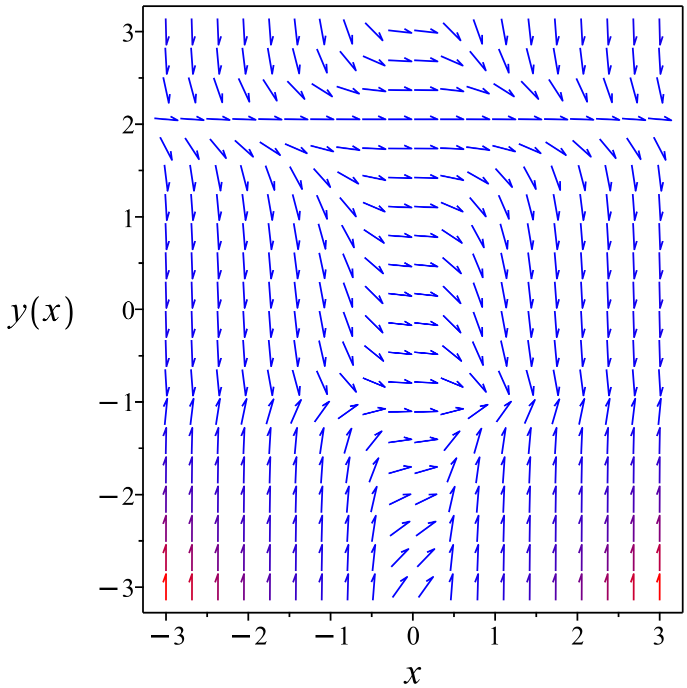


Figure 208: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} = \frac{\ln(1+y)}{9} - \frac{1}{3(y-2)} - \frac{\ln(y-2)}{9} + c_1$$

Verified OK.

3.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{(y+1)(y-2)^2}\right)\mathrm{d}y = (x^2)\,\mathrm{d}x$$
$$(-x^2)\,\mathrm{d}x + \left(-\frac{1}{(y+1)(y-2)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2$$
$$N(x, y) = -\frac{1}{(y+1)(y-2)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x^2)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{(y+1)(y-2)^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^2\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{(y+1)(y-2)^2}$. Therefore equation (4) becomes

$$-\frac{1}{(y+1)(y-2)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{(y+1)(y-2)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{(y+1)\,(y-2)^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y+1)}{9} + \frac{1}{3y-6} + \frac{\ln(y-2)}{9} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{\ln(y+1)}{9} + \frac{1}{3y-6} + \frac{\ln(y-2)}{9} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{\ln(y+1)}{9} + \frac{1}{3y-6} + \frac{\ln(y-2)}{9}$$

Summary
The solution(s) found are the following

$$-\frac{x^3}{3} - \frac{\ln(1+y)}{9} + \frac{1}{3y-6} + \frac{\ln(y-2)}{9} = c_1 \tag{1}$$

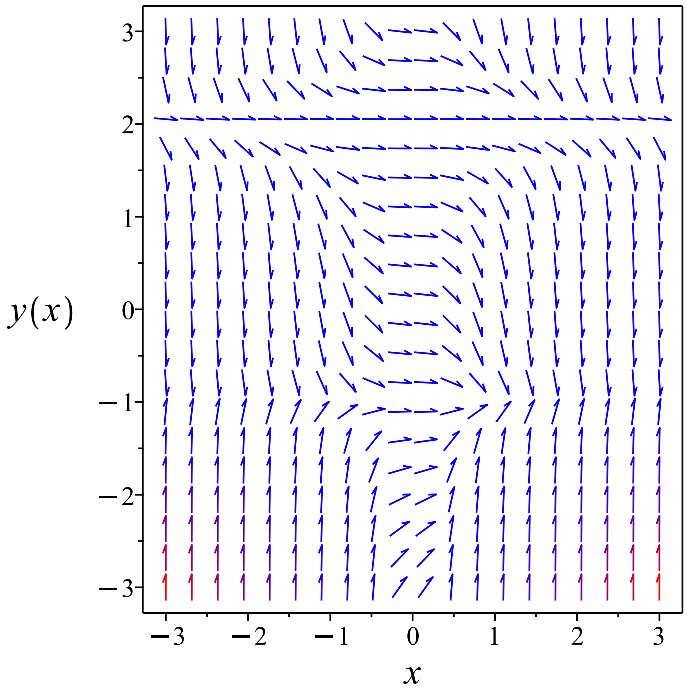


Figure 209: Slope field plot

Verification of solutions

$$-\frac{x^3}{3}-\frac{\ln(1+y)}{9}+\frac{1}{3y-6}+\frac{\ln(y-2)}{9}=c_1$$

Verified OK.

3.21.4 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2+f_3(x)y^3$$

Comparing the above to given ODE which is

$$y'=-y^3x^2+3x^2y^2-4x^2 \tag{1}$$

Therefore

$$\begin{aligned}f_0(x)&=-4x^2\\f_1(x)&=0\\f_2(x)&=3x^2\\f_3(x)&=-x^2\end{aligned}$$

Since $f_2(x) = 3x^2$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} y &= u(x) - \left(\frac{3x^2}{-3x^2}\right) \\ &= u(x) + 1 \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = 3x^2u(x) - 2x^2 - x^2u(x)^3 \tag{2}$$

The above ODE (2) can now be solved as separable.

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= x^2(-u^3 + 3u - 2) \end{aligned}$$

Where $f(x) = x^2$ and $g(u) = -u^3 + 3u - 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-u^3 + 3u - 2}\,du &= x^2\,dx \\ \int \frac{1}{-u^3 + 3u - 2}\,du &= \int x^2\,dx \\ \frac{1}{3u - 3} + \frac{\ln(u-1)}{9} - \frac{\ln(u+2)}{9} &= \frac{x^3}{3} + c_2 \end{aligned}$$

The solution is

$$\frac{1}{3u(x) - 3} + \frac{\ln(u(x) - 1)}{9} - \frac{\ln(2 + u(x))}{9} - \frac{x^3}{3} - c_2 = 0$$

Substituting $u = 1 + y$ in the above solution gives

$$\frac{1}{3y} + \frac{\ln(y)}{9} - \frac{\ln(3+y)}{9} - \frac{x^3}{3} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{1}{3y} + \frac{\ln(y)}{9} - \frac{\ln(3+y)}{9} - \frac{x^3}{3} - c_2 = 0 \tag{1}$$

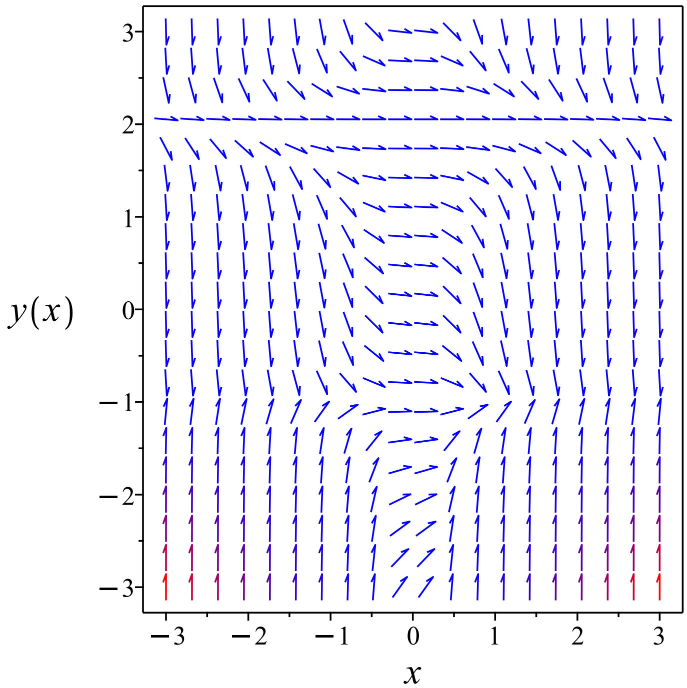


Figure 210: Slope field plot

Verification of solutions

$$\frac{1}{3y}+\frac{\ln(y)}{9}-\frac{\ln(3+y)}{9}-\frac{x^3}{3}-c_2=0$$

Verified OK.

3.21.5 Maple step by step solution

Let's solve

$y'+x^2(1+y)(y-2)^2=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1+y)(y-2)^2}=-x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y)(y-2)^2}dx=\int -x^2dx+c_1$

- Evaluate integral

$$\frac{\ln(1+y)}{9}-\frac{1}{3(y-2)}-\frac{\ln(y-2)}{9}=-\frac{x^3}{3}+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 35

```
dsolve(diff(y(x),x)+x^2*(y(x)+1)*(y(x)-2)^2=0,y(x), singsol=all)
```

$$y(x)=\mathrm{e}^{\mathrm{RootOf}\left(3x^3\mathrm{e}^{\_Z}+\ln\left(\mathrm{e}^{\_Z}+3\right)\mathrm{e}^{\_Z}+9c_1\mathrm{e}^{\_Z}-\_Z\mathrm{e}^{\_Z}-3\right)}+2$$

✓ Solution by Mathematica

Time used: 0.482 (sec). Leaf size: 52

```
DSolve[y'[x]+x^2*(y[x]+1)*(y[x]-2)^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \text{InverseFunction}\left[\frac{1}{9}\left(-\frac{3}{\#1-2}-\log(\#1-2)+\log(\#1+1)\right)\&\right]\left[-\frac{x^3}{3}+c_1\right]$$
$$y(x)\to -1$$
$$y(x)\to 2$$

3.22 problem 23

Internal problem ID [949]
Internal file name [`OUTPUT/949_Sunday_June_05_2022_01_54_58_AM_55960160/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x+1)(-2+x)\, y' + y = 0}$$

With initial conditions

$$[y(1) = -3]$$

3.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{(-2+x)(x+1)} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{(x+1)(-2+x)} = 0$$

The domain of $p(x) = \frac{1}{(-2+x)(x+1)}$ is

$$\{-\infty \leq x < -1, -1 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

3.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y}{(x+1)(-2+x)} \end{aligned}$$

Where $f(x) = -\frac{1}{(-2+x)(x+1)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= -\frac{1}{(-2+x)(x+1)}\,dx \\ \int \frac{1}{y}\,dy &= \int -\frac{1}{(-2+x)(x+1)}\,dx \\ \ln(y) &= \frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} + c_1 \\ y &= \mathrm{e}^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} + c_1} \\ &= c_1 \mathrm{e}^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3}} \end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1 (x+1)^{\frac{1}{3}}}{(-2+x)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{i 2^{\frac{1}{3}} \sqrt{3}\, c_1}{2} + \frac{2^{\frac{1}{3}} c_1}{2}$$

$$c_1 = \frac{3\, 2^{\frac{2}{3}}}{i\sqrt{3} - 1}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Verified OK.

3.22.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{(-2+x)(x+1)} dx} \\ &= e^{-\frac{\ln(x+1)}{3} + \frac{\ln(-2+x)}{3}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(-2+x)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-2+x)^{\frac{1}{3}} y}{(x+1)^{\frac{1}{3}}}\right) = 0$$

Integrating gives

$$\frac{(-2+x)^{\frac{1}{3}} y}{(x+1)^{\frac{1}{3}}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(-2+x)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}}$ results in

$$y = \frac{c_1 (x+1)^{\frac{1}{3}}}{(-2+x)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{i 2^{\frac{1}{3}} \sqrt{3}\, c_1}{2} + \frac{2^{\frac{1}{3}} c_1}{2}$$

$$c_1 = \frac{3\, 2^{\frac{2}{3}}}{i\sqrt{3} - 1}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Verified OK.

3.22.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+1)\,(-2+x)\,(u'(x)\,x+u(x))+u(x)\,x=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u(x^2-2)}{(x+1)\,(-2+x)\,x}
\end{aligned}$$

Where $f(x) = -\frac{x^2-2}{(-2+x)x(x+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^2-2}{(-2+x)\,x\,(x+1)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{x^2-2}{(-2+x)\,x\,(x+1)}\,dx\\
\ln(u) &= -\ln(x)+\frac{\ln(x+1)}{3}-\frac{\ln(-2+x)}{3}+c_2\\
u &= \mathrm{e}^{-\ln(x)+\frac{\ln(x+1)}{3}-\frac{\ln(-2+x)}{3}+c_2}\\
&= c_2\mathrm{e}^{-\ln(x)+\frac{\ln(x+1)}{3}-\frac{\ln(-2+x)}{3}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2(x+1)^{\frac{1}{3}}}{x\,(-2+x)^{\frac{1}{3}}}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux\\
&= \frac{c_2(x+1)^{\frac{1}{3}}}{(-2+x)^{\frac{1}{3}}}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{i2^{\frac{1}{3}}\sqrt{3}\,c_2}{2}+\frac{2^{\frac{1}{3}}c_2}{2}$$

$$
c_2 = \frac{3\,2^{\frac{2}{3}}}{i\sqrt{3}-1}
$$

Substituting c_2 found above in the general solution gives

$$
y = \frac{3\,2^{\frac{2}{3}}(x+1)^{\frac{1}{3}}}{i\left(-2+x\right)^{\frac{1}{3}}\sqrt{3}-\left(-2+x\right)^{\frac{1}{3}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{3\,2^{\frac{2}{3}}(x+1)^{\frac{1}{3}}}{i\left(-2+x\right)^{\frac{1}{3}}\sqrt{3}-\left(-2+x\right)^{\frac{1}{3}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{3\,2^{\frac{2}{3}}(x+1)^{\frac{1}{3}}}{i\left(-2+x\right)^{\frac{1}{3}}\sqrt{3}-\left(-2+x\right)^{\frac{1}{3}}}
$$

Verified OK.

3.22.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y}{(x+1)\left(-2+x\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 203: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln\left(\frac{1}{(x+1)^{\frac{1}{3}}}\right) + \ln\left((-2+x)^{\frac{1}{3}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{(x+1)(-2+x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{(x+1)^{\frac{4}{3}} (-2+x)^{\frac{2}{3}}} \\ S_y &= \frac{(-2+x)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(-2+x)^{\frac{1}{3}} y}{(x+1)^{\frac{1}{3}}} = c_1$$

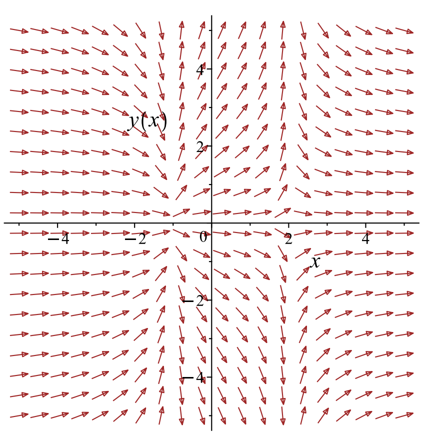
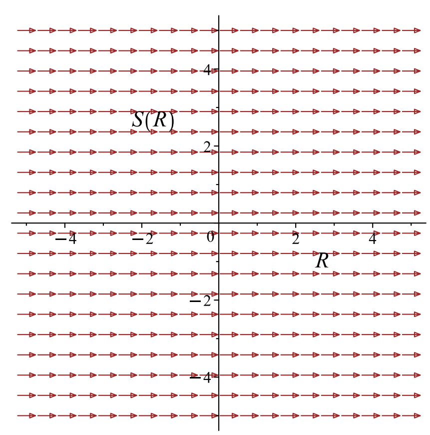
Which simplifies to

$$\frac{(-2+x)^{\frac{1}{3}} y}{(x+1)^{\frac{1}{3}}} = c_1$$

Which gives

$$y = \frac{c_1 (x+1)^{\frac{1}{3}}}{(-2+x)^{\frac{1}{3}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{(x+1)(-2+x)}$ 	$R = x$ $S = \frac{(-2+x)^{\frac{1}{3}} y}{(x+1)^{\frac{1}{3}}}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{i 2^{\frac{1}{3}} \sqrt{3}\, c_1}{2} + \frac{2^{\frac{1}{3}} c_1}{2}$$

$$c_1 = \frac{3\, 2^{\frac{2}{3}}}{i\sqrt{3} - 1}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{3\, 2^{\frac{2}{3}} (x+1)^{\frac{1}{3}}}{i\, (-2+x)^{\frac{1}{3}} \sqrt{3} - (-2+x)^{\frac{1}{3}}}$$

Verified OK.

3.22.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1}{(-2+x)(x+1)}\right)\mathrm{d}x$$
$$\left(-\frac{1}{(-2+x)(x+1)}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{(-2+x)(x+1)}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{(-2+x)(x+1)}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{(-2+x)(x+1)}\,\mathrm{d}x$$
$$\phi = \frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} - \ln(y)$$

The solution becomes

$$y = e^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} - c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -\frac{i 2^{\frac{1}{3}} \sqrt{3}\, e^{-c_1}}{2} + \frac{2^{\frac{1}{3}} e^{-c_1}}{2}$$

$$c_1 = -\frac{\ln\left(\frac{108}{\left(i\sqrt{3}-1\right)^3}\right)}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{(x+1)^{\frac{1}{3}}\, 108^{\frac{1}{3}} 8^{\frac{2}{3}}}{8\,(-2+x)^{\frac{1}{3}}}$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

3.22.7 Maple step by step solution

Let's solve

$[(x+1)(-2+x)\,y' + y = 0, y(1) = -3]$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{(-2+x)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{1}{(-2+x)(x+1)}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{\ln(x+1)}{3} - \frac{\ln(-2+x)}{3} + c_1}$

- Use initial condition $y(1) = -3$

  $-3 = -\mathrm{e}^{\frac{\ln(2)}{3} + \frac{2\,\mathrm{I}\pi}{3} + c_1}$

- Solve for c_1

  $c_1 = -\frac{2\,\mathrm{I}\pi}{3} - \frac{\ln(2)}{3} + \ln(3)$

- Substitute $c_1 = -\frac{2\,\mathrm{I}\pi}{3} - \frac{\ln(2)}{3} + \ln(3)$ into general solution and simplify

  $y = -\frac{3(x+1)^{\frac{1}{3}} 2^{\frac{2}{3}} \left(1+\mathrm{I}\sqrt{3}\right)}{4(-2+x)^{\frac{1}{3}}}$

- Solution to the IVP

  $y = -\frac{3(x+1)^{\frac{1}{3}} 2^{\frac{2}{3}} \left(1+\mathrm{I}\sqrt{3}\right)}{4(-2+x)^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 22

```
dsolve([(x+1)*(x-2)*diff(y(x),x)+y(x)=0,y(1) = -3],y(x), singsol=all)
```

$$y(x) = -\frac{3\,2^{\frac{2}{3}}\left(1+i\sqrt{3}\right)\left(x+1\right)^{\frac{1}{3}}}{4\left(-2+x\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 23

```
DSolve[{(x+1)*(x-2)*y'[x]+y[x]==0,y[1]==-3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3\sqrt[3]{x+1}}{\sqrt[3]{4-2x}}$$

3.23 problem 24

Internal problem ID [950]
Internal file name [`OUTPUT/950_Sunday_June_05_2022_01_54_59_AM_61564361/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{1 + y^2}{x^2 + 1} = 0}$$

3.23.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2 + 1}{x^2 + 1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2 + 1}\, dy = \frac{1}{x^2 + 1}\, dx$$

$$\int \frac{1}{y^2+1} dy = \int \frac{1}{x^2+1} dx$$

$$\arctan(y) = \arctan(x) + c_1$$

Which results in

$$y = \tan(\arctan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(\arctan(x) + c_1) \tag{1}$$

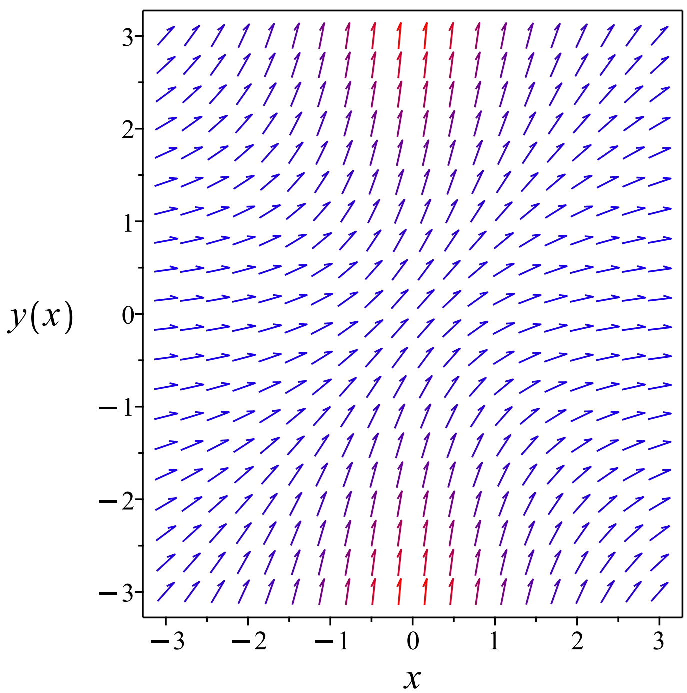


Figure 211: Slope field plot

Verification of solutions

$$y = \tan(\arctan(x) + c_1)$$

Verified OK.

3.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2+1 \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2+1} dx
\end{aligned}
$$

Which results in

$$
S = \arctan(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y^2+1}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\arctan(x) = \arctan(y) + c_1
$$

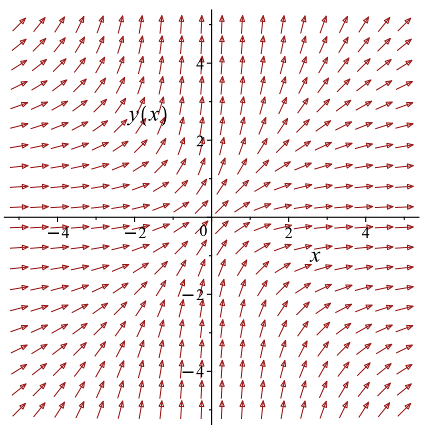
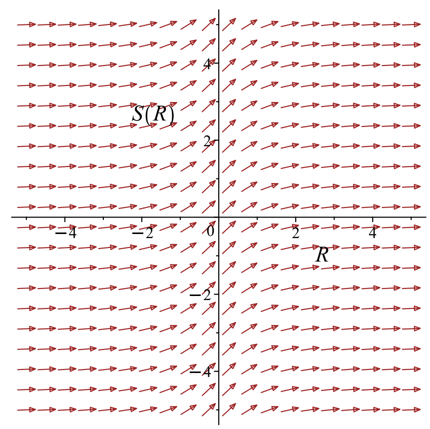
Which simplifies to

$$
\arctan(x) = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan(-\arctan(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary
The solution(s) found are the following

$$y = -\tan(-\arctan(x) + c_1) \tag{1}$$

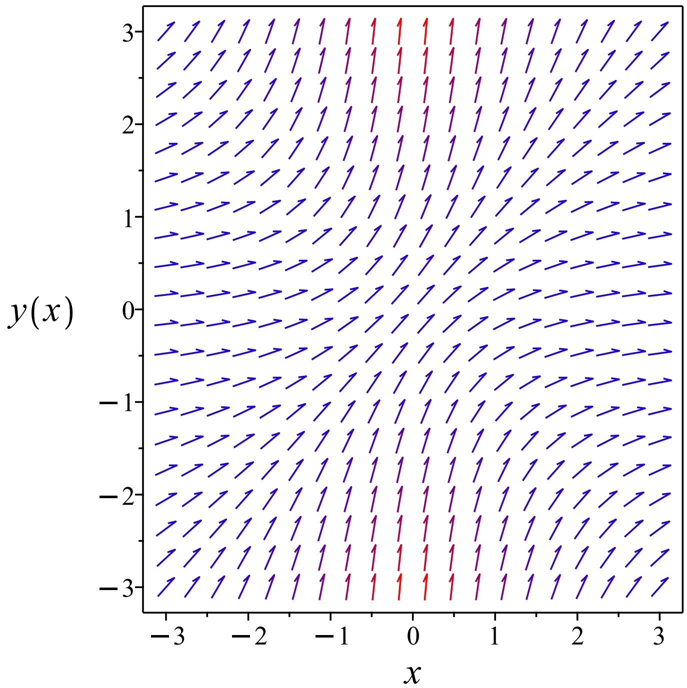


Figure 212: Slope field plot

Verification of solutions

$$y = -\tan(-\arctan(x) + c_1)$$

Verified OK.

3.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2+1}\right)\mathrm{d}y = \left(\frac{1}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2+1}$$
$$N(x, y) = \frac{1}{y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x^2+1} \mathrm{d}x$$
$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) + \arctan(y)$$

Summary

The solution(s) found are the following

$$-\arctan(x) + \arctan(y) = c_1 \tag{1}$$

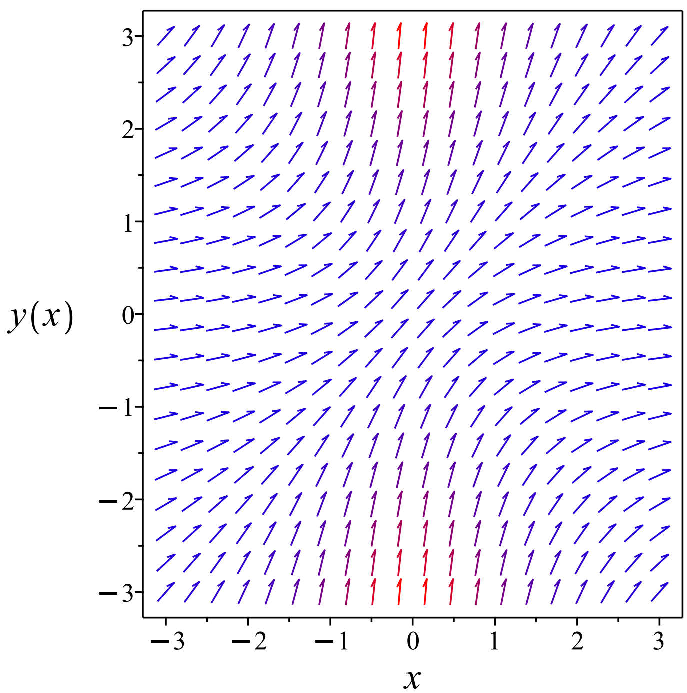


Figure 213: Slope field plot

Verification of solutions

$$-\arctan(x) + \arctan(y) = c_1$$

Verified OK.

3.23.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2+1}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2+1} + \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2x}{(x^2+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{(x^2+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2+1} + \frac{2xu'(x)}{(x^2+1)^2} + \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2+1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{(x^2+1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = -\frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 + x}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 + x}{c_3 x + 1} \tag{1}$$

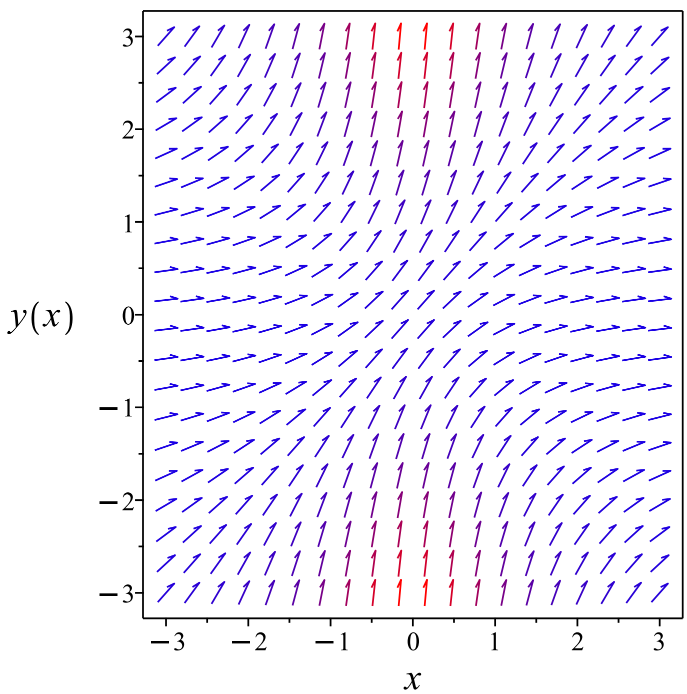


Figure 214: Slope field plot

Verification of solutions

$$y = \frac{-c_3 + x}{c_3 x + 1}$$

Verified OK.

3.23.5 Maple step by step solution

Let's solve

$y' - \frac{1+y^2}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = \tan(\arctan(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=(1+y(x)^2)/(1+x^2),y(x), singsol=all)
```

$$y(x) = \tan(\arctan(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.227 (sec). Leaf size: 25

```
DSolve[y'[x]==(1+y[x]^2)/(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(\arctan(x) + c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

3.24 problem 25

Internal problem ID [951]
Internal file name [`OUTPUT/951_Sunday_June_05_2022_01_55_01_AM_95367927/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'\sqrt{-x^2+1}+\sqrt{1-y^2}=0}$$

3.24.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}} \end{aligned}$$

Where $f(x)=-\frac{1}{\sqrt{-x^2+1}}$ and $g(y)=\sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{-y^2+1}}\,dy &= -\frac{1}{\sqrt{-x^2+1}}\,dx \\ \int \frac{1}{\sqrt{-y^2+1}}\,dy &= \int -\frac{1}{\sqrt{-x^2+1}}\,dx \end{aligned}$$

$$\arcsin(y) = \frac{\sqrt{-(x-1)^2 - 2x + 2}}{2} - \arcsin(x) - \frac{\sqrt{-(x+1)^2 + 2x + 2}}{2} + c_1$$

Which results in

$$y = \sin(-\arcsin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(-\arcsin(x) + c_1) \tag{1}$$

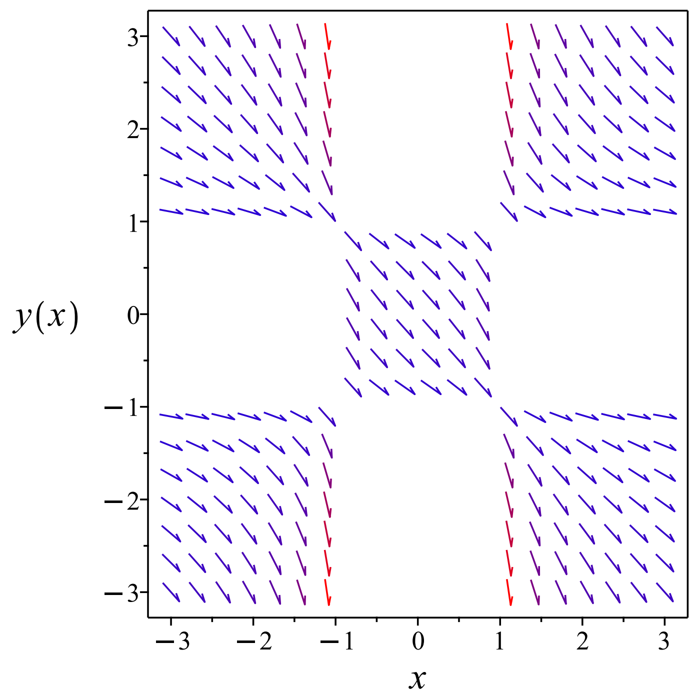


Figure 215: Slope field plot

Verification of solutions

$$y = \sin(-\arcsin(x) + c_1)$$

Verified OK.

3.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 209: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\sqrt{-x^2+1}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\sqrt{-x^2+1}} dx \end{aligned}$$

Which results in

$$S = -\arcsin(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{\sqrt{-x^2+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arcsin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\arcsin(x) = \arcsin(y) + c_1
$$

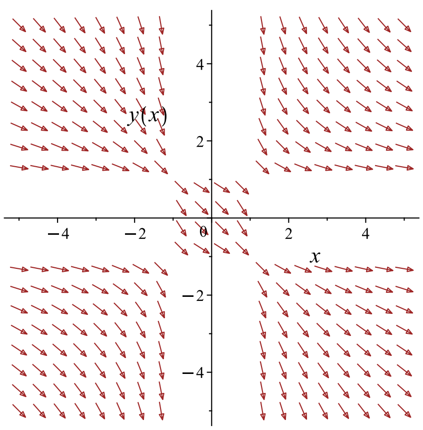
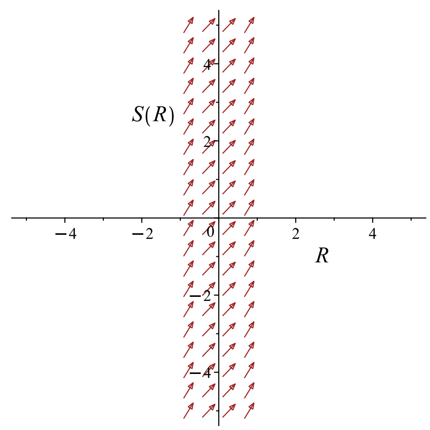
Which simplifies to

$$
-\arcsin(x) = \arcsin(y) + c_1
$$

Which gives

$$
y = -\sin(\arcsin(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{-y^2+1}}{\sqrt{-x^2+1}}$ 	$R = y$ $S = -\arcsin(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Summary

The solution(s) found are the following

$$y = -\sin(\arcsin(x) + c_1) \tag{1}$$

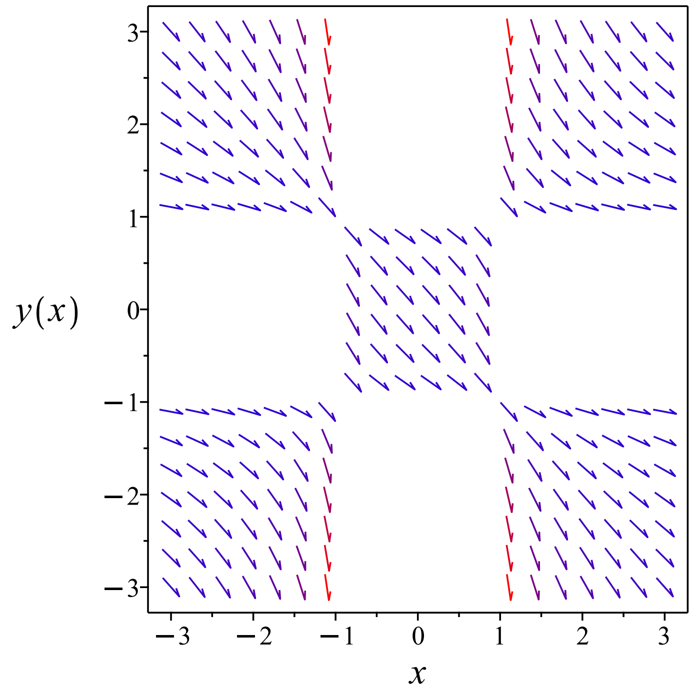


Figure 216: Slope field plot

Verification of solutions

$$y = -\sin(\arcsin(x) + c_1)$$

Verified OK.

3.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y = \left(\frac{1}{\sqrt{-x^2+1}}\right)\mathrm{d}x$$
$$\left(-\frac{1}{\sqrt{-x^2+1}}\right)\mathrm{d}x + \left(-\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{\sqrt{-x^2+1}}$$
$$N(x,y) = -\frac{1}{\sqrt{-y^2+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{-x^2+1}}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{-y^2+1}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{\sqrt{-x^2+1}} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\arcsin(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$
-\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{\sqrt{-y^2+1}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y \\
f(y) &= -\arcsin(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arcsin(x) - \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arcsin(x) - \arcsin(y)$$

Summary

The solution(s) found are the following

$$-\arcsin(x) - \arcsin(y) = c_1 \tag{1}$$

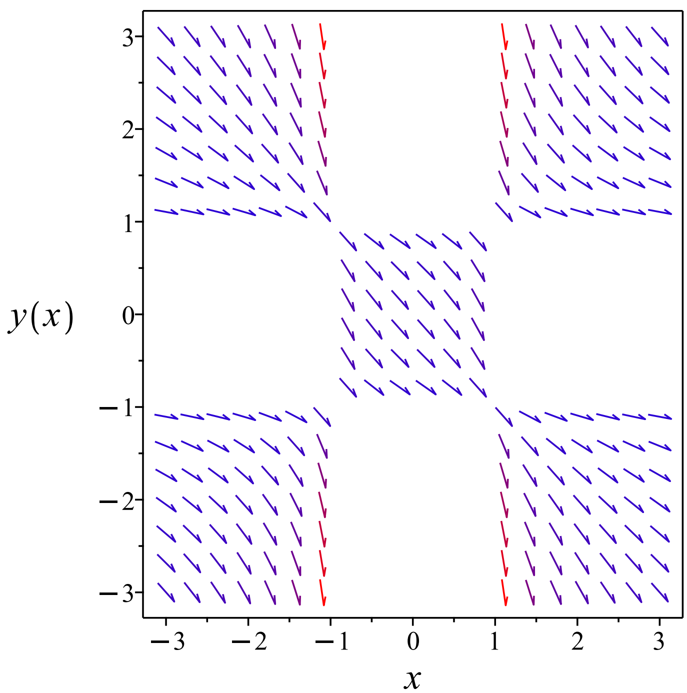


Figure 217: Slope field plot

Verification of solutions

$$-\arcsin(x) - \arcsin(y) = c_1$$

Verified OK.

3.24.4 Maple step by step solution

Let's solve

$y'\sqrt{-x^2+1}+\sqrt{1-y^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}}=-\frac{1}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx=\int -\frac{1}{\sqrt{-x^2+1}}dx+c_1$

- Evaluate integral

  $\arcsin(y)=-\arcsin(x)+c_1$

- Solve for y

  $y=\sin(-\arcsin(x)+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)*sqrt(1-x^2)+sqrt(1-y(x)^2)=0,y(x), singsol=all)
```

$$y(x)=-\sin(\arcsin(x)+c_1)$$

✓ Solution by Mathematica

Time used: 0.317 (sec). Leaf size: 47

```
DSolve[y'[x]*Sqrt[1-x^2]+Sqrt[1-y[x]^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \cos\left(2\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)+c_1\right)\\
y(x) &\to -1\\
y(x) &\to 1\\
y(x) &\to \text{Interval}[\{-1,1\}]
\end{aligned}
$$

3.25 problem 26

Internal problem ID [952]
Internal file name [`OUTPUT/952_Sunday_June_05_2022_01_55_03_AM_8578478/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\cos(x)}{\sin(y)} = 0}$$

With initial conditions

$$\left[y(\pi) = \frac{\pi}{2}\right]$$

3.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\cos(x)}{\sin(y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{\pi}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $f(x, y)$ when $x = \pi$ is

$$\{y < \pi\_Z54 \vee \pi\_Z54 < y\}$$

And the point $y_0 = \frac{\pi}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\cos(x)}{\sin(y)}\right)\\ &= -\frac{\cos(x)\cos(y)}{\sin(y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \pi$ is

$$\{y < \pi\_Z54 \vee \pi\_Z54 < y\}$$

And the point $y_0 = \frac{\pi}{2}$ is inside this domain. Therefore solution exists and is unique.

3.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{\cos(x)}{\sin(y)}\end{aligned}$$

Where $f(x) = \cos(x)$ and $g(y) = \frac{1}{\sin(y)}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{\sin(y)}}\,dy &= \cos(x)\,dx\\ \int \frac{1}{\frac{1}{\sin(y)}}\,dy &= \int \cos(x)\,dx\\ -\cos(y) &= \sin(x) + c_1\end{aligned}$$

Which results in

$$y = \pi - \arccos(\sin(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = \frac{\pi}{2} + \arcsin(c_1)$$

$$c_1 = 0$$

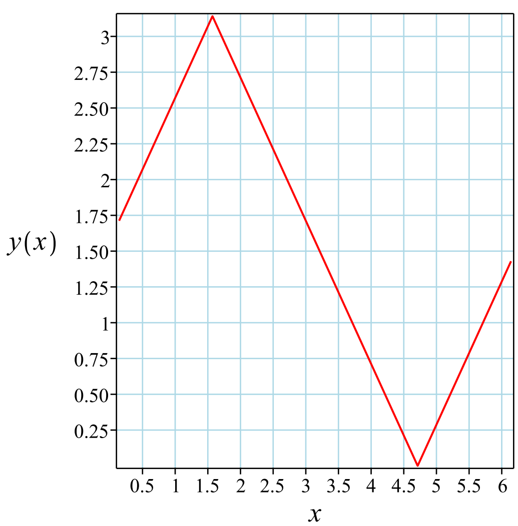
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

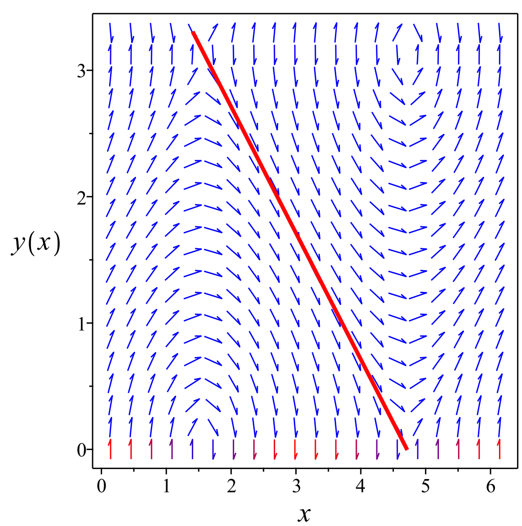
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} + \arcsin(\sin(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

Verified OK.

3.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\cos(x)}{\sin(y)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 212: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f(\frac{y}{x})$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F(\frac{y}{x})$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{\cos(x)} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dx \end{aligned}$$

Which results in

$$S = \sin(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(x)}{\sin(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \cos(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x) = -\cos(y) + c_1$$

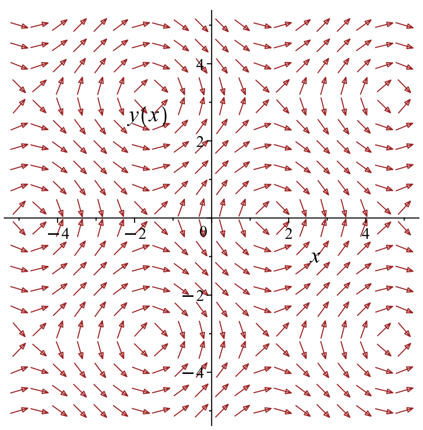
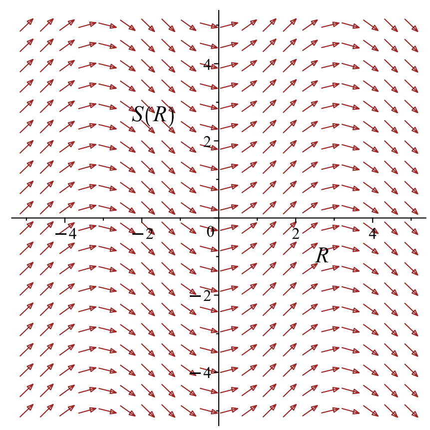
Which simplifies to

$$\sin(x) = -\cos(y) + c_1$$

Which gives

$$y = \arccos(-\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)}{\sin(y)}$ 	$R = y$ $S = \sin(x)$	$\frac{dS}{dR} = \sin(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = \frac{\pi}{2} - \arcsin(c_1)$$

$$c_1 = 0$$

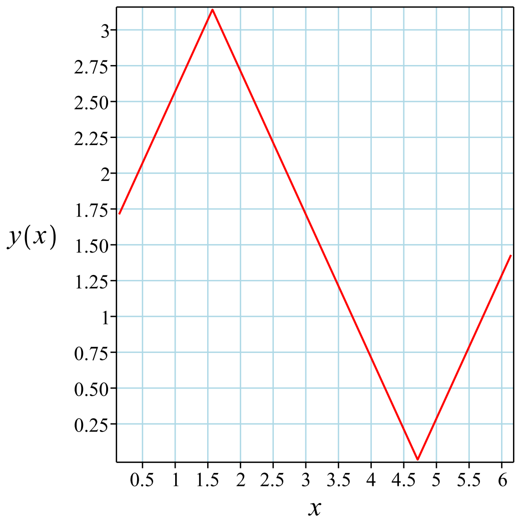
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

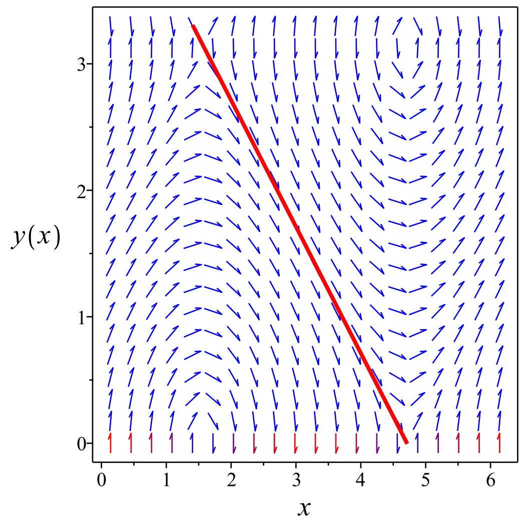
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} + \arcsin(\sin(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

Verified OK.

3.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(y))\,\mathrm{d}y&=(\cos(x))\,\mathrm{d}x\\ (-\cos(x))\,\mathrm{d}x+(\sin(y))\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\cos(x)\\ N(x,y)&=\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\cos(x))\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\sin(y))\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\cos(x) \mathrm{d}x$$
$$\phi = -\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(y)$. Therefore equation (4) becomes

$$\sin(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (\sin(y)) \mathrm{d}y$$
$$f(y) = -\cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - \cos(y)$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\sin(x) - \cos(y) = 0$$

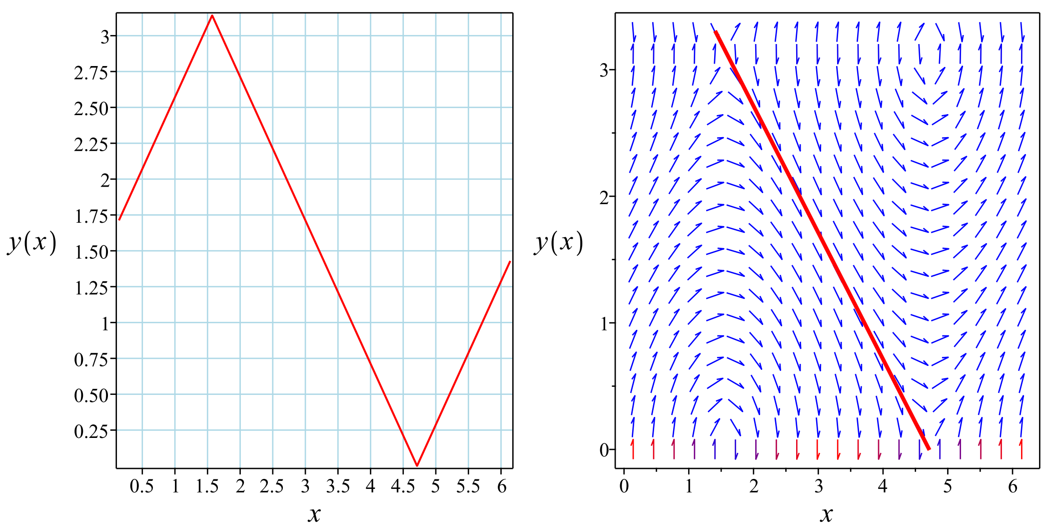
Solving for y from the above gives

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} + \arcsin(\sin(x)) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} + \arcsin(\sin(x))$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=cos(x)/sin(y(x)),y(Pi) = 1/2*Pi],y(x), singsol=all)
```

$$y(x) = \frac{\pi}{2} + \arcsin(\sin(x))$$

✓ Solution by Mathematica

Time used: 0.439 (sec). Leaf size: 10

```
DSolve[{y'[x]==Cos[x]/Sin[y[x]],y[Pi]==Pi/2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arccos(-\sin(x))$$

3.26 problem 27

Internal problem ID [953]
Internal file name [`OUTPUT/953_Sunday_June_05_2022_01_55_05_AM_25158409/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - ay + by^2 = 0}$$

With initial conditions

$$[y(0) = \text{y0}]$$

3.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -b\,y^2 + ay \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

But the point $y_0 = \text{y0}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.26.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-b\,y^2+ay}dy = \int dx$$
$$\frac{\ln(y)}{a} - \frac{\ln(by-a)}{a} = x + c_1$$

The above can be written as

$$\left(\frac{1}{a}\right)(\ln(y) - \ln(by-a)) = x + c_1$$
$$\ln(y) - \ln(by-a) = (a)(x+c_1)$$
$$= a(x+c_1)$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y)-\ln(by-a)} = ac_1\mathrm{e}^{ax}$$

Which simplifies to

$$-\frac{y}{-by+a} = c_2\mathrm{e}^{ax}$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=\mathrm{y0}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{y0} = \frac{c_2a}{bc_2-1}$$

$$c_2 = -\frac{\mathrm{y0}}{-b\,\mathrm{y0}+a}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\mathrm{y0}\ \mathrm{e}^{ax}a}{\mathrm{e}^{ax}b\,\mathrm{y0}-b\,\mathrm{y0}+a}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{y0}\ \mathrm{e}^{ax}a}{\mathrm{e}^{ax}b\,\mathrm{y0}-b\,\mathrm{y0}+a} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{y0}\ \mathrm{e}^{ax}a}{\mathrm{e}^{ax}b\,\mathrm{y0}-b\,\mathrm{y0}+a}$$

Verified OK.

3.26.3 Maple step by step solution

Let's solve

$[y' - ay + by^2 = 0, y(0) = y0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{ay - by^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{ay - by^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{\ln(y)}{a} - \frac{\ln(by - a)}{a} = x + c_1$

- Solve for y

  $y = \frac{e^{c_1 a + ax} a}{-1 + b\, e^{c_1 a + ax}}$

- Use initial condition $y(0) = y0$

  $y0 = \frac{e^{c_1 a} a}{-1 + b\, e^{c_1 a}}$

- Solve for c_1

  $c_1 = \frac{\ln\left(-\frac{y0}{-by0 + a}\right)}{a}$

- Substitute $c_1 = \frac{\ln\left(-\frac{y0}{-by0 + a}\right)}{a}$ into general solution and simplify

  $y = \frac{y0\, e^{ax} a}{e^{ax} by0 - by0 + a}$

- Solution to the IVP

  $y = \frac{y0\, e^{ax} a}{e^{ax} by0 - by0 + a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 25

```
dsolve([diff(y(x),x)=a*y(x)-b*y(x)^2,y(0) = y0],y(x), singsol=all)
```

$$y(x) = \frac{a\,\mathrm{y0}}{(-\,\mathrm{y0}\,b + a)\,\mathrm{e}^{-ax} + \mathrm{y0}\,b}$$

✓ Solution by Mathematica

Time used: 0.854 (sec). Leaf size: 27

```
DSolve[{y'[x]==a*y[x]-b*y[x]^2,y[0]==y0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{a\mathrm{y0}e^{ax}}{b\mathrm{y0}\left(e^{ax}-1\right)+a}$$

3.27 problem 35

Internal problem ID [954]
Internal file name [`OUTPUT/954_Sunday_June_05_2022_01_55_06_AM_83316683/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

$$\boxed{y' + y - \frac{2x\,\mathrm{e}^{-x}}{1 + \mathrm{e}^{x} y} = 0}$$

3.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(1+\mathrm{e}^x y)\,\mathrm{d}y &= \left(-y^2\mathrm{e}^x + 2x\,\mathrm{e}^{-x} - y\right)\mathrm{d}x\\ \left(y^2\mathrm{e}^x - 2x\,\mathrm{e}^{-x} + y\right)\mathrm{d}x + (1+\mathrm{e}^x y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2\mathrm{e}^x - 2x\,\mathrm{e}^{-x} + y\\ N(x,y) &= 1+\mathrm{e}^x y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\mathrm{e}^x - 2x\,\mathrm{e}^{-x} + y\right)\\ &= 2\,\mathrm{e}^x y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1+\mathrm{e}^x y)\\ &= \mathrm{e}^x y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{1+\mathrm{e}^x y}\left((2\,\mathrm{e}^x y + 1) - (\mathrm{e}^x y)\right)\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(y^{2}\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x} + y\right) \\
&= \mathrm{e}^{2x}y^{2} + \mathrm{e}^{x}y - 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1 + \mathrm{e}^{x}y) \\
&= (1 + \mathrm{e}^{x}y)\,\mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{2x}y^{2} + \mathrm{e}^{x}y - 2x\right) + \left((1 + \mathrm{e}^{x}y)\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \mathrm{e}^{2x}y^{2} + \mathrm{e}^{x}y - 2x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{x}y + \frac{\mathrm{e}^{2x}y^{2}}{2} - x^{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{x} + e^{2x}y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (1 + e^{x}y)\, e^{x}$. Therefore equation (4) becomes

$$(1 + e^{x}y)\, e^{x} = e^{x} + e^{2x}y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{x}y + \frac{e^{2x}y^{2}}{2} - x^{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{x}y + \frac{e^{2x}y^{2}}{2} - x^{2}$$

Summary

The solution(s) found are the following

$$e^{x}y + \frac{y^{2}e^{2x}}{2} - x^{2} = c_1 \tag{1}$$

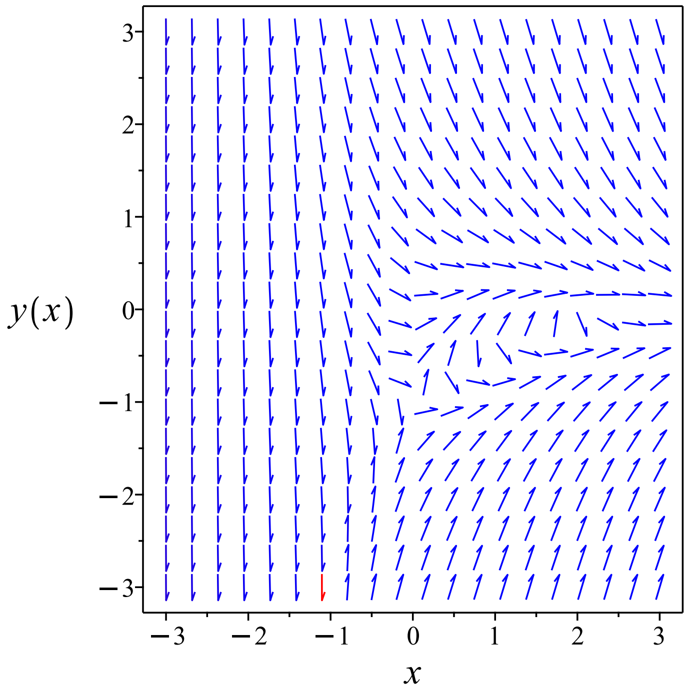


Figure 221: Slope field plot

Verification of solutions

$$e^{x}y+\frac{y^{2}e^{2x}}{2}-x^{2}=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve(diff(y(x),x)+y(x)=(2*x*exp(-x))/(1+y(x)*exp(x)),y(x), singsol=all)
```

$$y(x) = \left(-1 - \sqrt{2x^2 - 2c_1 + 1}\right) e^{-x}$$
$$y(x) = \left(-1 + \sqrt{2x^2 - 2c_1 + 1}\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 32.427 (sec). Leaf size: 70

```
DSolve[y'[x]+y[x]==(2*x*Exp[-x])/(1+y[x]*Exp[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-2x}\left(e^x + \sqrt{e^{2x}(2x^2 + 1 + c_1)}\right)$$
$$y(x) \to e^{-2x}\left(-e^x + \sqrt{e^{2x}(2x^2 + 1 + c_1)}\right)$$

3.28 problem 36

Internal problem ID [955]
Internal file name [`OUTPUT/955_Sunday_June_05_2022_01_55_09_AM_31201813/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y'x-2y-\frac{x^6}{x^2+y}=0}$$

3.28.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^6+2y\,x^2+2y^2}{x\,(x^2+y)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi=x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4+yb_5+b_2+\frac{(x^6+2y\,x^2+2y^2)\,(-2xa_4+xb_5-ya_5+2yb_6-a_2+b_3)}{x\,(x^2+y)} \\
&-\frac{(x^6+2y\,x^2+2y^2)^2\,(xa_5+2ya_6+a_3)}{x^2\,(x^2+y)^2} \\
&-\left(\frac{6x^5+4yx}{x\,(x^2+y)}-\frac{x^6+2y\,x^2+2y^2}{x^2\,(x^2+y)}-\frac{2(x^6+2y\,x^2+2y^2)}{(x^2+y)^2}\right)(x^2a_4 \\
&+yxa_5+y^2a_6+xa_2+ya_3+a_1)-\left(\frac{2x^2+4y}{x\,(x^2+y)}\right. \\
&\left.-\frac{x^6+2y\,x^2+2y^2}{x\,(x^2+y)^2}\right)(x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^{13}a_5+2x^{12}ya_6+x^{12}a_3+5x^{10}a_4-x^{10}b_5+8x^9ya_5-2x^9yb_6+11x^8y^2a_6+4x^9a_2-x^9b_3-x^9b_4+7x^8y}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^{13}a_5-2x^{12}ya_6-x^{12}a_3-5x^{10}a_4+x^{10}b_5-8x^9ya_5+2x^9yb_6-11x^8y^2a_6 \\
&-4x^9a_2+x^9b_3+x^9b_4-7x^8ya_3-7x^8ya_4+2x^8yb_5-10x^7y^2a_5+3x^7y^2b_6 \\
&-13x^6y^3a_6-3x^8a_1+x^8b_2-6x^7ya_2+2x^7yb_3-9x^6y^2a_3+x^7b_1 \\
&-5x^6ya_1-2x^6ya_4+x^6yb_5-4x^5y^2a_5+2x^5y^2b_6-6x^4y^3a_6-x^6b_2 \\
&-2x^4y^2a_3-4x^4y^2a_4+2x^4y^2b_5-8x^3y^3a_5+4x^3y^3b_6-12x^2y^4a_6-2x^5b_1 \\
&+2x^4ya_1-2x^4yb_2-4x^2y^3a_3-2x^2y^3a_4+x^2y^3b_5-4x\,y^4a_5+2x\,y^4b_6 \\
&-6y^5a_6-4x^3yb_1+4x^2y^2a_1-x^2y^2b_2-2y^4a_3-2x\,y^2b_1+2y^3a_1=0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_5 v_1^{13} - 2a_6 v_1^{12} v_2 - a_3 v_1^{12} - 5a_4 v_1^{10} - 8a_5 v_1^9 v_2 - 11a_6 v_1^8 v_2^2 + b_5 v_1^{10} + 2b_6 v_1^9 v_2 - 4a_2 v_1^9 \\
&- 7a_3 v_1^8 v_2 - 7a_4 v_1^8 v_2 - 10a_5 v_1^7 v_2^2 - 13a_6 v_1^6 v_2^3 + b_3 v_1^9 + b_4 v_1^9 + 2b_5 v_1^8 v_2 + 3b_6 v_1^7 v_2^2 - 3a_1 v_1^8 \\
&- 6a_2 v_1^7 v_2 - 9a_3 v_1^6 v_2^2 + b_2 v_1^8 + 2b_3 v_1^7 v_2 - 5a_1 v_1^6 v_2 - 2a_4 v_1^6 v_2 - 4a_5 v_1^5 v_2^2 - 6a_6 v_1^4 v_2^3 \\
&+ b_1 v_1^7 + b_5 v_1^6 v_2 + 2b_6 v_1^5 v_2^2 - 2a_3 v_1^4 v_2^2 - 4a_4 v_1^4 v_2^2 - 8a_5 v_1^3 v_2^3 - 12a_6 v_1^2 v_2^4 - b_2 v_1^6 + 2b_5 v_1^4 v_2^2 \\
&+ 4b_6 v_1^3 v_2^3 + 2a_1 v_1^4 v_2 - 4a_3 v_1^2 v_2^3 - 2a_4 v_1^2 v_2^3 - 4a_5 v_1 v_2^4 - 6a_6 v_2^5 - 2b_1 v_1^5 - 2b_2 v_1^4 v_2 \\
&+ b_5 v_1^2 v_2^3 + 2b_6 v_1 v_2^4 + 4a_1 v_1^2 v_2^2 - 2a_3 v_2^4 - 4b_1 v_1^3 v_2 - b_2 v_1^2 v_2^2 + 2a_1 v_2^3 - 2b_1 v_1 v_2^2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-8a_5 + 4b_6) v_1^3 v_2^3 + (-4a_3 - 2a_4 + b_5) v_1^2 v_2^3 + (4a_1 - b_2) v_1^2 v_2^2 \\
&+ (-4a_5 + 2b_6) v_1 v_2^4 + (-8a_5 + 2b_6) v_1^9 v_2 + (-7a_3 - 7a_4 + 2b_5) v_1^8 v_2 \\
&+ (-10a_5 + 3b_6) v_1^7 v_2^2 + (-6a_2 + 2b_3) v_1^7 v_2 + (-5a_1 - 2a_4 + b_5) v_1^6 v_2 \\
&+ (-4a_5 + 2b_6) v_1^5 v_2^2 + (-2a_3 - 4a_4 + 2b_5) v_1^4 v_2^2 + (2a_1 - 2b_2) v_1^4 v_2 - 6a_6 v_1^4 v_2^3 \\
&- 12a_6 v_1^2 v_2^4 - 4b_1 v_1^3 v_2 - 2b_1 v_1 v_2^2 + (-4a_2 + b_3 + b_4) v_1^9 + (-3a_1 + b_2) v_1^8 \\
&+ (-5a_4 + b_5) v_1^{10} - 2a_6 v_1^{12} v_2 - 11a_6 v_1^8 v_2^2 - 13a_6 v_1^6 v_2^3 - 9a_3 v_1^6 v_2^2 \\
&- a_5 v_1^{13} - a_3 v_1^{12} + b_1 v_1^7 - b_2 v_1^6 - 6a_6 v_2^5 - 2b_1 v_1^5 - 2a_3 v_2^4 + 2a_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
2a_1 &= 0 \\
-9a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-a_5 &= 0 \\
-13a_6 &= 0 \\
-12a_6 &= 0 \\
-11a_6 &= 0 \\
-6a_6 &= 0 \\
-2a_6 &= 0 \\
-4b_1 &= 0 \\
-2b_1 &= 0 \\
-b_2 &= 0 \\
-3a_1 + b_2 &= 0 \\
2a_1 - 2b_2 &= 0 \\
4a_1 - b_2 &= 0 \\
-6a_2 + 2b_3 &= 0 \\
-5a_4 + b_5 &= 0 \\
-10a_5 + 3b_6 &= 0 \\
-8a_5 + 2b_6 &= 0 \\
-8a_5 + 4b_6 &= 0 \\
-4a_5 + 2b_6 &= 0 \\
-5a_1 - 2a_4 + b_5 &= 0 \\
-4a_2 + b_3 + b_4 &= 0 \\
-7a_3 - 7a_4 + 2b_5 &= 0 \\
-4a_3 - 2a_4 + b_5 &= 0 \\
-2a_3 - 4a_4 + 2b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_4 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3b_4 \\
b_4 &= b_4 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= x^2 + 3y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x^2 + 3y - \left(\frac{x^6 + 2y\,x^2 + 2y^2}{x\,(x^2 + y)}\right)(x) \\
&= \frac{-x^6 + x^4 + 2y\,x^2 + y^2}{x^2 + y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^6+x^4+2y\,x^2+y^2}{x^2+y}} dy \end{aligned}$$

Which results in

$$S=\frac{\ln\left(-x^6+x^4+2y\,x^2+y^2\right)}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{x^6+2y\,x^2+2y^2}{x\left(x^2+y\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x(3x^4-2x^2-2y)}{\left(x^3+x^2+y\right)\left(x^3-x^2-y\right)} \\ S_y &= \frac{-x^2-y}{\left(x^3+x^2+y\right)\left(x^3-x^2-y\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

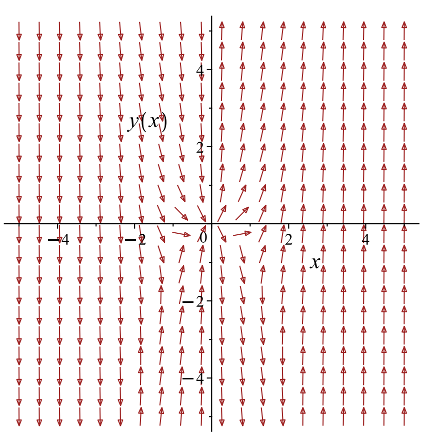
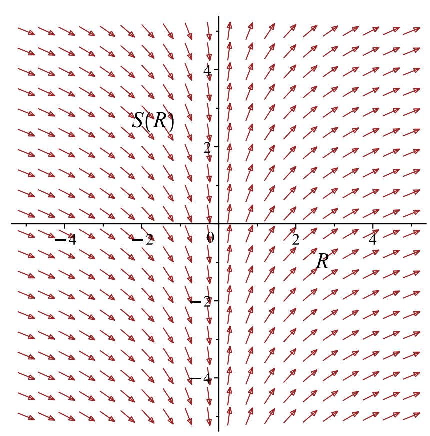
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-x^3 - x^2 - y)}{2} + \frac{\ln(x^3 - x^2 - y)}{2} = 2\ln(x) + c_1$$

Which simplifies to

$$\frac{\ln(-x^3 - x^2 - y)}{2} + \frac{\ln(x^3 - x^2 - y)}{2} = 2\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^6+2y\,x^2+2y^2}{x(x^2+y)}$ 	$R = x$ $S = \frac{\ln(-x^3-x^2-y)}{2} +$	$\frac{dS}{dR} = \frac{2}{R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(-x^3 - x^2 - y)}{2} + \frac{\ln(x^3 - x^2 - y)}{2} = 2\ln(x) + c_1 \tag{1}$$

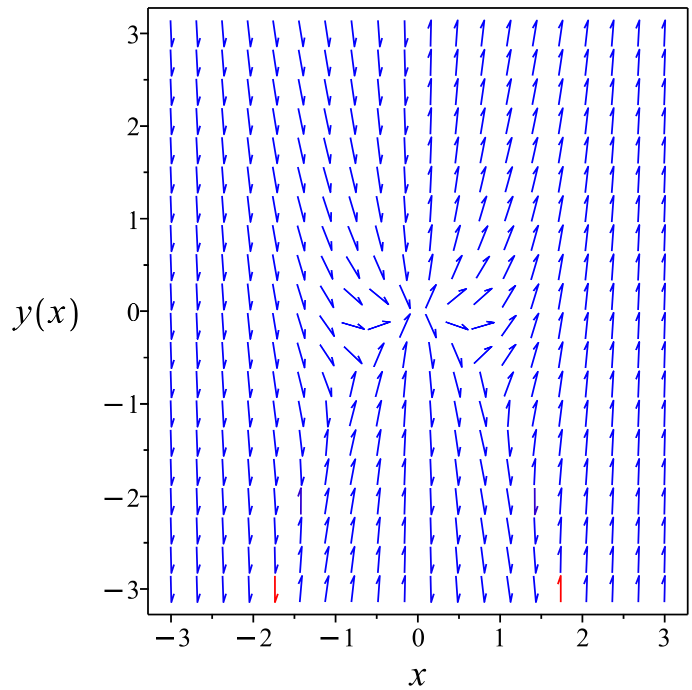


Figure 222: Slope field plot

Verification of solutions

$$\frac{\ln(-x^3 - x^2 - y)}{2} + \frac{\ln(x^3 - x^2 - y)}{2} = 2\ln(x) + c_1$$

Verified OK.

3.28.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(x^2 + y\right)\right)\mathrm{d}y &= \left(x^6 + 2y\,x^2 + 2y^2\right)\mathrm{d}x \\ \left(-x^6 - 2y\,x^2 - 2y^2\right)\mathrm{d}x + \left(x\left(x^2 + y\right)\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^6 - 2y\,x^2 - 2y^2 \\ N(x, y) &= x\left(x^2 + y\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^6 - 2y\,x^2 - 2y^2\right) \\ &= -2x^2 - 4y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^{2}+y\right)\right) \\
&= 3x^{2}+y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(x^{2}+y\right)}\left(\left(-2x^{2}-4y\right)-\left(3x^{2}+y\right)\right) \\
&= -\frac{5}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{5}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-5\ln(x)} \\
&= \frac{1}{x^{5}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^{5}}\left(-x^{6}-2y\,x^{2}-2y^{2}\right) \\
&= \frac{-x^{6}-2y\,x^{2}-2y^{2}}{x^{5}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{5}}\left(x\left(x^{2}+y\right)\right) \\
&= \frac{x^{2}+y}{x^{4}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^6 - 2y\,x^2 - 2y^2}{x^5}\right) + \left(\frac{x^2 + y}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\frac{-x^6 - 2y\,x^2 - 2y^2}{x^5}\,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + \frac{y^2}{2x^4} + \frac{y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{y}{x^4} + \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{x^2+y}{x^4}$. Therefore equation (4) becomes

$$\frac{x^2 + y}{x^4} = \frac{x^2 + y}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2x^4} + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2x^4} + \frac{y}{x^2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2x^4} + \frac{y}{x^2} = c_1 \tag{1}$$

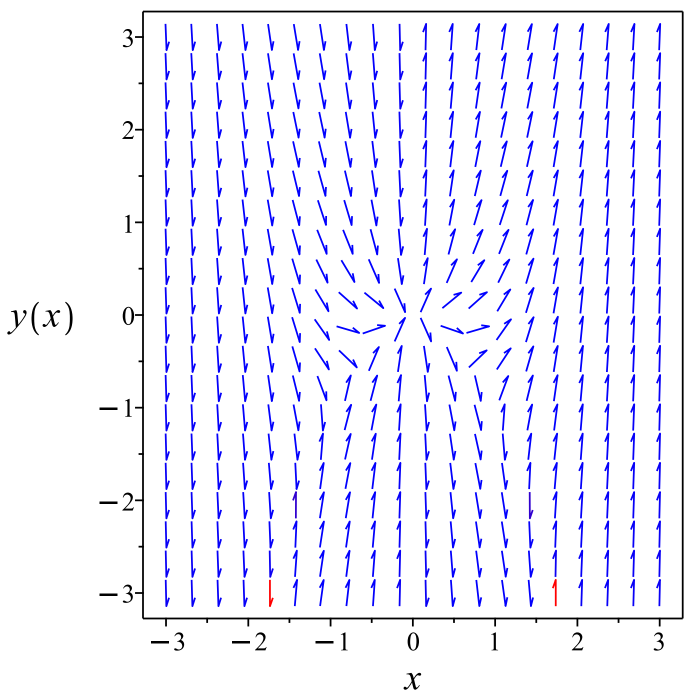


Figure 223: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2x^4} + \frac{y}{x^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(x*diff(y(x),x)-2*y(x)=x^6/(y(x)+x^2),y(x), singsol=all)
```

$$y(x) = \left(-1-\sqrt{x^2-2c_1+1}\right)x^2$$
$$y(x) = \left(-1+\sqrt{x^2-2c_1+1}\right)x^2$$

✓ Solution by Mathematica

Time used: 0.635 (sec). Leaf size: 70

```
DSolve[x*y'[x]-2*y[x]==x^6/(y[x]+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2\left(1+\sqrt{\frac{1}{x^5}}x^2\sqrt{x\left(x^2+1+c_1\right)}\right)$$
$$y(x) \to -x^2+\sqrt{\frac{1}{x^5}}x^4\sqrt{x\left(x^2+1+c_1\right)}$$

3.29 problem 37

Internal problem ID [956]
Internal file name [`OUTPUT/956_Sunday_June_05_2022_01_55_10_AM_76344571/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{y' - y - \frac{(x+1)\,\mathrm{e}^{4x}}{(y+\mathrm{e}^{x})^{2}} = 0}$$

3.29.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left((y+\mathrm{e}^{x})^{2}\right)\mathrm{d}y=\left(\mathrm{e}^{2x}y+2y^{2}\mathrm{e}^{x}+y^{3}+x\,\mathrm{e}^{4x}+\mathrm{e}^{4x}\right)\mathrm{d}x$$

$$\left(-\mathrm{e}^{2x}y-2y^{2}\mathrm{e}^{x}-y^{3}-x\,\mathrm{e}^{4x}-\mathrm{e}^{4x}\right)\mathrm{d}x+\left((y+\mathrm{e}^{x})^{2}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\mathrm{e}^{2x}y-2y^{2}\mathrm{e}^{x}-y^{3}-x\,\mathrm{e}^{4x}-\mathrm{e}^{4x}\\N(x,y)&=(y+\mathrm{e}^{x})^{2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\mathrm{e}^{2x}y-2y^{2}\mathrm{e}^{x}-y^{3}-x\,\mathrm{e}^{4x}-\mathrm{e}^{4x}\right)\\&=-3y^{2}-4\,\mathrm{e}^{x}y-\mathrm{e}^{2x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left((y+\mathrm{e}^{x})^{2}\right)\\&=2(y+\mathrm{e}^{x})\,\mathrm{e}^{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{(y+\mathrm{e}^{x})^{2}}\left(\left(-3y^{2}-4\,\mathrm{e}^{x}y-\mathrm{e}^{2x}\right)-\left(2(y+\mathrm{e}^{x})\,\mathrm{e}^{x}\right)\right)\\&=-3\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -3\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3x}\\ &= \mathrm{e}^{-3x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-3x}\left(-\mathrm{e}^{2x}y - 2y^2\mathrm{e}^{x} - y^3 - x\,\mathrm{e}^{4x} - \mathrm{e}^{4x}\right)\\ &= -\left(\mathrm{e}^{2x}y + 2y^2\mathrm{e}^{x} + y^3 + x\,\mathrm{e}^{4x} + \mathrm{e}^{4x}\right)\mathrm{e}^{-3x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-3x}\left((y+\mathrm{e}^{x})^2\right)\\ &= (y+\mathrm{e}^{x})^2\,\mathrm{e}^{-3x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\left(\mathrm{e}^{2x}y + 2y^2\mathrm{e}^{x} + y^3 + x\,\mathrm{e}^{4x} + \mathrm{e}^{4x}\right)\mathrm{e}^{-3x}\right) + \left((y+\mathrm{e}^{x})^2\,\mathrm{e}^{-3x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\left(\mathrm{e}^{2x}y + 2y^2\mathrm{e}^{x} + y^3 + x\,\mathrm{e}^{4x} + \mathrm{e}^{4x}\right)\mathrm{e}^{-3x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\mathrm{e}^{-3x}\left(x\,\mathrm{e}^{4x} - \frac{(3\,\mathrm{e}^{2x} + y(y+3\,\mathrm{e}^{x}))\,y}{3}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\mathrm{e}^{-3x}\left(-\frac{(2y+3\,\mathrm{e}^{x})\,y}{3}-\mathrm{e}^{2x}-\frac{y(y+3\,\mathrm{e}^{x})}{3}\right)+f'(y) \\ &= \left(y^{2}+2\,\mathrm{e}^{x} y+\mathrm{e}^{2x}\right)\mathrm{e}^{-3x}+f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y+\mathrm{e}^{x})^{2}\,\mathrm{e}^{-3x}$. Therefore equation (4) becomes

$$(y+\mathrm{e}^{x})^{2}\,\mathrm{e}^{-3x} = \left(y^{2}+2\,\mathrm{e}^{x} y+\mathrm{e}^{2x}\right)\mathrm{e}^{-3x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{-3x}\left(x\,\mathrm{e}^{4x}-\frac{(3\,\mathrm{e}^{2x}+y(y+3\,\mathrm{e}^{x}))\,y}{3}\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{-3x}\left(x\,\mathrm{e}^{4x}-\frac{(3\,\mathrm{e}^{2x}+y(y+3\,\mathrm{e}^{x}))\,y}{3}\right)$$

Summary

The solution(s) found are the following

$$-\mathrm{e}^{-3x}\left(x\,\mathrm{e}^{4x}-\frac{(3\,\mathrm{e}^{2x}+y(y+3\,\mathrm{e}^{x}))\,y}{3}\right) = c_1 \tag{1}$$

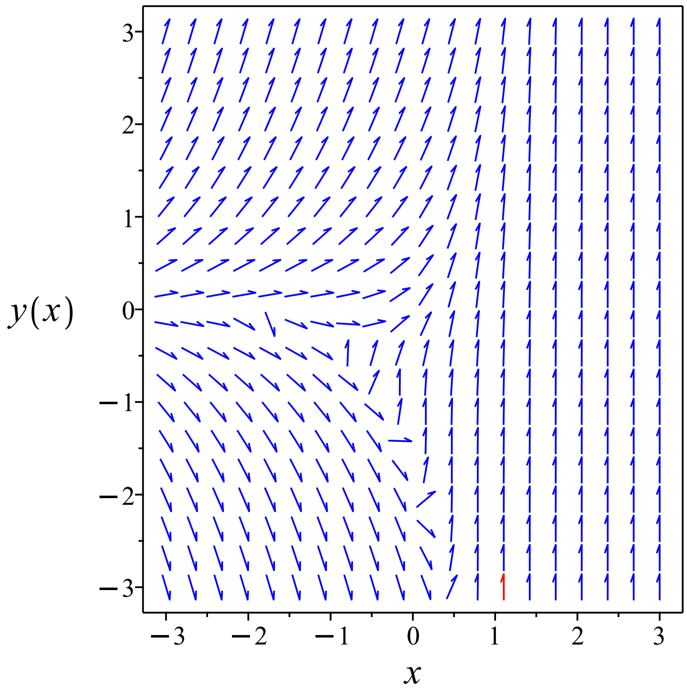


Figure 224: Slope field plot

Verification of solutions

$$-e^{-3x}\left(x\,e^{4x}-\frac{(3\,e^{2x}+y(y+3\,e^{x}))\,y}{3}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 92

```
dsolve(diff(y(x),x)-y(x)=((x+1)*exp(4*x))/(y(x)+exp(x))^2,y(x), singsol=all)
```

$$y(x) = e^{x}\left(-1+\left(3x\,e^{x}-3c_{1}+1\right)^{\frac{1}{3}}\right)$$
$$y(x) = -\frac{e^{x}\left(i\sqrt{3}\left(3x\,e^{x}-3c_{1}+1\right)^{\frac{1}{3}}+\left(3x\,e^{x}-3c_{1}+1\right)^{\frac{1}{3}}+2\right)}{2}$$
$$y(x) = \frac{e^{x}\left(i\sqrt{3}-1\right)\left(3x\,e^{x}-3c_{1}+1\right)^{\frac{1}{3}}}{2}-e^{x}$$

✓ Solution by Mathematica

Time used: 19.018 (sec). Leaf size: 143

```
DSolve[y'[x]-y[x]==((x+1)*Exp[4*x])/(y[x]+Exp[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{x}+e^{3x}\sqrt[3]{e^{-6x}\left(3e^{x}x+1+3c_{1}\right)}$$
$$y(x) \to -e^{x}+\frac{1}{2}i\left(\sqrt{3}+i\right)e^{3x}\sqrt[3]{e^{-6x}\left(3e^{x}x+1+3c_{1}\right)}$$
$$y(x) \to -e^{x}-\frac{1}{2}\left(1+i\sqrt{3}\right)e^{3x}\sqrt[3]{e^{-6x}\left(3e^{x}x+1+3c_{1}\right)}$$

3.30 problem 38

Internal problem ID [957]
Internal file name [OUTPUT/957_Sunday_June_05_2022_01_55_14_AM_57865975/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. separable equations. Section 2.2 Page 52
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class A`]]
```

$$\boxed{y' - 2y - \frac{x\,\mathrm{e}^{2x}}{1 - y\,\mathrm{e}^{-2x}} = 0}$$

3.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(y\,\mathrm{e}^{-2x} - 1\right)\mathrm{d}y &= \left(2y^2\mathrm{e}^{-2x} - x\,\mathrm{e}^{2x} - 2y\right)\mathrm{d}x \\
\left(-2y^2\mathrm{e}^{-2x} + x\,\mathrm{e}^{2x} + 2y\right)\mathrm{d}x + \left(y\,\mathrm{e}^{-2x} - 1\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -2y^2\mathrm{e}^{-2x} + x\,\mathrm{e}^{2x} + 2y \\
N(x, y) &= y\,\mathrm{e}^{-2x} - 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y^2\mathrm{e}^{-2x} + x\,\mathrm{e}^{2x} + 2y\right) \\
&= -4y\,\mathrm{e}^{-2x} + 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\,\mathrm{e}^{-2x} - 1\right) \\
&= -2y\,\mathrm{e}^{-2x}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{y\,\mathrm{e}^{-2x} - 1}\left(\left(-4y\,\mathrm{e}^{-2x} + 2\right) - \left(-2y\,\mathrm{e}^{-2x}\right)\right) \\
&= -2
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -2\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2x}\\ &= \mathrm{e}^{-2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2x}\left(-2y^2\mathrm{e}^{-2x} + x\,\mathrm{e}^{2x} + 2y\right)\\ &= \left(x\,\mathrm{e}^{4x} - 2y\left(y - \mathrm{e}^{2x}\right)\right)\mathrm{e}^{-4x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2x}\left(y\,\mathrm{e}^{-2x} - 1\right)\\ &= \left(y\,\mathrm{e}^{-2x} - 1\right)\mathrm{e}^{-2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(x\,\mathrm{e}^{4x} - 2y\left(y - \mathrm{e}^{2x}\right)\right)\mathrm{e}^{-4x}\right) + \left(\left(y\,\mathrm{e}^{-2x} - 1\right)\mathrm{e}^{-2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \left(x\,\mathrm{e}^{4x} - 2y\left(y - \mathrm{e}^{2x}\right)\right)\mathrm{e}^{-4x}\,\mathrm{d}x$$

$$\phi = \frac{x^2}{2} + \frac{\mathrm{e}^{-4x}y^2}{2} - y\,\mathrm{e}^{-2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-4x} y - \mathrm{e}^{-2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (y\,\mathrm{e}^{-2x} - 1)\,\mathrm{e}^{-2x}$. Therefore equation (4) becomes

$$\left(y\,\mathrm{e}^{-2x} - 1\right)\mathrm{e}^{-2x} = \mathrm{e}^{-4x} y - \mathrm{e}^{-2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} + \frac{\mathrm{e}^{-4x} y^2}{2} - y\,\mathrm{e}^{-2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} + \frac{\mathrm{e}^{-4x} y^2}{2} - y\,\mathrm{e}^{-2x}$$

Summary
The solution(s) found are the following

$$\frac{x^2}{2} + \frac{\mathrm{e}^{-4x} y^2}{2} - y\,\mathrm{e}^{-2x} = c_1 \tag{1}$$

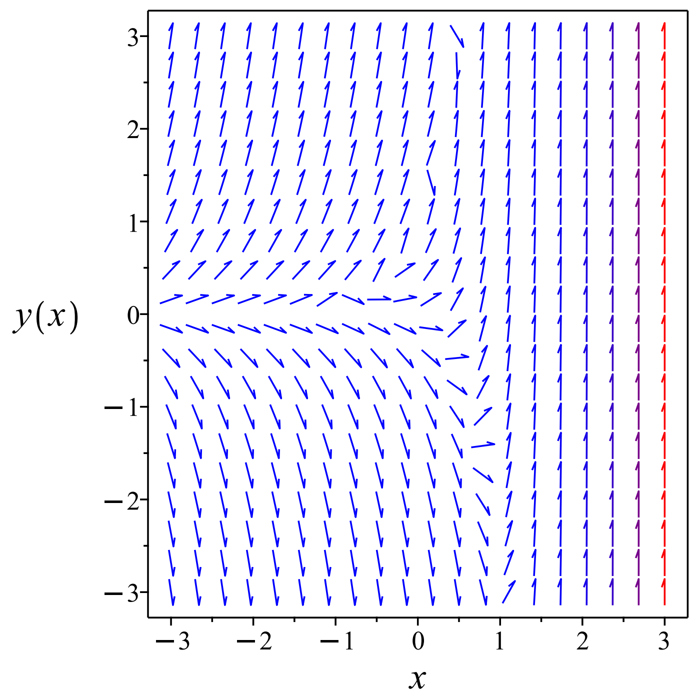


Figure 225: Slope field plot

Verification of solutions

$$\frac{x^2}{2}+\frac{e^{-4x}y^2}{2}-y\,e^{-2x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(diff(y(x),x)-2*y(x)=x*exp(2*x)/(1-y(x)*exp(-2*x)),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{4x}\sqrt{-\mathrm{e}^{-4x}\left(x^2 + 2c_1 - 1\right)} + \mathrm{e}^{2x}$$
$$y(x) = -\mathrm{e}^{4x}\sqrt{-\mathrm{e}^{-4x}\left(x^2 + 2c_1 - 1\right)} + \mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.777 (sec). Leaf size: 72

```
DSolve[y'[x]-2*y[x]==x*Exp[2*x]/(1-y[x]*Exp[-2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x} - \frac{\sqrt{x^2 - 1 - c_1}}{\sqrt{-e^{-4x}}}$$
$$y(x) \to e^{2x} + \frac{\sqrt{x^2 - 1 - c_1}}{\sqrt{-e^{-4x}}}$$

4 Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60

4.1 problem 1

Internal problem ID [958]
Internal file name [`OUTPUT/958_Sunday_June_05_2022_01_55_16_AM_8468720/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' - \frac{x^2 + y^2}{\sin(x)} = 0}$$

4.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{\sin(x)} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{x^2}{\sin(x)} + \frac{y^2}{\sin(x)}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x^2}{\sin(x)}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{\sin(x)}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{\sin(x)}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\cos(x)}{\sin(x)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{x^2}{\sin(x)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{\sin(x)} + \frac{\cos(x) u'(x)}{\sin(x)^2} + \frac{x^2 u(x)}{\sin(x)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \cot(x)\_Y'(x) + \csc(x)^2 x^2 \_Y(x)\right\}, \left\{\_Y(x)\right\}\right)$$

The above shows that

$$u'(x) = \frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) + \cot(x)\_Y'(x) + \csc(x)^2 x^2 \_Y(x)\right\}, \left\{\_Y(x)\right\}\right)$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{d}{dx}\mathrm{DESol}\left(\left\{\_Y''(x) + \cot(x)\_Y'(x) + \csc(x)^2 x^2 \_Y(x)\right\}, \left\{\_Y(x)\right\}\right)\right)\sin(x)}{\mathrm{DESol}\left(\left\{\_Y''(x) + \cot(x)\_Y'(x) + \csc(x)^2 x^2 \_Y(x)\right\}, \left\{\_Y(x)\right\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)\right)\sin(x)}{\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)\right)\sin(x)}{\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)} \quad (1)$$

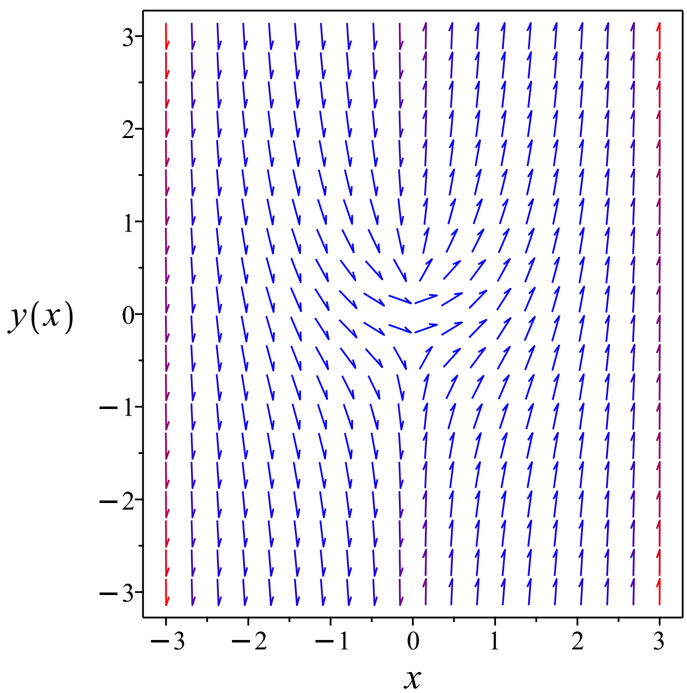


Figure 226: Slope field plot

Verification of solutions

$$y = -\frac{\left(\frac{d}{dx}\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)\right)\sin(x)}{\operatorname{DESol}\left(\left\{\_Y''(x)+\cot(x)\_Y'(x)+\csc(x)^{2}x^{2}\_Y(x)\right\},\left\{\_Y(x)\right\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -cos(x)*(diff(y(x), x))/sin(x)
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)/sin(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(x^2+y[x]^2)/Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.2 problem 2

Internal problem ID [959]
Internal file name [`OUTPUT/959_Sunday_June_05_2022_01_55_21_AM_90170821/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y' - \frac{y + e^{x}}{x^{2} + y^{2}} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(exp(x)*y(x)-K[1])/exp(x), y(x)`      ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-2*(y(x)*x+exp(x)*K[1])/x^2, y(x)`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(exp(x)*y(x)*x-2*exp(x)*y(x)+K[1]*x)/(x*e
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(exp(x)+y(x))/(x^2+y(x)^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(Exp[x]+y[x])/(x^2+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.3 problem 3

Internal problem ID [960]
Internal file name [OUTPUT/960_Sunday_June_05_2022_01_55_22_AM_80577424/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \tan(yx) = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
<- found 1 conformal symmetry. Proceeding with integration step
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=tan(x*y(x)),y(x), singsol=all)
```

$$y(x) = -i\,\mathrm{RootOf}\left(-\operatorname{erf}\left(\frac{(-x+\_Z)\sqrt{2}}{2}\right)\sqrt{\pi}-\sqrt{\pi}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+\_Z)}{2}\right)+\sqrt{2}\,c_1\right)$$

✓ Solution by Mathematica

Time used: 0.311 (sec). Leaf size: 69

```
DSolve[y'[x]==Tan[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{\frac{x^2}{2}}\left(\operatorname{erfi}\left(\frac{y(x)-ix}{\sqrt{2}}\right)+\operatorname{erfi}\left(\frac{y(x)+ix}{\sqrt{2}}\right)\right)=c_1e^{\frac{x^2}{2}},y(x)\right]$$

4.4 problem 4

Internal problem ID [961]
Internal file name [`OUTPUT/961_Sunday_June_05_2022_01_55_23_AM_35075872/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y' - \frac{x^2 + y^2}{\ln (yx)} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)/ln(x*y(x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(x^2+y[x]^2)/Log[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.5 problem 5

Internal problem ID [962]
Internal file name [`OUTPUT/962_Sunday_June_05_2022_01_55_25_AM_72143897/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - (x^2 + y^2)\, y^{\frac{1}{3}} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = (1/3)*y(x)/x, y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = (7/3)*y(x)/x, y(x)`      *** Sublevel 2
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2*y(x)/x, y(x)`      *** Sublevel 2 **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)*y(x)^(1/3),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(x^2+y[x]^2)*y[x]^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.6 problem 6

Internal problem ID [963]
Internal file name [`OUTPUT/963_Sunday_June_05_2022_01_55_26_AM_63170606/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{-2yx + y' = 0}$$

4.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2yx \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= 2x\,dx \\
\int \frac{1}{y}\,dy &= \int 2x\,dx \\
\ln(y) &= x^2 + c_1 \\
y &= e^{x^2+c_1} \\
&= c_1 e^{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

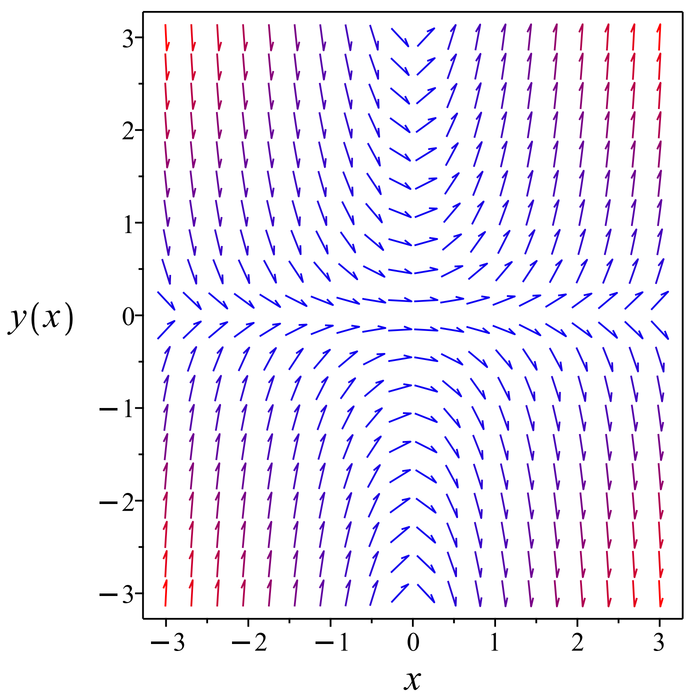


Figure 227: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

4.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$-2yx + y' = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -2x dx} \\ &= e^{-x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-x^2} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = c_1 e^{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

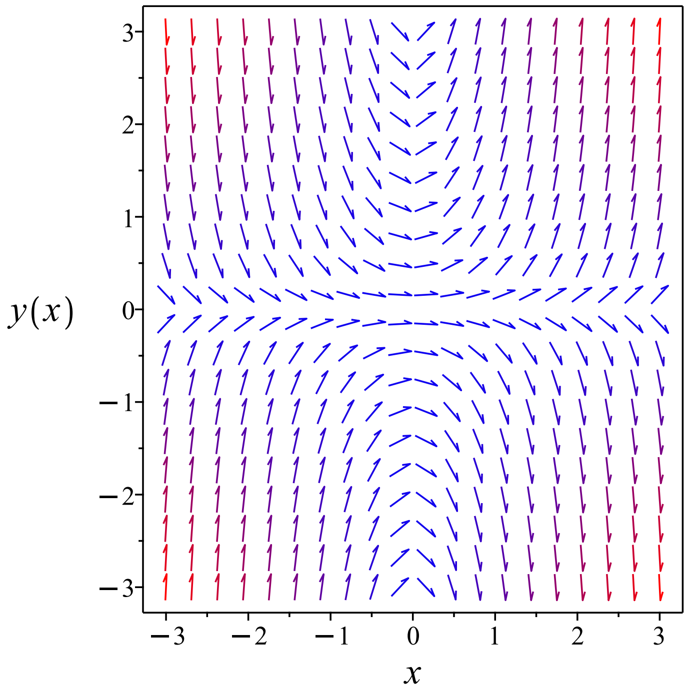


Figure 228: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

4.6.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-2u(x)\, x^2 + u'(x)\, x + u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(2x^2-1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{2x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x^2-1}{x}\,dx \\
\ln(u) &= x^2 - \ln(x) + c_2 \\
u &= e^{x^2-\ln(x)+c_2} \\
&= c_2 e^{x^2-\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 e^{x^2}}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 e^{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{x^2} \tag{1}$$

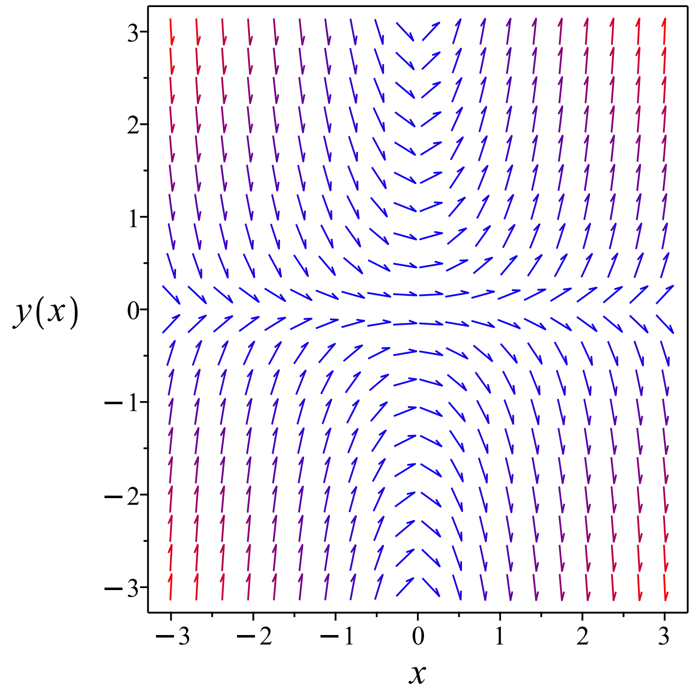


Figure 229: Slope field plot

Verification of solutions

$$y = c_2 e^{x^2}$$

Verified OK.

4.6.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2yx$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 215: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2}} dy \end{aligned}$$

Which results in

$$S = e^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2yx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x\, e^{-x^2} y \\ S_y &= e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = c_1$$

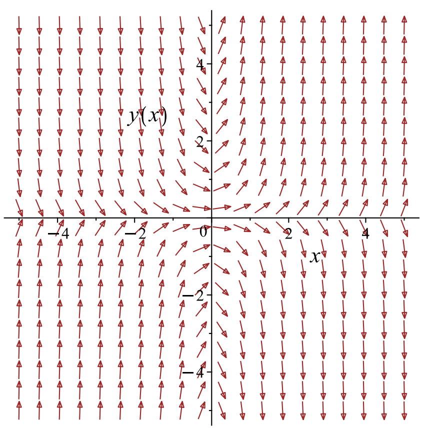
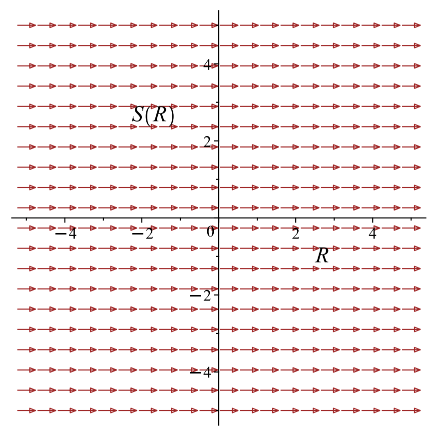
Which simplifies to

$$e^{-x^2} y = c_1$$

Which gives

$$y = c_1 e^{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2yx$ 	$R = x$ $S = e^{-x^2} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

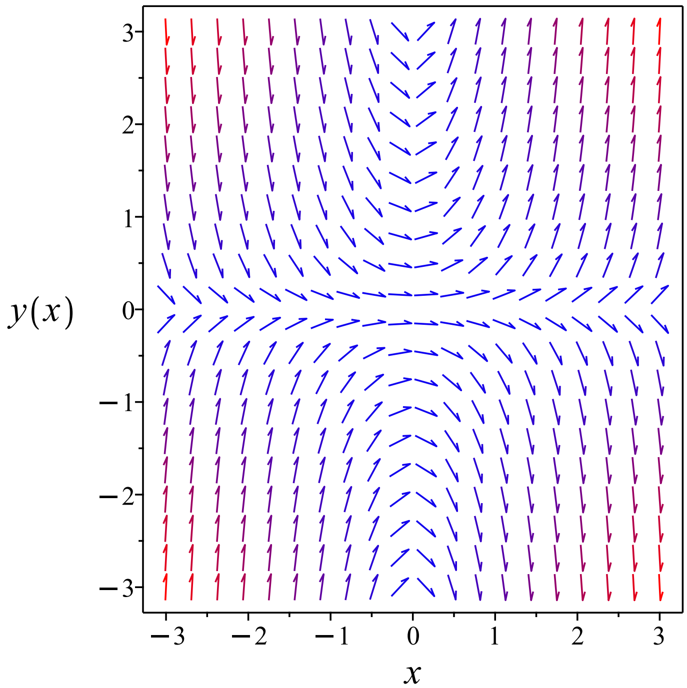


Figure 230: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

4.6.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y}\right) \mathrm{d}y &= (x) \, \mathrm{d}x \\
(-x) \, \mathrm{d}x + \left(\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$
\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{x^2+2c_1}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{x^2+2c_1} \tag{1}$$

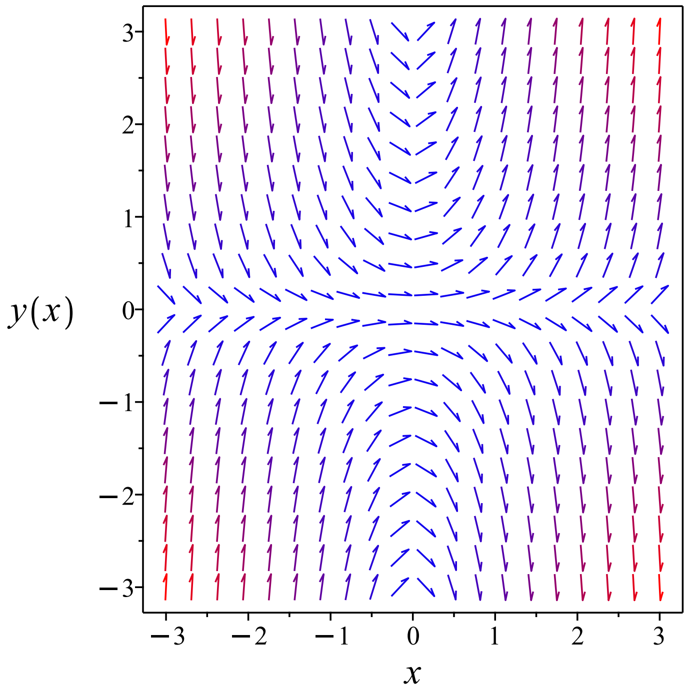


Figure 231: Slope field plot

Verification of solutions

$$y = e^{x^2+2c_1}$$

Verified OK.

4.6.6 Maple step by step solution

Let's solve

$-2yx + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

$$\ln(y) = x^2 + c_1$$

- Solve for y

$$y = e^{x^2 + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x) = e^{x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to 0$$

4.7 problem 7

Internal problem ID [964]
Internal file name [`OUTPUT/964_Sunday_June_05_2022_01_55_27_AM_80801569/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \ln\left(1 + x^2 + y^2\right) = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=ln(1+x^2+y(x)^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==Log[1+x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.8 problem 8

Internal problem ID [965]
Internal file name [`OUTPUT/965_Sunday_June_05_2022_01_55_28_AM_30487176/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x + 3y}{x - 4y} = 0}$$

4.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2x + 3u(x)\,x}{x - 4u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(2u^2 + u + 1)}{x\,(4u - 1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+u+1}{4u-1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u+1}{4u-1}} du = -\frac{2}{x} dx$$

$$\int \frac{1}{\frac{2u^2+u+1}{4u-1}} du = \int -\frac{2}{x} dx$$

$$\ln\left(2u^2+u+1\right) - \frac{4\sqrt{7}\,\arctan\left(\frac{(4u+1)\sqrt{7}}{7}\right)}{7} = -2\ln\left(x\right) + c_2$$

The solution is

$$\ln\left(2u(x)^2+u(x)+1\right) - \frac{4\sqrt{7}\,\arctan\left(\frac{(4u(x)+1)\sqrt{7}}{7}\right)}{7} + 2\ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{2y^2}{x^2} + \frac{y}{x} + 1\right) - \frac{4\sqrt{7}\,\arctan\left(\frac{\left(\frac{4y}{x}+1\right)\sqrt{7}}{7}\right)}{7} + 2\ln\left(x\right) - c_2 = 0$$

$$\ln\left(\frac{2y^2}{x^2} + \frac{y}{x} + 1\right) - \frac{4\sqrt{7}\,\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} + 2\ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{2y^2}{x^2} + \frac{y}{x} + 1\right) - \frac{4\sqrt{7}\,\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} + 2\ln\left(x\right) - c_2 = 0 \tag{1}$$

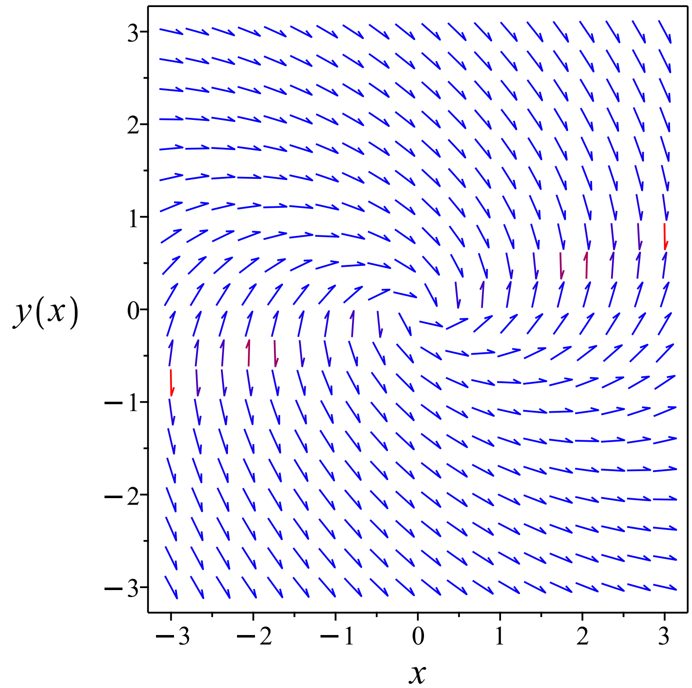


Figure 232: Slope field plot

Verification of solutions

$$\ln\left(\frac{2y^2}{x^2}+\frac{y}{x}+1\right)-\frac{4\sqrt{7}\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7}+2\ln(x)-c_2=0$$

Verified OK.

4.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2x+3y}{-x+4y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(2x+3y)(b_3-a_2)}{-x+4y} - \frac{(2x+3y)^2 a_3}{(-x+4y)^2} \\ & - \left(-\frac{2}{-x+4y} - \frac{2x+3y}{(-x+4y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{3}{-x+4y} + \frac{8x+12y}{(-x+4y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 + 10x^2b_2 - 2x^2b_3 - 16xya_2 + 12xya_3 + 8xyb_2 + 16xyb_3 - 12y^2a_2 - 2y^2a_3 - 16y^2b_2 + 12}{(x-4y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - 4x^2a_3 - 10x^2b_2 + 2x^2b_3 + 16xya_2 - 12xya_3 - 8xyb_2 \\ & - 16xyb_3 + 12y^2a_2 + 2y^2a_3 + 16y^2b_2 - 12y^2b_3 - 11xb_1 + 11ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2 + 16a_2v_1v_2 + 12a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 + 2a_3v_2^2 - 10b_2v_1^2 \\ & - 8b_2v_1v_2 + 16b_2v_2^2 + 2b_3v_1^2 - 16b_3v_1v_2 - 12b_3v_2^2 + 11a_1v_2 - 11b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-2a_2 - 4a_3 - 10b_2 + 2b_3)\, v_1^2 + (16a_2 - 12a_3 - 8b_2 - 16b_3)\, v_1 v_2 \\ - 11b_1 v_1 + (12a_2 + 2a_3 + 16b_2 - 12b_3)\, v_2^2 + 11a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 11a_1 &= 0 \\ -11b_1 &= 0 \\ -2a_2 - 4a_3 - 10b_2 + 2b_3 &= 0 \\ 12a_2 + 2a_3 + 16b_2 - 12b_3 &= 0 \\ 16a_2 - 12a_3 - 8b_2 - 16b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_2 + b_3 \\ a_3 &= -2b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{2x + 3y}{-x + 4y}\right)(x) \\ &= \frac{-2x^2 - 2yx - 4y^2}{x - 4y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2-2yx-4y^2}{x-4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + yx + 2y^2)}{2} - \frac{2\sqrt{7}\arctan\left(\frac{(x+4y)\sqrt{7}}{7x}\right)}{7}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + 3y}{-x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + 3y}{2x^2 + 2yx + 4y^2} \\ S_y &= \frac{-x + 4y}{2x^2 + 2yx + 4y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

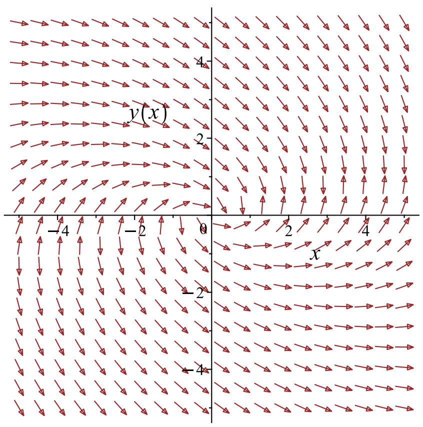
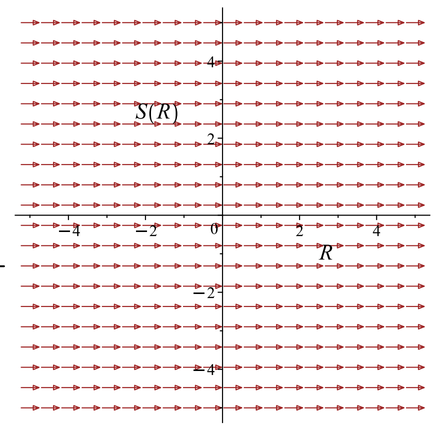
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(2y^2 + yx + x^2\right)}{2} - \frac{2\sqrt{7}\,\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} = c_1$$

Which simplifies to

$$\frac{\ln\left(2y^2 + yx + x^2\right)}{2} - \frac{2\sqrt{7}\,\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y}{-x+4y}$ 	$R = x$ $S = \frac{\ln(x^2 + yx + 2y^2)}{2} -$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2y^2 + yx + x^2)}{2} - \frac{2\sqrt{7}\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} = c_1 \tag{1}$$

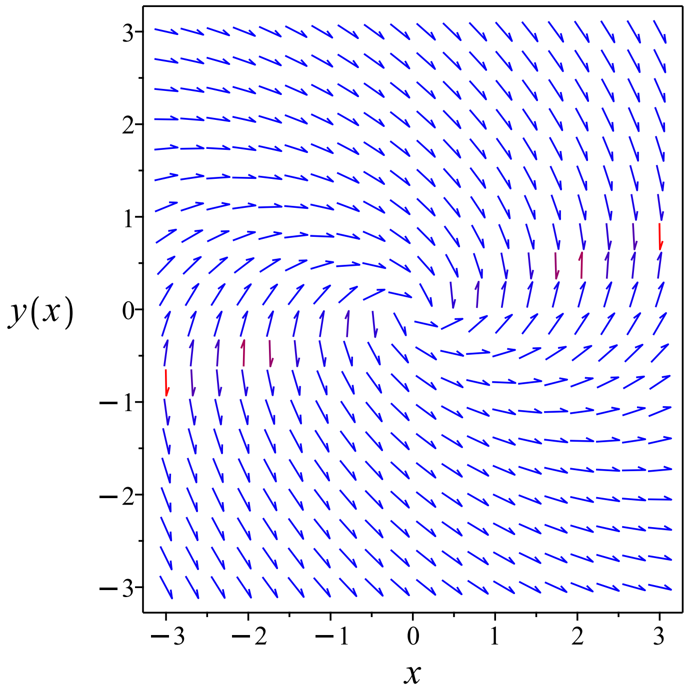


Figure 233: Slope field plot

Verification of solutions

$$\frac{\ln(2y^2 + yx + x^2)}{2} - \frac{2\sqrt{7}\,\arctan\left(\frac{(4y+x)\sqrt{7}}{7x}\right)}{7} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 51

```
dsolve(diff(y(x),x)=(2*x+3*y(x))/(x-4*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{x\left(\sqrt{7}\tan\left(\operatorname{RootOf}\left(\sqrt{7}\ln\left(\sec\left(\_Z\right)^{2}x^{2}\right)+\sqrt{7}\ln\left(7\right)-3\sqrt{7}\ln\left(2\right)+2\sqrt{7}\,c_1-4\_Z\right)\right)-1\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 53

```
DSolve[y'[x]==(2*x+3*y[x])/(x-4*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\log\left(\frac{2y(x)^2}{x^2}+\frac{y(x)}{x}+1\right)-\frac{4\arctan\left(\frac{\frac{4y(x)}{x}+1}{\sqrt{7}}\right)}{\sqrt{7}}=-2\log(x)+c_1,y(x)\right]$$

4.9 problem 9

Internal problem ID [966]
Internal file name [`OUTPUT/966_Sunday_June_05_2022_01_55_30_AM_71086193/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \sqrt{x^2 + y^2} = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)^(1/2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(x^2+y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.10 problem 10

Internal problem ID [967]
Internal file name [`OUTPUT/967_Sunday_June_05_2022_01_55_31_AM_4635800/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x\left(-1 + y^2\right)^{\frac{2}{3}} = 0}$$

4.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x\left(y^2 - 1\right)^{\frac{2}{3}} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \left(y^2 - 1\right)^{\frac{2}{3}}$. Integrating both sides gives

$$\frac{1}{\left(y^2 - 1\right)^{\frac{2}{3}}}\, dy = x\, dx$$

$$\int \frac{1}{(y^2-1)^{\frac{2}{3}}}\, dy = \int x\, dx$$

$$\frac{(-\operatorname{signum}(y^2-1))^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(y^2-1)^{\frac{2}{3}}} = \frac{x^2}{2} + c_1$$

The solution is

$$\frac{(-\operatorname{signum}(-1+y^2))^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(-1+y^2)^{\frac{2}{3}}} - \frac{x^2}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{(-\operatorname{signum}(-1+y^2))^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(-1+y^2)^{\frac{2}{3}}} - \frac{x^2}{2} - c_1 = 0 \tag{1}$$

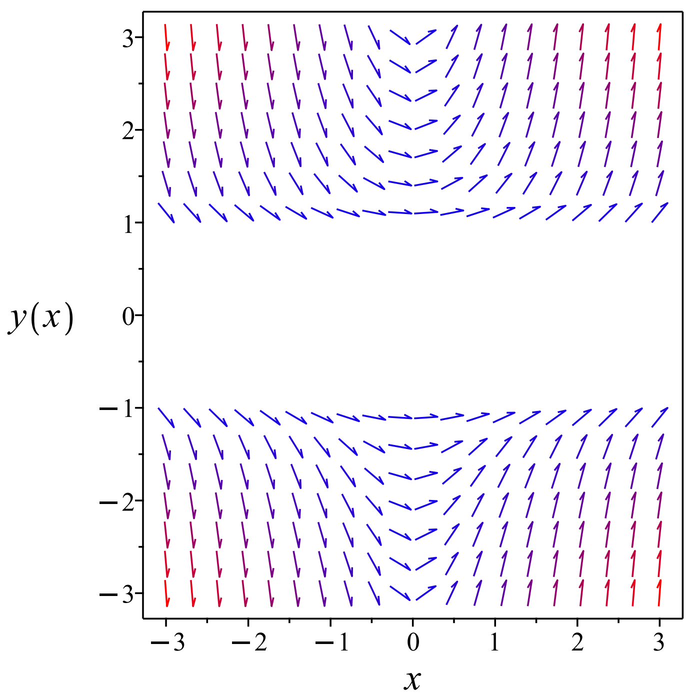


Figure 234: Slope field plot

Verification of solutions

$$\frac{\left(-\operatorname{signum}\left(-1+y^{2}\right)\right)^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right],\left[\frac{3}{2}\right], y^{2}\right)}{\operatorname{signum}\left(-1+y^{2}\right)^{\frac{2}{3}}}-\frac{x^{2}}{2}-c_{1}=0$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= x\left(y^{2}-1\right)^{\frac{2}{3}} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_{x}+\omega\left(\eta_{y}-\xi_{x}\right)-\omega^{2} \xi_{y}-\omega_{x} \xi-\omega_{y} \eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 218: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x(y^2 - 1)^{\frac{2}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y^2 - 1)^{\frac{2}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R^2 - 1)^{\frac{2}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{(-\operatorname{signum}(R^2-1))^{\frac{2}{3}} R \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], R^2\right)}{\operatorname{signum}(R^2-1)^{\frac{2}{3}}} + c_1 \tag{4}$$

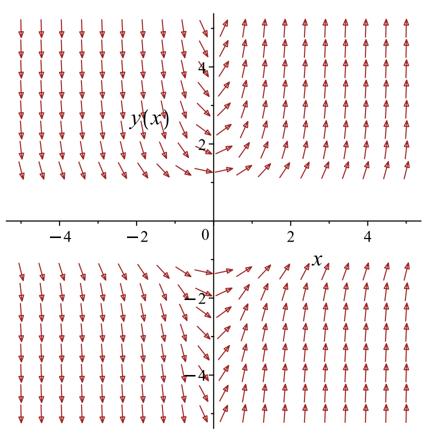
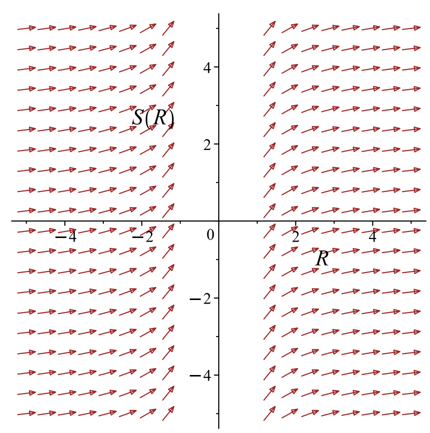
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{(-\operatorname{signum}(-1+y^2))^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(-1+y^2)^{\frac{2}{3}}} + c_1$$

Which simplifies to

$$\frac{y(1-i\sqrt{3}) \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{2} + \frac{x^2}{2} - c_1 = 0$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x(y^2-1)^{\frac{2}{3}}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{(R^2-1)^{\frac{2}{3}}}$ 

Summary

The solution(s) found are the following

$$\frac{y(1-i\sqrt{3}) \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{2} + \frac{x^2}{2} - c_1 = 0 \tag{1}$$

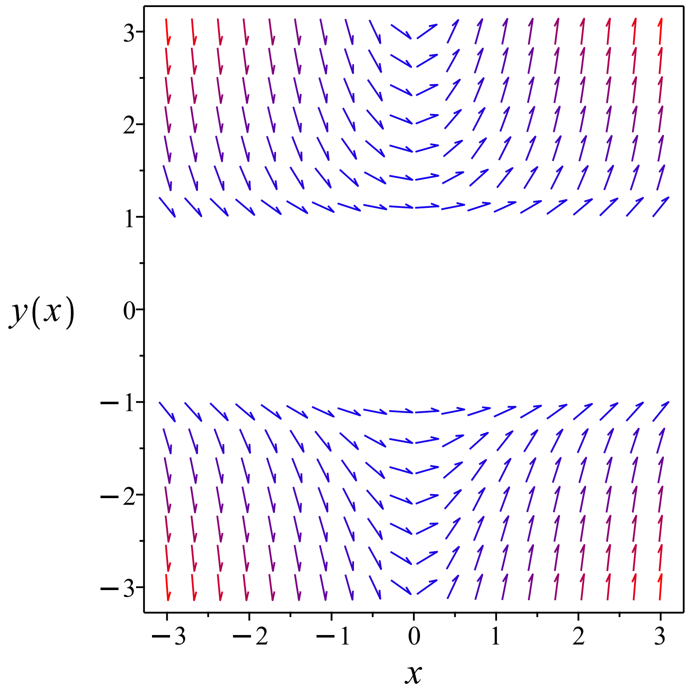


Figure 235: Slope field plot

Verification of solutions

$$\frac{y\left(1-i\sqrt{3}\right) \text{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{2} + \frac{x^2}{2} - c_1 = 0$$

Verified OK.

4.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{(y^2-1)^{\frac{2}{3}}}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{(y^2-1)^{\frac{2}{3}}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{(y^2-1)^{\frac{2}{3}}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{(y^2-1)^{\frac{2}{3}}}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(y^2-1)^{\frac{2}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{(y^2-1)^{\frac{2}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{(y^2-1)^{\frac{2}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{(y^2-1)^{\frac{2}{3}}}\right)\mathrm{d}y$$

$$f(y) = \frac{(-\operatorname{signum}(y^2-1))^{\frac{2}{3}}\, y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(y^2-1)^{\frac{2}{3}}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{(-\operatorname{signum}(y^2-1))^{\frac{2}{3}}\, y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(y^2-1)^{\frac{2}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{(-\operatorname{signum}(y^2-1))^{\frac{2}{3}}\, y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(y^2-1)^{\frac{2}{3}}}$$

Summary

The solution(s) found are the following

$$\frac{(-\operatorname{signum}(-1+y^2))^{\frac{2}{3}}\, y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}(-1+y^2)^{\frac{2}{3}}} - \frac{x^2}{2} = c_1 \tag{1}$$

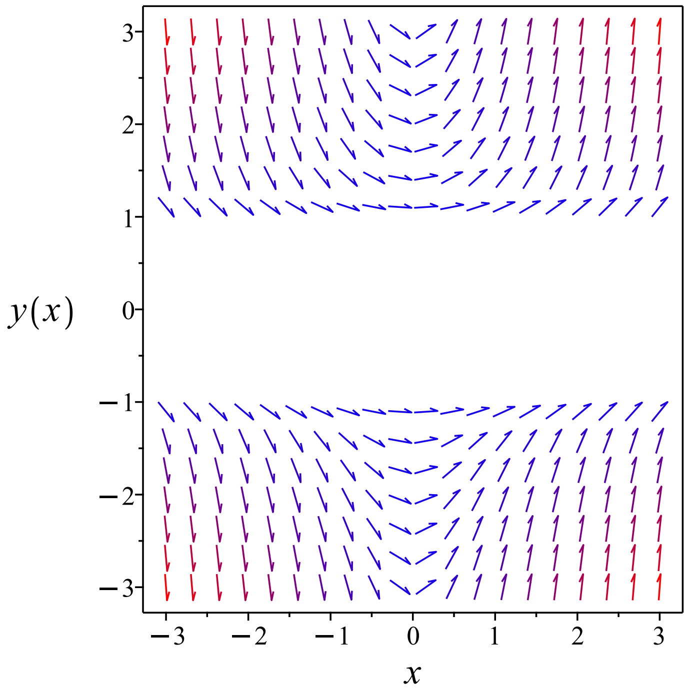


Figure 236: Slope field plot

Verification of solutions

$$\frac{\left(-\operatorname{signum}\left(-1+y^2\right)\right)^{\frac{2}{3}} y \operatorname{hypergeom}\left(\left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{3}{2}\right], y^2\right)}{\operatorname{signum}\left(-1+y^2\right)^{\frac{2}{3}}} - \frac{x^2}{2} = c_1$$

Verified OK.

4.10.4 Maple step by step solution

Let's solve

$y' - x(-1+y^2)^{\frac{2}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(-1+y^2)^{\frac{2}{3}}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{(-1+y^2)^{\frac{2}{3}}} dx = \int x dx + c_1$

- Cannot compute integral
  $\int \frac{y'}{(-1+y^2)^{\frac{2}{3}}}dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=x*(y(x)^2-1)^(2/3),y(x), singsol=all)
```

$$\frac{x^2}{2} - \frac{\left(-\operatorname{signum}\left(y(x)^2-1\right)\right)^{\frac{2}{3}} y(x) \operatorname{hypergeom}\left(\left[\frac{1}{2},\frac{2}{3}\right],\left[\frac{3}{2}\right],y(x)^2\right)}{\operatorname{signum}\left(y(x)^2-1\right)^{\frac{2}{3}}} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 6.768 (sec). Leaf size: 66

```
DSolve[y'[x]==x*(y[x]^2-1)^(2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\#1\left(1-\#1^2\right)^{2/3}\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{2}{3},\frac{3}{2},\#1^2\right)}{\left(\#1^2-1\right)^{2/3}}\&\right]\left[\frac{x^2}{2}+c_1\right]$$

$$y(x) \to -1$$

$$y(x) \to 1$$

4.11 problem 11

Internal problem ID [968]
Internal file name [`OUTPUT/968_Sunday_June_05_2022_01_55_33_AM_80678656/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \left(x^2 + y^2\right)^2 = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=(x^2+y(x)^2)^2,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==(x^2+y[x]^2)^2,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4.12 problem 12

Internal problem ID [969]
Internal file name [`OUTPUT/969_Sunday_June_05_2022_01_55_34_AM_74084635/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sqrt{x+y} = 0}$$

4.12.1 Solving as homogeneousTypeC ode

Let

$$z = x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = \sqrt{z}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{\sqrt{z}+1} dz$$
$$x + c_1 = 2\sqrt{z} + \ln\left(-1+\sqrt{z}\right) - \ln\left(\sqrt{z}+1\right) - \ln\left(z-1\right)$$

Replacing z back by its value from (1) then the above gives the solution as

$$2\sqrt{x+y} + \ln\left(-1+\sqrt{x+y}\right) - \ln\left(\sqrt{x+y}+1\right) - \ln\left(x-1+y\right) = x + c_1$$

Summary

The solution(s) found are the following

$$2\sqrt{x+y} + \ln\left(-1+\sqrt{x+y}\right) - \ln\left(\sqrt{x+y}+1\right) - \ln\left(x-1+y\right) = x + c_1 \tag{1}$$

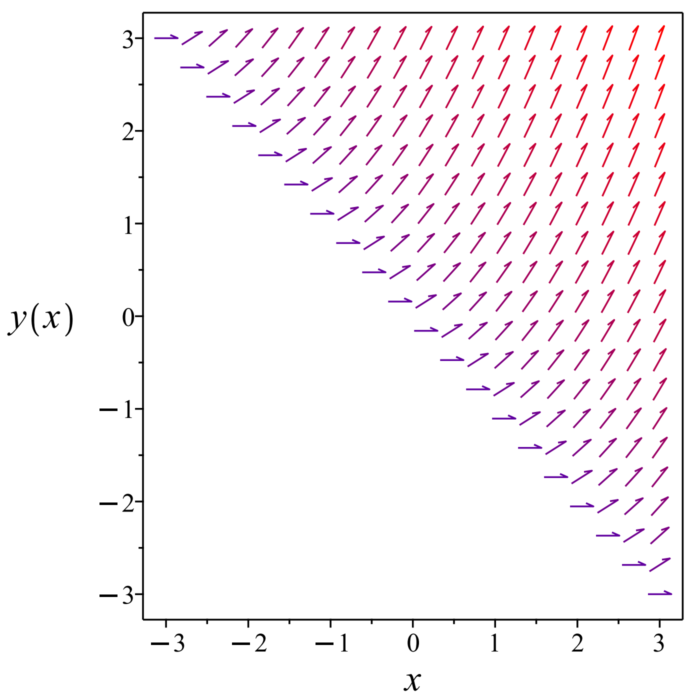


Figure 237: Slope field plot

Verification of solutions

$$2\sqrt{x+y} + \ln\left(-1+\sqrt{x+y}\right) - \ln\left(\sqrt{x+y}+1\right) - \ln\left(x-1+y\right) = x + c_1$$

Verified OK.

4.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \sqrt{x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 221: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 1\\ \eta(x,y) &= -1\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-1}{1}\\ &= -1\end{aligned}$$

This is easily solved to give

$$y = -x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sqrt{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{x+y}+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + \ln\left(-1+\sqrt{R}\right) - \ln\left(\sqrt{R}+1\right) - \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = 2\sqrt{x+y} + \ln\left(-1+\sqrt{x+y}\right) - \ln\left(\sqrt{x+y}+1\right) - \ln(x-1+y) + c_1$$

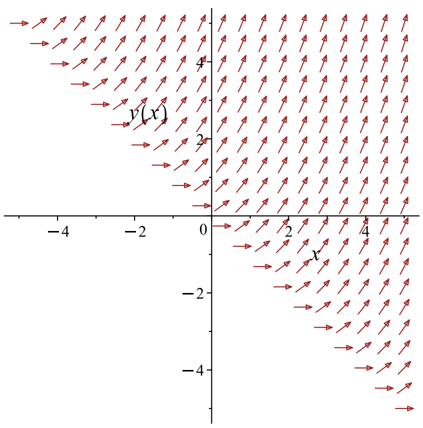
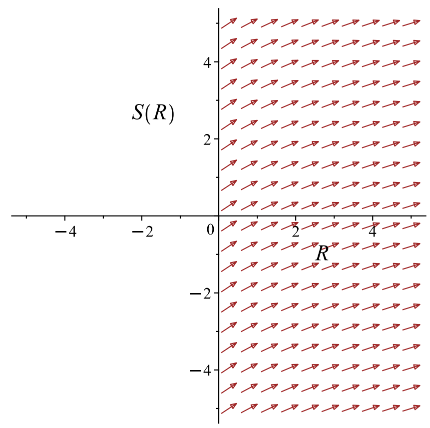
Which simplifies to

$$x = 2\sqrt{x+y} + \ln\left(-1+\sqrt{x+y}\right) - \ln\left(\sqrt{x+y}+1\right) - \ln(x-1+y) + c_1$$

Which gives

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,\mathrm{e}^{-\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{x+y}$ 	$R = x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}+1}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,\mathrm{e}^{-\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x + 1 \quad (1)$$

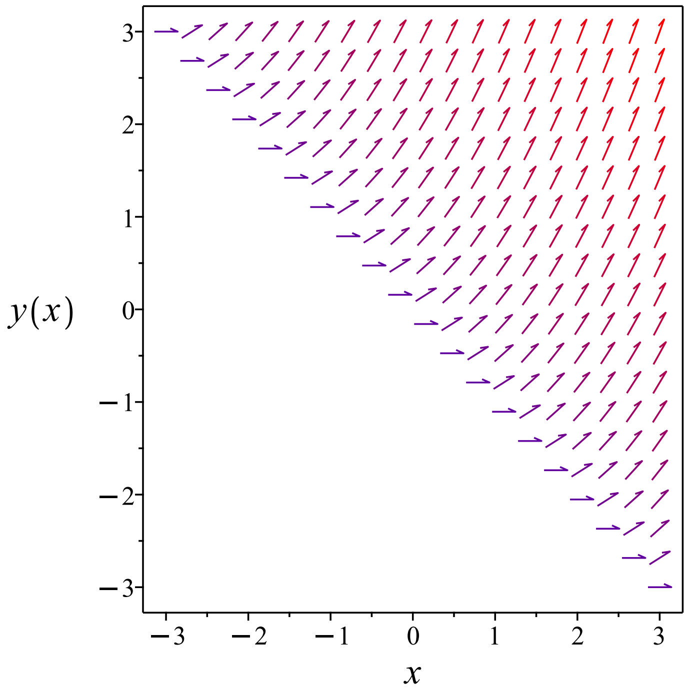


Figure 238: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,\mathrm{e}^{-\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(diff(y(x),x)=(x+y(x))^(1/2),y(x), singsol=all)
```

$$x-2\sqrt{x+y\left(x\right)}-\ln\left(-1+\sqrt{x+y\left(x\right)}\right)+\ln\left(1+\sqrt{x+y\left(x\right)}\right)+\ln\left(x+y(x)-1\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 9.242 (sec). Leaf size: 59

```
DSolve[y'[x]==(x+y[x])^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2+2W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)-x+1$$
$$y(x)\to 1-x$$

4.13 problem 13

Internal problem ID [970]
Internal file name [`OUTPUT/970_Sunday_June_05_2022_01_55_37_AM_91098866/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{\tan(y)}{x-1} = 0}$$

4.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\tan(y)}{x-1} \end{aligned}$$

Where $f(x) = \frac{1}{x-1}$ and $g(y) = \tan(y)$. Integrating both sides gives

$$\frac{1}{\tan(y)}\, dy = \frac{1}{x-1}\, dx$$

$$\int \frac{1}{\tan(y)}\,dy = \int \frac{1}{x-1}\,dx$$
$$\ln(\sin(y)) = \ln(x-1) + c_1$$

Raising both side to exponential gives

$$\sin(y) = \mathrm{e}^{\ln(x-1)+c_1}$$

Which simplifies to

$$\sin(y) = c_2(x-1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \arcsin\left(c_2 \mathrm{e}^{c_1}(x-1)\right) \tag{1}$$

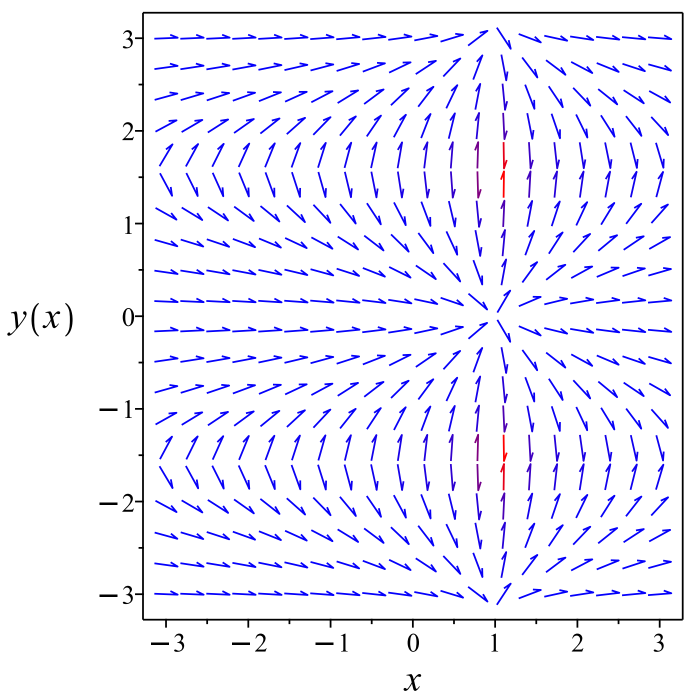


Figure 239: Slope field plot

<u>Verification of solutions</u>

$$y = \arcsin\left(c_2 \mathrm{e}^{c_1}(x-1)\right)$$

Verified OK.

4.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\tan(y)}{x-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 223: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x - 1 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x - 1} dx
\end{aligned}
$$

Which results in

$$
S = \ln (x - 1)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{\tan (y)}{x - 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x-1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cot(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cot(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\sin(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x-1) = \ln(\sin(y)) + c_1
$$

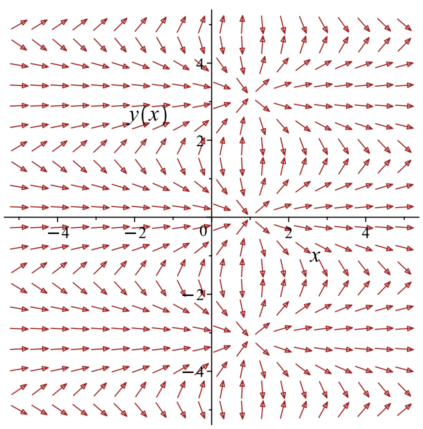
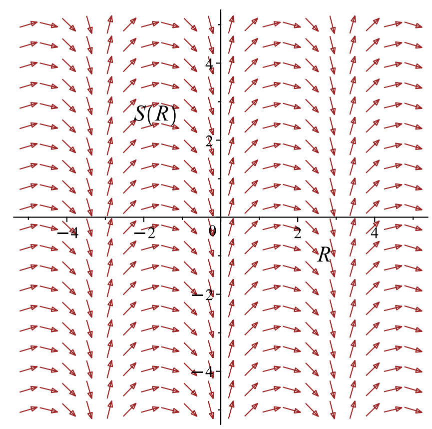
Which simplifies to

$$
\ln(x-1) = \ln(\sin(y)) + c_1
$$

Which gives

$$
y = \arcsin\left(\mathrm{e}^{-c_1}(x-1)\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{\tan(y)}{x-1}$
</td><td>$R = y$
$S = \ln(x-1)$</td><td>$\frac{dS}{dR} = \cot(R)$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \arcsin\left(\mathrm{e}^{-c_1}(x-1)\right) \tag{1}$$

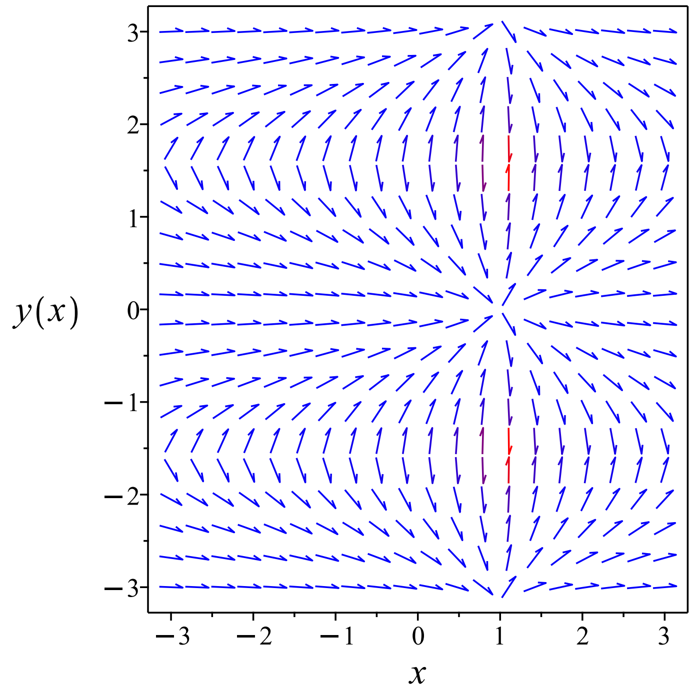


Figure 240: Slope field plot

Verification of solutions

$$y = \arcsin\left(\mathrm{e}^{-c_1}(x-1)\right)$$

Verified OK.

4.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\tan(y)}\right)\mathrm{d}y = \left(\frac{1}{x-1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x-1}\right)\mathrm{d}x + \left(\frac{1}{\tan(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x-1}$$
$$N(x, y) = \frac{1}{\tan(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x-1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\tan(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x-1} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x-1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\tan(y)}$. Therefore equation (4) becomes

$$
\frac{1}{\tan(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{\tan(y)} \\
&= \cot(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\cot(y))\,\mathrm{d}y$$
$$f(y) = \ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x-1) + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x-1) + \ln(\sin(y))$$

Summary

The solution(s) found are the following

$$-\ln(x-1) + \ln(\sin(y)) = c_1 \tag{1}$$

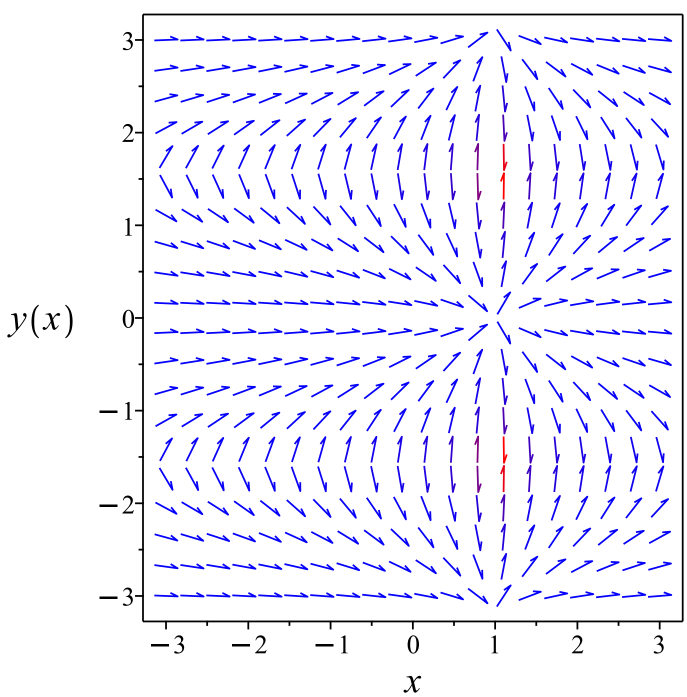


Figure 241: Slope field plot

Verification of solutions

$$-\ln(x-1)+\ln(\sin(y))=c_1$$

Verified OK.

4.13.4 Maple step by step solution

Let's solve

$y'-\frac{\tan(y)}{x-1}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\tan(y)}=\frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\tan(y)}dx=\int \frac{1}{x-1}dx+c_1$

- Evaluate integral

  $\ln(\sin(y))=\ln(x-1)+c_1$

- Solve for y

  $y=\arcsin(\mathrm{e}^{c_1}(x-1))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=tan(y(x))/(x-1),y(x), singsol=all)
```

$$y(x) = \arcsin\left(c_1(x-1)\right)$$

✓ Solution by Mathematica

Time used: 7.817 (sec). Leaf size: 19

```
DSolve[y'[x]==Tan[y[x]]/(x-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(e^{c_1}(x-1)\right)$$
$$y(x) \to 0$$

4.14 problem 16

Internal problem ID [971]
Internal file name [`OUTPUT/971_Sunday_June_05_2022_01_55_38_AM_93564178/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{2}{5}} = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{2}{5}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^{\frac{2}{5}}\right) = \frac{2}{5y^{\frac{3}{5}}}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.14.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{2}{5}}} dy = \int dx$$

$$\frac{5y^{\frac{3}{5}}}{3} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{5}{3} = c_1$$

$$c_1 = \frac{5}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{5y^{\frac{3}{5}}}{3} = x + \frac{5}{3}$$

Summary

The solution(s) found are the following

$$\frac{5y^{\frac{3}{5}}}{3} = x + \frac{5}{3} \tag{1}$$

Verification of solutions

$$\frac{5y^{\frac{3}{5}}}{3} = x + \frac{5}{3}$$

Verified OK.

4.14.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{2}{5}} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{2}{5}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{2}{5}}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{5y^{\frac{3}{5}}}{3} = x + c_1$

- Solve for y

  $y = \left(\frac{3x}{5} + \frac{3c_1}{5}\right)^{\frac{5}{3}}$

- Use initial condition $y(0) = 1$

  $1 = \frac{3\,3^{\frac{2}{3}} 5^{\frac{1}{3}} c_1^{\frac{5}{3}}}{25}$

- Solve for c_1

  $c_1 = \frac{5}{3}$

- Substitute $c_1 = \frac{5}{3}$ into general solution and simplify

  $y = \frac{(3x+5)\left(\frac{3x}{5}+1\right)^{\frac{2}{3}}}{5}$

- Solution to the IVP

  $y = \frac{(3x+5)\left(\frac{3x}{5}+1\right)^{\frac{2}{3}}}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=y(x)^(2/5),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(3x+5)\left(\frac{3x}{5}+1\right)^{\frac{2}{3}}}{5}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 23

```
DSolve[{y'[x]==y[x]^(2/5),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(3x+5)^{5/3}}{5\ 5^{2/3}}$$

4.15 problem 18

Internal problem ID [972]
Internal file name [`OUTPUT/972_Sunday_June_05_2022_01_55_40_AM_21455212/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3x(y-1)^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 3x(y-1)^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x(y-1)^{\frac{1}{3}}\right)\\ &= \frac{x}{(y-1)^{\frac{2}{3}}}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 1$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

4.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 3x(y-1)^{\frac{1}{3}}\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = (y-1)^{\frac{1}{3}}$. Integrating both sides gives

$$\frac{1}{(y-1)^{\frac{1}{3}}}\,dy = 3x\,dx$$

$$\int \frac{1}{(y-1)^{\frac{1}{3}}}\,dy = \int 3x\,dx$$

$$\frac{3(y-1)^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

The solution is

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0$$

Summary
The solution(s) found are the following

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0$$

Verified OK.

4.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x(y-1)^{\frac{1}{3}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 227: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{3x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x(y-1)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y-1)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3(R-1)^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

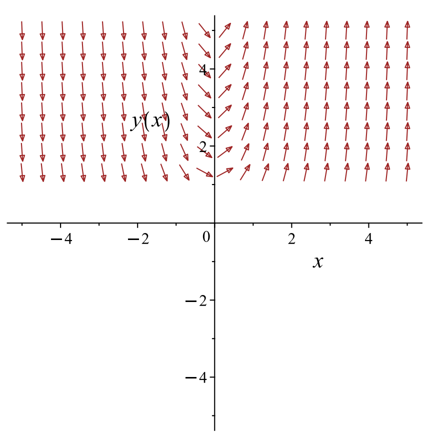
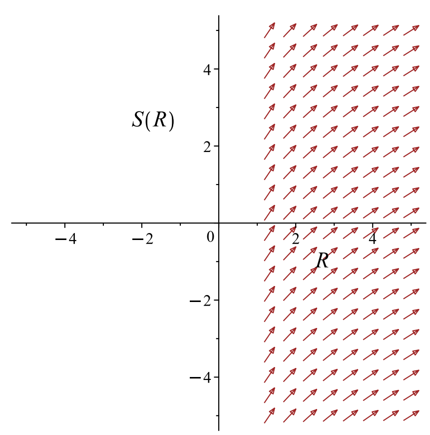
Which simplifies to

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27} + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x(y-1)^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{(-6c_1)^{\frac{3}{2}}}{27} + 1$$

$$c_1 = 0$$

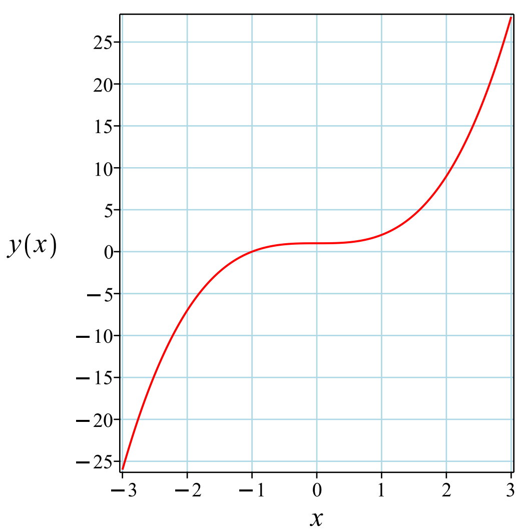
Substituting c_1 found above in the general solution gives

$$y = x^3 + 1$$

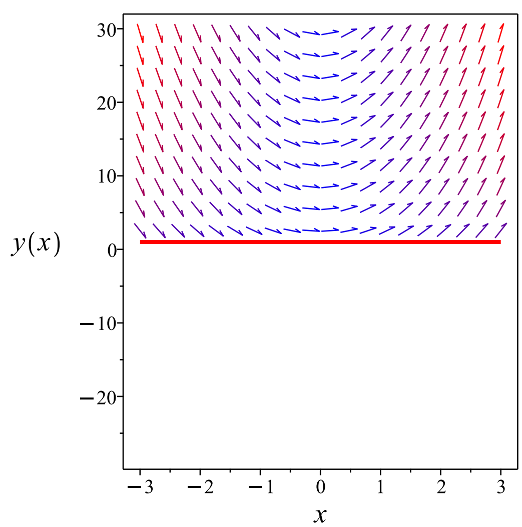
Summary

The solution(s) found are the following

$$y = x^3 + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^3 + 1$$

Verified OK. `{positive}`

4.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3(y-1)^{\frac{1}{3}}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{3(y-1)^{\frac{1}{3}}}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -x \\ N(x,y) &= \frac{1}{3\,(y-1)^{\frac{1}{3}}} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3\,(y-1)^{\frac{1}{3}}}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3(y-1)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3\left(y-1\right)^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3\left(y-1\right)^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{\left(y-1\right)^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\left(y-1\right)^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\left(y-1\right)^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = 1 + \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + 2c_1^{\frac{3}{2}}\sqrt{2}$$

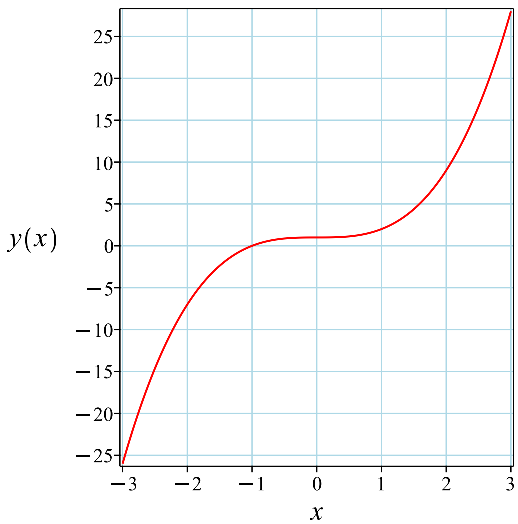
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

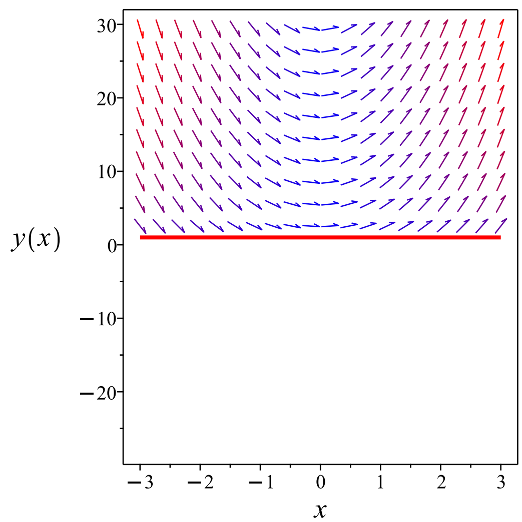
$$y = x^3 + 1$$

Summary
The solution(s) found are the following

$$y = x^3 + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^3 + 1$$

Verified OK. `{positive}`

4.15.5 Maple step by step solution

Let's solve

$$\left[y' - 3x(y-1)^{\frac{1}{3}} = 0, y(0) = 1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{(y-1)^{\frac{1}{3}}} = 3x$$

- Integrate both sides with respect to x

$$\int \frac{y'}{(y-1)^{\frac{1}{3}}} dx = \int 3x dx + c_1$$

- Evaluate integral

$$\frac{3(y-1)^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

- Solve for y

$$y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27} + 1$$

- Use initial condition $y(0) = 1$

$$1 = \frac{2\sqrt{6}\, c_1^{\frac{3}{2}}}{9} + 1$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = 1 + \operatorname{csgn}(x)\, x^3$$

- Solution to the IVP

$$y = 1 + \operatorname{csgn}(x)\, x^3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=3*x*(y(x)-1)^(1/3),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.296 (sec). Leaf size: 19

```
DSolve[{y'[x]==3*x*(y[x]-1)^(1/3),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$
$$y(x) \to \left(x^2\right)^{3/2} + 1$$

4.16 problem 19

Internal problem ID [973]
Internal file name [`OUTPUT/973_Sunday_June_05_2022_01_55_41_AM_2489202/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3x(y-1)^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 9]$$

4.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 3x(y-1)^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 9$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x(y-1)^{\frac{1}{3}}\right) \\
&= \frac{x}{(y-1)^{\frac{2}{3}}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 9$ is inside this domain. Therefore solution exists and is unique.

4.16.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= 3x(y-1)^{\frac{1}{3}}
\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = (y-1)^{\frac{1}{3}}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{(y-1)^{\frac{1}{3}}}\, dy &= 3x\, dx \\
\int \frac{1}{(y-1)^{\frac{1}{3}}}\, dy &= \int 3x\, dx \\
\frac{3(y-1)^{\frac{2}{3}}}{2} &= \frac{3x^2}{2} + c_1
\end{aligned}$$

The solution is

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 9$ in the above solution gives an equation to solve for the constant of integration.

$$6 - c_1 = 0$$

$$c_1 = 6$$

Substituting c_1 found above in the general solution gives

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - 6 = 0$$

Summary
The solution(s) found are the following

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - 6 = 0 \tag{1}$$

Verification of solutions

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - 6 = 0$$

Verified OK.

4.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x(y-1)^{\frac{1}{3}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 230: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x(y-1)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y-1)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3(R-1)^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

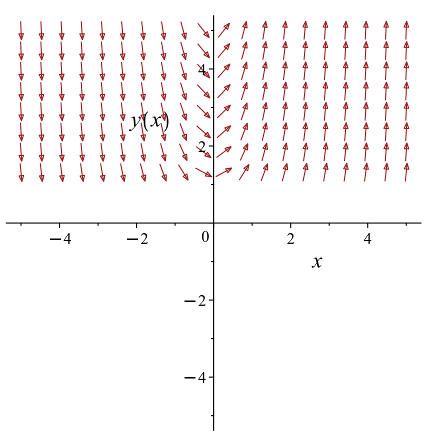
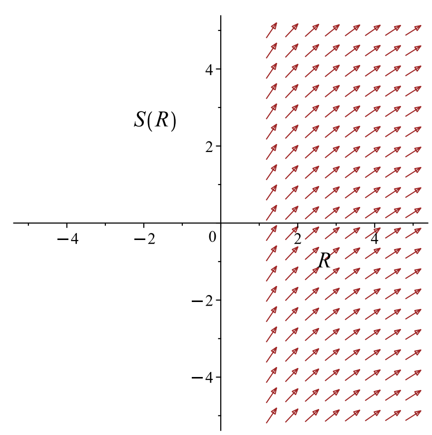
Which simplifies to

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27} + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x(y-1)^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 9$ in the above solution gives an equation to solve for the constant of integration.

$$9 = \frac{(-6c_1)^{\frac{3}{2}}}{27} + 1$$

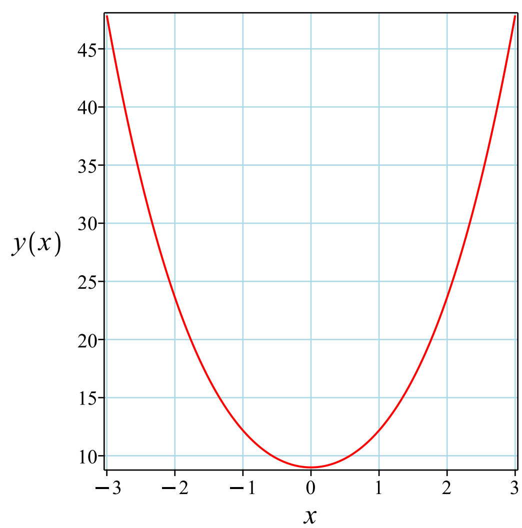
$$c_1 = -6$$

Substituting c_1 found above in the general solution gives

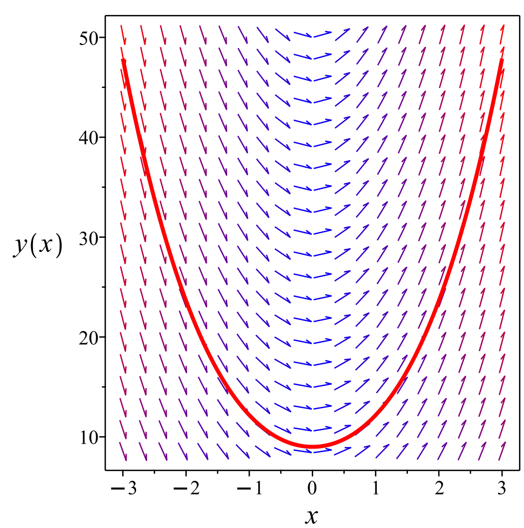
$$y = 1 + \left(x^2 + 4\right)^{\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$y = 1 + \left(x^2 + 4\right)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + \left(x^2 + 4\right)^{\frac{3}{2}}$$

Verified OK.

4.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right)\mathrm{d}y&=0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{3\left(y-1\right)^{\frac{1}{3}}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{3(y-1)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right)\mathrm{d}y$$

$$f(y)=\frac{\left(y-1\right)^{\frac{2}{3}}}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-\frac{x^2}{2}+\frac{\left(y-1\right)^{\frac{2}{3}}}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-\frac{x^2}{2}+\frac{\left(y-1\right)^{\frac{2}{3}}}{2}$$

The solution becomes

$$y=1+\left(x^2+2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=9$ in the above solution gives an equation to solve for the constant of integration.

$$9=1+2c_1^{\frac{3}{2}}\sqrt{2}$$

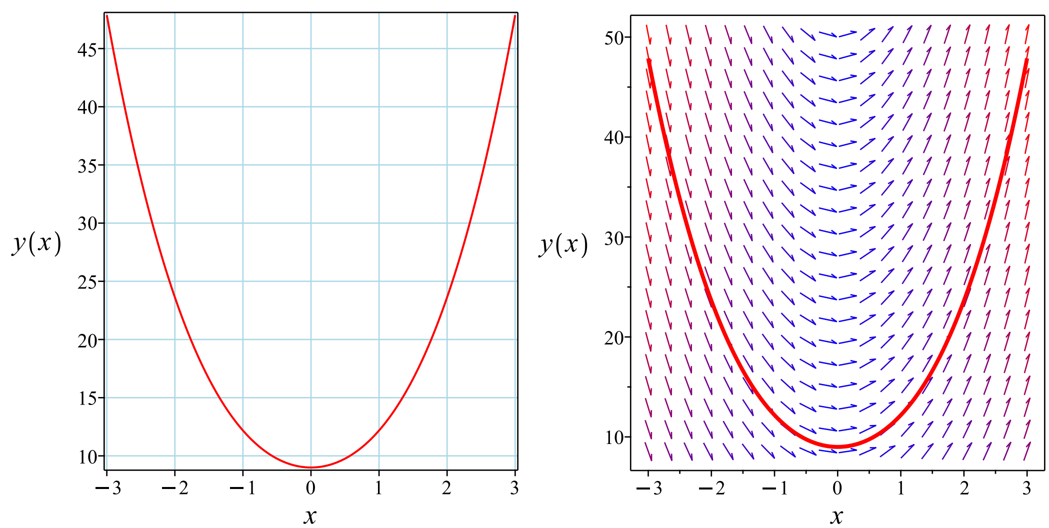
$$c_1=2$$

Substituting c_1 found above in the general solution gives

$$y = 1 + (x^2 + 4)^{\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$y = 1 + (x^2 + 4)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + (x^2 + 4)^{\frac{3}{2}}$$

Verified OK.

4.16.5 Maple step by step solution

Let's solve

$\left[y' - 3x(y-1)^{\frac{1}{3}} = 0, y(0) = 9\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{(y-1)^{\frac{1}{3}}} = 3x$$

- Integrate both sides with respect to x

$$\int \frac{y'}{(y-1)^{\frac{1}{3}}} dx = \int 3x dx + c_1$$

- Evaluate integral

$$\frac{3(y-1)^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

- Solve for y

$$y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27} + 1$$

- Use initial condition $y(0) = 9$

$$9 = \frac{2\sqrt{6}\, c_1^{\frac{3}{2}}}{9} + 1$$

- Solve for c_1

$$c_1 = 6$$

- Substitute $c_1 = 6$ into general solution and simplify

$$y = x^2\sqrt{x^2+4} + 4\sqrt{x^2+4} + 1$$

- Solution to the IVP

$$y = x^2\sqrt{x^2+4} + 4\sqrt{x^2+4} + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=3*x*(y(x)-1)^(1/3),y(0) = 9],y(x), singsol=all)
```

$$y(x) = x^2\sqrt{x^2+4} + 4\sqrt{x^2+4} + 1$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 16

```
DSolve[{y'[x]==3*x*(y[x]-1)^(1/3),y[0]==9},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(x^2+4\right)^{3/2}+1$$

4.17 problem 20(a)

Internal problem ID [974]
Internal file name [`OUTPUT/974_Sunday_June_05_2022_01_55_42_AM_64506206/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Existence and Uniqueness of Solutions of Nonlinear Equations. Section 2.3 Page 60
Problem number: 20(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3x(y-1)^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(3) = -7]$$

4.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3x(y-1)^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -7$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $f(x, y)$ when $x = 3$ is

$$\{1 \leq y\}$$

But the point $y_0 = -7$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= 3x(y-1)^{\frac{1}{3}} \end{aligned}$$

Where $f(x) = 3x$ and $g(y) = (y-1)^{\frac{1}{3}}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{(y-1)^{\frac{1}{3}}}\, dy &= 3x\, dx \\ \int \frac{1}{(y-1)^{\frac{1}{3}}}\, dy &= \int 3x\, dx \\ \frac{3(y-1)^{\frac{2}{3}}}{2} &= \frac{3x^2}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{3(y-1)^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -7$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{33}{2} + 3i\sqrt{3} - c_1 = 0$$

$$c_1 = -\frac{33}{2} + 3i\sqrt{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{3(y-1)^{\frac{2}{3}}}{2}-\frac{3x^2}{2}+\frac{33}{2}-3i\sqrt{3}=0$$

Solving for y from the above gives

$$y=1+\left(2i\sqrt{3}+x^2-11\right)^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y=1+\left(2i\sqrt{3}+x^2-11\right)^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y=1+\left(2i\sqrt{3}+x^2-11\right)^{\frac{3}{2}}$$

Verified OK.

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=3x(y-1)^{\frac{1}{3}}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 233: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x(y-1)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(y-1)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3(R-1)^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

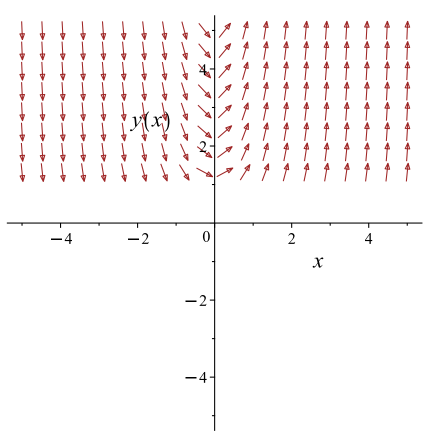
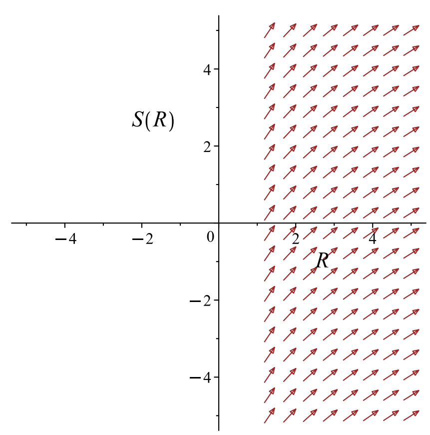
Which simplifies to

$$\frac{3x^2}{2} = \frac{3(y-1)^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27} + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x(y-1)^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{(R-1)^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -7$ in the above solution gives an equation to solve for the constant of integration.

$$-7 = \frac{(81 - 6c_1)^{\frac{3}{2}}}{27} + 1$$

$$c_1 = \frac{33}{2} + 3i\sqrt{3}$$

Substituting c_1 found above in the general solution gives

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$$

Summary

The solution(s) found are the following

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1 \tag{1}$$

Verification of solutions

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$$

Verified OK.

4.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3\,(y-1)^{\frac{1}{3}}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{3\,(y-1)^{\frac{1}{3}}}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{3\,(y-1)^{\frac{1}{3}}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3\,(y-1)^{\frac{1}{3}}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -x\,\mathrm{d}x$$
$$\phi=-\frac{x^2}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{3(y-1)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int \left(\frac{1}{3\left(y-1\right)^{\frac{1}{3}}}\right)\mathrm{d}y$$
$$f(y)=\frac{\left(y-1\right)^{\frac{2}{3}}}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{(y-1)^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{(y-1)^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = 1 + \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -7$ in the above solution gives an equation to solve for the constant of integration.

$$-7 = 1 + (9 + 2c_1)^{\frac{3}{2}}$$

$$c_1 = -\frac{11}{2} - i\sqrt{3}$$

Substituting c_1 found above in the general solution gives

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$$

Summary

The solution(s) found are the following

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1 \tag{1}$$

Verification of solutions

$$y = \left(-2i\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$$

Verified OK.

4.17.5 Maple step by step solution

Let's solve

$\left[y' - 3x(y-1)^{\frac{1}{3}} = 0, y(3) = -7\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(y-1)^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{(y-1)^{\frac{1}{3}}} dx = \int 3x dx + c_1$

- Evaluate integral

  $\frac{3(y-1)^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$

- Solve for y

  $y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27} + 1$

- Use initial condition $y(3) = -7$

  $-7 = \frac{(6c_1+81)^{\frac{3}{2}}}{27} + 1$

- Solve for c_1

  $c_1 = \left(-\frac{33}{2} - 3\,\mathrm{I}\sqrt{3}, -\frac{33}{2} + 3\,\mathrm{I}\sqrt{3}\right)$

- Substitute $c_1 = \left(-\frac{33}{2} - 3\,\mathrm{I}\sqrt{3}, -\frac{33}{2} + 3\,\mathrm{I}\sqrt{3}\right)$ into general solution and simplify

  $y = \left(-2\,\mathrm{I}\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$

- Solution to the IVP

  $y = \left(-2\,\mathrm{I}\sqrt{3} + x^2 - 11\right)^{\frac{3}{2}} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=3*x*(y(x)-1)^(1/3),y(3) = -7],y(x), singsol=all)
```

$$y(x) = 1 + \left(-11 + 2i\sqrt{3} + x^2\right)^{\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 49

```
DSolve[{y'[x]==3*x*(y[x]-1)^(1/3),y[3]==-7},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + \left(x^2 - 2i\sqrt{3} - 11\right)^{3/2}$$
$$y(x) \to 1 + \left(x^2 + 2i\sqrt{3} - 11\right)^{3/2}$$

5 Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68

5.1 problem Example 1

Internal problem ID [975]
Internal file name [`OUTPUT/975_Sunday_June_05_2022_01_55_45_AM_98582848/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: Example 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' - y - xy^2 = 0}$$

5.1.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x\,y^2 + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 236: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 \mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\,y^2 + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\mathrm{e}^x}{y} \\ S_y &= \frac{\mathrm{e}^x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\,\mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,\mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R - 1)\,e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{x}}{y} = (x - 1)\,e^{x} + c_1$$

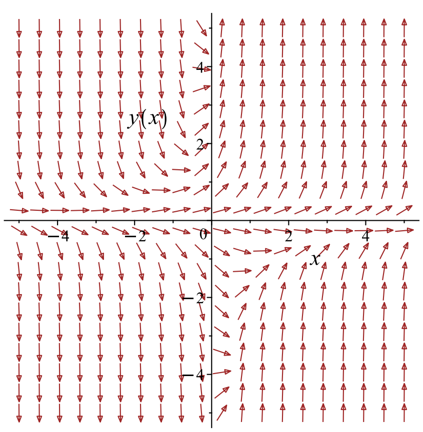
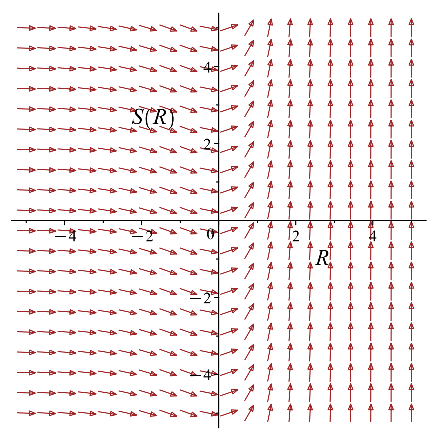
Which simplifies to

$$-\frac{e^{x}}{y} = (x - 1)\,e^{x} + c_1$$

Which gives

$$y = -\frac{e^{x}}{x\,e^{x} - e^{x} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,y^2 + y$ 	$R = x$ $S = -\frac{e^{x}}{y}$	$\frac{dS}{dR} = R\,e^{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{x}}{x\,e^{x}-e^{x}+c_1} \tag{1}$$

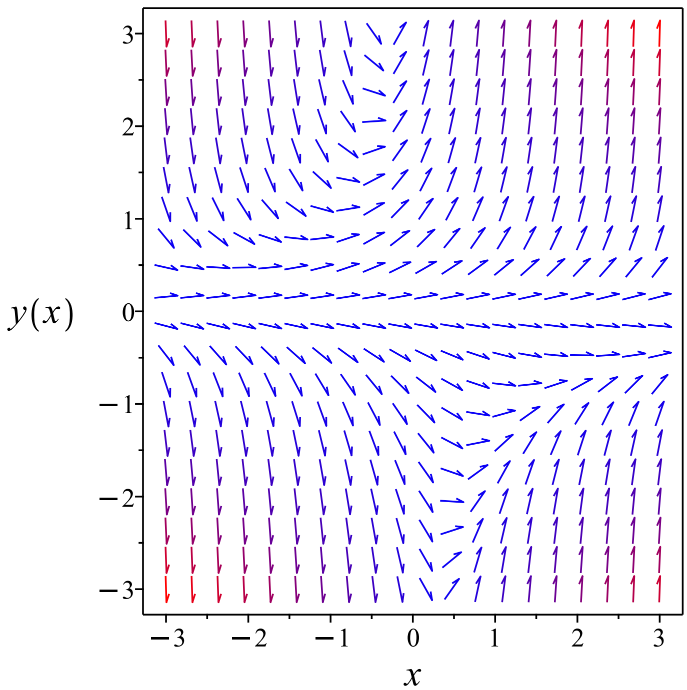


Figure 246: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}}{x\,e^{x}-e^{x}+c_1}$$

Verified OK.

5.1.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x\,y^2 + y \end{aligned}$$

This is a Bernoulli ODE.

$$y' = y + xy^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= 1\\ f_1(x) &= x\\ n &= 2\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{y} + x \tag{4}$$

Let

$$\begin{aligned}w &= y^{1-n}\\ &= \frac{1}{y}\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}-w'(x) &= w(x) + x\\ w' &= -w - x\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + w(x) = -x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} w) &= (\mathrm{e}^{x})(-x) \\
\mathrm{d}(\mathrm{e}^{x} w) &= (-x\,\mathrm{e}^{x})\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} w &= \int -x\,\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} w &= -(x-1)\,\mathrm{e}^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
w(x) = -\mathrm{e}^{-x}(x-1)\,\mathrm{e}^{x} + c_1\mathrm{e}^{-x}
$$

which simplifies to

$$
w(x) = 1 - x + c_1\mathrm{e}^{-x}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = 1 - x + c_1\mathrm{e}^{-x}
$$

Or

$$
y = \frac{1}{1 - x + c_1\mathrm{e}^{-x}}
$$

Summary

The solution(s) found are the following

$$y = \frac{1}{1 - x + c_1 e^{-x}} \tag{1}$$

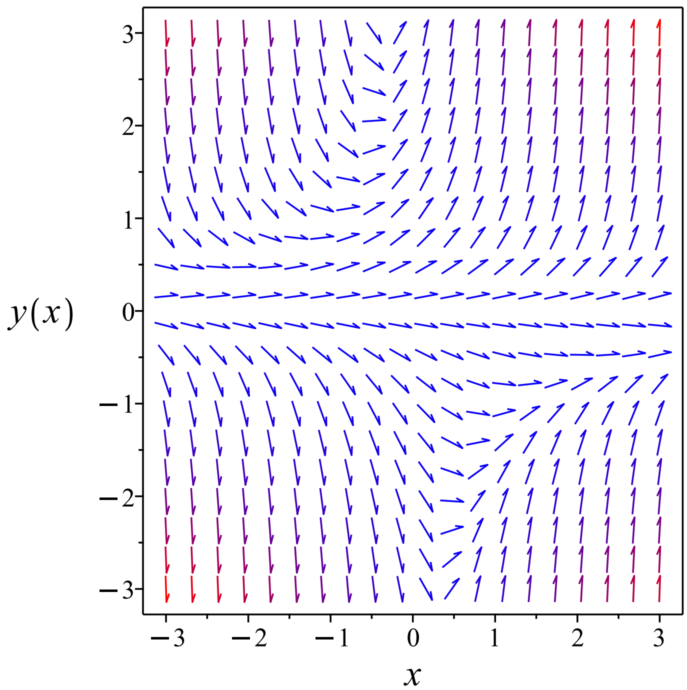


Figure 247: Slope field plot

Verification of solutions

$$y = \frac{1}{1 - x + c_1 e^{-x}}$$

Verified OK.

5.1.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x y^2 + y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x y^2 + y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 1$ and $f_2(x) = x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= x \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x u''(x) - (x+1) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + (x-1) e^x c_2$$

The above shows that

$$u'(x) = c_2 x e^x$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 e^x}{c_1 + (x-1) e^x c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{e^{x}}{c_3 + (x-1)e^{x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{x}}{c_3 + (x-1)e^{x}} \tag{1}$$

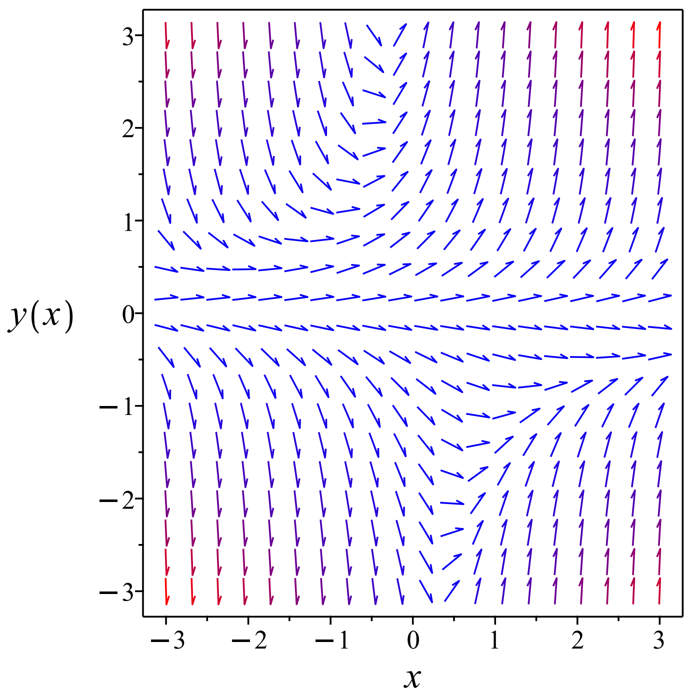


Figure 248: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}}{c_3 + (x-1)e^{x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)-y(x)=x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + e^{-x}c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 27

```
DSolve[y'[x]-y[x]==x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^x}{-e^x(x-1) + c_1}$$
$$y(x) \rightarrow 0$$

5.2 problem Example 2

Internal problem ID [976]
Internal file name [`OUTPUT/976_Sunday_June_05_2022_01_55_46_AM_27482921/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: Example 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y + x\,\mathrm{e}^{-\frac{y}{x}}}{x} = 0}$$

5.2.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \mathrm{e}^{-\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\,f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\,x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x+u &= u+g(x)\,f(bu)^{\frac{n}{m}}\\ u' &= \frac{1}{x}g(x)\,f(bu)^{\frac{n}{m}} \quad (2)\end{aligned}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y=ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned}g(x) &= 1\\ b &= 1\\ f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}}\end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x)=\frac{e^{-u(x)}}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{e^{-u}}{x}\end{aligned}$$

Where $f(x)=\frac{1}{x}$ and $g(u)=e^{-u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{e^{-u}}\,du &= \frac{1}{x}\,dx\\ \int\frac{1}{e^{-u}}\,du &= \int\frac{1}{x}\,dx\\ e^{u} &= \ln(x)+c_1\end{aligned}$$

The solution is

$$e^{u(x)}-\ln(x)-c_1=0$$

Therefore the solution is found using $y=ux$. Hence

$$e^{\frac{y}{x}}-\ln(x)-c_1=0$$

Summary

The solution(s) found are the following

$$e^{\frac{y}{x}}-\ln(x)-c_1=0 \tag{1}$$

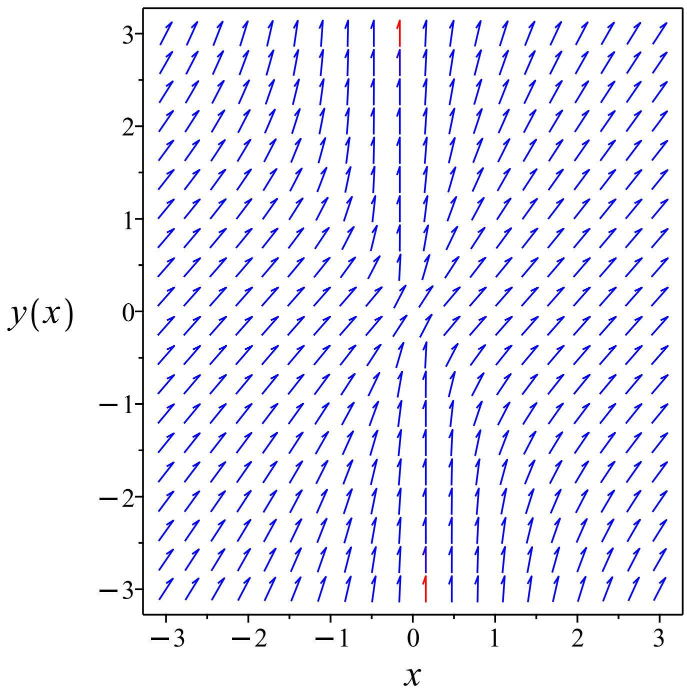


Figure 249: Slope field plot

Verification of solutions

$$e^{\frac{y}{x}}-\ln(x)-c_1=0$$

Verified OK.

5.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x+u(x)-\frac{u(x)\,x+x\,e^{-u(x)}}{x}=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{e^{-u}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{-u}$. Integrating both sides gives

$$\frac{1}{e^{-u}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{e^{-u}}\, du = \int \frac{1}{x}\, dx$$
$$e^{u} = \ln(x) + c_2$$

The solution is

$$e^{u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$e^{\frac{y}{x}} - \ln(x) - c_2 = 0$$
$$e^{\frac{y}{x}} - \ln(x) - c_2 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$e^{\frac{y}{x}} - \ln(x) - c_2 = 0 \tag{1}$$

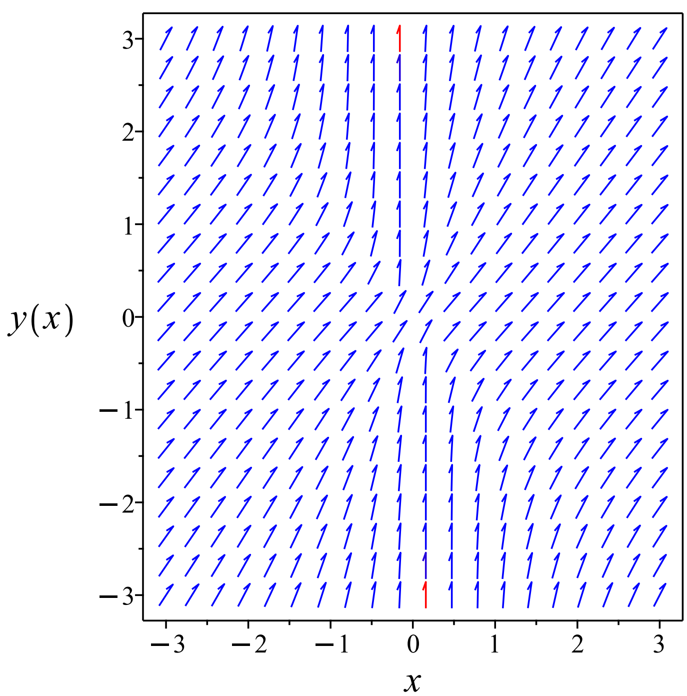


Figure 250: Slope field plot

Verification of solutions

$$
e^{\frac{y}{x}} - \ln(x) - c_2 = 0
$$

Verified OK.

5.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = \frac{y + x\,e^{-\frac{y}{x}}}{x}
$$
$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 238: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= yx
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{yx}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y + x\,\mathrm{e}^{-\frac{y}{x}}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\, e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{-e^{R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{-e^{\frac{y}{x}}}$$

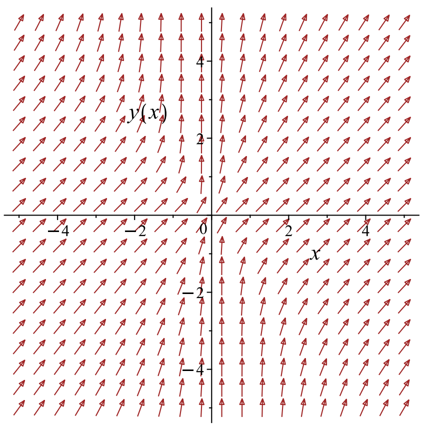
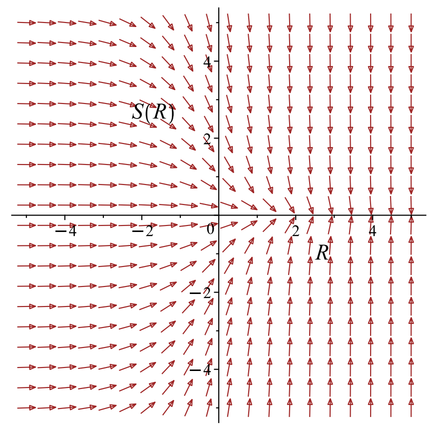
Which simplifies to

$$-\frac{1}{x} = c_1 e^{-e^{\frac{y}{x}}}$$

Which gives

$$y = \ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y + x\,\mathrm{e}^{-\frac{y}{x}}}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\,\mathrm{e}^{R}$ 

Summary

The solution(s) found are the following

$$y = \ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

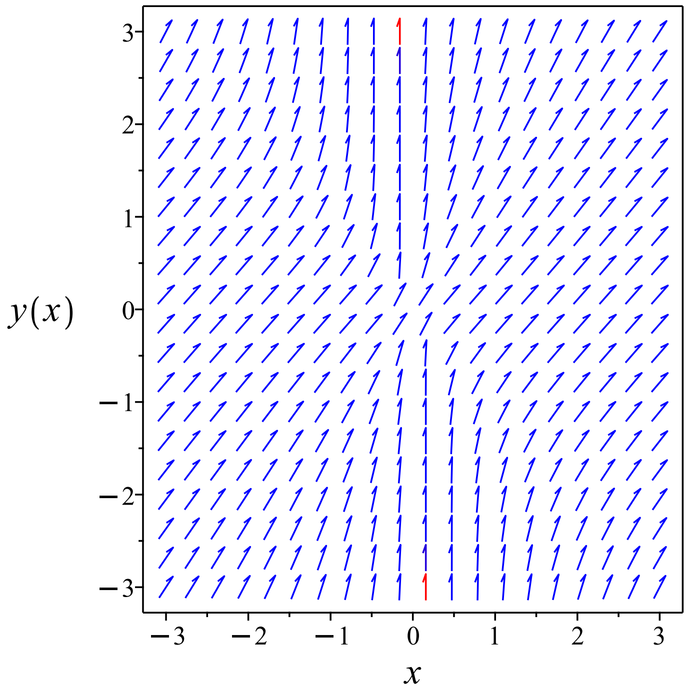


Figure 251: Slope field plot

Verification of solutions

$$y = \ln\left(-\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=(y(x)+x*exp(-y(x)/x))/x,y(x), singsol=all)
```

$$y(x) = \ln(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.368 (sec). Leaf size: 13

```
DSolve[y'[x]==(y[x]+x*Exp[-y[x]/x])/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \log(\log(x) + c_1)$$

5.3 problem Example 3(a) (As Riccati)

Internal problem ID [977]
Internal file name [`OUTPUT/977_Sunday_June_05_2022_01_55_47_AM_9394587/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: Example 3(a) (As Riccati).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - yx = -x^2}$$

5.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x^2 + yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -1 + \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= -\frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} - \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2x^2 + c_1}{x}$$

The above shows that

$$u'(x) = \frac{c_2x^2 - c_1}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{(c_2x^2 - c_1)\,x}{c_2x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-x^2 + c_3)\,x}{x^2 + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{(-x^2 + c_3)\,x}{x^2 + c_3} \tag{1}$$

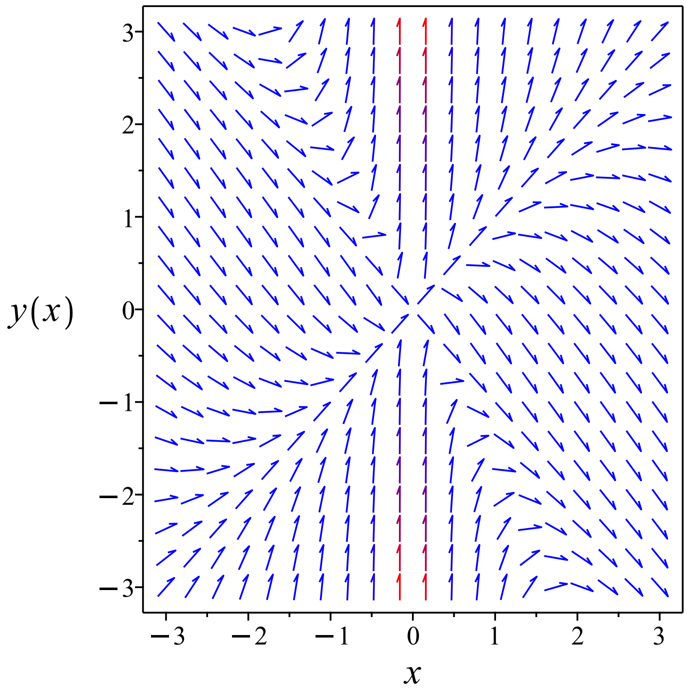


Figure 252: Slope field plot

Verification of solutions

$$y = \frac{(-x^2 + c_3)\, x}{x^2 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(x^2*diff(y(x),x)=y(x)^2+x*y(x)-x^2,y(x), singsol=all)
```

$$y(x) = -\tanh\left(\ln\left(x\right)+c_1\right)x$$

✓ Solution by Mathematica

Time used: 0.298 (sec). Leaf size: 298

```
DSolve[y'[x]==y[x]^2+x*y[x]-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5(\sqrt{5}-1)\,x\left(c_1\,\text{HermiteH}\left(\frac{1}{10}(-5+\sqrt{5}),\frac{\sqrt[4]{5}x}{\sqrt{2}}\right)-\text{Hypergeometric1F1}\left(\frac{5}{4}-\frac{1}{4\sqrt{5}},\frac{3}{2},\frac{\sqrt{5}x^2}{2}\right)+\text{Hyperg}\ldots\right.}{10\left(\text{Hypergeometric1F1}\left(\frac{1}{20}\left(5-\sqrt{5}\right),\frac{1}{2},\frac{\sqrt{5}x^2}{2}\right)\ldots\right.}$$

$$y(x) \to \frac{1}{2}\left(\sqrt{5}-1\right)x-\frac{\left(\sqrt{5}-5\right)\text{HermiteH}\left(\frac{1}{10}\left(-15+\sqrt{5}\right),\frac{\sqrt[4]{5}x}{\sqrt{2}}\right)}{\sqrt{2}5^{3/4}\,\text{HermiteH}\left(\frac{1}{10}\left(-5+\sqrt{5}\right),\frac{\sqrt[4]{5}x}{\sqrt{2}}\right)}$$

5.4 problem Example 3(b)

Internal problem ID [978]
Internal file name [`OUTPUT/979_Sunday_June_05_2022_01_55_50_AM_63671662/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: Example 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - yx = -x^2}$$

With initial conditions

$$[y(1) = 2]$$

5.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-x^2 + yx + y^2}{x^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x^2 + yx + y^2}{x^2}\right) \\ &= \frac{x + 2y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.4.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 - u(x)\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u^2 - 1}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2 - 1}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u^2 - 1}\,du &= \int \frac{1}{x}\,dx \\ -\operatorname{arctanh}(u) &= \ln(x) + c_2\end{aligned}$$

The solution is

$$-\operatorname{arctanh}\left(u(x)\right)-\ln\left(x\right)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\operatorname{arctanh}\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$
$$-\operatorname{arctanh}\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

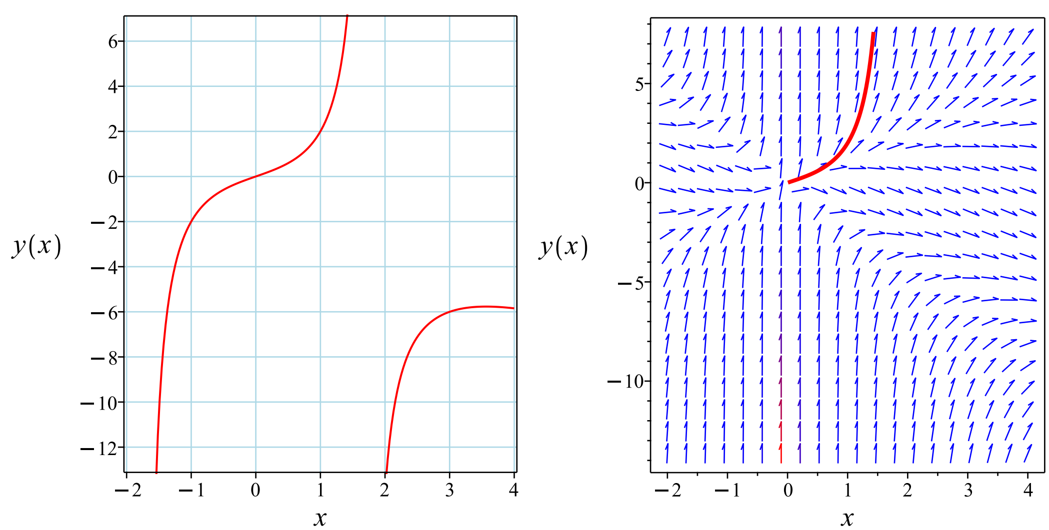
Substituting initial conditions and solving for c_2 gives $c_2=-\operatorname{arctanh}\left(\frac{1}{2}\right)+\frac{i\pi}{2}$. Hence the solution becomes Solving for y from the above gives

$$y=-\coth\left(-\operatorname{arctanh}\left(\frac{1}{2}\right)+\ln\left(x\right)\right)x$$

Summary

The solution(s) found are the following

$$y=-\coth\left(-\operatorname{arctanh}\left(\frac{1}{2}\right)+\ln\left(x\right)\right)x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=-\coth\left(-\operatorname{arctanh}\left(\frac{1}{2}\right)+\ln\left(x\right)\right)x$$

Verified OK.

5.4.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-x^2+yx+y^2}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{(-x^2+yx+y^2)(b_3-a_2)}{x^2} - \frac{(-x^2+yx+y^2)^2 a_3}{x^4} \\
&- \left(\frac{-2x+y}{x^2} - \frac{2(-x^2+yx+y^2)}{x^3}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{(x+2y)(x b_2 + y b_3 + b_1)}{x^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^4 a_2 - x^4 a_3 - x^4 b_3 + 2x^3 y a_3 - 2x^3 y b_2 + x^2 y^2 a_2 + 2x^2 y^2 a_3 - x^2 y^2 b_3 - y^4 a_3 - x^3 b_1 + x^2 y a_1 - 2x^2 y b_1 + 2x}{x^4} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
x^4 a_2 - x^4 a_3 - x^4 b_3 + 2x^3 y a_3 - 2x^3 y b_2 + x^2 y^2 a_2 + 2x^2 y^2 a_3 \\
- x^2 y^2 b_3 - y^4 a_3 - x^3 b_1 + x^2 y a_1 - 2x^2 y b_1 + 2x\, y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &a_2 v_1^4 + a_2 v_1^2 v_2^2 - a_3 v_1^4 + 2a_3 v_1^3 v_2 + 2a_3 v_1^2 v_2^2 - a_3 v_2^4 - 2b_2 v_1^3 v_2 \\ &- b_3 v_1^4 - b_3 v_1^2 v_2^2 + a_1 v_1^2 v_2 + 2a_1 v_1 v_2^2 - b_1 v_1^3 - 2b_1 v_1^2 v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(a_2 - a_3 - b_3)\, v_1^4 + (2a_3 - 2b_2)\, v_1^3 v_2 - b_1 v_1^3 \\ &+ (a_2 + 2a_3 - b_3)\, v_1^2 v_2^2 + (a_1 - 2b_1)\, v_1^2 v_2 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ a_1 - 2b_1 &= 0 \\ 2a_3 - 2b_2 &= 0 \\ a_2 - a_3 - b_3 &= 0 \\ a_2 + 2a_3 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{-x^2 + yx + y^2}{x^2}\right)(x) \\
&= \frac{x^2 - y^2}{x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 - y^2}{x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln (x + y)}{2} - \frac{\ln (-x + y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^2 + yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2 - y^2} \\ S_y &= \frac{x}{x^2 - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x+y)}{2} - \frac{\ln(-x+y)}{2} = -\ln(x) + c_1$$

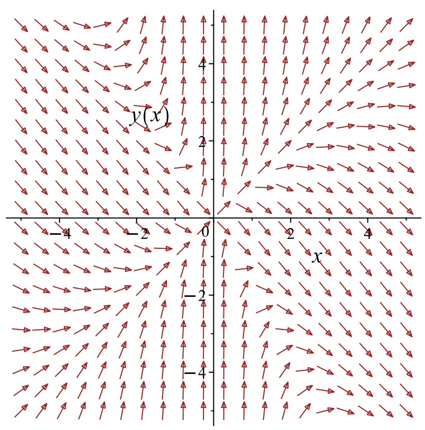
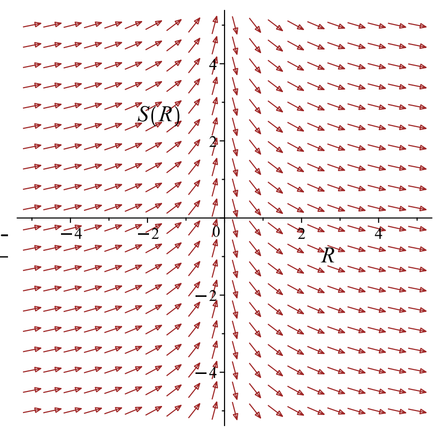
Which simplifies to

$$\frac{\ln(x+y)}{2} - \frac{\ln(-x+y)}{2} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(e^{2c_1} + x^2)}{e^{2c_1} - x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+yx+y^2}{x^2}$ 	$R = x$ $S = \frac{\ln(x+y)}{2} - \frac{\ln(-x -}{2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{e^{2c_1} + 1}{e^{2c_1} - 1}$$

$$c_1 = \frac{\ln(3)}{2}$$

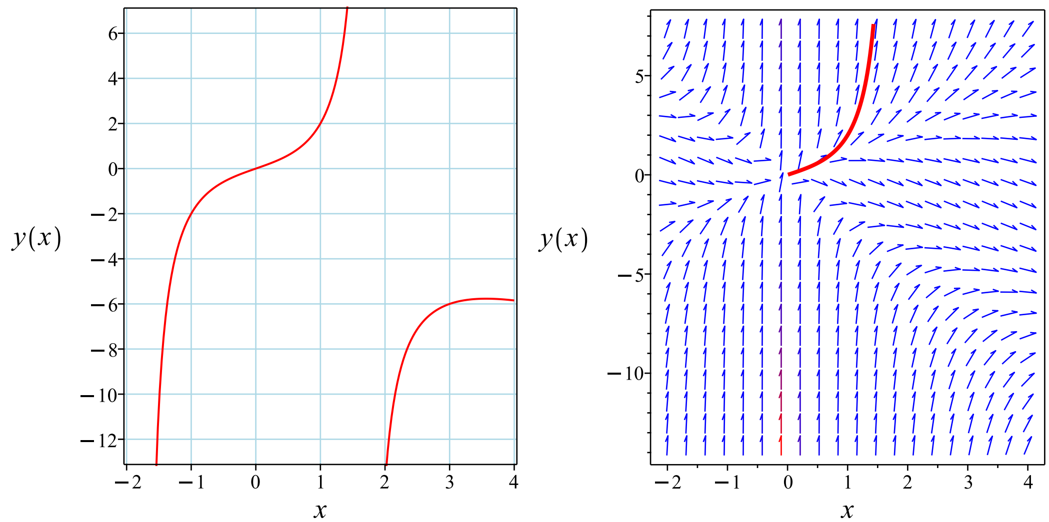
Substituting c_1 found above in the general solution gives

$$y = \frac{-x^3 - 3x}{x^2 - 3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{-x^3 - 3x}{x^2 - 3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x^3 - 3x}{x^2 - 3}$$

Verified OK.

5.4.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x^2 + yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -1 + \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= \frac{1}{x^3} \\ f_2^2f_0 &= -\frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} - \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2x^2 + c_1}{x}$$

The above shows that

$$u'(x) = \frac{c_2x^2 - c_1}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{(c_2x^2 - c_1)\,x}{c_2x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-x^2 + c_3)\,x}{x^2 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_3 - 1}{c_3 + 1}$$

$$c_3 = -3$$

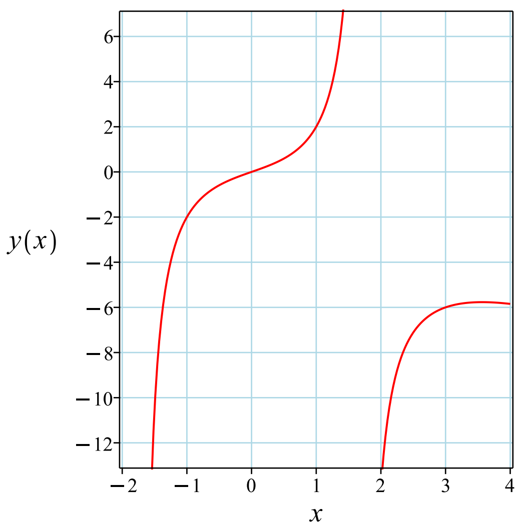
Substituting c_3 found above in the general solution gives

$$y = -\frac{(x^2+3)\,x}{x^2-3}$$

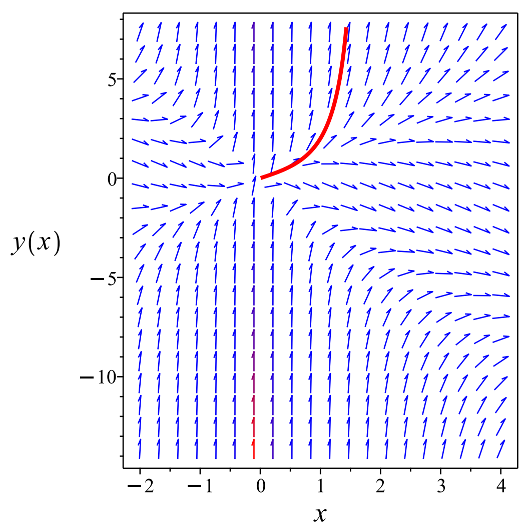
Summary

The solution(s) found are the following

$$y = -\frac{(x^2+3)\,x}{x^2-3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{(x^2+3)\,x}{x^2-3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 19

```
dsolve([x^2*diff(y(x),x)=y(x)^2+x*y(x)-x^2,y(1) = 2],y(x), singsol=all)
```

$$y(x) = -\frac{x(x^2+3)}{x^2-3}$$

✓ Solution by Mathematica

Time used: 0.576 (sec). Leaf size: 20

```
DSolve[{x^2*y'[x]==y[x]^2+x*y[x]-x^2,y[1]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x(x^2+3)}{x^2-3}$$

5.5 problem 1

Internal problem ID [979]
Internal file name [OUTPUT/980_Sunday_June_05_2022_01_55_51_AM_82695140/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y - y^2 = 0}$$

5.5.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y^2 - y \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -1 \\ f_1(x) &= 1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = -\frac{1}{y} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -w(x) + 1 \\ w' &= w - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -1 \end{aligned}$$

Hence the ode is

$$w'(x) - w(x) = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-1) \\ \frac{d}{dx}(e^{-x} w) &= (e^{-x})(-1) \\ d(e^{-x} w) &= (-e^{-x})\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x} w &= \int -e^{-x}\, dx \\ e^{-x} w &= e^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$w(x) = e^{-x} e^{x} + c_1 e^{x}$$

which simplifies to

$$w(x) = 1 + c_1 e^{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = 1 + c_1 e^{x}$$

Or

$$y = \frac{1}{1 + c_1 e^{x}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{1 + c_1 e^{x}} \tag{1}$$

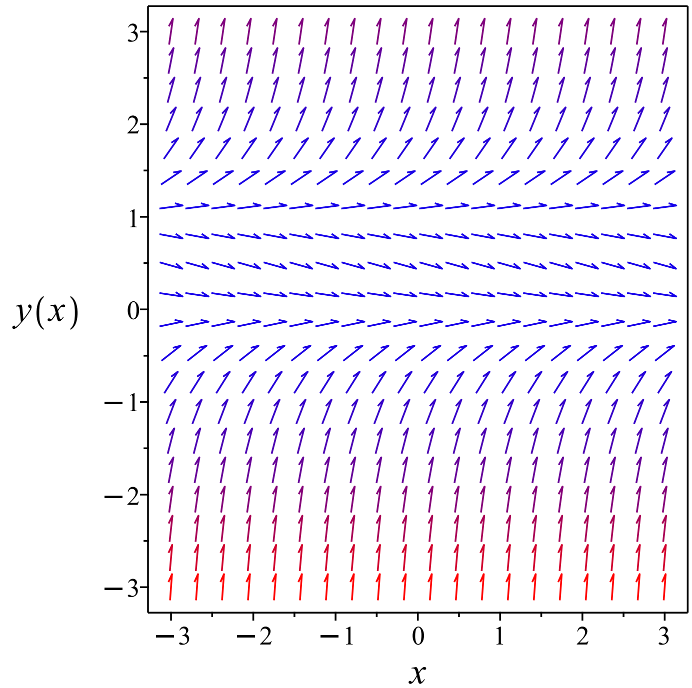


Figure 256: Slope field plot

Verification of solutions

$$y = \frac{1}{1 + c_1 e^x}$$

Verified OK.

5.5.2 Maple step by step solution

Let's solve

$y' + y - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2 - y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2 - y} dx = \int 1 dx + c_1$

- Evaluate integral

$$-\ln(y) + \ln(y-1) = x + c_1$$

- Solve for y

$$y = -\frac{1}{e^{x+c_1} - 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)+y(x)=y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + e^x c_1}$$

✓ Solution by Mathematica

Time used: 0.777 (sec). Leaf size: 54

```
DSolve[y'[x]+y[x]==y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{1 + e^{2(x+c_1)}}}$$
$$y(x) \to \frac{1}{\sqrt{1 + e^{2(x+c_1)}}}$$
$$y(x) \to -1$$
$$y(x) \to 0$$
$$y(x) \to 1$$

5.6 problem 2

Internal problem ID [980]
Internal file name [OUTPUT/981_Sunday_June_05_2022_01_55_53_AM_1156564/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{7y'x - 2y + \frac{x^2}{y^6} = 0}$$

5.6.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{2y^7 - x^2}{7x\,y^6} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{7x}y - \frac{x}{7}\frac{1}{y^6} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{7x} \\ f_1(x) &= -\frac{x}{7} \\ n &= -6 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^6}$ gives

$$y'y^6 = \frac{2y^7}{7x} - \frac{x}{7} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^7 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 7y^6y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{7} &= \frac{2w(x)}{7x} - \frac{x}{7} \\ w' &= \frac{2w}{x} - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(-x) \\ d\left(\frac{w}{x^2}\right) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -\frac{1}{x}\, dx \\ \frac{w}{x^2} &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = -\ln(x)\, x^2 + c_1 x^2$$

which simplifies to

$$w(x) = x^2(-\ln(x) + c_1)$$

Replacing w in the above by y^7 using equation (5) gives the final solution.

$$y^7 = x^2(-\ln(x) + c_1)$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \left(-x^2(\ln(x) - c_1)\right)^{\frac{1}{7}} \\
y(x) &= (-1)^{\frac{2}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \\
y(x) &= (-1)^{\frac{4}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \\
y(x) &= (-1)^{\frac{6}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \\
y(x) &= -(-1)^{\frac{1}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \\
y(x) &= -(-1)^{\frac{3}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \\
y(x) &= -(-1)^{\frac{5}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \left(-x^2(\ln(x) - c_1)\right)^{\frac{1}{7}} \tag{1}$$

$$y = (-1)^{\frac{2}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{2}$$

$$y = (-1)^{\frac{4}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{3}$$

$$y = (-1)^{\frac{6}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{4}$$

$$y = -(-1)^{\frac{1}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{5}$$

$$y = -(-1)^{\frac{3}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{6}$$

$$y = -(-1)^{\frac{5}{7}} \left(x^2(-\ln(x) + c_1)\right)^{\frac{1}{7}} \tag{7}$$

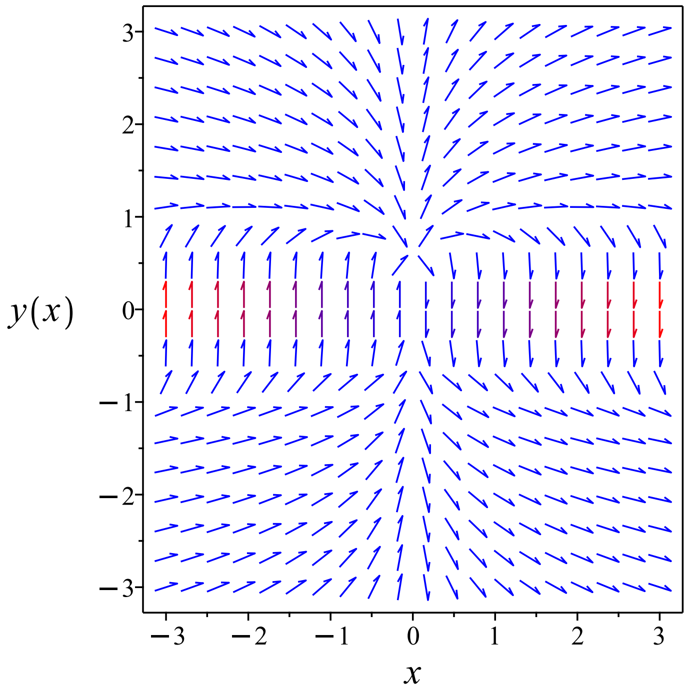


Figure 257: Slope field plot

Verification of solutions

$$y = \left(-x^2(\ln (x) - c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = (-1)^{\frac{2}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = (-1)^{\frac{4}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = (-1)^{\frac{6}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = -(-1)^{\frac{1}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = -(-1)^{\frac{3}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

$$y = -(-1)^{\frac{5}{7}} \left(x^2(-\ln (x) + c_1)\right)^{\frac{1}{7}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 134

```
dsolve(7*x*diff(y(x),x)-2*y(x)=-x^2/y(x)^6,y(x), singsol=all)
```

$$y(x) = \left(-x^2(\ln(x) - c_1)\right)^{\frac{1}{7}}$$
$$y(x) = -\left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{1}{7}}$$
$$y(x) = \left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{6}{7}}$$
$$y(x) = -\left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{5}{7}}$$
$$y(x) = \left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{2}{7}}$$
$$y(x) = -\left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{3}{7}}$$
$$y(x) = \left((c_1 - \ln(x))\,x^2\right)^{\frac{1}{7}} (-1)^{\frac{4}{7}}$$

✓ Solution by Mathematica

Time used: 0.238 (sec). Leaf size: 181

```
DSolve[7*x*y'[x]-2*y[x]==-x^2/y[x]^6,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to -\sqrt[7]{-1}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to (-1)^{2/7}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to -(-1)^{3/7}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to (-1)^{4/7}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to -(-1)^{5/7}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$
$$y(x) \to (-1)^{6/7}x^{2/7}\sqrt[7]{-\log(x) + c_1}$$

5.7 problem 3

Internal problem ID [981]
Internal file name [`OUTPUT/982_Sunday_June_05_2022_01_55_56_AM_71242951/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y'x^2 + 2y - 2\,\mathrm{e}^{\frac{1}{x}}\sqrt{y} = 0}$$

5.7.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{2\left(-\mathrm{e}^{\frac{1}{x}}\sqrt{y} + y\right)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x^2}y + \frac{2\,\mathrm{e}^{\frac{1}{x}}}{x^2}\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{2}{x^2} \\
f_1(x) &= \frac{2\,{\mathrm e}^{\frac{1}{x}}}{x^2} \\
n &= \frac{1}{2}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$
y'\frac{1}{\sqrt{y}} = -\frac{2\sqrt{y}}{x^2} + \frac{2\,{\mathrm e}^{\frac{1}{x}}}{x^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \sqrt{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = \frac{1}{2\sqrt{y}}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
2w'(x) &= -\frac{2w(x)}{x^2} + \frac{2\,{\mathrm e}^{\frac{1}{x}}}{x^2} \\
w' &= -\frac{w}{x^2} + \frac{{\mathrm e}^{\frac{1}{x}}}{x^2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x^2} \\
q(x) &= \frac{{\mathrm e}^{\frac{1}{x}}}{x^2}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x^2} = \frac{e^{\frac{1}{x}}}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x^2}dx} \\ &= e^{-\frac{1}{x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{e^{\frac{1}{x}}}{x^2}\right) \\ \frac{d}{dx}\left(e^{-\frac{1}{x}}w\right) &= \left(e^{-\frac{1}{x}}\right)\left(\frac{e^{\frac{1}{x}}}{x^2}\right) \\ d\left(e^{-\frac{1}{x}}w\right) &= \frac{1}{x^2}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-\frac{1}{x}}w &= \int \frac{1}{x^2}\, dx \\ e^{-\frac{1}{x}}w &= -\frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{1}{x}}$ results in

$$w(x) = -\frac{e^{\frac{1}{x}}}{x} + c_1 e^{\frac{1}{x}}$$

which simplifies to

$$w(x) = \frac{e^{\frac{1}{x}}(c_1 x - 1)}{x}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = \frac{e^{\frac{1}{x}}(c_1 x - 1)}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = \frac{e^{\frac{1}{x}}(c_1 x - 1)}{x} \tag{1}$$

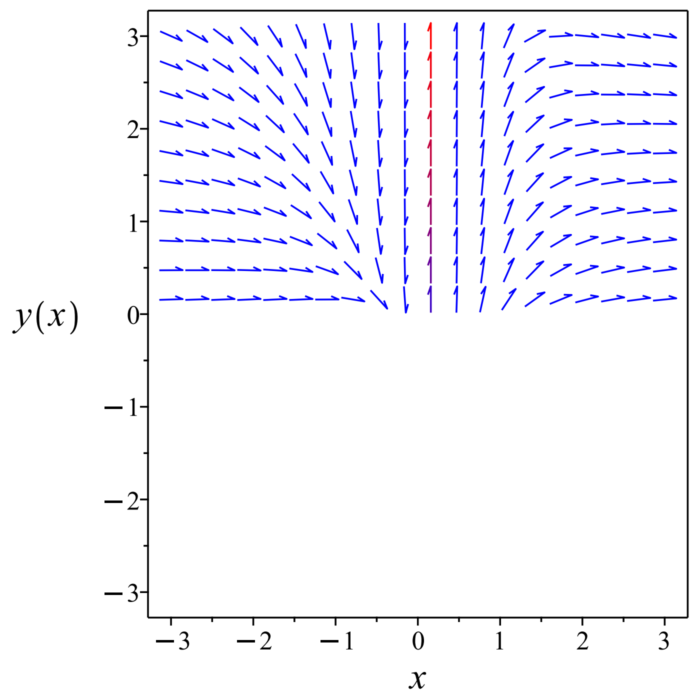


Figure 258: Slope field plot

Verification of solutions

$$\sqrt{y} = \frac{e^{\frac{1}{x}}(c_1 x - 1)}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x^2*diff(y(x),x)+2*y(x)=2*exp(1/x)*y(x)^(1/2),y(x), singsol=all)
```

$$\frac{-c_1 x \,\mathrm{e}^{\frac{1}{x}} + x\sqrt{y\left(x\right)} + \mathrm{e}^{\frac{1}{x}}}{x} = 0$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 39

```
DSolve[y'[x]+2*y[x]==2*Exp[1/x]*y[x]^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-2x} \left(\int_1^x 2e^{K[1]+\frac{1}{K[1]}} dK[1] + 2c_1 \right)^2$$

5.8 problem 4

Internal problem ID [982]
Internal file name [`OUTPUT/983_Sunday_June_05_2022_01_55_58_AM_35701149/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_rational, _Bernoulli]`

$$\boxed{\left(x^2+1\right)y'+2yx-\frac{1}{\left(x^2+1\right)y}=0}$$

5.8.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{2x^3y^2+2x\,y^2-1}{\left(x^2+1\right)^2 y}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2x^3+2x}{\left(x^2+1\right)^2}y+\frac{1}{\left(x^2+1\right)^2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{2x^3 + 2x}{(x^2+1)^2} \\
f_1(x) &= \frac{1}{(x^2+1)^2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{(2x^3 + 2x)\, y^2}{(x^2+1)^2} + \frac{1}{(x^2+1)^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{(2x^3 + 2x)\, w(x)}{(x^2+1)^2} + \frac{1}{(x^2+1)^2} \\
w' &= -\frac{2(2x^3 + 2x)\, w}{(x^2+1)^2} + \frac{2}{(x^2+1)^2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{4x}{x^2+1} \\
q(x) &= \frac{2}{(x^2+1)^2}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{4xw(x)}{x^2+1} = \frac{2}{\left(x^2+1\right)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{4x}{x^2+1}dx} \\ &= \left(x^2+1\right)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2}{\left(x^2+1\right)^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+1\right)^2 w\right) &= \left(\left(x^2+1\right)^2\right)\left(\frac{2}{\left(x^2+1\right)^2}\right) \\ \mathrm{d}\left(\left(x^2+1\right)^2 w\right) &= 2\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^2+1\right)^2 w &= \int 2\,\mathrm{d}x \\ \left(x^2+1\right)^2 w &= 2x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(x^2+1\right)^2$ results in

$$w(x) = \frac{2x}{\left(x^2+1\right)^2} + \frac{c_1}{\left(x^2+1\right)^2}$$

which simplifies to

$$w(x) = \frac{2x+c_1}{\left(x^2+1\right)^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{2x+c_1}{\left(x^2+1\right)^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2x+c_1}}{x^2+1} \\ y(x) &= -\frac{\sqrt{2x+c_1}}{x^2+1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2x + c_1}}{x^2 + 1} \tag{1}$$

$$y = -\frac{\sqrt{2x + c_1}}{x^2 + 1} \tag{2}$$

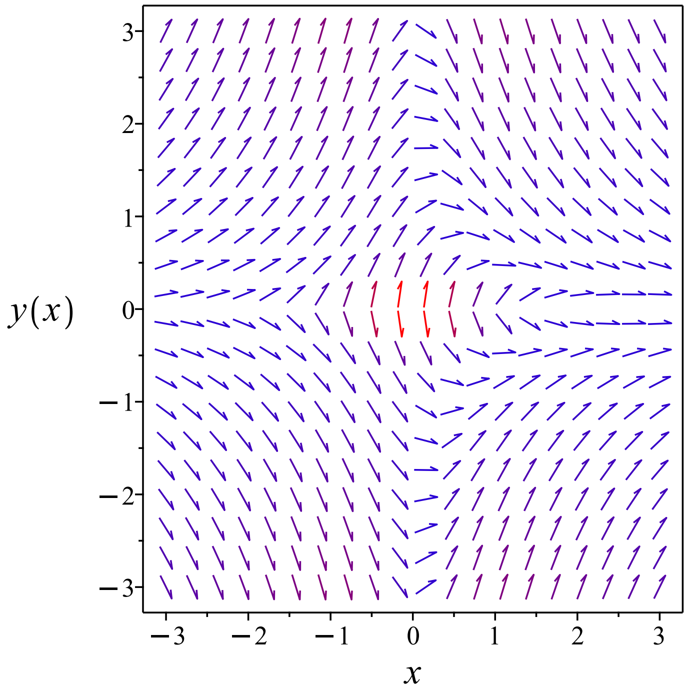


Figure 259: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2x + c_1}}{x^2 + 1}$$

Verified OK.

$$y = -\frac{\sqrt{2x + c_1}}{x^2 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve((1+x^2)*diff(y(x),x)+2*x*y(x)=1/((1+x^2)*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{2x + c_1}}{x^2 + 1}$$
$$y(x) = -\frac{\sqrt{2x + c_1}}{x^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 46

```
DSolve[(1+x^2)*y'[x]+2*x*y[x]==1/((1+x^2)*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2x + c_1}}{x^2 + 1}$$
$$y(x) \to \frac{\sqrt{2x + c_1}}{x^2 + 1}$$

5.9 problem 5

Internal problem ID [983]
Internal file name [`OUTPUT/984_Sunday_June_05_2022_01_55_59_AM_7349123/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{-yx + y' - x^3y^3 = 0}$$

5.9.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^3x^3 + yx \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy + x^3y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= x \\
f_1(x) &= x^3 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = \frac{x}{y^2} + x^3 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= w(x)\, x + x^3 \\
w' &= -2x^3 - 2xw
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x) w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= -2x^3
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 2w(x)\, x = -2x^3
$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(-2x^3\right)$$
$$\frac{d}{dx}\left(e^{x^2} w\right) = \left(e^{x^2}\right)\left(-2x^3\right)$$
$$d\left(e^{x^2} w\right) = \left(-2\,e^{x^2} x^3\right)\,dx$$

Integrating gives

$$e^{x^2} w = \int -2\,e^{x^2} x^3\,dx$$
$$e^{x^2} w = -\left(x^2-1\right)e^{x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$w(x) = -e^{-x^2}\left(x^2-1\right)e^{x^2} + c_1 e^{-x^2}$$

which simplifies to

$$w(x) = -x^2 + 1 + c_1 e^{-x^2}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = -x^2 + 1 + c_1 e^{-x^2}$$

Solving for y gives

$$y(x) = \frac{1}{\sqrt{-x^2 + 1 + c_1 e^{-x^2}}}$$
$$y(x) = -\frac{1}{\sqrt{-x^2 + 1 + c_1 e^{-x^2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{-x^2 + 1 + c_1 e^{-x^2}}} \tag{1}$$
$$y = -\frac{1}{\sqrt{-x^2 + 1 + c_1 e^{-x^2}}} \tag{2}$$

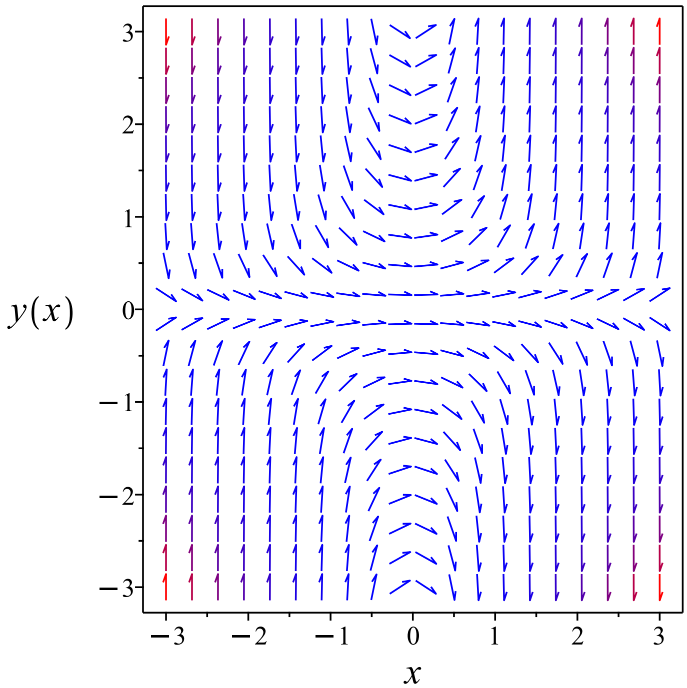


Figure 260: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-x^2 + 1 + c_1 \mathrm{e}^{-x^2}}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{-x^2 + 1 + c_1 \mathrm{e}^{-x^2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x)-x*y(x)=x^3*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{e^{-x^2} c_1 - x^2 + 1}}$$
$$y(x) = -\frac{1}{\sqrt{e^{-x^2} c_1 - x^2 + 1}}$$

✓ Solution by Mathematica

Time used: 1.866 (sec). Leaf size: 80

```
DSolve[y'[x]-x*y[x]==x^3*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{\frac{x^2}{2}}}{\sqrt{e^{x^2}(x^2-1) - c_1}}$$
$$y(x) \to \frac{ie^{\frac{x^2}{2}}}{\sqrt{e^{x^2}(x^2-1) - c_1}}$$
$$y(x) \to 0$$

5.10 problem 6

Internal problem ID [984]
Internal file name [`OUTPUT/985_Sunday_June_05_2022_01_56_01_AM_84307276/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y' - \frac{(x+1)\,y}{3x} - y^4 = 0}$$

5.10.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(3x\,y^3 + x + 1)}{3x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x+1}{3x}y + y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{x+1}{3x} \\
f_1(x) &= 1 \\
n &= 4
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$
y' \frac{1}{y^4} = \frac{x+1}{3x\, y^3} + 1 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^3}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{3}{y^4} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{3} &= \frac{(x+1)\, w(x)}{3x} + 1 \\
w' &= -\frac{(x+1)\, w}{x} - 3
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-x-1}{x} \\
q(x) &= -3
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{(-x-1)\,w(x)}{x} = -3$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-x-1}{x}dx} \\ &= \mathrm{e}^{x+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x\,\mathrm{e}^{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\,(-3) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x\,\mathrm{e}^{x} w) &= (x\,\mathrm{e}^{x})\,(-3) \\ \mathrm{d}(x\,\mathrm{e}^{x} w) &= (-3x\,\mathrm{e}^{x})\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x\,\mathrm{e}^{x} w &= \int -3x\,\mathrm{e}^{x}\,\mathrm{d}x \\ x\,\mathrm{e}^{x} w &= -3(x-1)\,\mathrm{e}^{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x\,\mathrm{e}^{x}$ results in

$$w(x) = -\frac{3\,\mathrm{e}^{-x}(x-1)\,\mathrm{e}^{x}}{x} + \frac{c_1 \mathrm{e}^{-x}}{x}$$

which simplifies to

$$w(x) = \frac{c_1 \mathrm{e}^{-x} - 3x + 3}{x}$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = \frac{c_1 \mathrm{e}^{-x} - 3x + 3}{x}$$

Solving for y gives

$$y(x) = \frac{\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{c_1 \mathrm{e}^{-x} - 3x + 3}$$

$$y(x) = \frac{\left(i\sqrt{3} - 1\right)\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{2c_1 \mathrm{e}^{-x} - 6x + 6}$$

$$y(x) = \frac{\left(1 + i\sqrt{3}\right)\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{-2c_1 \mathrm{e}^{-x} + 6x - 6}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{c_1 \mathrm{e}^{-x} - 3x + 3} \tag{1}$$

$$y = \frac{\left(i\sqrt{3} - 1\right)\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{2c_1 \mathrm{e}^{-x} - 6x + 6} \tag{2}$$

$$y = \frac{\left(1 + i\sqrt{3}\right)\left(x(c_1 \mathrm{e}^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{-2c_1 \mathrm{e}^{-x} + 6x - 6} \tag{3}$$

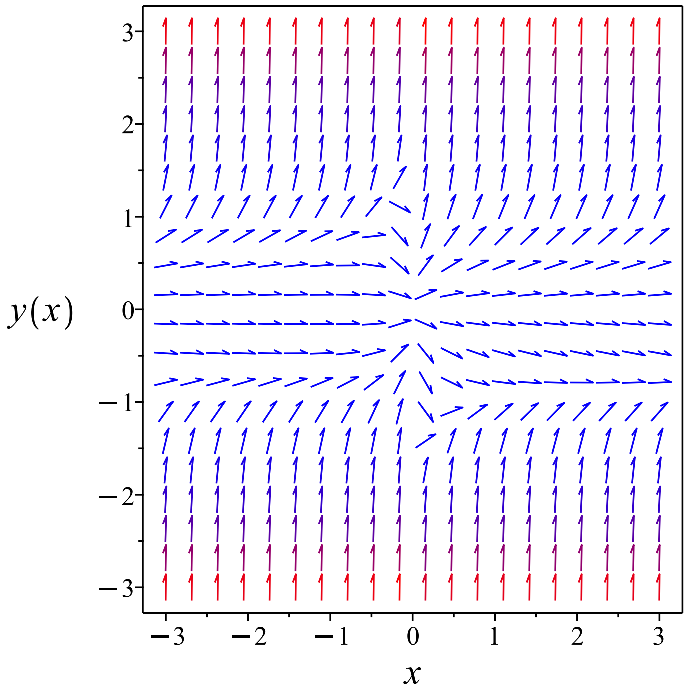


Figure 261: Slope field plot

Verification of solutions

$$y = \frac{\left(x(c_1 e^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{c_1 e^{-x} - 3x + 3}$$

Verified OK.

$$y = \frac{\left(i\sqrt{3} - 1\right)\left(x(c_1 e^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{2c_1 e^{-x} - 6x + 6}$$

Verified OK.

$$y = \frac{\left(1 + i\sqrt{3}\right)\left(x(c_1 e^{-x} - 3x + 3)^2\right)^{\frac{1}{3}}}{-2c_1 e^{-x} + 6x - 6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 121

```
dsolve(diff(y(x),x)-(1+x)/(3*x)*y(x)=y(x)^4,y(x), singsol=all)
```

$$y(x) = \frac{\left(x(\mathrm{e}^{-x}c_1 - 3x + 3)^2\right)^{\frac{1}{3}}}{\mathrm{e}^{-x}c_1 - 3x + 3}$$
$$y(x) = \frac{(1 + i\sqrt{3})\left(x(\mathrm{e}^{-x}c_1 - 3x + 3)^2\right)^{\frac{1}{3}}}{-2\,\mathrm{e}^{-x}c_1 + 6x - 6}$$
$$y(x) = \frac{(i\sqrt{3} - 1)\left(x(\mathrm{e}^{-x}c_1 - 3x + 3)^2\right)^{\frac{1}{3}}}{2\,\mathrm{e}^{-x}c_1 - 6x + 6}$$

✓ Solution by Mathematica

Time used: 1.88 (sec). Leaf size: 120

```
DSolve[y'[x]-(1+x)/(3*x)*y[x]==y[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{x/3}\sqrt[3]{x}}{\sqrt[3]{3e^x(x-1) - c_1}}$$
$$y(x) \to \frac{\sqrt[3]{-1}e^{x/3}\sqrt[3]{x}}{\sqrt[3]{3e^x(x-1) - c_1}}$$
$$y(x) \to -\frac{(-1)^{2/3}e^{x/3}\sqrt[3]{x}}{\sqrt[3]{3e^x(x-1) - c_1}}$$
$$y(x) \to 0$$

5.11 problem 7

Internal problem ID [985]
Internal file name [`OUTPUT/986_Sunday_June_05_2022_01_56_03_AM_17244321/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - 2y - xy^3 = 0}$$

With initial conditions

$$\left[y(0) = 2\sqrt{2}\right]$$

5.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= x\,y^3 + 2y \end{aligned}$$

The x domain of $f(x,y)$ when $y = 2\sqrt{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2\sqrt{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^3 + 2y\right) \\ &= 3x\,y^2 + 2\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2\sqrt{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2\sqrt{2}$ is inside this domain. Therefore solution exists and is unique.

5.11.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= x\,y^3 + 2y\end{aligned}$$

This is a Bernoulli ODE.

$$y' = 2y + xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= 2 \\ f_1(x) &= x \\ n &= 3\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{2}{y^2} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= 2w(x) + x \\ w' &= -4w - 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= -2x \end{aligned}$$

Hence the ode is

$$w'(x) + 4w(x) = -2x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 4dx} \\ &= e^{4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)(-2x) \\ \frac{d}{dx}\left(e^{4x} w\right) &= \left(e^{4x}\right)(-2x) \\ d\left(e^{4x} w\right) &= \left(-2x\, e^{4x}\right) dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{4x}w &= \int -2x\,e^{4x}\,dx \\
e^{4x}w &= -\frac{(4x-1)\,e^{4x}}{8} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$
w(x) = -\frac{e^{-4x}(4x-1)\,e^{4x}}{8} + c_1 e^{-4x}
$$

which simplifies to

$$
w(x) = -\frac{x}{2} + \frac{1}{8} + c_1 e^{-4x}
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = -\frac{x}{2} + \frac{1}{8} + c_1 e^{-4x}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2\sqrt{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{8} = \frac{1}{8} + c_1
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$
\frac{1}{y^2} = -\frac{x}{2} + \frac{1}{8}
$$

The above simplifies to

$$
4x\,y^2 - y^2 + 8 = 0
$$

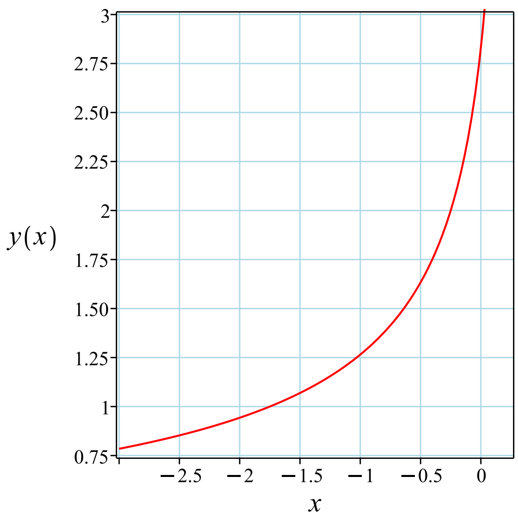
Solving for y from the above gives

$$
y = \frac{4}{\sqrt{-8x+2}}
$$

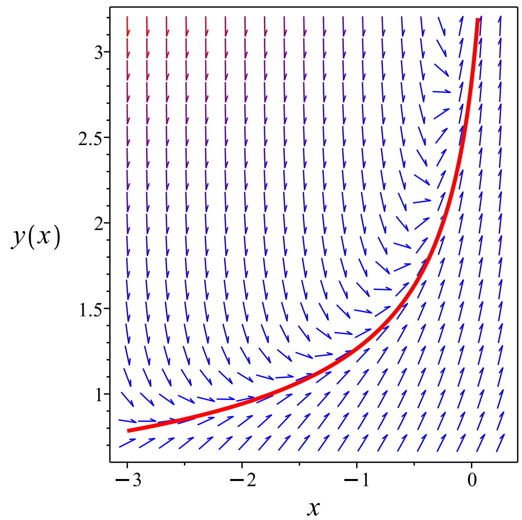
Summary

The solution(s) found are the following

$$
y = \frac{4}{\sqrt{-8x+2}} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{\sqrt{-8x+2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)-2*y(x)=x*y(x)^3,y(0) = 2*sqrt(2)],y(x), singsol=all)
```

$$y(x) = \frac{4}{\sqrt{-8x+2}}$$

✓ Solution by Mathematica

Time used: 1.869 (sec). Leaf size: 34

```
DSolve[{y'[x]-2*y[x]==x*y[x]^3,y[0]==2*Sqrt[2]},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2i\sqrt{2}e^{2x}}{\sqrt{e^{4x}(4x-1)}}$$

5.12 problem 8

Internal problem ID [986]
Internal file name [`OUTPUT/987_Sunday_June_05_2022_01_56_05_AM_12543053/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-yx + y' - x y^{\frac{3}{2}} = 0}$$

With initial conditions

$$[y(1) = 4]$$

5.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= yx + y^{\frac{3}{2}}x
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=4$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$
\{0 \leq y\}
$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(yx + y^{\frac{3}{2}}x\right) \\
&= x + \frac{3\sqrt{y}\,x}{2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=4$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$
\{0 \leq y\}
$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

5.12.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= x\left(y^{\frac{3}{2}} + y\right)
\end{aligned}
$$

Where $f(x) = x$ and $g(y) = y^{\frac{3}{2}} + y$. Integrating both sides gives

$$\frac{1}{y^{\frac{3}{2}} + y} \, dy = x \, dx$$

$$\int \frac{1}{y^{\frac{3}{2}} + y} \, dy = \int x \, dx$$

$$-2 \operatorname{arctanh}\left(\sqrt{y}\right) + \ln\left(y\right) - \ln\left(y - 1\right) = \frac{x^2}{2} + c_1$$

The solution is

$$-2 \operatorname{arctanh}\left(\sqrt{y}\right) + \ln\left(y\right) - \ln\left(y - 1\right) - \frac{x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$-2 \operatorname{arccoth}\left(2\right) + i\pi - \frac{1}{2} - c_1 - \ln\left(3\right) + 2\ln\left(2\right) = 0$$

$$c_1 = -\frac{1}{2} - 2 \operatorname{arccoth}\left(2\right) + i\pi - \ln\left(3\right) + 2\ln\left(2\right)$$

Substituting c_1 found above in the general solution gives

$$-2 \operatorname{arctanh}\left(\sqrt{y}\right) + \ln\left(y\right) - \ln\left(y - 1\right) - \frac{x^2}{2} + \frac{1}{2} + 2 \operatorname{arccoth}\left(2\right) - i\pi + \ln\left(3\right) - 2\ln\left(2\right) = 0$$

Summary

The solution(s) found are the following

$$-2 \operatorname{arctanh}\left(\sqrt{y}\right) + \ln\left(y\right) - \ln\left(y - 1\right) - \frac{x^2}{2} + \frac{1}{2} + 2 \operatorname{arccoth}\left(2\right) - i\pi + \ln\left(3\right) - 2\ln\left(2\right) = 0 \tag{1}$$

Verification of solutions

$$-2 \operatorname{arctanh}\left(\sqrt{y}\right) + \ln\left(y\right) - \ln\left(y - 1\right) - \frac{x^2}{2} + \frac{1}{2} + 2 \operatorname{arccoth}\left(2\right) - i\pi + \ln\left(3\right) - 2\ln\left(2\right) = 0$$

Verified OK.

5.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = yx + y^{\frac{3}{2}}x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 241: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = yx + y^{\frac{3}{2}} x
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\left(\sqrt{y}+1\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\left(\sqrt{R}+1\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R\right) - 2\ln\left(\sqrt{R}+1\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \ln\left(y\right) - 2\ln\left(1+\sqrt{y}\right) + c_1$$

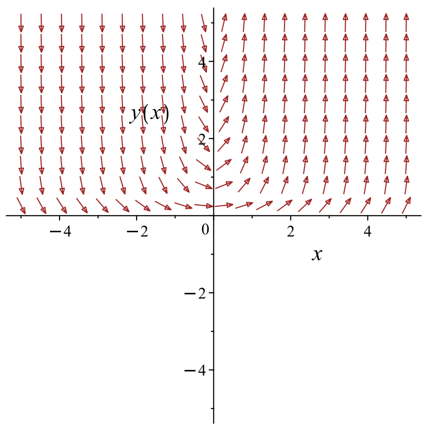
Which simplifies to

$$\frac{x^2}{2} = \ln\left(y\right) - 2\ln\left(1+\sqrt{y}\right) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = yx + y^{\frac{3}{2}}x$



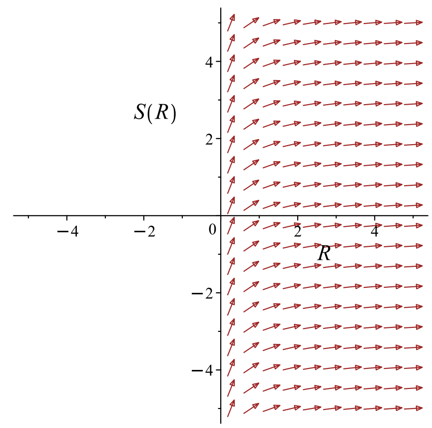
</td><td>

$R = y$

$S = \frac{x^2}{2}$

</td><td>

$\frac{dS}{dR} = \frac{1}{R\left(\sqrt{R}+1\right)}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = 2\ln(2) - 2\ln(3) + c_1$$

$$c_1 = -2\ln(2) + 2\ln(3) + \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{2} = \ln(y) - 2\ln(\sqrt{y} + 1) - 2\ln(2) + 2\ln(3) + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \ln(y) - 2\ln(1 + \sqrt{y}) - 2\ln(2) + 2\ln(3) + \frac{1}{2} \tag{1}$$

Verification of solutions

$$\frac{x^2}{2} = \ln(y) - 2\ln(1 + \sqrt{y}) - 2\ln(2) + 2\ln(3) + \frac{1}{2}$$

Verified OK.

5.12.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= yx + y^{\frac{3}{2}}x
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = xy + xy^{\frac{3}{2}} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= x \\
f_1(x) &= x \\
n &= \frac{3}{2}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{3}{2}}$ gives

$$
y'\frac{1}{y^{\frac{3}{2}}} = \frac{x}{\sqrt{y}} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{\sqrt{y}} \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{2y^{\frac{3}{2}}}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$-2w'(x) = w(x)\,x + x$$
$$w' = -\frac{1}{2}xw - \frac{1}{2}x \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = \frac{x}{2}$$
$$q(x) = -\frac{x}{2}$$

Hence the ode is

$$w'(x) + \frac{w(x)\,x}{2} = -\frac{x}{2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{x}{2}dx}$$
$$= e^{\frac{x^2}{4}}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(-\frac{x}{2}\right)$$
$$\frac{d}{dx}\left(e^{\frac{x^2}{4}} w\right) = \left(e^{\frac{x^2}{4}}\right)\left(-\frac{x}{2}\right)$$
$$d\left(e^{\frac{x^2}{4}} w\right) = \left(-\frac{x\,e^{\frac{x^2}{4}}}{2}\right) dx$$

Integrating gives

$$e^{\frac{x^2}{4}} w = \int -\frac{x\,e^{\frac{x^2}{4}}}{2}\, dx$$
$$e^{\frac{x^2}{4}} w = -e^{\frac{x^2}{4}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{4}}$ results in

$$w(x) = -e^{-\frac{x^2}{4}} e^{\frac{x^2}{4}} + c_1 e^{-\frac{x^2}{4}}$$

which simplifies to

$$w(x) = -1 + c_1 \mathrm{e}^{-\frac{x^2}{4}}$$

Replacing w in the above by $\frac{1}{\sqrt{y}}$ using equation (5) gives the final solution.

$$\frac{1}{\sqrt{y}} = -1 + c_1 \mathrm{e}^{-\frac{x^2}{4}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -1 + c_1 \mathrm{e}^{-\frac{1}{4}}$$

$$c_1 = \frac{3\,\mathrm{e}^{\frac{1}{4}}}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{\sqrt{y}} = -1 + \frac{3\,\mathrm{e}^{-\frac{(x-1)(x+1)}{4}}}{2}$$

The above simplifies to

$$-3\,\mathrm{e}^{-\frac{(x-1)(x+1)}{4}}\sqrt{y} + 2\sqrt{y} + 2 = 0$$

Summary

The solution(s) found are the following

$$-3\,\mathrm{e}^{-\frac{(x-1)(x+1)}{4}}\sqrt{y} + 2\sqrt{y} + 2 = 0 \tag{1}$$

Verification of solutions

$$-3\,\mathrm{e}^{-\frac{(x-1)(x+1)}{4}}\sqrt{y} + 2\sqrt{y} + 2 = 0$$

Verified OK.

5.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^{\frac{3}{2}} + y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{y^{\frac{3}{2}} + y}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= \frac{1}{y^{\frac{3}{2}} + y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^{\frac{3}{2}}+y}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x \\ \phi &= -\frac{x^2}{2}+f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{y^{\frac{3}{2}}+y}$. Therefore equation (4) becomes

$$\frac{1}{y^{\frac{3}{2}}+y} = 0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^{\frac{3}{2}} + y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y\left(\sqrt{y}+1\right)}\right) \mathrm{d}y$$
$$f(y) = \ln(y) - 2\ln\left(\sqrt{y}+1\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y) - 2\ln\left(\sqrt{y}+1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y) - 2\ln\left(\sqrt{y}+1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$2\ln(2) - 2\ln(3) - \frac{1}{2} = c_1$$

$$c_1 = 2\ln(2) - 2\ln(3) - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} + \ln(y) - 2\ln\left(\sqrt{y}+1\right) = 2\ln(2) - 2\ln(3) - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \ln(y) - 2\ln\left(1+\sqrt{y}\right) = 2\ln(2) - 2\ln(3) - \frac{1}{2} \tag{1}$$

Verification of solutions

$$-\frac{x^2}{2} + \ln(y) - 2\ln\left(1+\sqrt{y}\right) = 2\ln(2) - 2\ln(3) - \frac{1}{2}$$

Verified OK.

5.12.6 Maple step by step solution

Let's solve

$\left[-yx+y'-xy^{\frac{3}{2}}=0, y(1)=4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{3}{2}}+y}=x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{3}{2}}+y}dx=\int x dx+c_1$

- Evaluate integral

  $-2\,\mathrm{arctanh}\left(\sqrt{y}\right)+\ln(y)-\ln(y-1)=\frac{x^2}{2}+c_1$

- Use initial condition $y(1)=4$

  $-2\,\mathrm{arctanh}\left(\sqrt{4}\right)+2\ln(2)-\ln(3)=\frac{1}{2}+c_1$

- Solve for c_1

  $c_1=-\frac{1}{2}-2\,\mathrm{arctanh}\left(\sqrt{4}\right)+2\ln(2)-\ln(3)$

- Substitute $c_1=-\frac{1}{2}-2\mathrm{arctanh}\left(\sqrt{4}\right)+2\ln(2)-\ln(3)$ into general solution and simplify

  $-2\,\mathrm{arctanh}\left(\sqrt{y}\right)+\ln(y)-\ln(y-1)=\frac{x^2}{2}-\frac{1}{2}-2\,\mathrm{arctanh}\left(\frac{1}{2}\right)+\mathrm{I}\pi+2\ln(2)-\ln(3)$

- Solution to the IVP

  $-2\,\mathrm{arctanh}\left(\sqrt{y}\right)+\ln(y)-\ln(y-1)=\frac{x^2}{2}-\frac{1}{2}-2\,\mathrm{arctanh}\left(\frac{1}{2}\right)+\mathrm{I}\pi+2\ln(2)-\ln(3)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 21

```
dsolve([diff(y(x),x)-x*y(x)=x*y(x)^(3/2),y(1) = 4],y(x), singsol=all)
```

$$y(x) = \frac{4}{\left(-2 + 3\,\mathrm{e}^{-\frac{(x-1)(x+1)}{4}}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.405 (sec). Leaf size: 55

```
DSolve[{y'[x]-x*y[x]==x*y[x]^(3/2),y[1]==4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\tanh\left(\frac{1}{8}(-8\mathrm{arctanh}(3) - x^2 + 1)\right) - 1\right)^2$$
$$y(x) \to \frac{1}{4}\left(\tanh\left(\mathrm{arctanh}(5) - \frac{x^2}{8} + \frac{1}{8}\right) - 1\right)^2$$

5.13 problem 9

Internal problem ID [987]
Internal file name [OUTPUT/988_Sunday_June_05_2022_01_56_08_AM_70124900/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x + y - x^4y^4 = 0}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

5.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(x^4y^3 - 1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{1}{2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(x^4y^3 - 1)}{x}\right) \\ &= \frac{x^4y^3 - 1}{x} + 3y^3x^3\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

5.13.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= \frac{y(x^4y^3 - 1)}{x}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x^3y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= x^3 \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y' \frac{1}{y^4} = -\frac{1}{x\, y^3} + x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{y^4} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{3} &= -\frac{w(x)}{x} + x^3 \\ w' &= \frac{3w}{x} - 3x^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= -3x^3 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = -3x^3$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-3x^3\right) \\
\frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(-3x^3\right) \\
d\left(\frac{w}{x^3}\right) &= -3\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^3} &= \int -3\,dx \\
\frac{w}{x^3} &= -3x + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = c_1 x^3 - 3x^4$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = c_1 x^3 - 3x^4$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$8 = -3 + c_1$$

$$c_1 = 11$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^3} = -3x^4 + 11x^3$$

The above simplifies to

$$3x^4y^3 - 11y^3x^3 + 1 = 0$$

Summary

The solution(s) found are the following

$$1+\left(3x^{4}-11x^{3}\right)y^{3}=0 \tag{1}$$

Verification of solutions

$$1+\left(3x^{4}-11x^{3}\right)y^{3}=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 35

```
dsolve([x*diff(y(x),x)+y(x)=x^4*y(x)^4,y(1) = 1/2],y(x), singsol=all)
```

$$y(x)=\frac{\left(-(3x-11)^{2}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{6x^{2}-22x}$$

✓ Solution by Mathematica

Time used: 0.415 (sec). Leaf size: 19

```
DSolve[{x*y'[x]+y[x]==x^4*y[x]^4,y[1]==1/2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{\sqrt[3]{-x^{3}(3x-11)}}$$

5.14 problem 10

Internal problem ID [988]
Internal file name [`OUTPUT/989_Sunday_June_05_2022_01_56_09_AM_77334256/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y - 2\sqrt{y} = 0}$$

With initial conditions

$$[y(0) = 1]$$

5.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 2y + 2\sqrt{y} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(2y + 2\sqrt{y}) \\ &= 2 + \frac{1}{\sqrt{y}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.14.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2y + 2\sqrt{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 2y + 2\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 2 \\ f_1(x) &= 2 \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y' \frac{1}{\sqrt{y}} = 2\sqrt{y} + 2 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= 2w(x) + 2 \\ w' &= w + 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$w'(x) - w(x) = 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= \mu \\ \frac{d}{dx}\left(e^{-x}w\right) &= e^{-x} \\ d\left(e^{-x}w\right) &= e^{-x}dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x}w &= \int e^{-x}\, dx \\ e^{-x}w &= -e^{-x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$w(x) = -e^{-x}e^{x} + c_1e^{x}$$

which simplifies to

$$w(x) = c_1 e^x - 1$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = c_1 e^x - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y} = -1 + 2\,e^x$$

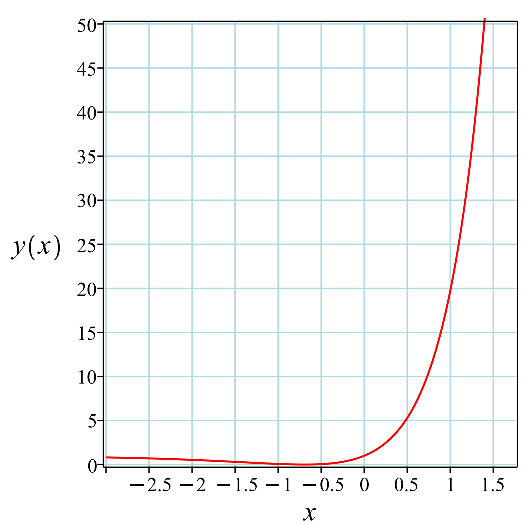
Solving for y from the above gives

$$y = 4\,e^{2x} - 4\,e^x + 1$$

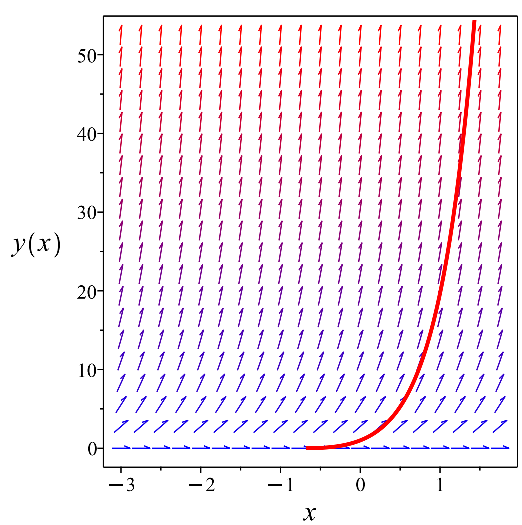
Summary

The solution(s) found are the following

$$y = 4\,e^{2x} - 4\,e^x + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\,e^{2x} - 4\,e^{x} + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve([diff(y(x),x)-2*y(x)=2*y(x)^(1/2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 4\,e^{2x} - 4\,e^{x} + 1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 14

```
DSolve[{y'[x]-2*y[x]==2*y[x]^(1/2),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (1 - 2e^{x})^{2}$$

5.15 problem 11

Internal problem ID [989]
Internal file name [OUTPUT/990_Sunday_June_05_2022_01_56_11_AM_82313958/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y' - 4y - \frac{48x}{y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

5.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{4y^3 + 48x}{y^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{4y^3 + 48x}{y^2}\right) \\ &= 12 - \frac{8(y^3 + 12x)}{y^3}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.15.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= \frac{4y^3 + 48x}{y^2}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = 4y + 48x\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 4 \\
f_1(x) &= 48x \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = 4y^3 + 48x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= 4w(x) + 48x \\
w' &= 12w + 144x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -12 \\
q(x) &= 144x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 12w(x) = 144x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-12)dx} \\
&= e^{-12x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(144x) \\
\frac{d}{dx}\left(e^{-12x} w\right) &= \left(e^{-12x}\right)(144x) \\
d\left(e^{-12x} w\right) &= \left(144x\, e^{-12x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-12x} w &= \int 144x\, e^{-12x}\, dx \\
e^{-12x} w &= -(12x+1)\, e^{-12x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-12x}$ results in

$$
w(x) = -e^{12x}(12x+1)\, e^{-12x} + c_1 e^{12x}
$$

which simplifies to

$$
w(x) = -12x - 1 + c_1 e^{12x}
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = -12x - 1 + c_1 e^{12x}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = c_1 - 1
$$

$$
c_1 = 2
$$

Substituting c_1 found above in the general solution gives

$$
y^3 = -1 + 2\, e^{12x} - 12x
$$

Summary
The solution(s) found are the following

$$
y^3 = -1 + 2\, e^{12x} - 12x \tag{1}
$$

Verification of solutions

$$
y^3 = -1 + 2\, e^{12x} - 12x
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)-4*y(x)=48*x/y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \left(2\,\mathrm{e}^{12x} - 12x - 1\right)^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 4.195 (sec). Leaf size: 21

```
DSolve[{y'[x]-4*y[x]==48*x/y[x]^2,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{-12x + 2e^{12x} - 1}$$

5.16 problem 12

Internal problem ID [990]
Internal file name [OUTPUT/991_Sunday_June_05_2022_01_56_13_AM_67970129/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x^2 + 2yx - y^3 = 0}$$

With initial conditions

$$\left[y(1) = \frac{\sqrt{2}}{2}\right]$$

5.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(y^2 - 2x)}{x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{\sqrt{2}}{2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\sqrt{2}}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(y^2 - 2x)}{x^2}\right)\\ &= \frac{y^2 - 2x}{x^2} + \frac{2y^2}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\sqrt{2}}{2}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\sqrt{2}}{2}$ is inside this domain. Therefore solution exists and is unique.

5.16.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= \frac{y(y^2 - 2x)}{x^2}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + \frac{1}{x^2}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = -\frac{2}{x\,y^2} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{2w(x)}{x} + \frac{1}{x^2} \\ w' &= \frac{4w}{x} - \frac{2}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= -\frac{2}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{4w(x)}{x} = -\frac{2}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{4}{x}dx} = \frac{1}{x^4}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(-\frac{2}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{w}{x^4}\right) = \left(\frac{1}{x^4}\right)\left(-\frac{2}{x^2}\right)$$
$$d\left(\frac{w}{x^4}\right) = \left(-\frac{2}{x^6}\right) dx$$

Integrating gives

$$\frac{w}{x^4} = \int -\frac{2}{x^6}\, dx$$
$$\frac{w}{x^4} = \frac{2}{5x^5} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$w(x) = \frac{2}{5x} + c_1 x^4$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{2}{5x} + c_1 x^4$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + \frac{2}{5}$$

$$c_1 = \frac{8}{5}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = \frac{\frac{2}{5} + \frac{8x^5}{5}}{x}$$

The above simplifies to

$$-8x^5y^2 - 2y^2 + 5x = 0$$

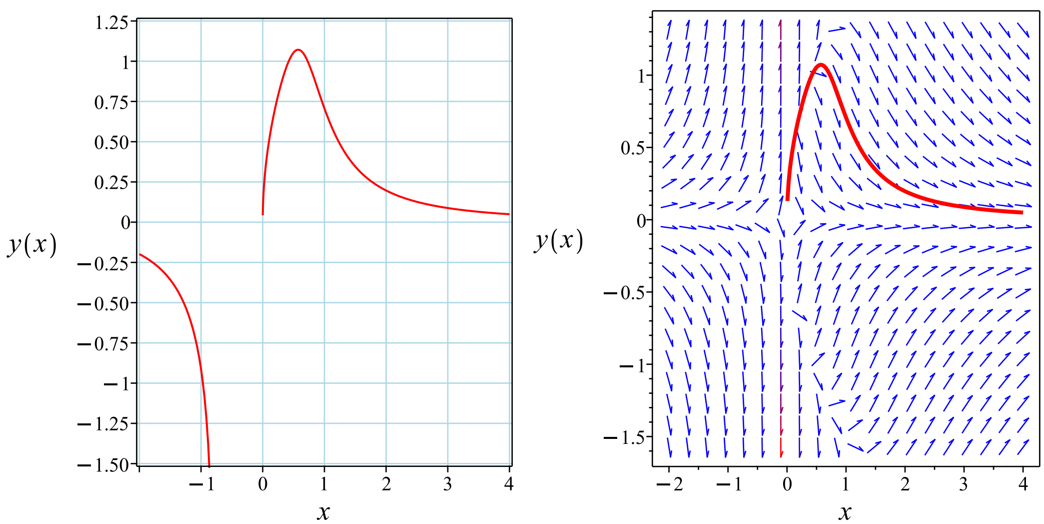
Solving for y from the above gives

$$y = \frac{\sqrt{10}\,\sqrt{4x^6 + x}}{8x^5 + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{10}\,\sqrt{4x^6 + x}}{8x^5 + 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{10}\,\sqrt{4x^6 + x}}{8x^5 + 2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 26

```
dsolve([x^2*diff(y(x),x)+2*x*y(x)=y(x)^3,y(1) = 1/sqrt(2)],y(x), singsol=all)
```

$$y(x)=\frac{\sqrt{10}\,\sqrt{4x^6+x}}{8x^5+2}$$

✓ Solution by Mathematica

Time used: 0.462 (sec). Leaf size: 29

```
DSolve[{x^2*y'[x]+2*x*y[x]==y[x]^3,y[1]==1/Sqrt[2]},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{\sqrt{\frac{5}{2}}\sqrt{x}}{\sqrt{4x^5+1}}$$

5.17 problem 13

Internal problem ID [991]
Internal file name [`OUTPUT/992_Sunday_June_05_2022_01_56_15_AM_45033092/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' - y - \sqrt{y}\, x = 0}$$

With initial conditions

$$[y(0) = 4]$$

5.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y + \sqrt{y}\, x \end{aligned}$$

The x domain of $f(x, y)$ when $y = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y + \sqrt{y}\, x) \\ &= 1 + \frac{x}{2\sqrt{y}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

5.17.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= y + \sqrt{y}\, x\end{aligned}$$

This is a Bernoulli ODE.

$$y' = y + x\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= 1 \\ f_1(x) &= x \\ n &= \frac{1}{2}\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y' \frac{1}{\sqrt{y}} = \sqrt{y} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= w(x) + x \\ w' &= \frac{w}{2} + \frac{x}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2} \\ q(x) &= \frac{x}{2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{2} = \frac{x}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{2} dx} \\ &= e^{-\frac{x}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{x}{2}\right) \\ \frac{d}{dx}\left(e^{-\frac{x}{2}} w\right) &= \left(e^{-\frac{x}{2}}\right)\left(\frac{x}{2}\right) \\ d\left(e^{-\frac{x}{2}} w\right) &= \left(\frac{x\, e^{-\frac{x}{2}}}{2}\right) dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\frac{x}{2}} w &= \int \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{2}\, \mathrm{d}x \\
\mathrm{e}^{-\frac{x}{2}} w &= -(2+x)\,\mathrm{e}^{-\frac{x}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x}{2}}$ results in

$$
w(x) = -\mathrm{e}^{\frac{x}{2}} (2+x)\,\mathrm{e}^{-\frac{x}{2}} + c_1 \mathrm{e}^{\frac{x}{2}}
$$

which simplifies to

$$
w(x) = -x - 2 + c_1 \mathrm{e}^{\frac{x}{2}}
$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$
\sqrt{y} = -x - 2 + c_1 \mathrm{e}^{\frac{x}{2}}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = c_1 - 2
$$

$$
c_1 = 4
$$

Substituting c_1 found above in the general solution gives

$$
\sqrt{y} = -2 + 4\,\mathrm{e}^{\frac{x}{2}} - x
$$

Summary

The solution(s) found are the following

$$
\sqrt{y} = -2 + 4\,\mathrm{e}^{\frac{x}{2}} - x \tag{1}
$$

Verification of solutions

$$
\sqrt{y} = -2 + 4\,\mathrm{e}^{\frac{x}{2}} - x
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 26

```
dsolve([diff(y(x),x)-y(x)=x*y(x)^(1/2),y(0) = 4],y(x), singsol=all)
```

$$y(x) = (-8x - 16)\,e^{\frac{x}{2}} + x^2 + 4x + 16\,e^{x} + 4$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 28

```
DSolve[{y'[x]-y[x]==x*y[x]^(1/2),y[0]==4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(x - 4e^{x/2} + 2\right)^2$$
$$y(x) \to (x + 2)^2$$

5.18 problem 15

Internal problem ID [992]
Internal file name [`OUTPUT/993_Sunday_June_05_2022_01_56_17_AM_34017998/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{x+y}{x} = 0}$$

5.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$
$$d\left(\frac{y}{x}\right) = \frac{1}{x}dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{1}{x}\, dx$$
$$\frac{y}{x} = \ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x + x\ln(x)$$

which simplifies to

$$y = x(\ln(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

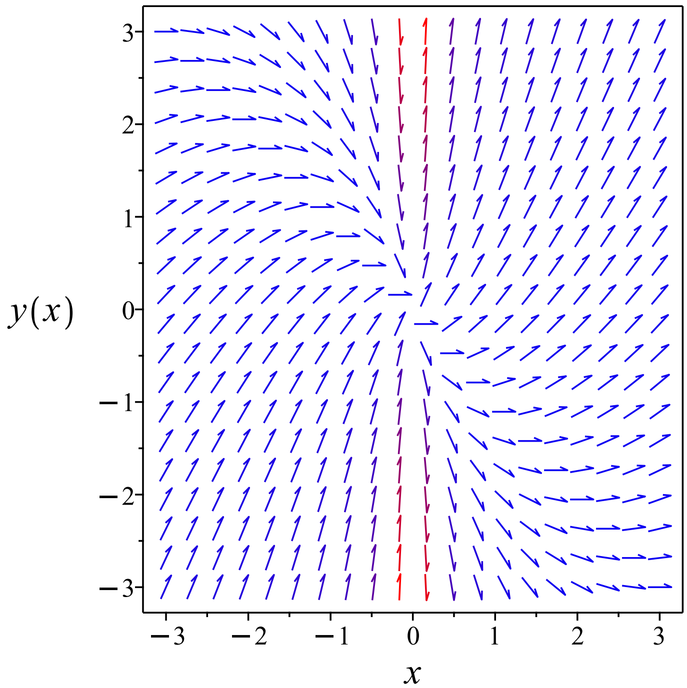


Figure 265: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

5.18.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{x + u(x)\, x}{x} = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_2) \tag{1}$$

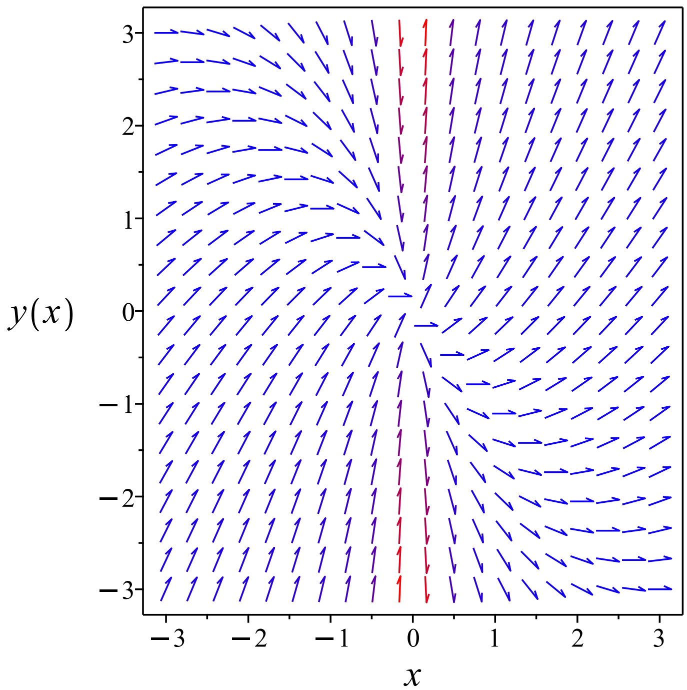


Figure 266: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_2)$$

Verified OK.

5.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 244: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

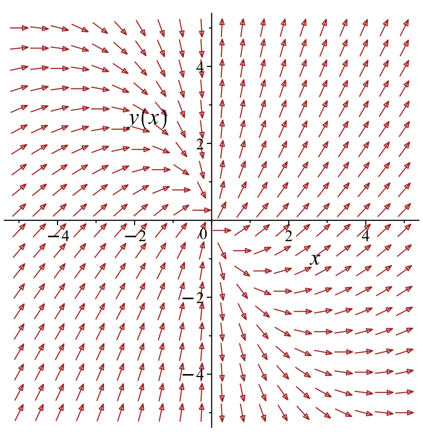
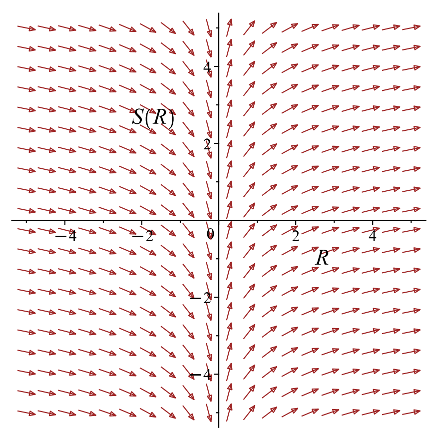
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

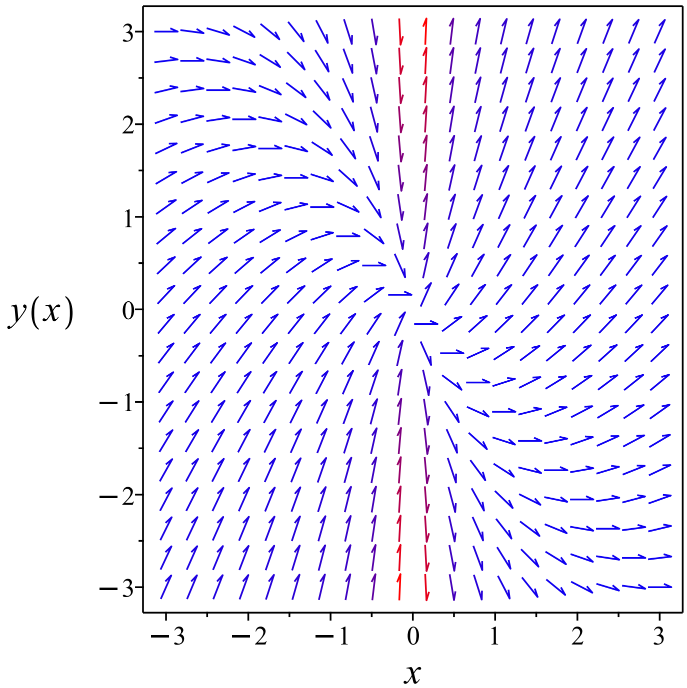


Figure 267: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

5.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{x + y}{x}\right) \mathrm{d}x$$
$$\left(-\frac{x + y}{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x + y}{x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x + y}{x}\right) \\ &= -\frac{1}{x}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(-\frac{x+y}{x}\right) \\
&= \frac{-x-y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}(1) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x-y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x-y}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + \frac{y}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{x}$$

The solution becomes

$$y = x(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

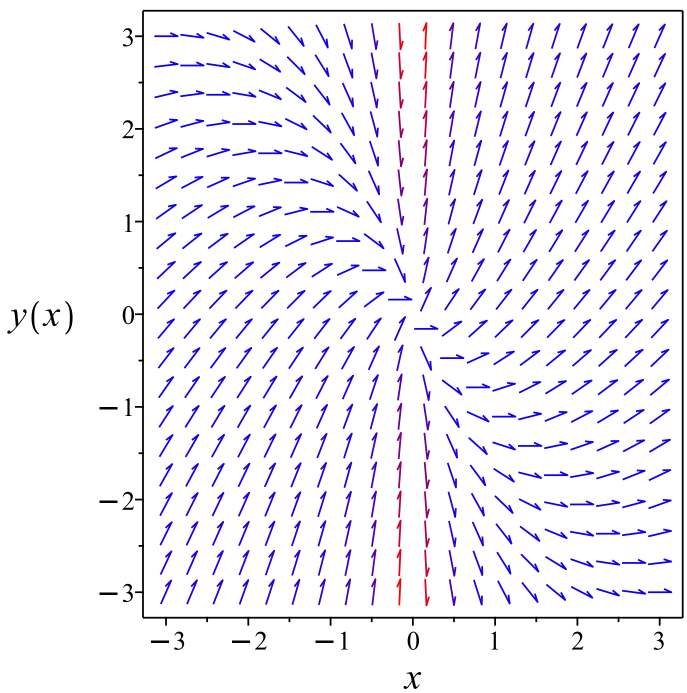


Figure 268: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

5.18.5 Maple step by step solution

Let's solve

$y' - \frac{x+y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=(y(x)+x)/x,y(x), singsol=all)
```

$$y(x) = (\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 12

```
DSolve[y'[x]==(y[x]+x)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + c_1)$$

5.19 problem 16

Internal problem ID [993]
Internal file name [`OUTPUT/994_Sunday_June_05_2022_01_56_18_AM_28134479/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{y^2 + 2yx}{x^2} = 0}$$

5.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)^2\,x^2 + 2u(x)\,x^2}{x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(u+1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u+1)$. Integrating both sides gives

$$\frac{1}{u(u+1)}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u(u+1)}\,du = \int \frac{1}{x}\,dx$$
$$\ln(u) - \ln(u+1) = \ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u)-\ln(u+1)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{u+1} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= -\frac{x^2 c_3}{c_3 x - 1} \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x^2 c_3}{c_3 x - 1} \tag{1}$$

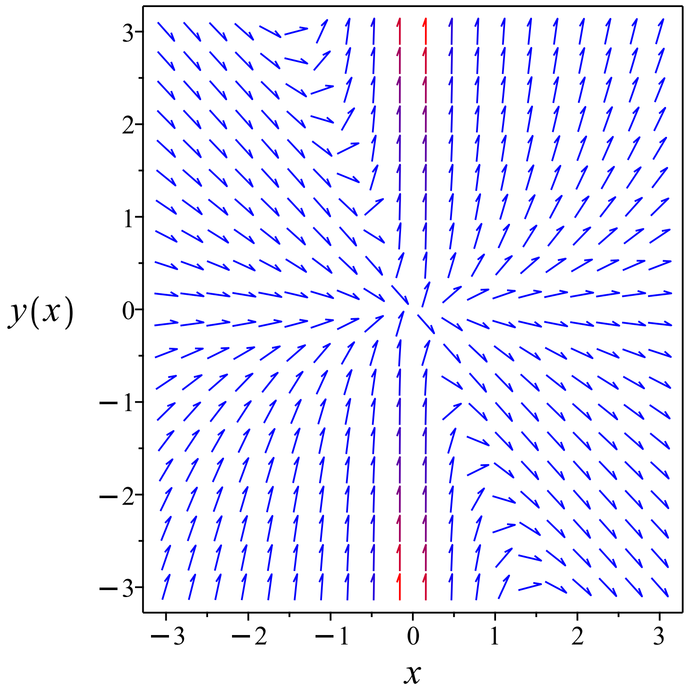


Figure 269: Slope field plot

Verification of solutions

$$y = -\frac{x^2 c_3}{c_3 x - 1}$$

Verified OK.

5.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2x + y)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 247: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(2x + y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x}{y} \\ S_y &= \frac{x^2}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{y} = x + c_1$$

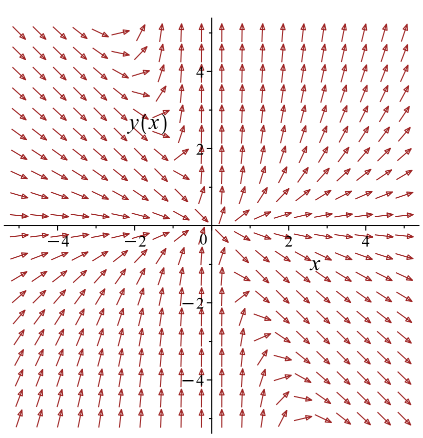
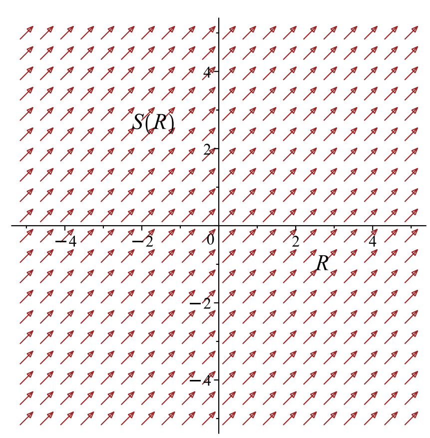
Which simplifies to

$$-\frac{x^2}{y} = x + c_1$$

Which gives

$$y = -\frac{x^2}{x + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x+y)}{x^2}$ 	$R = x$ $S = -\frac{x^2}{y}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{x + c_1} \tag{1}$$

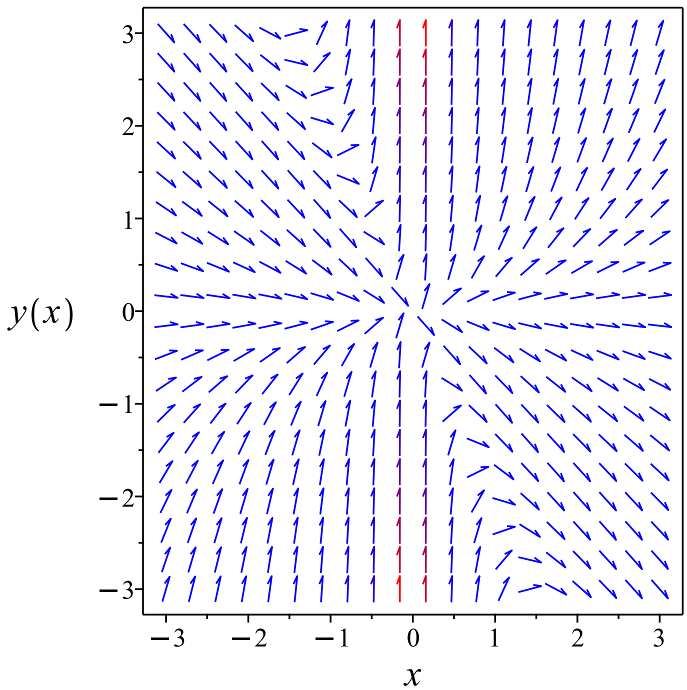


Figure 270: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{x + c_1}$$

Verified OK.

5.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{x}y + \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{2}{x} \\
f_1(x) &= \frac{1}{x^2} \\
n &= 2
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{2}{yx} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-w'(x) &= \frac{2w(x)}{x} + \frac{1}{x^2} \\
w' &= -\frac{2w}{x} - \frac{1}{x^2}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx}$$
$$= x^2$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(-\frac{1}{x^2}\right)$$
$$\frac{d}{dx}(x^2 w) = (x^2)\left(-\frac{1}{x^2}\right)$$
$$d(x^2 w) = -1\, dx$$

Integrating gives

$$x^2 w = \int -1\, dx$$
$$x^2 w = -x + c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{1}{x} + \frac{c_1}{x^2}$$

which simplifies to

$$w(x) = \frac{-x + c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-x + c_1}{x^2}$$

Or

$$y = \frac{x^2}{-x + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{-x + c_1} \tag{1}$$

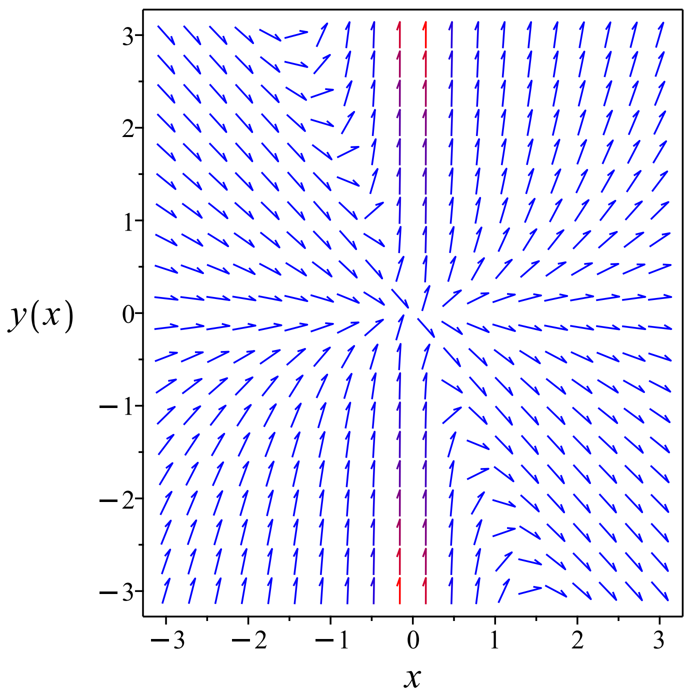


Figure 271: Slope field plot

Verification of solutions

$$y = \frac{x^2}{-x + c_1}$$

Verified OK.

5.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x^2)\,\mathrm{d}y &= (y(2x+y))\,\mathrm{d}x \\ (-y(2x+y))\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -y(2x+y) \\ N(x,y) &= x^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(2x+y)) \\ &= -2x - 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2}((-2x-2y)-(2x)) \\ &= \frac{-4x-2y}{x^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y\,(2x+y)}((2x)-(-2x-2y)) \\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2}(-y(2x+y))\\ &= \frac{-2x-y}{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^2}(x^2)\\ &= \frac{x^2}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-2x-y}{y}\right) + \left(\frac{x^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-2x-y}{y}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{(x+y)\,x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{x}{y} + \frac{(x+y)\,x}{y^2} + f'(y) \qquad (4)\\ &= \frac{x^2}{y^2} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2}{y^2}$. Therefore equation (4) becomes

$$\frac{x^2}{y^2} = \frac{x^2}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x+y)\,x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x+y)\,x}{y}$$

The solution becomes

$$y = -\frac{x^2}{x + c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{x^2}{x + c_1} \tag{1}$$

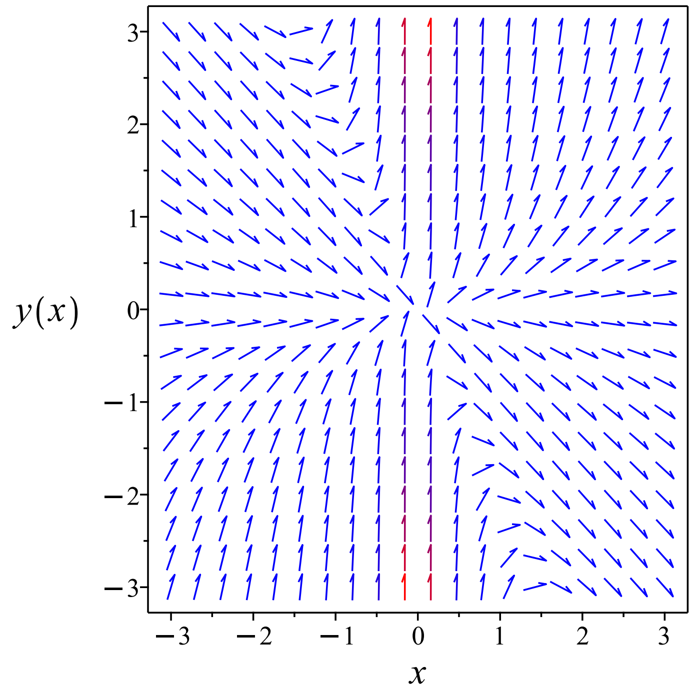


Figure 272: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{x + c_1}$$

Verified OK.

5.19.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x + y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{2y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{2}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{2}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 x^2}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{c_3 x^2}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3 x^2}{c_3 x + 1} \tag{1}$$

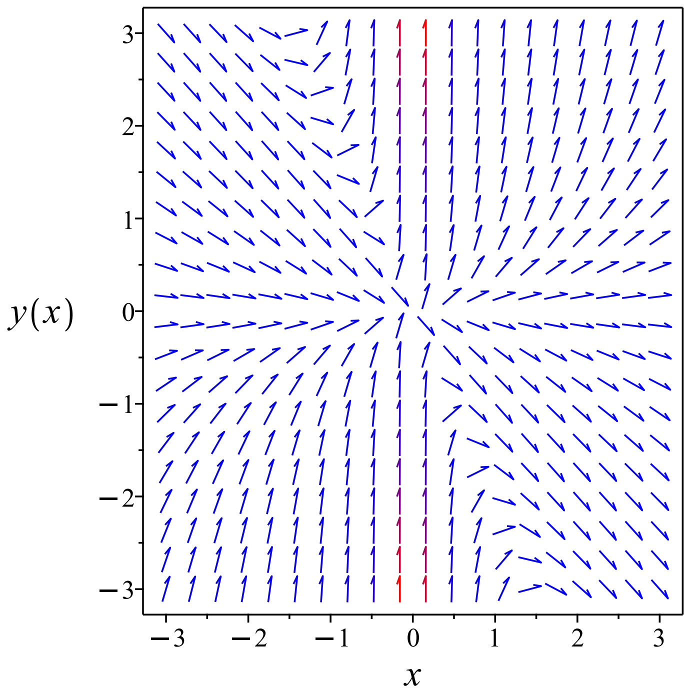


Figure 273: Slope field plot

Verification of solutions

$$y = -\frac{c_3 x^2}{c_3 x + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=(y(x)^2+2*x*y(x))/x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 23

```
DSolve[y'[x]==(y[x]^2+2*x*y[x])/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{x - c_1}$$
$$y(x) \to 0$$

5.20 problem 17

Internal problem ID [994]
Internal file name [`OUTPUT/995_Sunday_June_05_2022_01_56_19_AM_37774740/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xy^3y' - y^4 = x^4}$$

5.20.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^4u(x)^3\,(u'(x)\,x + u(x)) - u(x)^4\,x^4 = x^4$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{u^3x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u^3}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u^3}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{1}{u^3}}\,du = \int \frac{1}{x}\,dx$$
$$\frac{u^4}{4} = \ln(x) + c_2$$

The solution is

$$\frac{u(x)^4}{4} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^4}{4x^4} - \ln(x) - c_2 = 0$$
$$\frac{y^4}{4x^4} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^4}{4x^4} - \ln(x) - c_2 = 0 \tag{1}$$

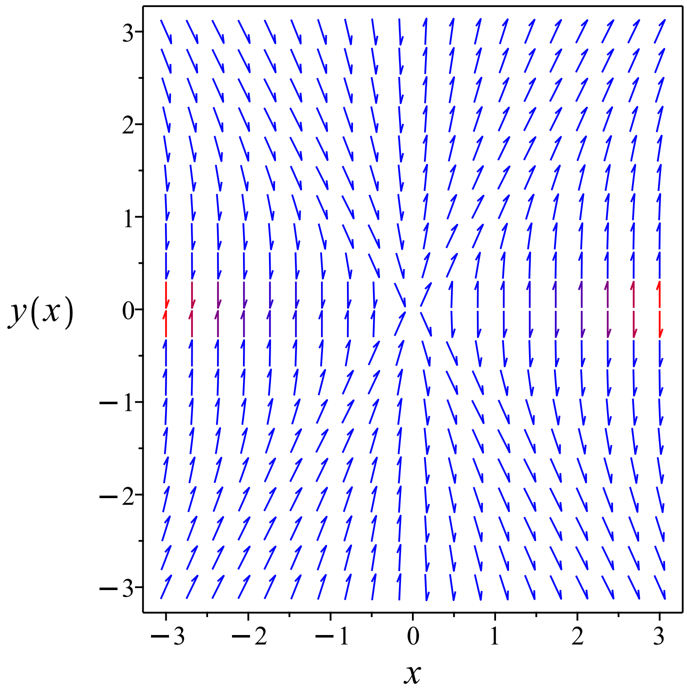


Figure 274: Slope field plot

Verification of solutions

$$\frac{y^4}{4x^4} - \ln(x) - c_2 = 0$$

Verified OK.

5.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^4 + y^4}{x\,y^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 249: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{x^4}{y^3} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4}{y^3}} dy \end{aligned}$$

Which results in

$$S = \frac{y^4}{4x^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^4 + y^4}{x\, y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^4}{x^5} \\ S_y &= \frac{y^3}{x^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

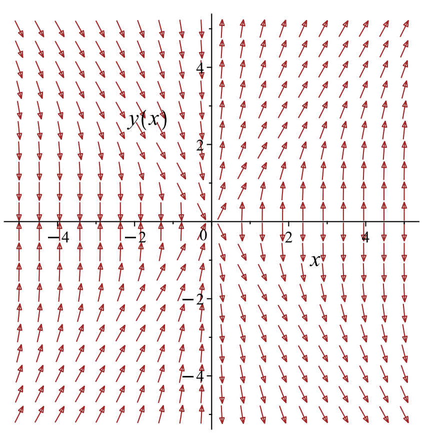
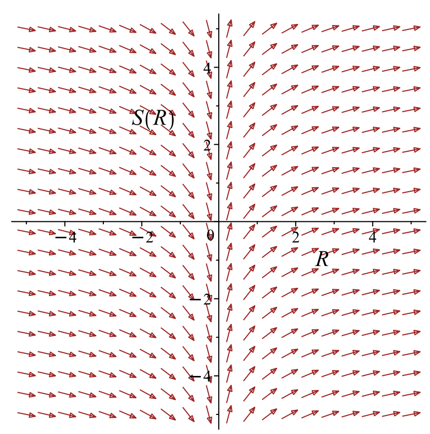
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^4}{4x^4} = \ln(x) + c_1$$

Which simplifies to

$$\frac{y^4}{4x^4} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4+y^4}{x\,y^3}$ 	$R = x$ $S = \frac{y^4}{4x^4}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{y^4}{4x^4} = \ln(x) + c_1 \tag{1}$$

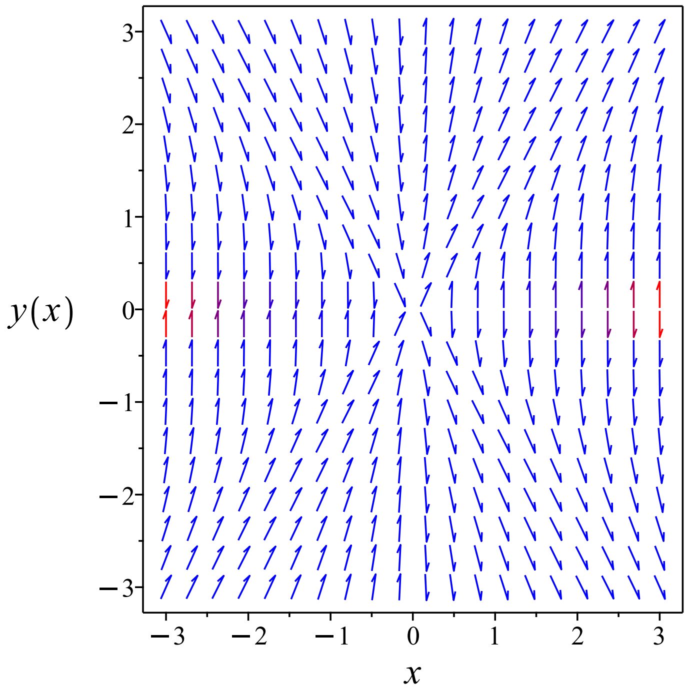


Figure 275: Slope field plot

Verification of solutions

$$\frac{y^4}{4x^4} = \ln(x) + c_1$$

Verified OK.

5.20.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^4 + y^4}{x\,y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x^3\frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= x^3 \\ n &= -3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$y' y^3 = \frac{y^4}{x} + x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^4 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 4y^3 y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{4} &= \frac{w(x)}{x} + x^3 \\ w' &= \frac{4w}{x} + 4x^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= 4x^3 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{4w(x)}{x} = 4x^3$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{4}{x}dx} \\
&= \frac{1}{x^4}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(4x^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(4x^3\right) \\
\mathrm{d}\left(\frac{w}{x^4}\right) &= \left(\frac{4}{x}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^4} &= \int \frac{4}{x}\,\mathrm{d}x \\
\frac{w}{x^4} &= 4\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$w(x) = 4\ln(x)\,x^4 + c_1 x^4$$

which simplifies to

$$w(x) = x^4(4\ln(x) + c_1)$$

Replacing w in the above by y^4 using equation (5) gives the final solution.

$$y^4 = x^4(4\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}
y(x) &= (4\ln(x) + c_1)^{\frac{1}{4}}\,x \\
y(x) &= i(4\ln(x) + c_1)^{\frac{1}{4}}\,x \\
y(x) &= -(4\ln(x) + c_1)^{\frac{1}{4}}\,x \\
y(x) &= -i(4\ln(x) + c_1)^{\frac{1}{4}}\,x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (4 \ln (x) + c_1)^{\frac{1}{4}} x \tag{1}$$
$$y = i(4 \ln (x) + c_1)^{\frac{1}{4}} x \tag{2}$$
$$y = -(4 \ln (x) + c_1)^{\frac{1}{4}} x \tag{3}$$
$$y = -i(4 \ln (x) + c_1)^{\frac{1}{4}} x \tag{4}$$

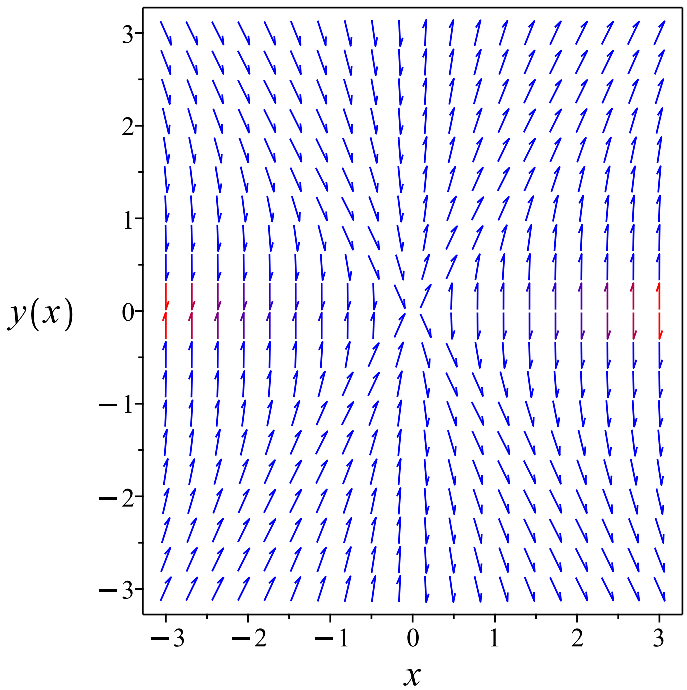


Figure 276: Slope field plot

Verification of solutions

$$y = (4\ln(x) + c_1)^{\frac{1}{4}} x$$

Verified OK.

$$y = i(4\ln(x) + c_1)^{\frac{1}{4}} x$$

Verified OK.

$$y = -(4\ln(x) + c_1)^{\frac{1}{4}} x$$

Verified OK.

$$y = -i(4\ln(x) + c_1)^{\frac{1}{4}} x$$

Verified OK.

5.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\,y^3)\,\mathrm{d}y&=(x^4+y^4)\,\mathrm{d}x\\ (-x^4-y^4)\,\mathrm{d}x+(x\,y^3)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^4-y^4\\ N(x,y)&=x\,y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x^4-y^4)\\ &=-4y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x\,y^3)\\ &=y^3\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{x\,y^3}\left((-4y^3)-(y^3)\right)\\ &=-\frac{5}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{5}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-5\ln(x)} \\ &= \frac{1}{x^5}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^5}\left(-x^4 - y^4\right) \\ &= \frac{-x^4 - y^4}{x^5}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^5}\left(x\, y^3\right) \\ &= \frac{y^3}{x^4}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^4 - y^4}{x^5}\right) + \left(\frac{y^3}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^4 - y^4}{x^5} \, \mathrm{d}x$$
$$\phi = \frac{y^4}{4x^4} - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y^3}{x^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^3}{x^4}$. Therefore equation (4) becomes

$$\frac{y^3}{x^4} = \frac{y^3}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^4}{4x^4} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^4}{4x^4} - \ln(x)$$

Summary

The solution(s) found are the following

$$\frac{y^4}{4x^4} - \ln(x) = c_1 \tag{1}$$

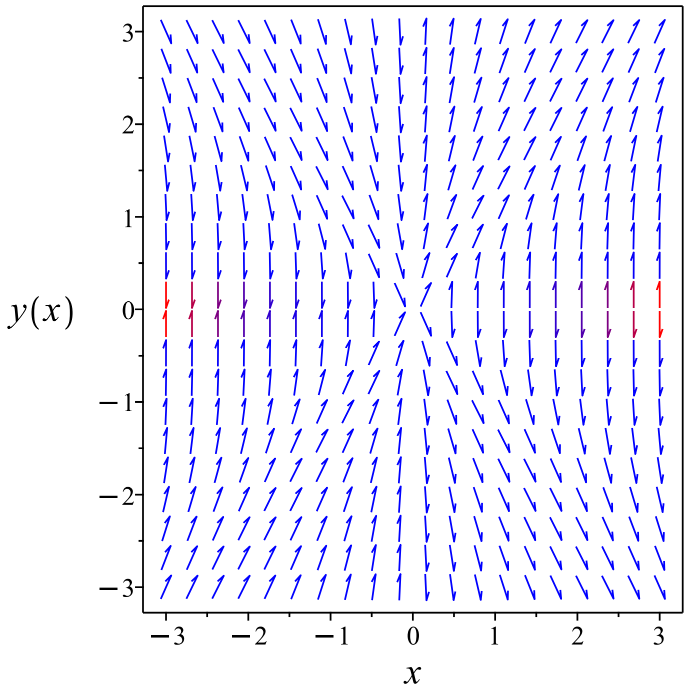


Figure 277: Slope field plot

Verification of solutions

$$\frac{y^4}{4x^4} - \ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
dsolve(x*y(x)^3*diff(y(x),x)=y(x)^4+x^4,y(x), singsol=all)
```

$$y(x) = (4\ln(x) + c_1)^{\frac{1}{4}} x$$
$$y(x) = -(4\ln(x) + c_1)^{\frac{1}{4}} x$$
$$y(x) = -i(4\ln(x) + c_1)^{\frac{1}{4}} x$$
$$y(x) = i(4\ln(x) + c_1)^{\frac{1}{4}} x$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 76

```
DSolve[x*y[x]^3*y'[x]==y[x]^4+x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt[4]{4\log(x) + c_1}$$
$$y(x) \to -ix\sqrt[4]{4\log(x) + c_1}$$
$$y(x) \to ix\sqrt[4]{4\log(x) + c_1}$$
$$y(x) \to x\sqrt[4]{4\log(x) + c_1}$$

5.21 problem 18

Internal problem ID [995]
Internal file name [`OUTPUT/996_Sunday_June_05_2022_01_56_21_AM_51119622/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} - \sec\left(\frac{y}{x}\right) = 0}$$

5.21.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} + \sec\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x+u &= u+g(x)\,f(bu)^{\frac{n}{m}}\\ u' &= \frac{1}{x}g(x)\,f(bu)^{\frac{n}{m}} \qquad (2)\end{aligned}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned}g(x) &= 1\\ b &= 1\\ f\left(\frac{bx}{y}\right) &= \sec\left(\frac{y}{x}\right)\end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = \frac{\sec(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{\sec(u)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sec(u)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sec(u)}\,du &= \frac{1}{x}\,dx\\ \int\frac{1}{\sec(u)}\,du &= \int\frac{1}{x}\,dx\\ \sin(u) &= \ln(x)+c_1\end{aligned}$$

The solution is

$$\sin(u(x)) - \ln(x) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_1 = 0 \tag{1}$$

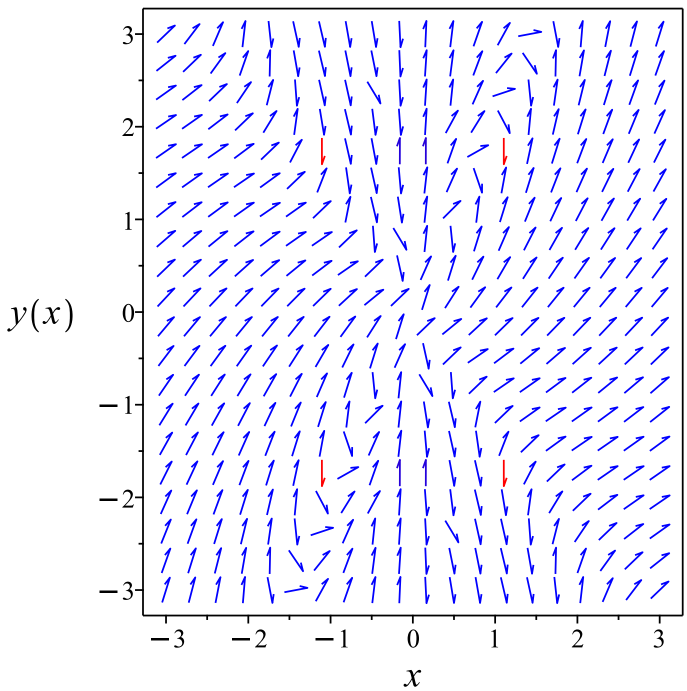


Figure 278: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_1 = 0$$

Verified OK.

5.21.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x - \sec(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sec(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sec(u)$. Integrating both sides gives

$$\frac{1}{\sec(u)}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\sec(u)}\, du = \int \frac{1}{x}\, dx$$

$$\sin(u) = \ln(x) + c_2$$

The solution is

$$\sin(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

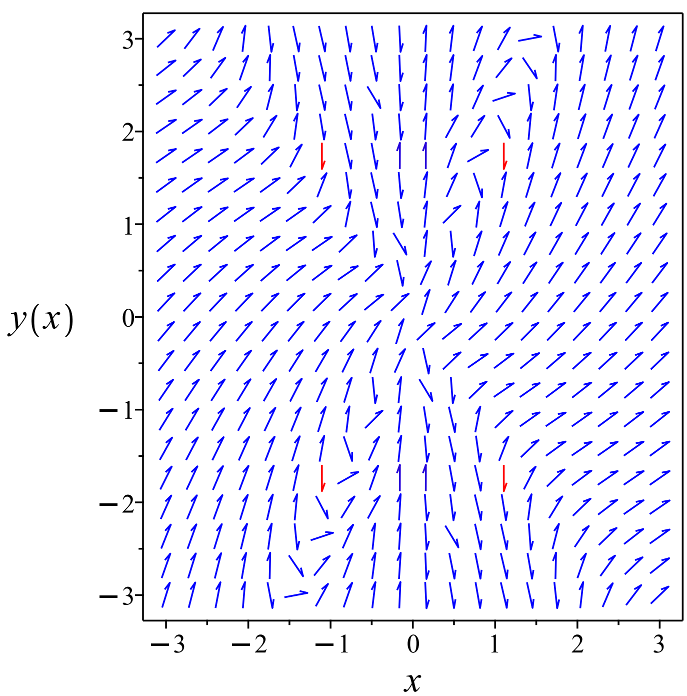


Figure 279: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right) - \ln\left(x\right) - c_2 = 0$$

Verified OK.

5.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sec\left(\frac{y}{x}\right) x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 251: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= yx \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{yx}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sec\left(\frac{y}{x}\right) x + y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{-\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{-\sin\left(\frac{y}{x}\right)}$$

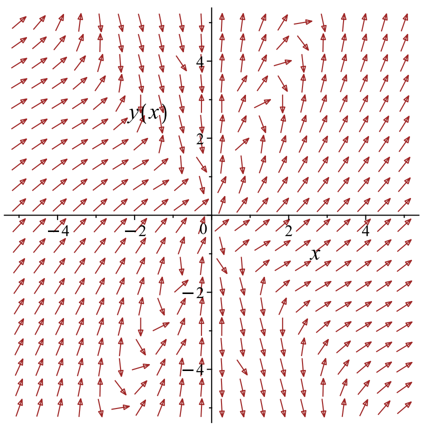
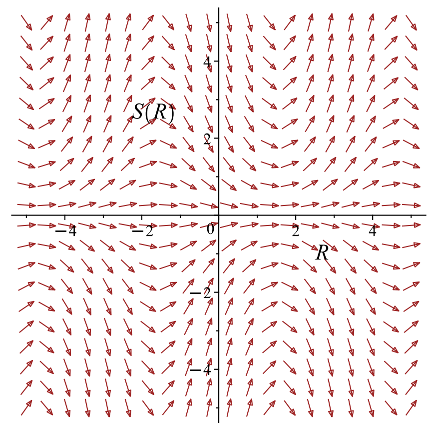
Which simplifies to

$$-\frac{1}{x} = c_1 e^{-\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sec\left(\frac{y}{x}\right)x+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\cos(R)$ 

Summary

The solution(s) found are the following

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right)x \tag{1}$$

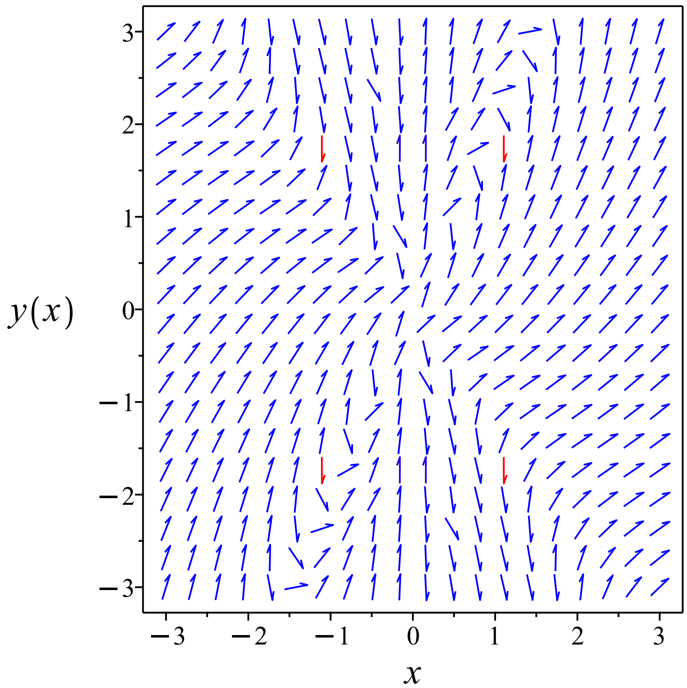


Figure 280: Slope field plot

Verification of solutions

$$y = -\arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=y(x)/x+sec(y(x)/x),y(x), singsol=all)
```

$$y(x) = \arcsin(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.409 (sec). Leaf size: 13

```
DSolve[y'[x]==y[x]/x+Sec[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(\log(x) + c_1)$$

5.22 problem 19

Internal problem ID [996]
Internal file name [`OUTPUT/997_Sunday_June_05_2022_01_56_23_AM_46290338/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - yx = x^2}$$

5.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 - u(x)\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 1$. Integrating both sides gives

$$\frac{1}{u^2+1}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u^2+1}\,du = \int \frac{1}{x}\,dx$$
$$\arctan(u) = \ln(x) + c_2$$

The solution is

$$\arctan(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$
$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

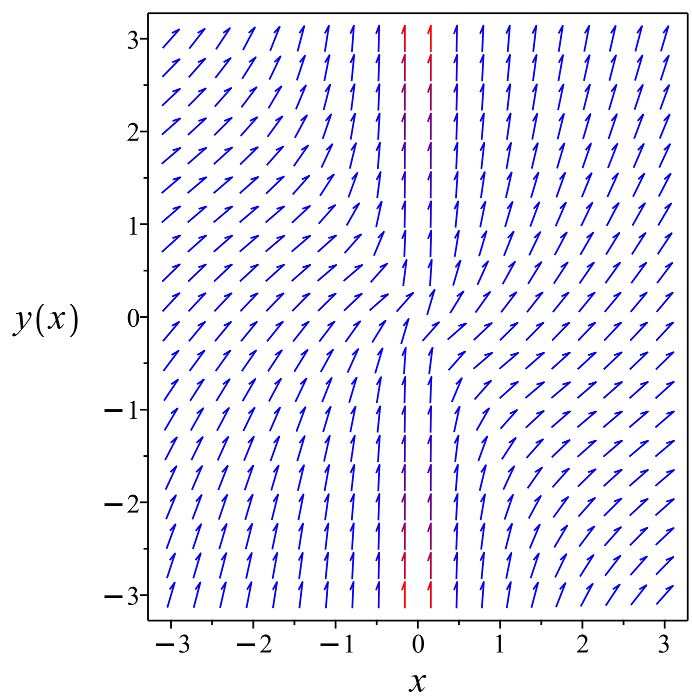


Figure 281: Slope field plot

Verification of solutions

$$\arctan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

5.22.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + yx + y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x^2 + yx + y^2)(b_3 - a_2)}{x^2} - \frac{(x^2 + yx + y^2)^2 a_3}{x^4} - \left(\frac{2x + y}{x^2} - \frac{2(x^2 + yx + y^2)}{x^3}\right)(x a_2 + y a_3 + a_1) - \frac{(x + 2y)(x b_2 + y b_3 + b_1)}{x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 a_2 + x^4 a_3 - x^4 b_3 + 2x^3 y a_3 + 2x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 a_3 + x^2 y^2 b_3 + y^4 a_3 + x^3 b_1 - x^2 y a_1 + 2x^2 y b_1 -}{x^4} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^4 a_2 - x^4 a_3 + x^4 b_3 - 2x^3 y a_3 - 2x^3 y b_2 + x^2 y^2 a_2 - 2x^2 y^2 a_3 \\
&- x^2 y^2 b_3 - y^4 a_3 - x^3 b_1 + x^2 y a_1 - 2x^2 y b_1 + 2x\, y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_2 v_1^4 + a_2 v_1^2 v_2^2 - a_3 v_1^4 - 2a_3 v_1^3 v_2 - 2a_3 v_1^2 v_2^2 - a_3 v_2^4 - 2b_2 v_1^3 v_2 \\
&+ b_3 v_1^4 - b_3 v_1^2 v_2^2 + a_1 v_1^2 v_2 + 2a_1 v_1 v_2^2 - b_1 v_1^3 - 2b_1 v_1^2 v_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 - a_3 + b_3)\, v_1^4 + (-2a_3 - 2b_2)\, v_1^3 v_2 - b_1 v_1^3 \\
&+ (a_2 - 2a_3 - b_3)\, v_1^2 v_2^2 + (a_1 - 2b_1)\, v_1^2 v_2 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
a_1 - 2b_1 &= 0 \\
-2a_3 - 2b_2 &= 0 \\
-a_2 - a_3 + b_3 &= 0 \\
a_2 - 2a_3 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^2 + yx + y^2}{x^2}\right)(x) \\
&= \frac{-x^2 - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - y^2}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2 + y^2} \\ S_y &= -\frac{x}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

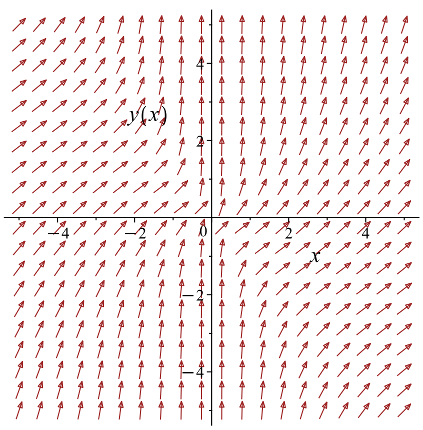
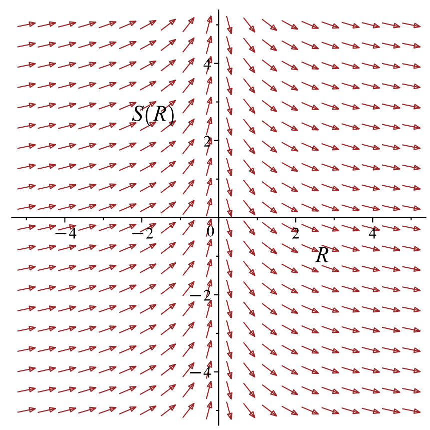
Which simplifies to

$$-\arctan\left(\frac{y}{x}\right) = -\ln(x) + c_1$$

Which gives

$$y = -\tan(-\ln(x) + c_1)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+yx+y^2}{x^2}$ 	$R = x$ $S = -\arctan\left(\frac{y}{x}\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\tan(-\ln(x) + c_1)\, x \tag{1}$$

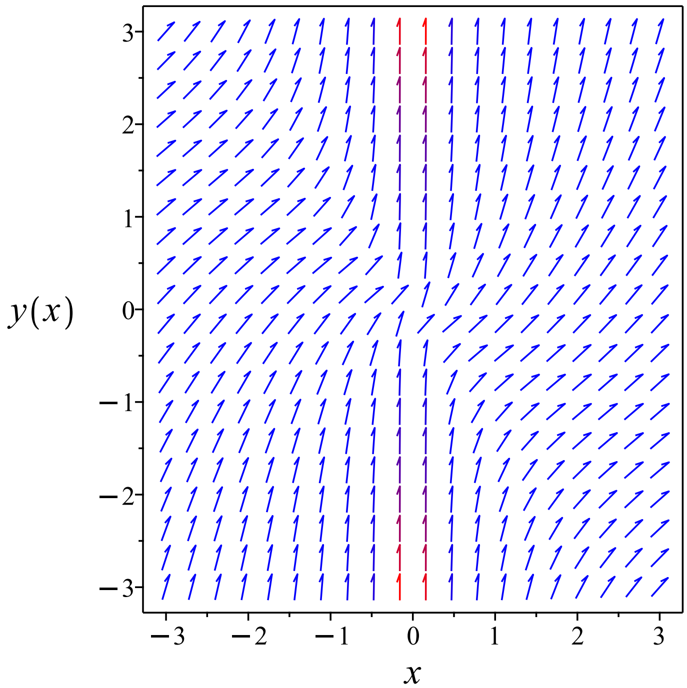


Figure 282: Slope field plot

Verification of solutions

$$
y = -\tan\left(-\ln\left(x\right) + c_1\right) x
$$

Verified OK.

5.22.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{x^2 + yx + y^2}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = 1 + \frac{y}{x} + \frac{y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 1$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= \frac{1}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} + \frac{u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sin(\ln(x))\, c_1 + c_2 \cos(\ln(x))$$

The above shows that

$$u'(x) = \frac{\cos(\ln(x))\, c_1 - c_2 \sin(\ln(x))}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{x(\cos(\ln(x))\, c_1 - c_2 \sin(\ln(x)))}{\sin(\ln(x))\, c_1 + c_2 \cos(\ln(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\cos\left(\ln\left(x\right)\right) c_{3}+\sin\left(\ln\left(x\right)\right)\right) x}{\sin\left(\ln\left(x\right)\right) c_{3}+\cos\left(\ln\left(x\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\cos\left(\ln\left(x\right)\right) c_{3}+\sin\left(\ln\left(x\right)\right)\right) x}{\sin\left(\ln\left(x\right)\right) c_{3}+\cos\left(\ln\left(x\right)\right)} \tag{1}$$

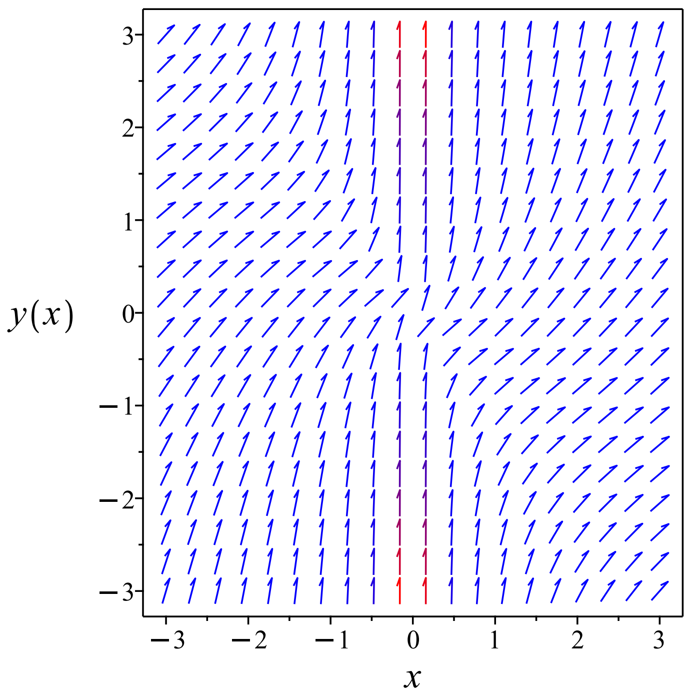


Figure 283: Slope field plot

Verification of solutions

$$y = \frac{\left(-\cos\left(\ln\left(x\right)\right) c_{3}+\sin\left(\ln\left(x\right)\right)\right) x}{\sin\left(\ln\left(x\right)\right) c_{3}+\cos\left(\ln\left(x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x^2*diff(y(x),x)=x*y(x)+x^2+y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan(\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.174 (sec). Leaf size: 13

```
DSolve[x^2*y'[x]==x*y[x]+x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan(\log(x) + c_1)$$

5.23 problem 20

Internal problem ID [997]
Internal file name [`OUTPUT/998_Sunday_June_05_2022_01_56_24_AM_63092252/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'xy - 2y^2 = x^2}$$

5.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2u(x) - 2u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+1}{u}}\, du = \int \frac{1}{x}\, dx$$

$$\frac{\ln (u^2+1)}{2} = \ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = c_3 x$$

Which simplifies to

$$\sqrt{u(x)^2+1} = c_3 \mathrm{e}^{c_2} x$$

The solution is

$$\sqrt{u(x)^2+1} = c_3 \mathrm{e}^{c_2} x$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = c_3 \mathrm{e}^{c_2} x$$

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3 \mathrm{e}^{c_2} x$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3 \mathrm{e}^{c_2} x \tag{1}$$

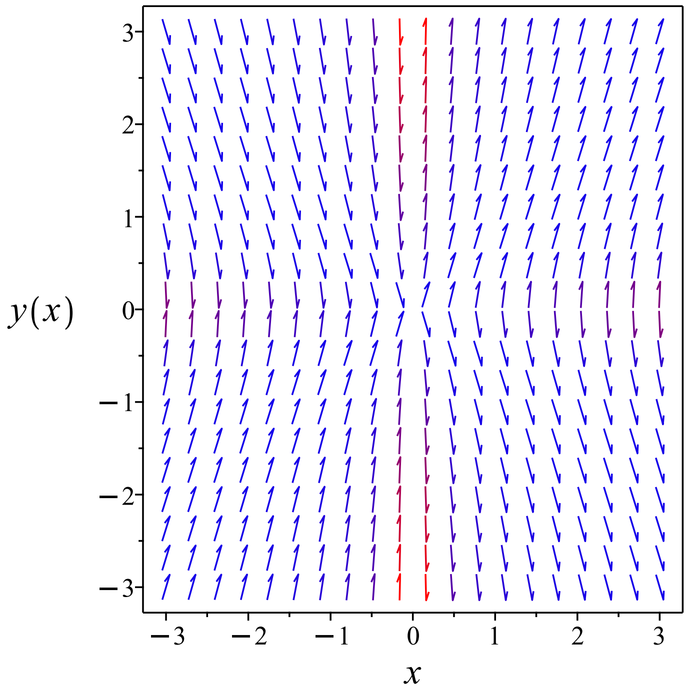


Figure 284: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3 e^{c_2} x$$

Verified OK.

5.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2+2y^2}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 253: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^4}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + 2y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y^2}{x^5} \\ S_y &= \frac{y}{x^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

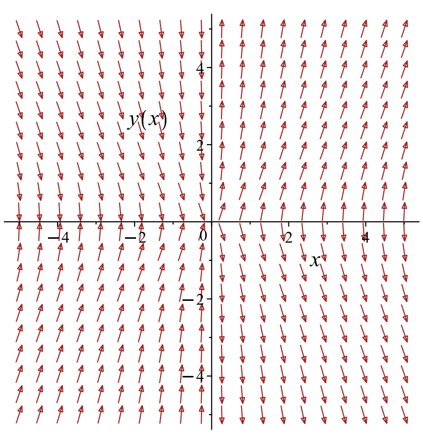
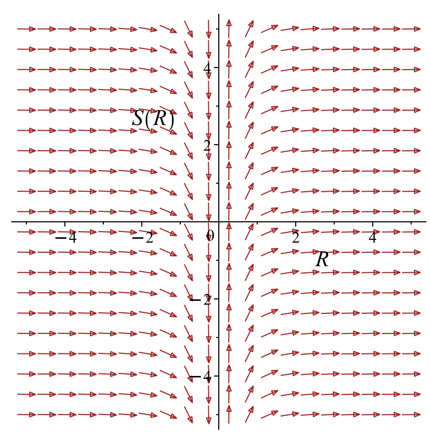
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^4} = -\frac{1}{2x^2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^4} = -\frac{1}{2x^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+2y^2}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^4}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x^4} = -\frac{1}{2x^2} + c_1 \tag{1}$$

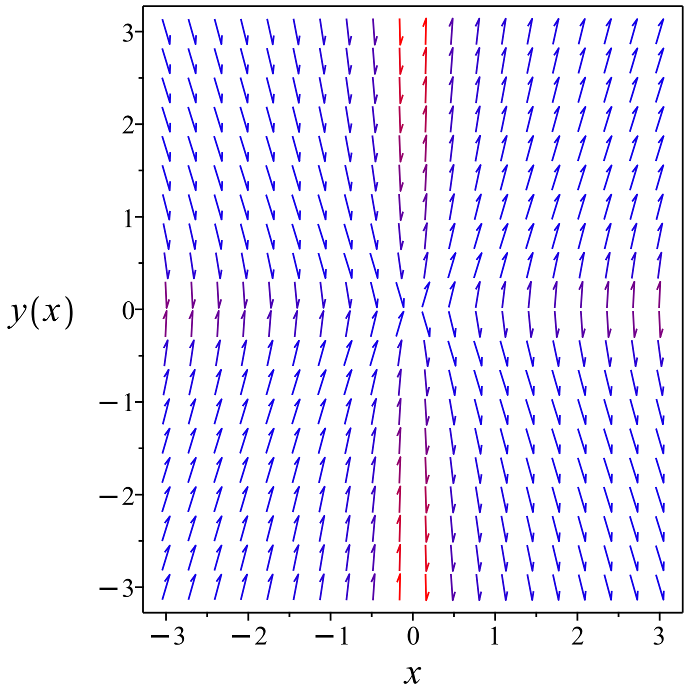


Figure 285: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^4} = -\frac{1}{2x^2} + c_1$$

Verified OK.

5.23.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 2y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{x} \\ f_1(x) &= x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{2y^2}{x} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{2w(x)}{x} + x \\ w' &= \frac{4w}{x} + 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= 2x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{4w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{4}{x}dx} \\
&= \frac{1}{x^4}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^4}\right) &= \left(\frac{1}{x^4}\right)(2x) \\
\mathrm{d}\left(\frac{w}{x^4}\right) &= \left(\frac{2}{x^3}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^4} &= \int \frac{2}{x^3}\,\mathrm{d}x \\
\frac{w}{x^4} &= -\frac{1}{x^2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$w(x) = c_1 x^4 - x^2$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x^4 - x^2$$

Solving for y gives

$$\begin{aligned}
y(x) &= \sqrt{c_1 x^2 - 1}\,x \\
y(x) &= -\sqrt{c_1 x^2 - 1}\,x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x^2 - 1}\,x \tag{1}$$

$$y = -\sqrt{c_1 x^2 - 1}\,x \tag{2}$$

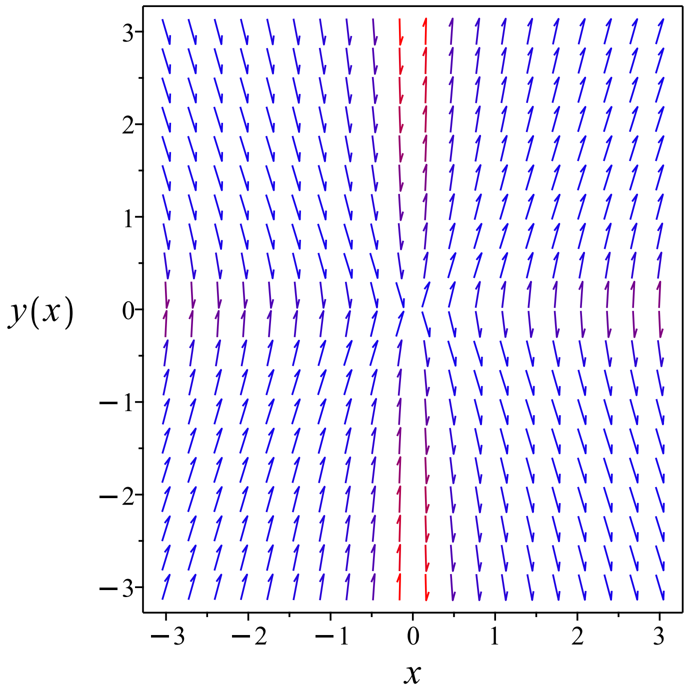


Figure 286: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x^2 - 1}\, x$$

Verified OK.

$$y = -\sqrt{c_1 x^2 - 1}\, x$$

Verified OK.

5.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(yx)\,\mathrm{d}y &= \left(x^2 + 2y^2\right)\mathrm{d}x\\ \left(-x^2 - 2y^2\right)\mathrm{d}x + (yx)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 - 2y^2\\ N(x, y) &= yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 2y^2\right)\\ &= -4y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yx}((-4y) - (y)) \\
&= -\frac{5}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{5}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-5\ln(x)} \\
&= \frac{1}{x^5}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^5}(-x^2 - 2y^2) \\
&= \frac{-x^2 - 2y^2}{x^5}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^5}(yx) \\
&= \frac{y}{x^4}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 - 2y^2}{x^5}\right) + \left(\frac{y}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 - 2y^2}{x^5}\,\mathrm{d}x$$
$$\phi = \frac{x^2 + y^2}{2x^4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^4}$. Therefore equation (4) becomes

$$\frac{y}{x^4} = \frac{y}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 + y^2}{2x^4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 + y^2}{2x^4}$$

Summary

The solution(s) found are the following

$$\frac{x^2 + y^2}{2x^4} = c_1 \tag{1}$$

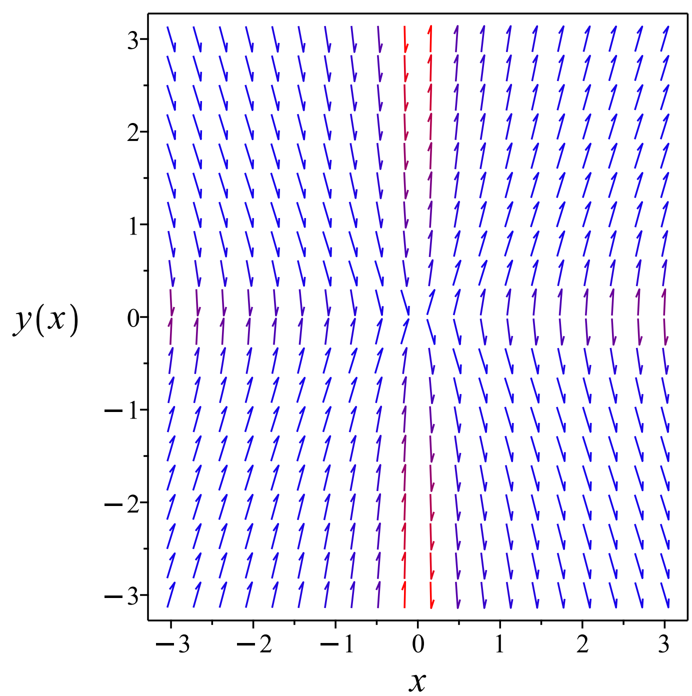


Figure 287: Slope field plot

Verification of solutions

$$\frac{x^2 + y^2}{2x^4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x*y(x)*diff(y(x),x)=x^2+2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 x^2 - 1}\, x$$
$$y(x) = -\sqrt{c_1 x^2 - 1}\, x$$

✓ Solution by Mathematica

Time used: 0.441 (sec). Leaf size: 38

```
DSolve[x*y[x]*y'[x]==x^2+2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{-1 + c_1 x^2}$$
$$y(x) \to x\sqrt{-1 + c_1 x^2}$$

5.24 problem 21

Internal problem ID [998]
Internal file name [`OUTPUT/999_Sunday_June_05_2022_01_56_25_AM_89065044/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`]]
```

$$\boxed{y' - \frac{2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2yx} = 0}$$

5.24.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2u(x)^2\, x^2 + x^2 \mathrm{e}^{-u(x)^2}}{2u\,(x)\, x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\mathrm{e}^{-u^2}}{2xu} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{e^{-u^2}}{u}$. Integrating both sides gives

$$\frac{1}{\frac{e^{-u^2}}{u}}\, du = \frac{1}{2x}\, dx$$

$$\int \frac{1}{\frac{e^{-u^2}}{u}}\, du = \int \frac{1}{2x}\, dx$$

$$\frac{e^{u^2}}{2} = \frac{\ln(x)}{2} + c_2$$

The solution is

$$\frac{e^{u(x)^2}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0 \tag{1}$$

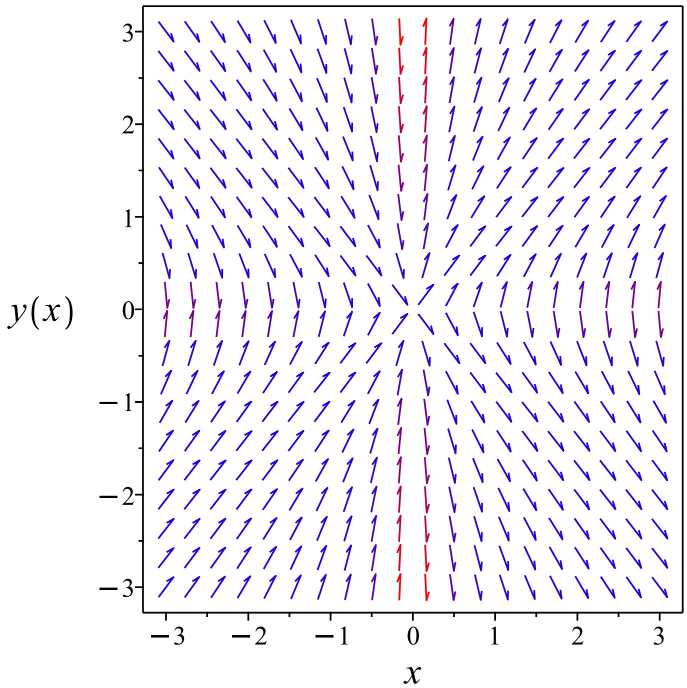


Figure 288: Slope field plot

Verification of solutions

$$\frac{e^{\frac{y^2}{x^2}}}{2} - \frac{\ln(x)}{2} - c_2 = 0$$

Verified OK.

5.24.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2y^2 + x^2 e^{-\frac{y^2}{x^2}}}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}\right)(b_3 - a_2)}{2yx} - \frac{\left(2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}\right)^2 a_3}{4y^2x^2} \\ &- \left(\frac{2x\,\mathrm{e}^{-\frac{y^2}{x^2}} + \frac{2y^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{x}}{2yx} - \frac{2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2y\,x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{4y - 2y\,\mathrm{e}^{-\frac{y^2}{x^2}}}{2yx} - \frac{2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2y^2 x}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 - 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 + 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^3 a_2 - 4}{4x^2y}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 - 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 \\ &+ 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^3 a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^3 b_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 b_1 \\ &- 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y a_1 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^2 b_1 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^3 a_1 - 4x\,y^2 b_1 + 4y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\mathrm{e}^{-\frac{2y^2}{x^2}} x^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^4 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 y b_3 - 6\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 a_3 \\ &+ 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y^2 b_2 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^3 a_2 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^3 b_3 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^4 a_3 + 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^3 b_1 \\ &- 2\,\mathrm{e}^{-\frac{y^2}{x^2}} x^2 y a_1 + 4\,\mathrm{e}^{-\frac{y^2}{x^2}} x\,y^2 b_1 - 4\,\mathrm{e}^{-\frac{y^2}{x^2}} y^3 a_1 - 4x\,y^2 b_1 + 4y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, e^{-\frac{2y^2}{x^2}}, e^{-\frac{y^2}{x^2}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, e^{-\frac{2y^2}{x^2}} = v_3, e^{-\frac{y^2}{x^2}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-4v_4v_1^3v_2a_2 - 4v_4v_1v_2^3a_2 - v_3v_1^4a_3 - 6v_4v_1^2v_2^2a_3 - 4v_4v_2^4a_3 \\
&+ 2v_4v_1^4b_2 + 4v_4v_1^2v_2^2b_2 + 4v_4v_1^3v_2b_3 + 4v_4v_1v_2^3b_3 - 2v_4v_1^2v_2a_1 \\
&- 4v_4v_2^3a_1 + 2v_4v_1^3b_1 + 4v_4v_1v_2^2b_1 + 4v_2^3a_1 - 4v_1v_2^2b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-v_3v_1^4a_3 + 2v_4v_1^4b_2 + (-4a_2 + 4b_3)\, v_1^3v_2v_4 + 2v_4v_1^3b_1 \\
&+ (-6a_3 + 4b_2)\, v_1^2v_2^2v_4 - 2v_4v_1^2v_2a_1 + (-4a_2 + 4b_3)\, v_1v_2^3v_4 \\
&+ 4v_4v_1v_2^2b_1 - 4v_1v_2^2b_1 - 4v_4v_2^4a_3 - 4v_4v_2^3a_1 + 4v_2^3a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 &= 0 \\
-2a_1 &= 0 \\
4a_1 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
-4b_1 &= 0 \\
2b_1 &= 0 \\
4b_1 &= 0 \\
2b_2 &= 0 \\
-4a_2 + 4b_3 &= 0 \\
-6a_3 + 4b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\a_2 &= b_3\\a_3 &= 0\\b_1 &= 0\\b_2 &= 0\\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\\eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\&= y - \left(\frac{2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2yx}\right)(x)\\&= -\frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2y}\\\xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2y}} dy
\end{aligned}
$$

Which results in

$$
S = -\mathrm{e}^{\frac{y^2}{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2y^2 + x^2 \mathrm{e}^{-\frac{y^2}{x^2}}}{2yx}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2y^2 \mathrm{e}^{\frac{y^2}{x^2}}}{x^3} \\
S_y &= -\frac{2\,\mathrm{e}^{\frac{y^2}{x^2}} y}{x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

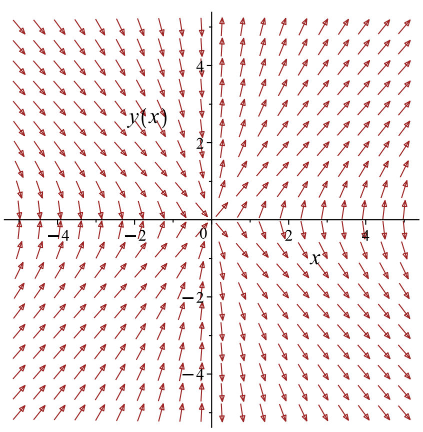
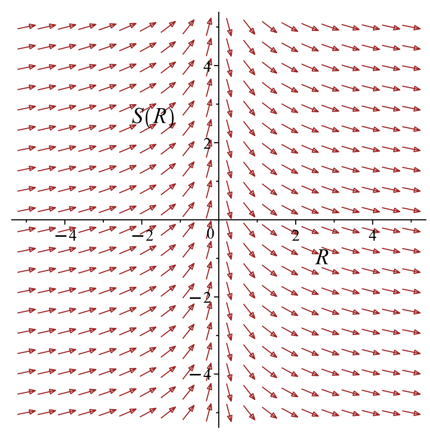
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

Which simplifies to

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y^2+x^2e^{-\frac{y^2}{x^2}}}{2yx}$ 	$R = x$ $S = -e^{\frac{y^2}{x^2}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

<u>Summary</u>
The solution(s) found are the following

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1 \tag{1}$$

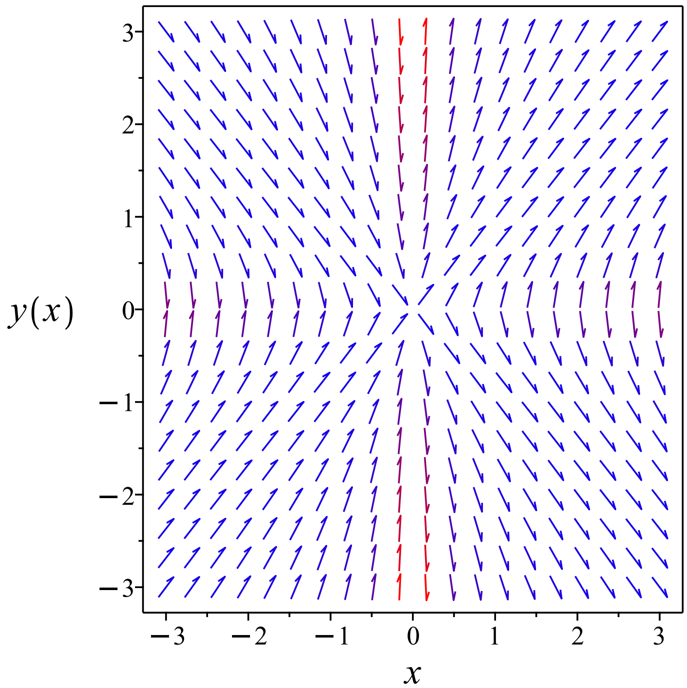


Figure 289: Slope field plot

Verification of solutions

$$-e^{\frac{y^2}{x^2}} = -\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)=(2*y(x)^2+x^2*exp(- (y(x)/x)^2 ))/(2*x*y(x)),y(x), singsol=all)
```

$$y(x) = \sqrt{\ln\left(\ln\left(x\right)+c_1\right)}\, x$$
$$y(x) = -\sqrt{\ln\left(\ln\left(x\right)+c_1\right)}\, x$$

✓ Solution by Mathematica

Time used: 2.166 (sec). Leaf size: 38

```
DSolve[y'[x]==(2*y[x]^2+x^2*Exp[- (y[x]/x)^2 ])/(2*x*y[x]),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -x\sqrt{\log(\log(x)+2c_1)}$$
$$y(x) \to x\sqrt{\log(\log(x)+2c_1)}$$

5.25 problem 22

Internal problem ID [999]
Internal file name [`OUTPUT/1000_Sunday_June_05_2022_01_56_27_AM_47396048/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{yx + y^2}{x^2} = 0}$$

With initial conditions

$$[y(-1) = 2]$$

5.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= \frac{y(x+y)}{x^2}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=2$ is

$$
\{x<0\vee 0<x\}
$$

And the point $x_0=-1$ is inside this domain. The y domain of $f(x,y)$ when $x=-1$ is

$$
\{-\infty<y<\infty\}
$$

And the point $y_0=2$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(x+y)}{x^2}\right) \\
&= \frac{x+y}{x^2}+\frac{y}{x^2}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=2$ is

$$
\{x<0\vee 0<x\}
$$

And the point $x_0=-1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$
\{-\infty<y<\infty\}
$$

And the point $y_0=2$ is inside this domain. Therefore solution exists and is unique.

5.25.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$
u'(x)\,x+u(x)-\frac{u(x)\,x^2+u(x)^2\,x^2}{x^2}=0
$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u^2}{x}
\end{aligned}
$$

Where $f(x)=\frac{1}{x}$ and $g(u)=u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u^2}\,du &= \int \frac{1}{x}\,dx \\
-\frac{1}{u} &= \ln(x)+c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x)}-\ln(x)-c_2=0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y}-\ln(x)-c_2=0 \\
-\frac{x}{y}-\ln(x)-c_2=0
\end{aligned}
$$

Substituting initial conditions and solving for c_2 gives $c_2=-i\pi+\frac{1}{2}$. Hence the solution becomes Solving for y from the above gives

$$
y=\frac{2x}{-2\ln(x)-1+2i\pi}
$$

Summary
The solution(s) found are the following

$$
y=\frac{2x}{-2\ln(x)-1+2i\pi} \tag{1}
$$

Verification of solutions

$$
y=\frac{2x}{-2\ln(x)-1+2i\pi}
$$

Verified OK.

5.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x+y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 255: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{y^2}{x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{y^2}{x}} dy\end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x}{y} = \ln(x) + c_1
$$

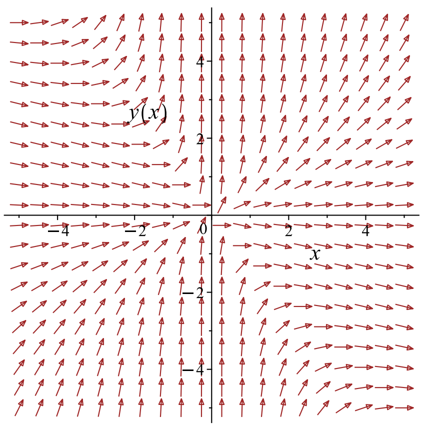
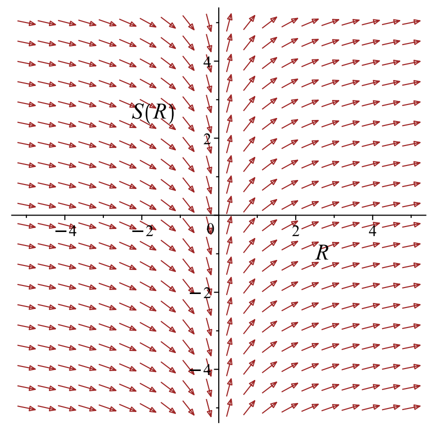
Which simplifies to

$$
-\frac{x}{y} = \ln(x) + c_1
$$

Which gives

$$
y = -\frac{x}{\ln(x) + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{i\pi + c_1}$$

$$c_1 = -i\pi + \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi} \tag{1}$$

Verification of solutions

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Verified OK.

5.25.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(x+y)}{x^2}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \frac{1}{x}y + \frac{1}{x^2}y^2 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= \frac{1}{x^2} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{1}{yx} + \frac{1}{x^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y} \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}-w'(x) &= \frac{w(x)}{x} + \frac{1}{x^2}\\ w' &= -\frac{w}{x} - \frac{1}{x^2}\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{1}{x^2}\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^2}\right)\\ \frac{d}{dx}(xw) &= (x)\left(-\frac{1}{x^2}\right)\\ d(xw) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -\frac{1}{x}\, dx\\ xw &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{-\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{-\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{i\pi - c_1}$$

$$c_1 = i\pi - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi} \tag{1}$$

Verification of solutions

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Verified OK.

5.25.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(\frac{yx+y^2}{x^2}\right)\mathrm{d}x\\ \left(-\frac{yx+y^2}{x^2}\right)\mathrm{d}x+\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{yx+y^2}{x^2} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{yx+y^2}{x^2}\right) \\
&= \frac{-2y-x}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{x}{y^2}$ is an integrating factor. Therefore by multiplying $M = -\frac{yx+y^2}{x^2}$ and $N = 1$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= -\frac{yx+y^2}{xy^2} \\
N &= \frac{x}{y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y^2}\right)\mathrm{d}y&=\left(\frac{yx+y^2}{x\,y^2}\right)\mathrm{d}x\\ \left(-\frac{yx+y^2}{x\,y^2}\right)\mathrm{d}x+\left(\frac{x}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{yx+y^2}{x\,y^2}\\ N(x,y)&=\frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{yx+y^2}{x\,y^2}\right)\\ &=\frac{1}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{yx + y^2}{x\, y^2} \, \mathrm{d}x \\
\phi &= -\ln(x) - \frac{x}{y} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$
\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{\ln(x) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{i\pi + c_1}$$

$$c_1 = -i\pi + \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi} \tag{1}$$

Verification of solutions

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Verified OK.

5.25.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \ln(x)$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 x}{c_1 + c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{c_3 + \ln(x)}$$

Initial conditions are used to solve for c_3. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{i\pi + c_3}$$

$$c_3 = -i\pi + \frac{1}{2}$$

Substituting c_3 found above in the general solution gives

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi} \tag{1}$$

Verification of solutions

$$y = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(x*y(x)+y(x)^2)/x^2,y(-1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{2x}{-2\ln(x) - 1 + 2i\pi}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 25

```
DSolve[{y'[x]==(x*y[x]+y[x]^2)/x^2,y[-1]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2ix}{2i\log(x) + 2\pi + i}$$

5.26 problem 23

Internal problem ID [1000]
Internal file name [`OUTPUT/1001_Sunday_June_05_2022_01_56_29_AM_53614540/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{x^3 + y^3}{x y^2} = 0}$$

With initial conditions

$$[y(1) = 3]$$

5.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x^3+y^3}{x\,y^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\{x<0\lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{y<0\lor 0<y\}$$

And the point $y_0=3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^3+y^3}{x\,y^2}\right) \\ &= \frac{3}{x}-\frac{2(x^3+y^3)}{x\,y^3} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=3$ is

$$\{x<0\lor 0<x\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\{y<0\lor 0<y\}$$

And the point $y_0=3$ is inside this domain. Therefore solution exists and is unique.

5.26.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x+u(x)-\frac{x^3+u(x)^3\,x^3}{x^3u\left(x\right)^2}=0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{1}{u^2 x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u^2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{u^2}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\frac{1}{u^2}}\, du &= \int \frac{1}{x}\, dx \\
\frac{u^3}{3} &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
\frac{u(x)^3}{3} - \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^3}{3x^3} - \ln(x) - c_2 = 0 \\
\frac{y^3}{3x^3} - \ln(x) - c_2 = 0
\end{aligned}
$$

Substituting initial conditions and solving for c_2 gives $c_2 = 9$. Hence the solution be-
Summary
The solution(s) found are the following
comes

$$
\frac{y^3}{3x^3} - \ln(x) - 9 = 0 \tag{1}
$$

Verification of solutions

$$
\frac{y^3}{3x^3} - \ln(x) - 9 = 0
$$

Verified OK.

5.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + y^3}{x\,y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 257: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^3}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^3}{y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y^3}{3x^3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^3 + y^3}{x\, y^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^3}{x^4} \\
S_y &= \frac{y^2}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

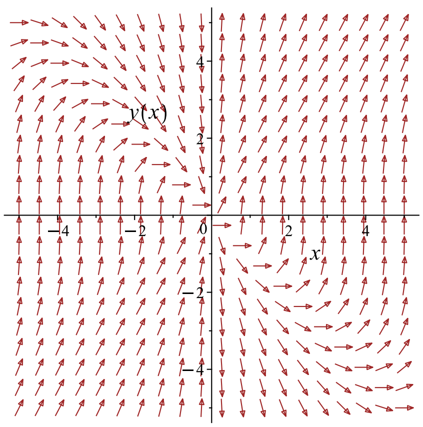
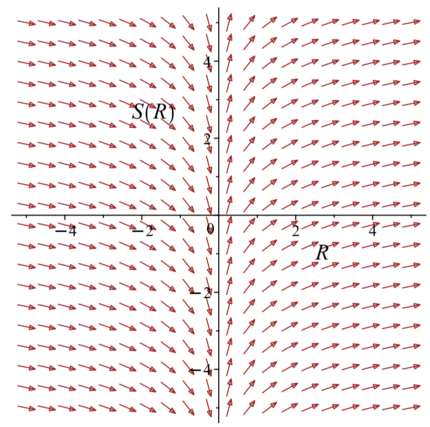
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y^3}{3x^3} = \ln(x) + c_1
$$

Which simplifies to

$$
\frac{y^3}{3x^3} = \ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y^3}{x\,y^2}$ 	$R = x$ $S = \frac{y^3}{3x^3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$9 = c_1$$

$$c_1 = 9$$

Substituting c_1 found above in the general solution gives

$$\frac{y^3}{3x^3} = \ln(x) + 9$$

The above simplifies to

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0$$

Summary

The solution(s) found are the following

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0 \tag{1}$$

Verification of solutions

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0$$

Verified OK.

5.26.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= \frac{x^3 + y^3}{x\,y^2}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x^2\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= \frac{1}{x}\\ f_1(x) &= x^2\\ n &= -2\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{y^3}{x} + x^2 \tag{4}$$

Let

$$\begin{aligned}w &= y^{1-n}\\ &= y^3\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= \frac{w(x)}{x} + x^2 \\
w' &= \frac{3w}{x} + 3x^2 \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= 3x^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{3w(x)}{x} = 3x^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(3x^2\right) \\
\frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(3x^2\right) \\
d\left(\frac{w}{x^3}\right) &= \left(\frac{3}{x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^3} &= \int \frac{3}{x}\, dx \\
\frac{w}{x^3} &= 3\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$
w(x) = 3x^3 \ln(x) + c_1 x^3
$$

which simplifies to

$$w(x) = x^3(3\ln(x) + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^3(3\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$27 = c_1$$

$$c_1 = 27$$

Substituting c_1 found above in the general solution gives

$$y^3 = 3x^3\ln(x) + 27x^3$$

Summary

The solution(s) found are the following

$$y^3 = 3(\ln(x) + 9)\,x^3 \tag{1}$$

Verification of solutions

$$y^3 = 3(\ln(x) + 9)\,x^3$$

Verified OK.

5.26.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\, y^2)\, \mathrm{d}y &= (x^3 + y^3)\, \mathrm{d}x \\ (-x^3 - y^3)\, \mathrm{d}x + (x\, y^2)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^3 - y^3$$
$$N(x, y) = x\, y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3 - y^3) \\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\, y^2) \\ &= y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\,y^2}\left((-3y^2) - (y^2)\right) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(-x^3 - y^3\right) \\ &= \frac{-x^3 - y^3}{x^4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(x\,y^2\right) \\ &= \frac{y^2}{x^3} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^3 - y^3}{x^4}\right) + \left(\frac{y^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^3 - y^3}{x^4} \, \mathrm{d}x$$

$$\phi = -\ln(x) + \frac{y^3}{3x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3}$. Therefore equation (4) becomes

$$\frac{y^2}{x^3} = \frac{y^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y^3}{3x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y^3}{3x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$9 = c_1$$

$$c_1 = 9$$

Substituting c_1 found above in the general solution gives

$$-\ln(x) + \frac{y^3}{3x^3} = 9$$

The above simplifies to

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0$$

Summary

The solution(s) found are the following

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0 \tag{1}$$

Verification of solutions

$$-3x^3 \ln(x) - 27x^3 + y^3 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=(x^3+y(x)^3)/(x*y(x)^2),y(1) = 3],y(x), singsol=all)
```

$$y(x) = (3\ln(x) + 27)^{\frac{1}{3}} x$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 20

```
DSolve[{y'[x]==(x^3+y[x]^3)/(x*y[x]^2),y[1]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{3}x\sqrt[3]{\log(x)+9}$$

5.27 problem 24

Internal problem ID [1001]
Internal file name [`OUTPUT/1002_Sunday_June_05_2022_01_56_30_AM_62428672/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'xy + y^2 = -x^2}$$

With initial conditions

$$[y(1) = 2]$$

5.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2+y^2}{xy}\right)\\ &= -\frac{2}{x} + \frac{x^2+y^2}{x\,y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.27.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2u(x) + u(x)^2\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{2u^2+1}{ux}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+1}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2+1}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(2u^2+1)}{4} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\left(2u^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(2u^2+1\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{9}{c_3^4}\right)}{4}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

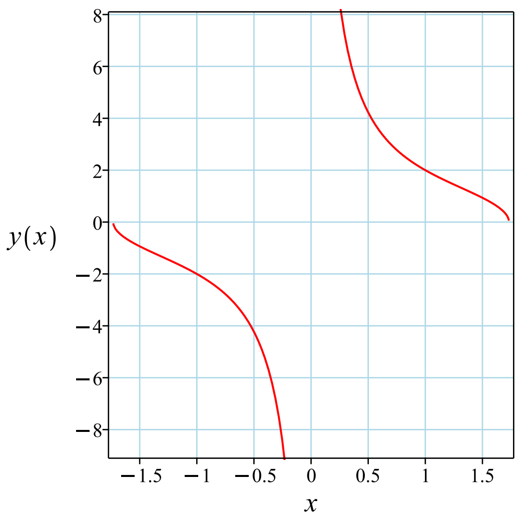
$$9^{\frac{1}{4}} = c_3\sqrt{3}\left(\frac{1}{c_3^4}\right)^{\frac{1}{4}}$$

Since $\lim_{c_1 \to \infty}$ gives $\left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\sqrt{3}\left(\frac{1}{c_3^4}\right)^{\frac{1}{4}}}{x} = \left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{\sqrt{3}}{x}$ and this result satisfies the given initial condition. Solving for y from the above gives

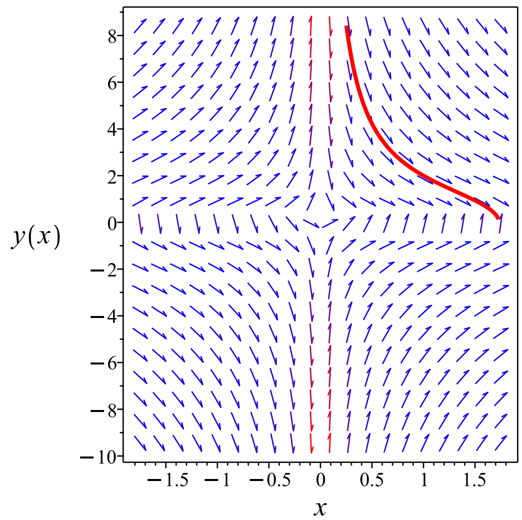
$$y = \frac{\sqrt{-2x^4+18}}{2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4+18}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

Verified OK.

5.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 + y^2}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 259: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2 y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2 y}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x\, y^2 \\ S_y &= y\, x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \tag{4}$$

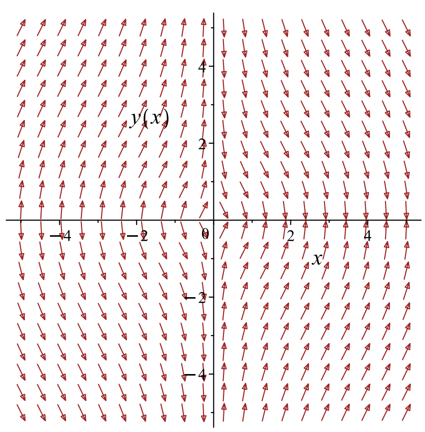
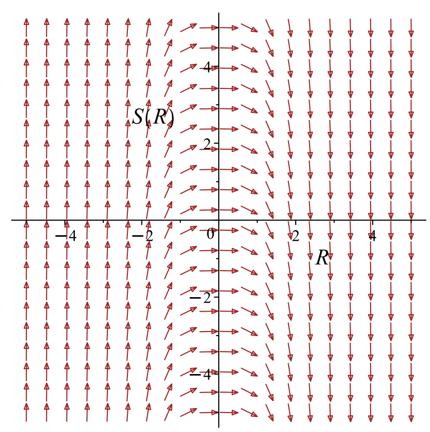
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{x^2y^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{1}{4} + c_1$$

$$c_1 = \frac{9}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2 y^2}{2} = -\frac{x^4}{4} + \frac{9}{4}$$

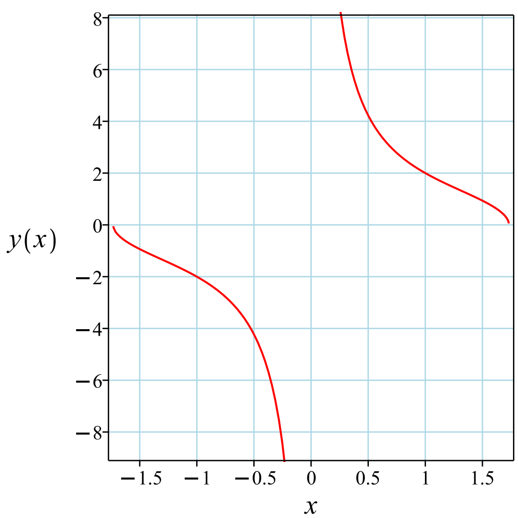
Solving for y from the above gives

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

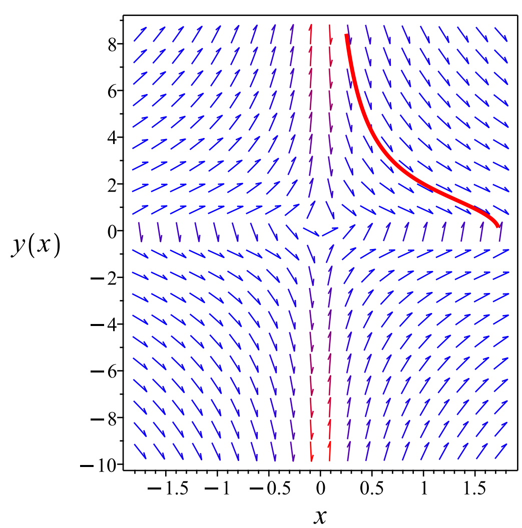
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 18}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

Verified OK.

5.27.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x^2+y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\
w' &= -\frac{2w}{x} - 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x} = -2x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\
\frac{d}{dx}(x^2 w) &= (x^2)(-2x) \\
d(x^2 w) &= (-2x^3)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 w &= \int -2x^3\, dx \\
x^2 w &= -\frac{x^4}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{9}{2}$$

Substituting c_1 found above in the general solution gives

$$y^2 = -\frac{x^4 - 9}{2x^2}$$

The above simplifies to

$$x^4 + 2x^2y^2 - 9 = 0$$

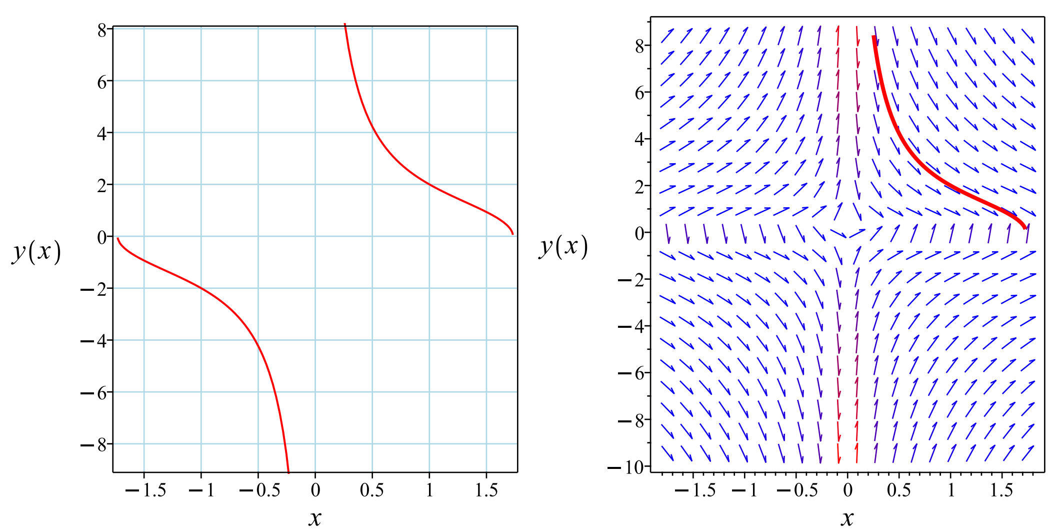
Solving for y from the above gives

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 18}}{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^4+18}}{2x}$$

Verified OK.

5.27.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(yx)\,\mathrm{d}y = \left(-x^2 - y^2\right)\mathrm{d}x$$
$$\left(x^2 + y^2\right)\mathrm{d}x + (yx)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x^2+y^2 \\
N(x,y) &= yx
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2+y^2) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yx}((2y)-(y)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(x^2+y^2\right) \\ &= x\left(x^2+y^2\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(yx) \\ &= y\,x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\left(x^2+y^2\right)\right)+\left(y\,x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int x\left(x^2+y^2\right)\mathrm{d}x\end{aligned}$$

$$\phi = \frac{\left(x^2+y^2\right)^2}{4}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = y\left(x^2+y^2\right)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,x^2$. Therefore equation (4) becomes

$$y\,x^2 = y(x^2 + y^2) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-y^3)\,\mathrm{d}y$$

$$f(y) = -\frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{9}{4} = c_1$$

$$c_1 = \frac{9}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = \frac{9}{4}$$

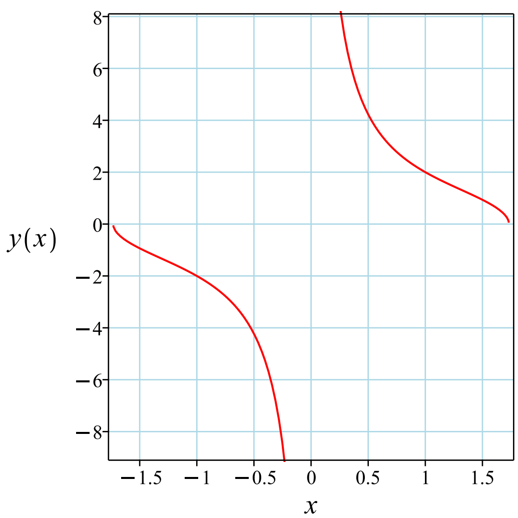
Solving for y from the above gives

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

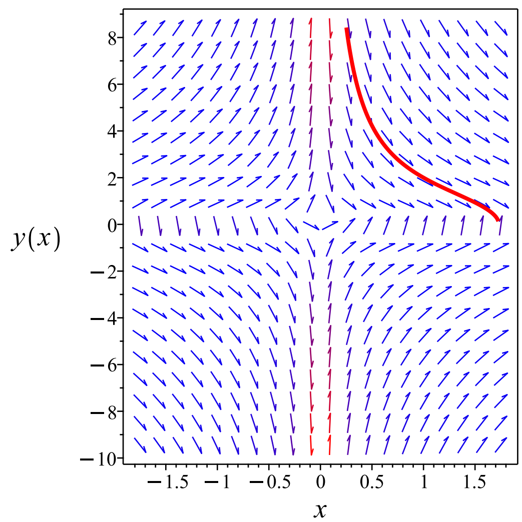
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 18}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^4 + 18}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve([x*y(x)*diff(y(x),x)+x^2+y(x)^2=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-2x^4 + 18}}{2x}$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 25

```
DSolve[{x*y[x]*y'[x]+x^2+y[x]^2==0,y[1]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{9 - x^4}}{\sqrt{2}x}$$

5.28 problem 25

Internal problem ID [1002]
Internal file name [`OUTPUT/1003_Sunday_June_05_2022_01_56_32_AM_34654589/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y' - \frac{y^2 - 3yx - 5x^2}{x^2} = 0}$$

With initial conditions

$$[y(1) = 1]$$

5.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-5x^2 - 3yx + y^2}{x^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-5x^2 - 3yx + y^2}{x^2}\right)\\ &= \frac{-3x + 2y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.28.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)^2\,x^2 - 3u(x)\,x^2 - 5x^2}{x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= \frac{u^2 - 4u - 5}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 - 4u - 5$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2 - 4u - 5}\,du &= \frac{1}{x}\,dx\\ \int \frac{1}{u^2 - 4u - 5}\,du &= \int \frac{1}{x}\,dx\\ \frac{\ln(u-5)}{6} - \frac{\ln(u+1)}{6} &= \ln(x) + c_2\end{aligned}$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{6}\right)(\ln (u-5)-\ln (u+1)) &= \ln (x)+2c_2 \\
\ln (u-5)-\ln (u+1) &= (6)(\ln (x)+2c_2) \\
&= 6\ln (x)+12c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-5)-\ln(u+1)} = e^{6\ln(x)+6c_2}
$$

Which simplifies to

$$
\begin{aligned}
\frac{u-5}{u+1} &= 6c_2x^6 \\
&= c_3x^6
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= -\frac{x(c_3x^6+5)}{c_3x^6-1}
\end{aligned}
$$

Initial conditions are used to solve for c_3. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\frac{c_3+5}{c_3-1}
$$

$$
c_3 = -2
$$

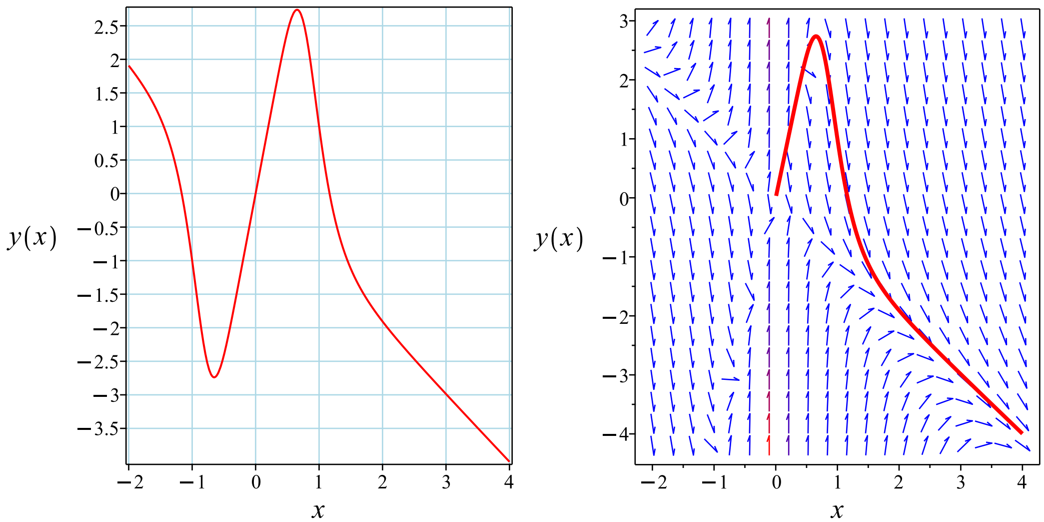
Substituting c_3 found above in the general solution gives

$$
y = -\frac{x(2x^6-5)}{2x^6+1}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x(2x^6-5)}{2x^6+1} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x(2x^6 - 5)}{2x^6 + 1}$$

Verified OK.

5.28.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-5x^2 - 3yx + y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{(-5x^2-3yx+y^2)\,(b_3-a_2)}{x^2}-\frac{(-5x^2-3yx+y^2)^2\,a_3}{x^4}\\
&-\left(\frac{-10x-3y}{x^2}-\frac{2(-5x^2-3yx+y^2)}{x^3}\right)(xa_2+ya_3+a_1)\\
&-\frac{(-3x+2y)\,(xb_2+yb_3+b_1)}{x^2}=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{5x^4a_2-25x^4a_3+4b_2x^4-5x^4b_3-30x^3ya_3-2x^3yb_2+x^2y^2a_2-2x^2y^2a_3-x^2y^2b_3+8x\,y^3a_3-y^4a_3+3x}{x^4}
$$
$=0$

Setting the numerator to zero gives

$$
\begin{aligned}
&5x^4a_2-25x^4a_3+4b_2x^4-5x^4b_3-30x^3ya_3-2x^3yb_2+x^2y^2a_2-2x^2y^2a_3\\
&-x^2y^2b_3+8x\,y^3a_3-y^4a_3+3x^3b_1-3x^2ya_1-2x^2yb_1+2x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&5a_2v_1^4+a_2v_1^2v_2^2-25a_3v_1^4-30a_3v_1^3v_2-2a_3v_1^2v_2^2+8a_3v_1v_2^3-a_3v_2^4+4b_2v_1^4\\
&-2b_2v_1^3v_2-5b_3v_1^4-b_3v_1^2v_2^2-3a_1v_1^2v_2+2a_1v_1v_2^2+3b_1v_1^3-2b_1v_1^2v_2=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(5a_2 - 25a_3 + 4b_2 - 5b_3)\, v_1^4 + (-30a_3 - 2b_2)\, v_1^3 v_2 + 3b_1 v_1^3 \\
&+ (a_2 - 2a_3 - b_3)\, v_1^2 v_2^2 + (-3a_1 - 2b_1)\, v_1^2 v_2 + 8a_3 v_1 v_2^3 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
-a_3 &= 0 \\
8a_3 &= 0 \\
3b_1 &= 0 \\
-3a_1 - 2b_1 &= 0 \\
-30a_3 - 2b_2 &= 0 \\
a_2 - 2a_3 - b_3 &= 0 \\
5a_2 - 25a_3 + 4b_2 - 5b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{-5x^2 - 3yx + y^2}{x^2}\right)(x) \\
&= \frac{5x^2 + 4yx - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{5x^2+4yx-y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y-5x)}{6} + \frac{\ln(x+y)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-5x^2 - 3yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+y)(5x-y)} \\ S_y &= \frac{x}{(x+y)(5x-y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y-5x)}{6} + \frac{\ln(x+y)}{6} = -\ln(x) + c_1$$

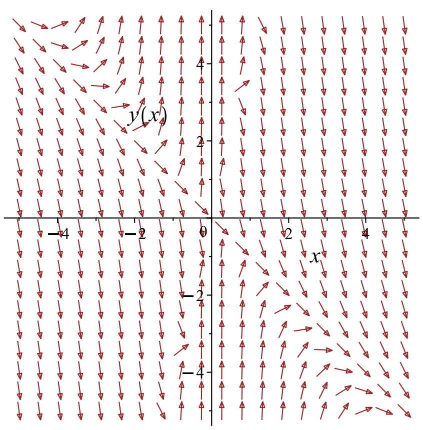
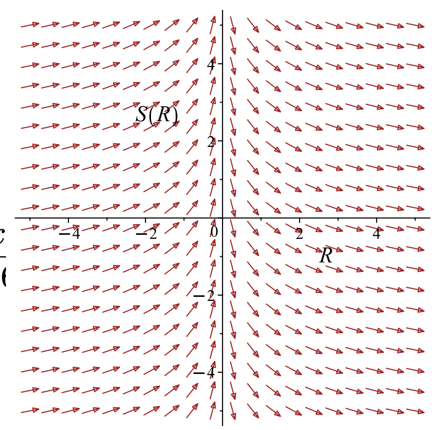
Which simplifies to

$$-\frac{\ln(y-5x)}{6} + \frac{\ln(x+y)}{6} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(5\,e^{6c_1} + x^6)}{e^{6c_1} - x^6}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-5x^2-3yx+y^2}{x^2}$ 	$R = x$ $S = -\frac{\ln (y-5x)}{6} + \frac{\ln (x}{(}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{5\,e^{6c_1} + 1}{e^{6c_1} - 1}$$

$$c_1 = -\frac{\ln (2)}{6} + \frac{i\pi}{6}$$

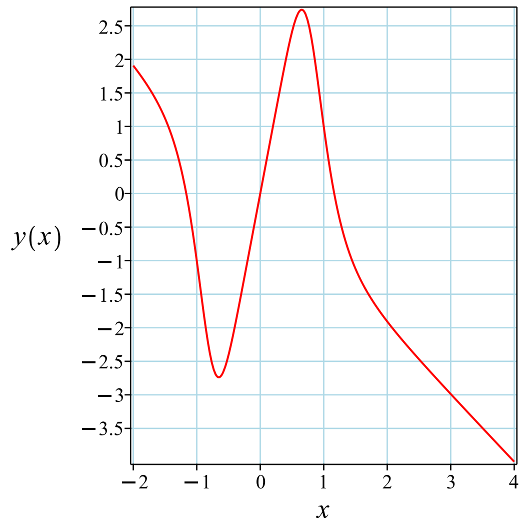
Substituting c_1 found above in the general solution gives

$$y = \frac{-2x^7 + 5x}{2x^6 + 1}$$

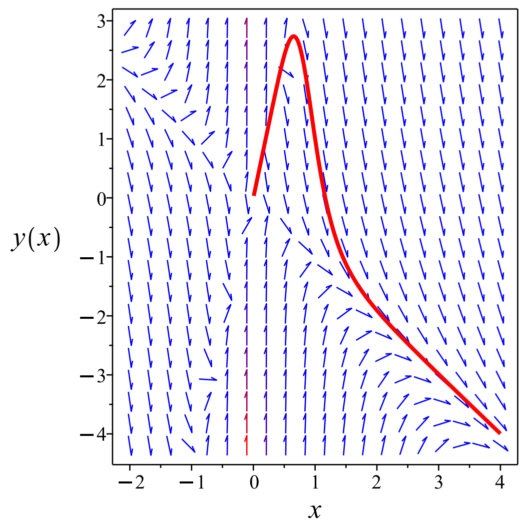
Summary

The solution(s) found are the following

$$y = \frac{-2x^7 + 5x}{2x^6 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2x^7 + 5x}{2x^6 + 1}$$

Verified OK.

5.28.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-5x^2 - 3yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -5 - \frac{3y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -5$, $f_1(x) = -\frac{3}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= -\frac{3}{x^3} \\ f_2^2f_0 &= -\frac{5}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{5u'(x)}{x^3} - \frac{5u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1x^6 + c_2}{x^5}$$

The above shows that

$$u'(x) = \frac{c_1x^6 - 5c_2}{x^6}$$

Using the above in (1) gives the solution

$$y = -\frac{(c_1x^6 - 5c_2)\,x}{c_1x^6 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{(c_3x^6 - 5)\,x}{c_3x^6 + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{c_3 - 5}{c_3 + 1}$$

$$c_3 = 2$$

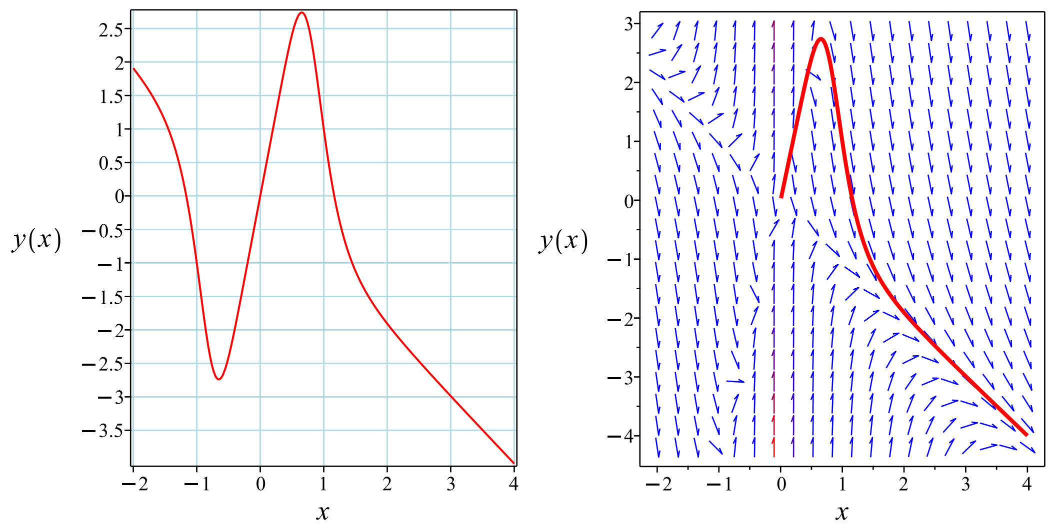
Substituting c_3 found above in the general solution gives

$$y = -\frac{x(2x^6 - 5)}{2x^6 + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(2x^6 - 5)}{2x^6 + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{x(2x^6 - 5)}{2x^6 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)=(y(x)^2-3*x*y(x)-5*x^2)/x^2,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-2x^7 + 5x}{2x^6 + 1}$$

✓ Solution by Mathematica

Time used: 1.947 (sec). Leaf size: 24

```
DSolve[{y'[x]==(y[x]^2-3*x*y[x]-5*x^2)/x^2,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5x - 2x^7}{2x^6 + 1}$$

5.29 problem 26

Internal problem ID [1003]
Internal file name [`OUTPUT/1004_Sunday_June_05_2022_01_56_34_AM_72577261/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - 4yx = 2x^2}$$

With initial conditions

$$[y(1) = 1]$$

5.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2x^2 + 4yx + y^2}{x^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x^2 + 4yx + y^2}{x^2}\right)\\ &= \frac{4x + 2y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.29.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 - 4u(x)\,x^2 = 2x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= \frac{u^2 + 3u + 2}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 + 3u + 2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2 + 3u + 2}\,du &= \frac{1}{x}\,dx\\ \int \frac{1}{u^2 + 3u + 2}\,du &= \int \frac{1}{x}\,dx\\ \ln(u + 1) - \ln(u + 2) &= \ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u+1)-\ln(u+2)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u+1}{u+2} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= -\frac{x(2c_3x-1)}{c_3x-1} \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2c_3-1}{c_3-1}$$

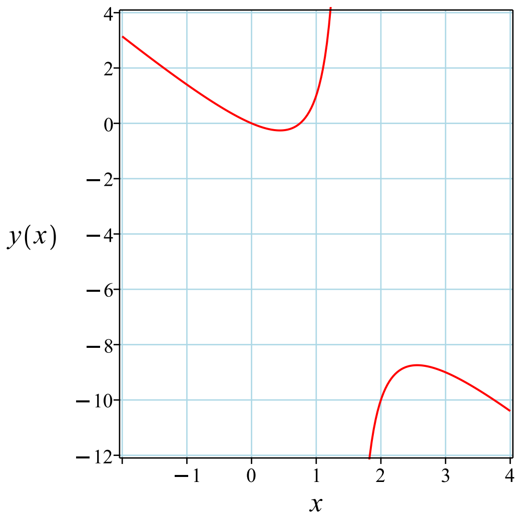
$$c_3 = \frac{2}{3}$$

Substituting c_3 found above in the general solution gives

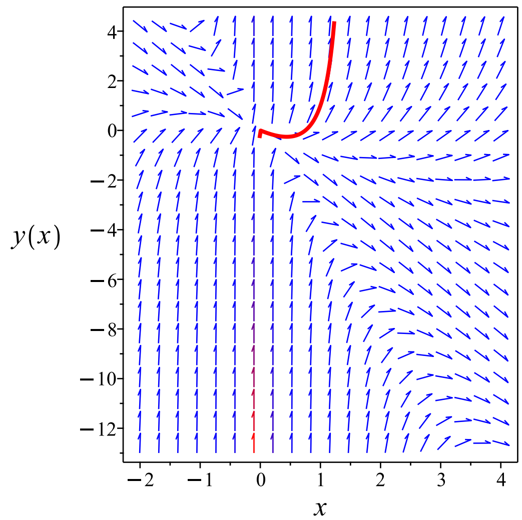
$$y = -\frac{x(4x-3)}{-3+2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x(4x-3)}{-3+2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{x(4x-3)}{-3+2x}$$

Verified OK.

5.29.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2x^2+4yx+y^2}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{(2x^2+4yx+y^2)(b_3-a_2)}{x^2}-\frac{(2x^2+4yx+y^2)^2 a_3}{x^4} \\
& -\left(\frac{4x+4y}{x^2}-\frac{2(2x^2+4yx+y^2)}{x^3}\right)(xa_2+ya_3+a_1) \\
& -\frac{(4x+2y)(xb_2+yb_3+b_1)}{x^2}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^4a_2+4x^4a_3+3b_2x^4-2x^4b_3+16x^3ya_3+2x^3yb_2-x^2y^2a_2+16x^2y^2a_3+x^2y^2b_3+6x\,y^3a_3+y^4a_3+4}{x^4}
$$
$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -2x^4a_2-4x^4a_3-3b_2x^4+2x^4b_3-16x^3ya_3-2x^3yb_2+x^2y^2a_2-16x^2y^2a_3 \\
& -x^2y^2b_3-6x\,y^3a_3-y^4a_3-4x^3b_1+4x^2ya_1-2x^2yb_1+2x\,y^2a_1=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -2a_2v_1^4+a_2v_1^2v_2^2-4a_3v_1^4-16a_3v_1^3v_2-16a_3v_1^2v_2^2-6a_3v_1v_2^3-a_3v_2^4-3b_2v_1^4 \\
& -2b_2v_1^3v_2+2b_3v_1^4-b_3v_1^2v_2^2+4a_1v_1^2v_2+2a_1v_1v_2^2-4b_1v_1^3-2b_1v_1^2v_2=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}&(-2a_2 - 4a_3 - 3b_2 + 2b_3)\, v_1^4 + (-16a_3 - 2b_2)\, v_1^3 v_2 - 4b_1 v_1^3 \\ &+ (a_2 - 16a_3 - b_3)\, v_1^2 v_2^2 + (4a_1 - 2b_1)\, v_1^2 v_2 - 6a_3 v_1 v_2^3 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0\\ -6a_3 &= 0\\ -a_3 &= 0\\ -4b_1 &= 0\\ 4a_1 - 2b_1 &= 0\\ -16a_3 - 2b_2 &= 0\\ a_2 - 16a_3 - b_3 &= 0\\ -2a_2 - 4a_3 - 3b_2 + 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= y - \left(\frac{2x^2 + 4yx + y^2}{x^2}\right)(x)\\ &= \frac{-2x^2 - 3yx - y^2}{x}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2-3yx-y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\ln (x + y) + \ln (2x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x^2 + 4yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{(x + y)(2x + y)} \\ S_y &= -\frac{x}{(x + y)(2x + y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x+y) + \ln(2x+y) = -\ln(x) + c_1$$

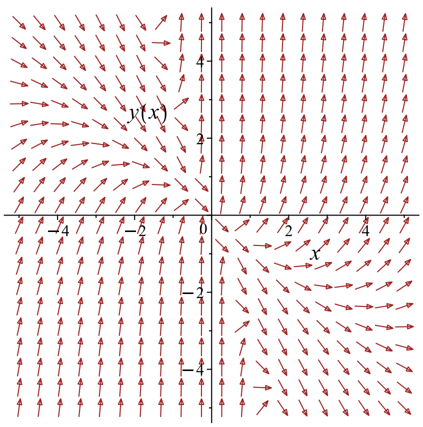
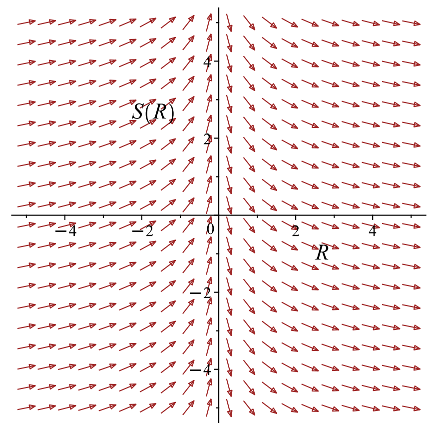
Which simplifies to

$$-\ln(x+y) + \ln(2x+y) = -\ln(x) + c_1$$

Which gives

$$y = -\frac{x(\mathrm{e}^{c_1} - 2x)}{\mathrm{e}^{c_1} - x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^2+4yx+y^2}{x^2}$ 	$R = x$ $S = -\ln(x+y) + \ln(2x$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{-e^{c_1} + 2}{-1 + e^{c_1}}$$

$$c_1 = \ln\left(\frac{3}{2}\right)$$

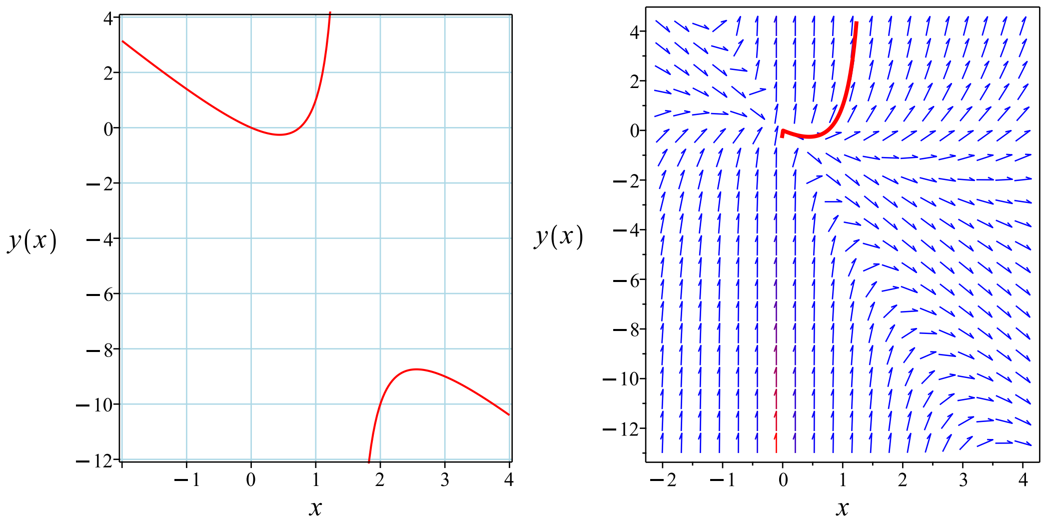
Substituting c_1 found above in the general solution gives

$$y = \frac{-4x^2 + 3x}{-3 + 2x}$$

Summary

The solution(s) found are the following

$$y = \frac{-4x^2 + 3x}{-3 + 2x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-4x^2 + 3x}{-3 + 2x}$$

Verified OK.

5.29.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2x^2 + 4yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2 + \frac{4y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2$, $f_1(x) = \frac{4}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= \frac{4}{x^3} \\ f_2^2f_0 &= \frac{2}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} - \frac{2u'(x)}{x^3} + \frac{2u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(c_1x + c_2)$$

The above shows that

$$u'(x) = 2c_1x + c_2$$

Using the above in (1) gives the solution

$$y = -\frac{(2c_1x + c_2)\,x}{c_1x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2c_3x^2 - x}{c_3x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2c_3 + 1}{c_3 + 1}$$

$$c_3 = -\frac{2}{3}$$

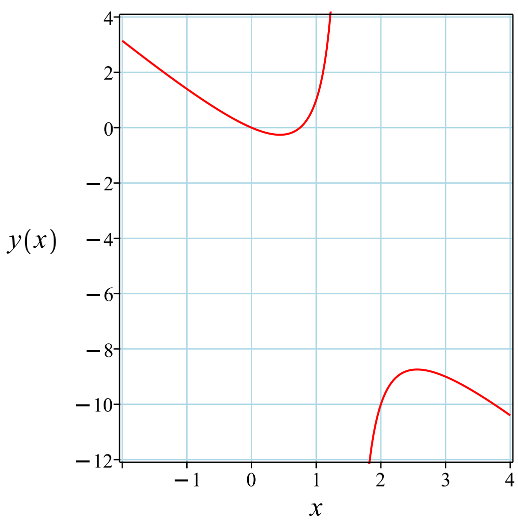
Substituting c_3 found above in the general solution gives

$$y = -\frac{x(4x-3)}{-3+2x}$$

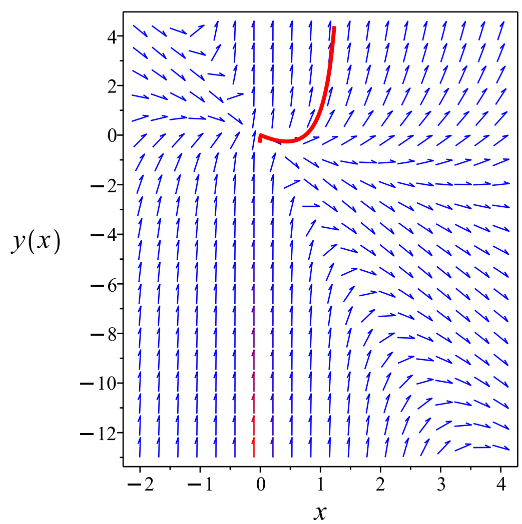
Summary

The solution(s) found are the following

$$y = -\frac{x(4x-3)}{-3+2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{x(4x-3)}{-3+2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve([x^2*diff(y(x),x)=2*x^2+y(x)^2+4*x*y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-4x^2 + 3x}{2x - 3}$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 20

```
DSolve[{x^2*y'[x]==2*x^2+y[x]^2+4*x*y[x],y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x(4x - 3)}{2x - 3}$$

5.30 problem 27

Internal problem ID [1004]
Internal file name [`OUTPUT/1005_Sunday_June_05_2022_01_56_35_AM_15507582/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'xy - 4y^2 = 3x^2}$$

With initial conditions

$$\left[y(1) = \sqrt{3}\right]$$

5.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{3x^2 + 4y^2}{xy} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \sqrt{3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = \sqrt{3}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x^2 + 4y^2}{xy}\right)\\ &= \frac{8}{x} - \frac{3x^2 + 4y^2}{x\,y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \sqrt{3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = \sqrt{3}$ is inside this domain. Therefore solution exists and is unique.

5.30.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 u(x) - 4u(x)^2\,x^2 = 3x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= \frac{3u^2 + 3}{ux}\end{aligned}$$

Where $f(x) = \frac{3}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2+1}{u}}\,du &= \frac{3}{x}\,dx\\ \int \frac{1}{\frac{u^2+1}{u}}\,du &= \int \frac{3}{x}\,dx\\ \frac{\ln(u^2 + 1)}{2} &= 3\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{3\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = c_3 x^3$$

Which simplifies to

$$\sqrt{u(x)^2+1} = c_3 \mathrm{e}^{c_2} x^3$$

The solution is

$$\sqrt{u(x)^2+1} = c_3 \mathrm{e}^{c_2} x^3$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = c_3 \mathrm{e}^{c_2} x^3$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3 \mathrm{e}^{c_2} x^3$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{4}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 2c_3\sqrt{\frac{1}{c_3^2}}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

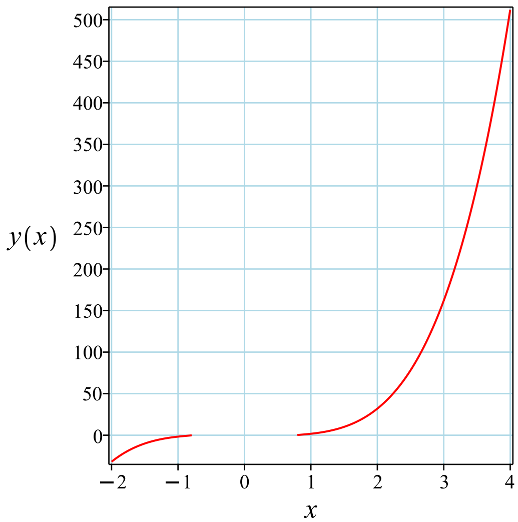
Solving for y from the above gives

$$y = \sqrt{4x^6-1}\, x$$

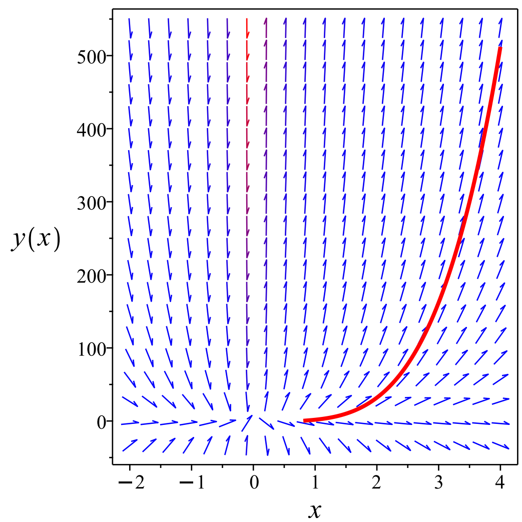
Summary

The solution(s) found are the following

$$y = \sqrt{4x^6-1}\, x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{4x^6 - 1}\, x$$

Verified OK. {positive}

5.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^2 + 4y^2}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 261: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{x^8}{y} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^8}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^8}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3x^2 + 4y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4y^2}{x^9} \\ S_y &= \frac{y}{x^8} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{x^7} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R^7}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^6} + c_1 \tag{4}$$

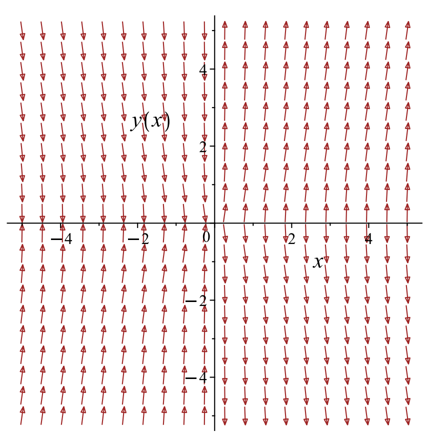
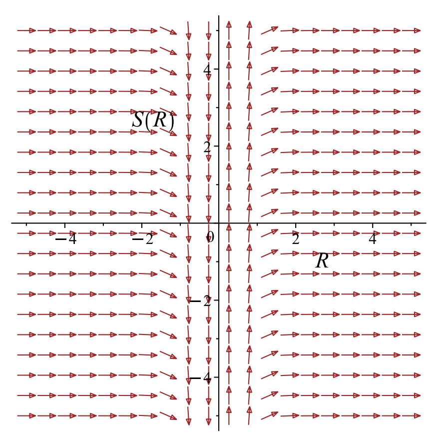
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^8} = -\frac{1}{2x^6} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^8} = -\frac{1}{2x^6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2+4y^2}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^8}$	$\frac{dS}{dR} = \frac{3}{R^7}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = -\frac{1}{2} + c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x^8} = \frac{4x^6 - 1}{2x^6}$$

The above simplifies to

$$-4x^8 + x^2 + y^2 = 0$$

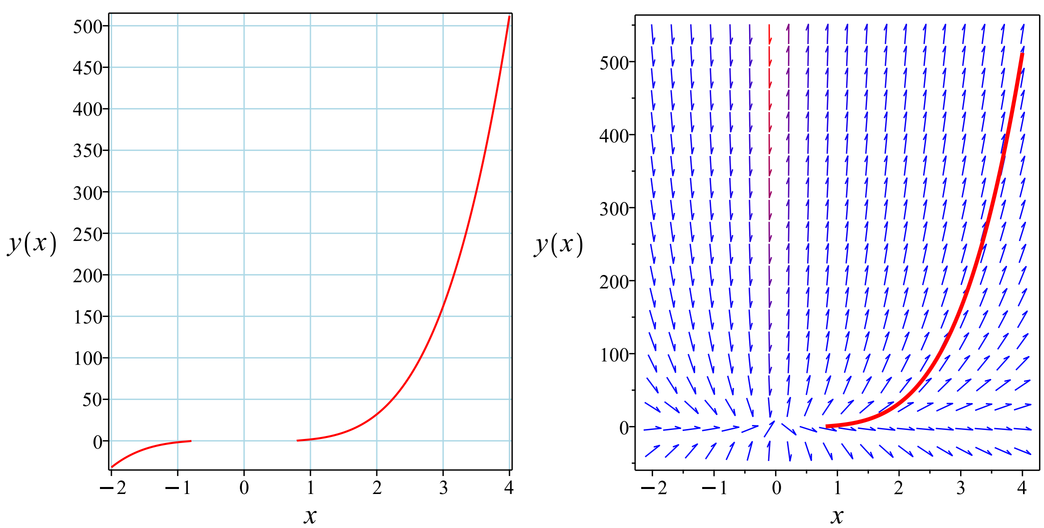
Solving for y from the above gives

$$y = \sqrt{4x^6 - 1}\, x$$

Summary

The solution(s) found are the following

$$y = \sqrt{4x^6 - 1}\, x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{4x^6 - 1}\, x$$

Verified OK. `{positive}`

5.30.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{3x^2+4y^2}{xy}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \frac{4}{x}y + 3x\frac{1}{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{4}{x}\\
f_1(x) &= 3x\\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{4y^2}{x} + 3x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n}\\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{4w(x)}{x} + 3x \\
w' &= \frac{8w}{x} + 6x
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{8}{x} \\
q(x) &= 6x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{8w(x)}{x} = 6x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{8}{x}dx} \\
&= \frac{1}{x^8}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(6x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^8}\right) &= \left(\frac{1}{x^8}\right)(6x) \\
\mathrm{d}\left(\frac{w}{x^8}\right) &= \left(\frac{6}{x^7}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^8} &= \int \frac{6}{x^7}\,\mathrm{d}x \\
\frac{w}{x^8} &= -\frac{1}{x^6} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^8}$ results in

$$
w(x) = c_1 x^8 - x^2
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1x^8 - x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1 - 1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$y^2 = 4x^8 - x^2$$

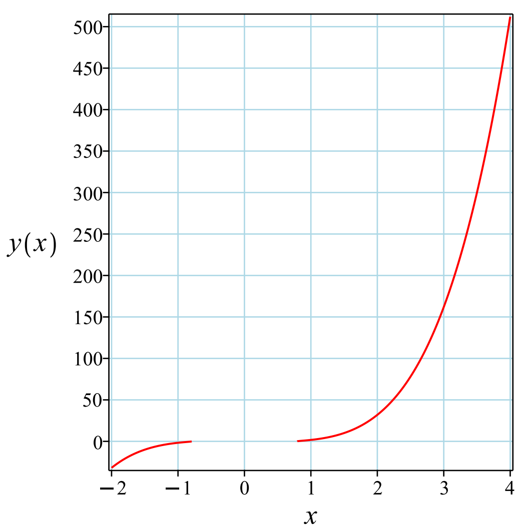
Solving for y from the above gives

$$y = \sqrt{4x^6 - 1}\, x$$

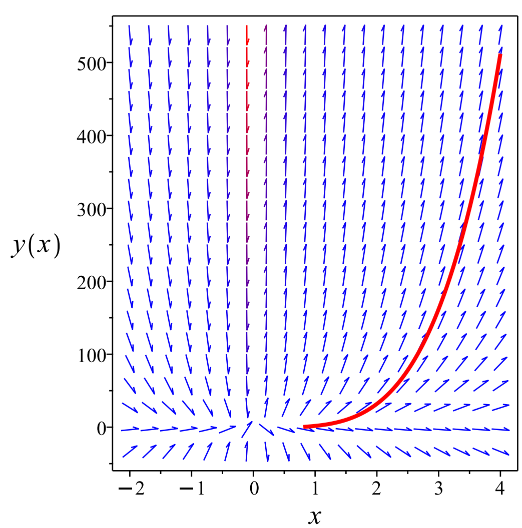
Summary

The solution(s) found are the following

$$y = \sqrt{4x^6 - 1}\, x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{4x^6 - 1}\, x$$

Verified OK. `{positive}`

5.30.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(yx)\,\mathrm{d}y&=\left(3x^2+4y^2\right)\mathrm{d}x\\ \left(-3x^2-4y^2\right)\mathrm{d}x+(yx)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3x^2-4y^2\\ N(x,y)&=yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2 - 4y^2\right)\\ &= -8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{yx}((-8y) - (y))\\ &= -\frac{9}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{9}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-9\ln(x)}\\ &= \frac{1}{x^9}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^9}\left(-3x^2 - 4y^2\right)\\ &= \frac{-3x^2 - 4y^2}{x^9}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^9}(yx) \\
&= \frac{y}{x^8}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-3x^2 - 4y^2}{x^9}\right) + \left(\frac{y}{x^8}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-3x^2 - 4y^2}{x^9}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2 + y^2}{2x^8} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{y}{x^8} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^8}$. Therefore equation (4) becomes

$$
\frac{y}{x^8} = \frac{y}{x^8} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 + y^2}{2x^8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 + y^2}{2x^8}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2 + y^2}{2x^8} = 2$$

The above simplifies to

$$-4x^8 + x^2 + y^2 = 0$$

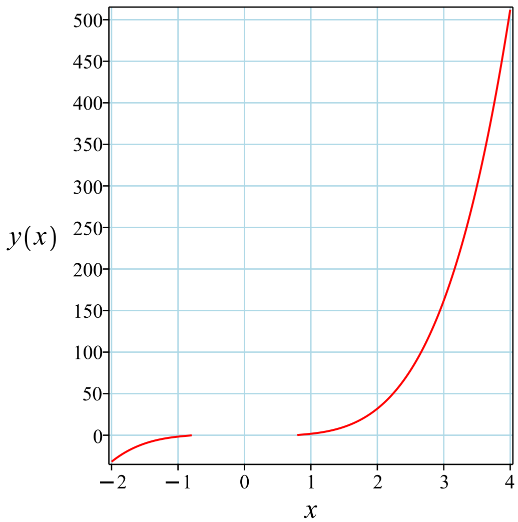
Solving for y from the above gives

$$y = \sqrt{4x^6 - 1}\, x$$

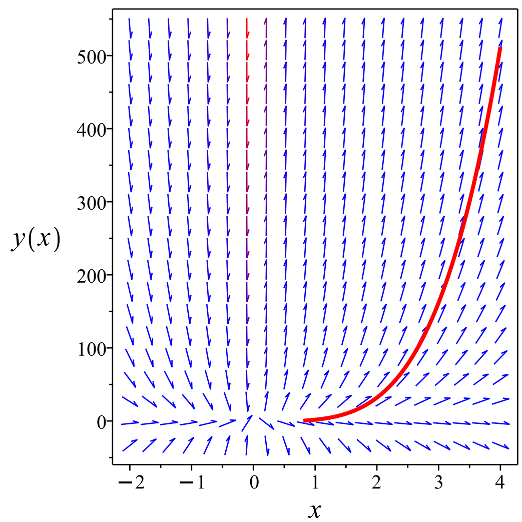
<u>Summary</u>

The solution(s) found are the following

$$y = \sqrt{4x^6 - 1}\, x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{4x^6 - 1}\, x$$

Verified OK. `{positive}`

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve([x*y(x)*diff(y(x),x)=3*x^2+4*y(x)^2,y(1) = sqrt(3)],y(x), singsol=all)
```

$$y(x) = \sqrt{4x^6 - 1}\, x$$

✓ Solution by Mathematica

Time used: 0.603 (sec). Leaf size: 18

```
DSolve[{x*y[x]*y'[x]==3*x^2+4*y[x]^2,y[1]==Sqrt[3]},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sqrt{4x^6 - 1}$$

5.31 problem 28

Internal problem ID [1005]
Internal file name [`OUTPUT/1006_Sunday_June_05_2022_01_56_37_AM_88901093/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x+y}{x-y} = 0}$$

5.31.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x + u(x)\,x}{x - u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

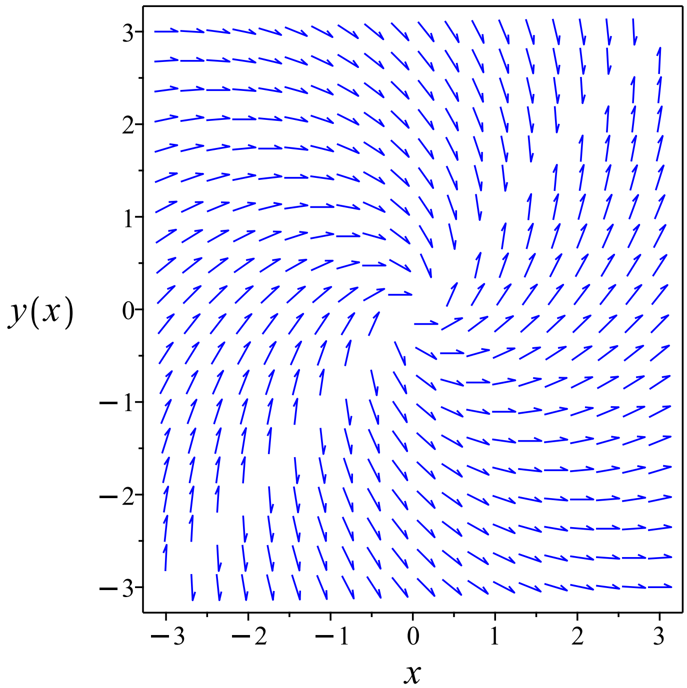


Figure 304: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

5.31.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y}{-x+y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x+y)(b_3 - a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\ & - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 + 2x b_1 - 2ya}{(x-y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 \\ & - 2xy b_3 + y^2 a_2 + y^2 a_3 + y^2 b_2 - y^2 b_3 - 2x b_1 + 2y a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2 v_1^2 + 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 + a_3 v_2^2 - b_2 v_1^2 \\ & - 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_1^2 - 2b_3 v_1 v_2 - b_3 v_2^2 + 2a_1 v_2 - 2b_1 v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-2b_1 &= 0 \\
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{x+y}{-x+y}\right)(x) \\
&= \frac{-x^2 - y^2}{x - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

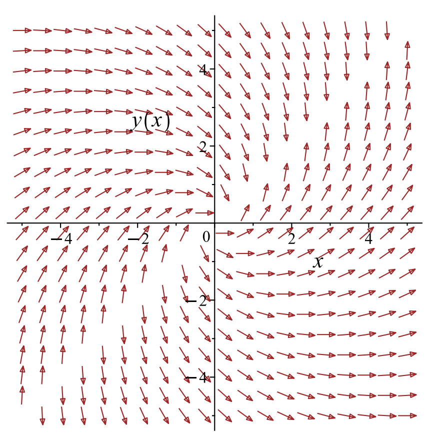
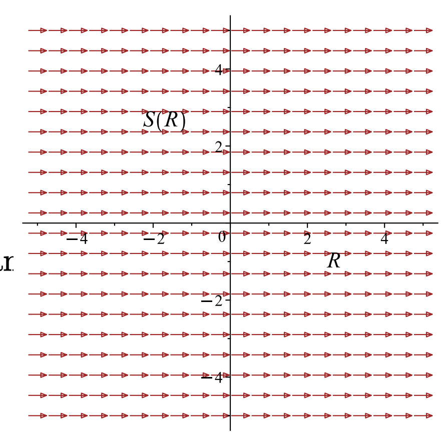
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \text{arctar}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1 \tag{1}$$

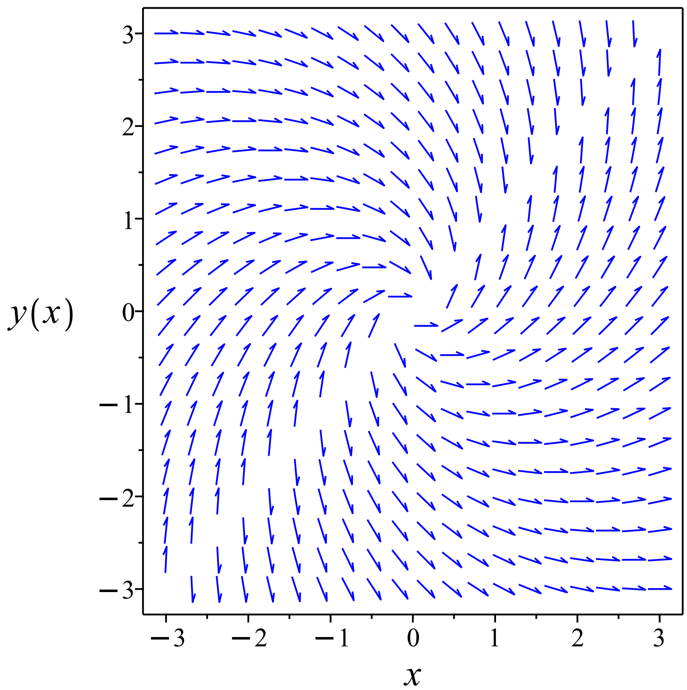


Figure 305: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Verified OK.

5.31.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x + y)\,\mathrm{d}y &= (-x - y)\,\mathrm{d}x \\ (x + y)\,\mathrm{d}x + (-x + y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + y \\ N(x, y) &= -x + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x+y}{x^2+y^2}\\ N &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x\\ \left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{x+y}{x^2+y^2}\\ N(x,y) &= \frac{-x+y}{x^2+y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{x+y}{x^2+y^2} \, \mathrm{d}x \\
\phi &= \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{-x+y}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)
$$

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

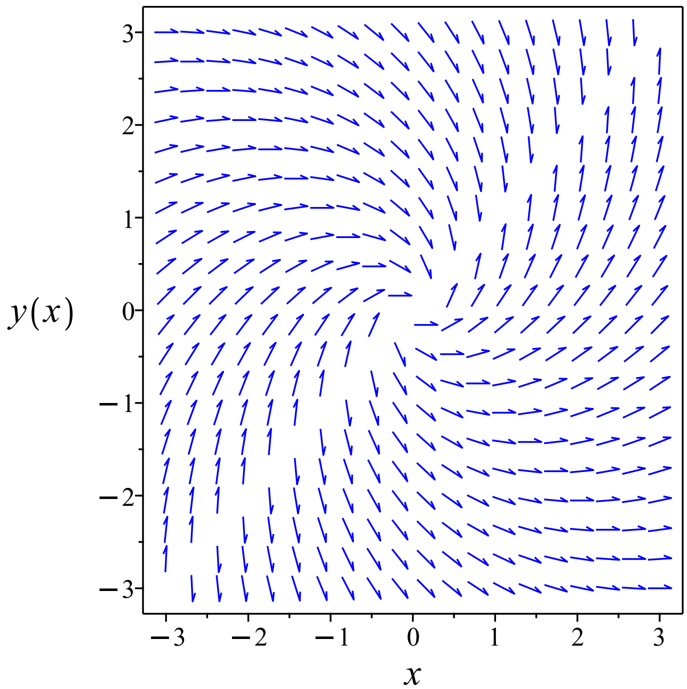


Figure 306: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=(x+y(x))/(x-y(x)),y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 36

```
DSolve[y'[x]==(x+y[x])/(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right)-\arctan\left(\frac{y(x)}{x}\right)=-\log(x)+c_1,y(x)\right]$$

5.32 problem 29

Internal problem ID [1006]
Internal file name [`OUTPUT/1007_Sunday_June_05_2022_01_56_38_AM_17805956/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`]]
```

$$\boxed{(-y + y'x)(\ln(y) - \ln(x)) = x}$$

5.32.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))\,x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(\ln(y)\,y - \ln(x)\,y + x)\,(b_3 - a_2)}{(\ln(y) - \ln(x))\,x} - \frac{(\ln(y)\,y - \ln(x)\,y + x)^2\,a_3}{(\ln(y) - \ln(x))^2\,x^2} - \left(\frac{-\frac{y}{x} + 1}{(\ln(y) - \ln(x))\,x} + \frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))^2\,x^2} - \frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))\,x^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{1 + \ln(y) - \ln(x)}{(\ln(y) - \ln(x))\,x} - \frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))^2\,xy}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(x)^2\,xyb_1 - \ln(x)^2\,y^2a_1 - 2\ln(x)\ln(y)\,xyb_1 + 2\ln(x)\ln(y)\,y^2a_1 - \ln(x)\,x^2ya_2 + \ln(x)\,x^2yb_3 - 2\ln(}{}$$

$$= 0$$

Setting the numerator to zero gives

$$-\ln(x)^2\,xyb_1 + \ln(x)^2\,y^2a_1 + 2\ln(x)\ln(y)\,xyb_1 - 2\ln(x)\ln(y)\,y^2a_1 + \ln(x)\,x^2ya_2 - \ln(x)\,x^2yb_3 + 2\ln(x)\,x\,y^2a_3 - \ln(y)^2\,xyb_1 + \ln(y)^2\,y^2a_1 - \ln(y)\,x^2ya_2 + \ln(y)\,x^2yb_3 - 2\ln(y)\,x\,y^2a_3 + x^3b_2 - x^2ya_2 - x^2ya_3 + x^2yb_3 - x\,y^2a_3 + x^2b_1 - xya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x), \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3, \ln(y) = v_4\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_3^2 v_2^2 a_1 - 2 v_3 v_4 v_2^2 a_1 + v_4^2 v_2^2 a_1 + v_3 v_1^2 v_2 a_2 - v_4 v_1^2 v_2 a_2 + 2 v_3 v_1 v_2^2 a_3 \\
&\quad - 2 v_4 v_1 v_2^2 a_3 - v_3^2 v_1 v_2 b_1 + 2 v_3 v_4 v_1 v_2 b_1 - v_4^2 v_1 v_2 b_1 - v_3 v_1^2 v_2 b_3 + v_4 v_1^2 v_2 b_3 \\
&\quad - v_1^2 v_2 a_2 - v_1^2 v_2 a_3 - v_1 v_2^2 a_3 + v_1^3 b_2 + v_1^2 v_2 b_3 - v_1 v_2 a_1 + v_1^2 b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&v_1^3 b_2 + (-b_3 + a_2) v_1^2 v_2 v_3 + (b_3 - a_2) v_1^2 v_2 v_4 + (-a_2 - a_3 + b_3) v_1^2 v_2 \\
&\quad + v_1^2 b_1 + 2 v_3 v_1 v_2^2 a_3 - 2 v_4 v_1 v_2^2 a_3 - v_1 v_2^2 a_3 - v_3^2 v_1 v_2 b_1 + 2 v_3 v_4 v_1 v_2 b_1 \\
&\quad - v_4^2 v_1 v_2 b_1 - v_1 v_2 a_1 + v_3^2 v_2^2 a_1 - 2 v_3 v_4 v_2^2 a_1 + v_4^2 v_2^2 a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-b_1 &= 0 \\
2b_1 &= 0 \\
-b_3 + a_2 &= 0 \\
b_3 - a_2 &= 0 \\
-a_2 - a_3 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))\,x}\right)(x) \\
&= \frac{x}{-\ln(y) + \ln(x)} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{-\ln(y)+\ln(x)}} dy
\end{aligned}$$

Which results in

$$S = \frac{-\ln(y)\,y + y + \ln(x)\,y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\ln(y)\,y - \ln(x)\,y + x}{(\ln(y) - \ln(x))\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y(\ln(y) - \ln(x))}{x^2} \\
S_y &= \frac{-\ln(y) + \ln(x)}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y(\ln(x) - \ln(y) + 1)}{x} = -\ln(x) + c_1$$

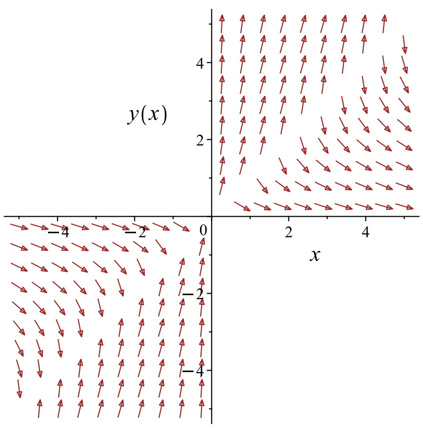
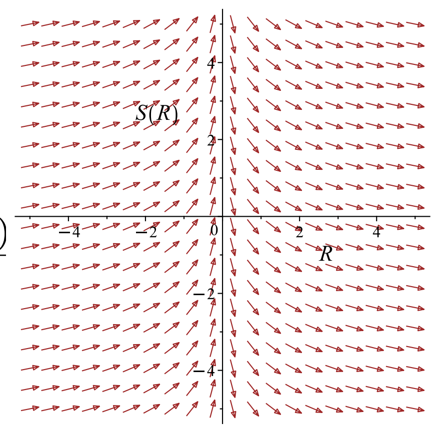
Which simplifies to

$$\frac{y(\ln(x) - \ln(y) + 1)}{x} = -\ln(x) + c_1$$

Which gives

$$y = e^{\operatorname{LambertW}((\ln(x) - c_1)e^{-1}) + 1} x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\ln(y)y - \ln(x)y + x}{(\ln(y) - \ln(x))x}$ 	$R = x$ $S = \frac{y(\ln(x) - \ln(y) + 1)}{x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{\operatorname{LambertW}((\ln(x) - c_1)e^{-1}) + 1} x \tag{1}$$

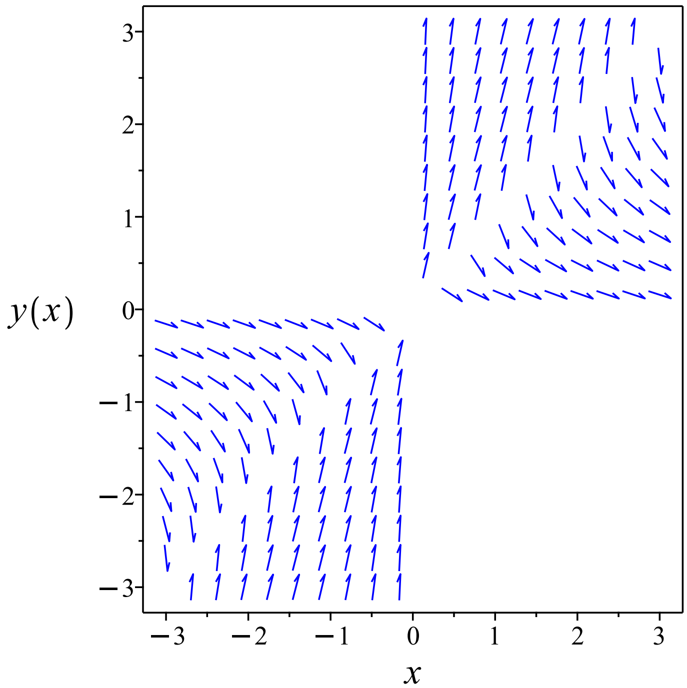


Figure 307: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left((\ln(x)-c_1)e^{-1}\right)+1}x$$

Verified OK.

5.32.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x(\ln (y) - \ln (x)))\, \mathrm{d}y = (x + (\ln (y) - \ln (x))\, y)\, \mathrm{d}x$$
$$(-x - (\ln (y) - \ln (x))\, y)\, \mathrm{d}x + (x(\ln (y) - \ln (x)))\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x - (\ln (y) - \ln (x))\, y$$
$$N(x, y) = x(\ln (y) - \ln (x))$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x - (\ln (y) - \ln (x))\, y)$$
$$= -1 - \ln (y) + \ln (x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x(\ln (y) - \ln (x)))$$
$$= \ln (y) - \ln (x) - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{(\ln(y) - \ln(x))\,x}((-1 - \ln(y) + \ln(x)) - (\ln(y) - \ln(x) - 1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(-x - (\ln(y) - \ln(x))\,y) \\
&= \frac{-x - (\ln(y) - \ln(x))\,y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x(\ln(y) - \ln(x))) \\
&= \frac{\ln(y) - \ln(x)}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x - (\ln(y) - \ln(x))\,y}{x^2}\right) + \left(\frac{\ln(y) - \ln(x)}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x - (\ln(y) - \ln(x))\, y}{x^2} \, \mathrm{d}x$$

$$\phi = \frac{(-x - y)\ln(x) + y(\ln(y) - 1)}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\ln(y) - \ln(x)}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\ln(y) - \ln(x)}{x}$. Therefore equation (4) becomes

$$\frac{\ln(y) - \ln(x)}{x} = \frac{\ln(y) - \ln(x)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-x - y)\ln(x) + y(\ln(y) - 1)}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-x - y)\ln(x) + y(\ln(y) - 1)}{x}$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left((\ln(x)+c_1)\mathrm{e}^{-1}\right)+1} x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left((\ln(x)+c_1)\mathrm{e}^{-1}\right)+1} x \tag{1}$$

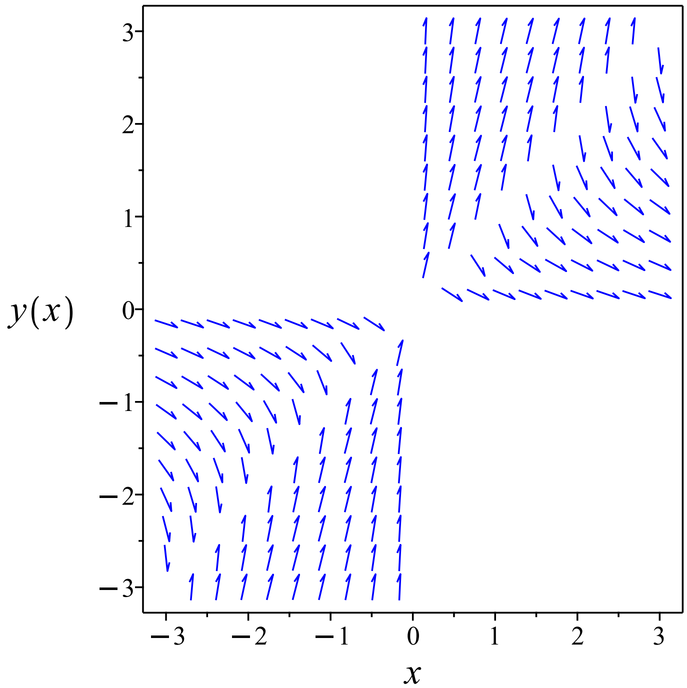


Figure 308: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left((\ln(x)+c_1)\mathrm{e}^{-1}\right)+1} x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve((diff(y(x),x)*x-y(x))*(ln(y(x))-ln(x))=x,y(x), singsol=all)
```

$$y(x) = \frac{x \ln \left(\frac{x}{c_1}\right)}{\text{LambertW}\left(\ln \left(\frac{x}{c_1}\right) \mathrm{e}^{-1}\right)}$$

✓ Solution by Mathematica

Time used: 60.147 (sec). Leaf size: 24

```
DSolve[(y'[x]*x-y[x])*(Log[y[x]]-Log[x])==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) + c_1)}{W\left(\frac{\log(x)+c_1}{e}\right)}$$

5.33 problem 30

Internal problem ID [1007]
Internal file name [`OUTPUT/1008_Sunday_June_05_2022_01_56_41_AM_39474354/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{y^3 + 2xy^2 + x^2y + x^3}{x\left(x+y\right)^2} = 0}$$

5.33.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)^3\,x^3 + 2x^3u(x)^2 + x^3u(x) + x^3}{x\left(x + u\left(x\right)x\right)^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{x\left(u+1\right)^2} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{(u+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{(u+1)^2}}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{1}{(u+1)^2}}\, du = \int \frac{1}{x}\, dx$$

$$\frac{(u+1)^3}{3} = \ln(x) + c_2$$

The solution is

$$\frac{(u(x)+1)^3}{3} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(1+\frac{y}{x}\right)^3}{3} - \ln(x) - c_2 = 0$$

$$\frac{(x+y)^3}{3x^3} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{(x+y)^3}{3x^3} - \ln(x) - c_2 = 0 \tag{1}$$

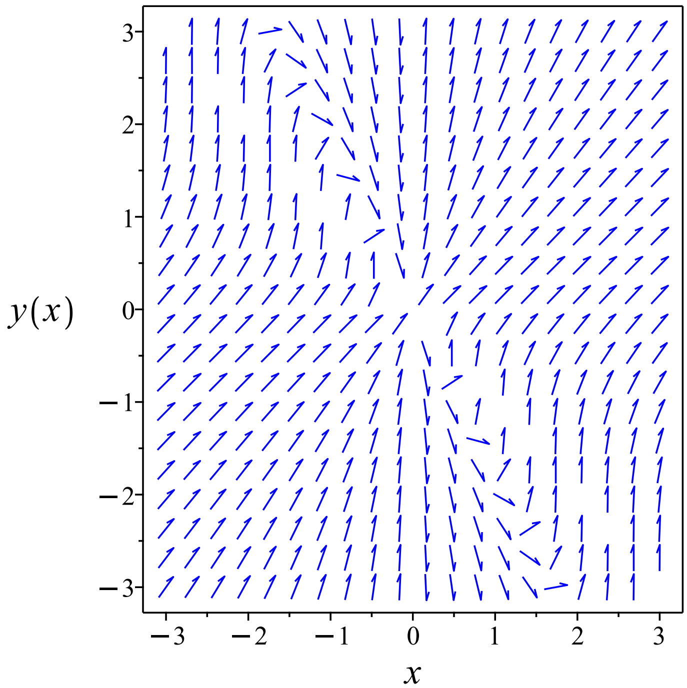


Figure 309: Slope field plot

Verification of solutions

$$\frac{(x+y)^3}{3x^3} - \ln(x) - c_2 = 0$$

Verified OK.

5.33.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^3 + y\,x^2 + 2x\,y^2 + y^3}{x\,(x+y)^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(x^3 + y\,x^2 + 2x\,y^2 + y^3)\,(b_3 - a_2)}{x\,(x+y)^2} - \frac{(x^3 + y\,x^2 + 2x\,y^2 + y^3)^2\,a_3}{x^2\,(x+y)^4} \\
&- \left(\frac{3x^2 + 2yx + 2y^2}{x\,(x+y)^2} - \frac{x^3 + y\,x^2 + 2x\,y^2 + y^3}{x^2\,(x+y)^2}\right. \\
&\left.- \frac{2(x^3 + y\,x^2 + 2x\,y^2 + y^3)}{x\,(x+y)^3}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{x^2 + 4yx + 3y^2}{x\,(x+y)^2} - \frac{2(x^3 + y\,x^2 + 2x\,y^2 + y^3)}{x\,(x+y)^3}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^6a_2 + x^6a_3 - 2x^6b_2 - x^6b_3 + 4x^5ya_2 + 2x^5ya_3 - 2x^5yb_2 - 4x^5yb_3 + 3x^4y^2a_2 + 6x^4y^2a_3 - 3x^4y^2b_3 + 4x}{x^2\,(x+}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^6a_2 - x^6a_3 + 2x^6b_2 + x^6b_3 - 4x^5ya_2 - 2x^5ya_3 + 2x^5yb_2 + 4x^5yb_3 \\
&- 3x^4y^2a_2 - 6x^4y^2a_3 + 3x^4y^2b_3 - 4x^3y^3a_3 + x^5b_1 - x^4ya_1 - 2x^4yb_1 \\
&+ 2x^3y^2a_1 - 6x^3y^2b_1 + 6x^2y^3a_1 - 4x^2y^3b_1 + 4x\,y^4a_1 - x\,y^4b_1 + y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^6 - 4a_2v_1^5v_2 - 3a_2v_1^4v_2^2 - a_3v_1^6 - 2a_3v_1^5v_2 - 6a_3v_1^4v_2^2 - 4a_3v_1^3v_2^3 + 2b_2v_1^6 \\
&+ 2b_2v_1^5v_2 + b_3v_1^6 + 4b_3v_1^5v_2 + 3b_3v_1^4v_2^2 - a_1v_1^4v_2 + 2a_1v_1^3v_2^2 + 6a_1v_1^2v_2^3 \\
&+ 4a_1v_1v_2^4 + a_1v_2^5 + b_1v_1^5 - 2b_1v_1^4v_2 - 6b_1v_1^3v_2^2 - 4b_1v_1^2v_2^3 - b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-a_2 - a_3 + 2b_2 + b_3)\, v_1^6 + (-4a_2 - 2a_3 + 2b_2 + 4b_3)\, v_1^5 v_2 + b_1 v_1^5 \\ &+ (-3a_2 - 6a_3 + 3b_3)\, v_1^4 v_2^2 + (-a_1 - 2b_1)\, v_1^4 v_2 - 4a_3 v_1^3 v_2^3 \\ &+ (2a_1 - 6b_1)\, v_1^3 v_2^2 + (6a_1 - 4b_1)\, v_1^2 v_2^3 + (4a_1 - b_1)\, v_1 v_2^4 + a_1 v_2^5 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -4a_3 &= 0 \\ -a_1 - 2b_1 &= 0 \\ 2a_1 - 6b_1 &= 0 \\ 4a_1 - b_1 &= 0 \\ 6a_1 - 4b_1 &= 0 \\ -3a_2 - 6a_3 + 3b_3 &= 0 \\ -4a_2 - 2a_3 + 2b_2 + 4b_3 &= 0 \\ -a_2 - a_3 + 2b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x^{3}+y\,x^{2}+2x\,y^{2}+y^{3}}{x\left(x+y\right)^{2}}\right)(x) \\
&= -\frac{x^{3}}{x^{2}+2yx+y^{2}} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{-\frac{x^{3}}{x^{2}+2yx+y^{2}}}dy
\end{aligned}$$

Which results in

$$S = -\frac{\left(x+y\right)^{3}}{3x^{3}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^{3}+y\,x^{2}+2x\,y^{2}+y^{3}}{x\left(x+y\right)^{2}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y(x+y)^2}{x^4} \\
S_y &= -\frac{(x+y)^2}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

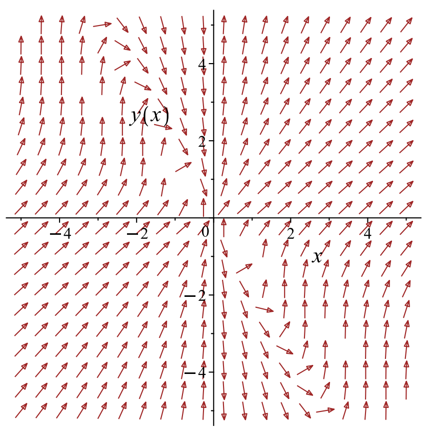
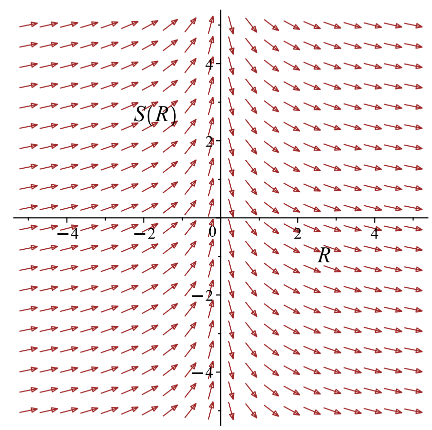
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{(x+y)^3}{3x^3} = -\ln(x) + c_1
$$

Which simplifies to

$$
-\frac{(x+y)^3}{3x^3} = -\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y\,x^2+2x\,y^2+y^3}{x(x+y)^2}$ 	$R = x$ $S = -\frac{(x+y)^3}{3x^3}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{(x+y)^3}{3x^3} = -\ln(x) + c_1 \tag{1}$$

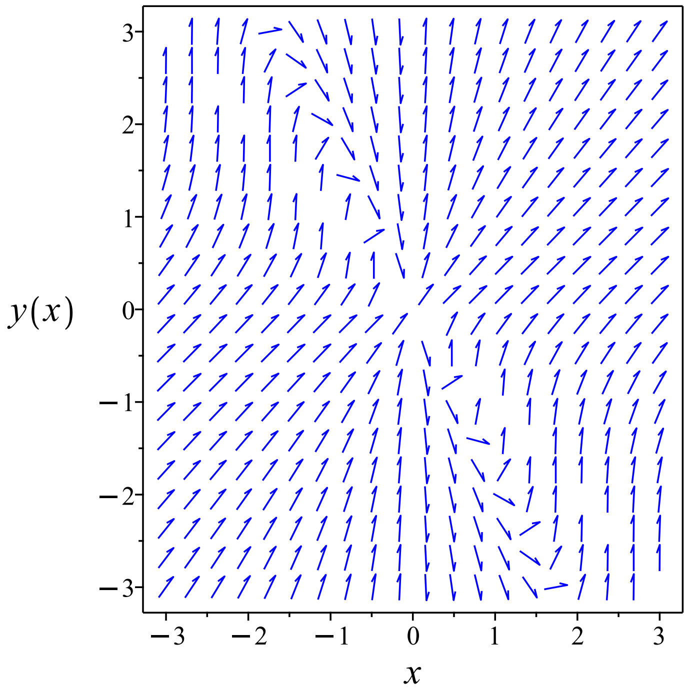


Figure 310: Slope field plot

Verification of solutions

$$-\frac{(x+y)^3}{3x^3} = -\ln(x) + c_1$$

Verified OK.

5.33.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x(x+y)^2\right)\mathrm{d}y = \left(x^3 + y\,x^2 + 2x\,y^2 + y^3\right)\mathrm{d}x$$
$$\left(-x^3 - y\,x^2 - 2x\,y^2 - y^3\right)\mathrm{d}x + \left(x(x+y)^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^3 - y\,x^2 - 2x\,y^2 - y^3$$
$$N(x, y) = x(x+y)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 - y\,x^2 - 2x\,y^2 - y^3\right)\\ &= -(3y + x)(x + y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x(x+y)^2\right)\\ &= 3x^2 + 4yx + y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(x+y)^2}\left((-x^2 - 4yx - 3y^2) - \left((x+y)^2 + 2x(x+y)\right)\right) \\
&= -\frac{4}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{4}{x} \,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-4\ln(x)} \\
&= \frac{1}{x^4}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^4}\left(-x^3 - y\,x^2 - 2x\,y^2 - y^3\right) \\
&= \frac{-x^3 - y\,x^2 - 2x\,y^2 - y^3}{x^4}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4}\left(x(x+y)^2\right) \\
&= \frac{(x+y)^2}{x^3}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 - y\,x^2 - 2x\,y^2 - y^3}{x^4}\right) + \left(\frac{(x+y)^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^3 - y\,x^2 - 2x\,y^2 - y^3}{x^4} \mathrm{d}x$$
$$\phi = -\ln(x) + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{3x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + \frac{2y}{x^2} + \frac{y^2}{x^3} + f'(y) \tag{4}$$
$$= \frac{(x+y)^2}{x^3} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x+y)^2}{x^3}$. Therefore equation (4) becomes

$$\frac{(x+y)^2}{x^3} = \frac{(x+y)^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{3x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{3x^3}$$

<u>Summary</u>
The solution(s) found are the following

$$-\ln(x) + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{3x^3} = c_1 \tag{1}$$

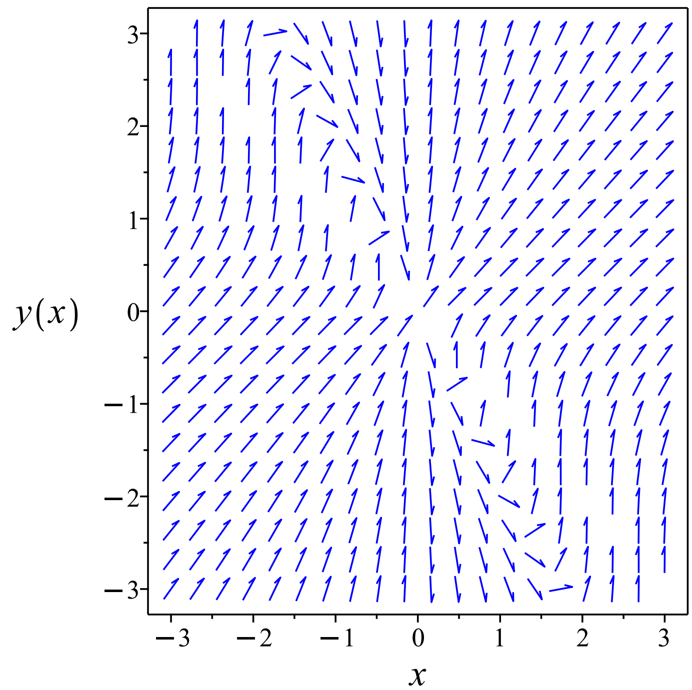


Figure 311: Slope field plot

<u>Verification of solutions</u>

$$-\ln(x) + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{3x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 80

```
dsolve(diff(y(x),x)=(y(x)^3+2*x*y(x)^2+x^2*y(x)+x^3)/(x*(y(x)+x)^2),y(x), singsol=all)
```

$$y(x) = x\left(-1+(3\ln(x)+3c_1)^{\frac{1}{3}}\right)$$
$$y(x) = -\frac{x\left(i\sqrt{3}\,(3\ln(x)+3c_1)^{\frac{1}{3}}+(3\ln(x)+3c_1)^{\frac{1}{3}}+2\right)}{2}$$
$$y(x) = \frac{x(i\sqrt{3}-1)\,(3\ln(x)+3c_1)^{\frac{1}{3}}}{2}-x$$

✓ Solution by Mathematica

Time used: 1.409 (sec). Leaf size: 109

```
DSolve[y'[x]==(y[x]^3+2*x*y[x]^2+x^2*y[x]+x^3)/(x*(y[x]+x)^2),y[x],x,IncludeSingularSolution
```

$$y(x) \to -x+\sqrt[3]{x^3(3\log(x)+1+3c_1)}$$
$$y(x) \to -x+\frac{1}{2}i\left(\sqrt{3}+i\right)\sqrt[3]{x^3(3\log(x)+1+3c_1)}$$
$$y(x) \to -x-\frac{1}{2}\left(1+i\sqrt{3}\right)\sqrt[3]{x^3(3\log(x)+1+3c_1)}$$

5.34 problem 31

Internal problem ID [1008]
Internal file name [`OUTPUT/1009_Sunday_June_05_2022_01_56_42_AM_67943606/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x + 2y}{2x + y} = 0}$$

5.34.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x + 2u(x)\,x}{2x + u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 1}{x\,(u + 2)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u+2}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2-1}{u+2}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{u^2-1}{u+2}}\, du &= \int -\frac{1}{x}\, dx \\
\frac{3\ln(u-1)}{2} - \frac{\ln(u+1)}{2} &= -\ln(x) + c_2
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\frac{3\ln(u-1) - \ln(u+1)}{2} &= -\ln(x) + c_2 \\
3\ln(u-1) - \ln(u+1) &= (2)\left(-\ln(x) + c_2\right) \\
&= -2\ln(x) + 2c_2
\end{aligned}$$

Raising both side to exponential gives

$$e^{3\ln(u-1)-\ln(u+1)} = e^{-2\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}
\frac{(u-1)^3}{u+1} &= \frac{2c_2}{x^2} \\
&= \frac{c_3}{x^2}
\end{aligned}$$

Which simplifies to

$$\frac{(u(x)-1)^3}{u(x)+1} = \frac{c_3 e^{2c_2}}{x^2}$$

The solution is

$$\frac{(u(x)-1)^3}{u(x)+1} = \frac{c_3 e^{2c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}
\frac{\left(\frac{y}{x}-1\right)^3}{1+\frac{y}{x}} &= \frac{c_3 e^{2c_2}}{x^2} \\
\frac{(-x+y)^3}{x^2(x+y)} &= \frac{c_3 e^{2c_2}}{x^2}
\end{aligned}$$

Which simplifies to

$$-\frac{(x-y)^3}{x+y} = c_3 e^{2c_2}$$

Summary
The solution(s) found are the following

$$-\frac{(x-y)^3}{x+y} = c_3 e^{2c_2} \tag{1}$$

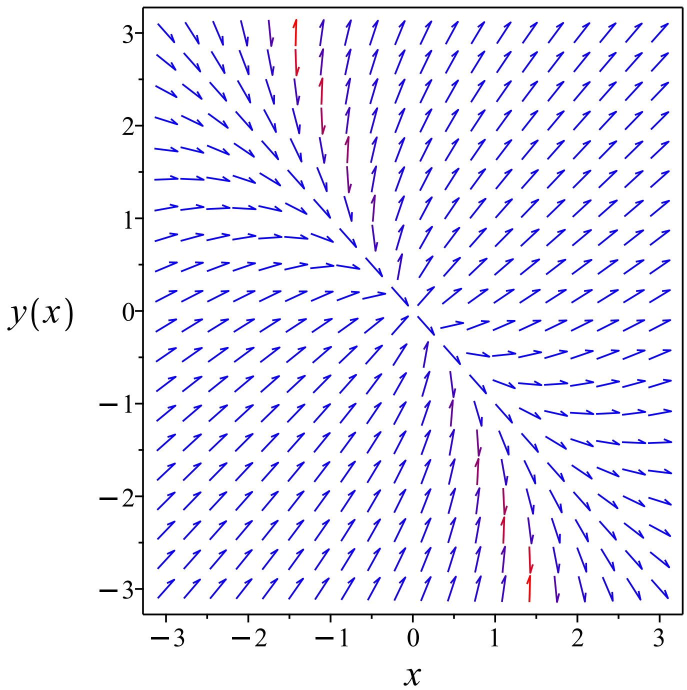


Figure 312: Slope field plot

Verification of solutions

$$-\frac{(x-y)^3}{x+y} = c_3 e^{2c_2}$$

Verified OK.

5.34.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x+2y}{2x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(x+2y)(b_3 - a_2)}{2x+y} - \frac{(x+2y)^2 a_3}{(2x+y)^2} \\
&- \left(\frac{1}{2x+y} - \frac{2(x+2y)}{(2x+y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{2}{2x+y} - \frac{x+2y}{(2x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^2a_2 + x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 - 4xyb_2 - 2xyb_3 + 2y^2a_2 + y^2a_3 - y^2b_2 - 2y^2b_3 + 3xb_1 -}{(2x+y)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^2a_2 - x^2a_3 + x^2b_2 + 2x^2b_3 - 2xya_2 - 4xya_3 + 4xyb_2 \\
&+ 2xyb_3 - 2y^2a_2 - y^2a_3 + y^2b_2 + 2y^2b_3 - 3xb_1 + 3ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 \\ & + 4b_2v_1v_2 + b_2v_2^2 + 2b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 + 3a_1v_2 - 3b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (-2a_2 - a_3 + b_2 + 2b_3)\, v_1^2 + (-2a_2 - 4a_3 + 4b_2 + 2b_3)\, v_1v_2 \\ & - 3b_1v_1 + (-2a_2 - a_3 + b_2 + 2b_3)\, v_2^2 + 3a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 3a_1 &= 0 \\ -3b_1 &= 0 \\ -2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0 \\ -2a_2 - a_3 + b_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x+2y}{2x+y}\right)(x) \\
&= \frac{-x^2+y^2}{2x+y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2+y^2}{2x+y}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(x+y)}{2} + \frac{3\ln(-x+y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x + 2y}{2x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + 2y}{x^2 - y^2} \\ S_y &= \frac{-2x - y}{x^2 - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

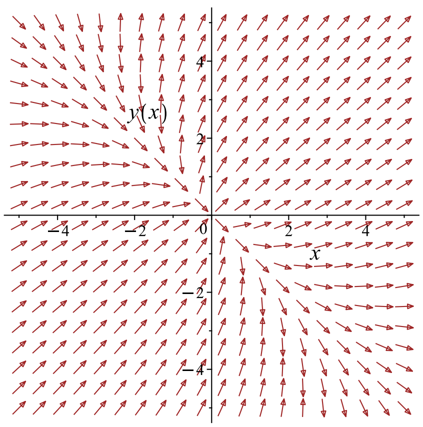
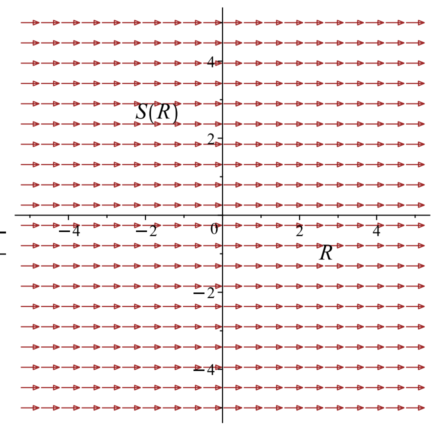
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x + y)}{2} + \frac{3\ln(-x + y)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln(x + y)}{2} + \frac{3\ln(-x + y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+2y}{2x+y}$ 	$R = x$ $S = -\frac{\ln(x+y)}{2} + \frac{3\ln(-}{}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x+y)}{2} + \frac{3\ln(-x+y)}{2} = c_1 \tag{1}$$

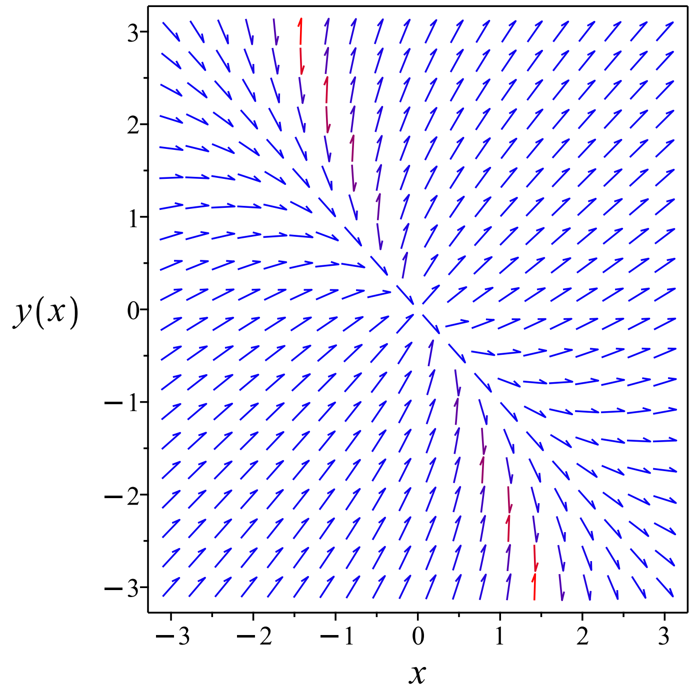


Figure 313: Slope field plot

<u>Verification of solutions</u>

$$-\frac{\ln(x+y)}{2}+\frac{3\ln(-x+y)}{2}=c_1$$

Verified OK.

5.34.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x + y)\,\mathrm{d}y &= (x + 2y)\,\mathrm{d}x \\ (-2y - x)\,\mathrm{d}x + (2x + y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2y - x \\ N(x, y) &= 2x + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2y - x) \\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + y) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2-y^2}$ is an integrating factor. Therefore by multiplying $M = -x - 2y$ and $N = 2x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{-x - 2y}{x^2 - y^2}$$
$$N = \frac{2x + y}{x^2 - y^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2x+y}{x^2-y^2}\right)\mathrm{d}y = \left(-\frac{-2y-x}{x^2-y^2}\right)\mathrm{d}x$$

$$\left(\frac{-2y-x}{x^2-y^2}\right)\mathrm{d}x + \left(\frac{2x+y}{x^2-y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{-2y-x}{x^2-y^2}$$
$$N(x,y) = \frac{2x+y}{x^2-y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-2y-x}{x^2-y^2}\right)\\ &= \frac{-2x^2-2yx-2y^2}{(x^2-y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x+y}{x^2-y^2}\right)\\ &= \frac{-2x^2-2yx-2y^2}{(x^2-y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-2y-x}{x^2-y^2}\,\mathrm{d}x$$
$$\phi = \frac{\ln(x+y)}{2} - \frac{3\ln(x-y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{2x+2y} + \frac{3}{2(x-y)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x+y}{x^2-y^2}$. Therefore equation (4) becomes

$$\frac{2x+y}{x^2-y^2} = \frac{2x+y}{x^2-y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x+y)}{2} - \frac{3\ln(x-y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x+y)}{2} - \frac{3\ln(x-y)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x+y)}{2} - \frac{3\ln(x-y)}{2} = c_1 \tag{1}$$

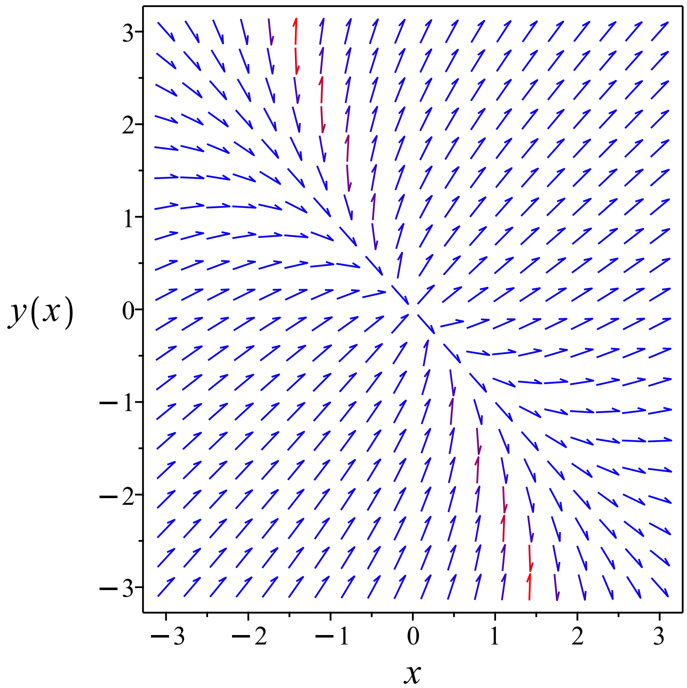


Figure 314: Slope field plot

Verification of solutions

$$\frac{\ln(x+y)}{2} - \frac{3\ln(x-y)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 271

```
dsolve(diff(y(x),x)=(x+2*y(x))/(2*x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{x\left(\frac{c_1\left(\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{2}{3}}+3\right)}{3x\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{1}{3}}}+c_1^2\right)}{c_1^2}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{2}{3}}-6xc_1\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{1}{3}}-3i\sqrt{3}+3}{6\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{1}{3}}c_1}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{2}{3}}+6xc_1\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{1}{3}}-3i\sqrt{3}-3}{6\left(27c_1x+3\sqrt{3}\,\sqrt{27c_1^2x^2-1}\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 30.082 (sec). Leaf size: 382

```
DSolve[y'[x]==(x+2*y[x])/(2*x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}}{3^{2/3}} - \frac{e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}} + x$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}}{2\ 3^{2/3}} + \frac{\left(1+i\sqrt{3}\right)e^{2c_1}}{2\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}} + x$$

$$y(x) \to -\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}}{2\ 3^{2/3}} + \frac{\left(1-i\sqrt{3}\right)e^{2c_1}}{2\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{27e^{4c_1}x^2+e^{6c_1}}-9e^{2c_1}x}} + x$$

5.35 problem 32

Internal problem ID [1009]
Internal file name [`OUTPUT/1010_Sunday_June_05_2022_01_56_44_AM_88150377/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{-2x+y} = 0}$$

5.35.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x}{-2x + u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u-3)}{x\,(u-2)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-3)}{u-2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u(u-3)}{u-2}} du &= -\frac{1}{x} dx \\
\int \frac{1}{\frac{u(u-3)}{u-2}} du &= \int -\frac{1}{x} dx \\
\frac{2\ln(u)}{3} + \frac{\ln(u-3)}{3} &= -\ln(x) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{2\ln(u) + \ln(u-3)}{3} &= -\ln(x) + c_2 \\
2\ln(u) + \ln(u-3) &= (3)(-\ln(x) + c_2) \\
&= -3\ln(x) + 3c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{2\ln(u)+\ln(u-3)} = e^{-3\ln(x)+3c_2}
$$

Which simplifies to

$$
\begin{aligned}
u^2(u-3) &= \frac{3c_2}{x^3} \\
&= \frac{c_3}{x^3}
\end{aligned}
$$

Which simplifies to

$$
u(x)^2 (u(x) - 3) = \frac{c_3 e^{3c_2}}{x^3}
$$

The solution is

$$
u(x)^2 (u(x) - 3) = \frac{c_3 e^{3c_2}}{x^3}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2\left(\frac{y}{x} - 3\right)}{x^2} &= \frac{c_3 e^{3c_2}}{x^3} \\
\frac{-3xy^2 + y^3}{x^3} &= \frac{c_3 e^{3c_2}}{x^3}
\end{aligned}
$$

Which simplifies to

$$-3xy^2 + y^3 = c_3 e^{3c_2}$$

Summary
The solution(s) found are the following

$$-3xy^2 + y^3 = c_3 e^{3c_2} \tag{1}$$

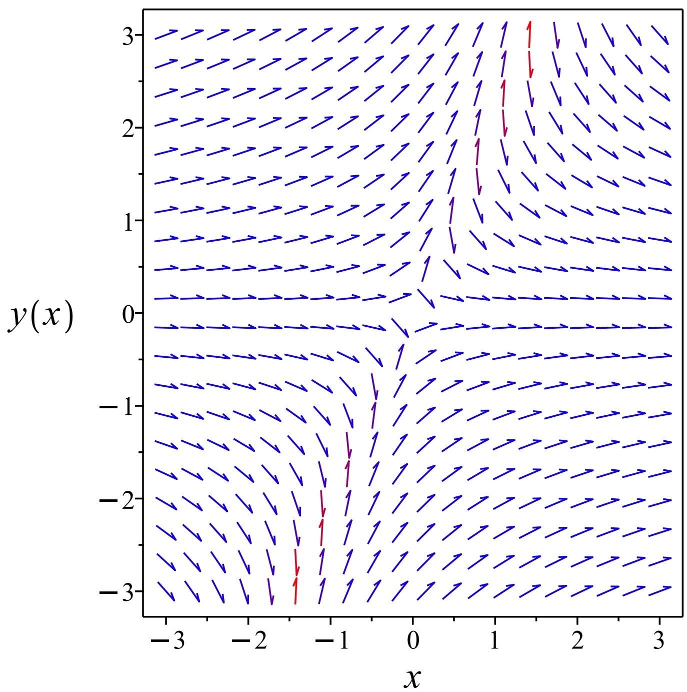


Figure 315: Slope field plot

Verification of solutions

$$-3xy^2 + y^3 = c_3 e^{3c_2}$$

Verified OK.

5.35.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{-2x + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{-2x + y} - \frac{y^2 a_3}{(-2x + y)^2} - \frac{2y(x a_2 + y a_3 + a_1)}{(-2x + y)^2} - \left(\frac{1}{-2x + y} - \frac{y}{(-2x + y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2 b_2 - 4xy b_2 - y^2 a_2 - 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 2x b_1 - 2y a_1}{(2x - y)^2} = 0$$

Setting the numerator to zero gives

$$6x^2 b_2 - 4xy b_2 - y^2 a_2 - 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 2x b_1 - 2y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_2^2 - 3a_3v_2^2 + 6b_2v_1^2 - 4b_2v_1v_2 + b_2v_2^2 + b_3v_2^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^2 - 4b_2v_1v_2 + 2b_1v_1 + (-a_2 - 3a_3 + b_2 + b_3)\,v_2^2 - 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0\\
2b_1 &= 0\\
-4b_2 &= 0\\
6b_2 &= 0\\
-a_2 - 3a_3 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= -3a_3 + b_3\\
a_3 &= a_3\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{y}{-2x+y}\right)(x) \\ &= \frac{3yx - y^2}{2x - y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3yx - y^2}{2x - y}} dy\end{aligned}$$

Which results in

$$S = \frac{2\ln(y)}{3} + \frac{\ln(-3x + y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{-2x + y}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{3x - y} \\
S_y &= \frac{2x - y}{y\,(3x - y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

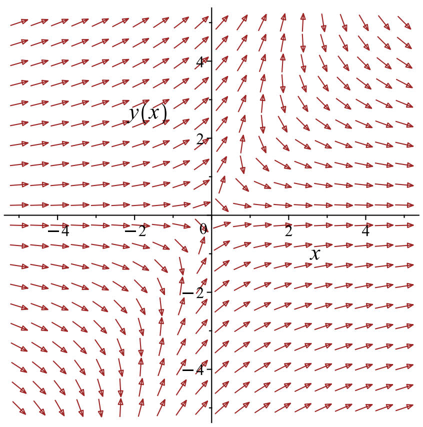
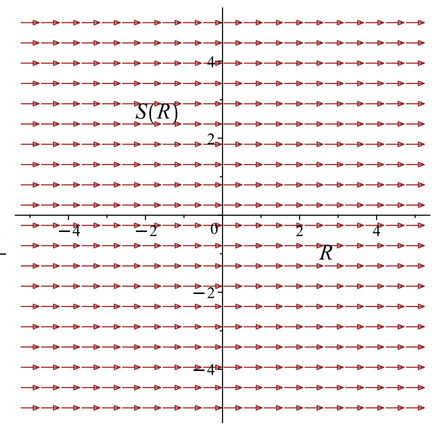
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(y)}{3} + \frac{\ln(-3x + y)}{3} = c_1
$$

Which simplifies to

$$
\frac{2\ln(y)}{3} + \frac{\ln(-3x + y)}{3} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{-2x+y}$ 	$R = x$ $S = \frac{2\ln(y)}{3} + \frac{\ln(-3x +}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y)}{3} + \frac{\ln(-3x + y)}{3} = c_1 \tag{1}$$

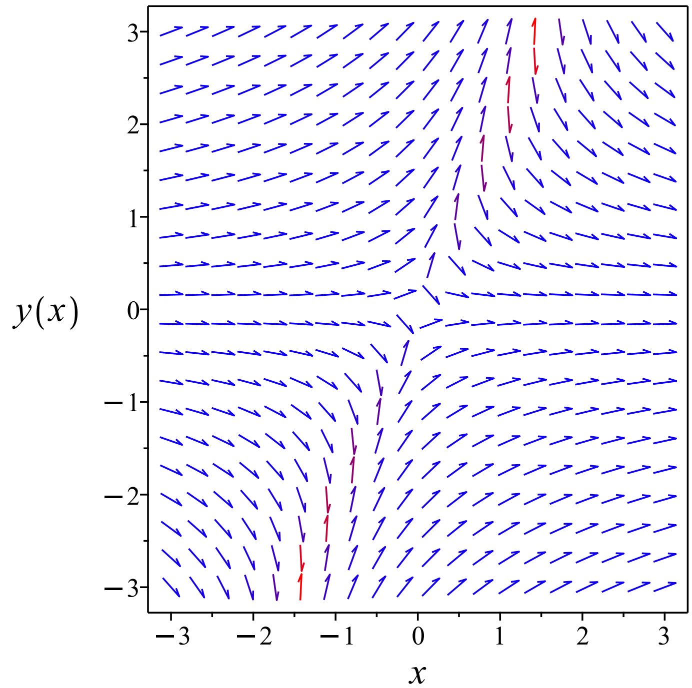


Figure 316: Slope field plot

Verification of solutions

$$\frac{2\ln(y)}{3} + \frac{\ln(-3x+y)}{3} = c_1$$

Verified OK.

5.35.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-2x + y)\, \mathrm{d}y = (y)\, \mathrm{d}x$$
$$(-y)\, \mathrm{d}x + (-2x + y)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y$$
$$N(x, y) = -2x + y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x + y) \\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-2x+y}((-1) - (-2)) \\
&= \frac{1}{-2x+y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y}((-2) - (-1)) \\
&= \frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y(-y) \\
&= -y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y(-2x+y) \\
&= (-2x+y)\, y
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-y^2\right) + \left(\left(-2x + y\right) y\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -y^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x\, y^2 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -2yx + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (-2x + y)\, y$. Therefore equation (4) becomes

$$
(-2x + y)\, y = -2yx + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(y^2\right) \mathrm{d}y \\
f(y) &= \frac{y^3}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\,y^2 + \frac{1}{3}y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\,y^2 + \frac{1}{3}y^3$$

Summary

The solution(s) found are the following

$$-xy^2 + \frac{y^3}{3} = c_1 \tag{1}$$

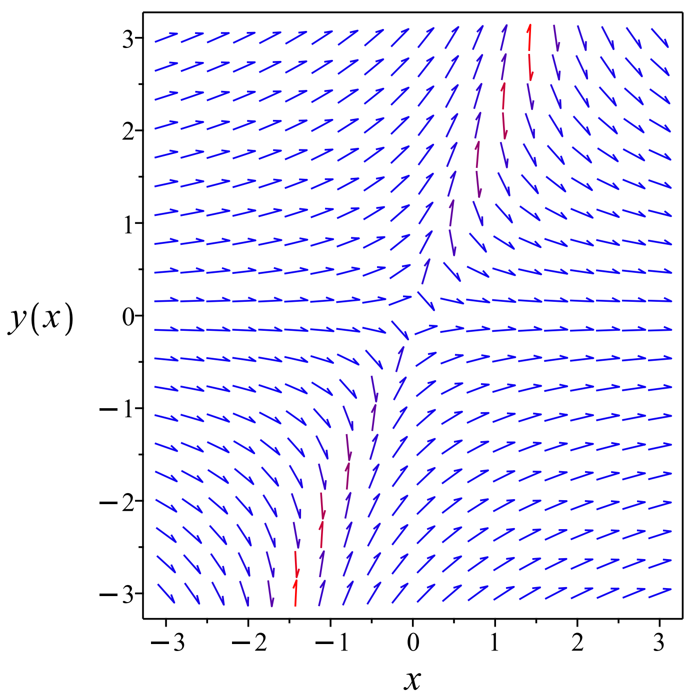


Figure 317: Slope field plot

Verification of solutions

$$-xy^2 + \frac{y^3}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 306

```
dsolve(diff(y(x),x)=y(x)/(y(x)-2*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{c_1\left(-4x^3 + 3c_1\right)}\right)^{\frac{1}{3}}}{2} + \frac{2x^2}{\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{c_1\left(-4x^3 + 3c_1\right)}\right)^{\frac{1}{3}}} + x$$

$$y(x) = \frac{\left(-1 - i\sqrt{3}\right)\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}}{4} + \frac{x\left(ix\sqrt{3} - x + \left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}\right)}{\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3} - 1\right)\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}}{4} - \frac{x\left(ix\sqrt{3} + x - \left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}\right)}{\left(-12c_1 + 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 37.127 (sec). Leaf size: 479

```
DSolve[y'[x]==y[x]/(y[x]-2*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}x^2}{\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}} + x$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}}{2\sqrt[3]{2}} - \frac{(1+i\sqrt{3})x^2}{2^{2/3}\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}} + x$$

$$y(x) \to -\frac{(1+i\sqrt{3})\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}}{2\sqrt[3]{2}} + \frac{i(\sqrt{3}+i)x^2}{2^{2/3}\sqrt[3]{2x^3 + \sqrt{e^{6c_1} - 4e^{3c_1}x^3} - e^{3c_1}}} + x$$

$$y(x) \to 0$$

$$y(x) \to -\frac{i\left(\sqrt[3]{x^3} - x\right)\left((\sqrt{3}-i)\sqrt[3]{x^3} - 2ix\right)}{2x}$$

$$y(x) \to \frac{i\left(\sqrt[3]{x^3} - x\right)\left((\sqrt{3}+i)\sqrt[3]{x^3} + 2ix\right)}{2x}$$

$$y(x) \to \sqrt[3]{x^3} + \frac{(x^3)^{2/3}}{x} + x$$

5.36 problem 33

Internal problem ID [1010]
Internal file name [`OUTPUT/1011_Sunday_June_05_2022_01_56_45_AM_37605449/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{x y^2 + 2y^3}{x^3 + x^2 y + x y^2} = 0}$$

5.36.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^3 u(x)^2 + 2u(x)^3\,x^3}{x^3 + x^3 u\,(x) + x^3 u\,(x)^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x) g(u) \\ &= \frac{u^3 - u}{x\,(u^2 + u + 1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2+u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{u^2+u+1}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3-u}{u^2+u+1}}\,du = \int \frac{1}{x}\,dx$$

$$-\ln(u) + \frac{\ln(u+1)}{2} + \frac{3\ln(u-1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u)+\frac{\ln(u+1)}{2}+\frac{3\ln(u-1)}{2}} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{\sqrt{u+1}\,(u-1)^{\frac{3}{2}}}{u} = c_3 x$$

Therefore the solution y is

$$y = xu$$

$$= -\frac{x\left(-\mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_Z^6 - 2\_Z^2 c_3^2 x^2 - c_3^2 x^2\right)^4 c_3^2 x^2 + \mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_Z\right.\right.}{\mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_\right.}$$

Summary

The solution(s) found are the following

$$y = -\frac{x\left(-\mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_Z^6 - 2\_Z^2 c_3^2 x^2 - c_3^2 x^2\right)^4 c_3^2 x^2 + \mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_Z^6\right.\right.}{\mathrm{RootOf}\left(\_Z^8 - \_Z^4 c_3^2 x^2 + 2\_\right.} \quad (1)$$

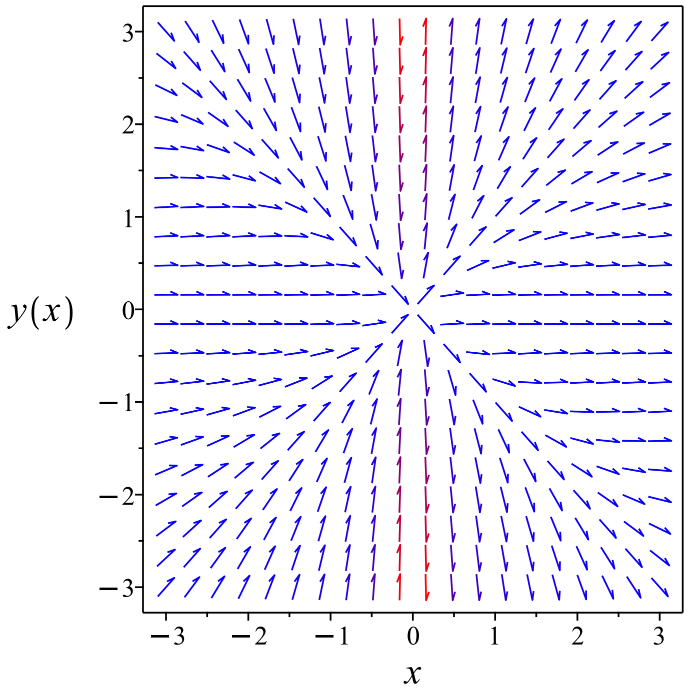


Figure 318: Slope field plot

Verification of solutions

$y =$

$$-\frac{x\left(-\operatorname{RootOf}\left(\_Z^8-\_Z^4c_3^2x^2+2\_Z^6-2\_Z^2c_3^2x^2-c_3^2x^2\right)^4c_3^2x^2+\operatorname{RootOf}\left(\_Z^8-\_Z^4c_3^2x^2+2\_Z^6\right.\right.}{\operatorname{RootOf}\left(\_Z^8-\_Z^4c_3^2x^2+2\_\right.}$$

Warning, solution could not be verified

5.36.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2(x+2y)}{x\,(x^2+yx+y^2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y^2 (x+2y)(b_3 - a_2)}{x (x^2+yx+y^2)} - \frac{y^4 (x+2y)^2 a_3}{x^2 (x^2+yx+y^2)^2} \\ & - \left(\frac{y^2}{x (x^2+yx+y^2)} - \frac{y^2 (x+2y)}{x^2 (x^2+yx+y^2)} \right. \\ & \left. - \frac{y^2 (x+2y)(2x+y)}{x (x^2+yx+y^2)^2}\right) (x a_2 + y a_3 + a_1) - \left(\frac{2y(x+2y)}{x (x^2+yx+y^2)} \right. \\ & \left. + \frac{2y^2}{x (x^2+yx+y^2)} - \frac{y^2 (x+2y)^2}{x (x^2+yx+y^2)^2}\right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6 b_2 + x^4 y^2 a_2 - 4x^4 y^2 b_2 - x^4 y^2 b_3 + 4x^3 y^3 a_2 + 2x^3 y^3 a_3 - 2x^3 y^3 b_2 - 4x^3 y^3 b_3 + x^2 y^4 a_2 + 6x^2 y^4 a_3 - x^2 y^4 b_2}{x^2 (x^2 + yx} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^6 b_2 + x^4 y^2 a_2 - 4x^4 y^2 b_2 - x^4 y^2 b_3 + 4x^3 y^3 a_2 + 2x^3 y^3 a_3 - 2x^3 y^3 b_2 - 4x^3 y^3 b_3 \\ & + x^2 y^4 a_2 + 6x^2 y^4 a_3 - x^2 y^4 b_2 - x^2 y^4 b_3 - 2y^6 a_3 - 2x^4 y b_1 + 2x^3 y^2 a_1 \\ & - 7x^3 y^2 b_1 + 7x^2 y^3 a_1 - 4x^2 y^3 b_1 + 4x y^4 a_1 - 2x y^4 b_1 + 2y^5 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2 v_1^4 v_2^2 + 4a_2 v_1^3 v_2^3 + a_2 v_1^2 v_2^4 + 2a_3 v_1^3 v_2^3 + 6a_3 v_1^2 v_2^4 - 2a_3 v_2^6 + b_2 v_1^6 - 4b_2 v_1^4 v_2^2 \\ & - 2b_2 v_1^3 v_2^3 - b_2 v_1^2 v_2^4 - b_3 v_1^4 v_2^2 - 4b_3 v_1^3 v_2^3 - b_3 v_1^2 v_2^4 + 2a_1 v_1^3 v_2^2 + 7a_1 v_1^2 v_2^3 \\ & + 4a_1 v_1 v_2^4 + 2a_1 v_2^5 - 2b_1 v_1^4 v_2 - 7b_1 v_1^3 v_2^2 - 4b_1 v_1^2 v_2^3 - 2b_1 v_1 v_2^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & b_2 v_1^6 + (a_2 - 4b_2 - b_3) v_1^4 v_2^2 - 2b_1 v_1^4 v_2 + (4a_2 + 2a_3 - 2b_2 - 4b_3) v_1^3 v_2^3 \\ & + (2a_1 - 7b_1) v_1^3 v_2^2 + (a_2 + 6a_3 - b_2 - b_3) v_1^2 v_2^4 \\ & + (7a_1 - 4b_1) v_1^2 v_2^3 + (4a_1 - 2b_1) v_1 v_2^4 - 2a_3 v_2^6 + 2a_1 v_2^5 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ 2a_1 - 7b_1 &= 0 \\ 4a_1 - 2b_1 &= 0 \\ 7a_1 - 4b_1 &= 0 \\ a_2 - 4b_2 - b_3 &= 0 \\ a_2 + 6a_3 - b_2 - b_3 &= 0 \\ 4a_2 + 2a_3 - 2b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^2(x+2y)}{x\,(x^2+yx+y^2)}\right)(x) \\
&= \frac{y\,x^2 - y^3}{x^2+yx+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y\,x^2-y^3}{x^2+yx+y^2}} dy
\end{aligned}
$$

Which results in

$$S = \ln(y) - \frac{\ln(x+y)}{2} - \frac{3\ln(-x+y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2(x+2y)}{x(x^2+yx+y^2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-2x-y}{x^2-y^2} \\ S_y &= \frac{1}{y} - \frac{1}{2x+2y} + \frac{3}{2x-2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

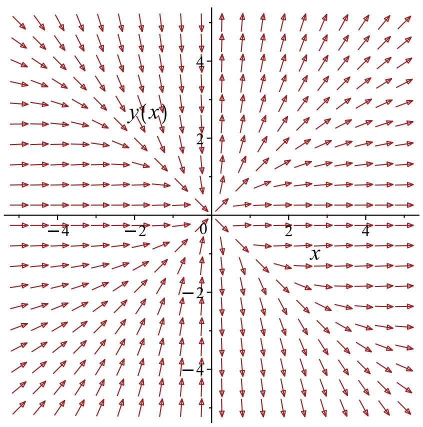
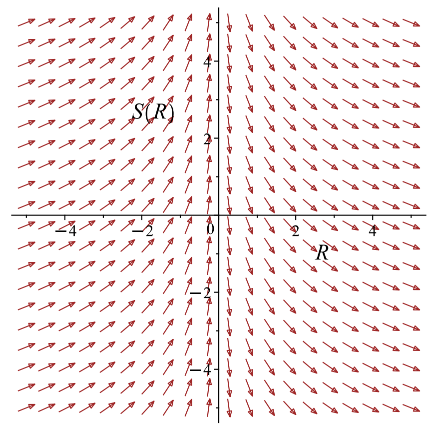
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{\ln(x+y)}{2} - \frac{3\ln(-x+y)}{2} = -2\ln(x) + c_1$$

Which simplifies to

$$\ln(y) - \frac{\ln(x+y)}{2} - \frac{3\ln(-x+y)}{2} = -2\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2(x+2y)}{x(x^2+yx+y^2)}$ 	$R = x$ $S = \ln(y) - \frac{\ln(x+y)}{2} -$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$\ln(y) - \frac{\ln(x+y)}{2} - \frac{3\ln(-x+y)}{2} = -2\ln(x) + c_1 \tag{1}$$

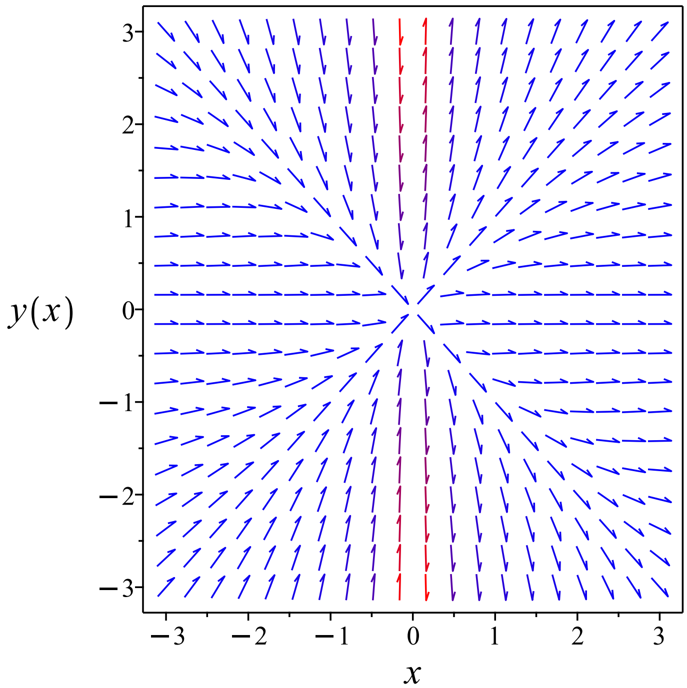


Figure 319: Slope field plot

Verification of solutions

$$\ln(y) - \frac{\ln(x+y)}{2} - \frac{3\ln(-x+y)}{2} = -2\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 2.969 (sec). Leaf size: 129

```
dsolve(diff(y(x),x)=(x*y(x)^2+2*y(x)^3)/(x^3+x^2*y(x)+x*y(x)^2),y(x), singsol=all)
```

$$y(x) = x\Big(\operatorname{RootOf}\left(\_Z^8 c_1 x^2 + 2\_Z^6 c_1 x^2 + \_Z^4 c_1 x^2 - 2\_Z^2 - 1\right)^6 c_1 x^2 + 2\operatorname{RootOf}\left(\_Z^8 c_1 x^2 + 2\_Z^6 c_1 x^2 + \_Z^4 c_1 x^2 - 2\_Z^2 - 1\right)^4 c_1 x^2 + \operatorname{RootOf}\left(\_Z^8 c_1 x^2 + 2\_Z^6 c_1 x^2 + \_Z^4 c_1 x^2 - 2\_Z^2 - 1\right)^2 c_1 x^2 - 1\Big)$$

✓ Solution by Mathematica

Time used: 60.157 (sec). Leaf size: 1989

```
DSolve[y'[x]==(x*y[x]^2+2*y[x]^3)/(x^3+x^2*y[x]+x*y[x]^2),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{6}\left(-\sqrt{3}\sqrt{-2e^{2c_1}x^4+3x^2+\frac{e^{4c_1}x^8}{\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+6\sqrt{3}\sqrt{e^{4c_1}x^{16}\left(27+e^{4c_1}x^4\right)}}}+\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8 \right.$$

$$-\sqrt{3}\sqrt{-4e^{2c_1}x^4+6x^2-\frac{e^{4c_1}x^8}{\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+6\sqrt{3}\sqrt{e^{4c_1}x^{16}\left(27+e^{4c_1}x^4\right)}}}-\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+}$$

$$\left. +3x\right)$$

$$y(x) \to \frac{1}{6}\left(-\sqrt{3}\sqrt{-2e^{2c_1}x^4+3x^2+\frac{e^{4c_1}x^8}{\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+6\sqrt{3}\sqrt{e^{4c_1}x^{16}\left(27+e^{4c_1}x^4\right)}}}+\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8 \right.$$

$$+\sqrt{3}\sqrt{-4e^{2c_1}x^4+6x^2-\frac{e^{4c_1}x^8}{\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+6\sqrt{3}\sqrt{e^{4c_1}x^{16}\left(27+e^{4c_1}x^4\right)}}}-\sqrt[3]{e^{6c_1}x^{12}+54e^{2c_1}x^8+}$$

$$\left. +3x\right)$$

$y(x)$

$$\frac{1}{}\left(- \sqrt{\frac{e^{4c_1}x^8}{}}\right.$$

5.37 problem 34

Internal problem ID [1011]
Internal file name [`OUTPUT/1012_Sunday_June_05_2022_01_56_48_AM_18891282/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{x^3 + x^2y + 3y^3}{x^3 + 3xy^2} = 0}$$

5.37.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^3 + x^3u(x) + 3u(x)^3\,x^3}{x^3 + 3x^3u\,(x)^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{x\,(3u^2+1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{3u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{1}{3u^2+1}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{1}{3u^2+1}}\,du = \int \frac{1}{x}\,dx$$

$$u^3 + u = \ln(x) + c_2$$

The solution is

$$u(x)^3 + u(x) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^3}{x^3} + \frac{y}{x} - \ln(x) - c_2 = 0$$

$$\frac{y^3}{x^3} + \frac{y}{x} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^3}{x^3} + \frac{y}{x} - \ln(x) - c_2 = 0 \tag{1}$$

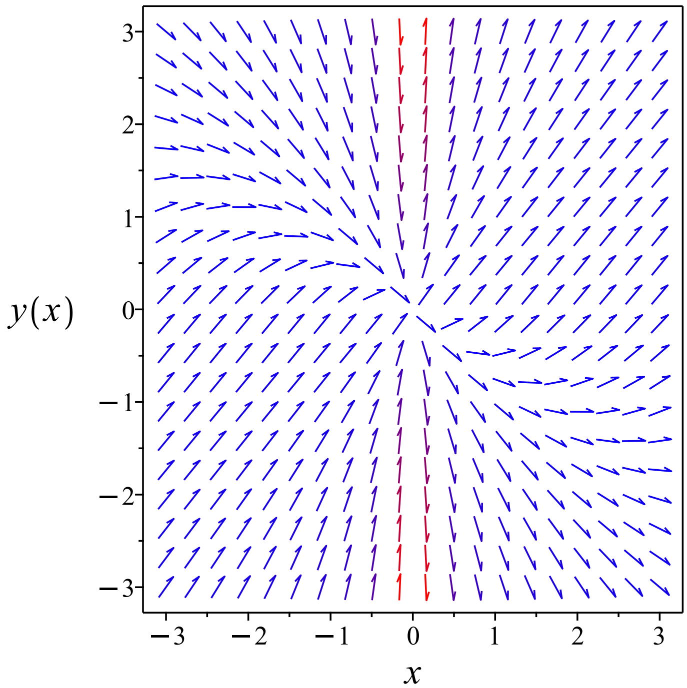


Figure 320: Slope field plot

Verification of solutions

$$\frac{y^3}{x^3} + \frac{y}{x} - \ln(x) - c_2 = 0$$

Verified OK.

5.37.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^3 + y\,x^2 + 3y^3}{x\,(x^2 + 3y^2)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(x^3 + y\,x^2 + 3y^3)(b_3 - a_2)}{x\,(x^2 + 3y^2)} - \frac{(x^3 + y\,x^2 + 3y^3)^2 a_3}{x^2\,(x^2 + 3y^2)^2} \\
&- \left(\frac{3x^2 + 2yx}{x\,(x^2 + 3y^2)} - \frac{x^3 + y\,x^2 + 3y^3}{x^2\,(x^2 + 3y^2)} - \frac{2(x^3 + y\,x^2 + 3y^3)}{(x^2 + 3y^2)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{x^2 + 9y^2}{x\,(x^2 + 3y^2)} - \frac{6(x^3 + y\,x^2 + 3y^3)\,y}{x\,(x^2 + 3y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^6a_2 + x^6a_3 - x^6b_3 + 2x^5ya_3 - 6x^5yb_2 + 9x^4y^2a_2 - 9x^4y^2b_3 + 12x^3y^3a_3 + x^5b_1 - x^4ya_1 - 6x^4yb_1 + 6x^3}{x^2\,(x^2 + 3y^2)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^6a_2 - x^6a_3 + x^6b_3 - 2x^5ya_3 + 6x^5yb_2 - 9x^4y^2a_2 + 9x^4y^2b_3 - 12x^3y^3a_3 \\
&- x^5b_1 + x^4ya_1 + 6x^4yb_1 - 6x^3y^2a_1 - 6x^3y^2b_1 + 6x^2y^3a_1 - 9x\,y^4b_1 + 9y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^6 - 9a_2v_1^4v_2^2 - a_3v_1^6 - 2a_3v_1^5v_2 - 12a_3v_1^3v_2^3 + 6b_2v_1^5v_2 + b_3v_1^6 + 9b_3v_1^4v_2^2 \\
&+ a_1v_1^4v_2 - 6a_1v_1^3v_2^2 + 6a_1v_1^2v_2^3 + 9a_1v_2^5 - b_1v_1^5 + 6b_1v_1^4v_2 - 6b_1v_1^3v_2^2 - 9b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + b_3) v_1^6 + (-2a_3 + 6b_2) v_1^5 v_2 - b_1 v_1^5 + (-9a_2 + 9b_3) v_1^4 v_2^2 + (a_1 + 6b_1) v_1^4 v_2 - 12a_3 v_1^3 v_2^3 + (-6a_1 - 6b_1) v_1^3 v_2^2 + 6a_1 v_1^2 v_2^3 - 9b_1 v_1 v_2^4 + 9a_1 v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0 \\
9a_1 &= 0 \\
-12a_3 &= 0 \\
-9b_1 &= 0 \\
-b_1 &= 0 \\
-6a_1 - 6b_1 &= 0 \\
a_1 + 6b_1 &= 0 \\
-9a_2 + 9b_3 &= 0 \\
-2a_3 + 6b_2 &= 0 \\
-a_2 - a_3 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x^3 + y\,x^2 + 3y^3}{x\,(x^2+3y^2)}\right)(x) \\
&= -\frac{x^3}{x^2+3y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x^3}{x^2+3y^2}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{y\,x^2 + y^3}{x^3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x^3 + y\,x^2 + 3y^3}{x\,(x^2+3y^2)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y(x^2 + 3y^2)}{x^4} \\
S_y &= \frac{-x^2 - 3y^2}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

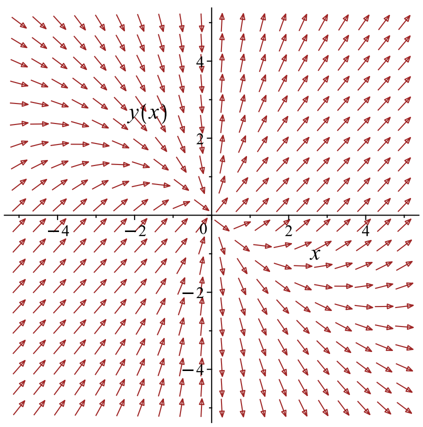
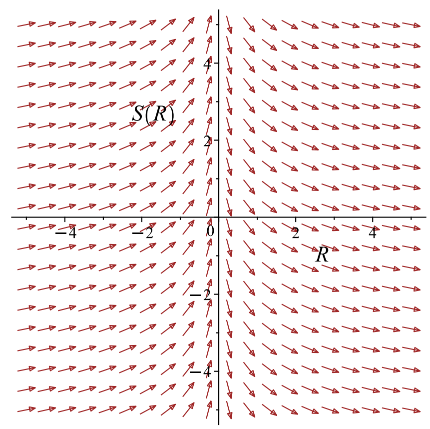
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{y(x^2 + y^2)}{x^3} = -\ln(x) + c_1
$$

Which simplifies to

$$
-\frac{y(x^2 + y^2)}{x^3} = -\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y\,x^2+3y^3}{x(x^2+3y^2)}$ 	$R = x$ $S = -\frac{y(x^2+y^2)}{x^3}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{y(x^2+y^2)}{x^3} = -\ln(x) + c_1 \tag{1}$$

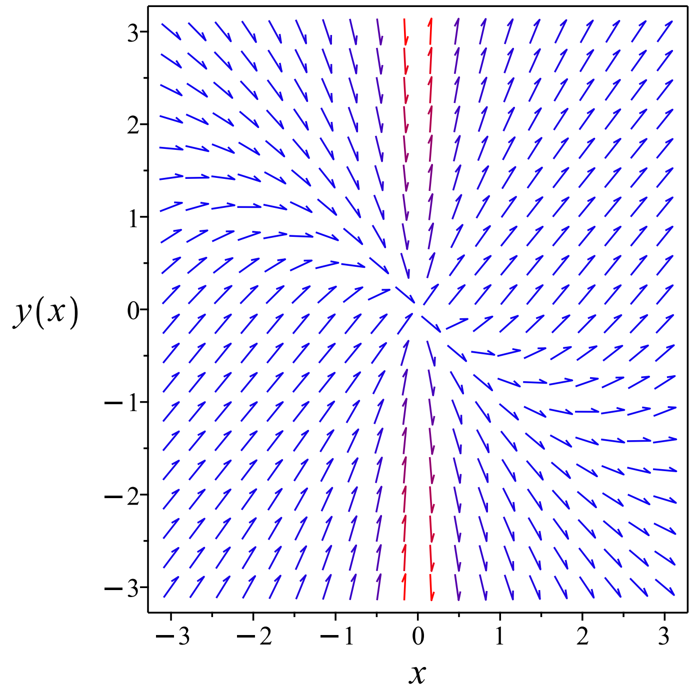


Figure 321: Slope field plot

Verification of solutions

$$-\frac{y(x^2+y^2)}{x^3} = -\ln(x) + c_1$$

Verified OK.

5.37.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x\left(x^2+3y^2\right)\right)\mathrm{d}y &= \left(x^3+y\,x^2+3y^3\right)\mathrm{d}x \\
\left(-x^3-y\,x^2-3y^3\right)\mathrm{d}x + \left(x\left(x^2+3y^2\right)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^3 - y\,x^2 - 3y^3 \\
N(x, y) &= x\left(x^2+3y^2\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 - y\,x^2 - 3y^3\right) \\
&= -x^2 - 9y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^2+3y^2\right)\right) \\
&= 3x^2 + 3y^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\,(x^2+3y^2)}\left(\left(-x^2-9y^2\right) - \left(3x^2+3y^2\right)\right) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(-x^3 - y\,x^2 - 3y^3\right) \\ &= \frac{-x^3 - y\,x^2 - 3y^3}{x^4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(x\left(x^2+3y^2\right)\right) \\ &= \frac{x^2+3y^2}{x^3} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^3 - y\,x^2 - 3y^3}{x^4}\right) + \left(\frac{x^2+3y^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^3 - y\,x^2 - 3y^3}{x^4} \, \mathrm{d}x$$

$$\phi = -\ln(x) + \frac{y}{x} + \frac{y^3}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + \frac{3y^2}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+3y^2}{x^3}$. Therefore equation (4) becomes

$$\frac{x^2 + 3y^2}{x^3} = \frac{1}{x} + \frac{3y^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{x} + \frac{y^3}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{x} + \frac{y^3}{x^3}$$

Summary
The solution(s) found are the following

$$\frac{y^3}{x^3}+\frac{y}{x}-\ln(x)=c_1 \tag{1}$$

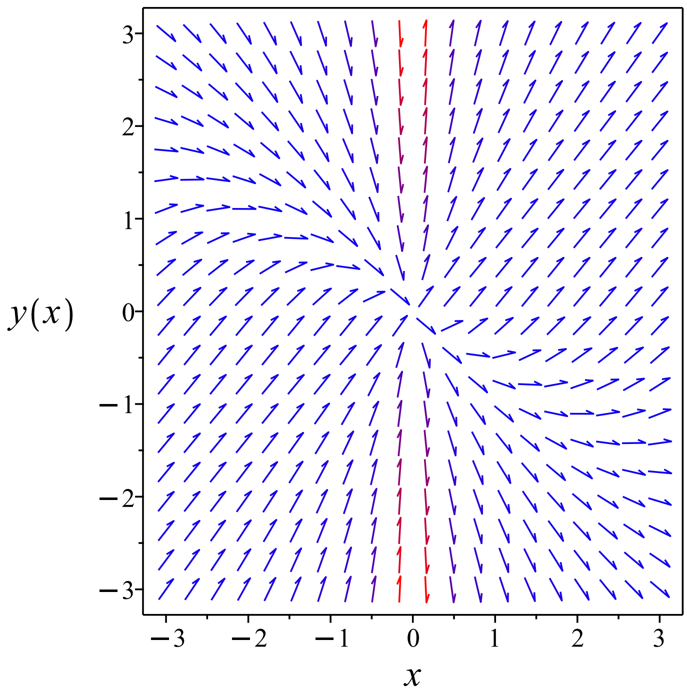


Figure 322: Slope field plot

Verification of solutions

$$\frac{y^3}{x^3}+\frac{y}{x}-\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 276

```
dsolve(diff(y(x),x)=(x^3+x^2*y(x)+3*y(x)^3)/(x^3+3*x*y(x)^2),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(x)c_1 + 81c_1^2}\right)^{\frac{2}{3}} - 12\right)x}{6\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(x)c_1 + 81c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(i\sqrt{3}\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(x)c_1 + 81c_1^2}\right)^{\frac{2}{3}} + 12i\sqrt{3} + \left(108\ln(x) + 1\ldots\right.\right.}{12\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(\ldots\right.}$$

$$y(x) = \frac{x\left(\left(i\sqrt{3} - 1\right)\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(x)c_1 + 81c_1^2}\right)^{\frac{2}{3}} + 12i\sqrt{3} + 12\right)}{12\left(108\ln(x) + 108c_1 + 12\sqrt{12 + 81\ln(x)^2 + 162\ln(x)c_1 + 81c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 27.532 (sec). Leaf size: 398

```
DSolve[y'[x]==(x^3+x^2*y[x]+3*y[x]^3)/(x^3+3*x*y[x]^2),y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{-2\sqrt[3]{3}x^2 + \sqrt[3]{2}\left(\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3\right)^{2/3}}{6^{2/3}\sqrt[3]{\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3}}$$

$$y(x) \to \frac{2\sqrt[3]{3}\left(1+i\sqrt{3}\right)x^2 + i\sqrt[3]{2}\left(\sqrt{3}+i\right)\left(\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3\right)^{2/3}}{2\ 6^{2/3}\sqrt[3]{\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3}}$$

$$y(x) \to \frac{2\sqrt[3]{3}\left(1-i\sqrt{3}\right)x^2 - \sqrt[3]{2}\left(1+i\sqrt{3}\right)\left(\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3\right)^{2/3}}{2\ 6^{2/3}\sqrt[3]{\sqrt{3}\sqrt{x^6\left(4+27(\log(x)+c_1)^2\right)} + 9x^3\log(x) + 9c_1x^3}}$$

5.38 problem 35(a)

Internal problem ID [1012]
Internal file name [`OUTPUT/1013_Sunday_June_05_2022_01_56_49_AM_26280276/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 35(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{y'x^2 - y^2 - yx = -4x^2}$$

With initial conditions

$$[y(-1) = 0]$$

5.38.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{-4x^2 + yx + y^2}{x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-4x^2 + yx + y^2}{x^2}\right) \\ &= \frac{x + 2y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.38.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 - u(x)\,x^2 = -4x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u^2 - 4}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 - 4$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2 - 4}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u^2 - 4}\,du &= \int \frac{1}{x}\,dx \\ \frac{\ln(u - 2)}{4} - \frac{\ln(u + 2)}{4} &= \ln(x) + c_2\end{aligned}$$

The above can be written as

$$\left(\frac{1}{4}\right)(\ln(u-2)-\ln(u+2)) = \ln(x)+2c_2$$

$$\begin{aligned}\ln(u-2)-\ln(u+2) &= (4)(\ln(x)+2c_2)\\ &= 4\ln(x)+8c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-2)-\ln(u+2)} = e^{4\ln(x)+4c_2}$$

Which simplifies to

$$\begin{aligned}\frac{u-2}{u+2} &= 4c_2x^4\\ &= c_3x^4\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= xu\\ &= -\frac{2x(c_3x^4+1)}{c_3x^4-1}\end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x=-1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2c_3+2}{c_3-1}$$

$$c_3 = -1$$

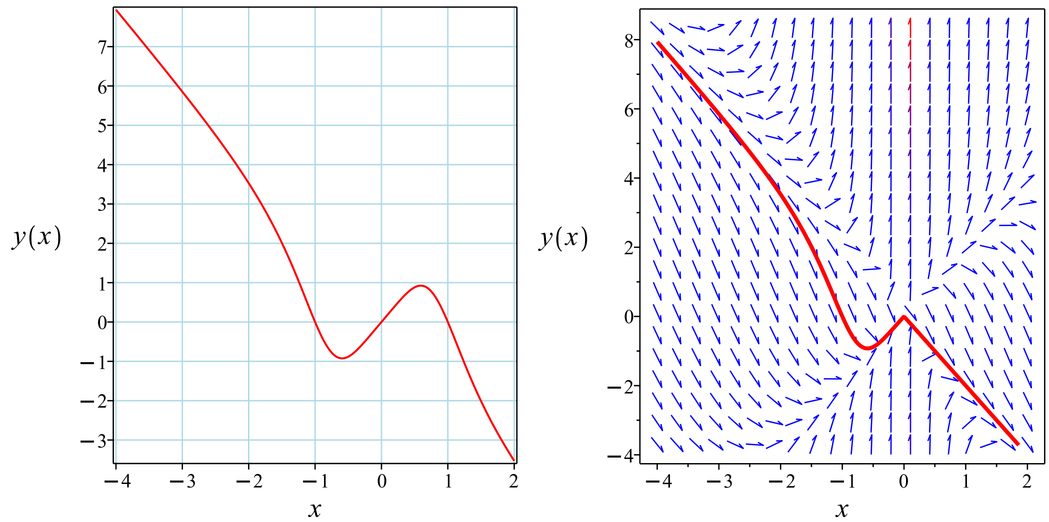
Substituting c_3 found above in the general solution gives

$$y = -\frac{2x(x^4-1)}{x^4+1}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x(x^4-1)}{x^4+1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2x(x^4 - 1)}{x^4 + 1}$$

Verified OK.

5.38.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-4x^2 + yx + y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{(-4x^2+yx+y^2)(b_3-a_2)}{x^2}-\frac{(-4x^2+yx+y^2)^2 a_3}{x^4} \\
&-\left(\frac{-8x+y}{x^2}-\frac{2(-4x^2+yx+y^2)}{x^3}\right)(xa_2+ya_3+a_1) \\
&-\frac{(x+2y)(xb_2+yb_3+b_1)}{x^2}=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{4x^4a_2-16x^4a_3-4x^4b_3+8x^3ya_3-2x^3yb_2+x^2y^2a_2+8x^2y^2a_3-x^2y^2b_3-y^4a_3-x^3b_1+x^2ya_1-2x^2yb_1}{x^4}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4x^4a_2-16x^4a_3-4x^4b_3+8x^3ya_3-2x^3yb_2+x^2y^2a_2+8x^2y^2a_3 \\
&-x^2y^2b_3-y^4a_3-x^3b_1+x^2ya_1-2x^2yb_1+2x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4a_2v_1^4+a_2v_1^2v_2^2-16a_3v_1^4+8a_3v_1^3v_2+8a_3v_1^2v_2^2-a_3v_2^4-2b_2v_1^3v_2 \\
&-4b_3v_1^4-b_3v_1^2v_2^2+a_1v_1^2v_2+2a_1v_1v_2^2-b_1v_1^3-2b_1v_1^2v_2=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}(4a_2 - 16a_3 - 4b_3)\, v_1^4 + (8a_3 - 2b_2)\, v_1^3 v_2 - b_1 v_1^3 \\ + (a_2 + 8a_3 - b_3)\, v_1^2 v_2^2 + (a_1 - 2b_1)\, v_1^2 v_2 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ a_1 - 2b_1 &= 0 \\ 8a_3 - 2b_2 &= 0 \\ a_2 + 8a_3 - b_3 &= 0 \\ 4a_2 - 16a_3 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{-4x^2 + yx + y^2}{x^2}\right)(x) \\ &= \frac{4x^2 - y^2}{x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x^2 - y^2}{x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x + y)}{4} - \frac{\ln(-2x + y)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-4x^2 + yx + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{4x^2 - y^2} \\ S_y &= \frac{x}{4x^2 - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2x+y)}{4} - \frac{\ln(-2x+y)}{4} = -\ln(x) + c_1$$

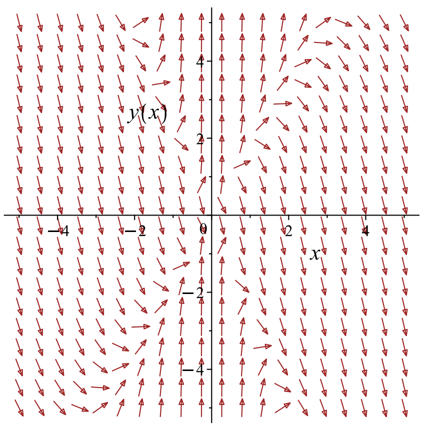
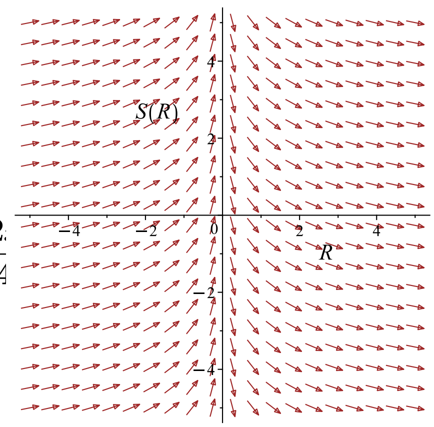
Which simplifies to

$$\frac{\ln(2x+y)}{4} - \frac{\ln(-2x+y)}{4} = -\ln(x) + c_1$$

Which gives

$$y = -\frac{2x(-e^{4c_1} - x^4)}{e^{4c_1} - x^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-4x^2+yx+y^2}{x^2}$ 	$R = x$ $S = \frac{\ln\left(2x+y\right)}{4} - \frac{\ln\left(-2\right.}{4}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-2\,e^{4c_1} - 2}{e^{4c_1} - 1}$$

$$c_1 = \frac{i\pi}{4}$$

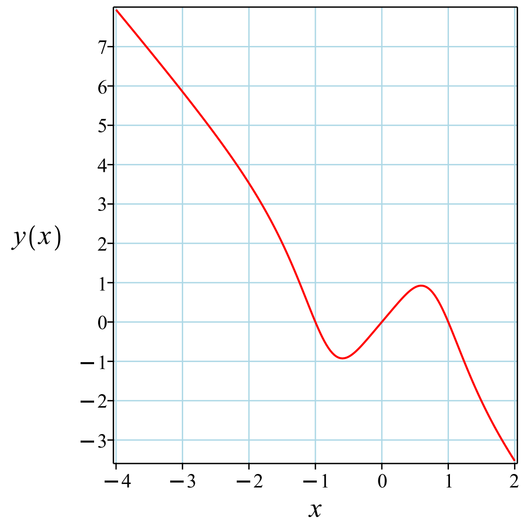
Substituting c_1 found above in the general solution gives

$$y = \frac{-2x^5 + 2x}{x^4 + 1}$$

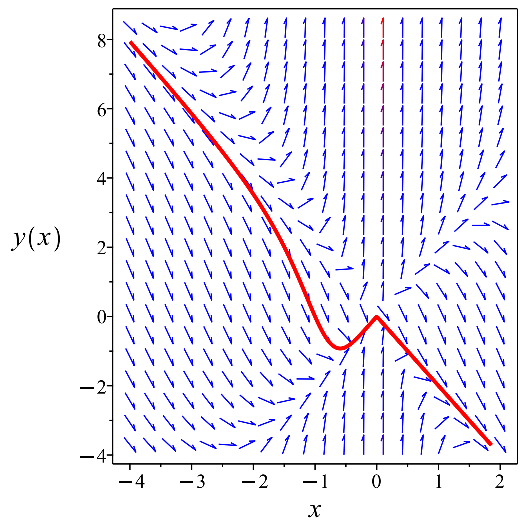
Summary

The solution(s) found are the following

$$y = \frac{-2x^5 + 2x}{x^4 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2x^5 + 2x}{x^4 + 1}$$

Verified OK.

5.38.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-4x^2 + yx + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -4 + \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -4$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= \frac{1}{x^3} \\ f_2^2f_0 &= -\frac{4}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} - \frac{4u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2x^4 + c_1}{x^2}$$

The above shows that

$$u'(x) = \frac{2c_2x^4 - 2c_1}{x^3}$$

Using the above in (1) gives the solution

$$y = -\frac{(2c_2x^4 - 2c_1)\,x}{c_2x^4 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2(-x^4 + c_3)\,x}{x^4 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{2(c_3 - 1)}{c_3 + 1}$$

$$c_3 = 1$$

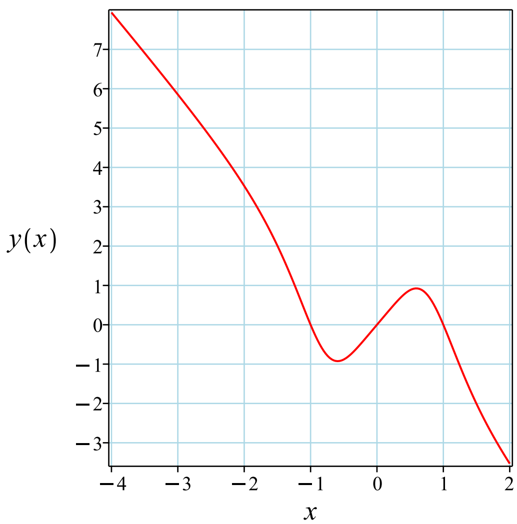
Substituting c_3 found above in the general solution gives

$$y = -\frac{2x(x^4 - 1)}{x^4 + 1}$$

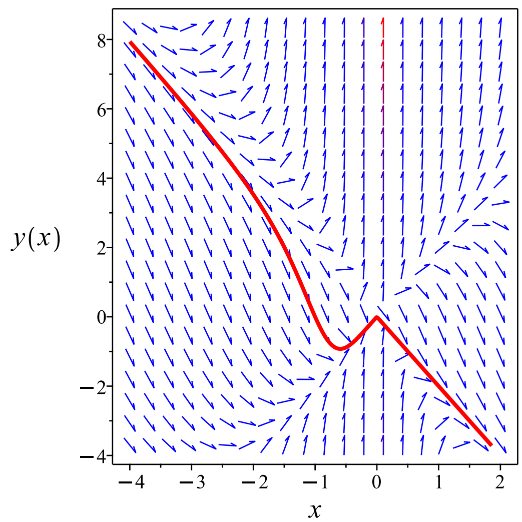
Summary

The solution(s) found are the following

$$y = -\frac{2x(x^4 - 1)}{x^4 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2x(x^4 - 1)}{x^4 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 19

```
dsolve([x^2*diff(y(x),x)=y(x)^2+x*y(x)-4*x^2,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{-2x^5 + 2x}{x^4 + 1}$$

✓ Solution by Mathematica

Time used: 2.186 (sec). Leaf size: 20

```
DSolve[{x^2*y'[x]==y[x]^2+x*y[x]-4*x^2,y[-1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x(x^4 - 1)}{x^4 + 1}$$

5.39 problem 36(a)

Internal problem ID [1013]
Internal file name [`OUTPUT/1014_Sunday_June_05_2022_01_56_51_AM_97637980/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 36(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y'xy - y^2 + yx = x^2}$$

5.39.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2u(x) - u(x)^2\,x^2 + u(x)\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u-1}{ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u-1}{u}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u-1}{u}}\,du = \int -\frac{1}{x}\,dx$$
$$u + \ln(u-1) = -\ln(x) + c_2$$

The solution is

$$u(x) + \ln(u(x)-1) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} + \ln\left(\frac{y}{x} - 1\right) + \ln(x) - c_2 = 0$$
$$\frac{y}{x} + \ln\left(\frac{-x+y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} + \ln\left(\frac{-x+y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

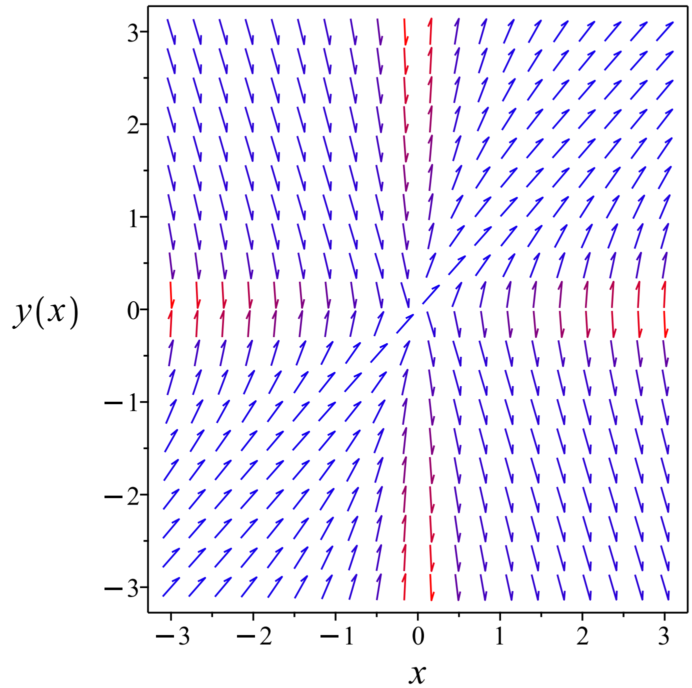


Figure 326: Slope field plot

Verification of solutions

$$\frac{y}{x}+\ln\left(\frac{-x+y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

5.39.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2-yx+y^2}{xy}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x^2 - yx + y^2)(b_3 - a_2)}{xy} - \frac{(x^2 - yx + y^2)^2 a_3}{x^2 y^2} - \left(\frac{2x - y}{xy} - \frac{x^2 - yx + y^2}{x^2 y}\right)(x a_2 + y a_3 + a_1) - \left(\frac{-x + 2y}{xy} - \frac{x^2 - yx + y^2}{x\,y^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 a_3 - x^4 b_2 + 2x^3 y a_2 - 2x^3 y a_3 - 2x^3 y b_3 - x^2 y^2 a_2 + 4x^2 y^2 a_3 + x^2 y^2 b_3 - 2x\,y^3 a_3 - x^3 b_1 + x^2 y a_1 + x\,y^2}{x^2 y^2} = 0$$

Setting the numerator to zero gives

$$-x^4 a_3 + x^4 b_2 - 2x^3 y a_2 + 2x^3 y a_3 + 2x^3 y b_3 + x^2 y^2 a_2 - 4x^2 y^2 a_3 - x^2 y^2 b_3 + 2x\,y^3 a_3 + x^3 b_1 - x^2 y a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_1^3 v_2 + a_2 v_1^2 v_2^2 - a_3 v_1^4 + 2a_3 v_1^3 v_2 - 4a_3 v_1^2 v_2^2 + 2a_3 v_1 v_2^3 + b_2 v_1^4 + 2b_3 v_1^3 v_2 - b_3 v_1^2 v_2^2 - a_1 v_1^2 v_2 + a_1 v_2^3 + b_1 v_1^3 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_3+b_2)\,v_1^4+(-2a_2+2a_3+2b_3)\,v_1^3v_2+b_1v_1^3\\ +(a_2-4a_3-b_3)\,v_1^2v_2^2-a_1v_1^2v_2+2a_3v_1v_2^3-b_1v_1v_2^2+a_1v_2^3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1&=0\\ b_1&=0\\ -a_1&=0\\ 2a_3&=0\\ -b_1&=0\\ -a_3+b_2&=0\\ -2a_2+2a_3+2b_3&=0\\ a_2-4a_3-b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=0\\ a_2&=b_3\\ a_3&=0\\ b_1&=0\\ b_2&=0\\ b_3&=b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi&=x\\ \eta&=y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{x^2 - yx + y^2}{xy}\right)(x) \\ &= \frac{-x^2 + yx}{y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 + yx}{y}} dy\end{aligned}$$

Which results in

$$S = \ln(-x + y) + \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 - yx + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^2 - yx + y^2}{x^2 (x - y)} \\
S_y &= -\frac{y}{x (x - y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln (-x + y) x + y}{x} = c_1
$$

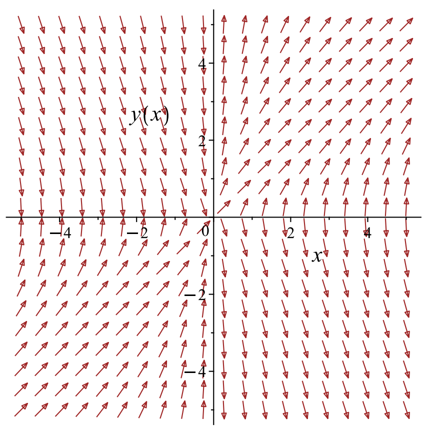
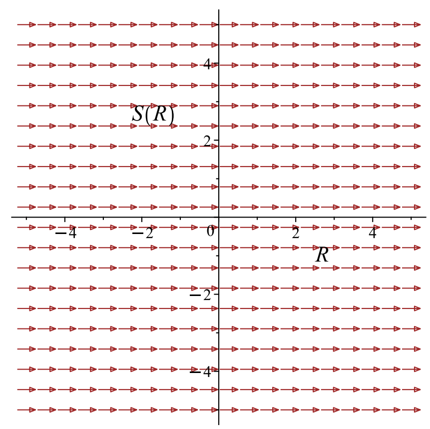
Which simplifies to

$$
\frac{\ln (-x + y) x + y}{x} = c_1
$$

Which gives

$$
y = x \operatorname{LambertW}\left(\frac{e^{c_1 - 1}}{x}\right) + x
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2-yx+y^2}{xy}$ 	$R = x$ $S = \frac{\ln(-x+y)\,x+y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = x \,\mathrm{LambertW}\left(\frac{\mathrm{e}^{c_1-1}}{x}\right) + x \tag{1}$$

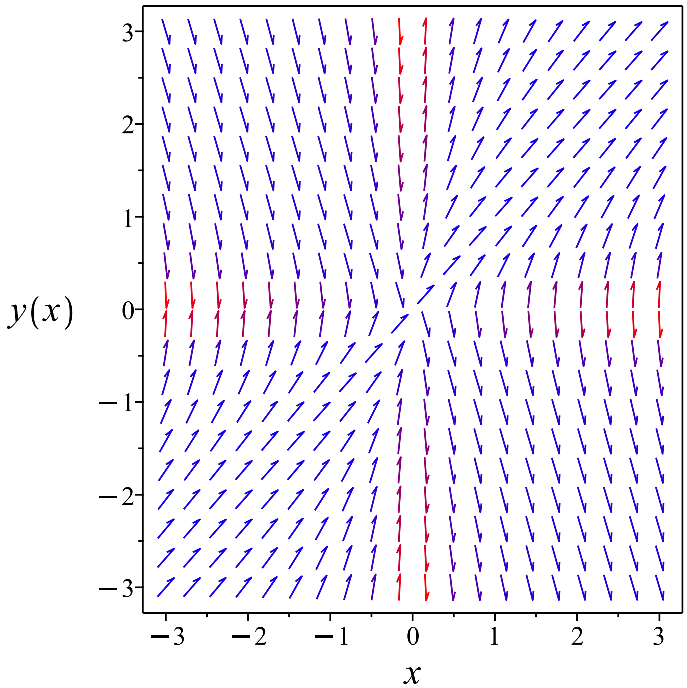


Figure 327: Slope field plot

Verification of solutions

$$y = x \operatorname{LambertW}\left(\frac{e^{c_1 - 1}}{x}\right) + x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(x*y(x)*diff(y(x),x)=x^2-x*y(x)+y(x)^2,y(x), singsol=all)
```

$$y(x) = x\left(1 + \mathrm{LambertW}\left(\frac{\mathrm{e}^{-1-c_1}}{x}\right)\right)$$

✓ Solution by Mathematica

Time used: 3.649 (sec). Leaf size: 25

```
DSolve[x*y[x]*y'[x]==x^2-x*y[x]+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(1 + W\left(\frac{e^{-1+c_1}}{x}\right)\right)$$
$$y(x) \to x$$

5.40 problem 37(a)

Internal problem ID [1014]
Internal file name [`OUTPUT/1015_Sunday_June_05_2022_01_56_53_AM_8622130/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 37(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{2y^2 - yx + 2x^2}{yx + 2x^2} = 0}$$

5.40.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2u(x)^2\,x^2 - u(x)\,x^2 + 2x^2}{u\,(x)\,x^2 + 2x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2 - 3u + 2}{x\,(u+2)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^2-3u+2}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-3u+2}{u+2}}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-3u+2}{u+2}}\,du = \int \frac{1}{x}\,dx$$

$$-3\ln(u-1) + 4\ln(u-2) = \ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-3\ln(u-1)+4\ln(u-2)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u-2)^4}{(u-1)^3} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \operatorname{RootOf}\left(\_Z^4 + (-c_3 x - 8)\_Z^3 + (3c_3 x + 24)\_Z^2 + (-3c_3 x - 32)\_Z + c_3 x + 16\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \operatorname{RootOf}\left(\_Z^4 + (-c_3 x - 8)\_Z^3 + (3c_3 x + 24)\_Z^2 + (-3c_3 x - 32)\_Z + c_3 x + 16\right) \tag{1}$$

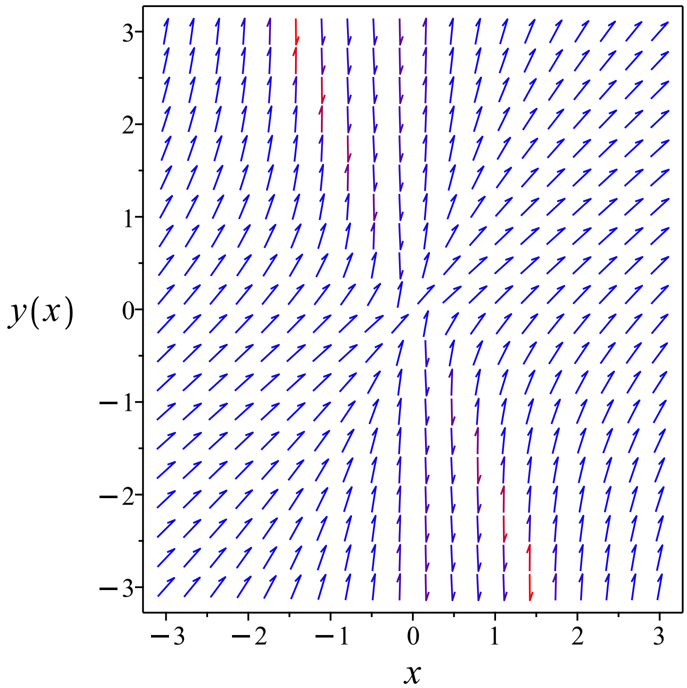


Figure 328: Slope field plot

Verification of solutions

$$y = x \operatorname{RootOf}\left(\_Z^4 + (-c_3x - 8)\_Z^3 + (3c_3x + 24)\_Z^2 + (-3c_3x - 32)\_Z + c_3x + 16\right)$$

Verified OK.

5.40.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2x^2 - yx + 2y^2}{x(2x + y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(2x^2 - yx + 2y^2)(b_3 - a_2)}{x(2x + y)} - \frac{(2x^2 - yx + 2y^2)^2 a_3}{x^2 (2x + y)^2} \\
&- \left(\frac{4x - y}{x(2x + y)} - \frac{2x^2 - yx + 2y^2}{x^2 (2x + y)} - \frac{2(2x^2 - yx + 2y^2)}{x(2x + y)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{-x + 4y}{x(2x + y)} - \frac{2x^2 - yx + 2y^2}{x(2x + y)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^4a_2 + 4x^4a_3 - 8x^4b_2 - 4x^4b_3 + 4x^3ya_2 - 4x^3ya_3 + 4x^3yb_2 - 4x^3yb_3 - 5x^2y^2a_2 + 13x^2y^2a_3 + x^2y^2b_2 +}{x^2 (2x + y)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-4x^4a_2 - 4x^4a_3 + 8x^4b_2 + 4x^4b_3 - 4x^3ya_2 + 4x^3ya_3 - 4x^3yb_2 \\
&+ 4x^3yb_3 + 5x^2y^2a_2 - 13x^2y^2a_3 - x^2y^2b_2 - 5x^2y^2b_3 + 12x\,y^3a_3 \\
&- 2y^4a_3 + 4x^3b_1 - 4x^2ya_1 - 8x^2yb_1 + 8x\,y^2a_1 - 2x\,y^2b_1 + 2y^3a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-4a_2v_1^4 - 4a_2v_1^3v_2 + 5a_2v_1^2v_2^2 - 4a_3v_1^4 + 4a_3v_1^3v_2 - 13a_3v_1^2v_2^2 + 12a_3v_1v_2^3 \\
&- 2a_3v_2^4 + 8b_2v_1^4 - 4b_2v_1^3v_2 - b_2v_1^2v_2^2 + 4b_3v_1^4 + 4b_3v_1^3v_2 - 5b_3v_1^2v_2^2 \\
&- 4a_1v_1^2v_2 + 8a_1v_1v_2^2 + 2a_1v_2^3 + 4b_1v_1^3 - 8b_1v_1^2v_2 - 2b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-4a_2 - 4a_3 + 8b_2 + 4b_3)\, v_1^4 + (-4a_2 + 4a_3 - 4b_2 + 4b_3)\, v_1^3 v_2 \\
&+ 4b_1 v_1^3 + (5a_2 - 13a_3 - b_2 - 5b_3)\, v_1^2 v_2^2 + (-4a_1 - 8b_1)\, v_1^2 v_2 \\
&+ 12a_3 v_1 v_2^3 + (8a_1 - 2b_1)\, v_1 v_2^2 - 2a_3 v_2^4 + 2a_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-2a_3 &= 0 \\
12a_3 &= 0 \\
4b_1 &= 0 \\
-4a_1 - 8b_1 &= 0 \\
8a_1 - 2b_1 &= 0 \\
-4a_2 - 4a_3 + 8b_2 + 4b_3 &= 0 \\
-4a_2 + 4a_3 - 4b_2 + 4b_3 &= 0 \\
5a_2 - 13a_3 - b_2 - 5b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{2x^2 - yx + 2y^2}{x\,(2x+y)}\right)(x) \\ &= \frac{-2x^2 + 3yx - y^2}{2x+y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2+3yx-y^2}{2x+y}} dy\end{aligned}$$

Which results in

$$S = 3\ln(-x+y) - 4\ln(-2x+y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x^2 - yx + 2y^2}{x\,(2x+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3}{-x+y} + \frac{8}{-2x+y} \\
S_y &= \frac{3}{-x+y} - \frac{4}{-2x+y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -2\ln(R) + c_1 \tag{4}
$$

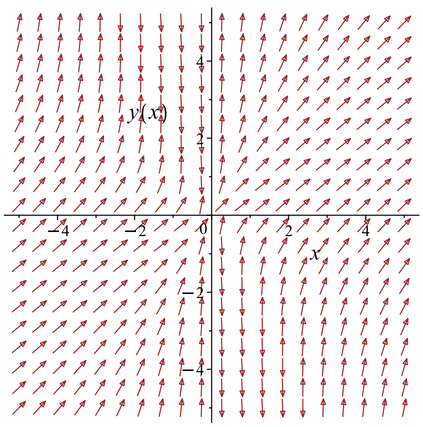
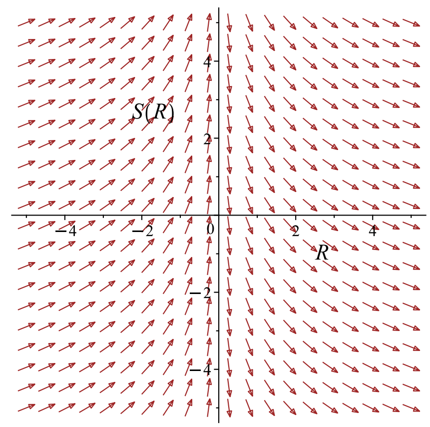
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
3\ln(-x+y) - 4\ln(-2x+y) = -2\ln(x) + c_1
$$

Which simplifies to

$$
3\ln(-x+y) - 4\ln(-2x+y) = -2\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^2-yx+2y^2}{x(2x+y)}$ 	$R = x$ $S = 3\ln(-x+y) - 4\ln($	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$3\ln(-x+y) - 4\ln(-2x+y) = -2\ln(x) + c_1 \tag{1}$$

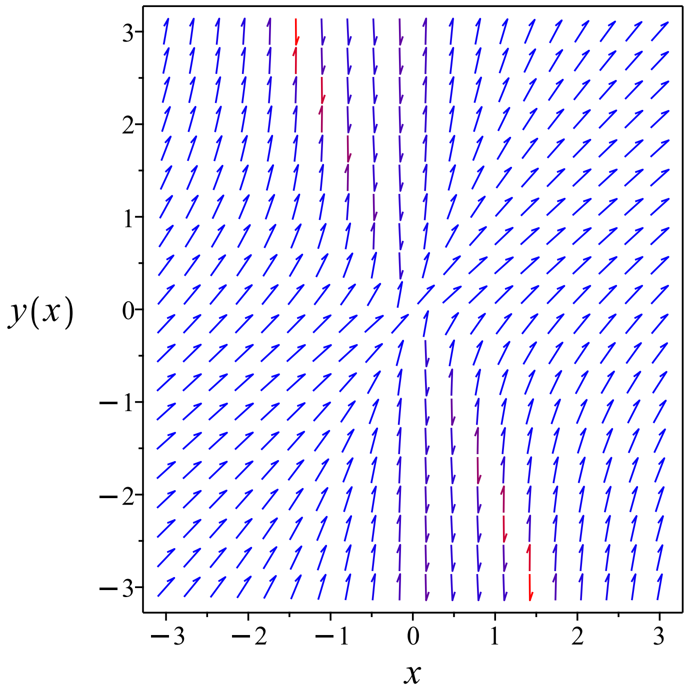


Figure 329: Slope field plot

Verification of solutions

$$3\ln(-x+y) - 4\ln(-2x+y) = -2\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(diff(y(x),x)=(2*y(x)^2-x*y(x)+2*x^2)/(x*y(x)+2*x^2),y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(\_Z^4 + c_1x + 16 + (-3c_1x - 32)\_Z + (3c_1x + 24)\_Z^2 + (-c_1x - 8)\_Z^3\right) x$$

✓ Solution by Mathematica

Time used: 60.166 (sec). Leaf size: 1913

```
DSolve[y'[x]==(2*y[x]^2-x*y[x]+2*x^2)/(x*y[x]+2*x^2),y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{12}\left(-\sqrt{9e^{2c_1}x^4 + 6\ 2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{e^{3c_1}x^{15}\left(-256+27e^{c_1}x\right)}+27e^{2c_1}x^8} + 24e^{c_1}x^3\left(-3+\frac{\cdots}{\sqrt[3]{\sqrt{3}\sqrt{e^{3c_1}x^{15}\cdots}}}\right.}\right.$$

$$-6\sqrt{\frac{1}{2}x^2\left(-8+e^{c_1}x\right)^2 + 4x^2\left(-8+e^{c_1}x\right) - \frac{\sqrt[3]{27e^{2c_1}x^8+\sqrt{729e^{4c_1}x^{16}-6912e^{3c_1}x^{15}}}}{3\sqrt[3]{2}} - \frac{\cdots}{\sqrt[3]{\sqrt{3}\sqrt{e^{3c_1}x^{15}\cdots}}}}$$

$$\left. - 3x\left(-8+e^{c_1}x\right)\right)$$

$$y(x) \to \frac{1}{12}\left(-\sqrt{9e^{2c_1}x^4 + 6\ 2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{e^{3c_1}x^{15}\left(-256+27e^{c_1}x\right)}+27e^{2c_1}x^8} + 24e^{c_1}x^3\left(-3+\frac{\cdots}{\sqrt[3]{\sqrt{3}\sqrt{e^{3c_1}x^{15}\cdots}}}\right.}\right.$$

$$+6\sqrt{\frac{1}{2}x^2\left(-8+e^{c_1}x\right)^2 + 4x^2\left(-8+e^{c_1}x\right) - \frac{\sqrt[3]{27e^{2c_1}x^8+\sqrt{729e^{4c_1}x^{16}-6912e^{3c_1}x^{15}}}}{3\sqrt[3]{2}} - \frac{\cdots}{\sqrt[3]{\sqrt{3}\sqrt{e^{3c_1}x^{15}\cdots}}}}$$

5.41 problem 38

Internal problem ID [1015]
Internal file name [`OUTPUT/1016_Sunday_June_05_2022_01_56_55_AM_38015959/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{x^2 + yx + y^2}{yx} = 0}$$

5.41.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 + u(x)\,x^2 + u(x)^2\,x^2}{u\,(x)\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u+1}{xu} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u+1}{u}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u+1}{u}}\,du = \int \frac{1}{x}\,dx$$
$$u - \ln(u+1) = \ln(x) + c_2$$

The solution is

$$u(x) - \ln(u(x)+1) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} - \ln\left(1 + \frac{y}{x}\right) - \ln(x) - c_2 = 0$$
$$\frac{y}{x} - \ln\left(\frac{x+y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{x+y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

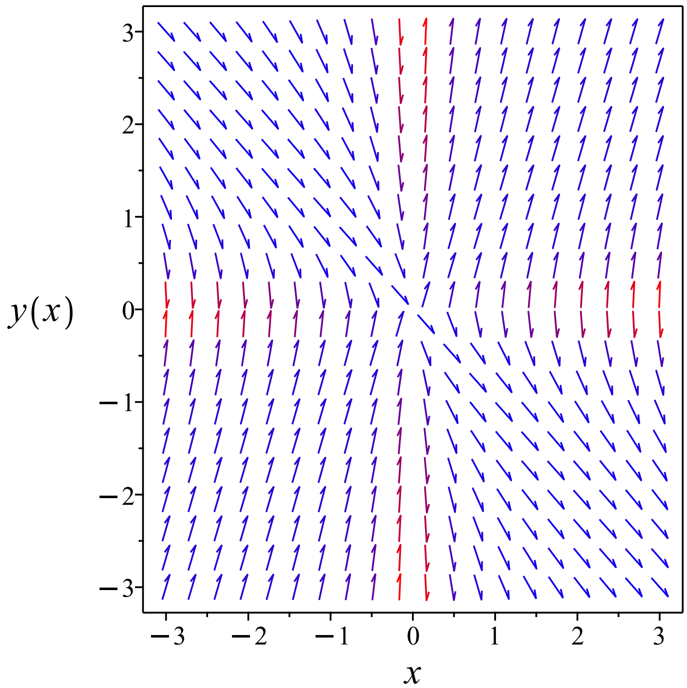


Figure 330: Slope field plot

Verification of solutions

$$\frac{y}{x} - \ln\left(\frac{x+y}{x}\right) - \ln(x) - c_2 = 0$$

Verified OK.

5.41.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + yx + y^2}{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{\left(x^2+y x+y^2\right)\left(b_3-a_2\right)}{y x}-\frac{\left(x^2+y x+y^2\right)^2 a_3}{y^2 x^2} \\ & -\left(\frac{2 x+y}{y x}-\frac{x^2+y x+y^2}{y\, x^2}\right)\left(x a_2+y a_3+a_1\right) \\ & -\left(\frac{x+2 y}{y x}-\frac{x^2+y x+y^2}{y^2 x}\right)\left(x b_2+y b_3+b_1\right)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 a_3-x^4 b_2+2 x^3 y a_2+2 x^3 y a_3-2 x^3 y b_3+x^2 y^2 a_2+4 x^2 y^2 a_3-x^2 y^2 b_3+2 x\, y^3 a_3-x^3 b_1+x^2 y a_1+x\, y^2}{x^2 y^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4 a_3+x^4 b_2-2 x^3 y a_2-2 x^3 y a_3+2 x^3 y b_3-x^2 y^2 a_2-4 x^2 y^2 a_3 \\ & +x^2 y^2 b_3-2 x\, y^3 a_3+x^3 b_1-x^2 y a_1-x\, y^2 b_1+y^3 a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2 a_2 v_1^3 v_2-a_2 v_1^2 v_2^2-a_3 v_1^4-2 a_3 v_1^3 v_2-4 a_3 v_1^2 v_2^2-2 a_3 v_1 v_2^3 \\ & +b_2 v_1^4+2 b_3 v_1^3 v_2+b_3 v_1^2 v_2^2-a_1 v_1^2 v_2+a_1 v_2^3+b_1 v_1^3-b_1 v_1 v_2^2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_3+b_2)\,v_1^4+(-2a_2-2a_3+2b_3)\,v_1^3v_2+b_1v_1^3\\ +(-a_2-4a_3+b_3)\,v_1^2v_2^2-a_1v_1^2v_2-2a_3v_1v_2^3-b_1v_1v_2^2+a_1v_2^3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1&=0\\ b_1&=0\\ -a_1&=0\\ -2a_3&=0\\ -b_1&=0\\ -a_3+b_2&=0\\ -2a_2-2a_3+2b_3&=0\\ -a_2-4a_3+b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=0\\ a_2&=b_3\\ a_3&=0\\ b_1&=0\\ b_2&=0\\ b_3&=b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi&=x\\ \eta&=y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 + yx + y^2}{yx}\right)(x) \\
&= \frac{-x^2 - yx}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - yx}{y}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(x + y) - \frac{y}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^2 + yx + y^2}{yx}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^2 + yx + y^2}{x^2 (x + y)} \\
S_y &= -\frac{y}{x (x + y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (x + y) x - y}{x} = c_1$$

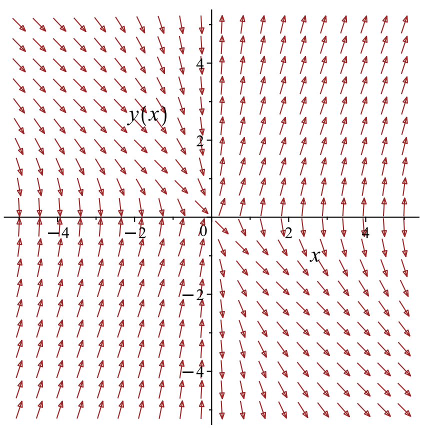
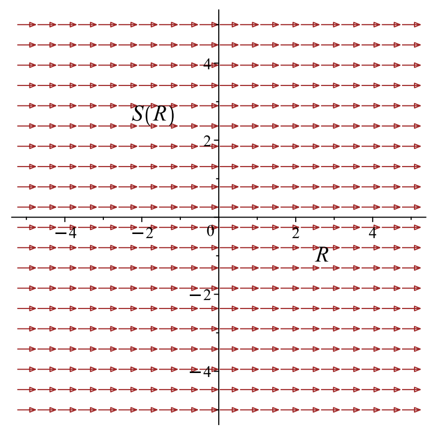
Which simplifies to

$$\frac{\ln (x + y) x - y}{x} = c_1$$

Which gives

$$y = -x \operatorname{LambertW}\left(-\frac{e^{c_1 - 1}}{x}\right) - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+yx+y^2}{yx}$ 	$R = x$ $S = \frac{\ln(x+y)\,x - y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = -x \operatorname{LambertW}\left(-\frac{e^{c_1-1}}{x}\right) - x \tag{1}$$

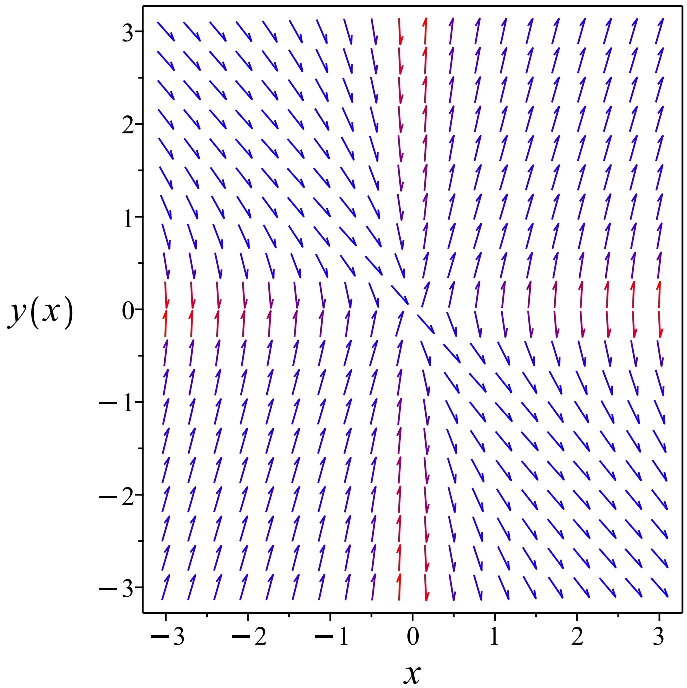


Figure 331: Slope field plot

Verification of solutions

$$y = -x \operatorname{LambertW}\left(-\frac{e^{c_1-1}}{x}\right) - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x)=(x*y(x)+x^2+y(x)^2)/(x*y(x)),y(x), singsol=all)
```

$$y(x) = x\left(-\operatorname{LambertW}\left(-\frac{e^{-1-c_1}}{x}\right)-1\right)$$

✓ Solution by Mathematica

Time used: 4.422 (sec). Leaf size: 31

```
DSolve[y'[x]==(x*y[x]+x^2+y[x]^2)/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\left(1+W\left(-\frac{e^{-1-c_1}}{x}\right)\right)$$

$$y(x) \to -x$$

5.42 problem 41

Internal problem ID [1016]
Internal file name [`OUTPUT/1017_Sunday_June_05_2022_01_56_56_AM_54647075/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-6x + y - 3}{2x - y - 1} = 0}$$

5.42.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -6, b_1 = 1, c_1 = -3, a_2 = 2, b_2 = -1, c_2 = -1$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
-6x_0 + y_0 - 3 &= 0 \\
2x_0 - y_0 - 1 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= -3
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x + 1 \\
Y &= y + 3
\end{aligned}
$$

Using this transformation in $y' - \frac{-6x+y-3}{2x-y-1} = 0$ result in

$$
\frac{dY}{dX} = \frac{-6X + Y}{2X - Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{-6X + Y}{-2X + Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -6X + Y$ and $N = 2X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u+6}{u-2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)+6}{u(X)-2} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)+6}{u(X)-2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 - u(X) - 6 = 0$$

Or

$$X(u(X) - 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - u(X) - 6 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - u - 6}{X\,(u-2)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-u-6}{u-2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-u-6}{u-2}}\,du &= -\frac{1}{X}\,dX \\ \int \frac{1}{\frac{u^2-u-6}{u-2}}\,du &= \int -\frac{1}{X}\,dX \\ \frac{\ln(u-3)}{5} + \frac{4\ln(u+2)}{5} &= -\ln(X) + c_3\end{aligned}$$

The above can be written as

$$\frac{\ln (u-3)+4 \ln (u+2)}{5}=-\ln (X)+c_3$$
$$\ln (u-3)+4 \ln (u+2)=(5)(-\ln (X)+c_3)$$
$$=-5 \ln (X)+5 c_3$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln (u-3)+4 \ln (u+2)}=\mathrm{e}^{-5 \ln (X)+5 c_3}$$

Which simplifies to

$$(u-3)(u+2)^4=\frac{5 c_3}{X^5}$$
$$=\frac{c_4}{X^5}$$

Which simplifies to

$$u(X)=\operatorname{RootOf}\left(\_Z^5+5\_Z^4-\frac{c_4 \mathrm{e}^{5 c_3}}{X^5}-40\_Z^2-80\_Z-48\right)$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$Y(X)=X \operatorname{RootOf}\left(\_Z^5 X^5+5\_Z^4 X^5-40\_Z^2 X^5-c_4 \mathrm{e}^{5 c_3}-80\_Z X^5-48 X^5\right)$$

The solution is

$$Y(X)=X \operatorname{RootOf}\left(\_Z^5 X^5+5\_Z^4 X^5-40\_Z^2 X^5-c_4 \mathrm{e}^{5 c_3}-80\_Z X^5-48 X^5\right)$$

Replacing $Y=y-y_0, X=x-x_0$ gives

$$3+y=(x+1) \operatorname{RootOf}\left(\_Z^5(x+1)^5+5\_Z^4(x+1)^5-40\_Z^2(x+1)^5-c_4 \mathrm{e}^{5 c_3}-80\_Z(x+1)^5-48(x+\right.$$

Or

$$y=(x+1) \operatorname{RootOf}\left(\_Z^5(x+1)^5+5\_Z^4(x+1)^5-40\_Z^2(x+1)^5-c_4 \mathrm{e}^{5 c_3}-80\_Z(x+1)^5-48(x+1)\right.$$

Summary

The solution(s) found are the following

$$\begin{aligned} y=(x+1) \operatorname{RootOf}\Big(&\left(x^5+5 x^4+10 x^3+10 x^2+5 x+1\right)\_Z^5 \\ &+\left(5 x^5+25 x^4+50 x^3+50 x^2+25 x+5\right)\_Z^4 \\ &+\left(-40 x^5-200 x^4-400 x^3-400 x^2-200 x-40\right)\_Z^2 \\ &+\left(-80 x^5-400 x^4-800 x^3-800 x^2-400 x-80\right)\_Z-48 x^5-240 x^4-480 x^3 \\ &-c_4 \mathrm{e}^{5 c_3}-480 x^2-240 x-48\Big)-3 \end{aligned} \tag{1}$$

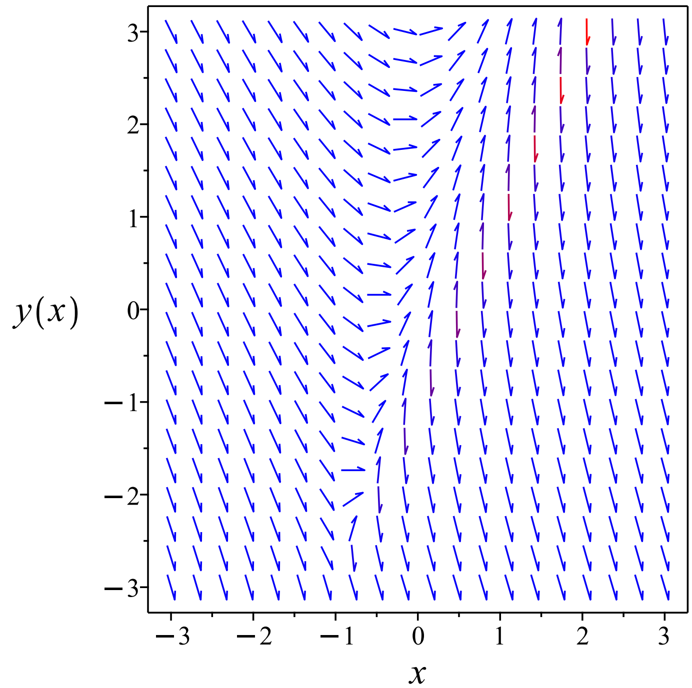


Figure 332: Slope field plot

Verification of solutions

$$y = (x+1)\,\mathrm{RootOf}\Big(\big(x^5+5x^4+10x^3+10x^2+5x+1\big)\_Z^5 + \big(5x^5+25x^4+50x^3+50x^2+25x+5\big)\_Z^4 + \big(-40x^5-200x^4-400x^3-400x^2-200x-40\big)\_Z^2 + \big(-80x^5-400x^4-800x^3-800x^2-400x-80\big)\_Z - 48x^5 - 240x^4 - 480x^3 - c_4\mathrm{e}^{5c_3} - 480x^2 - 240x - 48\Big) - 3$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.422 (sec). Leaf size: 99

```
dsolve(diff(y(x),x)=(-6*x+y(x)-3)/(2*x-y(x)-1),y(x), singsol=all)
```

$$y(x) = \frac{-\operatorname{RootOf}\left(\_Z^{25} + \left(-5c_1x^5 - 25c_1x^4 - 50c_1x^3 - 50c_1x^2 - 25c_1x - 5c_1\right)\_Z^5 - c_1x^5 - 5c_1x^4 - 10c_1x^3 -}{c_1\left(x+1\right)^4}$$

✓ Solution by Mathematica

Time used: 60.095 (sec). Leaf size: 3011

```
DSolve[y'[x]==(-6*x+y[x]-3)/(2*x-y[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

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5.43 problem 42

Internal problem ID [1017]
Internal file name [`OUTPUT/1018_Sunday_June_05_2022_01_56_58_AM_44778598/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x + y + 1}{x + 2y - 4} = 0}$$

5.43.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = 2, b_1 = 1, c_1 = 1, a_2 = 1, b_2 = 2, c_2 = -4$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$
\begin{aligned}
X &= x - x_0 \\
Y &= y - y_0
\end{aligned}
$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$
\begin{aligned}
a_1 x_0 + b_1 y_0 + c_1 &= 0 \\
a_2 x_0 + b_2 y_0 + c_2 &= 0
\end{aligned}
$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$
\begin{aligned}
2x_0 + y_0 + 1 &= 0 \\
x_0 + 2y_0 - 4 &= 0
\end{aligned}
$$

Solving for x_0, y_0 from the above gives

$$
\begin{aligned}
x_0 &= -2 \\
y_0 &= 3
\end{aligned}
$$

Therefore the transformation becomes

$$
\begin{aligned}
X &= x + 2 \\
Y &= y - 3
\end{aligned}
$$

Using this transformation in $y' - \frac{2x+y+1}{x+2y-4} = 0$ result in

$$
\frac{dY}{dX} = \frac{2X + Y}{X + 2Y}
$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{2X + Y}{X + 2Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 2X + Y$ and $N = X + 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u+2}{2u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+2}{2u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+2}{2u(X)+1} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + 2u(X)^2 - 2 = 0$$

Or

$$-2 + X(2u(X)+1)\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{2(u^2-1)}{X\,(2u+1)}\end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = \frac{u^2-1}{2u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-1}{2u+1}}\,du &= -\frac{2}{X}\,dX\\ \int\frac{1}{\frac{u^2-1}{2u+1}}\,du &= \int -\frac{2}{X}\,dX\\ \frac{3\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -2\ln(X) + c_3\end{aligned}$$

The above can be written as

$$\begin{aligned}\frac{3\ln(u-1)+\ln(u+1)}{2} &= -2\ln(X)+c_3\\ 3\ln(u-1)+\ln(u+1) &= (2)(-2\ln(X)+c_3)\\ &= -4\ln(X)+2c_3\end{aligned}$$

Raising both side to exponential gives

$$e^{3\ln(u-1)+\ln(u+1)} = e^{-4\ln(X)+2c_3}$$

Which simplifies to

$$\begin{aligned}(u-1)^3(u+1) &= \frac{2c_3}{X^4}\\ &= \frac{c_4}{X^4}\end{aligned}$$

Which simplifies to

$$u(X) = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - \frac{c_4 e^{2c_3}}{X^4} + 2\_Z - 1\right)$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X\,\text{RootOf}\left(\_Z^4 X^4 - 2\_Z^3 X^4 + 2\_Z X^4 - X^4 - c_4 e^{2c_3}\right)$$

The solution is

$$Y(X) = X\,\text{RootOf}\left(\_Z^4 X^4 - 2\_Z^3 X^4 + 2\_Z X^4 - X^4 - c_4 e^{2c_3}\right)$$

Replacing $Y = y - y_0, X = x - x_0$ gives

$$-3+y = (2+x)\,\text{RootOf}\left(\_Z^4(2+x)^4 - 2\_Z^3(2+x)^4 + 2\_Z(2+x)^4 - (2+x)^4 - c_4 e^{2c_3}\right)$$

Or

$$y = (2+x)\,\text{RootOf}\left(\_Z^4(2+x)^4 - 2\_Z^3(2+x)^4 + 2\_Z(2+x)^4 - (2+x)^4 - c_4 e^{2c_3}\right)+3$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}y = (2+x)\,\text{RootOf}\Big(&\left(x^4+8x^3+24x^2+32x+16\right)\_Z^4\\ &+\left(-2x^4-16x^3-48x^2-64x-32\right)\_Z^3+\left(2x^4+16x^3+48x^2+64x+32\right)\_Z\\ &-x^4-8x^3-c_4e^{2c_3}-24x^2-32x-16\Big)+3\end{aligned} \tag{1}$$

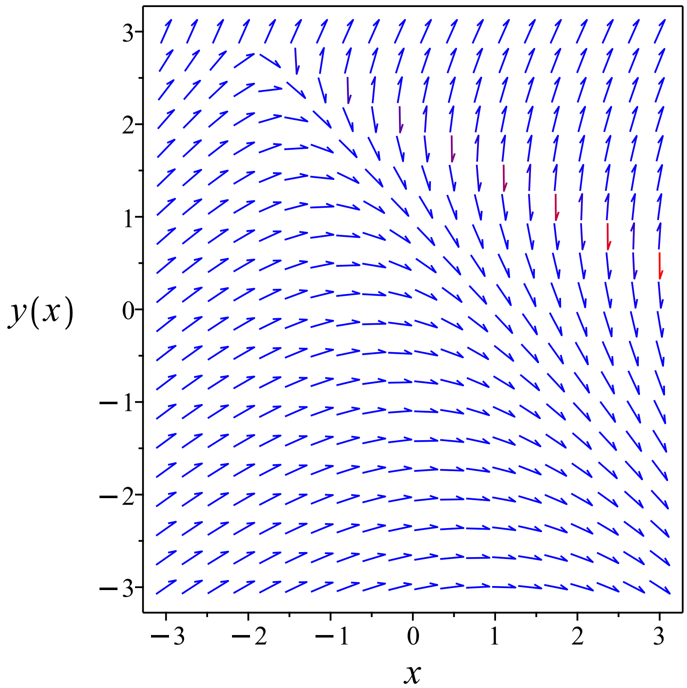

Figure 333: Slope field plot

Verification of solutions

$$y = (2+x)\,\mathrm{RootOf}\left(\left(x^4+8x^3+24x^2+32x+16\right)\_Z^4 + \left(-2x^4-16x^3-48x^2-64x-32\right)\_Z^3 + \left(2x^4+16x^3+48x^2+64x+32\right)\_Z - x^4-8x^3-c_4 e^{2c_3}-24x^2-32x-16\right)+3$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 4.359 (sec). Leaf size: 138

```
dsolve(diff(y(x),x)=(2*x+y(x)+1)/(x+2*y(x)-4),y(x), singsol=all)
```

$$y(x) = \frac{(x+5)\operatorname{RootOf}\left(\_Z^{16} + (2c_1x^4 + 16c_1x^3 + 48c_1x^2 + 64c_1x + 32c_1)\_Z^4 - c_1x^4 - 8c_1x^3 - 24c_1x^2 - 32c_1\right.}{\operatorname{RootOf}\left(\_Z^{16} + (2c_1x^4 + 16c_1x^3 + 48c_1x^2 + 64c_1x + 32c_1)\_Z^4 - c_1x^4 - 8c_1x^3 - 24c_1x^2 - 32c\right.}$$

✓ Solution by Mathematica

Time used: 60.312 (sec). Leaf size: 8077

```
DSolve[y'[x]==(2*x+y[x]+1)/(x+2*y[x]-4),y[x],x,IncludeSingularSolutions -> True]
```

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5.44 problem 43

Internal problem ID [1018]
Internal file name [`OUTPUT/1019_Sunday_June_05_2022_01_57_02_AM_87412369/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-x + 3y - 14}{x + y - 2} = 0}$$

5.44.1 Solving as polynomial ode

This is ODE of type polynomial. Where the RHS of the ode is ratio of equations of two lines. Writing the ODE in the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_3}$$

Where $a_1 = -1, b_1 = 3, c_1 = -14, a_2 = 1, b_2 = 1, c_2 = -2$. There are now two possible solution methods. The first case is when the two lines $a_1x + b_1y + c_1$,$a_2x + b_2y + c_3$ are not parallel and the second case is if they are parallel. If they are not parallel, then the transformation $X = x - x_0$, $Y = y - y_0$ converts the ODE to a homogeneous ODE. The values x_0, y_0 have to be determined. If they are parallel then a transformation $U(x) = a_1x + b_1y$ converts the given ODE in y to a separable ODE in $U(x)$. The first case is when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ and the second case when $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. From the above we see that

$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Hence this is case one where lines are not parallel. Using the transformation

$$\begin{aligned} X &= x - x_0 \\ Y &= y - y_0 \end{aligned}$$

Where the constants x_0, y_0 are obtained by solving the following two linear algebraic equations

$$\begin{aligned} a_1 x_0 + b_1 y_0 + c_1 &= 0 \\ a_2 x_0 + b_2 y_0 + c_2 &= 0 \end{aligned}$$

Substituting the values for $a_1, b_1, c_1, a_2, b_2, c_2$ gives

$$\begin{aligned} -x_0 + 3y_0 - 14 &= 0 \\ x_0 + y_0 - 2 &= 0 \end{aligned}$$

Solving for x_0, y_0 from the above gives

$$\begin{aligned} x_0 &= -2 \\ y_0 &= 4 \end{aligned}$$

Therefore the transformation becomes

$$\begin{aligned} X &= x + 2 \\ Y &= y - 4 \end{aligned}$$

Using this transformation in $y' - \frac{-x+3y-14}{x+y-2} = 0$ result in

$$\frac{dY}{dX} = \frac{-X + 3Y}{X + Y}$$

This is now a homogeneous ODE which will now be solved for $Y(X)$. In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-X + 3Y}{X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X + 3Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{3u - 1}{u + 1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{3u(X)-1}{u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{3u(X)-1}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + u(X)^2 - 2u(X) + 1 = 0$$

Or

$$X(u(X) + 1)\left(\frac{d}{dX}u(X)\right) + (u(X) - 1)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X, u)\\ &= f(X)g(u)\\ &= -\frac{(u - 1)^2}{X\,(u + 1)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u-1)^2}{u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{(u-1)^2}{u+1}}\,du &= -\frac{1}{X}\,dX\\ \int \frac{1}{\frac{(u-1)^2}{u+1}}\,du &= \int -\frac{1}{X}\,dX\\ \ln{(u - 1)} - \frac{2}{u - 1} &= -\ln{(X)} + c_3\end{aligned}$$

The solution is

$$\ln \left(u(X)-1\right)-\frac{2}{u \left(X \right)-1}+\ln \left(X \right)-c_3 = 0$$

Now u in the above solution is replaced back by Y using $u=\frac{Y}{X}$ which results in the solution

$$\ln \left(\frac{Y(X)}{X}-1\right)-\frac{2}{\frac{Y(X)}{X}-1}+\ln \left(X \right)-c_3 = 0$$

The solution is implicit $\ln \left(\frac{Y(X)-X}{X}\right)+\frac{2X}{-Y(X)+X}+\ln \left(X \right)-c_3 = 0$. Replacing $Y = y-y_0, X = x-x_0$ gives

$$\ln \left(\frac{y-6-x}{2+x}\right)+\frac{2x+4}{-y+6+x}+\ln \left(2+x \right)-c_3 = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\ln \left(\frac{y-6-x}{2+x}\right)+\frac{2x+4}{-y+6+x}+\ln \left(2+x \right)-c_3 = 0 \tag{1}$$

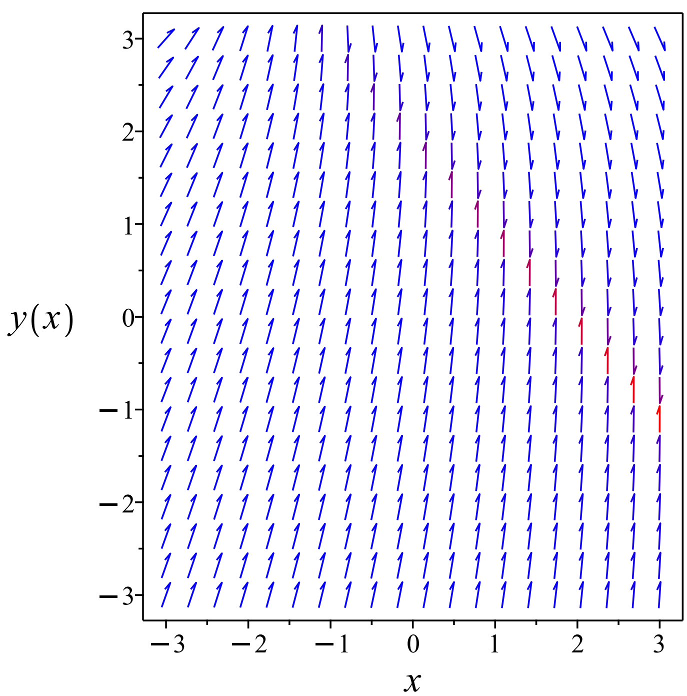


Figure 334: Slope field plot

Verification of solutions

$$\ln\left(\frac{y-6-x}{2+x}\right)+\frac{2x+4}{-y+6+x}+\ln(2+x)-c_3=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)=(-x+3*y(x)-14)/(x+y(x)-2),y(x), singsol=all)
```

$$y(x)=\frac{(x+6)\,\mathrm{LambertW}\left(-2c_1(2+x)\right)+2x+4}{\mathrm{LambertW}\left(-2c_1\left(2+x\right)\right)}$$

✓ Solution by Mathematica

Time used: 1.05 (sec). Leaf size: 144

```
DSolve[y'[x]==(-x+3*y[x]-14)/(x+y[x]-2),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[-\frac{2^{2/3}\left(x\log\left(\frac{y(x)-x-6}{y(x)+x-2}\right)-(x+6)\log\left(\frac{x+2}{y(x)+x-2}\right)+6\log\left(\frac{y(x)-x-6}{y(x)+x-2}\right)+y(x)\left(\log\left(\frac{x+2}{y(x)+x-2}\right)-\log\left(\right.\right.\right.}{9(-y(x)+x+6)}\right.$$

5.45 problem 44

Internal problem ID [1019]
Internal file name [`OUTPUT/1020_Sunday_June_05_2022_01_57_03_AM_92182999/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3y'y^2x - y^3 = x}$$

5.45.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^3 + x}{3y^2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 263: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^3 + x}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{3x^2} \\ S_y &= \frac{y^2}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{3} + c_1 \tag{4}$$

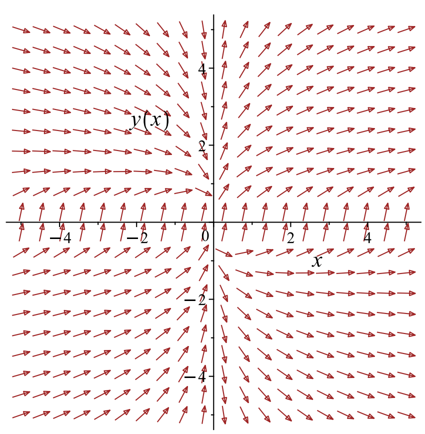
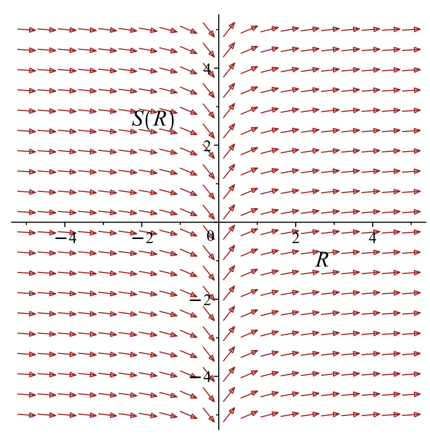
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x} = \frac{\ln(x)}{3} + c_1$$

Which simplifies to

$$\frac{y^3}{3x} = \frac{\ln(x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^3+x}{3y^2x}$ 	$R = x$ $S = \frac{y^3}{3x}$	$\frac{dS}{dR} = \frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$\frac{y^3}{3x} = \frac{\ln(x)}{3} + c_1 \tag{1}$$

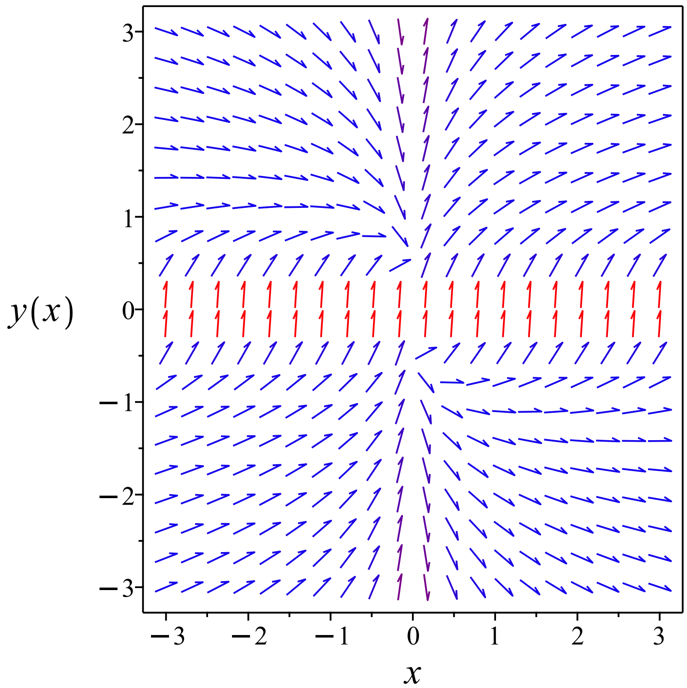


Figure 335: Slope field plot

Verification of solutions

$$\frac{y^3}{3x} = \frac{\ln(x)}{3} + c_1$$

Verified OK.

5.45.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^3 + x}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{3x}y + \frac{1}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{3x} \\ f_1(x) &= \frac{1}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y' y^2 = \frac{y^3}{3x} + \frac{1}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2 y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{w(x)}{3x} + \frac{1}{3} \\ w' &= \frac{w}{x} + 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = 1$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = \mu$$
$$\frac{d}{dx}\left(\frac{w}{x}\right) = \frac{1}{x}$$
$$d\left(\frac{w}{x}\right) = \frac{1}{x}dx$$

Integrating gives

$$\frac{w}{x} = \int \frac{1}{x}\, dx$$
$$\frac{w}{x} = \ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x\ln(x)$$

which simplifies to

$$w(x) = x(\ln(x) + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x(\ln(x) + c_1)$$

Solving for y gives

$$y(x) = (x(\ln(x) + c_1))^{\frac{1}{3}}$$
$$y(x) = \frac{(x(\ln(x) + c_1))^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$
$$y(x) = -\frac{(x(\ln(x) + c_1))^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$

Summary
The solution(s) found are the following

$$y = (x(\ln(x) + c_1))^{\frac{1}{3}} \tag{1}$$

$$y = \frac{(x(\ln(x) + c_1))^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2} \tag{2}$$

$$y = -\frac{(x(\ln(x) + c_1))^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

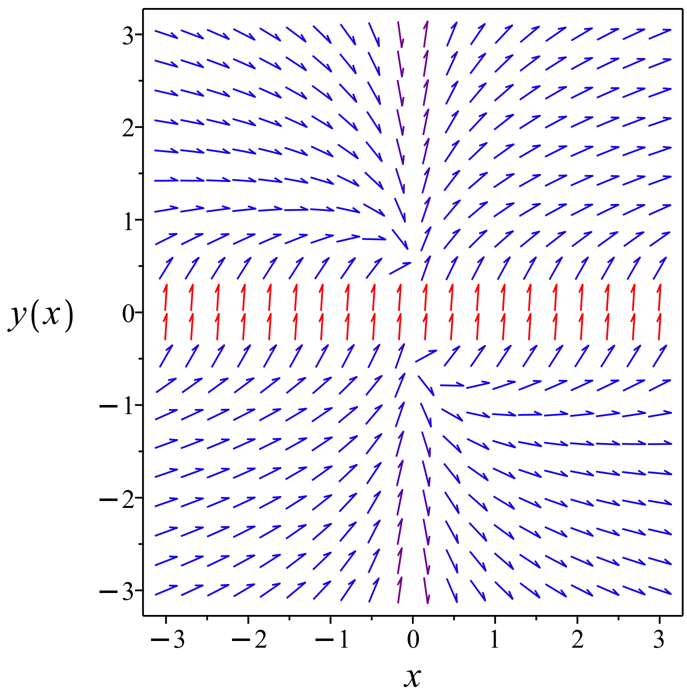


Figure 336: Slope field plot

Verification of solutions

$$y = (x(\ln(x) + c_1))^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{(x(\ln(x) + c_1))^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

$$y = -\frac{(x(\ln(x) + c_1))^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

5.45.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3x\,y^2)\,\mathrm{d}y &= (y^3 + x)\,\mathrm{d}x \\
(-y^3 - x)\,\mathrm{d}x + (3x\,y^2)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -y^3 - x \\
N(x,y) &= 3x\,y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^3 - x) \\
&= -3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x\,y^2) \\
&= 3y^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3x\,y^2}\left((-3y^2) - (3y^2)\right) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-y^3 - x)\\ &= \frac{-y^3 - x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(3x\,y^2)\\ &= \frac{3y^2}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-y^3 - x}{x^2}\right) + \left(\frac{3y^2}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-y^3 - x}{x^2} \mathrm{d}x$$

$$\phi = -\ln(x) + \frac{y^3}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x}$. Therefore equation (4) becomes

$$\frac{3y^2}{x} = \frac{3y^2}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y^3}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y^3}{x}$$

Summary

The solution(s) found are the following

$$-\ln(x) + \frac{y^3}{x} = c_1 \tag{1}$$

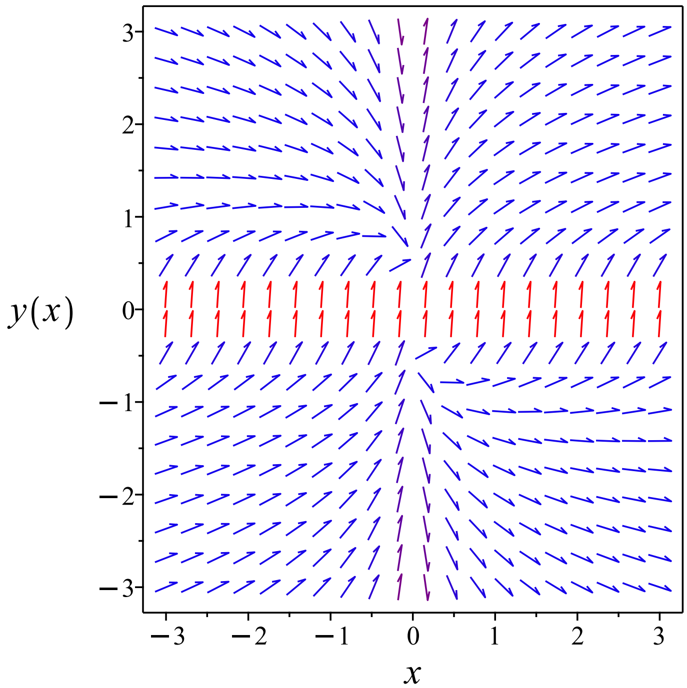


Figure 337: Slope field plot

Verification of solutions

$$-\ln(x) + \frac{y^3}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(3*x*y(x)^2*diff(y(x),x)=y(x)^3+x,y(x), singsol=all)
```

$$y(x) = ((\ln(x) + c_1)\, x)^{\frac{1}{3}}$$
$$y(x) = -\frac{((\ln(x) + c_1)\, x)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{((\ln(x) + c_1)\, x)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 69

```
DSolve[3*x*y[x]^2*y'[x]==y[x]^3+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x}\sqrt[3]{\log(x) + c_1}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{x}\sqrt[3]{\log(x) + c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{x}\sqrt[3]{\log(x) + c_1}$$

5.46 problem 45

Internal problem ID [1020]
Internal file name [`OUTPUT/1021_Sunday_June_05_2022_01_57_06_AM_9504560/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'xy - 6y^2 = 3x^6}$$

5.46.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^6 + 6y^2}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 265: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^{12}}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^{12}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^{12}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3x^6 + 6y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6y^2}{x^{13}} \\ S_y &= \frac{y}{x^{12}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{x^7} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R^7}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^6} + c_1 \tag{4}$$

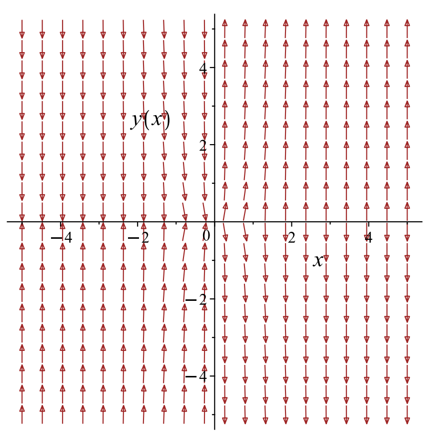
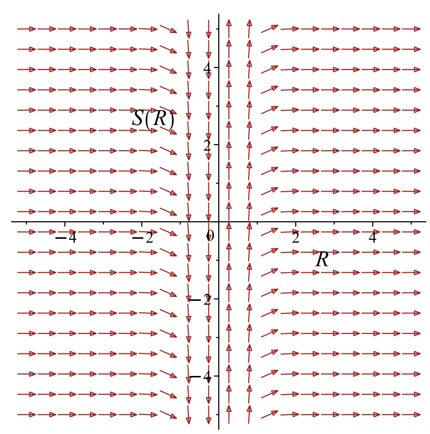
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^{12}} = -\frac{1}{2x^6} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^{12}} = -\frac{1}{2x^6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^6+6y^2}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^{12}}$	$\frac{dS}{dR} = \frac{3}{R^7}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^{12}} = -\frac{1}{2x^6} + c_1 \tag{1}$$

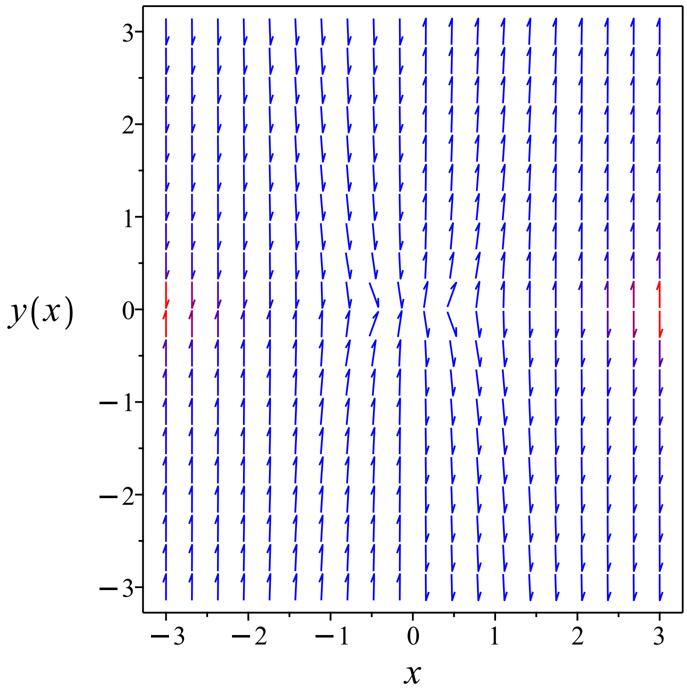


Figure 338: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^{12}} = -\frac{1}{2x^6} + c_1$$

Verified OK.

5.46.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{3x^6 + 6y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{6}{x}y + 3x^5\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{6}{x} \\ f_1(x) &= 3x^5 \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{6y^2}{x} + 3x^5 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{6w(x)}{x} + 3x^5 \\ w' &= \frac{12w}{x} + 6x^5 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{12}{x} \\ q(x) &= 6x^5 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{12w(x)}{x} = 6x^5$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{12}{x}dx} \\ &= \frac{1}{x^{12}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(6x^5\right) \\ \frac{d}{dx}\left(\frac{w}{x^{12}}\right) &= \left(\frac{1}{x^{12}}\right)\left(6x^5\right) \\ d\left(\frac{w}{x^{12}}\right) &= \left(\frac{6}{x^7}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^{12}} &= \int \frac{6}{x^7} dx \\ \frac{w}{x^{12}} &= -\frac{1}{x^6} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{12}}$ results in

$$w(x) = c_1 x^{12} - x^6$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x^{12} - x^6$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{c_1 x^6 - 1}\, x^3 \\ y(x) &= -\sqrt{c_1 x^6 - 1}\, x^3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x^6 - 1}\, x^3 \tag{1}$$

$$y = -\sqrt{c_1 x^6 - 1}\, x^3 \tag{2}$$

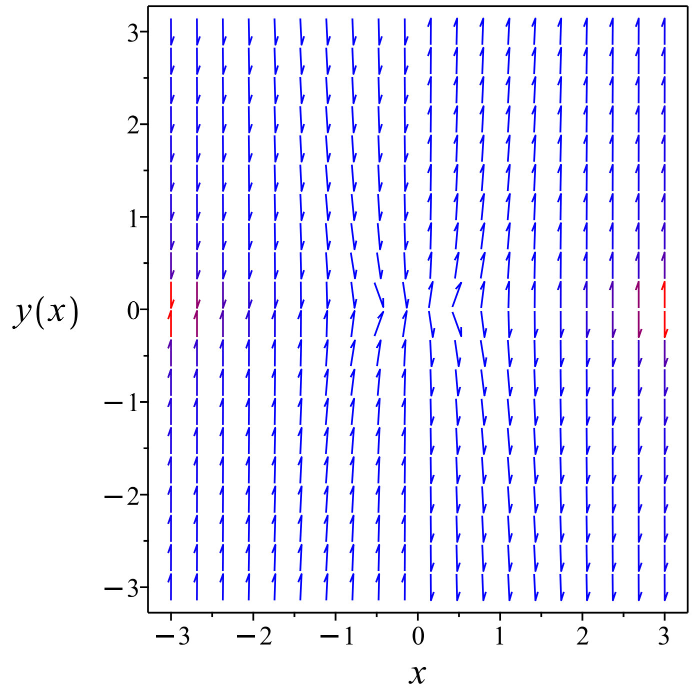


Figure 339: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x^6 - 1}\, x^3$$

Verified OK.

$$y = -\sqrt{c_1 x^6 - 1}\, x^3$$

Verified OK.

5.46.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(yx)\,\mathrm{d}y&=\left(3x^6+6y^2\right)\mathrm{d}x\\ \left(-3x^6-6y^2\right)\mathrm{d}x+(yx)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3x^6-6y^2\\ N(x,y)&=yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-3x^6-6y^2\right)\\ &=-12y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yx}((-12y) - (y)) \\
&= -\frac{13}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{13}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-13\ln(x)} \\
&= \frac{1}{x^{13}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^{13}}\left(-3x^6 - 6y^2\right) \\
&= \frac{-3x^6 - 6y^2}{x^{13}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{13}}(yx) \\
&= \frac{y}{x^{12}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-3x^6 - 6y^2}{x^{13}}\right) + \left(\frac{y}{x^{12}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-3x^6 - 6y^2}{x^{13}} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^6 + y^2}{2x^{12}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{y}{x^{12}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^{12}}$. Therefore equation (4) becomes

$$
\frac{y}{x^{12}} = \frac{y}{x^{12}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^6 + y^2}{2x^{12}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^6 + y^2}{2x^{12}}$$

Summary

The solution(s) found are the following

$$\frac{x^6 + y^2}{2x^{12}} = c_1 \tag{1}$$

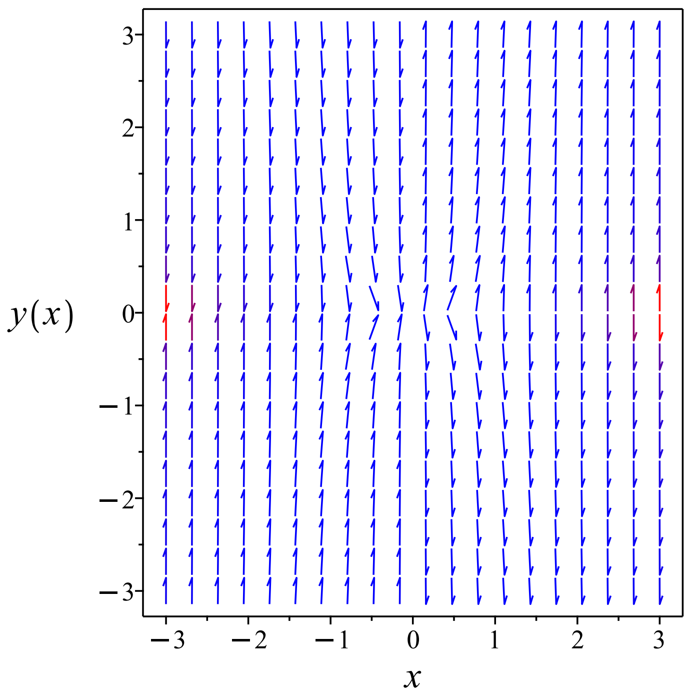


Figure 340: Slope field plot

Verification of solutions

$$\frac{x^6 + y^2}{2x^{12}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x*y(x)*diff(y(x),x)=3*x^6+6*y(x)^2,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 x^6 - 1}\, x^3$$
$$y(x) = -\sqrt{c_1 x^6 - 1}\, x^3$$

✓ Solution by Mathematica

Time used: 0.672 (sec). Leaf size: 42

```
DSolve[x*y[x]*y'[x]==3*x^6+6*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^3 \sqrt{-1 + c_1 x^6}$$
$$y(x) \to x^3 \sqrt{-1 + c_1 x^6}$$

5.47 problem 46

Internal problem ID [1021]
Internal file name [OUTPUT/1022_Sunday_June_05_2022_01_57_07_AM_99991387/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{x^3y' - 2y^2 - 2x^2y = -2x^4}$$

5.47.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-2x^4 + 2y\,x^2 + 2y^2}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{2(-x^4 + y\,x^2 + y^2)\,(b_3 - a_2)}{x^3} - \frac{4(-x^4 + y\,x^2 + y^2)^2\,a_3}{x^6} - \left(\frac{-8x^3 + 4yx}{x^3} - \frac{6(-x^4 + y\,x^2 + y^2)}{x^4}\right)(xa_2 + ya_3 + a_1) - \frac{2(x^2 + 2y)\,(xb_2 + yb_3 + b_1)}{x^3} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^8a_3 - 4x^7a_2 + 2x^7b_3 - 10x^6ya_3 - 2x^6a_1 + b_2x^6 - 6x^4y^2a_3 + 2x^5b_1 - 2x^4ya_1 + 4x^4yb_2 - 4x^3y^2a_2 + 2x}{x^6} = 0$$

Setting the numerator to zero gives

$$-4x^8a_3 + 4x^7a_2 - 2x^7b_3 + 10x^6ya_3 + 2x^6a_1 - b_2x^6 + 6x^4y^2a_3 - 2x^5b_1 + 2x^4ya_1 - 4x^4yb_2 + 4x^3y^2a_2 - 2x^3y^2b_3 - 2x^2y^3a_3 - 4x^3yb_1 + 6x^2y^2a_1 - 4y^4a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-4a_3v_1^8 + 4a_2v_1^7 + 10a_3v_1^6v_2 - 2b_3v_1^7 + 2a_1v_1^6 + 6a_3v_1^4v_2^2 - b_2v_1^6 + 2a_1v_1^4v_2 + 4a_2v_1^3v_2^2 - 2a_3v_1^2v_2^3 - 2b_1v_1^5 - 4b_2v_1^4v_2 - 2b_3v_1^3v_2^2 + 6a_1v_1^2v_2^2 - 4a_3v_2^4 - 4b_1v_1^3v_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-4a_3v_1^8 + (4a_2 - 2b_3)\,v_1^7 + 10a_3v_1^6v_2 + (2a_1 - b_2)\,v_1^6 - 2b_1v_1^5 + 6a_3v_1^4v_2^2 \\ + (2a_1 - 4b_2)\,v_1^4v_2 + (4a_2 - 2b_3)\,v_1^3v_2^2 - 4b_1v_1^3v_2 - 2a_3v_1^2v_2^3 + 6a_1v_1^2v_2^2 - 4a_3v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0\\
-4a_3 &= 0\\
-2a_3 &= 0\\
6a_3 &= 0\\
10a_3 &= 0\\
-4b_1 &= 0\\
-2b_1 &= 0\\
2a_1 - 4b_2 &= 0\\
2a_1 - b_2 &= 0\\
4a_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= 2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y - \left(\frac{-2x^4 + 2y\,x^2 + 2y^2}{x^3}\right)(x) \\
&= \frac{2x^4 - 2y^2}{x^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^4 - 2y^2}{x^2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(-x^2 + y)}{4} + \frac{\ln(x^2 + y)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-2x^4 + 2y\,x^2 + 2y^2}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{yx}{x^4 - y^2} \\
S_y &= \frac{x^2}{2x^4 - 2y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(-x^2 + y)}{4} + \frac{\ln(x^2 + y)}{4} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{\ln(-x^2 + y)}{4} + \frac{\ln(x^2 + y)}{4} = -\ln(x) + c_1$$

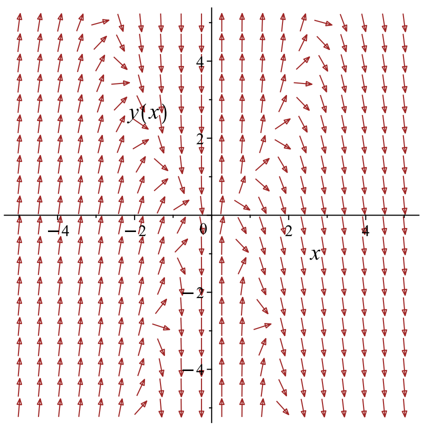
Which gives

$$y = -\frac{x^2(-\mathrm{e}^{4c_1} - x^4)}{\mathrm{e}^{4c_1} - x^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-2x^4+2y\,x^2+2y^2}{x^3}$



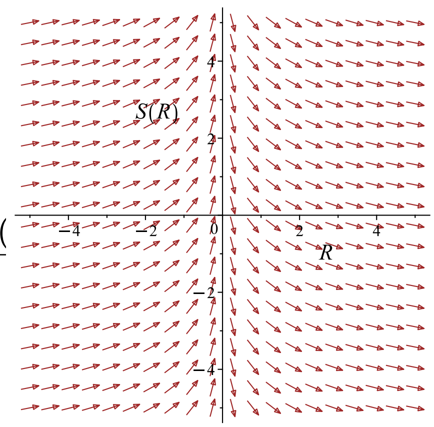
</td><td>

$R = x$

$S = -\frac{\ln(-x^2+y)}{4} + \ln($

</td><td>

$\frac{dS}{dR} = -\frac{1}{R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{x^2(-e^{4c_1} - x^4)}{e^{4c_1} - x^4} \tag{1}$$

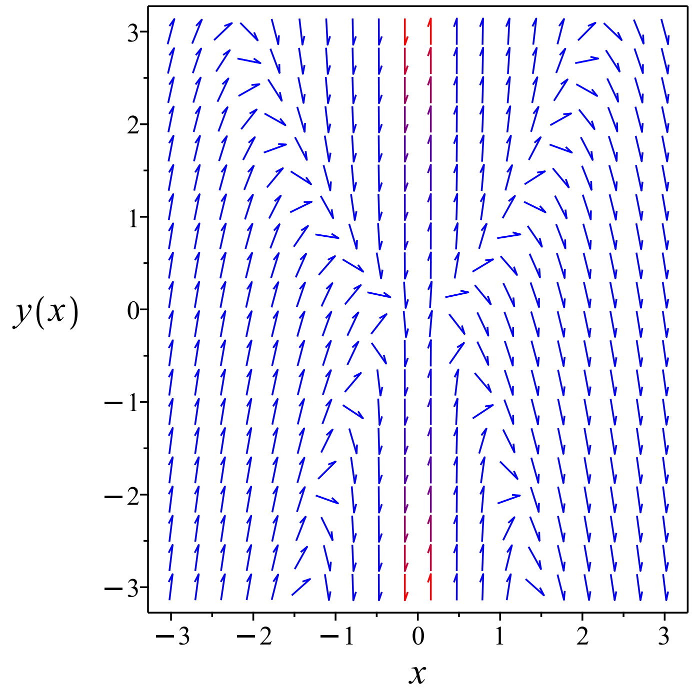


Figure 341: Slope field plot

Verification of solutions

$$y = -\frac{x^2(-e^{4c_1} - x^4)}{e^{4c_1} - x^4}$$

Verified OK.

5.47.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-2x^4 + 2y\,x^2 + 2y^2}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -2x + \frac{2y}{x} + \frac{2y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -2x$, $f_1(x) = \frac{2}{x}$ and $f_2(x) = \frac{2}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{2u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{6}{x^4} \\ f_1 f_2 &= \frac{4}{x^4} \\ f_2^2 f_0 &= -\frac{8}{x^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{2u''(x)}{x^3} + \frac{2u'(x)}{x^4} - \frac{8u(x)}{x^5} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x^4 + c_2}{x^2}$$

The above shows that

$$u'(x) = \frac{2c_1 x^4 - 2c_2}{x^3}$$

Using the above in (1) gives the solution

$$y = -\frac{(2c_1 x^4 - 2c_2)\, x^2}{2\,(c_1 x^4 + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{(c_3 x^4 - 1)\, x^2}{c_3 x^4 + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{(c_3 x^4 - 1)\, x^2}{c_3 x^4 + 1} \tag{1}$$

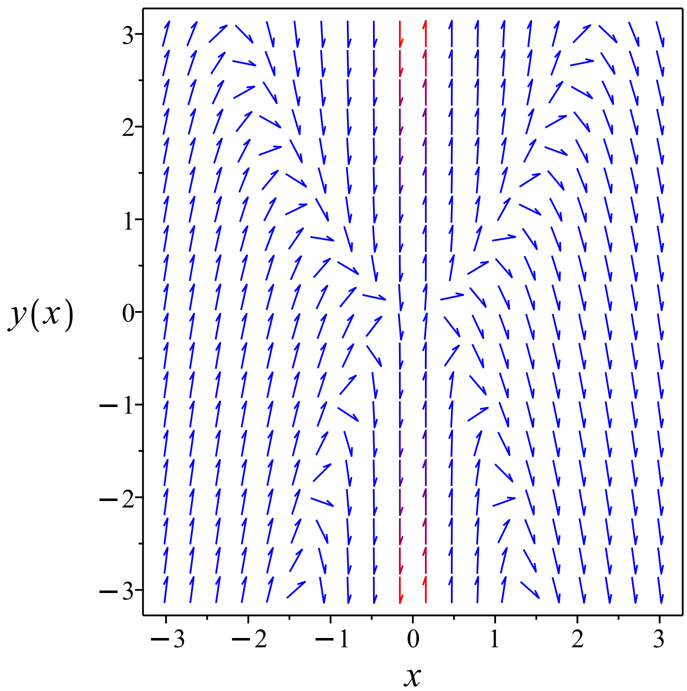


Figure 342: Slope field plot

Verification of solutions

$$y = -\frac{(c_3 x^4 - 1)\, x^2}{c_3 x^4 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^3*diff(y(x),x)=2*(y(x)^2+x^2*y(x)-x^4),y(x), singsol=all)
```

$$y(x) = \tanh(-2\ln(x) + 2c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 1.137 (sec). Leaf size: 62

```
DSolve[x^3*y'[x]==2*(y[x]^2+x^2*y[x]-x^4),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to ix^2 \tan(2i\log(x) + c_1)$$
$$y(x) \to \frac{x^2\left(-x^4 + e^{2i\text{Interval}[\{0,\pi\}]}\right)}{x^4 + e^{2i\text{Interval}[\{0,\pi\}]}}$$

5.48 problem 47

Internal problem ID [1022]
Internal file name [`OUTPUT/1023_Sunday_June_05_2022_01_57_08_AM_74272329/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Riccati]
```

$$\boxed{y' - y^2 e^{-x} - 4y = 2\,e^{x}}$$

5.48.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= y^2 e^{-x} + 4y + 2\,e^{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \left(y^2 \mathrm{e}^{-x} + 4y + 2\,\mathrm{e}^{x}\right)(b_3 - a_2) - \left(y^2 \mathrm{e}^{-x} + 4y + 2\,\mathrm{e}^{x}\right)^2 a_3 \\ - \left(-y^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{x}\right)(x a_2 + y a_3 + a_1) - \left(4 + 2\,\mathrm{e}^{-x} y\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} -\mathrm{e}^{-2x} y^4 a_3 - 4\,\mathrm{e}^{-x} \mathrm{e}^{x} y^2 a_3 + \mathrm{e}^{-x} x\, y^2 a_2 - 7\,\mathrm{e}^{-x} y^3 a_3 - 2\,\mathrm{e}^{-x} x y b_2 + \mathrm{e}^{-x} y^2 a_1 \\ - \mathrm{e}^{-x} y^2 a_2 - \mathrm{e}^{-x} y^2 b_3 - 2\,\mathrm{e}^{-x} y b_1 - 4\,\mathrm{e}^{2x} a_3 - 2\,\mathrm{e}^{x} x a_2 - 18\,\mathrm{e}^{x} y a_3 \\ - 16 y^2 a_3 - 2\,\mathrm{e}^{x} a_1 - 2\,\mathrm{e}^{x} a_2 + 2\,\mathrm{e}^{x} b_3 - 4 x b_2 - 4 y a_2 - 4 b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} -\mathrm{e}^{-2x} y^4 a_3 - 4\,\mathrm{e}^{-x} \mathrm{e}^{x} y^2 a_3 + \mathrm{e}^{-x} x\, y^2 a_2 - 7\,\mathrm{e}^{-x} y^3 a_3 - 2\,\mathrm{e}^{-x} x y b_2 + \mathrm{e}^{-x} y^2 a_1 \\ - \mathrm{e}^{-x} y^2 a_2 - \mathrm{e}^{-x} y^2 b_3 - 2\,\mathrm{e}^{-x} y b_1 - 4\,\mathrm{e}^{2x} a_3 - 2\,\mathrm{e}^{x} x a_2 - 18\,\mathrm{e}^{x} y a_3 \\ - 16 y^2 a_3 - 2\,\mathrm{e}^{x} a_1 - 2\,\mathrm{e}^{x} a_2 + 2\,\mathrm{e}^{x} b_3 - 4 x b_2 - 4 y a_2 - 4 b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} -\mathrm{e}^{-2x} y^4 a_3 - 20 y^2 a_3 + \mathrm{e}^{-x} x\, y^2 a_2 - 7\,\mathrm{e}^{-x} y^3 a_3 - 2\,\mathrm{e}^{-x} x y b_2 + \mathrm{e}^{-x} y^2 a_1 \\ - \mathrm{e}^{-x} y^2 a_2 - \mathrm{e}^{-x} y^2 b_3 - 2\,\mathrm{e}^{-x} y b_1 - 4\,\mathrm{e}^{2x} a_3 - 2\,\mathrm{e}^{x} x a_2 - 18\,\mathrm{e}^{x} y a_3 \\ - 2\,\mathrm{e}^{x} a_1 - 2\,\mathrm{e}^{x} a_2 + 2\,\mathrm{e}^{x} b_3 - 4 x b_2 - 4 y a_2 - 4 b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{x}, \mathrm{e}^{-2x}, \mathrm{e}^{-x}, \mathrm{e}^{2x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{x} = v_3, \mathrm{e}^{-2x} = v_4, \mathrm{e}^{-x} = v_5, \mathrm{e}^{2x} = v_6\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-v_4 v_2^4 a_3 + v_5 v_1 v_2^2 a_2 - 7 v_5 v_2^3 a_3 + v_5 v_2^2 a_1 - v_5 v_2^2 a_2 - 2 v_5 v_1 v_2 b_2 \\
- v_5 v_2^2 b_3 - 2 v_3 v_1 a_2 - 20 v_2^2 a_3 - 18 v_3 v_2 a_3 - 2 v_5 v_2 b_1 - 2 v_3 a_1 \\
- 4 v_2 a_2 - 2 v_3 a_2 - 4 v_6 a_3 - 4 v_1 b_2 + 2 v_3 b_3 - 4 b_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5, v_6\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
v_5 v_1 v_2^2 a_2 - 2 v_5 v_1 v_2 b_2 - 2 v_3 v_1 a_2 - 4 v_1 b_2 - v_4 v_2^4 a_3 - 7 v_5 v_2^3 a_3 \\
+ (a_1 - a_2 - b_3) v_2^2 v_5 - 20 v_2^2 a_3 - 18 v_3 v_2 a_3 - 2 v_5 v_2 b_1 \\
- 4 v_2 a_2 + (-2a_1 - 2a_2 + 2b_3) v_3 - 4 v_6 a_3 - 4 b_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_2 &= 0 \\
-4a_2 &= 0 \\
-2a_2 &= 0 \\
-20a_3 &= 0 \\
-18a_3 &= 0 \\
-7a_3 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
-2b_1 &= 0 \\
-4b_2 &= 0 \\
-2b_2 &= 0 \\
-4b_1 + b_2 &= 0 \\
-2a_1 - 2a_2 + 2b_3 &= 0 \\
a_1 - a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= b_3\\ a_2 &= 0\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= y - \left(y^2 \mathrm{e}^{-x} + 4y + 2\,\mathrm{e}^{x}\right)(1)\\ &= -y^2 \mathrm{e}^{-x} - 2\,\mathrm{e}^{x} - 3y\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{-y^2 \mathrm{e}^{-x} - 2\,\mathrm{e}^{x} - 3y} dy\end{aligned}$$

Which results in

$$S = -\frac{2\,\mathrm{e}^{x} \arctan\left(\frac{2y+3\,\mathrm{e}^{x}}{\sqrt{-\mathrm{e}^{2x}}}\right)}{\sqrt{-\mathrm{e}^{2x}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y^2\mathrm{e}^{-x} + 4y + 2\,\mathrm{e}^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{-x}y}{y^2\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-x}y + 2} \\ S_y &= -\frac{\mathrm{e}^{-x}}{y^2\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-x}y + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ \mathrm{arctanh}\left(2y\,\mathrm{e}^{-x} + 3\right) = -x + c_1$$

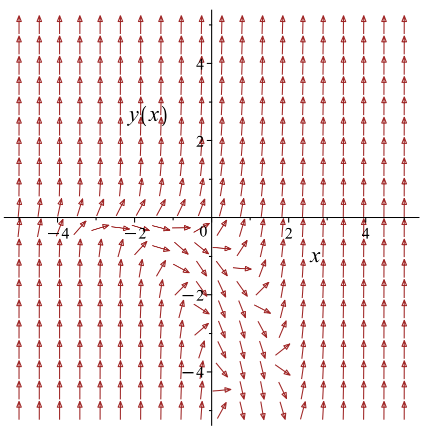
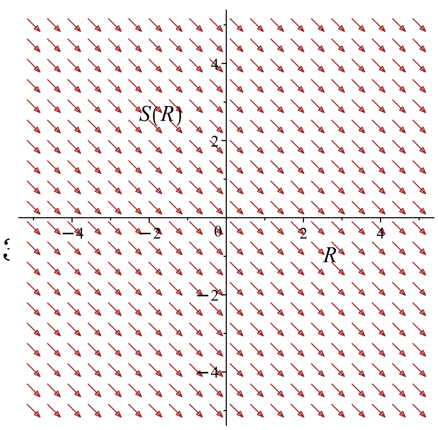
Which simplifies to

$$2\ \operatorname{arctanh}\left(2y\,\mathrm{e}^{-x}+3\right)=-x+c_1$$

Which gives

$$y=\frac{\left(-3+\tanh\left(-\frac{x}{2}+\frac{c_1}{2}\right)\right)\mathrm{e}^{x}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=y^2\mathrm{e}^{-x}+4y+2\,\mathrm{e}^{x}$ 	$R=x$ $S=2\ \operatorname{arctanh}\left(2\,\mathrm{e}^{-x}y+3\right.$	$\frac{dS}{dR}=-1$ 

Summary

The solution(s) found are the following

$$y=\frac{\left(-3+\tanh\left(-\frac{x}{2}+\frac{c_1}{2}\right)\right)\mathrm{e}^{x}}{2} \tag{1}$$

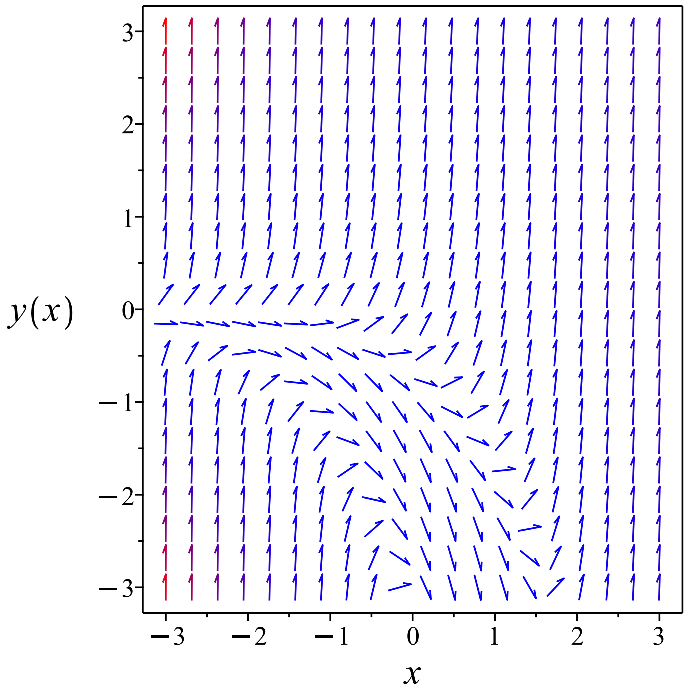


Figure 343: Slope field plot

Verification of solutions

$$y = \frac{\left(-3+\tanh\left(-\frac{x}{2}+\frac{c_1}{2}\right)\right) e^{x}}{2}$$

Verified OK.

5.48.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y^2 e^{-x} + 4y + 2 e^{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 e^{-x} + 4y + 2 e^{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 2\,e^{x}$, $f_1(x) = 4$ and $f_2(x) = e^{-x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{e^{-x} u}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -e^{-x} \\
f_1 f_2 &= 4\,e^{-x} \\
f_2^2 f_0 &= 2\,e^{-2x} e^{x}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
e^{-x} u''(x) - 3\,e^{-x} u'(x) + 2\,e^{-2x} e^{x} u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 e^{2x} + c_2 e^{x}
$$

The above shows that

$$
u'(x) = 2c_1 e^{2x} + c_2 e^{x}
$$

Using the above in (1) gives the solution

$$
y = -\frac{(2c_1 e^{2x} + c_2 e^{x})\, e^{x}}{c_1 e^{2x} + c_2 e^{x}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{-2c_3 e^{2x} - e^{x}}{c_3 e^{x} + 1}
$$

Summary

The solution(s) found are the following

$$y = \frac{-2c_3 e^{2x} - e^{x}}{c_3 e^{x} + 1} \tag{1}$$

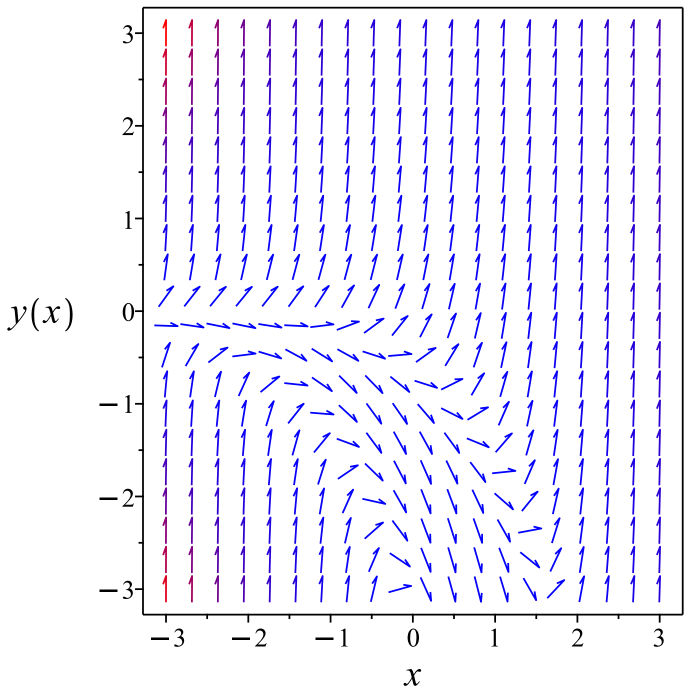


Figure 344: Slope field plot

Verification of solutions

$$y = \frac{-2c_3 e^{2x} - e^{x}}{c_3 e^{x} + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 22

```
dsolve(diff(y(x),x)=y(x)^2*exp(-x)+4*y(x)+2*exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{2\,\mathrm{e}^{x}(\mathrm{e}^{x}c_1 - 1)}{-2 + \mathrm{e}^{x}c_1}$$

✓ Solution by Mathematica

Time used: 0.265 (sec). Leaf size: 30

```
DSolve[y'[x]==y[x]^2*Exp[-x]+4*y[x]+2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to -2e^x + \frac{1}{e^{-x} + c_1} \\ y(x) &\to -2e^x \end{aligned}$$

5.49 problem 48

Internal problem ID [1023]
Internal file name [`OUTPUT/1024_Sunday_June_05_2022_01_57_10_AM_73978969/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - \frac{y^2 + \tan(x)\, y + \tan(x)^2}{\sin(x)^2} = 0}$$

5.49.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2 + \tan(x)\, y + \tan(x)^2}{\sin(x)^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\tan(x)^2}{\sin(x)^2} + \frac{\tan(x)\, y}{\sin(x)^2} + \frac{y^2}{\sin(x)^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{\tan(x)^2}{\sin(x)^2}$, $f_1(x) = \frac{\tan(x)}{\sin(x)^2}$ and $f_2(x) = \frac{1}{\sin(x)^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{\sin(x)^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2\cos(x)}{\sin(x)^3} \\ f_1 f_2 &= \frac{\tan(x)}{\sin(x)^4} \\ f_2^2 f_0 &= \frac{\tan(x)^2}{\sin(x)^6} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{\sin(x)^2} - \left(-\frac{2\cos(x)}{\sin(x)^3} + \frac{\tan(x)}{\sin(x)^4}\right) u'(x) + \frac{\tan(x)^2 u(x)}{\sin(x)^6} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$\begin{aligned} u(x) = c_1 \sin\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right) \\ + c_2 \cos\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right) \end{aligned}$$

The above shows that

$$\begin{aligned} u'(x) = \sec(x)\csc(x)\Bigg(c_2 \sin\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right) \\ - c_1 \cos\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right)\Bigg) \end{aligned}$$

Using the above in (1) gives the solution

$$y = -\frac{\sec(x)\csc(x)\left(c_2\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)-c_1\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\right.\right.}{c_1\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)+c_2\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(s}{}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$y = \frac{\left(-\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)+c_3\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)\right)\tan}{c_3\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)+\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)+c_3\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)\right)\tan}{c_3\sin\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)+\cos\left(-\ln(\sin(x))+\frac{\ln(\sin(x)+1)}{2}+\frac{\ln(\sin(x)-1)}{2}\right)} \quad (1)$$

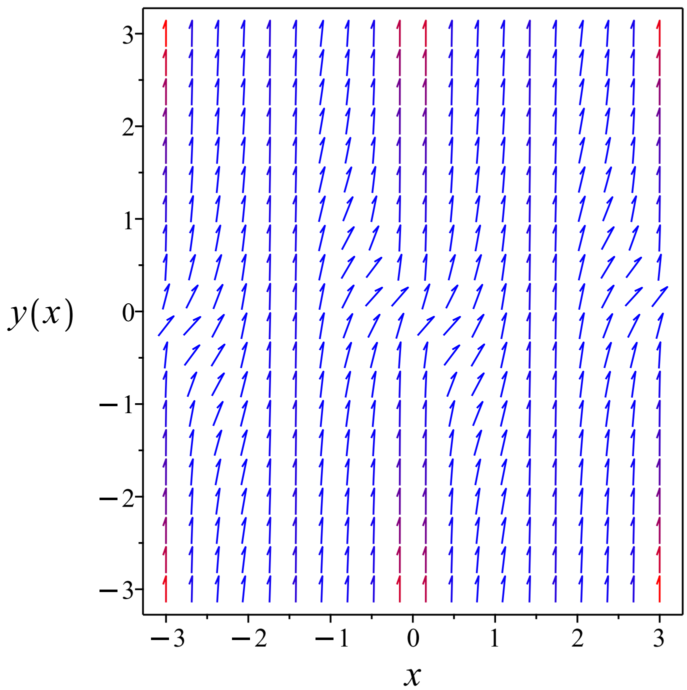


Figure 345: Slope field plot

Verification of solutions

$$y = \frac{\left(-\sin\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right) + c_3 \cos\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right)\right) \tan}{c_3 \sin\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right) + \cos\left(-\ln(\sin(x)) + \frac{\ln(\sin(x)+1)}{2} + \frac{\ln(\sin(x)-1)}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-2*cos(x)*sin(x)+tan(x))*(dif
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         <- linear_1 successful
         Change of variables used:
            [x = arcsin(t)]
         Linear ODE actually solved:
            -8*t^2*u(t)+(-24*t^7+32*t^5-8*t^3)*diff(u(t),t)+(-8*t^8+16*t^6-8*t^4)*diff(diff(
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 102

```
dsolve(diff(y(x),x)=(y(x)^2+y(x)*tan(x)+tan(x)^2)/sin(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{\tan(x)\left(c_1 \sin\left(\frac{\ln(\sin(x)-1)}{2} + \frac{\ln(\sin(x)+1)}{2} - \ln(\sin(x))\right) - \cos\left(\frac{\ln(\sin(x)-1)}{2} + \frac{\ln(\sin(x)+1)}{2} - \ln(\sin(x))\right)\right)}{c_1 \cos\left(\frac{\ln(\sin(x)-1)}{2} + \frac{\ln(\sin(x)+1)}{2} - \ln(\sin(x))\right) + \sin\left(\frac{\ln(\sin(x)-1)}{2} + \frac{\ln(\sin(x)+1)}{2} - \ln(\sin(x))\right)}$$

✓ Solution by Mathematica

Time used: 0.678 (sec). Leaf size: 20

```
DSolve[y'[x]==(y[x]^2+y[x]*Tan[x]+Tan[x]^2)/Sin[x]^2,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \tan(x)\tan(\log(\sin(x)) - \log(\cos(x)) + c_1)$$

5.50 problem 49

Internal problem ID [1024]
Internal file name [`OUTPUT/1025_Sunday_June_05_2022_01_57_12_AM_91127049/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`], _Riccati]
```

$$\boxed{x \ln(x)^2 y' - \ln(x) y - y^2 = -4 \ln(x)^2}$$

5.50.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{4 \ln(x)^2 - \ln(x) y - y^2}{x \ln(x)^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{4}{x} + \frac{y}{x \ln(x)} + \frac{y^2}{x \ln(x)^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{4}{x}$, $f_1(x) = \frac{1}{x \ln(x)}$ and $f_2(x) = \frac{1}{x \ln(x)^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x \ln(x)^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2\ln(x)^2} - \frac{2}{x^2\ln(x)^3} \\ f_1f_2 &= \frac{1}{x^2\ln(x)^3} \\ f_2^2f_0 &= -\frac{4}{x^3\ln(x)^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x\ln(x)^2} - \left(-\frac{1}{x^2\ln(x)^2} - \frac{1}{x^2\ln(x)^3}\right)u'(x) - \frac{4u(x)}{x^3\ln(x)^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_2\ln(x)^4 + c_1}{\ln(x)^2}$$

The above shows that

$$u'(x) = \frac{2c_2\ln(x)^4 - 2c_1}{x\ln(x)^3}$$

Using the above in (1) gives the solution

$$y = -\frac{(2c_2\ln(x)^4 - 2c_1)\ln(x)}{c_2\ln(x)^4 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2(\ln(x)^4 - c_3)\ln(x)}{\ln(x)^4 + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{2(\ln(x)^4 - c_3)\ln(x)}{\ln(x)^4 + c_3} \tag{1}$$

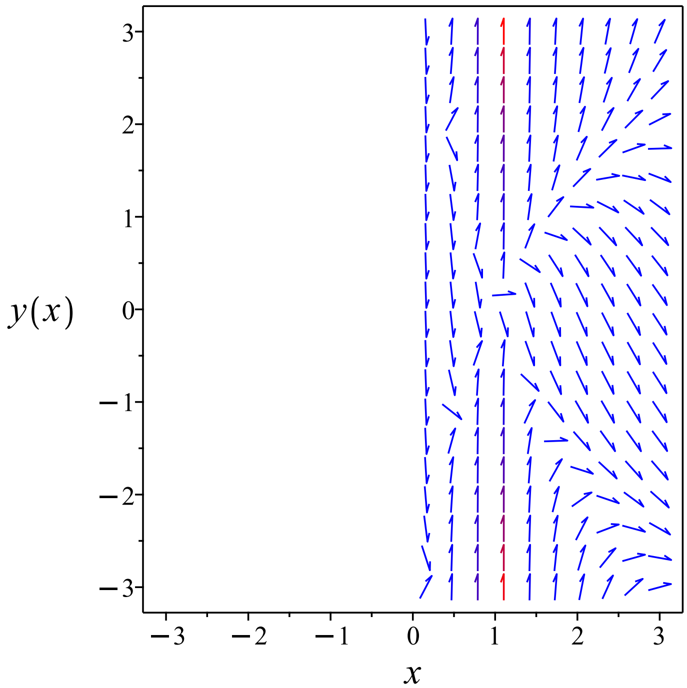


Figure 346: Slope field plot

Verification of solutions

$$y = -\frac{2(\ln(x)^4 - c_3)\ln(x)}{\ln(x)^4 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*(ln(x))^2*diff(y(x),x)=-4*(ln(x))^2+y(x)*ln(x)+y(x)^2,y(x), singsol=all)
```

$$y(x) = 2i\tan\left(2i\ln\left(\ln\left(x\right)\right)+c_1\right)\ln\left(x\right)$$

✓ Solution by Mathematica

Time used: 1.259 (sec). Leaf size: 64

```
DSolve[x*(Log[x])^2*y'[x]==-4*(Log[x])^2+y[x]*Log[x]+y[x]^2,y[x],x,IncludeSingularSolutions
```

$$y(x) \to 2i\log(x)\tan(2i\log(\log(x))+c_1)$$
$$y(x) \to \frac{2\log(x)\left(-\log^4(x)+e^{2i\text{Interval}[\{0,\pi\}]}\right)}{\log^4(x)+e^{2i\text{Interval}[\{0,\pi\}]}}$$

5.51 problem 50

Internal problem ID [1025]
Internal file name [`OUTPUT/1026_Sunday_June_05_2022_01_57_13_AM_3413717/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$
\boxed{2x\left(y+2\sqrt{x}\right)y'-\left(y+\sqrt{x}\right)^{2}=0}
$$

5.51.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2y\sqrt{x}+y^2+x}{4x^{\frac{3}{2}}+2yx} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(2y\sqrt{x}+y^2+x)(b_3-a_2)}{4x^{\frac{3}{2}}+2yx} - \frac{(2y\sqrt{x}+y^2+x)^2 a_3}{4\left(yx+2x^{\frac{3}{2}}\right)^2} \\ & - \left(\frac{\frac{y}{\sqrt{x}}+1}{4x^{\frac{3}{2}}+2yx} - \frac{(2y\sqrt{x}+y^2+x)(y+3\sqrt{x})}{2\left(yx+2x^{\frac{3}{2}}\right)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(\frac{2y+2\sqrt{x}}{4x^{\frac{3}{2}}+2yx} - \frac{(2y\sqrt{x}+y^2+x)x}{2\left(yx+2x^{\frac{3}{2}}\right)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^{\frac{5}{2}}y^2b_2+4x^{\frac{5}{2}}yb_3-2x^{\frac{3}{2}}y^2b_1+8x^3yb_2+10x^{\frac{7}{2}}b_2-6x^{\frac{5}{2}}b_1+2x^2a_1+2\sqrt{x}\,y^3a_1+2x^{\frac{3}{2}}y^2a_3+\sqrt{x}\,y^4a_3+8x}{4\sqrt{x}\left(yx+2x^{\frac{3}{2}}\right)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^{\frac{5}{2}}y^2b_2+4x^{\frac{5}{2}}yb_3-2x^{\frac{3}{2}}y^2b_1+8x^3yb_2+10x^{\frac{7}{2}}b_2-6x^{\frac{5}{2}}b_1+2x^2a_1 \\ & +2\sqrt{x}\,y^3a_1+2x^{\frac{3}{2}}y^2a_3+\sqrt{x}\,y^4a_3+8x^{\frac{3}{2}}ya_1-2x^3a_2+4x^3b_3 \\ & -2x^2ya_3-8x^2yb_1-2x^{\frac{5}{2}}ya_2+4x\,y^3a_3+8x\,y^2a_1-x^{\frac{5}{2}}a_3=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x}, x^{\frac{3}{2}}, x^{\frac{5}{2}}, x^{\frac{7}{2}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x} = v_3, x^{\frac{3}{2}} = v_4, x^{\frac{5}{2}} = v_5, x^{\frac{7}{2}} = v_6\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_3v_2^4a_3 + 2v_3v_2^3a_1 + 4v_1v_2^3a_3 + 8v_1^3v_2b_2 + 8v_1v_2^2a_1 - 2v_1^3a_2 - 2v_1^2v_2a_3 \\
&+ 2v_4v_2^2a_3 - 8v_1^2v_2b_1 - 2v_4v_2^2b_1 + 2v_5v_2^2b_2 + 4v_1^3b_3 + 2v_1^2a_1 \\
&+ 8v_4v_2a_1 - 2v_5v_2a_2 + 4v_5v_2b_3 - v_5a_3 - 6v_5b_1 + 10v_6b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5, v_6\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&8v_1^3v_2b_2 + (-2a_2 + 4b_3)\, v_1^3 + (-2a_3 - 8b_1)\, v_1^2v_2 + 2v_1^2a_1 + 4v_1v_2^3a_3 \\
&+ 8v_1v_2^2a_1 + v_3v_2^4a_3 + 2v_3v_2^3a_1 + (2a_3 - 2b_1)\, v_2^2v_4 + 2v_5v_2^2b_2 \\
&+ 8v_4v_2a_1 + (-2a_2 + 4b_3)\, v_2v_5 + (-a_3 - 6b_1)\, v_5 + 10v_6b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
2a_1 &= 0 \\
8a_1 &= 0 \\
4a_3 &= 0 \\
2b_2 &= 0 \\
8b_2 &= 0 \\
10b_2 &= 0 \\
-2a_2 + 4b_3 &= 0 \\
-2a_3 - 8b_1 &= 0 \\
-a_3 - 6b_1 &= 0 \\
2a_3 - 2b_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 2x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{2y\sqrt{x}+y^2+x}{4x^{\frac{3}{2}}+2yx}\right)(2x) \\ &= -\frac{x^2}{yx+2x^{\frac{3}{2}}} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{x^2}{yx+2x^{\frac{3}{2}}}} dy\end{aligned}$$

Which results in

$$S = -\frac{\frac{x\,y^2}{2} + 2x^{\frac{3}{2}}y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y\sqrt{x} + y^2 + x}{4x^{\frac{3}{2}} + 2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y\left(y\sqrt{x} + 2x\right)}{2x^{\frac{5}{2}}} \\
S_y &= \frac{-y - 2\sqrt{x}}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-y\sqrt{x} - 2x}{x^{\frac{3}{2}}\left(2y + 4\sqrt{x}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

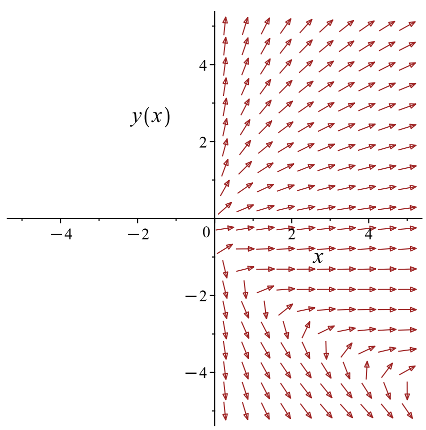
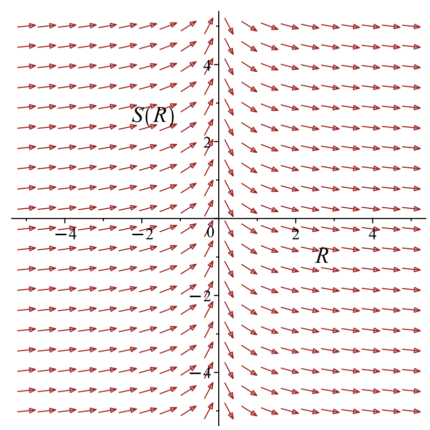
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y\left(y + 4\sqrt{x}\right)}{2x} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$-\frac{y\left(y+4\sqrt{x}\right)}{2x} = -\frac{\ln\left(x\right)}{2}+c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y\sqrt{x}+y^2+x}{4x^{\frac{3}{2}}+2yx}$ 	$R = x$ $S = -\frac{y\left(y+4\sqrt{x}\right)}{2x}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$-\frac{y\left(y+4\sqrt{x}\right)}{2x} = -\frac{\ln\left(x\right)}{2}+c_1 \tag{1}$$

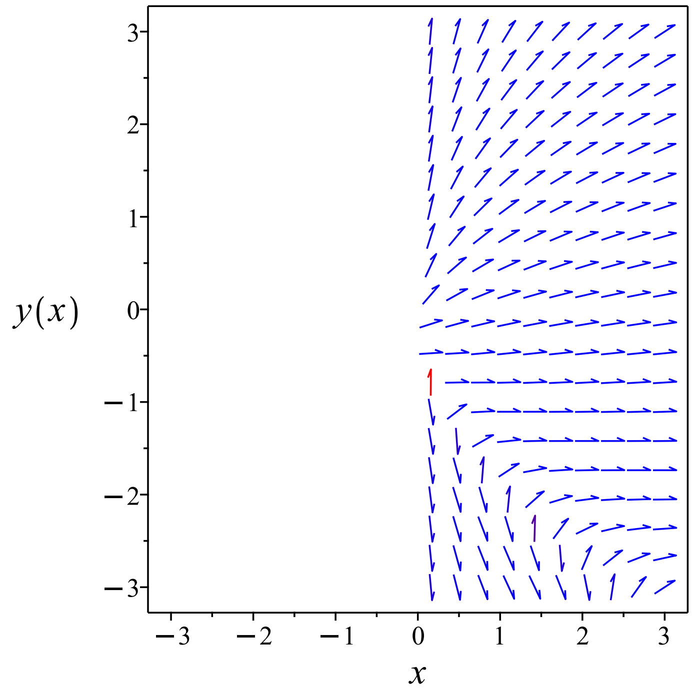


Figure 347: Slope field plot

Verification of solutions

$$-\frac{y(y+4\sqrt{x})}{2x} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

5.51.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(2x\left(y+2\sqrt{x}\right)\right)\mathrm{d}y = \left(\left(y+\sqrt{x}\right)^2\right)\mathrm{d}x$$
$$\left(-\left(y+\sqrt{x}\right)^2\right)\mathrm{d}x + \left(2x\left(y+2\sqrt{x}\right)\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\left(y+\sqrt{x}\right)^2$$
$$N(x,y) = 2x\left(y+2\sqrt{x}\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\left(y+\sqrt{x}\right)^2\right)\\ &= -2y-2\sqrt{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x\left(y+2\sqrt{x}\right)\right)\\ &= 2y+6\sqrt{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2x\left(y+2\sqrt{x}\right)}\left(\left(-2y-2\sqrt{x}\right)-\left(2y+6\sqrt{x}\right)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-\left(y+\sqrt{x}\right)^2\right) \\
&= -\frac{\left(y+\sqrt{x}\right)^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(2x\left(y+2\sqrt{x}\right)\right) \\
&= \frac{2y+4\sqrt{x}}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{\left(y+\sqrt{x}\right)^2}{x^2}\right) + \left(\frac{2y+4\sqrt{x}}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{\left(y+\sqrt{x}\right)^2}{x^2} \,\mathrm{d}x$$
$$\phi = -\ln\left(x\right) + \frac{4y}{\sqrt{x}} + \frac{y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{4}{\sqrt{x}} + \frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y+4\sqrt{x}}{x}$. Therefore equation (4) becomes

$$\frac{2y+4\sqrt{x}}{x} = \frac{4}{\sqrt{x}} + \frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(x\right) + \frac{4y}{\sqrt{x}} + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{4y}{\sqrt{x}} + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$-\ln(x) + \frac{4y}{\sqrt{x}} + \frac{y^2}{x} = c_1 \tag{1}$$

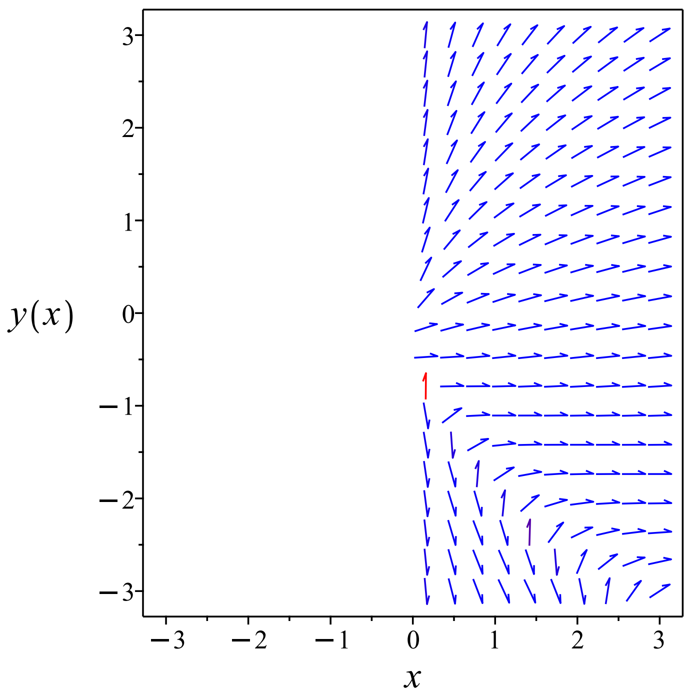


Figure 348: Slope field plot

Verification of solutions

$$-\ln(x) + \frac{4y}{\sqrt{x}} + \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 50

```
dsolve(2*x*(y(x)+2*sqrt(x))*diff(y(x),x)=(y(x)+sqrt(x))^2,y(x), singsol=all)
```

$$y(x) = \frac{-2x + \sqrt{x^2 (\ln(x) - c_1 + 4)}}{\sqrt{x}}$$
$$y(x) = -\frac{2x + \sqrt{x^2 (\ln(x) - c_1 + 4)}}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.61 (sec). Leaf size: 68

```
DSolve[2*x*(y[x]+2*Sqrt[x])*y'[x]==(y[x]+Sqrt[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\sqrt{x} - \sqrt{\frac{1}{x^2}} x \sqrt{x(\log(x) + 4 + c_1)}$$
$$y(x) \to -2\sqrt{x} + \sqrt{\frac{1}{x^2}} x \sqrt{x(\log(x) + 4 + c_1)}$$

5.52 problem 51

Internal problem ID [1026]
Internal file name [`OUTPUT/1027_Sunday_June_05_2022_01_57_15_AM_76060930/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`], [_Abel, `2nd type
   `, `class A`]]
```

$$\boxed{\left(y+{\mathrm e}^{x^{2}}\right) y'-2x\left(y^{2}+{\mathrm e}^{x^{2}} y+{\mathrm e}^{2x^{2}}\right)=0}$$

5.52.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y + \mathrm{e}^{x^2}\right) \mathrm{d}y = \left(2x\left(y^2 + \mathrm{e}^{x^2} y + \mathrm{e}^{2x^2}\right)\right) \mathrm{d}x$$
$$\left(-2x\left(y^2 + \mathrm{e}^{x^2} y + \mathrm{e}^{2x^2}\right)\right) \mathrm{d}x + \left(y + \mathrm{e}^{x^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2x\left(y^2 + \mathrm{e}^{x^2} y + \mathrm{e}^{2x^2}\right)$$
$$N(x, y) = y + \mathrm{e}^{x^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x\left(y^2 + \mathrm{e}^{x^2} y + \mathrm{e}^{2x^2}\right)\right)\\ &= -2x\left(2y + \mathrm{e}^{x^2}\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y+\mathrm{e}^{x^2}\right)\\ &= 2\,\mathrm{e}^{x^2}x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y+\mathrm{e}^{x^2}}\left(\left(-2x\left(2y+\mathrm{e}^{x^2}\right)\right) - \left(2\,\mathrm{e}^{x^2}x\right)\right)\\ &= -4x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -4x\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2x^2}\\ &= \mathrm{e}^{-2x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2x^2}\left(-2x\left(y^2+\mathrm{e}^{x^2}y+\mathrm{e}^{2x^2}\right)\right)\\ &= -2x\left(y^2+\mathrm{e}^{x^2}y+\mathrm{e}^{2x^2}\right)\mathrm{e}^{-2x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2x^2}\left(y+\mathrm{e}^{x^2}\right)\\ &= \left(y+\mathrm{e}^{x^2}\right)\mathrm{e}^{-2x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2x\left(y^2 + \mathrm{e}^{x^2}y + \mathrm{e}^{2x^2}\right)\mathrm{e}^{-2x^2}\right) + \left(\left(y + \mathrm{e}^{x^2}\right)\mathrm{e}^{-2x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -2x\left(y^2 + \mathrm{e}^{x^2}y + \mathrm{e}^{2x^2}\right)\mathrm{e}^{-2x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^2 + \frac{\mathrm{e}^{-2x^2}y^2}{2} + \mathrm{e}^{-x^2}y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x^2}y + \mathrm{e}^{-x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(y + \mathrm{e}^{x^2}\right)\mathrm{e}^{-2x^2}$. Therefore equation (4) becomes

$$
\left(y + \mathrm{e}^{x^2}\right)\mathrm{e}^{-2x^2} = \mathrm{e}^{-2x^2}y + \mathrm{e}^{-x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + \frac{e^{-2x^2} y^2}{2} + e^{-x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + \frac{e^{-2x^2} y^2}{2} + e^{-x^2} y$$

<u>Summary</u>

The solution(s) found are the following

$$-x^2 + \frac{e^{-2x^2} y^2}{2} + e^{-x^2} y = c_1 \tag{1}$$

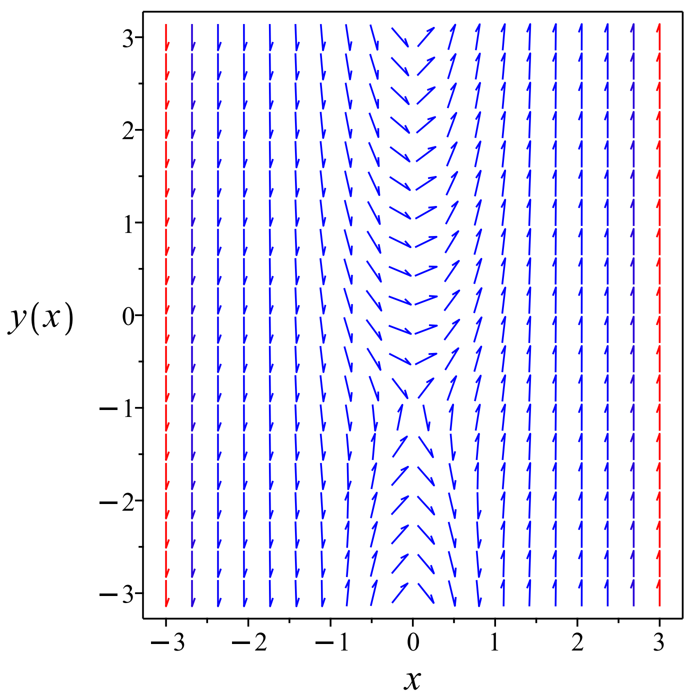


Figure 349: Slope field plot

<u>Verification of solutions</u>

$$-x^2 + \frac{e^{-2x^2} y^2}{2} + e^{-x^2} y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve((y(x)+exp(x^2))*diff(y(x),x)=2*x*(y(x)^2+y(x)*exp(x^2)+exp(2*x^2)),y(x), singsol=all)
```

$$y(x) = \left(-1-\sqrt{2x^2-2c_1+1}\right)\mathrm{e}^{x^2}$$
$$y(x) = \left(-1+\sqrt{2x^2-2c_1+1}\right)\mathrm{e}^{x^2}$$

✓ Solution by Mathematica

Time used: 0.744 (sec). Leaf size: 76

```
DSolve[(y[x]+Exp[x^2])*y'[x]==2*x*(y[x]^2+y[x]*Exp[x^2]+Exp[2*x^2]),y[x],x,IncludeSingularSo
```

$$y(x) \to -e^{x^2} - \frac{\sqrt{2x^2+1+c_1}}{\sqrt{e^{-2x^2}}}$$
$$y(x) \to -e^{x^2} + \frac{\sqrt{2x^2+1+c_1}}{\sqrt{e^{-2x^2}}}$$

5.53 problem 52

Internal problem ID [1027]
Internal file name [`OUTPUT/1028_Sunday_June_05_2022_01_57_16_AM_139120/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' + \frac{2y}{x} - \frac{3x^2y^2 + 6yx + 2}{x^2(2yx + 3)} = 0}$$

With initial conditions

$$[y(2) = 2]$$

5.53.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x^2y^2 - 2}{x^2(2yx + 3)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{4}, -\frac{3}{4} < x \leq \infty\right\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\left\{y < -\frac{3}{4} \vee -\frac{3}{4} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2y^2 - 2}{x^2(2yx + 3)}\right) \\ &= -\frac{2y}{2yx + 3} + \frac{2x^2y^2 - 4}{x(2yx + 3)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{4}, -\frac{3}{4} < x \leq \infty\right\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\left\{y < -\frac{3}{4} \vee -\frac{3}{4} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.53.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{x^2y^2 - 2}{x^2(2yx + 3)} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x^2y^2-2)(b_3-a_2)}{x^2(2yx+3)} - \frac{(x^2y^2-2)^2 a_3}{x^4(2yx+3)^2} - \left(-\frac{2y^2}{x(2yx+3)} + \frac{2x^2y^2-4}{x^3(2yx+3)} + \frac{2(x^2y^2-2)y}{x^2(2yx+3)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2y}{2yx+3} + \frac{2x^2y^2-4}{x(2yx+3)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^6y^2b_2 - 3x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 + 18x^5yb_2 + 3x^4y^2a_2 + 3x^4y^2b_3 + 6x^4yb_1 + 13b_2x^4 + 8x^3ya_2 + 8x}{x^4(2yx+3)^2} = 0$$

Setting the numerator to zero gives

$$6x^6y^2b_2 - 3x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 + 18x^5yb_2 + 3x^4y^2a_2 + 3x^4y^2b_3 + 6x^4yb_1 + 13b_2x^4 + 8x^3ya_2 + 8x^3yb_3 + 16x^2y^2a_3 + 4x^3b_1 + 12x^2ya_1 + 6x^2a_2 + 6x^2b_3 + 12xya_3 + 12xa_1 - 4a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_3v_1^4v_2^4 + 6b_2v_1^6v_2^2 - 2a_1v_1^4v_2^3 + 2b_1v_1^5v_2^2 + 3a_2v_1^4v_2^2 + 18b_2v_1^5v_2 + 3b_3v_1^4v_2^2 + 6b_1v_1^4v_2 + 8a_2v_1^3v_2 + 16a_3v_1^2v_2^2 + 13b_2v_1^4 + 8b_3v_1^3v_2 + 12a_1v_1^2v_2 + 4b_1v_1^3 + 6a_2v_1^2 + 12a_3v_1v_2 + 6b_3v_1^2 + 12a_1v_1 - 4a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^6v_2^2 + 2b_1v_1^5v_2^2 + 18b_2v_1^5v_2 - 3a_3v_1^4v_2^4 - 2a_1v_1^4v_2^3 + (3a_2 + 3b_3)\, v_1^4v_2^2 + 6b_1v_1^4v_2 + 13b_2v_1^4 + (8a_2 + 8b_3)\, v_1^3v_2 + 4b_1v_1^3 + 16a_3v_1^2v_2^2 + 12a_1v_1^2v_2 + (6a_2 + 6b_3)\, v_1^2 + 12a_3v_1v_2 + 12a_1v_1 - 4a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0\\
12a_1 &= 0\\
-4a_3 &= 0\\
-3a_3 &= 0\\
12a_3 &= 0\\
16a_3 &= 0\\
2b_1 &= 0\\
4b_1 &= 0\\
6b_1 &= 0\\
6b_2 &= 0\\
13b_2 &= 0\\
18b_2 &= 0\\
3a_2 + 3b_3 &= 0\\
6a_2 + 6b_3 &= 0\\
8a_2 + 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x^2y^2-2}{x^2(2yx+3)}\right)(-x) \\
&= \frac{x^2y^2+3yx+2}{2y\,x^2+3x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2y^2+3yx+2}{2y\,x^2+3x}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(x^2y^2+3yx+2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2y^2 - 2}{x^2 (2yx + 3)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x\, y^2 + 3y}{(yx + 2)\,(yx + 1)} \\ S_y &= \frac{2y\, x^2 + 3x}{(yx + 2)\,(yx + 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln (R) + c_1 \tag{4}$$

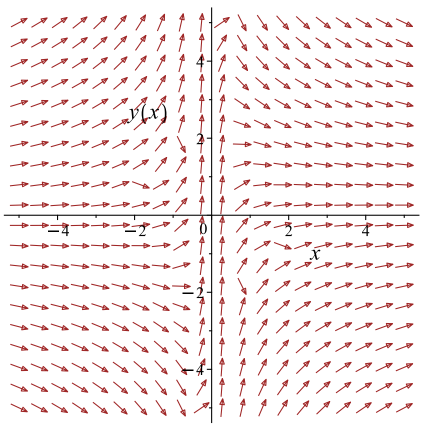
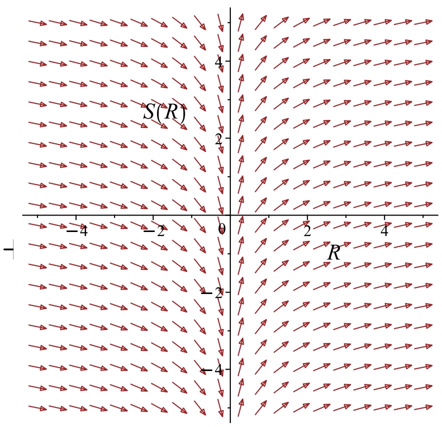
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln (yx + 2) + \ln (yx + 1) = \ln (x) + c_1$$

Which simplifies to

$$\ln (yx + 2) + \ln (yx + 1) = \ln (x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2y^2-2}{x^2(2yx+3)}$ 	$R = x$ $S = \ln(yx+2) + \ln(yx +$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) + \ln(3) + \ln(5) = \ln(2) + c_1$$

$$c_1 = \ln(3) + \ln(5)$$

Substituting c_1 found above in the general solution gives

$$\ln(yx+2) + \ln(yx+1) = \ln(x) + \ln(3) + \ln(5)$$

Summary

The solution(s) found are the following

$$\ln(yx+2) + \ln(yx+1) = \ln(x) + \ln(3) + \ln(5) \tag{1}$$

Verification of solutions

$$\ln(yx+2) + \ln(yx+1) = \ln(x) + \ln(3) + \ln(5)$$

Verified OK.

5.53.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(-\frac{2y}{x}+\frac{3x^2y^2+6yx+2}{x^2(2yx+3)}\right)\mathrm{d}x$$

$$\left(\frac{2y}{x}-\frac{3x^2y^2+6yx+2}{x^2(2yx+3)}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{2y}{x} - \frac{3x^2y^2 + 6yx + 2}{x^2 (2yx + 3)} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{x} - \frac{3x^2y^2 + 6yx + 2}{x^2 (2yx + 3)}\right) \\
&= \frac{2x^2y^2 + 6yx + 4}{x (2yx + 3)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{2}{x} - \frac{6y\,x^2 + 6x}{x^2 (2yx + 3)} + \frac{6x^2y^2 + 12yx + 4}{x (2yx + 3)^2}\right) - (0)\right) \\
&= \frac{2x^2y^2 + 6yx + 4}{x (2yx + 3)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{x^2(2yx + 3)}{x^2y^2 - 2}\left((0) - \left(\frac{2}{x} - \frac{6y\,x^2 + 6x}{x^2 (2yx + 3)} + \frac{6x^2y^2 + 12yx + 4}{x (2yx + 3)^2}\right)\right) \\
&= -\frac{2(x^2y^2 + 3yx + 2)\,x}{(2yx + 3)\,(x^2y^2 - 2)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(0) - \left(\frac{2}{x} - \frac{6y\,x^2+6x}{x^2(2yx+3)} + \frac{6x^2y^2+12yx+4}{x(2yx+3)^2}\right)}{x\left(\frac{2y}{x} - \frac{3x^2y^2+6yx+2}{x^2(2yx+3)}\right) - y\,(1)} \\ &= \frac{2}{2yx+3} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{2}{2t+3}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(\frac{2}{2t+3}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(2t+3)} \\ &= 2t+3 \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = 2yx+3$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= 2yx+3\left(\frac{2y}{x} - \frac{3x^2y^2+6yx+2}{x^2\,(2yx+3)}\right) \\ &= \frac{x^2y^2-2}{x^2} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= 2yx + 3(1) \\
&= 2yx + 3
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2y^2 - 2}{x^2}\right) + (2yx + 3)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2y^2 - 2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\,y^2 + \frac{2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx + 3$. Therefore equation (4) becomes

$$
2yx + 3 = 2yx + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 3
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (3)\,\mathrm{d}y$$
$$f(y) = 3y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^2 + \frac{2}{x} + 3y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 + \frac{2}{x} + 3y$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$15 = c_1$$

$$c_1 = 15$$

Substituting c_1 found above in the general solution gives

$$x\,y^2 + \frac{2}{x} + 3y = 15$$

The above simplifies to

$$x^2y^2 + 3yx - 15x + 2 = 0$$

Summary

The solution(s) found are the following

$$2 + x^2y^2 + 3(y - 5)\,x = 0 \tag{1}$$

Verification of solutions

$$2 + x^2y^2 + 3(y - 5)\,x = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)+2/x*y(x)=(3*x^2*y(x)^2+6*x*y(x)+2)/(x^2*(2*x*y(x)+3)),y(2) = 2],y(x), s
```

$$y(x) = \frac{-3 + \sqrt{60x + 1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.725 (sec). Leaf size: 35

```
DSolve[{y'[x]+2/x*y[x]==(3*x^2*y[x]^2+6*x*y[x]+2)/(x^2*(2*x*y[x]+3)),y[2]==2},y[x],x,Include
```

$$y(x) \to \frac{\sqrt{\frac{1}{x^2}}\sqrt{x^2(60x + 1)} - 3}{2x}$$

5.54 problem 53

Internal problem ID [1028]
Internal file name [`OUTPUT/1029_Sunday_June_05_2022_01_57_17_AM_39879720/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' + \frac{3y}{x} - \frac{3x^4y^2 + 10x^2y + 6}{x^3\,(2x^2y + 5)} = 0}$$

With initial conditions

$$[y(1) = 1]$$

5.54.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{3x^4y^2 + 5y\,x^2 - 6}{x^3\,(2y\,x^2 + 5)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\left\{y < -\frac{5}{2} \lor -\frac{5}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x^4y^2 + 5y\,x^2 - 6}{x^3\,(2y\,x^2 + 5)}\right)\\ &= -\frac{6x^4y + 5x^2}{x^3\,(2y\,x^2 + 5)} + \frac{6x^4y^2 + 10y\,x^2 - 12}{x\,(2y\,x^2 + 5)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{5}{2} \lor -\frac{5}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

5.54.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{3x^4y^2 + 5y\,x^2 - 6}{x^3\,(2y\,x^2 + 5)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(3x^4y^2+5y\,x^2-6)\,(b_3-a_2)}{x^3\,(2y\,x^2+5)} - \frac{(3x^4y^2+5y\,x^2-6)^2\,a_3}{x^6\,(2y\,x^2+5)^2} - \left(-\frac{12x^3y^2+10yx}{x^3\,(2y\,x^2+5)} + \frac{9x^4y^2+15y\,x^2-18}{x^4\,(2y\,x^2+5)} + \frac{4(3x^4y^2+5y\,x^2-6)\,y}{x^2\,(2y\,x^2+5)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{6x^4y+5x^2}{x^3\,(2y\,x^2+5)} + \frac{6x^4y^2+10y\,x^2-12}{x\,(2y\,x^2+5)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^{10}y^2b_2 - 15x^8y^4a_3 + 6x^9y^2b_1 - 6x^8y^3a_1 + 50x^8yb_2 + 10x^7y^2a_2 + 5x^7y^2b_3 - 45x^6y^3a_3 + 30x^7yb_1 - 15x}{x} = 0$$

Setting the numerator to zero gives

$$10x^{10}y^2b_2 - 15x^8y^4a_3 + 6x^9y^2b_1 - 6x^8y^3a_1 + 50x^8yb_2 + 10x^7y^2a_2 + 5x^7y^2b_3 - 45x^6y^3a_3 + 30x^7yb_1 - 15x^6y^2a_1 + 62b_2x^6 + 48x^5ya_2 + 24x^5yb_3 + 46x^4y^2a_3 + 37x^5b_1 + 35x^4ya_1 + 60x^3a_2 + 30x^3b_3 + 150x^2ya_3 + 90x^2a_1 - 36a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-15a_3v_1^8v_2^4 + 10b_2v_1^{10}v_2^2 - 6a_1v_1^8v_2^3 + 6b_1v_1^9v_2^2 + 10a_2v_1^7v_2^2 - 45a_3v_1^6v_2^3 + 50b_2v_1^8v_2 + 5b_3v_1^7v_2^2 - 15a_1v_1^6v_2^2 + 30b_1v_1^7v_2 + 48a_2v_1^5v_2 + 46a_3v_1^4v_2^2 + 62b_2v_1^6 + 24b_3v_1^5v_2 + 35a_1v_1^4v_2 + 37b_1v_1^5 + 60a_2v_1^3 + 150a_3v_1^2v_2 + 30b_3v_1^3 + 90a_1v_1^2 - 36a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&10b_2v_1^{10}v_2^2 + 6b_1v_1^9v_2^2 - 15a_3v_1^8v_2^4 - 6a_1v_1^8v_2^3 + 50b_2v_1^8v_2 + (10a_2 + 5b_3)\, v_1^7v_2^2 \\
&+ 30b_1v_1^7v_2 - 45a_3v_1^6v_2^3 - 15a_1v_1^6v_2^2 + 62b_2v_1^6 + (48a_2 + 24b_3)\, v_1^5v_2 + 37b_1v_1^5 \\
&+ 46a_3v_1^4v_2^2 + 35a_1v_1^4v_2 + (60a_2 + 30b_3)\, v_1^3 + 150a_3v_1^2v_2 + 90a_1v_1^2 - 36a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-15a_1 &= 0 \\
-6a_1 &= 0 \\
35a_1 &= 0 \\
90a_1 &= 0 \\
-45a_3 &= 0 \\
-36a_3 &= 0 \\
-15a_3 &= 0 \\
46a_3 &= 0 \\
150a_3 &= 0 \\
6b_1 &= 0 \\
30b_1 &= 0 \\
37b_1 &= 0 \\
10b_2 &= 0 \\
50b_2 &= 0 \\
62b_2 &= 0 \\
10a_2 + 5b_3 &= 0 \\
48a_2 + 24b_3 &= 0 \\
60a_2 + 30b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(-\frac{3x^4y^2 + 5y\,x^2 - 6}{x^3\,(2y\,x^2 + 5)}\right)(x) \\
&= \frac{-x^4y^2 - 5y\,x^2 - 6}{2x^4y + 5x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^4y^2 - 5y\,x^2 - 6}{2x^4y + 5x^2}} dy
\end{aligned}
$$

Which results in

$$S = -\ln\left(x^4y^2 + 5y\,x^2 + 6\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^4y^2 + 5y\,x^2 - 6}{x^3\,(2y\,x^2 + 5)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{2yx}{y\,x^2 + 3} - \frac{2yx}{y\,x^2 + 2}\\
S_y &= \frac{-2x^4y - 5x^2}{(y\,x^2 + 3)\,(y\,x^2 + 2)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

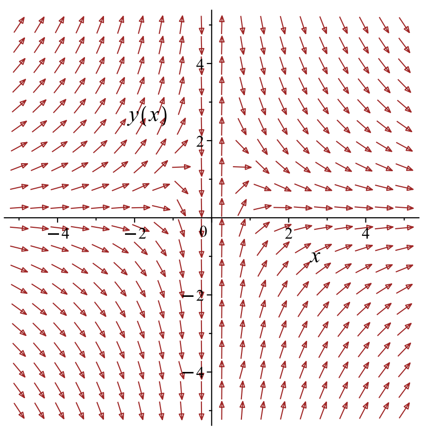
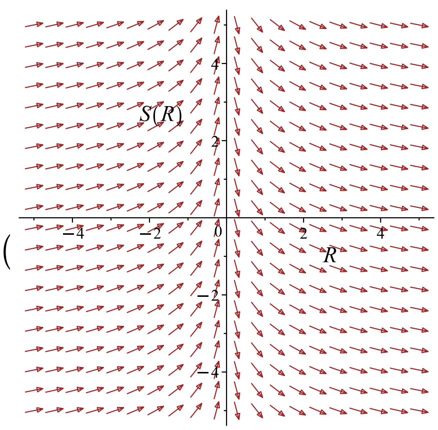
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(x^2y + 3\right) - \ln\left(x^2y + 2\right) = -\ln(x) + c_1$$

Which simplifies to

$$-\ln\left(x^2 y + 3\right) - \ln\left(x^2 y + 2\right) = -\ln\left(x\right) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^4y^2+5y\,x^2-6}{x^3(2y\,x^2+5)}$ 	$R = x$ $S = -\ln\left(y\,x^2+3\right) - \ln\left(\right.$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-2\ln\left(2\right) - \ln\left(3\right) = c_1$$

$$c_1 = -2\ln\left(2\right) - \ln\left(3\right)$$

Substituting c_1 found above in the general solution gives

$$-\ln\left(y\,x^2 + 3\right) - \ln\left(y\,x^2 + 2\right) = -\ln\left(x\right) - 2\ln\left(2\right) - \ln\left(3\right)$$

Summary

The solution(s) found are the following

$$-\ln\left(x^2 y + 3\right) - \ln\left(x^2 y + 2\right) = -\ln\left(x\right) - 2\ln\left(2\right) - \ln\left(3\right) \tag{1}$$

Verification of solutions

$$-\ln\left(x^2 y + 3\right) - \ln\left(x^2 y + 2\right) = -\ln\left(x\right) - 2\ln\left(2\right) - \ln\left(3\right)$$

Verified OK.

5.54.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3\left(2y\,x^2+5\right)\right)\mathrm{d}y&=\left(-3x^4y^2-5y\,x^2+6\right)\mathrm{d}x\\ \left(3x^4y^2+5y\,x^2-6\right)\mathrm{d}x+\left(x^3\left(2y\,x^2+5\right)\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^4y^2+5y\,x^2-6\\ N(x,y)&=x^3\left(2y\,x^2+5\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^4y^2 + 5y\,x^2 - 6\right)\\ &= 6x^4y + 5x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3\left(2y\,x^2 + 5\right)\right)\\ &= 10x^4y + 15x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y\,x^5 + 5x^3}\left(\left(6x^4y + 5x^2\right) - \left(3x^2\left(2y\,x^2 + 5\right) + 4x^4y\right)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(3x^4y^2 + 5y\,x^2 - 6\right)\\ &= \frac{3x^4y^2 + 5y\,x^2 - 6}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(x^3\left(2y\,x^2+5\right)\right) \\
&= 2y\,x^3+5x
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x^4y^2+5y\,x^2-6}{x^2}\right)+\left(2y\,x^3+5x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{3x^4y^2+5y\,x^2-6}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=x^3y^2+5yx+\frac{6}{x}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=2y\,x^3+5x+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=2y\,x^3+5x$. Therefore equation (4) becomes

$$
2y\,x^3+5x=2y\,x^3+5x+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3y^2 + 5yx + \frac{6}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3y^2 + 5yx + \frac{6}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$12 = c_1$$

$$c_1 = 12$$

Substituting c_1 found above in the general solution gives

$$x^3y^2 + 5yx + \frac{6}{x} = 12$$

The above simplifies to

$$x^4y^2 + 5y\,x^2 - 12x + 6 = 0$$

Summary
The solution(s) found are the following

$$x^4y^2 + 5x^2y - 12x + 6 = 0 \tag{1}$$

Verification of solutions

$$x^4y^2 + 5x^2y - 12x + 6 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.078 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)+3/x*y(x)=(3*x^4*y(x)^2+10*x^2*y(x)+6)/(x^3*(2*x^2*y(x)+5)),y(1) = 1],y(
```

$$y(x) = \frac{-5 + \sqrt{48x + 1}}{2x^2}$$

✓ Solution by Mathematica
Time used: 0.815 (sec). Leaf size: 37

```
DSolve[{y'[x]+3/x*y[x]==(3*x^4*y[x]^2+10*x^2*y[x]+6)/(x^3*(2*x^2*y[x]+5)),y[1]==1},y[x],x,In
```

$$y(x) \to \frac{\sqrt{\frac{1}{x^2}}\sqrt{x^4(48x + 1)} - 5x}{2x^3}$$

5.55 problem 56

Internal problem ID [1029]
Internal file name [`OUTPUT/1030_Sunday_June_05_2022_01_57_19_AM_22874992/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Transformation of Nonlinear Equations into Separable Equations. Section 2.4 Page 68
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + (1+2x)\,y - x y^2 = x + 1}$$

5.55.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = x\,y^2 - 2yx + x - y + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + yxa_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + yxb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4+yb_5+b_2+\left(x\,y^2-2yx+x-y+1\right)\left(-2xa_4+xb_5-ya_5+2yb_6-a_2+b_3\right)\\
&-\left(x\,y^2-2yx+x-y+1\right)^2\left(xa_5+2ya_6+a_3\right)\\
&-\left(y^2-2y+1\right)\left(x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1\right)\\
&-\left(2yx-2x-1\right)\left(x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1\right)=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
&-x^3y^4a_5-2x^2y^5a_6+4x^3y^3a_5-x^2y^4a_3+8x^2y^4a_6-6x^3y^2a_5+4x^2y^3a_3\\
&+2x^2y^3a_5-12x^2y^3a_6+4x\,y^4a_6+4x^3ya_5-2x^3yb_4-6x^2y^2a_3-3x^2y^2a_4\\
&-6x^2y^2a_5+8x^2y^2a_6-x^2y^2b_5+2x\,y^3a_3-2x\,y^3a_5-12x\,y^3a_6-y^4a_6\\
&-x^3a_5+2x^3b_4+4x^2ya_3+6x^2ya_4+6x^2ya_5-2x^2ya_6-2x^2yb_2-2x\,y^2a_2\\
&-6x\,y^2a_3+3x\,y^2a_5+12x\,y^2a_6-x\,y^2b_3-2x\,y^2b_6-y^3a_3-x^2a_3\\
&-3x^2a_4-2x^2a_5+2x^2b_2+x^2b_4+x^2b_5+4xya_2+6xya_3+2xya_4\\
&-4xya_6-2xyb_1+2xyb_6-y^2a_1+y^2a_3+y^2a_5+3y^2a_6-y^2b_6-2xa_2\\
&-2xa_3-2xa_4-xa_5+2xb_1+xb_2+xb_3+2xb_4+xb_5+2ya_1+ya_2\\
&+ya_3-ya_5-2ya_6+yb_5+2yb_6-a_1-a_2-a_3+b_1+b_2+b_3=0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^3y^4a_5-2x^2y^5a_6+4x^3y^3a_5-x^2y^4a_3+8x^2y^4a_6-6x^3y^2a_5\\
&+4x^2y^3a_3+2x^2y^3a_5-12x^2y^3a_6+4x\,y^4a_6+4x^3ya_5-2x^3yb_4\\
&-6x^2y^2a_3-3x^2y^2a_4-6x^2y^2a_5+8x^2y^2a_6-x^2y^2b_5+2x\,y^3a_3-2x\,y^3a_5\\
&-12x\,y^3a_6-y^4a_6-x^3a_5+2x^3b_4+4x^2ya_3+6x^2ya_4+6x^2ya_5-2x^2ya_6\\
&-2x^2yb_2-2x\,y^2a_2-6x\,y^2a_3+3x\,y^2a_5+12x\,y^2a_6-x\,y^2b_3-2x\,y^2b_6\\
&-y^3a_3-x^2a_3-3x^2a_4-2x^2a_5+2x^2b_2+x^2b_4+x^2b_5+4xya_2+6xya_3\\
&+2xya_4-4xya_6-2xyb_1+2xyb_6-y^2a_1+y^2a_3+y^2a_5+3y^2a_6-y^2b_6\\
&-2xa_2-2xa_3-2xa_4-xa_5+2xb_1+xb_2+xb_3+2xb_4+xb_5+2ya_1\\
&+ya_2+ya_3-ya_5-2ya_6+yb_5+2yb_6-a_1-a_2-a_3+b_1+b_2+b_3=0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_5 v_1^3 v_2^4 - 2a_6 v_1^2 v_2^5 - a_3 v_1^2 v_2^4 + 4a_5 v_1^3 v_2^3 + 8a_6 v_1^2 v_2^4 + 4a_3 v_1^2 v_2^3 \\ &- 6a_5 v_1^3 v_2^2 + 2a_5 v_1^2 v_2^3 - 12a_6 v_1^2 v_2^3 + 4a_6 v_1 v_2^4 - 6a_3 v_1^2 v_2^2 + 2a_3 v_1 v_2^3 \\ &- 3a_4 v_1^2 v_2^2 + 4a_5 v_1^3 v_2 - 6a_5 v_1^2 v_2^2 - 2a_5 v_1 v_2^3 + 8a_6 v_1^2 v_2^2 - 12a_6 v_1 v_2^3 \\ &- a_6 v_2^4 - 2b_4 v_1^3 v_2 - b_5 v_1^2 v_2^2 - 2a_2 v_1 v_2^2 + 4a_3 v_1^2 v_2 - 6a_3 v_1 v_2^2 - a_3 v_2^3 \\ &+ 6a_4 v_1^2 v_2 - a_5 v_1^3 + 6a_5 v_1^2 v_2 + 3a_5 v_1 v_2^2 - 2a_6 v_1^2 v_2 + 12a_6 v_1 v_2^2 - 2b_2 v_1^2 v_2 \\ &- b_3 v_1 v_2^2 + 2b_4 v_1^3 - 2b_6 v_1 v_2^2 - a_1 v_2^2 + 4a_2 v_1 v_2 - a_3 v_1^2 + 6a_3 v_1 v_2 + a_3 v_2^2 \\ &- 3a_4 v_1^2 + 2a_4 v_1 v_2 - 2a_5 v_1^2 + a_5 v_2^2 - 4a_6 v_1 v_2 + 3a_6 v_2^2 - 2b_1 v_1 v_2 \\ &+ 2b_2 v_1^2 + b_4 v_1^2 + b_5 v_1^2 + 2b_6 v_1 v_2 - b_6 v_2^2 + 2a_1 v_2 - 2a_2 v_1 + a_2 v_2 \\ &- 2a_3 v_1 + a_3 v_2 - 2a_4 v_1 - a_5 v_1 - a_5 v_2 - 2a_6 v_2 + 2b_1 v_1 + b_2 v_1 + b_3 v_1 \\ &+ 2b_4 v_1 + b_5 v_1 + b_5 v_2 + 2b_6 v_2 - a_1 - a_2 - a_3 + b_1 + b_2 + b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(4a_5 - 2b_4) v_1^3 v_2 + (-a_3 + 8a_6) v_1^2 v_2^4 + (4a_3 + 2a_5 - 12a_6) v_1^2 v_2^3 \\ &+ (-6a_3 - 3a_4 - 6a_5 + 8a_6 - b_5) v_1^2 v_2^2 + (4a_3 + 6a_4 + 6a_5 - 2a_6 - 2b_2) v_1^2 v_2 \\ &+ (2a_3 - 2a_5 - 12a_6) v_1 v_2^3 + (-2a_2 - 6a_3 + 3a_5 + 12a_6 - b_3 - 2b_6) v_1 v_2^2 \\ &+ (4a_2 + 6a_3 + 2a_4 - 4a_6 - 2b_1 + 2b_6) v_1 v_2 + b_1 + b_2 + b_3 - a_1 \\ &- a_2 - a_3 + (-a_3 - 3a_4 - 2a_5 + 2b_2 + b_4 + b_5) v_1^2 + (-a_5 + 2b_4) v_1^3 \\ &- a_5 v_1^3 v_2^4 - 2a_6 v_1^2 v_2^5 + 4a_6 v_1 v_2^4 - 6a_5 v_1^3 v_2^2 + 4a_5 v_1^3 v_2^3 - a_6 v_2^4 - a_3 v_2^3 \\ &+ (-a_1 + a_3 + a_5 + 3a_6 - b_6) v_2^2 + (2a_1 + a_2 + a_3 - a_5 - 2a_6 + b_5 + 2b_6) v_2 \\ &+ (-2a_2 - 2a_3 - 2a_4 - a_5 + 2b_1 + b_2 + b_3 + 2b_4 + b_5) v_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_3 &= 0 \\
-6a_5 &= 0 \\
-a_5 &= 0 \\
4a_5 &= 0 \\
-2a_6 &= 0 \\
-a_6 &= 0 \\
4a_6 &= 0 \\
-a_3 + 8a_6 &= 0 \\
-a_5 + 2b_4 &= 0 \\
4a_5 - 2b_4 &= 0 \\
2a_3 - 2a_5 - 12a_6 &= 0 \\
4a_3 + 2a_5 - 12a_6 &= 0 \\
-a_1 + a_3 + a_5 + 3a_6 - b_6 &= 0 \\
-6a_3 - 3a_4 - 6a_5 + 8a_6 - b_5 &= 0 \\
4a_3 + 6a_4 + 6a_5 - 2a_6 - 2b_2 &= 0 \\
-a_1 - a_2 - a_3 + b_1 + b_2 + b_3 &= 0 \\
-2a_2 - 6a_3 + 3a_5 + 12a_6 - b_3 - 2b_6 &= 0 \\
4a_2 + 6a_3 + 2a_4 - 4a_6 - 2b_1 + 2b_6 &= 0 \\
-a_3 - 3a_4 - 2a_5 + 2b_2 + b_4 + b_5 &= 0 \\
2a_1 + a_2 + a_3 - a_5 - 2a_6 + b_5 + 2b_6 &= 0 \\
-2a_2 - 2a_3 - 2a_4 - a_5 + 2b_1 + b_2 + b_3 + 2b_4 + b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_6 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_6 \\
b_2 &= 0 \\
b_3 &= -2b_6 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= y^2 - 2y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y^2 - 2y + 1 - \left(x\, y^2 - 2yx + x - y + 1\right)(-1) \\
&= x\, y^2 - 2yx + y^2 + x - 3y + 2 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2 - 2yx + y^2 + x - 3y + 2} dy \end{aligned}$$

Which results in

$$S = -\ln(y-1) + \ln(yx - x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\,y^2 - 2yx + x - y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y-1}{(y-1)\,x + y - 2} \\ S_y &= \frac{1}{(yx - x + y - 2)\,(y-1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y-1) + \ln((y-1)x + y - 2) = x + c_1$$

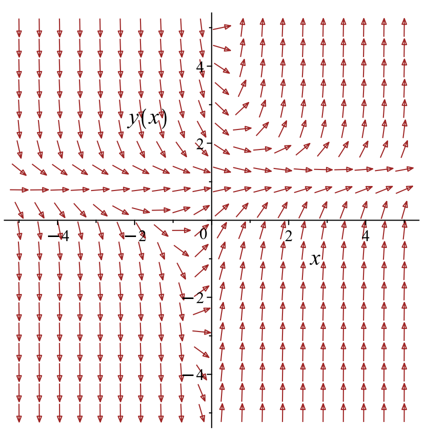
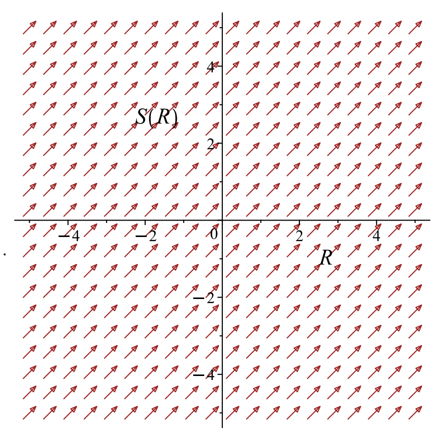
Which simplifies to

$$-\ln(y-1) + \ln((y-1)x + y - 2) = x + c_1$$

Which gives

$$y = \frac{-2 + e^{x+c_1} - x}{e^{x+c_1} - x - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,y^2 - 2yx + x - y + 1$ 	$R = x$ $S = -\ln(y-1) + \ln((y$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{-2 + e^{x+c_1} - x}{e^{x+c_1} - x - 1} \tag{1}$$

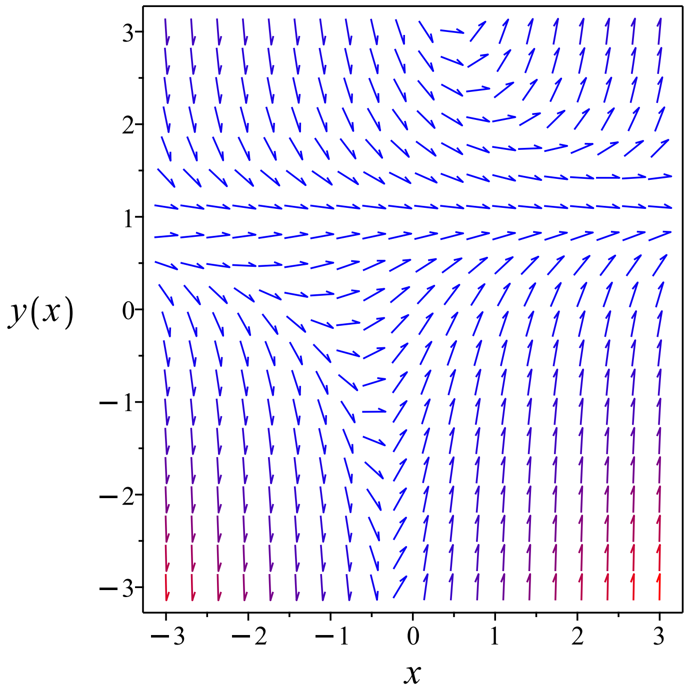


Figure 350: Slope field plot

Verification of solutions

$$y = \frac{-2 + e^{x+c_1} - x}{e^{x+c_1} - x - 1}$$

Verified OK.

5.55.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x\,y^2 - 2yx + x - y + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x\,y^2 - 2yx + x - y + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x+1$, $f_1(x) = -1-2x$ and $f_2(x) = x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1 f_2 &= (-1-2x) x \\ f_2^2 f_0 &= x^2 (x+1) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$xu''(x) - (1 + (-1-2x) x) u'(x) + x^2 (x+1) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{-\frac{x^2}{2}} + c_2 (x+1) e^{-\frac{x(2+x)}{2}}$$

The above shows that

$$u'(x) = -\left(c_1 e^{-\frac{x^2}{2}} + c_2 e^{-\frac{x(2+x)}{2}} (2+x)\right) x$$

Using the above in (1) gives the solution

$$y = \frac{c_1 e^{-\frac{x^2}{2}} + c_2 e^{-\frac{x(2+x)}{2}} (2+x)}{c_1 e^{-\frac{x^2}{2}} + c_2 (x+1) e^{-\frac{x(2+x)}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + 2 e^{-\frac{x(2+x)}{2}}}{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + e^{-\frac{x(2+x)}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + 2\,e^{-\frac{x(2+x)}{2}}}{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + e^{-\frac{x(2+x)}{2}}} \tag{1}$$

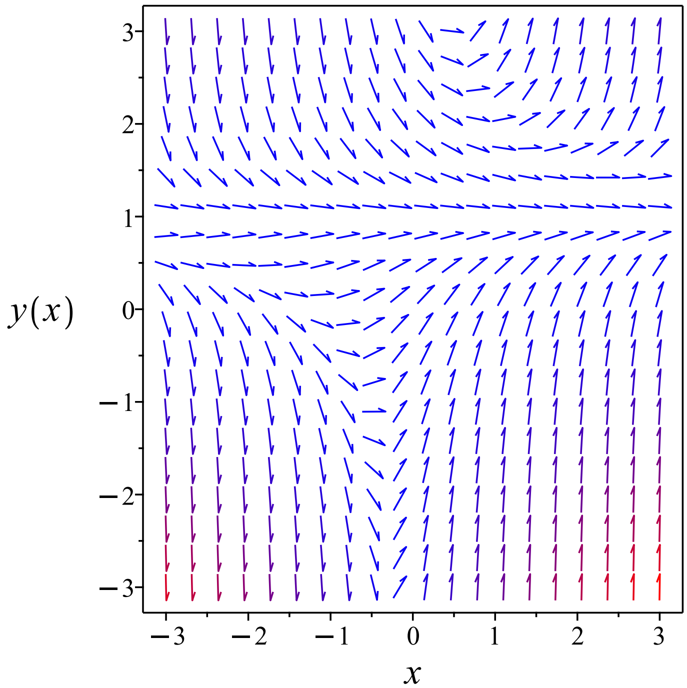


Figure 351: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + 2\,e^{-\frac{x(2+x)}{2}}}{e^{-\frac{x(2+x)}{2}} x + c_3 e^{-\frac{x^2}{2}} + e^{-\frac{x(2+x)}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (a) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x)=1+x-(1+2*x)*y(x)+x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(2x+4)\,\mathrm{e}^{-x} - c_1}{(2+2x)\,\mathrm{e}^{-x} - c_1}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 31

```
DSolve[y'[x]==1+x-(1+2*x)*y[x]+x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_1 e^x + 2}{x + c_1 e^x + 1}$$
$$y(x) \to 1$$

6 Chapter 2, First order equations. Exact equations. Section 2.5 Page 79

6.1 problem 1

Internal problem ID [1030]
Internal file name [`OUTPUT/1031_Sunday_June_05_2022_01_57_20_AM_80314616/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{6x^2y^2 + 4y'yx^3 = 0}$$

6.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{3y}{2x} \end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{3}{2x}\,dx$$
$$\int \frac{1}{y}\,dy = \int -\frac{3}{2x}\,dx$$
$$\ln(y) = -\frac{3\ln(x)}{2} + c_1$$
$$y = e^{-\frac{3\ln(x)}{2}+c_1}$$
$$= \frac{c_1}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

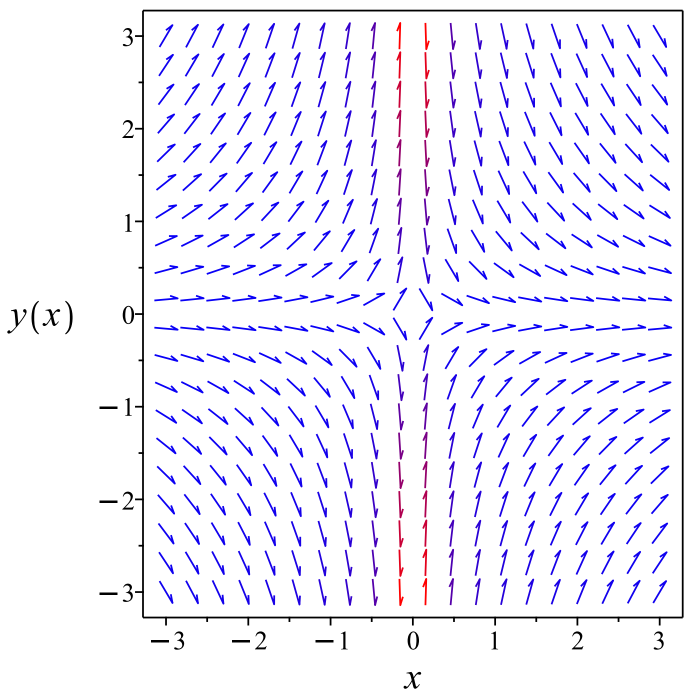


Figure 352: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

6.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{2x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{2x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{3}{2x}dx} \\ &= x^{\frac{3}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{3}{2}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$x^{\frac{3}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = x^{\frac{3}{2}}$ results in

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

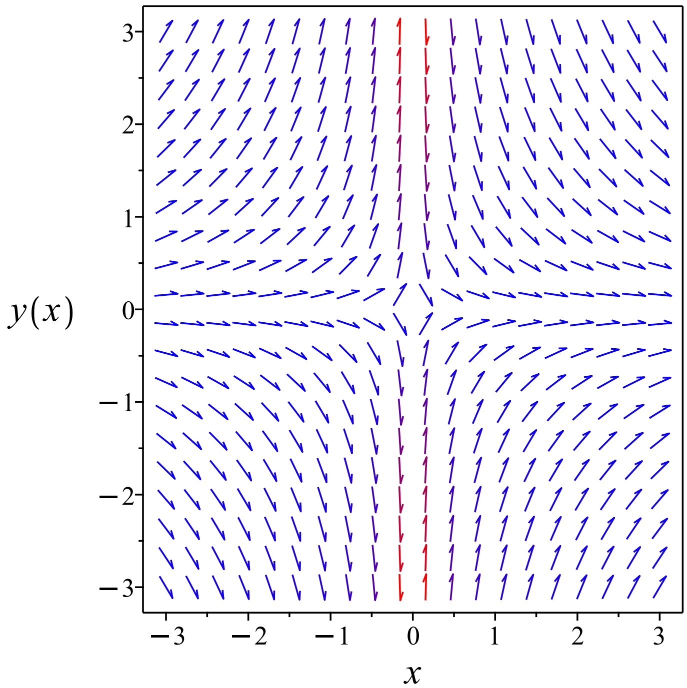


Figure 353: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

6.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$6x^4 u(x)^2 + 4(u'(x)\,x + u(x))\,u(x)\,x^4 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\frac{1}{u}\,du = -\frac{5}{2x}\,dx$$
$$\int \frac{1}{u}\,du = \int -\frac{5}{2x}\,dx$$
$$\ln(u) = -\frac{5\ln(x)}{2} + c_2$$
$$u = e^{-\frac{5\ln(x)}{2}+c_2}$$
$$= \frac{c_2}{x^{\frac{5}{2}}}$$

Therefore the solution y is

$$y = xu$$
$$= \frac{c_2}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{x^{\frac{3}{2}}} \tag{1}$$

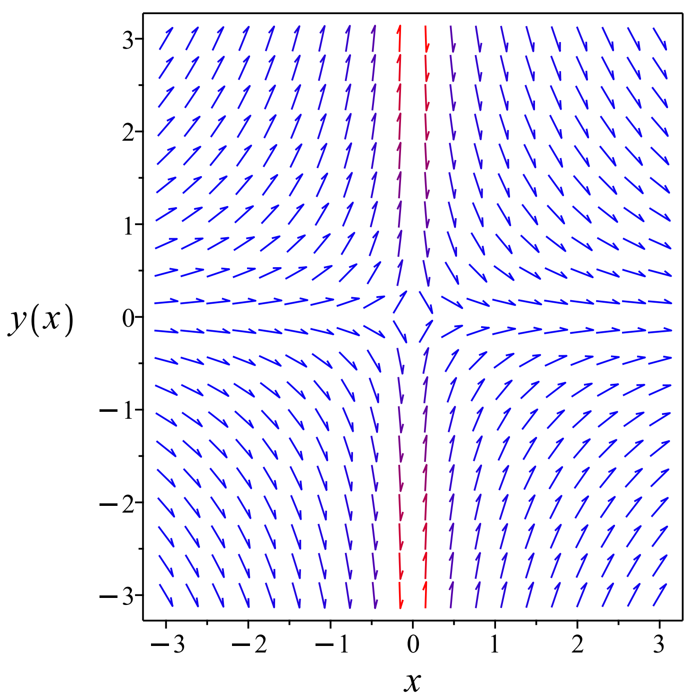


Figure 354: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^{\frac{3}{2}}}$$

Verified OK.

6.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 267: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^{\frac{3}{2}}}} dy \end{aligned}$$

Which results in

$$S = x^{\frac{3}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3y\sqrt{x}}{2} \\ S_y &= x^{\frac{3}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^{\frac{3}{2}} = c_1$$

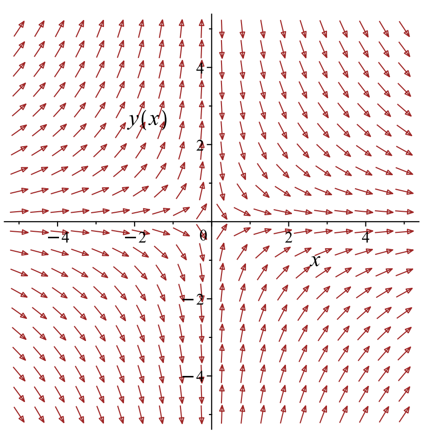
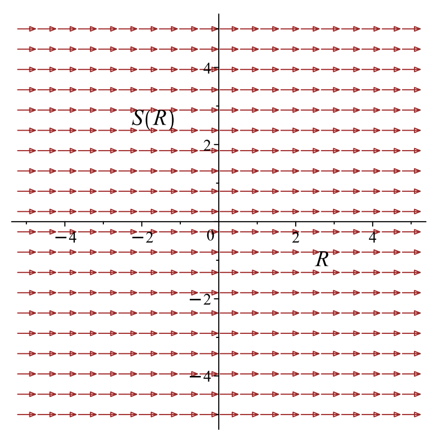
Which simplifies to

$$y x^{\frac{3}{2}} = c_1$$

Which gives

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y}{2x}$ 	$R = x$ $S = x^{\frac{3}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

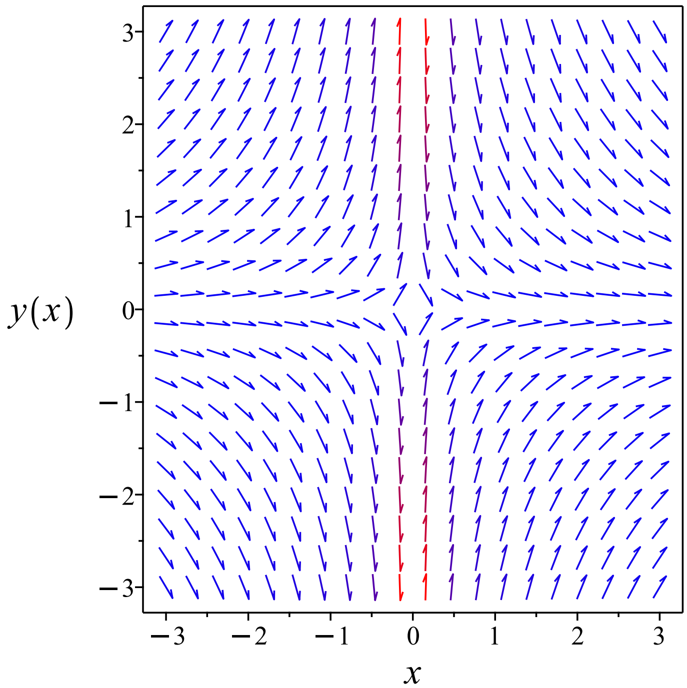


Figure 355: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

6.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{2}{3y}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{2}{3y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x}$$
$$N(x,y) = -\frac{2}{3y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2}{3y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{1}{x} \,\mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2}{3y}$. Therefore equation (4) becomes

$$-\frac{2}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{3y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-\frac{2}{3y}\right) \mathrm{d}y$$
$$f(y) = -\frac{2\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{2\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{2\ln(y)}{3}$$

The solution becomes

$$y = e^{-\frac{3\ln(x)}{2} - \frac{3c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{3\ln(x)}{2} - \frac{3c_1}{2}} \tag{1}$$

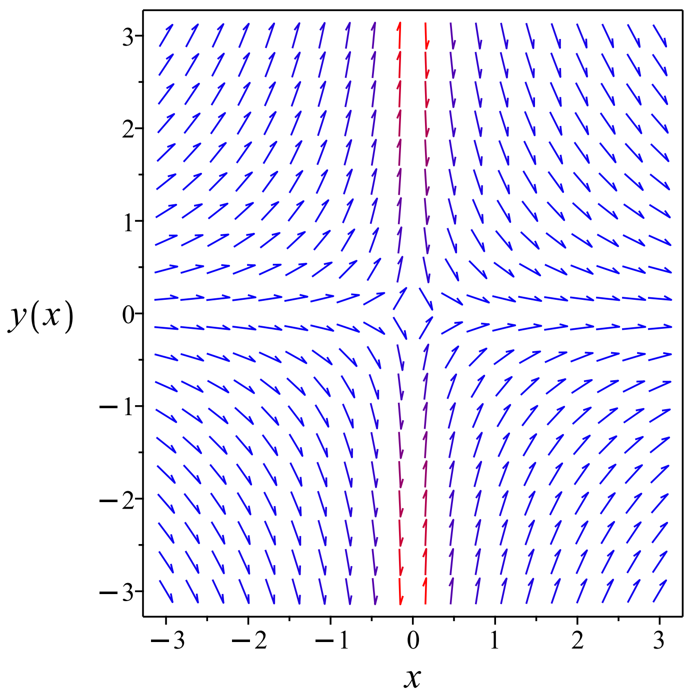


Figure 356: Slope field plot

Verification of solutions

$$y = e^{-\frac{3\ln(x)}{2}-\frac{3c_1}{2}}$$

Verified OK.

6.1.6 Maple step by step solution

Let's solve

$6x^2y^2 + 4y'yx^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (6x^2y^2 + 4y'yx^3)\,dx = \int 0dx + c_1$

- Evaluate integral

  $2x^3y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{2}\,\sqrt{c_1x}}{2x^2}, y = \frac{\sqrt{2}\,\sqrt{c_1x}}{2x^2}\right\}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(6*x^2*y(x)^2+4*x^3*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= -\frac{\sqrt{2}\,\sqrt{-c_1 x}}{2x^2} \\
y(x) &= \frac{\sqrt{2}\,\sqrt{-c_1 x}}{2x^2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 23

```
DSolve[6*x^2*y[x]^2+4*x^3*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to 0 \\
y(x) &\to \frac{c_1}{x^{3/2}} \\
y(x) &\to 0
\end{aligned}
$$

6.2 problem 2

Internal problem ID [1031]
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Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3\cos(x)\,y + 2yx^3 + (3\sin(x) + 3)\,y' = -4x\,\mathrm{e}^x}$$

6.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x^3 - 3\cos(x)}{3\sin(x) + 3}$$
$$q(x) = -\frac{4x\,\mathrm{e}^x}{3\sin(x) + 3}$$

Hence the ode is

$$y' - \frac{(-2x^3 - 3\cos(x))\,y}{3\sin(x) + 3} = -\frac{4x\,\mathrm{e}^x}{3\sin(x) + 3}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{4x\,e^{x}}{3\sin(x)+3}\right)$$

$$\frac{d}{dx}\left(e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx} y\right) = \left(e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx}\right)\left(-\frac{4x\,e^{x}}{3\sin(x)+3}\right)$$

$$d\left(e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx} y\right) = \left(-\frac{4x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{3\sin(x)+3}\right) dx$$

Integrating gives

$$e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx} y = \int -\frac{4x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{3\sin(x)+3}\, dx$$

$$e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx} y = \int -\frac{4x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{3\sin(x)+3} dx + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-2x^3-3\cos(x)}{3\sin(x)+3}dx}$ results in

$$y = e^{-\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}\left(\int -\frac{4x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{3\sin(x)+3} dx\right) + c_1 e^{-\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}$$

which simplifies to

$$y = -\frac{e^{-\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}\left(4\left(\int \frac{x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{\sin(x)+1} dx\right) - 3c_1\right)}{3}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}\left(4\left(\int \frac{x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{\sin(x)+1} dx\right) - 3c_1\right)}{3} \tag{1}$$

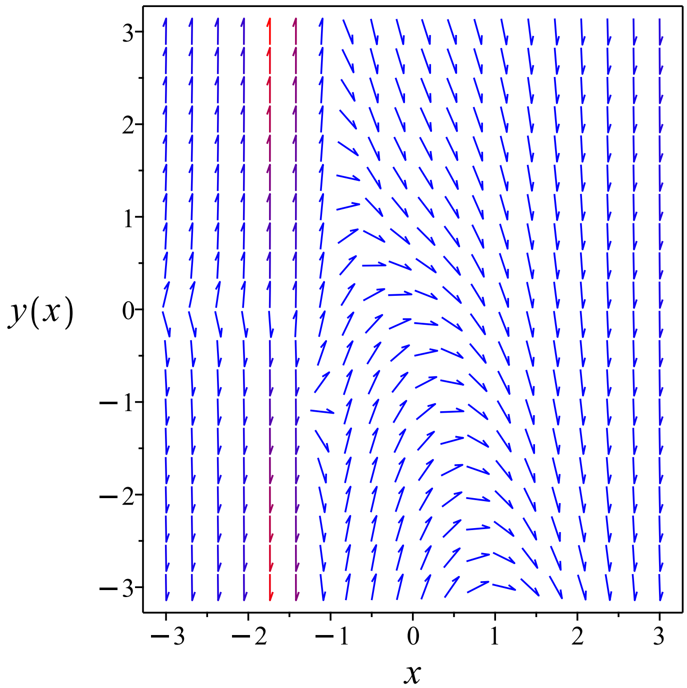


Figure 357: Slope field plot

Verification of solutions

$$y = -\frac{e^{-\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}\left(4\left(\int \frac{x\,e^{x+\frac{\left(\int \frac{2x^3+3\cos(x)}{\sin(x)+1}dx\right)}{3}}}{\sin(x)+1}dx\right)-3c_1\right)}{3}$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y\,x^3 + 4x\,e^x + 3\cos(x)\,y}{3(\sin(x)+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 270: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-ix+\frac{4x^3}{3(\mathrm{e}^{ix}+i)}-2\ln(\mathrm{e}^{ix}+i)+2\ln(\mathrm{e}^{ix})+\frac{4ix^3}{3}-4x^2\ln(1-i\mathrm{e}^{ix})+8ix\,\mathrm{polylog}(2,i\mathrm{e}^{ix})-8\,\mathrm{polylog}(3,i\mathrm{e}^{ix})}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-ix+\frac{4x^3}{3(e^{ix}+i)}-2\ln(e^{ix}+i)+2\ln(e^{ix})+\frac{4ix^3}{3}-4x^2\ln(1-ie^{ix})+8ix\,\mathrm{polylog}(2,ie^{ix})-8\,\mathrm{polylog}(3,ie^{ix})}} dy \end{aligned}$$

Which results in

$$S = \left(e^{ix}+i\right)^2 e^{-ix-\frac{4x^3}{3(e^{ix}+i)}-\frac{4ix^3}{3}+4x^2\ln(1-ie^{ix})-8ix\,\mathrm{polylog}(2,ie^{ix})+8\,\mathrm{polylog}(3,ie^{ix})} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y\,x^3 + 4x\,e^x + 3\cos(x)\,y}{3(\sin(x)+1)}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$

$$R_y = 0$$

$$S_x = \frac{4(e^{ix}+i)^{\frac{4x^2\left(e^{2ix}-1+2ie^{ix}\right)}{\left(e^{ix}+i\right)^2}} y\left(\left(-3ix\pi + 6x\ln(e^{ix}+i) - 6x\ln(1-ie^{ix}) + \frac{3i}{4}\right) e^{\frac{\left(-24ix\,e^{ix}+24x\right)\mathrm{polylog}\left(2,ie^{ix}\right)+}{}}\right.}{}$$

$$S_y = e^{\frac{\left(-24ix\,e^{ix}+24x\right)\mathrm{polylog}\left(2,ie^{ix}\right)+\left(24i+24\,e^{ix}\right)\mathrm{polylog}\left(3,ie^{ix}\right)+12\left(x^2+\frac{1}{2}\right)\left(e^{ix}+i\right)\ln\left(e^{ix}+i\right)-4ix\left(x^2+\frac{3}{2}\pi x+\frac{3}{4}\right)e^{ix}+6\pi\,x^2+3x}{3\,e^{ix}+3i}}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{4x\sec(x)(\sec(x)-\tan(x))\,e^{\frac{24\left(e^{ix}+i\right)x\,\mathrm{polylog}\left(2,ie^{ix}\right)+\left(24ie^{ix}-24\right)\mathrm{polylog}\left(3,ie^{ix}\right)+12\left(x^2+\frac{1}{2}\right)\left(ie^{ix}-1\right)\ln\left(e^{ix}+i\right)+6x\left(\left(\frac{2}{3}x^2\right.\right.}{3ie^{ix}-3}}}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{4\left(\mathrm{e}^{iR}+i\right)^{\frac{2\left(2R^2-1\right)\left(\mathrm{e}^{2iR}-1+2i\mathrm{e}^{iR}\right)}{\left(\mathrm{e}^{iR}+i\right)^2}} R\left(4i\mathrm{e}^{\frac{\left(24i+24\,\mathrm{e}^{iR}\right)\operatorname{polylog}\left(3, i\mathrm{e}^{iR}\right)-4\left(\left(6i\mathrm{e}^{iR}-6\right)\operatorname{polylog}\left(2, i\mathrm{e}^{iR}\right)+\left(iR^2+\frac{3}{2}iR\pi-\frac{3}{4}-\frac{3}{2}i\right)\mathrm{e}^{iR}-3\cdots}{3\,\mathrm{e}^{iR}+3i}}\cdots\right.}{\cdots}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{4\left(-4i\mathrm{e}^{\frac{-6iR^2\mathrm{e}^{iR}\pi-4iR^3\mathrm{e}^{iR}-24i\mathrm{e}^{iR}\operatorname{polylog}\left(2, i\mathrm{e}^{iR}\right)R+6iR\,\mathrm{e}^{iR}+6\pi R^2+24\operatorname{polylog}\left(2, i\mathrm{e}^{iR}\right)R+24i\operatorname{polylog}\left(3, i\mathrm{e}^{iR}\right)+3iR+24\operatorname{polylog}\left(3, i\mathrm{e}^{i\cdots}\right)\cdots}{3\,\mathrm{e}^{iR}+3i}}\cdots\right.}{\cdots} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{\frac{\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2, i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3, i\mathrm{e}^{ix}\right)+12\left(x^2+\frac{1}{2}\right)\left(\mathrm{e}^{ix}+i\right)\ln\left(\mathrm{e}^{ix}+i\right)-4ix\left(x^2+\frac{3}{2}\pi x+\frac{3}{4}\right)\mathrm{e}^{ix}+6\pi x^2+3x}{3\,\mathrm{e}^{ix}+3i}} = \int \frac{4\left(-4i\mathrm{e}^{\frac{-6i\cdots}{\cdots}}\cdots\right.}{\cdots}$$

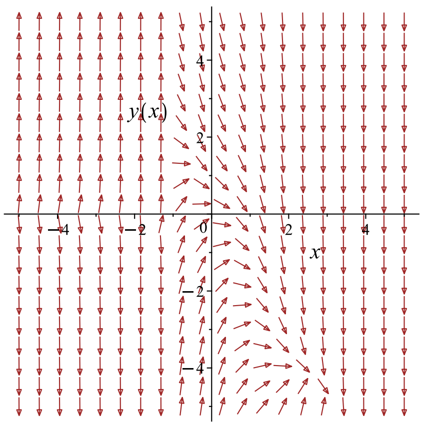
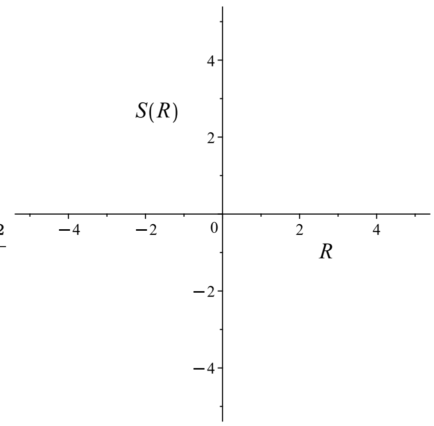
Which simplifies to

$$y\,\mathrm{e}^{\frac{\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2, i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3, i\mathrm{e}^{ix}\right)+12\left(x^2+\frac{1}{2}\right)\left(\mathrm{e}^{ix}+i\right)\ln\left(\mathrm{e}^{ix}+i\right)-4ix\left(x^2+\frac{3}{2}\pi x+\frac{3}{4}\right)\mathrm{e}^{ix}+6\pi x^2+3x}{3\,\mathrm{e}^{ix}+3i}} = -\frac{4\left(\int -\frac{\left(\mathrm{e}^{ix}+i\right)^{\frac{2\cdots}{\cdots}}\cdots}{\cdots}\cdots\right.}{\cdots}$$

Which gives

Expression too large to display

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y\,x^3+4x\,e^x+3\cos(x)y}{3(\sin(x)+1)}$ 	$R = x$ $S = y\,e^{\frac{(-24ix\,e^{ix}+24x)\,\mathrm{polylog}(2}{}}$ $(e^{ix}+i)\ln($	$\frac{dS}{dR} = -\frac{4(e^{iR}+i)^{\frac{2(2R^2-1)(e^{2iR}-1+2ie^{iR})}{(e^{iR}+i)^2}}\,R\left(4ie^{\frac{(24i+24\,e^{iR}}{}}\right.}{}$ 

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

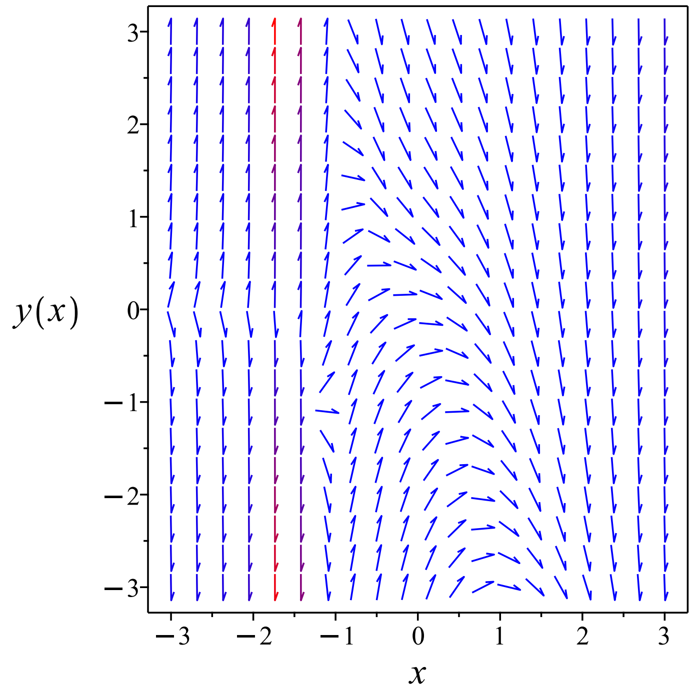


Figure 358: Slope field plot

Verification of solutions

Expression too large to display

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(3\sin(x) + 3)\,\mathrm{d}y = \left(-2y\,x^3 - 3\cos(x)\,y - 4x\,\mathrm{e}^x\right)\mathrm{d}x$$
$$\left(2y\,x^3 + 4x\,\mathrm{e}^x + 3\cos(x)\,y\right)\mathrm{d}x + (3\sin(x) + 3)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2y\,x^3 + 4x\,\mathrm{e}^x + 3\cos(x)\,y$$
$$N(x,y) = 3\sin(x) + 3$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y\,x^3 + 4x\,\mathrm{e}^x + 3\cos(x)\,y\right) \\ &= 2x^3 + 3\cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3\sin(x) + 3) \\ &= 3\cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\&= \frac{1}{3\sin(x)+3}\left(\left(2x^3+3\cos(x)\right) - \left(3\cos(x)\right)\right)\\&= \frac{2x^3}{3\sin(x)+3}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\&= e^{\int \frac{2x^3}{3\sin(x)+3}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{4x^3}{3(\mathrm{e}^{ix}+i)} - \frac{4ix^3}{3} + 4x^2\ln(1-i\mathrm{e}^{ix}) - 8ix\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + 8\,\mathrm{polylog}(3, i\mathrm{e}^{ix})}\\&= \mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\&= \mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\left(2y\,x^3+4x\,\mathrm{e}^{x}+3\cos(x)\,y\right)\\&= \left(2y\,x^3+4x\,\mathrm{e}^{x}+3\cos(x)\,y\right)\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\&= \mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\left(3\sin(x)+3\right)\\&= 3(\sin(x)+1)\,\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\left(\left(2y\,x^3+4x\,\mathrm{e}^{x}+3\cos(x)\,y\right)\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix}) + (-24ix\,\mathrm{e}^{ix}+24x)\,\mathrm{polylog}(2, i\mathrm{e}^{ix}) + (24i+24\,\mathrm{e}^{ix})\,\mathrm{polylog}(3, i\mathrm{e}^{ix}) - 4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\right) + \Big(3$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \left(2y\,x^3 + 4x\,\mathrm{e}^x + 3\cos(x)\,y\right)\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix})+(-24ix\,\mathrm{e}^{ix}+24x)\operatorname{polylog}(2,i\mathrm{e}^{ix})+(24i+24\,\mathrm{e}^{ix})\operatorname{polylog}(3,i\mathrm{e}^{ix})-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\,\mathrm{d}x$$

$$\phi = \int^x \left(2y\_a^3 + 4\_a\,\mathrm{e}^{\_a} + 3\cos(\_a)\,y\right)\mathrm{e}^{\frac{12(\mathrm{e}^{i\_a}+i)\_a^2\ln(1-i\mathrm{e}^{i\_a})+(-24i\_a\,\mathrm{e}^{i\_a}+24\_a)\operatorname{polylog}(2,i\mathrm{e}^{i\_a})+(24i+24\,\mathrm{e}^{i\_a})\operatorname{polylog}(3,i\mathrm{e}^{i\_a})-4i\mathrm{e}^{i\_a}\_a^3}{3\,\mathrm{e}^{i\_a}+3i}}\,d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int^x \left(2\_a^3 + 3\cos(\_a)\right)\mathrm{e}^{\frac{12(\mathrm{e}^{i\_a}+i)\_a^2\ln(1-i\mathrm{e}^{i\_a})+(-24i\_a\,\mathrm{e}^{i\_a}+24\_a)\operatorname{polylog}(2,i\mathrm{e}^{i\_a})+(24i+24\,\mathrm{e}^{i\_a})\operatorname{polylog}(3,i\mathrm{e}^{i\_a})-4i\mathrm{e}^{i\_a}\_a^3}{3\,\mathrm{e}^{i\_a}+3i}}\,d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3(\sin(x)+1)\,\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix})+(-24ix\,\mathrm{e}^{ix}+24x)\operatorname{polylog}(2,i\mathrm{e}^{ix})+(24i+24\,\mathrm{e}^{ix})\operatorname{polylog}(3,i\mathrm{e}^{i}}{3\,\mathrm{e}^{ix}+3i}}$
Therefore equation (4) becomes

$$3(\sin(x)+1)\,\mathrm{e}^{\frac{12(\mathrm{e}^{ix}+i)x^2\ln(1-i\mathrm{e}^{ix})+(-24ix\,\mathrm{e}^{ix}+24x)\operatorname{polylog}(2,i\mathrm{e}^{ix})+(24i+24\,\mathrm{e}^{ix})\operatorname{polylog}(3,i\mathrm{e}^{ix})-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}} = \int^x \left(2\_a^3 + 3\cos(\_a)\right)\mathrm{e}^{\frac{12(\mathrm{e}^{i\_a}+i)\_a^2\ln(1-i\mathrm{e}^{i\_a})+(-24i\_a\,\mathrm{e}^{i\_a}+24\_a)\operatorname{polylog}(2,i\mathrm{e}^{i\_a})+(24i+24\,\mathrm{e}^{i\_a})\operatorname{polylog}(3,i\mathrm{e}^{i\_a})-4i\mathrm{e}^{i\_a}\_a^3}{3\,\mathrm{e}^{i\_a}+3i}}\,d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3\,\mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^2 \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}} \sin(x) - \left(\int^x \left(2\_a^3 + 3\cos(\_a)\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^2 \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{i\_a}\right)-4i\mathrm{e}^{i\_a}\_a^3}{3\,\mathrm{e}^{i\_a}+3i}} d\_a\right) + 3\,\mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^2 \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}$$

$$= -\left(\int^x \left(2\_a^3 + 3\cos(\_a)\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^2 \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{i\_a}\right)-4i\mathrm{e}^{i\_a}\_a^3}{3\,\mathrm{e}^{i\_a}+3i}} d\_a\right) + (3\sin(x) + 3)\,\mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^2 \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(-\left(\int^x \left(2\_a^3+3\cos(\_a)\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^2 \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{i\_a}\right)-}{3\,\mathrm{e}^{i\_a}+3i}} \right. \right. + (3\sin(x) + 3)\,\mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^2 \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\Bigg)\,\mathrm{d}y$$

$$f(y) = \left(-\left(\int^x \left(2\_a^3+3\cos(\_a)\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^2 \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{i\_a}\right.}{3\,\mathrm{e}^{i\_a}+3i}} \right. \right. + (3\sin(x) + 3)\,\mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^2 \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right)\operatorname{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right)\operatorname{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^3}{3\,\mathrm{e}^{ix}+3i}}\Bigg)\,y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\begin{aligned}
\phi = \int^{x} & \left(2y\_a^{3} + 4\_a\,e^{-\_a}\right. \\
& \left. +3\cos\left(\_a\right) y\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4ie^{i\_a}\_a^{3}}{3e^{i\_a}+3i}} d\_a \\
& + \left(-\left(\int^{x} \left(2\_a^{3}+3\cos\left(\_a\right)\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4}{3e^{i\_a}+3i}}\right.\right. \\
& \qquad + \left(3\sin\left(x\right)\right. \\
& \qquad \left.\left. + 3\right) e^{\frac{12\left(e^{ix}+i\right)x^{2}\ln\left(1-ie^{ix}\right)+\left(-24ix e^{ix}+24x\right)\operatorname{polylog}\left(2,ie^{ix}\right)+\left(24i+24e^{ix}\right)\operatorname{polylog}\left(3,ie^{ix}\right)-4ie^{ix}x^{3}}{3e^{ix}+3i}}\right) y + c_1
\end{aligned}
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
\begin{aligned}
c_1 = \int^{x} & \left(2y\_a^{3} + 4\_a\,e^{-\_a}\right. \\
& \left. +3\cos\left(\_a\right) y\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4ie^{i\_a}\_a^{3}}{3e^{i\_a}+3i}} d\_a \\
& + \left(-\left(\int^{x} \left(2\_a^{3}+3\cos\left(\_a\right)\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4}{3e^{i\_a}+3i}}\right.\right. \\
& \qquad + \left(3\sin\left(x\right)\right. \\
& \qquad \left.\left. + 3\right) e^{\frac{12\left(e^{ix}+i\right)x^{2}\ln\left(1-ie^{ix}\right)+\left(-24ix e^{ix}+24x\right)\operatorname{polylog}\left(2,ie^{ix}\right)+\left(24i+24e^{ix}\right)\operatorname{polylog}\left(3,ie^{ix}\right)-4ie^{ix}x^{3}}{3e^{ix}+3i}}\right) y
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
\int^{x} & \left(2y\_a^{3} + 4\_a\,e^{-\_a}\right. \\
& \left. +3\cos\left(\_a\right) y\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4ie^{i\_a}\_a^{3}}{3e^{i\_a}+3i}} d\_a \\
& + \left(-\left(\int^{x} \left(2\_a^{3}+3\cos\left(\_a\right)\right) e^{\frac{12\left(e^{i\_a}+i\right)\_a^{2}\ln\left(1-ie^{i\_a}\right)+\left(-24i\_a e^{i\_a}+24\_a\right)\operatorname{polylog}\left(2,ie^{i\_a}\right)+\left(24i+24e^{i\_a}\right)\operatorname{polylog}\left(3,ie^{i\_a}\right)-4ie^{i\_a}-}{3e^{i\_a}+3i}}\right.\right. \\
& \left.\left. + \left(3\sin\left(x\right)+3\right) e^{\frac{12\left(e^{ix}+i\right)x^{2}\ln\left(1-ie^{ix}\right)+\left(-24ix e^{ix}+24x\right)\operatorname{polylog}\left(2,ie^{ix}\right)+\left(24i+24e^{ix}\right)\operatorname{polylog}\left(3,ie^{ix}\right)-4ie^{ix}x^{3}}{3e^{ix}+3i}}\right) y \\
& = c_1
\end{aligned} \tag{1}
$$

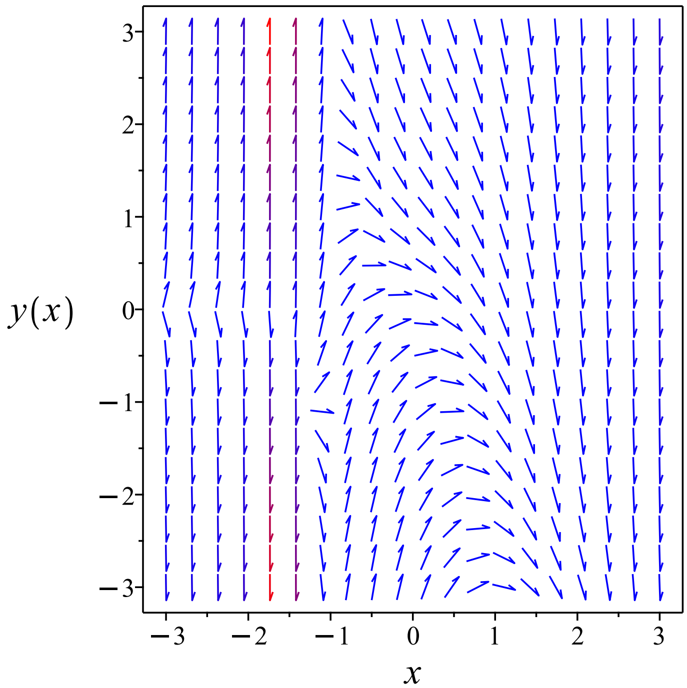


Figure 359: Slope field plot

Verification of solutions

$$
\int^{x} \left(2y\_a^{3} + 4\_a \,\mathrm{e}^{-\_a} + 3\cos\left(\_a\right) y\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^{2} \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right) \mathrm{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right) \mathrm{polylog}\left(3,i\mathrm{e}^{i\_a}\right)-4i\mathrm{e}^{i\_a}\_a^{3}}{3\,\mathrm{e}^{i\_a}+3i}} d\_a
$$

$$
+ \left(-\left(\int^{x} \left(2\_a^{3}+3\cos\left(\_a\right)\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{i\_a}+i\right)\_a^{2} \ln\left(1-i\mathrm{e}^{i\_a}\right)+\left(-24i\_a\,\mathrm{e}^{i\_a}+24\_a\right) \mathrm{polylog}\left(2,i\mathrm{e}^{i\_a}\right)+\left(24i+24\,\mathrm{e}^{i\_a}\right) \mathrm{polylog}\left(3,i\mathrm{e}^{i\_a}\right)-4i\mathrm{e}^{i\_}}{3\,\mathrm{e}^{i\_a}+3i}}\right.\right.
$$

$$
\left. + \left(3\sin\left(x\right)+3\right) \mathrm{e}^{\frac{12\left(\mathrm{e}^{ix}+i\right)x^{2} \ln\left(1-i\mathrm{e}^{ix}\right)+\left(-24ix\,\mathrm{e}^{ix}+24x\right) \mathrm{polylog}\left(2,i\mathrm{e}^{ix}\right)+\left(24i+24\,\mathrm{e}^{ix}\right) \mathrm{polylog}\left(3,i\mathrm{e}^{ix}\right)-4i\mathrm{e}^{ix}x^{3}}{3\,\mathrm{e}^{ix}+3i}}\right) y
$$

$$
= c_1
$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$3\cos(x)\, y + 2yx^3 + (3\sin(x) + 3)\, y' = -4x\, \mathrm{e}^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(2x^3+3\cos(x))y}{3(\sin(x)+1)} - \frac{4x\,\mathrm{e}^x}{3(\sin(x)+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(2x^3+3\cos(x))y}{3(\sin(x)+1)} = -\frac{4x\,\mathrm{e}^x}{3(\sin(x)+1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(2x^3+3\cos(x))y}{3(\sin(x)+1)}\right) = -\frac{4\mu(x)x\,\mathrm{e}^x}{3(\sin(x)+1)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{(2x^3+3\cos(x))y}{3(\sin(x)+1)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(2x^3+3\cos(x))}{3(\sin(x)+1)}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\frac{-4\,\mathrm{I}\,\mathrm{e}^{\mathrm{I}x}x^3+12x^2\ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)\right)\mathrm{e}^{\mathrm{I}x}-24\,\mathrm{I}\,\mathrm{e}^{\mathrm{I}x}\mathit{polylog}\left(2,\mathrm{I}\,\mathrm{e}^{\mathrm{I}x}\right)x+12\,\mathrm{I}x^2\ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)\right)-3\,\mathrm{I}x\,\mathrm{e}^{\mathrm{I}x}+6\,\mathrm{e}^{\mathrm{I}x}\ln\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)+24\,\mathrm{e}^{\mathrm{I}x}\mathit{polylog}\left(3,\mathrm{I}\,\mathrm{e}^{\mathrm{I}x}\right)+6}{3\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{4\mu(x)x\,\mathrm{e}^x}{3(\sin(x)+1)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{4\mu(x)x\,\mathrm{e}^x}{3(\sin(x)+1)} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{4\mu(x)x\,\mathrm{e}^x}{3(\sin(x)+1)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\frac{-4\,\mathrm{I}\mathrm{e}^{\mathrm{I}x}x^3+12x^2\ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)\right)\mathrm{e}^{\mathrm{I}x}-24\,\mathrm{I}\mathrm{e}^{\mathrm{I}x}\mathit{polylog}\left(2,\mathrm{I}\mathrm{e}^{\mathrm{I}x}\right)x+12\,\mathrm{I}x^2\ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)\right)-3\,\mathrm{I}x\mathrm{e}^{\mathrm{I}x}+6\mathrm{e}^{\mathrm{I}x}\ln\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)+24\mathrm{e}^{\mathrm{I}x}\mathit{poly}}{3\left(\mathrm{e}^{\mathrm{I}x}+\mathrm{I}\right)}}$

$$y = \frac{\int -\frac{4 \,\mathrm{e}^{\frac{-4 \,\mathrm{I}\, \mathrm{e}^{\mathrm{I} x} x^{3}+12 x^{2} \ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)\right) \mathrm{e}^{\mathrm{I} x}-24 \,\mathrm{I}\, \mathrm{e}^{\mathrm{I} x} \mathit{polylog}\left(2, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right) x+12 \,\mathrm{I} x^{2} \ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)\right)-3 \,\mathrm{I} x \,\mathrm{e}^{\mathrm{I} x}+6 \,\mathrm{e}^{\mathrm{I} x} \ln\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)+24 \,\mathrm{e}^{\mathrm{I} x} \mathit{polylog}\left(3, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right)+6}{3\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)}}}{3(\sin(x)+1)}}{\mathrm{e}^{\frac{-4 \,\mathrm{I}\, \mathrm{e}^{\mathrm{I} x} x^{3}+12 x^{2} \ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)\right) \mathrm{e}^{\mathrm{I} x}-24 \,\mathrm{I}\, \mathrm{e}^{\mathrm{I} x} \mathit{polylog}\left(2, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right) x+12 \,\mathrm{I} x^{2} \ln\left(-\mathrm{I}\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)\right)-3 \,\mathrm{I} x \,\mathrm{e}^{\mathrm{I} x}+6 \,\mathrm{e}^{\mathrm{I} x} \ln\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)+24 \,\mathrm{e}^{\mathrm{I} x} \mathit{polylog}\left(3, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right)}{3\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)}}}$$

- Simplify

$$y = -\frac{\left(4\left(\int \frac{x \,\mathrm{e}^{\frac{4\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right) x^{2} \ln\left(1-\mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right)+8 x\left(1-\mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right) \mathit{polylog}\left(2, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right)+8 \mathit{polylog}\left(3, \mathrm{I}\, \mathrm{e}^{\mathrm{I} x}\right)\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)+2 \ln\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)\left(\mathrm{e}^{\mathrm{I} x}+\mathrm{I}\right)+\left(\left(-\frac{4 \,\mathrm{I} x^{2}}{3}+1-\mathrm{I}\right) \mathrm{e}^{\mathrm{I} x}+1+\mathrm{I}\right.}{\mathrm{e}^{\mathrm{I} x}+\mathrm{I}}}}{\sin(x)+1}\right.\right.}{}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 395

```
dsolve((3*y(x)*cos(x)+4*x*exp(x)+2*x^3*y(x))+(3*sin(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{\frac{\left(-24 x+24 i x \,\mathrm{e}^{i x}\right) \operatorname{polylog}\left(2, i \mathrm{e}^{i x}\right)+\left(-24 i-24 \,\mathrm{e}^{i x}\right) \operatorname{polylog}\left(3, i \mathrm{e}^{i x}\right)+4 i x\left(x^{2}+\frac{3}{4}\right) \mathrm{e}^{i x}-3 x}{3 \,\mathrm{e}^{i x}+3 i}}\left(4\left(\int \frac{\left(-2 i \mathrm{e}^{\frac{24 x\left(\mathrm{e}^{i x}+i\right) \operatorname{polylog}\left(2, i \mathrm{e}^{i x}\right)+4 \,\mathrm{e}^{i x} x^{3}+3 i x \,\mathrm{e}^{i x}+2}{3 i \mathrm{e}^{i x}}}\right.}{}\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 27.47 (sec). Leaf size: 193

```
DSolve[(3*y[x]*Cos[x]+4*x*Exp[x]+2*x^3*y[x])+(3*Sin[x]+3)*y'[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{\left(1+i e^{-i x}\right)^{-4 x^{2}} \exp\left(-8 i x \operatorname{PolyLog}\left(2,-i e^{-i x}\right)-8 \operatorname{PolyLog}\left(3,-i e^{-i x}\right)+\frac{2}{3} x^{3}\left(\cot\left(\frac{1}{4}(2 x+\pi)\right)-i\right)\right)\left(\int_{1}\right.}{}$$

6.3 problem 3

Internal problem ID [1032]
Internal file name [`OUTPUT/1033_Sunday_June_05_2022_01_57_23_AM_85532474/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{14y^3x^2 + 21x^2y^2y' = 0}$$

6.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{3}{2y}dy = \int dx$$
$$-\frac{3\ln(y)}{2} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y^{\frac{3}{2}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{y^{\frac{3}{2}}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{1}{(c_2 e^x)^{\frac{2}{3}}} \tag{1}$$

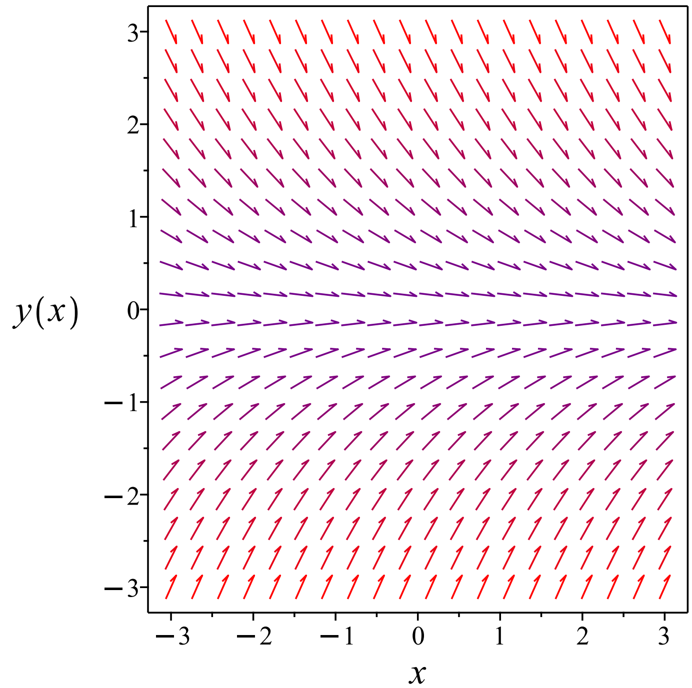


Figure 360: Slope field plot

Verification of solutions

$$y = \frac{1}{(c_2 e^x)^{\frac{2}{3}}}$$

Verified OK.

6.3.2 Maple step by step solution

Let's solve

$14y^3x^2 + 21x^2y^2y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{2}{3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{2}{3} dx + c_1$

- Evaluate integral

$$\ln(y) = -\frac{2x}{3} + c_1$$

- Solve for y

$$y = e^{-\frac{2x}{3} + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((14*x^2*y(x)^3)+(21*x^2*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1 e^{-\frac{2x}{3}}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 25

```
DSolve[(14*x^2*y[x]^3)+(21*x^2*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1 e^{-2x/3}$$
$$y(x) \to 0$$

6.4 problem 4

Internal problem ID [1033]
Internal file name [`OUTPUT/1034_Sunday_June_05_2022_01_57_25_AM_50789681/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{-2y^2 + (12y^2 - 4yx)\, y' = -2x}$$

6.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-4yx+12y^2\right)\mathrm{d}y &= \left(2y^2-2x\right)\mathrm{d}x\\ \left(-2y^2+2x\right)\mathrm{d}x + \left(-4yx+12y^2\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2y^2+2x\\ N(x,y) &= -4yx+12y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y^2+2x\right)\\ &= -4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-4yx+12y^2\right)\\ &= -4y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2y^2 + 2x \, \mathrm{d}x$$

$$\phi = x(-2y^2 + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -4yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -4yx + 12y^2$. Therefore equation (4) becomes

$$-4yx + 12y^2 = -4yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 12y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (12y^2) \, \mathrm{d}y$$

$$f(y) = 4y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(-2y^2 + x) + 4y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(-2y^2 + x) + 4y^3$$

Summary
The solution(s) found are the following

$$x(-2y^2+x)+4y^3=c_1 \tag{1}$$

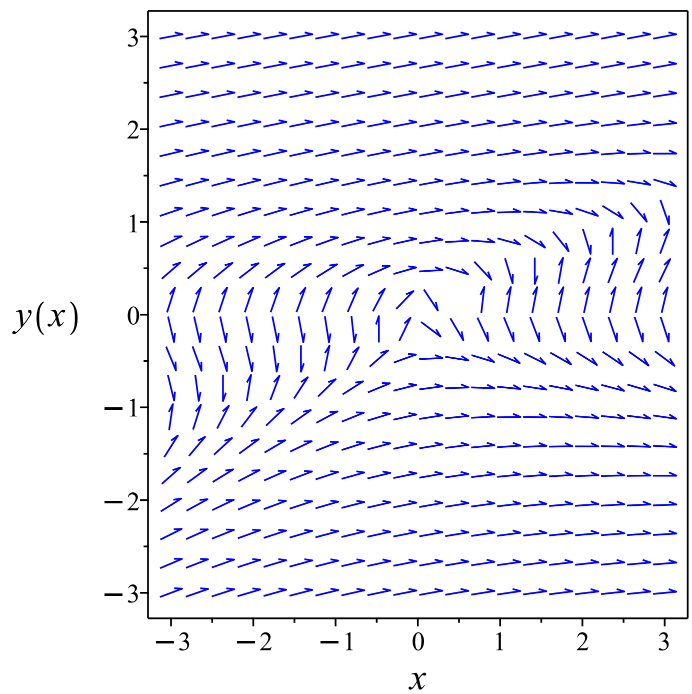


Figure 361: Slope field plot

Verification of solutions

$$x(-2y^2+x)+4y^3=c_1$$

Verified OK.

6.4.2 Maple step by step solution

Let's solve

$-2y^2+(12y^2-4yx)\,y'=-2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-4y = -4y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-2y^2 + 2x) \, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -2x\, y^2 + x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-4yx + 12y^2 = -4yx + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 12y^2$

- Solve for $f_1(y)$

  $f_1(y) = 4y^3$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -2x\, y^2 + 4y^3 + x^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $-2x\, y^2 + 4y^3 + x^2 = c_1$

- Solve for y

$$\left\{y=\frac{\left(-27x^{2}+27c_{1}+x^{3}+3\sqrt{-6x^{5}+6c_{1}x^{3}+81x^{4}-162c_{1}x^{2}+81c_{1}^{2}}\right)^{\frac{1}{3}}}{6}+\frac{x^{2}}{6\left(-27x^{2}+27c_{1}+x^{3}+3\sqrt{-6x^{5}+6c_{1}x^{3}+81x^{4}-162c_{1}x^{2}\right.}}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 429

```
dsolve((2*x-2*y(x)^2)+(12*y(x)^2-4*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-6x^5 - 6c_1x^3 + 81x^4 + 162c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{x^2}{6\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-6x^5 - 6c_1x^3 + 81x^4 + 162c_1x^2 + 81c_1^2}\right)^{\frac{1}{3}}} + \frac{x}{6}$$

$$y(x) = \frac{\left(-\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-3\left(x^2 + c_1\right)\left(2x^3 - 27x^2 - 27c_1\right)}\right)^{\frac{1}{3}} + x\right)\left(i\left(\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{\right.\right.\right.}{12\left(-27x^2 - 27c_1 + x^3 + \right.}$$

$$y(x) = \frac{\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-3\left(x^2 + c_1\right)\left(2x^3 - 27x^2 - 27c_1\right)}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{12} - \frac{x\left(ix\sqrt{3} + x - 2\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-3\left(x^2 + c_1\right)\left(2x^3 - 27x^2 - 27c_1\right)}\right)^{\frac{1}{3}}\right)}{12\left(-27x^2 - 27c_1 + x^3 + 3\sqrt{-3\left(x^2 + c_1\right)\left(2x^3 - 27x^2 - 27c_1\right)}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.8 (sec). Leaf size: 414

```
DSolve[(2*x-2*y[x]^2)+(12*y[x]^2-4*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{3\ 2^{2/3}\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}} - \frac{\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}}{6\sqrt[3]{2}}+\frac{x}{6}$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)x^2}{6\ 2^{2/3}\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}} + \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}}{12\sqrt[3]{2}}+\frac{x}{6}$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)x^2}{6\ 2^{2/3}\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}} + \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-2x^3+54x^2+\sqrt{-4x^6+4\left(x^3-27x^2-54c_1\right)^2}+108c_1}}{12\sqrt[3]{2}}+\frac{x}{6}$$

6.5 problem 5

Internal problem ID [1034]
Internal file name [`OUTPUT/1035_Sunday_June_05_2022_01_57_26_AM_68709905/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(x+y)^2+(x+y)^2 y'=0}$$

6.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -1 \, \mathrm{d}x \\ &= -x+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y=-x+c_1 \tag{1}$$

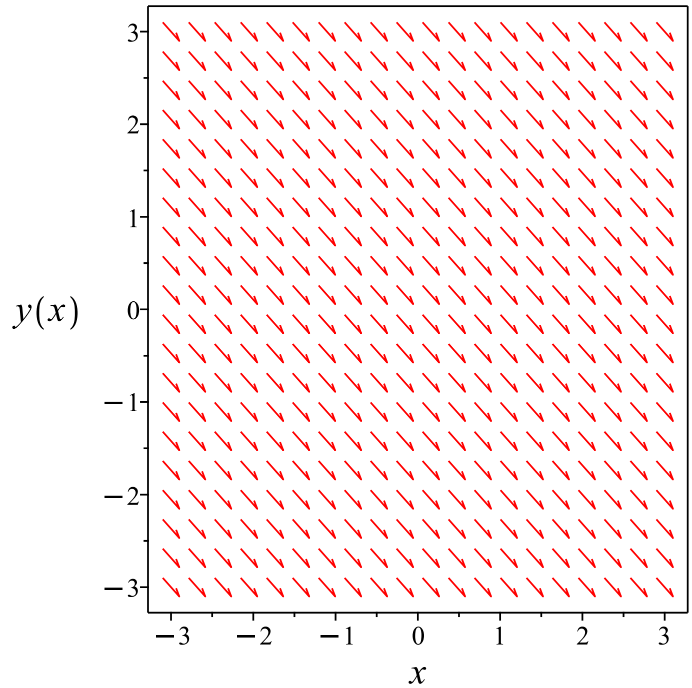


Figure 362: Slope field plot

Verification of solutions

$$y = -x + c_1$$

Verified OK.

6.5.2 Maple step by step solution

Let's solve

$(x+y)^2 + (x+y)^2 y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left((x+y)^2 + (x+y)^2 y'\right) dx = \int 0 dx + c_1$

- Evaluate integral

  $\frac{(x+y)^3}{3} = c_1$

- Solve for y

$$y = c_1^{\frac{1}{3}} 3^{\frac{1}{3}} - x$$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve((x+y(x))^2+(x+y(x))^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = c_1 - x$$
$$y(x) = -\frac{c_1}{2} - \frac{i\sqrt{3}\, c_1}{2} - x$$
$$y(x) = -\frac{c_1}{2} + \frac{i\sqrt{3}\, c_1}{2} - x$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 18

```
DSolve[(x+y[x])^2+(x+y[x])^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x$$
$$y(x) \to -x + c_1$$

6.6 problem 6

Internal problem ID [1035]
Internal file name [`OUTPUT/1036_Sunday_June_05_2022_01_57_27_AM_85796263/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{7y+(3x+4y)\,y'=-4x}$$

6.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$7u(x)\,x+(3x+4u(x)\,x)\,(u'(x)\,x+u(x))=-4x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2(2u^2+5u+2)}{x\,(4u+3)}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+5u+2}{4u+3}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^2+5u+2}{4u+3}}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{\frac{2u^2+5u+2}{4u+3}}\,du &= \int -\frac{2}{x}\,dx \\
\frac{\ln(2u+1)}{3} + \frac{5\ln(u+2)}{3} &= -2\ln(x) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{\ln(2u+1) + 5\ln(u+2)}{3} &= -2\ln(x) + c_2 \\
\ln(2u+1) + 5\ln(u+2) &= (3)\left(-2\ln(x) + c_2\right) \\
&= -6\ln(x) + 3c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(2u+1)+5\ln(u+2)} = e^{-6\ln(x)+3c_2}
$$

Which simplifies to

$$
\begin{aligned}
(2u+1)(u+2)^5 &= \frac{3c_2}{x^6} \\
&= \frac{c_3}{x^6}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \operatorname{RootOf}\left(2\_Z^6 + 21\_Z^5 + 90\_Z^4 + 200\_Z^3 - \frac{c_3 e^{3c_2}}{x^6} + 240\_Z^2 + 144\_Z + 32\right)
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x \operatorname{RootOf}\left(2\_Z^6 x^6 + 21\_Z^5 x^6 + 90x^6\_Z^4 + 200\_Z^3 x^6 + 240\_Z^2 x^6 + 144\_Z x^6 + 32x^6 - c_3 e^{3c_2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = x \operatorname{RootOf}\left(2\_Z^6 x^6 + 21\_Z^5 x^6 + 90x^6\_Z^4 + 200\_Z^3 x^6 + 240\_Z^2 x^6 \right. \\
\left. + 144\_Z x^6 + 32x^6 - c_3 e^{3c_2}\right)
\end{aligned} \tag{1}
$$

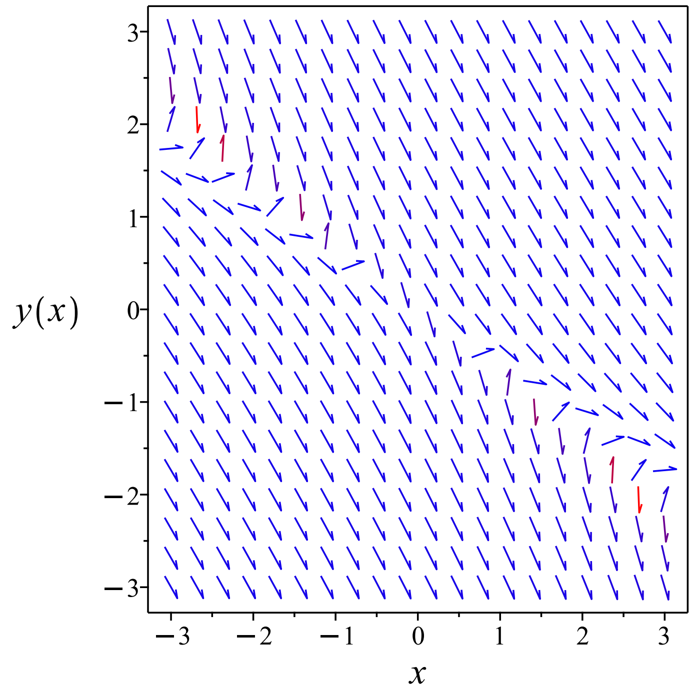


Figure 363: Slope field plot

Verification of solutions

$$y = x\,\mathrm{RootOf}\left(2\_Z^6x^6 + 21\_Z^5x^6 + 90x^6\_Z^4 + 200\_Z^3x^6 + 240\_Z^2x^6 + 144\_Zx^6 + 32x^6 - c_3\mathrm{e}^{3c_2}\right)$$

Verified OK.

6.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{4x+7y}{3x+4y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(4x+7y)(b_3-a_2)}{3x+4y} - \frac{(4x+7y)^2 a_3}{(3x+4y)^2} - \left(-\frac{4}{3x+4y} + \frac{12x+21y}{(3x+4y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{7}{3x+4y} + \frac{16x+28y}{(3x+4y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{12x^2a_2 - 16x^2a_3 + 14x^2b_2 - 12x^2b_3 + 32xya_2 - 56xya_3 + 24xyb_2 - 32xyb_3 + 28y^2a_2 - 54y^2a_3 + 16y^2b_2 -}{(3x+4y)^2} = 0$$

Setting the numerator to zero gives

$$12x^2a_2 - 16x^2a_3 + 14x^2b_2 - 12x^2b_3 + 32xya_2 - 56xya_3 + 24xyb_2 - 32xyb_3 + 28y^2a_2 - 54y^2a_3 + 16y^2b_2 - 28y^2b_3 + 5xb_1 - 5ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$12a_2v_1^2 + 32a_2v_1v_2 + 28a_2v_2^2 - 16a_3v_1^2 - 56a_3v_1v_2 - 54a_3v_2^2 + 14b_2v_1^2 + 24b_2v_1v_2 + 16b_2v_2^2 - 12b_3v_1^2 - 32b_3v_1v_2 - 28b_3v_2^2 - 5a_1v_2 + 5b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\left(12a_2 - 16a_3 + 14b_2 - 12b_3\right) v_1^2 + \left(32a_2 - 56a_3 + 24b_2 - 32b_3\right) v_1 v_2 + 5b_1 v_1 + \left(28a_2 - 54a_3 + 16b_2 - 28b_3\right) v_2^2 - 5a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-5a_1 &= 0 \\
5b_1 &= 0 \\
12a_2 - 16a_3 + 14b_2 - 12b_3 &= 0 \\
28a_2 - 54a_3 + 16b_2 - 28b_3 &= 0 \\
32a_2 - 56a_3 + 24b_2 - 32b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{5a_3}{2} + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= -a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{4x + 7y}{3x + 4y}\right) (x) \\
&= \frac{4x^2 + 10yx + 4y^2}{3x + 4y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x^2+10yx+4y^2}{3x+4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (x + 2y)}{6} + \frac{5 \ln (2x + y)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4x + 7y}{3x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4x + 7y}{4x^2 + 10yx + 4y^2} \\ S_y &= \frac{3x + 4y}{4x^2 + 10yx + 4y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

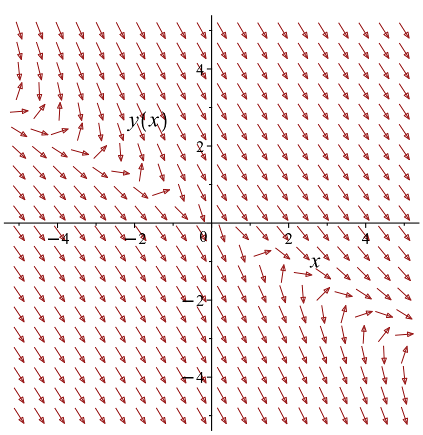
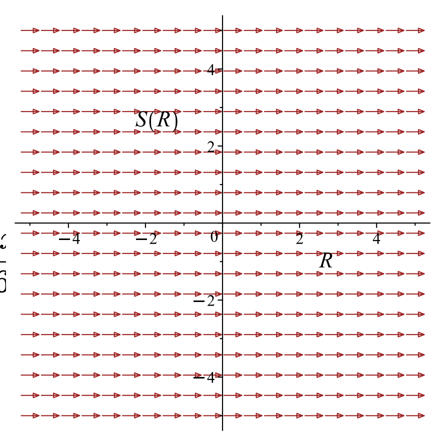
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x+2y)}{6} + \frac{5\ln(2x+y)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(x+2y)}{6} + \frac{5\ln(2x+y)}{6} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x+7y}{3x+4y}$ 	$R = x$ $S = \frac{\ln(x+2y)}{6} + \frac{5\ln(2x+y)}{6}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x+2y)}{6}+\frac{5\ln(2x+y)}{6}=c_1 \tag{1}$$

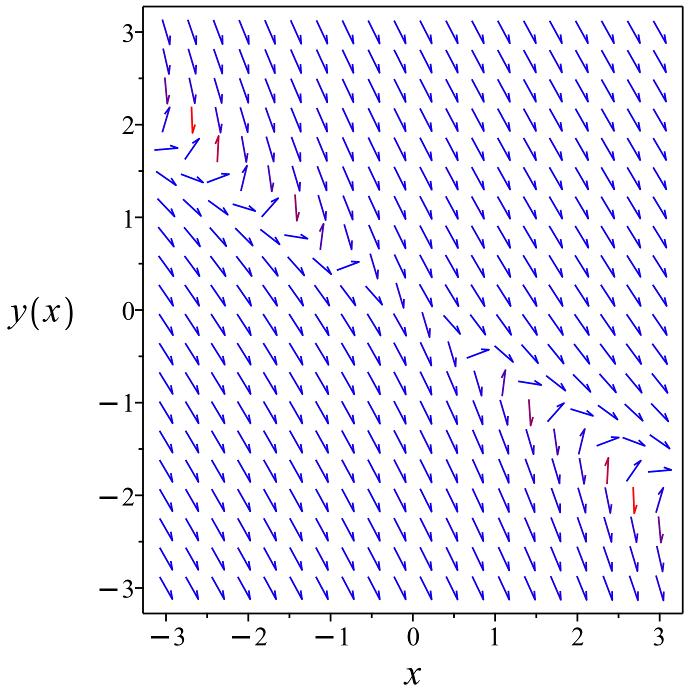


Figure 364: Slope field plot

Verification of solutions

$$\frac{\ln(x+2y)}{6}+\frac{5\ln(2x+y)}{6}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 54

```
dsolve((4*x+7*y(x))+(3*x+4*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x\left(-2\,\mathrm{RootOf}\left(\_Z^{36}+3\_Z^{6}c_1x^6-2c_1x^6\right)^6+1\right)}{\mathrm{RootOf}\left(\_Z^{36}+3\_Z^{6}c_1x^6-2c_1x^6\right)^6}$$

✓ Solution by Mathematica

Time used: 3.222 (sec). Leaf size: 409

```
DSolve[(4*x+7*y[x])+(3*x+4*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 1\right]$$
$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 2\right]$$
$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 3\right]$$
$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 4\right]$$
$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 5\right]$$
$$y(x) \to \text{Root}\left[2\#1^6 + 21\#1^5x + 90\#1^4x^2 + 200\#1^3x^3 + 240\#1^2x^4 + 144\#1x^5 + 32x^6 - e^{3c_1}\&, 6\right]$$

6.7 problem 7

Internal problem ID [1036]
Internal file name [`OUTPUT/1037_Sunday_June_05_2022_01_57_28_AM_49692152/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2\sin(x)\, y^2 + 3y^3 + \left(4\cos(x)\, y + 9x y^2\right) y' = 2x}$$

6.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(4\cos(x)\,y + 9x\,y^2\right)\mathrm{d}y &= \left(2\sin(x)\,y^2 - 3y^3 + 2x\right)\mathrm{d}x \\
\left(3y^3 - 2\sin(x)\,y^2 - 2x\right)\mathrm{d}x + \left(4\cos(x)\,y + 9x\,y^2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 3y^3 - 2\sin(x)\,y^2 - 2x \\
N(x,y) &= 4\cos(x)\,y + 9x\,y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y^3 - 2\sin(x)\,y^2 - 2x\right) \\
&= y(9y - 4\sin(x))
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4\cos(x)\,y + 9x\,y^2\right) \\
&= y(9y - 4\sin(x))
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 3y^3 - 2\sin(x)\,y^2 - 2x \,\mathrm{d}x$$
$$\phi = 3x\,y^3 - x^2 + 2\cos(x)\,y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4\cos(x)\,y + 9x\,y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4\cos(x)\,y + 9x\,y^2$. Therefore equation (4) becomes

$$4\cos(x)\,y + 9x\,y^2 = 4\cos(x)\,y + 9x\,y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 3x\,y^3 - x^2 + 2\cos(x)\,y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3x\,y^3 - x^2 + 2\cos(x)\,y^2$$

Summary

The solution(s) found are the following

$$3xy^3 - x^2 + 2\cos(x)\,y^2 = c_1 \tag{1}$$

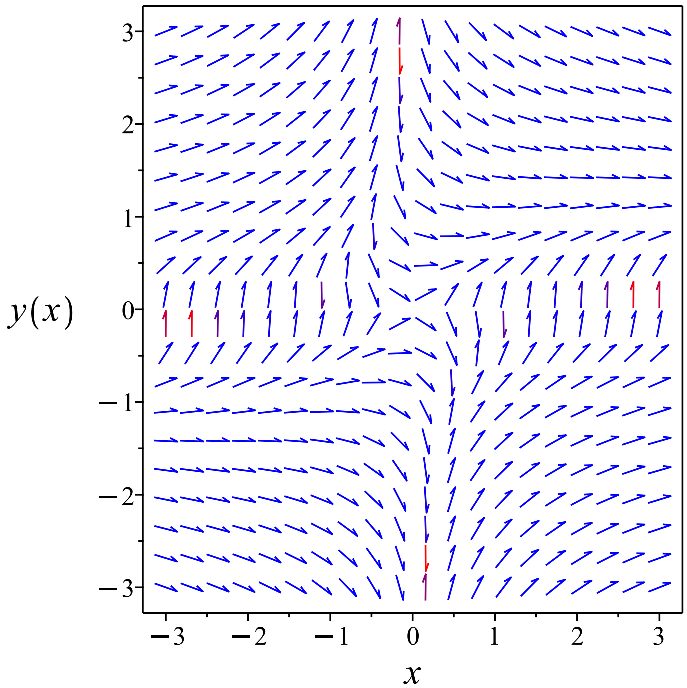


Figure 365: Slope field plot

Verification of solutions

$$3xy^3 - x^2 + 2\cos(x)\, y^2 = c_1$$

Verified OK.

6.7.2 Maple step by step solution

Let's solve

$-2\sin(x)\, y^2 + 3y^3 + (4\cos(x)\, y + 9xy^2)\, y' = 2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $9y^2 - 4\sin(x)\, y = 9y^2 - 4\sin(x)\, y$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (3y^3 - 2\sin(x)\, y^2 - 2x)\, dx + f_1(y)$
- Evaluate integral

  $F(x,y) = 3x\, y^3 - x^2 + 2\cos(x)\, y^2 + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $4\cos(x)\, y + 9x\, y^2 = 9x\, y^2 + 4\cos(x)\, y + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 3x\, y^3 - x^2 + 2\cos(x)\, y^2$
- Substitute $F(x,y)$ into the solution of the ODE

  $3x\, y^3 - x^2 + 2\cos(x)\, y^2 = c_1$
- Solve for y

  $$\left\{ y = \frac{\left(972x^4 - 64\cos(x)^3 + 972c_1x^2 + 36\sqrt{729x^6 - 96\cos(x)^3x^2 + 1458c_1x^4 - 96\cos(x)^3c_1 + 729c_1^2x^2}\, x\right)^{\frac{1}{3}}}{18x} + \frac{}{9x\left(972x^4 - 64\cos(x)^3\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 638

```
dsolve((-2*y(x)^2*sin(x)+3*y(x)^3-2*x)+(4*y(x)*cos(x)+9*x*y(x)^2)*diff(y(x),x)=0,y(x), sings
```

$$y(x) = \frac{\frac{\left(972x^4+36\sqrt{3}\,\sqrt{(-x^2+c_1)\left(-243x^4+32\cos(x)^3+243c_1x^2\right)}\,x-64\cos(x)^3-972c_1x^2\right)^{\frac{1}{3}}}{2}+\frac{8\cos(x)^2}{\left(972x^4+36\sqrt{3}\,\sqrt{(-x^2+c_1)\left(-243x^4+32\cos(x)^3\right.}\right.}}{9x}$$

$$y(x) = -\frac{\left(\left(972x^4+36\sqrt{3}\,\sqrt{32\cos(x)^3(-x^2+c_1)+243(x^2-c_1)^2x^2}\,x-64\cos(x)^3-972c_1x^2\right)^{\frac{1}{3}}+4\cos(x\right.}{}$$

$$y(x) = \frac{\left(i\left(\left(972x^4+36\sqrt{3}\,\sqrt{32\cos(x)^3(-x^2+c_1)+243(x^2-c_1)^2x^2}\,x-64\cos(x)^3-972c_1x^2\right)^{\frac{1}{3}}-4\cos(x\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 34.362 (sec). Leaf size: 520

```
DSolve[(-2*y[x]^2*Sin[x]+3*y[x]^3-2*x)+(4*y[x]*Cos[x]+9*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSin
```

$$y(x) \to \frac{2^{2/3}\sqrt[3]{-16\cos^3(x)+9\left(27x^4+27c_1x^2+\sqrt{3}\sqrt{x^2\left(x^2+c_1\right)\left(-32\cos^3(x)+243x^2\left(x^2+c_1\right)\right)}\right)}+\frac{\quad}{\sqrt[3]{-8\cos\ldots}}}{18x}$$

$$y(x) \to \frac{i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{-16\cos^3(x)+9\left(27x^4+27c_1x^2+\sqrt{3}\sqrt{x^2\left(x^2+c_1\right)\left(-32\cos^3(x)+243x^2\left(x^2+c_1\right)\right)}\right)}\ldots}{36\ldots}$$

$$y(x) \to -\frac{2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{-16\cos^3(x)+9\left(27x^4+27c_1x^2+\sqrt{3}\sqrt{x^2\left(x^2+c_1\right)\left(-32\cos^3(x)+243x^2\left(x^2+c_1\right)\right)}\ldots}}{3\ldots}$$

6.8 problem 8

Internal problem ID [1037]
Internal file name [`OUTPUT/1038_Sunday_June_05_2022_01_57_34_AM_71905172/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2y + 2x)\, y' = -2x}$$

6.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (2u(x)\, x + 2x)\, (u'(x)\, x + u(x)) = -2x$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2u^2 + 3u + 2}{2x\,(u + 1)}\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{2u^2+3u+2}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+3u+2}{u+1}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{2u^2+3u+2}{u+1}}\,du = \int -\frac{1}{2x}\,dx$$

$$\frac{\ln\left(2u^2+3u+2\right)}{4} + \frac{\sqrt{7}\arctan\left(\frac{(4u+3)\sqrt{7}}{7}\right)}{14} = -\frac{\ln\left(x\right)}{2} + c_2$$

The solution is

$$\frac{\ln\left(2u(x)^2+3u(x)+2\right)}{4} + \frac{\sqrt{7}\arctan\left(\frac{(4u(x)+3)\sqrt{7}}{7}\right)}{14} + \frac{\ln\left(x\right)}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{2y^2}{x^2}+\frac{3y}{x}+2\right)}{4} + \frac{\sqrt{7}\arctan\left(\frac{\left(\frac{4y}{x}+3\right)\sqrt{7}}{7}\right)}{14} + \frac{\ln\left(x\right)}{2} - c_2 = 0$$

$$\frac{\ln\left(\frac{2y^2}{x^2}+\frac{3y}{x}+2\right)}{4} + \frac{\sqrt{7}\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{14} + \frac{\ln\left(x\right)}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{2y^2}{x^2}+\frac{3y}{x}+2\right)}{4} + \frac{\sqrt{7}\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{14} + \frac{\ln\left(x\right)}{2} - c_2 = 0 \tag{1}$$

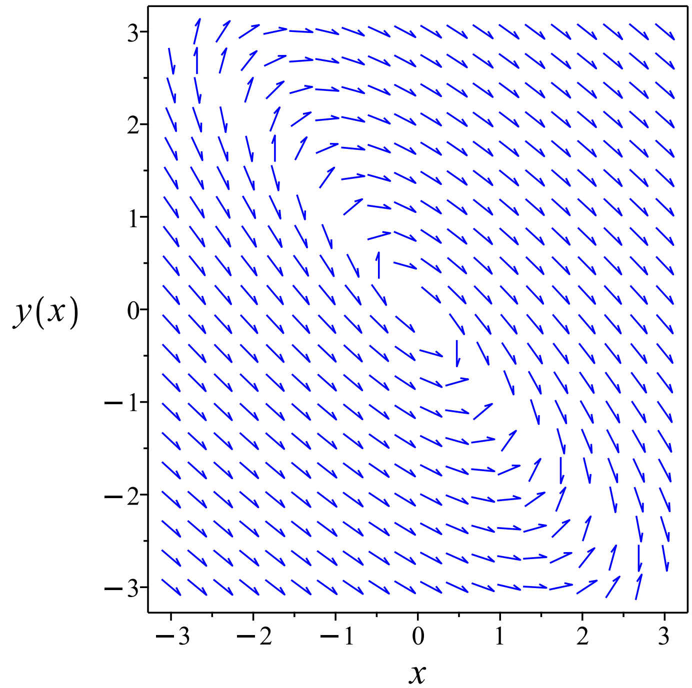


Figure 366: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{2y^2}{x^2}+\frac{3y}{x}+2\right)}{4}+\frac{\sqrt{7}\,\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{14}+\frac{\ln(x)}{2}-c_2=0$$

Verified OK.

6.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2x+y}{2(x+y)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(2x+y)(b_3-a_2)}{2(x+y)} - \frac{(2x+y)^2 a_3}{4(x+y)^2} \\ &- \left(-\frac{1}{x+y} + \frac{2x+y}{2(x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{1}{2(x+y)} + \frac{2x+y}{2(x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^2a_2 - 4x^2a_3 + 2x^2b_2 - 4x^2b_3 + 8xya_2 - 4xya_3 + 8xyb_2 - 8xyb_3 + 2y^2a_2 + y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xb_1}{4(x+y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4x^2a_2 - 4x^2a_3 + 2x^2b_2 - 4x^2b_3 + 8xya_2 - 4xya_3 + 8xyb_2 \\ &- 8xyb_3 + 2y^2a_2 + y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2v_1^2 + 8a_2v_1v_2 + 2a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 + a_3v_2^2 + 2b_2v_1^2 \\ &+ 8b_2v_1v_2 + 4b_2v_2^2 - 4b_3v_1^2 - 8b_3v_1v_2 - 2b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(4a_2 - 4a_3 + 2b_2 - 4b_3) v_1^2 + (8a_2 - 4a_3 + 8b_2 - 8b_3) v_1 v_2 \\ - 2b_1 v_1 + (2a_2 + a_3 + 4b_2 - 2b_3) v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0\\ -2b_1 &= 0\\ 2a_2 + a_3 + 4b_2 - 2b_3 &= 0\\ 4a_2 - 4a_3 + 2b_2 - 4b_3 &= 0\\ 8a_2 - 4a_3 + 8b_2 - 8b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= \frac{3a_3}{2} + b_3\\ a_3 &= a_3\\ b_1 &= 0\\ b_2 &= -a_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\ &= y - \left(-\frac{2x + y}{2\,(x + y)}\right)(x)\\ &= \frac{2x^2 + 3yx + 2y^2}{2x + 2y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2+3yx+2y^2}{2x+2y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x^2 + 3yx + 2y^2)}{2} + \frac{\sqrt{7} \arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{7}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + y}{2(x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + y}{2x^2 + 3yx + 2y^2} \\ S_y &= \frac{2x + 2y}{2x^2 + 3yx + 2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

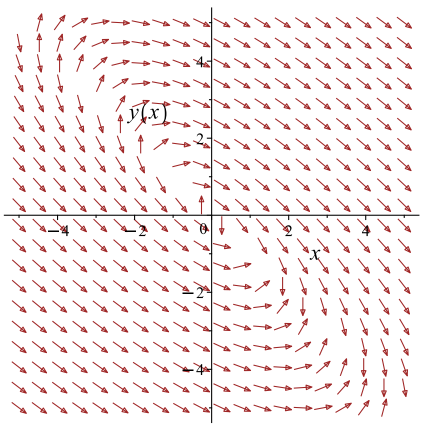
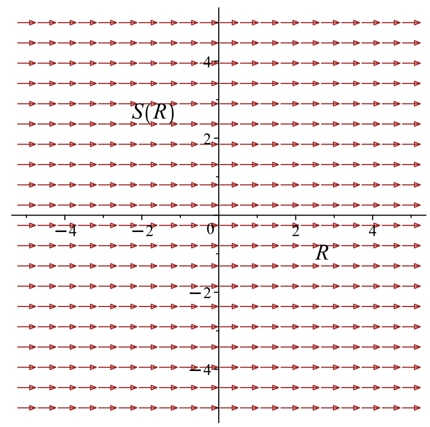
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^2 + 3yx + 2x^2)}{2} + \frac{\sqrt{7}\,\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{7} = c_1$$

Which simplifies to

$$\frac{\ln(2y^2 + 3yx + 2x^2)}{2} + \frac{\sqrt{7}\,\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{7} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+y}{2(x+y)}$ 	$R = x$ $S = \frac{\ln(2x^2 + 3yx + 2y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2y^2 + 3yx + 2x^2)}{2} + \frac{\sqrt{7}\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{7} = c_1 \tag{1}$$

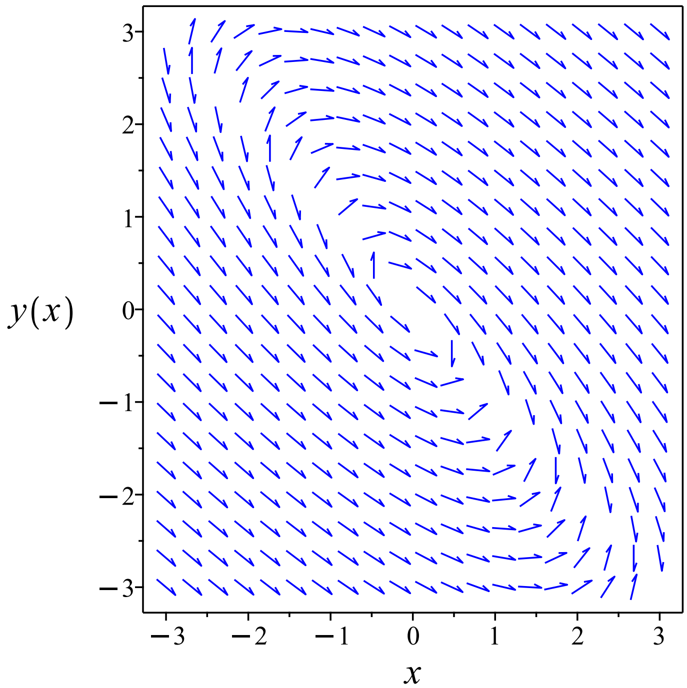


Figure 367: Slope field plot

Verification of solutions

$$\frac{\ln\left(2y^2+3yx+2x^2\right)}{2}+\frac{\sqrt{7}\,\arctan\left(\frac{(3x+4y)\sqrt{7}}{7x}\right)}{7}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 51

```
dsolve((2*x+y(x))+(2*y(x)+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x\left(\sqrt{7}\,\tan\left(\operatorname{RootOf}\left(\sqrt{7}\,\ln\left(\sec\left(\_Z\right)^{2} x^{2}\right)+\sqrt{7}\,\ln\left(7\right)-3\sqrt{7}\,\ln\left(2\right)+2\sqrt{7}\,c_{1}+2\_Z\right)\right)-3\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 62

```
DSolve[(2*x+y[x])+(2*y[x]+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{\frac{4y(x)}{x}+3}{\sqrt{7}}\right)}{2\sqrt{7}}+\frac{1}{4}\log\left(\frac{2y(x)^2}{x^2}+\frac{3y(x)}{x}+2\right)=-\frac{\log(x)}{2}+c_1,y(x)\right]$$

6.9 problem 9

Internal problem ID [1038]
Internal file name [`OUTPUT/1039_Sunday_June_05_2022_01_57_36_AM_71641712/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{2yx + 4y^2 + \left(x^2 + 8yx + 18y\right) y' = -3x^2}$$

6.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2+8yx+18y\right)\mathrm{d}y = \left(-3x^2-2yx-4y^2\right)\mathrm{d}x$$
$$\left(3x^2+2yx+4y^2\right)\mathrm{d}x + \left(x^2+8yx+18y\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 3x^2+2yx+4y^2$$
$$N(x,y) = x^2+8yx+18y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3x^2+2yx+4y^2\right)$$
$$= 2x+8y$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^2+8yx+18y\right)$$
$$= 2x+8y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$
$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3x^2 + 2yx + 4y^2 \,\mathrm{d}x$$
$$\phi = x^3 + y\,x^2 + 4x\,y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + 8yx + f'(y) \tag{4}$$
$$= x(x + 8y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 8yx + 18y$. Therefore equation (4) becomes

$$x^2 + 8yx + 18y = x(x + 8y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 18y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (18y) \,\mathrm{d}y$$
$$f(y) = 9y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3 + y\,x^2 + 4x\,y^2 + 9y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3 + y\,x^2 + 4x\,y^2 + 9y^2$$

Summary
The solution(s) found are the following

$$x^3 + x^2y + 4xy^2 + 9y^2 = c_1 \tag{1}$$

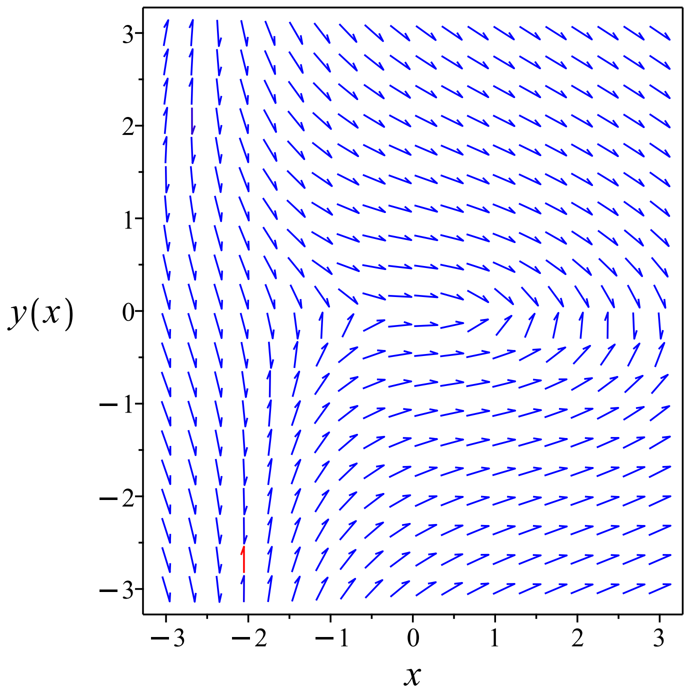


Figure 368: Slope field plot

Verification of solutions

$$x^3 + x^2y + 4xy^2 + 9y^2 = c_1$$

Verified OK.

6.9.2 Maple step by step solution

Let's solve

$2yx + 4y^2 + (x^2 + 8yx + 18y)\, y' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2x + 8y = 2x + 8y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x^2 + 2yx + 4y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^3 + y\, x^2 + 4x\, y^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $x^2 + 8yx + 18y = x^2 + 8yx + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 18y$

- Solve for $f_1(y)$

  $f_1(y) = 9y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^3 + y\, x^2 + 4x\, y^2 + 9y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^3 + y\, x^2 + 4x\, y^2 + 9y^2 = c_1$

- Solve for y

  $\left\{y = \frac{-x^2 + \sqrt{-15x^4 - 36x^3 + 16c_1 x + 36c_1}}{2(4x+9)}, y = -\frac{x^2 + \sqrt{-15x^4 - 36x^3 + 16c_1 x + 36c_1}}{2(4x+9)}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve((3*x^2+2*x*y(x)+4*y(x)^2)+(x^2+8*x*y(x)+18*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^2 + \sqrt{-15x^4 - 36x^3 - 16c_1x - 36c_1}}{8x + 18}$$
$$y(x) = \frac{-x^2 - \sqrt{-15x^4 - 36x^3 - 16c_1x - 36c_1}}{8x + 18}$$

✓ Solution by Mathematica

Time used: 0.616 (sec). Leaf size: 84

```
DSolve[(3*x^2+2*x*y[x]+4*y[x]^2)+(x^2+8*x*y[x]+18*y[x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to -\frac{x^2 + \sqrt{-15x^4 - 36x^3 + 16c_1x + 36c_1}}{8x + 18}$$
$$y(x) \to \frac{-x^2 + \sqrt{-15x^4 - 36x^3 + 16c_1x + 36c_1}}{8x + 18}$$

6.10 problem 10

Internal problem ID [1039]
Internal file name [`OUTPUT/1040_Sunday_June_05_2022_01_57_38_AM_50380762/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{8yx + y^2 + \left(2x^2 + \frac{xy^3}{3}\right) y' = -2x^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((2*x^2+8*x*y(x)+y(x)^2)+(2*x^2+x*y(x)^3/3)*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*x^2+8*x*y[x]+y[x]^2)+(2*x^2+x*y[x]^3/3)*y'[x]==0,y[x],x,IncludeSingularSolutions -
```

Not solved

6.11 problem 11

Internal problem ID [1040]
Internal file name [`OUTPUT/1041_Sunday_June_05_2022_01_57_40_AM_87212688/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\left(\frac{1}{y}+2y\right)y' = -\frac{1}{x}-2x}$$

6.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{y(2x^2+1)}{(2y^2+1)\,x}
\end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x}$ and $g(y) = \frac{y}{2y^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{2y^2+1}}\,dy = -\frac{2x^2+1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{2y^2+1}}\,dy = \int -\frac{2x^2+1}{x}\,dx$$

$$y^2 + \ln(y) = -x^2 - \ln(x) + c_1$$

Which results in

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2+2c_1}}{x^2}\right)}{2} - x^2 + c_1}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2+2c_1}}{x^2}\right)}{2} - x^2 + c_1}}{x} \tag{1}$$

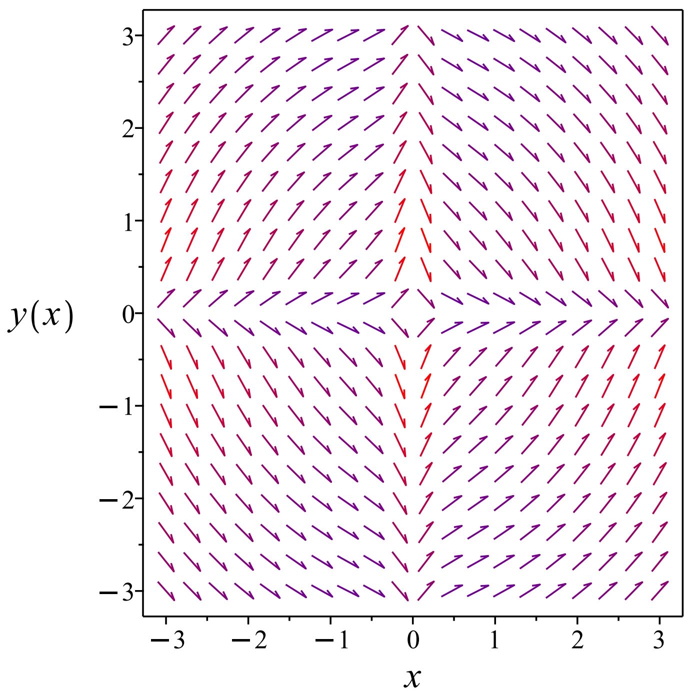


Figure 369: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2+2c_1}}{x^2}\right)}{2} - x^2 + c_1}}{x}$$

Verified OK.

6.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(2x^2+1)}{(2y^2+1)\,x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 278: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= -\frac{x}{2x^2+1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{x}{2x^2+1}} dx
\end{aligned}$$

Which results in

$$S = -x^2 - \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(2x^2+1)}{(2y^2+1)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{x} - 2x \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2y^2 + 1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R^2 + 1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x^2 - \ln(x) = y^2 + \ln(y) + c_1$$

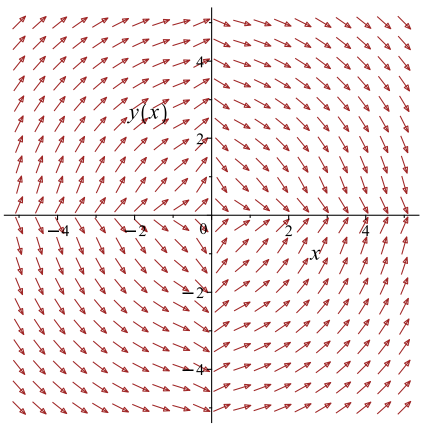
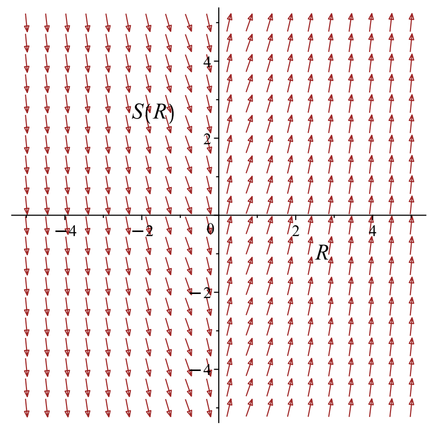
Which simplifies to

$$-x^2 - \ln(x) = y^2 + \ln(y) + c_1$$

Which gives

$$y = \frac{e^{-\frac{\mathrm{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2} - x^2 - c_1}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x^2+1)}{(2y^2+1)x}$ 	$R = y$ $S = -x^2 - \ln(x)$	$\frac{dS}{dR} = \frac{2R^2+1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{\mathrm{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2}-x^2-c_1}}{x} \tag{1}$$

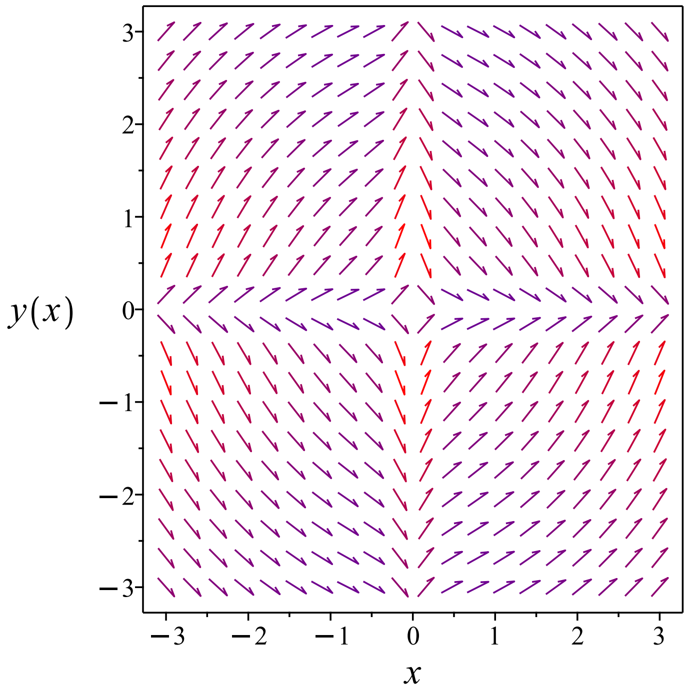


Figure 370: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2}-x^2-c_1}}{x}$$

Verified OK.

6.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{2y^2+1}{y}\right)\mathrm{d}y = \left(\frac{2x^2+1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{2x^2+1}{x}\right)\mathrm{d}x + \left(-\frac{2y^2+1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{2x^2+1}{x}$$
$$N(x,y) = -\frac{2y^2+1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x^2+1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2y^2+1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{2x^2+1}{x} \,\mathrm{d}x$$
$$\phi = -x^2 - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y^2+1}{y}$. Therefore equation (4) becomes

$$-\frac{2y^2+1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2y^2+1}{y}$$
$$= \frac{-2y^2-1}{y}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-2y^2-1}{y}\right)\mathrm{d}y$$
$$f(y) = -y^2 - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - \ln(x) - y^2 - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - \ln(x) - y^2 - \ln(y)$$

The solution becomes

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2}-x^2-c_1}}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2}-x^2-c_1}}{x} \tag{1}$$

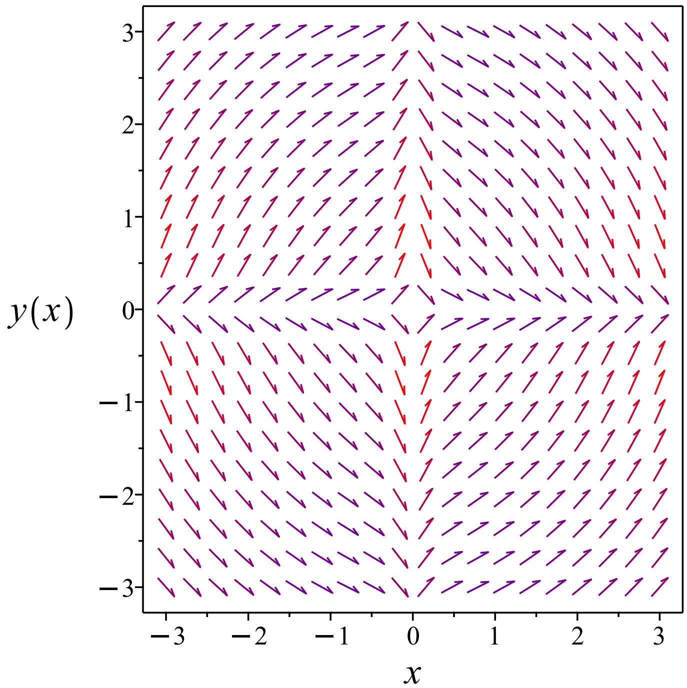


Figure 371: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{\mathrm{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}{2}-x^2-c_1}}{x}$$

Verified OK.

6.11.4 Maple step by step solution

Let's solve

$\left(\frac{1}{y}+2y\right)y' = -\frac{1}{x}-2x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{1}{y}+2y\right)y'dx = \int \left(-\frac{1}{x}-2x\right)dx + c_1$

- Evaluate integral

  $y^2+\ln(y) = -x^2-\ln(x)+c_1$

- Solve for y

$$y = \frac{e^{-\frac{\mathit{LambertW}\left(\frac{2\,e^{-2x^2+2c_1}}{x^2}\right)}{2}-x^2+c_1}}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve((1/x+2*x)+(1/y(x)+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-x^2-c_1}\sqrt{2}}{2\sqrt{\frac{e^{-2x^2-2c_1}}{x^2\,\mathrm{LambertW}\left(\frac{2\,e^{-2x^2-2c_1}}{x^2}\right)}}\,x}$$

✓ Solution by Mathematica

Time used: 9.517 (sec). Leaf size: 71

```
DSolve[(1/x+2*x)+(1/y[x]+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{W\left(\frac{2e^{-2x^2+2c_1}}{x^2}\right)}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{W\left(\frac{2e^{-2x^2+2c_1}}{x^2}\right)}}{\sqrt{2}}$$
$$y(x) \to 0$$

6.12 problem 12

Internal problem ID [1041]
Internal file name [OUTPUT/1042_Sunday_June_05_2022_01_57_41_AM_93331707/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[NONE]`

Unable to solve or complete the solution.

$$\boxed{\sin (x) y^2 + x y^3 \cos (x) + \left(\sin (x) x y + x y^3 \cos (x)\right) y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)+(sin(2*x)-2*y(x))/sin(2*x), y(x)`      **
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)*sin(2*x)+2*x*sin(2*x)-2*y(x)*x)/(si
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 5

```
dsolve((y(x)^2*sin(x)+x*y(x)^3*cos(x))+(x*sin(x)*y(x)+x*y(x)^3*cos(x))*diff(y(x),x)=0,y(x),
```

$$y(x) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(y[x]^2*Sin[x]+x*y[x]^3*Cos[x])+(x*Sin[x]*y[x]+x*y[x]^3*Cos[x])*y'[x]==0,y[x],x,Inclu
```

Not solved

6.13 problem 13

Internal problem ID [1042]
Internal file name [`OUTPUT/1043_Sunday_June_05_2022_01_58_44_AM_54770693/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}=0}$$

6.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{x}{y}
\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{-x^2 + 2c_1}$$

$$y = -\sqrt{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{-x^2 + 2c_1} \tag{2}$$

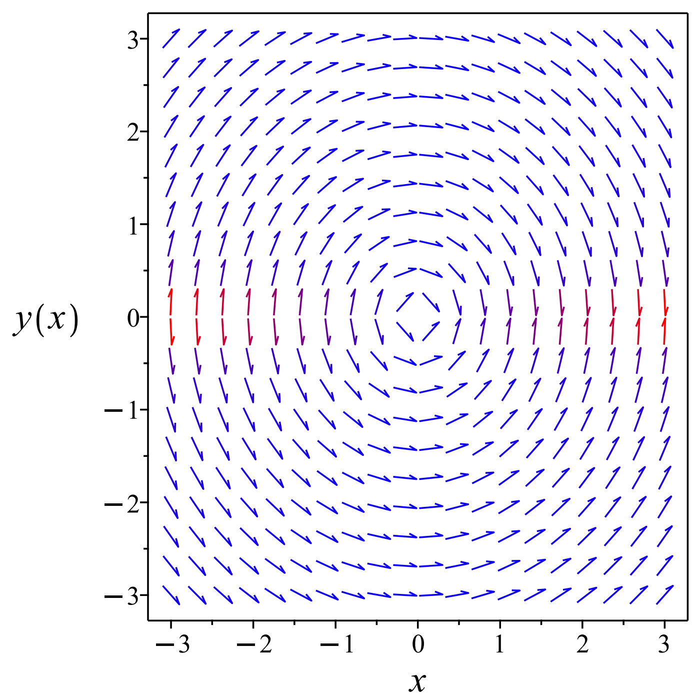


Figure 372: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1}$$

Verified OK.

6.13.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x}{\left(x^2 + u(x)^2\, x^2\right)^{\frac{3}{2}}} + \frac{u(x)\, x(u'(x)\, x + u(x))}{\left(x^2 + u(x)^2\, x^2\right)^{\frac{3}{2}}} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 + 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{\frac{u^2+1}{u}}\, du &= \int -\frac{1}{x}\, dx \\ \frac{\ln(u^2 + 1)}{2} &= -\ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 + 1} = e^{-\ln(x) + c_2}$$

Which simplifies to

$$\sqrt{u^2 + 1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2 + 1} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Summary
The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

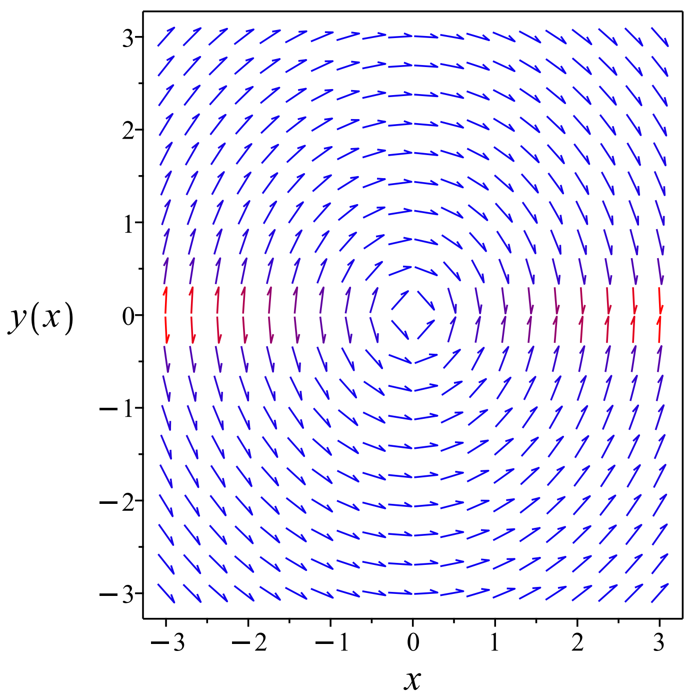


Figure 373: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.13.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy = (-x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dx = d\left(-\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(-\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$
$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -\sqrt{-x^2 + 2c_1} + c_1 \tag{2}$$

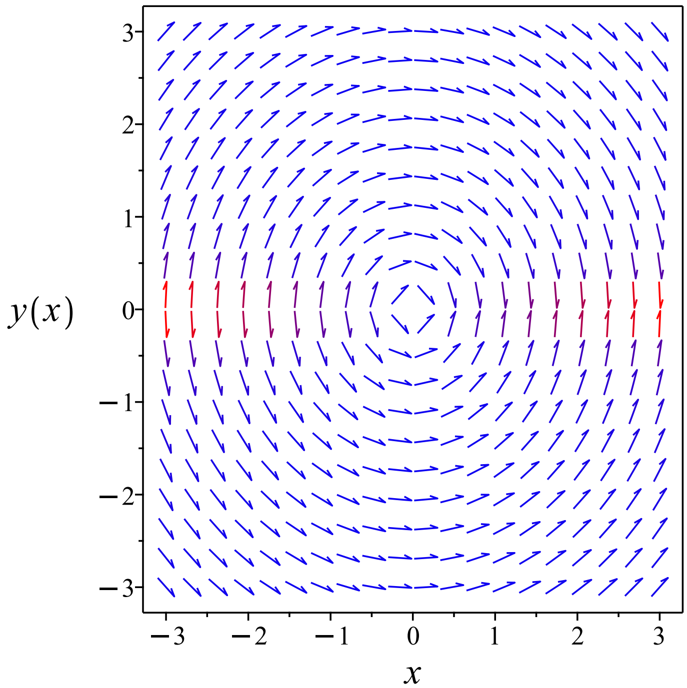


Figure 374: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

6.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 281: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{x} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

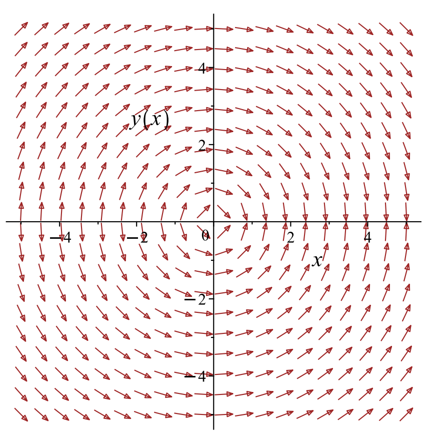
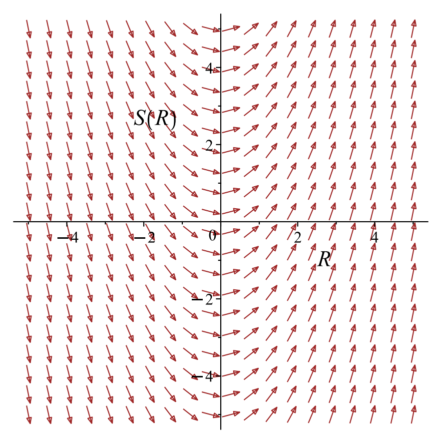
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

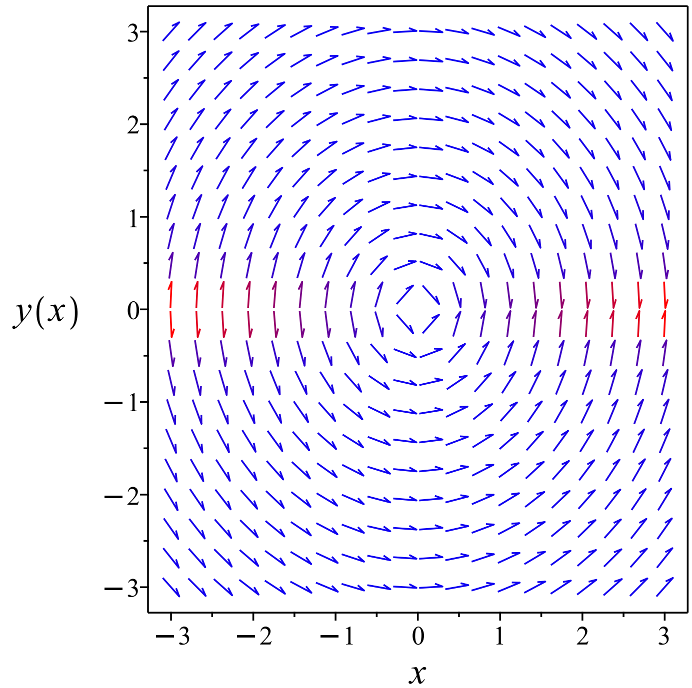


Figure 375: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

6.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + (-y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

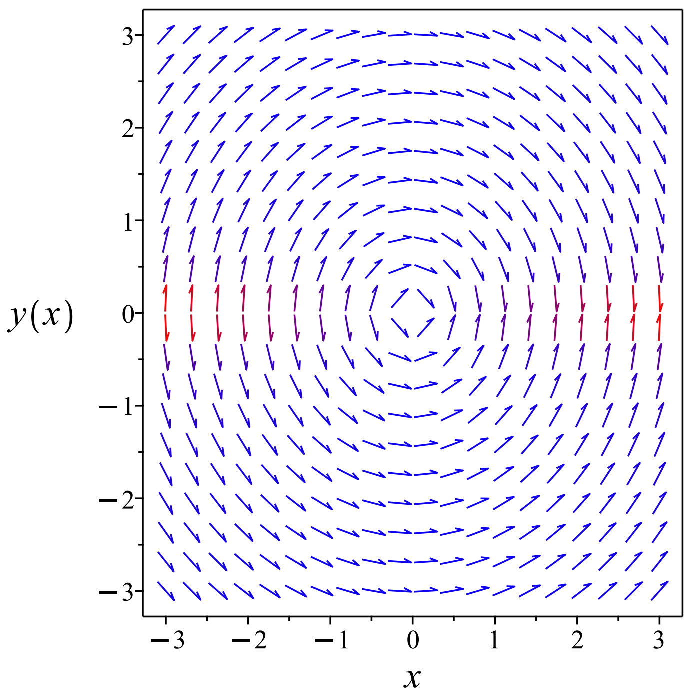


Figure 376: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

6.13.6 Maple step by step solution

Let's solve

$\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+\frac{yy'}{(x^2+y^2)^{\frac{3}{2}}}\right)dx=\int 0dx+c_1$

- Evaluate integral

  $-\frac{1}{\sqrt{x^2+y^2}}=c_1$

- Solve for y

  $\left\{y=\frac{\sqrt{-c_1^2x^2+1}}{c_1}, y=-\frac{\sqrt{-c_1^2x^2+1}}{c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve((x/(x^2+y(x)^2)^(3/2))+(y(x)/(x^2+y(x)^2)^(3/2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\sqrt{-x^2+c_1}$$
$$y(x)=-\sqrt{-x^2+c_1}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 39

```
DSolve[(x/(x^2+y[x]^2)^(3/2))+(y[x]/(x^2+y[x]^2)^(3/2))*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to -\sqrt{-x^2 + 2c_1}$$
$$y(x) \to \sqrt{-x^2 + 2c_1}$$

6.14 problem 14

Internal problem ID [1043]
Internal file name [`OUTPUT/1044_Sunday_June_05_2022_01_58_45_AM_33933115/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{e^{x}\left(x^{2}y^{2}+2xy^{2}\right)+\left(2x^{2}y\,e^{x}+2\right)y'=-6x}$$

6.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2x^2y\,\mathrm{e}^x + 2\right)\mathrm{d}y &= \left(-\mathrm{e}^x\left(x^2y^2 + 2x\,y^2\right) - 6x\right)\mathrm{d}x \\
\left(\mathrm{e}^x\left(x^2y^2 + 2x\,y^2\right) + 6x\right)\mathrm{d}x + \left(2x^2y\,\mathrm{e}^x + 2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \mathrm{e}^x\left(x^2y^2 + 2x\,y^2\right) + 6x \\
N(x,y) &= 2x^2y\,\mathrm{e}^x + 2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x\left(x^2y^2 + 2x\,y^2\right) + 6x\right) \\
&= 2x\,\mathrm{e}^x y(2 + x)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2y\,\mathrm{e}^x + 2\right) \\
&= 2x\,\mathrm{e}^x y(2 + x)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{x}\left(x^2y^2+2x\,y^2\right)+6x\,\mathrm{d}x$$
$$\phi = x^2\left(3+y^2\mathrm{e}^{x}\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^2y\,\mathrm{e}^{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^2y\,\mathrm{e}^{x}+2$. Therefore equation (4) becomes

$$2x^2y\,\mathrm{e}^{x}+2 = 2x^2y\,\mathrm{e}^{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2)\,\mathrm{d}y$$
$$f(y) = 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2\left(3+y^2\mathrm{e}^{x}\right)+2y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2\left(3+y^2\mathrm{e}^{x}\right)+2y$$

Summary
The solution(s) found are the following

$$x^2(3+y^2e^x)+2y=c_1 \tag{1}$$

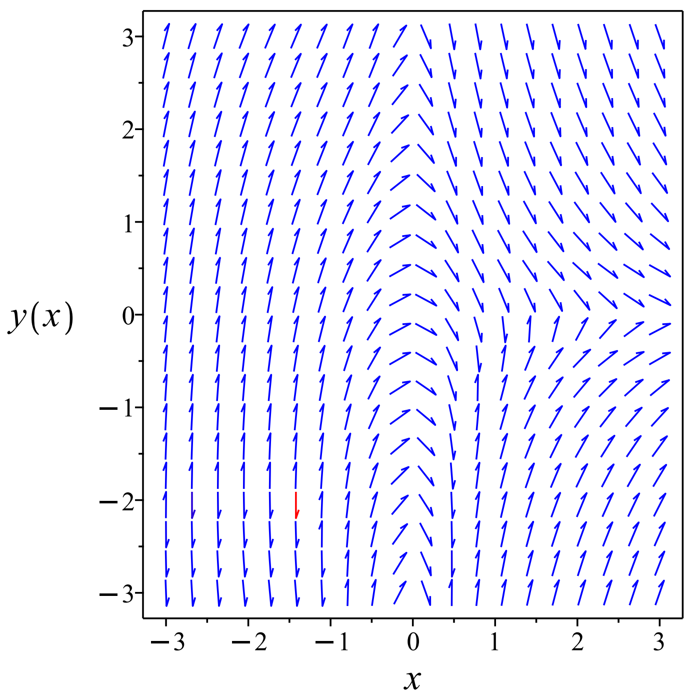


Figure 377: Slope field plot

Verification of solutions

$$x^2(3+y^2e^x)+2y=c_1$$

Verified OK.

6.14.2 Maple step by step solution

Let's solve

$e^x(x^2y^2+2xy^2)+(2x^2y\,e^x+2)\,y'=-6x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $e^{x}(2y\, x^{2} + 4yx) = 4x\, e^{x} y + 2x^{2} y\, e^{x}$

- Simplify

  $2x\, e^{x} y(2 + x) = 2x\, e^{x} y(2 + x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(e^{x}(x^{2}y^{2} + 2x\, y^{2}) + 6x\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y^{2}(x^{2}e^{x} - 2x\, e^{x} + 2\, e^{x}) + 2y^{2}(x\, e^{x} - e^{x}) + 3x^{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2x^{2}y\, e^{x} + 2 = 2y(x^{2}e^{x} - 2x\, e^{x} + 2\, e^{x}) + 4y(x\, e^{x} - e^{x}) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2x^{2}y\, e^{x} + 2 - 2y(x^{2}e^{x} - 2x\, e^{x} + 2\, e^{x}) - 4y(x\, e^{x} - e^{x})$

- Solve for $f_1(y)$

  $f_1(y) = 2y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y^{2}(x^{2}e^{x} - 2x\, e^{x} + 2\, e^{x}) + 2y^{2}(x\, e^{x} - e^{x}) + 3x^{2} + 2y$

- Substitute $F(x, y)$ into the solution of the ODE

  $y^{2}(x^{2}e^{x} - 2x\, e^{x} + 2\, e^{x}) + 2y^{2}(x\, e^{x} - e^{x}) + 3x^{2} + 2y = c_1$

- Solve for y

  $\left\{y = \frac{-1+\sqrt{-3\, e^{x}x^{4} + e^{x}c_1x^{2} + 1}}{e^{x}x^{2}}, y = -\frac{1+\sqrt{-3\, e^{x}x^{4} + e^{x}c_1x^{2} + 1}}{e^{x}x^{2}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve((exp(x)*(x^2*y(x)^2+2*x*y(x)^2)+6*x)+(2*x^2*y(x)*exp(x)+2)*diff(y(x),x)=0,y(x), sings
```

$$y(x) = \frac{\left(-1+\sqrt{-3\,\mathrm{e}^{x} x^{4}-\mathrm{e}^{x} c_{1} x^{2}+1}\right) \mathrm{e}^{-x}}{x^{2}}$$
$$y(x) = \frac{\left(-1-\sqrt{-3\,\mathrm{e}^{x} x^{4}-\mathrm{e}^{x} c_{1} x^{2}+1}\right) \mathrm{e}^{-x}}{x^{2}}$$

✓ Solution by Mathematica

Time used: 36.472 (sec). Leaf size: 76

```
DSolve[(Exp[x]*(x^2*y[x]^2+2*x*y[x]^2)+6*x)+(2*x^2*y[x]*Exp[x]+2)*y'[x]==0,y[x],x,IncludeSin
```

$$y(x) \to -\frac{e^{-x}\left(1+\sqrt{1+e^{x}\left(-3x^{4}+c_{1}x^{2}\right)}\right)}{x^{2}}$$
$$y(x) \to \frac{e^{-x}\left(-1+\sqrt{1+e^{x}\left(-3x^{4}+c_{1}x^{2}\right)}\right)}{x^{2}}$$

6.15 problem 15

Internal problem ID [1044]
Internal file name [`OUTPUT/1045_Sunday_June_05_2022_01_58_48_AM_12936598/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$
\boxed{x^2 e^{x^2+y}(2x^2+3)+\left(x^3 e^{x^2+y}-12y^2\right)y'=-4x}
$$

6.15.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x}&=M\\
\frac{\partial\phi}{\partial y}&=N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(x^3\mathrm{e}^{x^2+y}-12y^2\right)\mathrm{d}y=\left(-x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)-4x\right)\mathrm{d}x$$
$$\left(x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)+4x\right)\mathrm{d}x+\left(x^3\mathrm{e}^{x^2+y}-12y^2\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)+4x$$
$$N(x,y)=x^3\mathrm{e}^{x^2+y}-12y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)+4x\right)\\&=x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^3\mathrm{e}^{x^2+y}-12y^2\right)\\&=x^2\mathrm{e}^{x^2+y}\left(2x^2+3\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x^2 \mathrm{e}^{x^2+y} (2x^2+3) + 4x \, \mathrm{d}x$$
$$\phi = x^2 \left(\mathrm{e}^{x^2+y} x + 2 \right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 \mathrm{e}^{x^2+y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 \mathrm{e}^{x^2+y} - 12y^2$. Therefore equation (4) becomes

$$x^3 \mathrm{e}^{x^2+y} - 12y^2 = x^3 \mathrm{e}^{x^2+y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -12y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-12y^2) \, \mathrm{d}y$$
$$f(y) = -4y^3 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 \left(\mathrm{e}^{x^2+y} x + 2 \right) - 4y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 \left(\mathrm{e}^{x^2+y} x + 2 \right) - 4y^3$$

Summary
The solution(s) found are the following

$$x^2\left(e^{x^2+y}x+2\right)-4y^3=c_1 \tag{1}$$

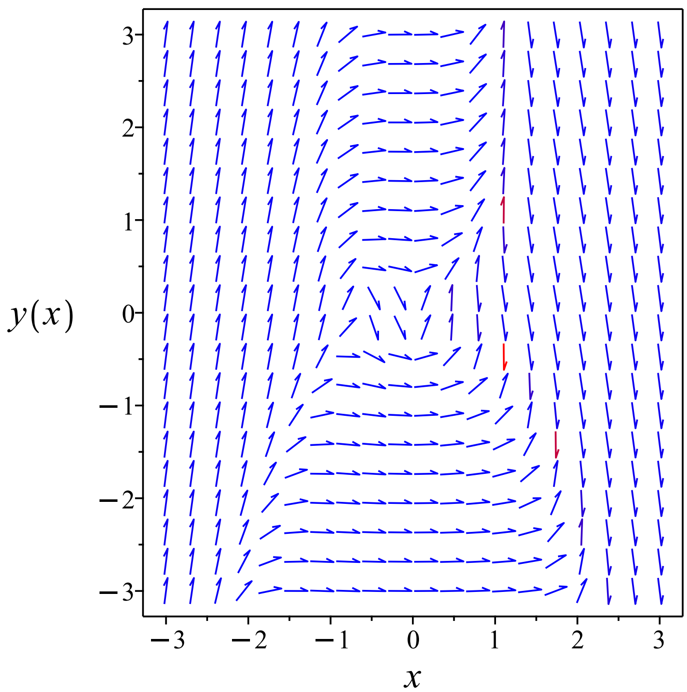


Figure 378: Slope field plot

Verification of solutions

$$x^2\left(e^{x^2+y}x+2\right)-4y^3=c_1$$

Verified OK.

6.15.2 Maple step by step solution

Let's solve

$x^2e^{x^2+y}(2x^2+3)+\left(x^3e^{x^2+y}-12y^2\right)y'=-4x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y} F(x,y)\right) y' = 0$

- Evaluate derivatives

  $x^2 e^{x^2+y}(2x^2+3) = 3x^2 e^{x^2+y} + 2x^4 e^{x^2+y}$

- Simplify

  $x^2 e^{x^2+y}(2x^2+3) = x^2 e^{x^2+y}(2x^2+3)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y} F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \left(x^2 e^{x^2+y}(2x^2+3) + 4x\right) dx + f_1(y)$

- Evaluate integral

  $F(x,y) = 3\,e^y \left(\frac{e^{x^2} x}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{4}\right) + 2\,e^y \left(\frac{e^{x^2} x^3}{2} - \frac{3\,e^{x^2} x}{4} + \frac{3\sqrt{\pi}\,\mathrm{erfi}(x)}{8}\right) + 2x^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y} F(x,y)$

- Compute derivative

  $x^3 e^{x^2+y} - 12y^2 = 3\,e^y \left(\frac{e^{x^2} x}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{4}\right) + 2\,e^y \left(\frac{e^{x^2} x^3}{2} - \frac{3\,e^{x^2} x}{4} + \frac{3\sqrt{\pi}\,\mathrm{erfi}(x)}{8}\right) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = x^3 e^{x^2+y} - 12y^2 - 3\,e^y \left(\frac{e^{x^2} x}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{4}\right) - 2\,e^y \left(\frac{e^{x^2} x^3}{2} - \frac{3\,e^{x^2} x}{4} + \frac{3\sqrt{\pi}\,\mathrm{erfi}(x)}{8}\right)$

- Solve for $f_1(y)$

  $f_1(y) = -x^3 e^{x^2} e^y + x^3 e^{x^2+y} - 4y^3$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 3\,e^y \left(\frac{e^{x^2} x}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{4}\right) + 2\,e^y \left(\frac{e^{x^2} x^3}{2} - \frac{3\,e^{x^2} x}{4} + \frac{3\sqrt{\pi}\,\mathrm{erfi}(x)}{8}\right) + 2x^2 - x^3 e^{x^2} e^y + x^3 e^{x^2+y} - 4$

- Substitute $F(x,y)$ into the solution of the ODE

  $3\,e^y \left(\frac{e^{x^2} x}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{4}\right) + 2\,e^y \left(\frac{e^{x^2} x^3}{2} - \frac{3\,e^{x^2} x}{4} + \frac{3\sqrt{\pi}\,\mathrm{erfi}(x)}{8}\right) + 2x^2 - x^3 e^{x^2} e^y + x^3 e^{x^2+y} - 4y^3 = c_1$

- Solve for y

  $y = RootOf\left(-x^3 e^{x^2+\_Z} + 4\_Z^3 - 2x^2 + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve((x^2*exp(x^2+y(x))*(2*x^2+3)+4*x)+(x^3*exp(x^2+y(x))-12*y(x)^2)*diff(y(x),x)=0,y(x),
```

$$x^3 e^{x^2+y(x)} - 4y(x)^3 + 2x^2 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.428 (sec). Leaf size: 30

```
DSolve[(x^2*Exp[x^2+y[x]]*(2*x^2+3)+4*x)+(x^3*Exp[x^2+y[x]]-12*y[x]^2)*y'[x]==0,y[x],x,Inclu
```

$$\text{Solve}\left[2x^2 + x^3 e^{x^2+y(x)} - 4y(x)^3 = c_1, y(x)\right]$$

6.16 problem 16

Internal problem ID [1045]
Internal file name [`OUTPUT/1046_Sunday_June_05_2022_01_58_50_AM_38794570/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{e^{yx}\left(y x^{4}+4 x^{3}\right)+3 y+\left(x^{5} e^{yx}+3 x\right) y^{\prime}=0}$$

6.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y^{\prime} &= -\frac{e^{yx} x^{4} y+4\, e^{yx} x^{3}+3 y}{x\left(e^{yx} x^{4}+3\right)} \\ y^{\prime} &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_{x}+\omega\left(\eta_{y}-\xi_{x}\right)-\omega^{2} \xi_{y}-\omega_{x} \xi-\omega_{y} \eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi=x^{3} a_{7}+y\, x^{2} a_{8}+x\, y^{2} a_{9}+y^{3} a_{10}+x^{2} a_{4}+y x a_{5}+y^{2} a_{6}+x a_{2}+y a_{3}+a_{1} \tag{1E}$$

$$\eta=x^{3} b_{7}+y\, x^{2} b_{8}+x\, y^{2} b_{9}+y^{3} b_{10}+x^{2} b_{4}+y x b_{5}+y^{2} b_{6}+x b_{2}+y b_{3}+b_{1} \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \tag{5E}$$
$$- \frac{(e^{yx}x^4y + 4\,e^{yx}x^3 + 3y)\,(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 -}{x\,(e^{yx}x^4 + 3)}$$
$$- \frac{(e^{yx}x^4y + 4\,e^{yx}x^3 + 3y)^2\,(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{x^2\,(e^{yx}x^4 + 3)^2}$$
$$- \left(-\frac{e^{yx}y^2x^4 + 8\,e^{yx}x^3y + 12\,e^{yx}x^2}{x\,(e^{yx}x^4 + 3)} + \frac{e^{yx}x^4y + 4\,e^{yx}x^3 + 3y}{x^2\,(e^{yx}x^4 + 3)}\right.$$
$$\left. + \frac{(e^{yx}x^4y + 4\,e^{yx}x^3 + 3y)\,(e^{yx}x^4y + 4\,e^{yx}x^3)}{x\,(e^{yx}x^4 + 3)^2}\right)(x^3a_7 + y\,x^2a_8$$
$$+ x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1)$$
$$- \left(-\frac{5\,e^{yx}x^4 + x^5e^{yx}y + 3}{x\,(e^{yx}x^4 + 3)} + \frac{(e^{yx}x^4y + 4\,e^{yx}x^3 + 3y)\,x^4e^{yx}}{(e^{yx}x^4 + 3)^2}\right)(x^3b_7$$
$$+ y\,x^2b_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0$$

Putting the above in normal form gives

$$\frac{-27y^3a_6 - 12\,e^{yx}x^4y^4a_{10} - 48\,e^{yx}x^3y^3a_{10} + 18x^3yb_8 + 2\,e^{2yx}x^{10}b_2 + e^{2yx}x^9b_1 - 4\,e^{2yx}x^8a_2 - 4\,e^{2yx}x^8b_3 - 8}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-27y^3a_6 - 12\,\mathrm{e}^{yx}x^4y^4a_{10} - 48\,\mathrm{e}^{yx}x^3y^3a_{10} + 18x^3yb_8 \\
&+ 2\,\mathrm{e}^{2yx}x^{10}b_2 + \mathrm{e}^{2yx}x^9b_1 - 4\,\mathrm{e}^{2yx}x^8a_2 - 4\,\mathrm{e}^{2yx}x^8b_3 \\
&- 8\,\mathrm{e}^{2yx}x^7a_1 - 16\,\mathrm{e}^{2yx}x^6a_3 + 24\,\mathrm{e}^{yx}x^6b_2 + 18\,\mathrm{e}^{yx}x^5b_1 \\
&+ 36\,\mathrm{e}^{yx}x^4a_2 - 12\,\mathrm{e}^{yx}x^4b_3 + 24\,\mathrm{e}^{yx}x^3a_1 + 4\,\mathrm{e}^{2yx}x^{12}b_7 \\
&+ 4\,\mathrm{e}^{2yx}x^{10}a_7 - 4\,\mathrm{e}^{2yx}x^{10}b_8 - 16\,\mathrm{e}^{2yx}x^8a_8 + 27x^3b_4 + 9x^2ya_4 \\
&- 9x\,y^2a_5 - 9x\,y^2b_6 + 30\,\mathrm{e}^{yx}x^7b_4 + 48\,\mathrm{e}^{yx}x^5a_4 - 12\,\mathrm{e}^{yx}x^5b_5 \\
&- 36y^4a_{10} + 18b_2x^2 + 9yb_5x^2 + 24\,\mathrm{e}^{yx}x^7yb_8 + 12\,\mathrm{e}^{yx}x^6y^2b_9 \\
&+ 24\,\mathrm{e}^{yx}x^7ya_7 + 12\,\mathrm{e}^{yx}x^6y^2a_8 + 24\,\mathrm{e}^{yx}x^5ya_8 - 24\,\mathrm{e}^{yx}x^5yb_9 \\
&- 12\,\mathrm{e}^{yx}x^4y^2a_9 - 36\,\mathrm{e}^{yx}x^4y^2b_{10} - 12\,\mathrm{e}^{2yx}x^8ya_5 \\
&- 8\,\mathrm{e}^{2yx}x^8yb_6 - 24\,\mathrm{e}^{2yx}x^7y^2a_6 - 32\,\mathrm{e}^{2yx}x^6ya_6 + \mathrm{e}^{2yx}x^{10}ya_4 \\
&+ \mathrm{e}^{2yx}x^{10}yb_5 - \mathrm{e}^{2yx}x^9y^2a_5 - \mathrm{e}^{2yx}x^9y^2b_6 - 3\,\mathrm{e}^{2yx}x^8y^3a_6 \\
&+ 3\,\mathrm{e}^{2yx}x^{11}b_4 - 4\,\mathrm{e}^{2yx}x^9b_5 - 16\,\mathrm{e}^{2yx}x^7a_5 - 48\,\mathrm{e}^{2yx}x^6y^2a_{10} \\
&+ 2\,\mathrm{e}^{2yx}x^{11}yb_8 + 2\,\mathrm{e}^{2yx}x^{11}ya_7 - 2\,\mathrm{e}^{2yx}x^9y^3a_9 - 2\,\mathrm{e}^{2yx}x^9y^3b_{10} \\
&- 8\,\mathrm{e}^{2yx}x^9ya_8 - 8\,\mathrm{e}^{2yx}x^9yb_9 - 20\,\mathrm{e}^{2yx}x^8y^2a_9 \\
&- 12\,\mathrm{e}^{2yx}x^8y^2b_{10} - 4\,\mathrm{e}^{2yx}x^8y^4a_{10} - 32\,\mathrm{e}^{2yx}x^7y^3a_{10} \\
&- 32\,\mathrm{e}^{2yx}x^7ya_9 + 18\,\mathrm{e}^{yx}x^6ya_4 + 18\,\mathrm{e}^{yx}x^6yb_5 + 6\,\mathrm{e}^{yx}x^5y^2a_5 \\
&+ 6\,\mathrm{e}^{yx}x^5y^2b_6 - 6\,\mathrm{e}^{yx}x^4y^3a_6 + 12\,\mathrm{e}^{yx}x^4ya_5 - 24\,\mathrm{e}^{yx}x^4yb_6 \\
&- 24\,\mathrm{e}^{yx}x^3y^2a_6 + 36x^4b_7 + 18x^3ya_7 - 18x\,y^3a_9 \\
&- 18x\,y^3b_{10} + 12\,\mathrm{e}^{yx}x^5ya_2 + 12\,\mathrm{e}^{yx}x^5yb_3 + 6\,\mathrm{e}^{yx}x^4ya_1 \\
&- 2\,\mathrm{e}^{2yx}x^8y^2a_3 - \mathrm{e}^{2yx}x^8ya_1 - 16\,\mathrm{e}^{2yx}x^7ya_3 - 18y^2a_3 \\
&+ 9xb_1 - 9ya_1 + 36\,\mathrm{e}^{yx}x^8b_7 + 60\,\mathrm{e}^{yx}x^6a_7 - 12\,\mathrm{e}^{yx}x^6b_8 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$\begin{aligned}
&-27y^3a_6 - 12\,\mathrm{e}^{yx}x^4y^4a_{10} - 48\,\mathrm{e}^{yx}x^3y^3a_{10} + 18x^3yb_8 \\
&+ 2\,\mathrm{e}^{2yx}x^{10}b_2 + \mathrm{e}^{2yx}x^9b_1 - 4\,\mathrm{e}^{2yx}x^8a_2 - 4\,\mathrm{e}^{2yx}x^8b_3 \\
&- 8\,\mathrm{e}^{2yx}x^7a_1 - 16\,\mathrm{e}^{2yx}x^6a_3 + 24\,\mathrm{e}^{yx}x^6b_2 + 18\,\mathrm{e}^{yx}x^5b_1 \\
&+ 36\,\mathrm{e}^{yx}x^4a_2 - 12\,\mathrm{e}^{yx}x^4b_3 + 24\,\mathrm{e}^{yx}x^3a_1 + 4\,\mathrm{e}^{2yx}x^{12}b_7 \\
&+ 4\,\mathrm{e}^{2yx}x^{10}a_7 - 4\,\mathrm{e}^{2yx}x^{10}b_8 - 16\,\mathrm{e}^{2yx}x^8a_8 + 27x^3b_4 + 9x^2ya_4 \\
&- 9x\,y^2a_5 - 9x\,y^2b_6 + 30\,\mathrm{e}^{yx}x^7b_4 + 48\,\mathrm{e}^{yx}x^5a_4 - 12\,\mathrm{e}^{yx}x^5b_5 \\
&- 36y^4a_{10} + 18b_2x^2 + 9yb_5x^2 + 24\,\mathrm{e}^{yx}x^7yb_8 + 12\,\mathrm{e}^{yx}x^6y^2b_9 \\
&+ 24\,\mathrm{e}^{yx}x^7ya_7 + 12\,\mathrm{e}^{yx}x^6y^2a_8 + 24\,\mathrm{e}^{yx}x^5ya_8 - 24\,\mathrm{e}^{yx}x^5yb_9 \\
&- 12\,\mathrm{e}^{yx}x^4y^2a_9 - 36\,\mathrm{e}^{yx}x^4y^2b_{10} - 12\,\mathrm{e}^{2yx}x^8ya_5 \\
&- 8\,\mathrm{e}^{2yx}x^8yb_6 - 24\,\mathrm{e}^{2yx}x^7y^2a_6 - 32\,\mathrm{e}^{2yx}x^6ya_6 + \mathrm{e}^{2yx}x^{10}ya_4 \\
&+ \mathrm{e}^{2yx}x^{10}yb_5 - \mathrm{e}^{2yx}x^9y^2a_5 - \mathrm{e}^{2yx}x^9y^2b_6 - 3\,\mathrm{e}^{2yx}x^8y^3a_6 \\
&+ 3\,\mathrm{e}^{2yx}x^{11}b_4 - 4\,\mathrm{e}^{2yx}x^9b_5 - 16\,\mathrm{e}^{2yx}x^7a_5 - 48\,\mathrm{e}^{2yx}x^6y^2a_{10} \\
&+ 2\,\mathrm{e}^{2yx}x^{11}yb_8 + 2\,\mathrm{e}^{2yx}x^{11}ya_7 - 2\,\mathrm{e}^{2yx}x^9y^3a_9 - 2\,\mathrm{e}^{2yx}x^9y^3b_{10} \\
&- 8\,\mathrm{e}^{2yx}x^9ya_8 - 8\,\mathrm{e}^{2yx}x^9yb_9 - 20\,\mathrm{e}^{2yx}x^8y^2a_9 \\
&- 12\,\mathrm{e}^{2yx}x^8y^2b_{10} - 4\,\mathrm{e}^{2yx}x^8y^4a_{10} - 32\,\mathrm{e}^{2yx}x^7y^3a_{10} \\
&- 32\,\mathrm{e}^{2yx}x^7ya_9 + 18\,\mathrm{e}^{yx}x^6ya_4 + 18\,\mathrm{e}^{yx}x^6yb_5 + 6\,\mathrm{e}^{yx}x^5y^2a_5 \\
&+ 6\,\mathrm{e}^{yx}x^5y^2b_6 - 6\,\mathrm{e}^{yx}x^4y^3a_6 + 12\,\mathrm{e}^{yx}x^4ya_5 - 24\,\mathrm{e}^{yx}x^4yb_6 \\
&- 24\,\mathrm{e}^{yx}x^3y^2a_6 + 36x^4b_7 + 18x^3ya_7 - 18x\,y^3a_9 \\
&- 18x\,y^3b_{10} + 12\,\mathrm{e}^{yx}x^5ya_2 + 12\,\mathrm{e}^{yx}x^5yb_3 + 6\,\mathrm{e}^{yx}x^4ya_1 \\
&- 2\,\mathrm{e}^{2yx}x^8y^2a_3 - \mathrm{e}^{2yx}x^8ya_1 - 16\,\mathrm{e}^{2yx}x^7ya_3 - 18y^2a_3 \\
&+ 9xb_1 - 9ya_1 + 36\,\mathrm{e}^{yx}x^8b_7 + 60\,\mathrm{e}^{yx}x^6a_7 - 12\,\mathrm{e}^{yx}x^6b_8 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{yx}, \mathrm{e}^{2yx}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{yx} = v_3, \mathrm{e}^{2yx} = v_4\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&18v_1^3v_2b_8 + 2v_4v_1^{10}b_2 + v_4v_1^9b_1 - 4v_4v_1^8a_2 - 4v_4v_1^8b_3 \\
&- 8v_4v_1^7a_1 - 16v_4v_1^6a_3 + 24v_3v_1^6b_2 + 18v_3v_1^5b_1 + 36v_3v_1^4a_2 \\
&- 12v_3v_1^4b_3 + 24v_3v_1^3a_1 + 4v_4v_1^{12}b_7 + 4v_4v_1^{10}a_7 - 4v_4v_1^{10}b_8 \\
&- 16v_4v_1^8a_8 + 9v_1^2v_2a_4 - 9v_1v_2^2a_5 - 9v_1v_2^2b_6 + 30v_3v_1^7b_4 \\
&+ 48v_3v_1^5a_4 - 12v_3v_1^5b_5 + 9v_2b_5v_1^2 + 3v_4v_1^{11}b_4 - 4v_4v_1^9b_5 \\
&- 16v_4v_1^7a_5 + 18v_1^3v_2a_7 - 18v_1v_2^3a_9 - 18v_1v_2^3b_{10} + 36v_3v_1^8b_7 \\
&+ 60v_3v_1^6a_7 - 12v_3v_1^6b_8 - 27v_2^3a_6 + 27v_1^3b_4 - 36v_2^4a_{10} \\
&+ 18b_2v_1^2 + 36v_1^4b_7 - 18v_2^2a_3 + 9v_1b_1 - 9v_2a_1 - 12v_3v_1^4v_2^4a_{10} \\
&- 48v_3v_1^3v_2^3a_{10} + 24v_3v_1^7v_2b_8 + 12v_3v_1^6v_2^2b_9 + 24v_3v_1^7v_2a_7 \\
&+ 12v_3v_1^6v_2^2a_8 + 24v_3v_1^5v_2a_8 - 24v_3v_1^5v_2b_9 - 12v_3v_1^4v_2^2a_9 \\
&- 36v_3v_1^4v_2^2b_{10} - 12v_4v_1^8v_2a_5 - 8v_4v_1^8v_2b_6 - 24v_4v_1^7v_2^2a_6 \\
&- 32v_4v_1^6v_2a_6 + v_4v_1^{10}v_2a_4 + v_4v_1^{10}v_2b_5 - v_4v_1^9v_2^2a_5 \\
&- v_4v_1^9v_2^2b_6 - 3v_4v_1^8v_2^3a_6 - 48v_4v_1^6v_2^2a_{10} + 2v_4v_1^{11}v_2b_8 \\
&+ 2v_4v_1^{11}v_2a_7 - 2v_4v_1^9v_2^3a_9 - 2v_4v_1^9v_2^3b_{10} - 8v_4v_1^9v_2a_8 \\
&- 8v_4v_1^9v_2b_9 - 20v_4v_1^8v_2^2a_9 - 12v_4v_1^8v_2^2b_{10} - 4v_4v_1^8v_2^4a_{10} \\
&- 32v_4v_1^7v_2^3a_{10} - 32v_4v_1^7v_2a_9 + 18v_3v_1^6v_2a_4 + 18v_3v_1^6v_2b_5 \\
&+ 6v_3v_1^5v_2^2a_5 + 6v_3v_1^5v_2^2b_6 - 6v_3v_1^4v_2^3a_6 + 12v_3v_1^4v_2a_5 \\
&- 24v_3v_1^4v_2b_6 - 24v_3v_1^3v_2^2a_6 + 12v_3v_1^5v_2a_2 + 12v_3v_1^5v_2b_3 \\
&+ 6v_3v_1^4v_2a_1 - 2v_4v_1^8v_2^2a_3 - v_4v_1^8v_2a_1 - 16v_4v_1^7v_2a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(18b_8 + 18a_7)\, v_2 v_1^3 + (2b_2 + 4a_7 - 4b_8)\, v_1^{10} v_4 + (b_1 - 4b_5)\, v_1^9 v_4 \\
&+ (-4a_2 - 4b_3 - 16a_8)\, v_1^8 v_4 + (-8a_1 - 16a_5)\, v_1^7 v_4 \\
&+ (24b_2 + 60a_7 - 12b_8)\, v_1^6 v_3 + (18b_1 + 48a_4 - 12b_5)\, v_1^5 v_3 \\
&+ (36a_2 - 12b_3)\, v_1^4 v_3 + (9a_4 + 9b_5)\, v_2 v_1^2 + (-9a_5 - 9b_6)\, v_2^2 v_1 \\
&+ (-18a_9 - 18b_{10})\, v_2^3 v_1 + (-20a_9 - 12b_{10} - 2a_3)\, v_2^2 v_1^8 v_4 \\
&+ (-32a_9 - 16a_3)\, v_2 v_1^7 v_4 + (18a_4 + 18b_5)\, v_2 v_1^6 v_3 \\
&+ (6a_5 + 6b_6)\, v_2^2 v_1^5 v_3 + (12a_5 - 24b_6 + 6a_1)\, v_2 v_1^4 v_3 - 16 v_4 v_1^6 a_3 \\
&+ 24 v_3 v_1^3 a_1 + 4 v_4 v_1^{12} b_7 + 30 v_3 v_1^7 b_4 + 3 v_4 v_1^{11} b_4 + 36 v_3 v_1^8 b_7 \\
&- 27 v_2^3 a_6 + 27 v_1^3 b_4 - 36 v_2^4 a_{10} + 18 b_2 v_1^2 + 36 v_1^4 b_7 - 18 v_2^2 a_3 \\
&+ 9 v_1 b_1 - 9 v_2 a_1 + (24b_8 + 24a_7)\, v_2 v_1^7 v_3 + (12b_9 + 12a_8)\, v_2^2 v_1^6 v_3 \\
&+ (24a_8 - 24b_9 + 12a_2 + 12b_3)\, v_2 v_1^5 v_3 + (-12a_9 - 36b_{10})\, v_2^2 v_1^4 v_3 \\
&+ (-8b_6 - a_1 - 12a_5)\, v_2 v_1^8 v_4 + (a_4 + b_5)\, v_2 v_1^{10} v_4 \\
&+ (-a_5 - b_6)\, v_2^2 v_1^9 v_4 + (2b_8 + 2a_7)\, v_2 v_1^{11} v_4 + (-2a_9 - 2b_{10})\, v_2^3 v_1^9 v_4 \\
&+ (-8a_8 - 8b_9)\, v_2 v_1^9 v_4 - 12 v_3 v_1^4 v_2^4 a_{10} - 48 v_3 v_1^3 v_2^3 a_{10} \\
&- 24 v_4 v_1^7 v_2^2 a_6 - 32 v_4 v_1^6 v_2 a_6 - 3 v_4 v_1^8 v_2^3 a_6 - 48 v_4 v_1^6 v_2^2 a_{10} \\
&- 4 v_4 v_1^8 v_2^4 a_{10} - 32 v_4 v_1^7 v_2^3 a_{10} - 6 v_3 v_1^4 v_2^3 a_6 - 24 v_3 v_1^3 v_2^2 a_6 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-9a_1 &= 0\\
24a_1 &= 0\\
-18a_3 &= 0\\
-16a_3 &= 0\\
-32a_6 &= 0\\
-27a_6 &= 0\\
-24a_6 &= 0\\
-6a_6 &= 0\\
-3a_6 &= 0\\
-48a_{10} &= 0\\
-36a_{10} &= 0\\
-32a_{10} &= 0\\
-12a_{10} &= 0\\
-4a_{10} &= 0\\
9b_1 &= 0\\
18b_2 &= 0\\
3b_4 &= 0\\
27b_4 &= 0\\
30b_4 &= 0\\
4b_7 &= 0\\
36b_7 &= 0\\
-8a_1 - 16a_5 &= 0\\
36a_2 - 12b_3 &= 0\\
a_4 + b_5 &= 0\\
9a_4 + 9b_5 &= 0\\
18a_4 + 18b_5 &= 0\\
-9a_5 - 9b_6 &= 0\\
-a_5 - b_6 &= 0\\
6a_5 + 6b_6 &= 0\\
-8a_8 - 8b_9 &= 0\\
-32a_9 - 16a_3 &= 0\\
-18a_9 - 18b_{10} &= 0\\
-12a_9 - 36b_{10} &= 0\\
-2a_9 - 2b_{10} &= 0\\
b_1 - 4b_5 &= 0\\
2b_8 + 2a_7 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_9\\
a_3 &= 0\\
a_4 &= 0\\
a_5 &= 0\\
a_6 &= 0\\
a_7 &= 0\\
a_8 &= -b_9\\
a_9 &= 0\\
a_{10} &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 3b_9\\
b_4 &= 0\\
b_5 &= 0\\
b_6 &= 0\\
b_7 &= 0\\
b_8 &= 0\\
b_9 &= b_9\\
b_{10} &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y\,x^2 + x\\
\eta &= x\,y^2 + 3y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= x\,y^2 + 3y - \left(-\frac{\mathrm{e}^{yx}x^4y + 4\,\mathrm{e}^{yx}x^3 + 3y}{x\,(\mathrm{e}^{yx}x^4 + 3)}\right)(-y\,x^2 + x)\\
&= \frac{4\,\mathrm{e}^{yx}x^4 + 12yx}{x^5\mathrm{e}^{yx} + 3x}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4\,\mathrm{e}^{yx} x^4 + 12yx}{x^5 \mathrm{e}^{yx} + 3x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln \left(\mathrm{e}^{yx} x^4 + 3yx\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\mathrm{e}^{yx} x^4 y + 4\,\mathrm{e}^{yx} x^3 + 3y}{x\left(\mathrm{e}^{yx} x^4 + 3\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{yx} x^4 y + 4\,\mathrm{e}^{yx} x^3 + 3y}{4x\left(\mathrm{e}^{yx} x^3 + 3y\right)} \\ S_y &= \frac{\mathrm{e}^{yx} x^4 + 3}{4\,\mathrm{e}^{yx} x^3 + 12y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)}{4} + \frac{\ln(e^{yx}x^3 + 3y)}{4} = c_1$$

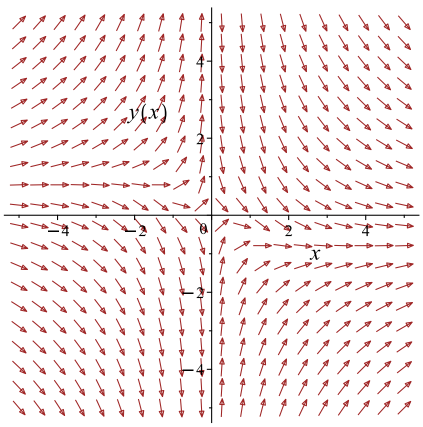
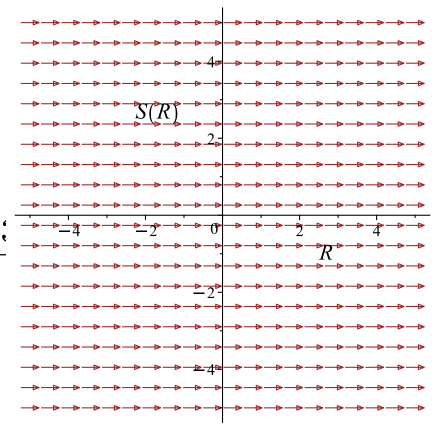
Which simplifies to

$$\frac{\ln(x)}{4} + \frac{\ln(e^{yx}x^3 + 3y)}{4} = c_1$$

Which gives

$$y = -\frac{3\,\text{LambertW}\left(\frac{x^4 e^{\frac{e^{4c_1}}{3}}}{3}\right) - e^{4c_1}}{3x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{yx}x^4y+4\,e^{yx}x^3+3y}{x(e^{yx}x^4+3)}$ 	$R = x$ $S = \frac{\ln(x)}{4} + \frac{\ln(e^{yx}x^3 + 3}{4}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{LambertW}\left(\frac{x^4 e^{\frac{e^{4c_1}}{3}}}{3}\right) - e^{4c_1}}{3x} \tag{1}$$

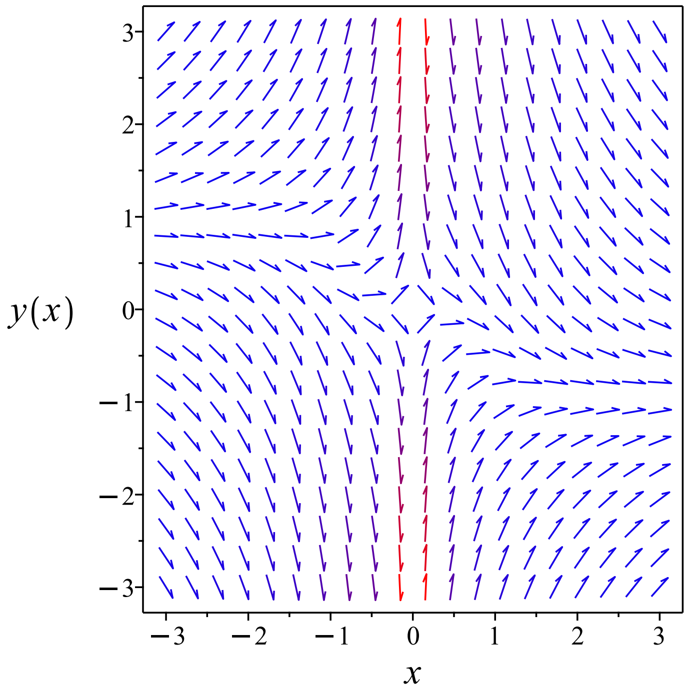


Figure 379: Slope field plot

Verification of solutions

$$y = -\frac{3\,\mathrm{LambertW}\left(\frac{x^4 \mathrm{e}^{\frac{\mathrm{e}^{4c_1}}{3}}}{3}\right) - \mathrm{e}^{4c_1}}{3x}$$

Verified OK.

6.16.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^5 \mathrm{e}^{yx} + 3x\right) \mathrm{d}y = \left(-\mathrm{e}^{yx}\left(x^4 y + 4x^3\right) - 3y\right) \mathrm{d}x$$
$$\left(\mathrm{e}^{yx}\left(x^4 y + 4x^3\right) + 3y\right) \mathrm{d}x + \left(x^5 \mathrm{e}^{yx} + 3x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \mathrm{e}^{yx}\left(x^4 y + 4x^3\right) + 3y$$
$$N(x, y) = x^5 \mathrm{e}^{yx} + 3x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{yx}\left(x^4 y + 4x^3\right) + 3y\right) \\ &= 5 \, \mathrm{e}^{yx} x^4 + x^5 \mathrm{e}^{yx} y + 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^5 \mathrm{e}^{yx} + 3x\right) \\ &= 5 \, \mathrm{e}^{yx} x^4 + x^5 \mathrm{e}^{yx} y + 3\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \mathrm{e}^{yx}\left(x^4y+4x^3\right)+3y\,\mathrm{d}x$$
$$\phi=x\left(\mathrm{e}^{yx}x^3+3y\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x\left(\mathrm{e}^{yx}x^4+3\right)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^5\mathrm{e}^{yx}+3x$. Therefore equation (4) becomes

$$x^5\mathrm{e}^{yx}+3x=x\left(\mathrm{e}^{yx}x^4+3\right)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=x\left(\mathrm{e}^{yx}x^3+3y\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=x\left(\mathrm{e}^{yx}x^3+3y\right)$$

The solution becomes

$$y = -\frac{3\,\mathrm{LambertW}\left(\frac{x^4 e^{\frac{c_1}{3}}}{3}\right) - c_1}{3x}$$

Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{LambertW}\left(\frac{x^4 e^{\frac{c_1}{3}}}{3}\right) - c_1}{3x} \tag{1}$$

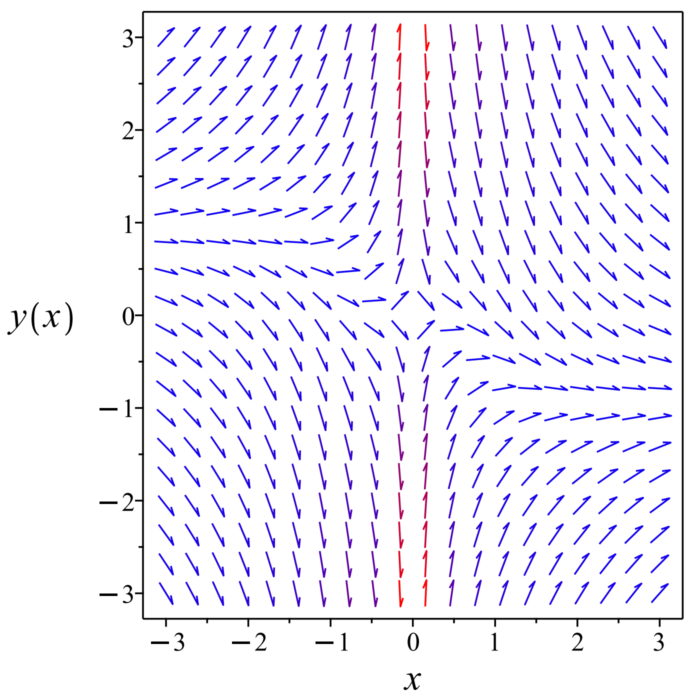


Figure 380: Slope field plot

Verification of solutions

$$y = -\frac{3\,\mathrm{LambertW}\left(\frac{x^4 e^{\frac{c_1}{3}}}{3}\right) - c_1}{3x}$$

Verified OK.

6.16.3 Maple step by step solution

Let's solve

$e^{yx}(yx^4+4x^3)+3y+(x^5e^{yx}+3x)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $e^{yx}x(x^4y+4x^3)+e^{yx}x^4+3=5\,e^{yx}x^4+x^5e^{yx}y+3$

  - Simplify

    $5\,e^{yx}x^4+x^5e^{yx}y+3=5\,e^{yx}x^4+x^5e^{yx}y+3$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\,, N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(e^{yx}(x^4y+4x^3)+3y\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{\frac{e^{yx}y^4x^4-4y^3x^3e^{yx}+12\,e^{yx}y^2x^2-24\,e^{yx}yx+24\,e^{yx}}{y^3}+\frac{4\left(y^3x^3e^{yx}-3\,e^{yx}y^2x^2+6\,e^{yx}yx-6\,e^{yx}\right)}{y^3}}{y}+3yx+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^5e^{yx}+3x=-\frac{\frac{e^{yx}y^4x^4-4y^3x^3e^{yx}+12\,e^{yx}y^2x^2-24\,e^{yx}yx+24\,e^{yx}}{y^3}+\frac{4\left(y^3x^3e^{yx}-3\,e^{yx}y^2x^2+6\,e^{yx}yx-6\,e^{yx}\right)}{y^3}}{y^2}+\frac{-\frac{3\left(e^{yx}y^4x^4-4y^3x^3\right.}{}}{}$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=x^5e^{yx}+\frac{\frac{e^{yx}y^4x^4-4y^3x^3e^{yx}+12\,e^{yx}y^2x^2-24\,e^{yx}yx+24\,e^{yx}}{y^3}+\frac{4\left(y^3x^3e^{yx}-3\,e^{yx}y^2x^2+6\,e^{yx}yx-6\,e^{yx}\right)}{y^3}}{y^2}-\frac{-\frac{3\left(e^{yx}y^4x^4-4y\right.}{}}{}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y) = \frac{\frac{e^{yx}y^4x^4-4y^3x^3e^{yx}+12\,e^{yx}y^2x^2-24\,e^{yx}yx+24\,e^{yx}}{y^3}+\frac{4\left(y^3x^3e^{yx}-3\,e^{yx}y^2x^2+6\,e^{yx}yx-6\,e^{yx}\right)}{y^3}}{y} + 3yx$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{\frac{e^{yx}y^4x^4-4y^3x^3e^{yx}+12\,e^{yx}y^2x^2-24\,e^{yx}yx+24\,e^{yx}}{y^3}+\frac{4\left(y^3x^3e^{yx}-3\,e^{yx}y^2x^2+6\,e^{yx}yx-6\,e^{yx}\right)}{y^3}}{y} + 3yx = c_1$$

- Solve for y

  $$y = -\frac{3LambertW\left(\frac{x^4e^{-\frac{c_1}{3}}}{3}\right)-c_1}{3x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve((exp(x*y(x))*(x^4*y(x)+4*x^3)+3*y(x))+( x^5*exp(x*y(x))+3*x)*diff(y(x),x)=0,y(x), sin
```

$$y(x) = \frac{-3\,\mathrm{LambertW}\left(\frac{x^4e^{-\frac{c_1}{3}}}{3}\right)-c_1}{3x}$$

✓ Solution by Mathematica

Time used: 4.486 (sec). Leaf size: 33

```
DSolve[(Exp[x*y[x]]*(x^4*y[x]+4*x^3)+3*y[x])+(x^5*Exp[x*y[x]]+3*x)*y'[x]==0,y[x],x,IncludeSi
```

$$y(x) \to \frac{c_1 - 3W\left(\frac{1}{3}e^{\frac{c_1}{3}}x^4\right)}{3x}$$

6.17 problem 17

Internal problem ID [1046]
Internal file name [`OUTPUT/1047_Sunday_June_05_2022_01_58_53_AM_46735432/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$3\cos(x)\,yx^2 - x^3y^2\sin(x) + \left(8y - x^4\sin(x)\,y\right)y' = -4x$$

Unable to determine ODE type.

✗ Solution by Maple

```
dsolve((3*x^2*cos(x)*y(x)-x^3*y(x)*sin(x)*y(x)+4*x)+(8*y(x)-x^4*sin(x)*y(x))*diff(y(x),x)=0,
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(3*x^2*Cos[x]*y[x]-x^3*y[x]*Sin[x]*y[x]+4*x)+(8*y[x]-x^4*Sin[x]*y[x])*y'[x]==0,y[x],x
```

Not solved

6.18 problem 18

Internal problem ID [1047]
Internal file name [`OUTPUT/1048_Sunday_June_05_2022_02_01_17_AM_9722049/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{4x^3y^2-6x^2y+\left(2yx^4-2x^3\right)y'=2x+3}$$

With initial conditions

$$[y(1)=3]$$

6.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{4x^3y^2-6y\,x^2-2x-3}{2x^3\,(yx-1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\left\{-\infty\leq x<0, 0<x<\frac{1}{3}, \frac{1}{3}<x\leq\infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{4x^3y^2 - 6y\,x^2 - 2x - 3}{2x^3\,(yx - 1)}\right) \\ &= -\frac{8y\,x^3 - 6x^2}{2x^3\,(yx - 1)} + \frac{4x^3y^2 - 6y\,x^2 - 2x - 3}{2x^2\,(yx - 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 3$ is

$$\left\{-\infty \le x < 0, 0 < x < \frac{1}{3}, \frac{1}{3} < x \le \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

6.18.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2x^4y - 2x^3\right)\mathrm{d}y &= \left(-4x^3y^2 + 6y\,x^2 + 2x + 3\right)\mathrm{d}x\\
\left(4x^3y^2 - 6y\,x^2 - 2x - 3\right)\mathrm{d}x + \left(2x^4y - 2x^3\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 4x^3y^2 - 6y\,x^2 - 2x - 3\\
N(x,y) &= 2x^4y - 2x^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4x^3y^2 - 6y\,x^2 - 2x - 3\right)\\
&= 8y\,x^3 - 6x^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^4y - 2x^3\right)\\
&= 8y\,x^3 - 6x^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 4x^3y^2 - 6y\,x^2 - 2x - 3 \, \mathrm{d}x$$

$$\phi = x^4y^2 - 2y\,x^3 - x^2 - 3x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^4y - 2x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^4y - 2x^3$. Therefore equation (4) becomes

$$2x^4y - 2x^3 = 2x^4y - 2x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^4y^2 - 2y\,x^3 - x^2 - 3x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^4y^2 - 2y\,x^3 - x^2 - 3x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x^4y^2 - 2y\,x^3 - x^2 - 3x = -1$$

Summary
The solution(s) found are the following

$$x^4y^2 - 2yx^3 - x^2 - 3x = -1 \tag{1}$$

Verification of solutions

$$x^4y^2 - 2yx^3 - x^2 - 3x = -1$$

Verified OK.

6.18.3 Maple step by step solution

Let's solve

$[4x^3y^2 - 6x^2y + (2yx^4 - 2x^3)\,y' = 2x + 3, y(1) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$
  - Evaluate derivatives

    $8y\,x^3 - 6x^2 = 8y\,x^3 - 6x^2$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (4x^3y^2 - 6y\,x^2 - 2x - 3)\,dx + f_1(y)$
- Evaluate integral

$F(x, y) = x^4 y^2 - 2y\, x^3 - x^2 - 3x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2x^4 y - 2x^3 = 2x^4 y - 2x^3 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^4 y^2 - 2y\, x^3 - x^2 - 3x$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^4 y^2 - 2y\, x^3 - x^2 - 3x = c_1$

- Solve for y

  $\left\{y = \frac{x - \sqrt{2x^2 + c_1 + 3x}}{x^2}, y = \frac{x + \sqrt{2x^2 + c_1 + 3x}}{x^2}\right\}$

- Use initial condition $y(1) = 3$

  $3 = 1 - \sqrt{c_1 + 5}$

- Solution does not satisfy initial condition

- Use initial condition $y(1) = 3$

  $3 = 1 + \sqrt{c_1 + 5}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = \frac{x + \sqrt{2x^2 + 3x - 1}}{x^2}$

- Solution to the IVP

  $y = \frac{x + \sqrt{2x^2 + 3x - 1}}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 22

```
dsolve([(4*x^3*y(x)^2-6*x^2*y(x)-2*x-3)+(2*x^4*y(x)-2*x^3)*diff(y(x),x)=0,y(1) = 3],y(x), si
```

$$y(x) = \frac{x + \sqrt{2x^2 + 3x - 1}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.743 (sec). Leaf size: 31

```
DSolve[{(4*x^3*y[x]^2-6*x^2*y[x]-2*x-3)+(2*x^4*y[x]-2*x^3)*y'[x]==0,y[1]==3},y[x],x,IncludeS
```

$$y(x) \to \frac{x^3 + \sqrt{x^4 (2x^2 + 3x - 1)}}{x^4}$$

6.19 problem 19

Internal problem ID [1048]
Internal file name [`OUTPUT/1049_Sunday_June_05_2022_02_01_18_AM_62428574/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel, `2
   nd type`, `class A`]]
```

$$
\boxed{-4\cos(x)\, y + (4y - 4\sin(x))\, y' = -4\cos(x)\sin(x) - \sec(x)^2}
$$

With initial conditions

$$
\left[y\left(\frac{\pi}{4}\right) = 0\right]
$$

6.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x}=M$$
$$\frac{\partial \phi}{\partial y}=N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(4y-4\sin(x))\,\mathrm{d}y=\left(4\cos(x)\,y-4\cos(x)\sin(x)-\sec(x)\right.$$

$$\left(\sec(x)^2-4\cos(x)\,y+4\cos(x)\sin(x)\right)\mathrm{d}x+(4y-4\sin(x))\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\sec(x)^2-4\cos(x)\,y+4\cos(x)\sin(x)\\N(x,y)&=4y-4\sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\sec(x)^2-4\cos(x)\,y+4\cos(x)\sin(x)\right)\\&=-4\cos(x)\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4y - 4\sin(x)) \\
&= -4\cos(x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \sec(x)^2 - 4\cos(x)\, y + 4\cos(x)\sin(x) \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \tan(x) - 4\sin(x)\, y + 2\sin(x)^2 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -4\sin(x) + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4y - 4\sin(x)$. Therefore equation (4) becomes

$$
4y - 4\sin(x) = -4\sin(x) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 4y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (4y) \,\mathrm{d}y \\
f(y) &= 2y^2 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \tan(x) - 4\sin(x)y + 2\sin(x)^2 + 2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \tan(x) - 4\sin(x)y + 2\sin(x)^2 + 2y^2$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\tan(x) - 4\sin(x)y + 2\sin(x)^2 + 2y^2 = 2$$

Summary

The solution(s) found are the following

$$\tan(x) - 4\sin(x)y + 2\sin(x)^2 + 2y^2 = 2 \tag{1}$$

Verification of solutions

$$\tan(x) - 4\sin(x)y + 2\sin(x)^2 + 2y^2 = 2$$

Verified OK.

6.19.2 Maple step by step solution

Let's solve

$\left[-4\cos(x)y + (4y - 4\sin(x))y' = -4\cos(x)\sin(x) - \sec(x)^2, y\left(\frac{\pi}{4}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-4\cos(x) = -4\cos(x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(\sec(x)^2 - 4\cos(x) y + 4\cos(x)\sin(x)\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \tan(x) - 4\sin(x) y + 2\sin(x)^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $4y - 4\sin(x) = -4\sin(x) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 4y$

- Solve for $f_1(y)$

  $f_1(y) = 2y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \tan(x) - 4\sin(x) y + 2\sin(x)^2 + 2y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\tan(x) - 4\sin(x) y + 2\sin(x)^2 + 2y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{2\sin(x)^3 - 2\sin(x) - \sqrt{4\cos(x)^2\sin(x)^4 + 4\sin(x)^6 + 2\cos(x)^4 c_1 + 4\cos(x)^4 - 2\sin(x)\cos(x)^3 - 8\sin(x)^4 - 4\cos(x)^2 + 4\sin(x)}}{2\cos(x)^2}\right.$

- Use initial condition $y\left(\frac{\pi}{4}\right) = 0$

$$0 = \frac{\sqrt{2}}{2} + \sqrt{-\frac{1}{2} + \frac{c_1}{2}}$$

- Solution does not satisfy initial condition
- Use initial condition $y\left(\frac{\pi}{4}\right) = 0$

$$0 = \frac{\sqrt{2}}{2} - \sqrt{-\frac{1}{2} + \frac{c_1}{2}}$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$y = \sin(x) - \frac{\sec(x)^2 \sqrt{2}\, \sqrt{\cos(x)^3 (-\sin(x) + 2\cos(x))}}{2}$$

- Solution to the IVP

$$y = \sin(x) - \frac{\sec(x)^2 \sqrt{2}\, \sqrt{\cos(x)^3 (-\sin(x) + 2\cos(x))}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 1.719 (sec). Leaf size: 32

```
dsolve([(-4*y(x)*cos(x)+4*sin(x)*cos(x)+sec(x)^2)+(4*y(x)-4*sin(x))*diff(y(x),x)=0,y(1/4*Pi)
```

$$y(x) = \sin(x) - \frac{\sec(x)^2 \sqrt{2}\, \sqrt{\cos(x)^3 (2\cos(x) - \sin(x))}}{2}$$

✓ Solution by Mathematica

Time used: 11.693 (sec). Leaf size: 38

```
DSolve[{(-4*y[x]*Cos[x]+4*Sin[x]*Cos[x]+Sec[x]^2)+(4*y[x]-4*Sin[x])*y'[x]==0,y[Pi/4]==0},y[x
```

$$y(x) \to \sin(x) + \frac{1}{2}\sqrt{-\sec^2(x)}\sqrt{\sin(2x) - 2\cos(2x) - 2}$$

6.20 problem 20

Internal problem ID [1049]
Internal file name [`OUTPUT/1050_Sunday_June_05_2022_02_01_25_AM_47202873/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(y^3-1)\,\mathrm{e}^x+3y^2(1+\mathrm{e}^x)\,y'=0}$$

With initial conditions

$$[y(0)=0]$$

6.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{(y^3-1)\,\mathrm{e}^x}{3y^2\,(1+\mathrm{e}^x)} \end{aligned}$$

$f(x,y)$ is not defined at $y=0$ therefore existence and uniqueness theorem do not apply.

6.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{(y^3-1)\,\mathrm{e}^x}{3y^2\,(1+\mathrm{e}^x)} \end{aligned}$$

Where $f(x) = -\frac{\mathrm{e}^x}{3(1+\mathrm{e}^x)}$ and $g(y) = \frac{y^3-1}{y^2}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^3-1}{y^2}}\,dy &= -\frac{\mathrm{e}^x}{3\,(1+\mathrm{e}^x)}\,dx \\ \int \frac{1}{\frac{y^3-1}{y^2}}\,dy &= \int -\frac{\mathrm{e}^x}{3\,(1+\mathrm{e}^x)}\,dx \\ \frac{\ln(y^3-1)}{3} &= -\frac{\ln(1+\mathrm{e}^x)}{3} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$(y^3-1)^{\frac{1}{3}} = \mathrm{e}^{-\frac{\ln(1+\mathrm{e}^x)}{3}+c_1}$$

Which simplifies to

$$(y^3-1)^{\frac{1}{3}} = \frac{c_2}{(1+\mathrm{e}^x)^{\frac{1}{3}}}$$

Which can be simplified to become

$$(y^3-1)^{\frac{1}{3}} = \frac{c_2\mathrm{e}^{c_1}}{(1+\mathrm{e}^x)^{\frac{1}{3}}}$$

The solution is

$$(y^3-1)^{\frac{1}{3}} = \frac{c_2\mathrm{e}^{c_1}}{(1+\mathrm{e}^x)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} + \frac{i\sqrt{3}}{2} = \frac{2^{\frac{2}{3}}c_2\mathrm{e}^{c_1}}{2}$$

$$
c_1 = \frac{\ln\left(\frac{\left(1+i\sqrt{3}\right)^3}{4c_2^3}\right)}{3}
$$

Substituting c_1 found above in the general solution gives

$$
\left(y^3-1\right)^{\frac{1}{3}} = \frac{c_2 4^{\frac{2}{3}} \left(-\frac{8}{c_2^3}\right)^{\frac{1}{3}}}{4\left(1+\mathrm{e}^x\right)^{\frac{1}{3}}}
$$

The above simplifies to

$$
-c_2 4^{\frac{2}{3}} \left(-\frac{8}{c_2^3}\right)^{\frac{1}{3}} + 4\left(y^3-1\right)^{\frac{1}{3}} \left(1+\mathrm{e}^x\right)^{\frac{1}{3}} = 0
$$

Summary

The solution(s) found are the following

$$
-4c_2 2^{\frac{1}{3}} \left(-\frac{1}{c_2^3}\right)^{\frac{1}{3}} + 4\left(y^3-1\right)^{\frac{1}{3}} \left(1+\mathrm{e}^x\right)^{\frac{1}{3}} = 0 \tag{1}
$$

Verification of solutions

$$
-4c_2 2^{\frac{1}{3}} \left(-\frac{1}{c_2^3}\right)^{\frac{1}{3}} + 4\left(y^3-1\right)^{\frac{1}{3}} \left(1+\mathrm{e}^x\right)^{\frac{1}{3}} = 0
$$

Verified OK.

6.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\left(y^3-1\right)\mathrm{e}^x}{3y^2\left(1+\mathrm{e}^x\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 289: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -3(1 + e^x)\, e^{-x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-3\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{-x}} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln\left(1+\mathrm{e}^{x}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\left(y^3-1\right)\mathrm{e}^{x}}{3y^2\left(1+\mathrm{e}^{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{\mathrm{e}^{x}}{3+3\,\mathrm{e}^{x}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2}{y^3-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{R^3-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln\left(\left(R-1\right)\left(R^2+R+1\right)\right)}{3} + c_1 \tag{4}$$

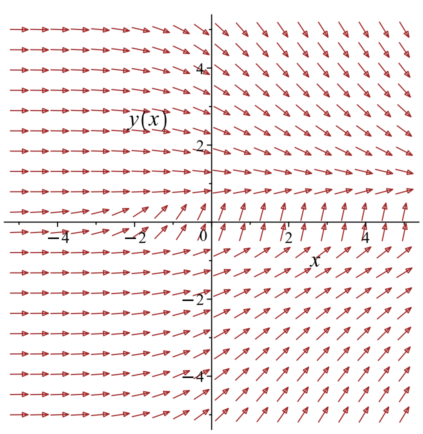
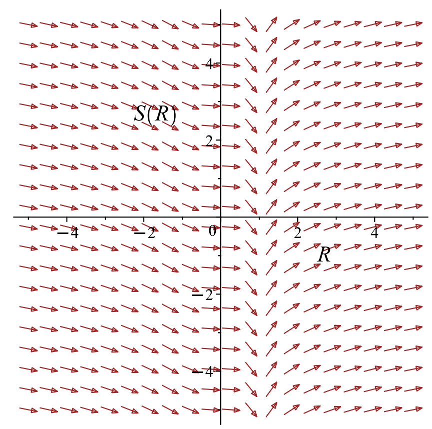
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(1+\mathrm{e}^x\right)}{3} = \frac{\ln\left(\left(y-1\right)\left(y^2+y+1\right)\right)}{3} + c_1$$

Which simplifies to

$$-\frac{\ln\left(1+\mathrm{e}^x\right)}{3} = \frac{\ln\left(\left(y-1\right)\left(y^2+y+1\right)\right)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(y^3-1)\mathrm{e}^x}{3y^2(1+\mathrm{e}^x)}$ 	$R = y$ $S = -\frac{\ln(1+\mathrm{e}^x)}{3}$	$\frac{dS}{dR} = \frac{R^2}{R^3-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(2)}{3} = \frac{i\pi}{3} + c_1$$

$$
c_1 = -\frac{i\pi}{3} - \frac{\ln(2)}{3}
$$

Substituting c_1 found above in the general solution gives

$$
-\frac{\ln(1+\mathrm{e}^x)}{3} = \frac{\ln((y-1)(y^2+y+1))}{3} - \frac{i\pi}{3} - \frac{\ln(2)}{3}
$$

Summary

The solution(s) found are the following

$$
-\frac{\ln(1+\mathrm{e}^x)}{3} = \frac{\ln((y-1)(y^2+y+1))}{3} - \frac{i\pi}{3} - \frac{\ln(2)}{3} \tag{1}
$$

Verification of solutions

$$
-\frac{\ln(1+\mathrm{e}^x)}{3} = \frac{\ln((y-1)(y^2+y+1))}{3} - \frac{i\pi}{3} - \frac{\ln(2)}{3}
$$

Verified OK.

6.20.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{(y^3-1)\,\mathrm{e}^x}{3y^2\,(1+\mathrm{e}^x)}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -\frac{\mathrm{e}^x}{3\,(1+\mathrm{e}^x)}y + \frac{\mathrm{e}^x}{3+3\,\mathrm{e}^x}\frac{1}{y^2} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{e^x}{3\left(1+e^x\right)} \\
f_1(x) &= \frac{e^x}{3+3\,e^x} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = -\frac{e^x y^3}{3\left(1+e^x\right)} + \frac{e^x}{3+3\,e^x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= -\frac{e^x w(x)}{3\left(1+e^x\right)} + \frac{e^x}{3+3\,e^x} \\
w' &= -\frac{e^x w}{1+e^x} + \frac{e^x}{1+e^x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{e^x}{1+e^x} \\
q(x) &= \frac{e^x}{1+e^x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{e^x w(x)}{1+e^x} = \frac{e^x}{1+e^x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{e^x}{1+e^x}dx} \\
&= 1 + e^x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{e^x}{1+e^x}\right) \\
\frac{d}{dx}((1+e^x)\, w) &= (1+e^x)\left(\frac{e^x}{1+e^x}\right) \\
d((1+e^x)\, w) &= e^x\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(1+e^x)\, w &= \int e^x\, dx \\
(1+e^x)\, w &= e^x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = 1 + e^x$ results in

$$
w(x) = \frac{e^x}{1+e^x} + \frac{c_1}{1+e^x}
$$

which simplifies to

$$
w(x) = \frac{e^x + c_1}{1+e^x}
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = \frac{e^x + c_1}{1+e^x}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = \frac{1}{2} + \frac{c_1}{2}
$$

$$
c_1 = -1
$$

Substituting c_1 found above in the general solution gives

$$
y^3 = \frac{e^x - 1}{1+e^x}
$$

The above simplifies to

$$1+y^3\mathrm{e}^x+y^3-\mathrm{e}^x=0$$

Summary
The solution(s) found are the following

$$1+y^3\mathrm{e}^x+y^3-\mathrm{e}^x=0 \tag{1}$$

Verification of solutions

$$1+y^3\mathrm{e}^x+y^3-\mathrm{e}^x=0$$

Verified OK.

6.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work

and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{3y^2}{y^3-1}\right)\mathrm{d}y &= \left(\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)\mathrm{d}x\\ \left(-\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)\mathrm{d}x+\left(-\frac{3y^2}{y^3-1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\\ N(x,y) &= -\frac{3y^2}{y^3-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{3y^2}{y^3-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{\mathrm{e}^x}{1+\mathrm{e}^x} \mathrm{d}x$$
$$\phi = -\ln(1+\mathrm{e}^x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3y^2}{y^3-1}$. Therefore equation (4) becomes

$$-\frac{3y^2}{y^3-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{3y^2}{y^3-1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{3y^2}{y^3-1}\right) \mathrm{d}y$$
$$f(y) = -\ln\left(y^3-1\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(1+\mathrm{e}^x) - \ln\left(y^3-1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+\mathrm{e}^x) - \ln\left(y^3-1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) - i\pi = c_1$$

$$c_1 = -\ln(2) - i\pi$$

Substituting c_1 found above in the general solution gives

$$-\ln(1 + e^x) - \ln(y^3 - 1) = -\ln(2) - i\pi$$

Summary
The solution(s) found are the following

$$-\ln(1 + e^x) - \ln(y^3 - 1) = -\ln(2) - i\pi \tag{1}$$

Verification of solutions

$$-\ln(1 + e^x) - \ln(y^3 - 1) = -\ln(2) - i\pi$$

Verified OK.

6.20.6 Maple step by step solution

Let's solve

$\left[(y^3 - 1) e^x + 3y^2(1 + e^x) y' = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int ((y^3 - 1) e^x + 3y^2(1 + e^x) y') dx = \int 0 dx + c_1$

- Evaluate integral

  $y^3(1 + e^x) - e^x = c_1$

- Solve for y

  $y = \frac{\left((e^x + c_1)(1 + e^x)^2\right)^{\frac{1}{3}}}{1 + e^x}$

- Use initial condition $y(0) = 0$

  $0 = \frac{(4 + 4c_1)^{\frac{1}{3}}}{2}$

- Solve for c_1

$c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

$$y = \frac{\left((\mathrm{e}^x-1)(1+\mathrm{e}^x)^2\right)^{\frac{1}{3}}}{1+\mathrm{e}^x}$$

- Solution to the IVP

$$y = \frac{\left((\mathrm{e}^x-1)(1+\mathrm{e}^x)^2\right)^{\frac{1}{3}}}{1+\mathrm{e}^x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 91

```
dsolve([((y(x)^3-1)*exp(x))+(3*y(x)^2*(exp(x)+1))*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all
```

$$y(x) = \frac{\left((\mathrm{e}^x-1)(1+\mathrm{e}^x)^2\right)^{\frac{1}{3}}}{1+\mathrm{e}^x}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left((\mathrm{e}^x-1)(1+\mathrm{e}^x)^2\right)^{\frac{1}{3}}}{2+2\,\mathrm{e}^x}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left((\mathrm{e}^x-1)(1+\mathrm{e}^x)^2\right)^{\frac{1}{3}}}{2+2\,\mathrm{e}^x}$$

✓ Solution by Mathematica

Time used: 0.984 (sec). Leaf size: 77

```
DSolve[{((y[x]^3-1)*Exp[x])+(3*y[x]^2*(Exp[x]+1))*y'[x]==0,y[0]==0},y[x],x,IncludeSingularSo
```

$$y(x) \to \sqrt[3]{\frac{e^x - 1}{e^x + 1}}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{\frac{e^x - 1}{e^x + 1}}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{\frac{e^x - 1}{e^x + 1}}$$

6.21 problem 21

Internal problem ID [1050]
Internal file name [`OUTPUT/1051_Sunday_June_05_2022_02_01_26_AM_60860642/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-\sin(x)\,y + y'\cos(x) = -\sin(x) + 2\cos(x)}$$

With initial conditions

$$[y(0) = 1]$$

6.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\tan(x)$$
$$q(x) = -\tan(x) + 2$$

Hence the ode is

$$y' - \tan(x)\,y = -\tan(x) + 2$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z50 \lor \frac{1}{2}\pi + \pi\_Z50 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\tan(x) + 2$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z50 \lor \frac{1}{2}\pi + \pi\_Z50 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\tan(x)dx} \\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-\tan(x) + 2\right) \\ \frac{d}{dx}(\cos(x)\,y) &= (\cos(x))\left(-\tan(x) + 2\right) \\ d(\cos(x)\,y) &= (-\sin(x) + 2\cos(x))\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\cos(x)\,y &= \int -\sin(x) + 2\cos(x)\,dx \\ \cos(x)\,y &= \cos(x) + 2\sin(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)\left(\cos(x) + 2\sin(x)\right) + c_1\sec(x)$$

which simplifies to

$$y = 1 + 2\tan(x) + c_1\sec(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

$$c_1 = 0$$

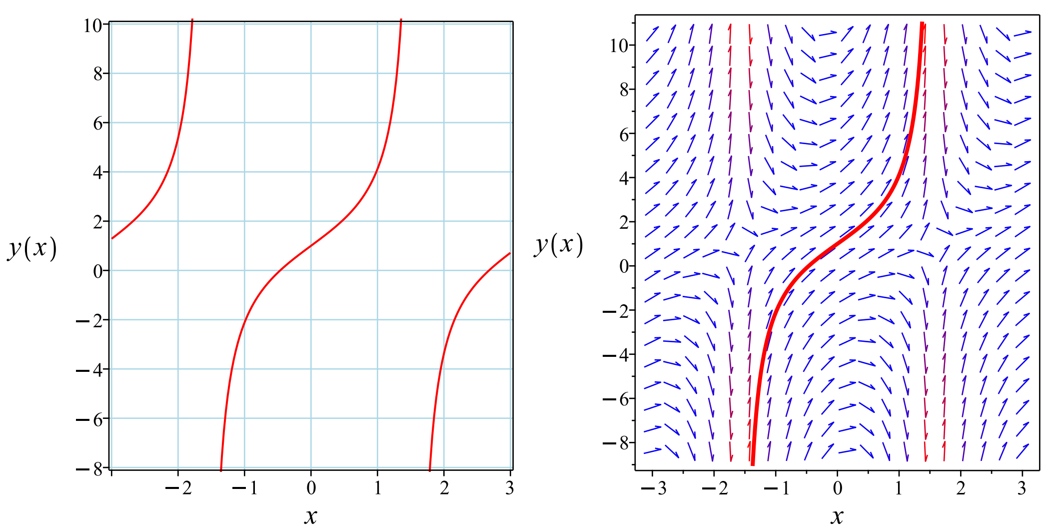
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\tan(x)$$

Summary

The solution(s) found are the following

$$y = 1 + 2\tan(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + 2\tan(x)$$

Verified OK.

6.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-\sin(x) + \sin(x)\, y + 2\cos(x)}{\cos(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 292: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\cos(x)}} dy
\end{aligned}
$$

Which results in

$$
S = \cos(x)\, y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-\sin(x) + \sin(x)\, y + 2\cos(x)}{\cos(x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\sin(x)\, y \\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\sin(x) + 2\cos(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\sin(R) + 2\cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \cos(R) + 2\sin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\cos(x)\, y = \cos(x) + 2\sin(x) + c_1
$$

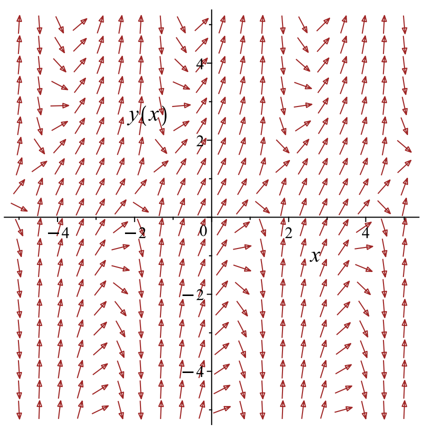
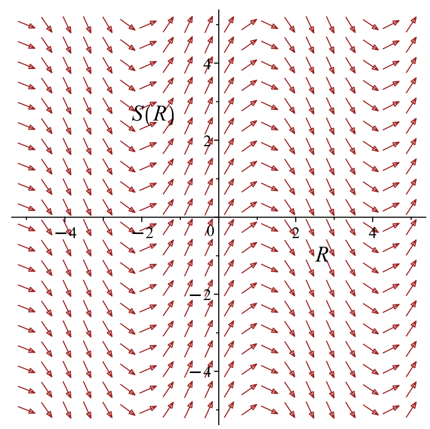
Which simplifies to

$$
\cos(x)\, y = \cos(x) + 2\sin(x) + c_1
$$

Which gives

$$
y = \frac{\cos(x) + 2\sin(x) + c_1}{\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{-\sin(x)+\sin(x)y+2\cos(x)}{\cos(x)}$

</td><td>$R = x$
$S = \cos(x)\, y$</td><td>$\frac{dS}{dR} = -\sin(R) + 2\cos(R)$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

$$c_1 = 0$$

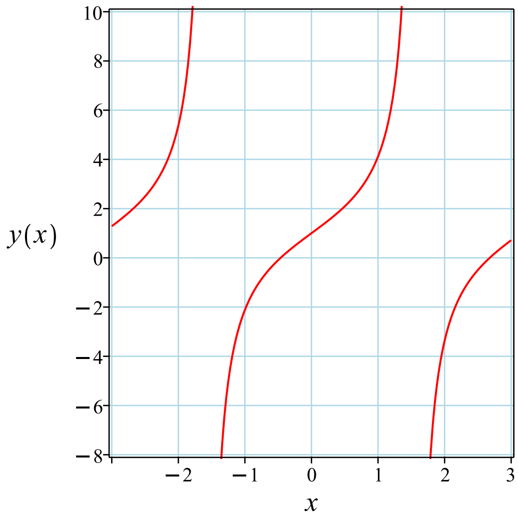
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\tan(x)$$

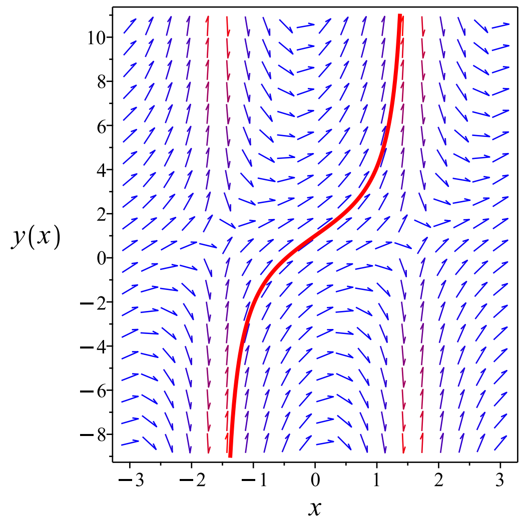
Summary

The solution(s) found are the following

$$y = 1 + 2\tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + 2\tan(x)$$

Verified OK.

6.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(\cos(x))\,\mathrm{d}y &= (-\sin(x) + \sin(x)\,y + 2\cos(x))\,\mathrm{d}x \\
(\sin(x) - \sin(x)\,y - 2\cos(x))\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \sin(x) - \sin(x)\,y - 2\cos(x) \\
N(x, y) &= \cos(x)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(x) - \sin(x)\,y - 2\cos(x)) \\
&= -\sin(x)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x)) \\
&= -\sin(x)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \sin(x) - \sin(x) \, y - 2\cos(x) \, \mathrm{d}x$$

$$\phi = \cos(x)(y-1) - 2\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x)(y-1) - 2\sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x)(y-1) - 2\sin(x)$$

The solution becomes

$$y = \frac{\cos(x) + 2\sin(x) + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

$$c_1 = 0$$

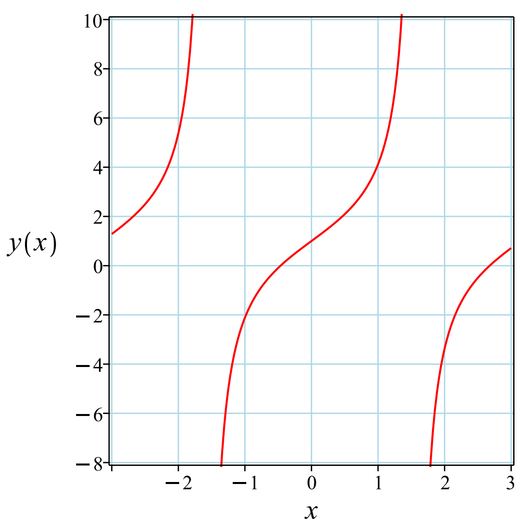
Substituting c_1 found above in the general solution gives

$$y = 1 + 2\tan(x)$$

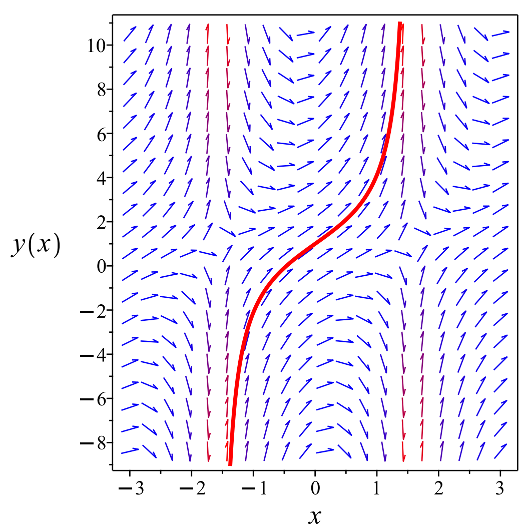
Summary

The solution(s) found are the following

$$y = 1 + 2\tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + 2\tan(x)$$

Verified OK.

6.21.5 Maple step by step solution

Let's solve

$[-\sin(x)\, y + y' \cos(x) = -\sin(x) + 2\cos(x)\, , y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{\sin(x) y}{\cos(x)} + \frac{-\sin(x)+2\cos(x)}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{\sin(x) y}{\cos(x)} = \frac{-\sin(x)+2\cos(x)}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{\sin(x) y}{\cos(x)}\right) = \frac{\mu(x)(-\sin(x)+2\cos(x))}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{\sin(x) y}{\cos(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)\sin(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(-\sin(x)+2\cos(x))}{\cos(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(-\sin(x)+2\cos(x))}{\cos(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(-\sin(x)+2\cos(x))}{\cos(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)$

  $y = \frac{\int (-\sin(x)+2\cos(x)) dx + c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\cos(x)+2\sin(x)+c_1}{\cos(x)}$

- Simplify

  $y = 1 + 2\tan(x) + c_1 \sec(x)$
- Use initial condition $y(0) = 1$

  $1 = 1 + c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = 1 + 2\tan(x)$
- Solution to the IVP

  $y = 1 + 2\tan(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([(sin(x)-y(x)*sin(x)-2*cos(x))+(cos(x))*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = 2\tan(x) + 1$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 11

```
DSolve[{(Sin[x]-y[x]*Sin[x]-2*Cos[x])+(Cos[x])*y'[x]==0,y[0]==1},y[x],x,IncludeSingularSolut
```

$$y(x) \to 2\tan(x) + 1$$

6.22 problem 22

Internal problem ID [1051]
Internal file name [`OUTPUT/1052_Sunday_June_05_2022_02_01_28_AM_20778865/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(2x-1)(y-1)+(2+x)(x-3)y'=0}$$

With initial conditions

$$[y(1)=-1]$$

6.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1-2x}{(2+x)(x-3)} \\ q(x) &= \frac{2x-1}{(2+x)(x-3)} \end{aligned}$$

Hence the ode is

$$y' - \frac{(1-2x)\,y}{(2+x)(x-3)} = \frac{2x-1}{(2+x)(x-3)}$$

The domain of $p(x) = -\frac{1-2x}{(2+x)(x-3)}$ is

$$\{-\infty \le x < -2, -2 < x < 3, 3 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2x-1}{(2+x)(x-3)}$ is

$$\{-\infty \le x < -2, -2 < x < 3, 3 < x \le \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

6.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(2x-1)(-y+1)}{(2+x)(x-3)} \end{aligned}$$

Where $f(x) = \frac{2x-1}{(2+x)(x-3)}$ and $g(y) = -y+1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-y+1}\,dy &= \frac{2x-1}{(2+x)(x-3)}\,dx \\ \int \frac{1}{-y+1}\,dy &= \int \frac{2x-1}{(2+x)(x-3)}\,dx \\ -\ln(y-1) &= \ln((2+x)(x-3)) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-1} = e^{\ln((2+x)(x-3))+c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2(2+x)(x-3)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{6\,e^{-c_1}e^{c_1}c_2 - e^{-c_1}}{6c_2}$$

$$c_1 = -\ln(12c_2)$$

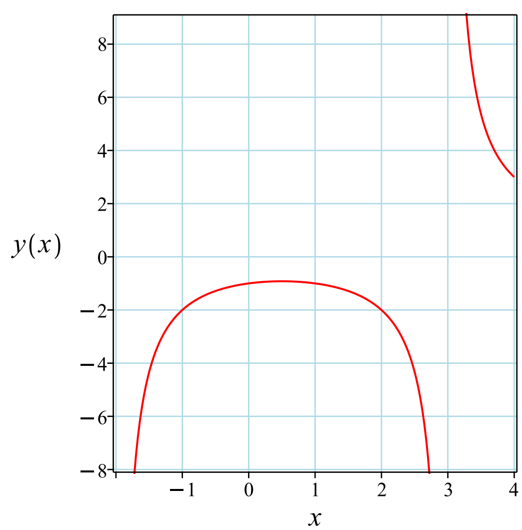
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - x + 6}{x^2 - x - 6}$$

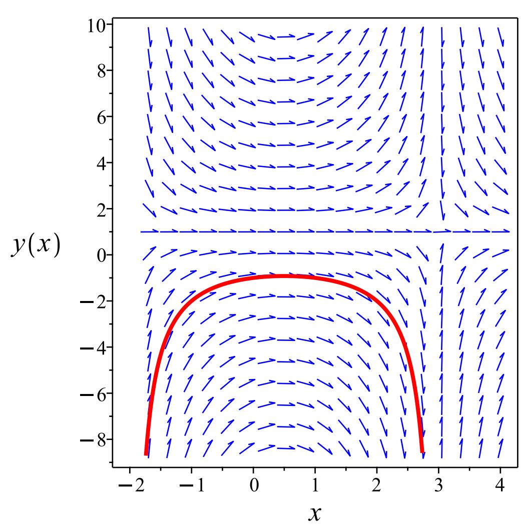
Summary

The solution(s) found are the following

$$y = \frac{x^2 - x + 6}{x^2 - x - 6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - x + 6}{x^2 - x - 6}$$

Verified OK.

6.22.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{1-2x}{(2+x)(x-3)}dx} \\
&= (2+x)(x-3)
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x-1}{(2+x)(x-3)}\right) \\
\frac{d}{dx}((2+x)(x-3)y) &= ((2+x)(x-3))\left(\frac{2x-1}{(2+x)(x-3)}\right) \\
d((2+x)(x-3)y) &= (2x-1)\ dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(2+x)(x-3)y &= \int 2x - 1\, dx \\
(2+x)(x-3)y &= x^2 - x + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (2+x)(x-3)$ results in

$$y = \frac{x^2 - x}{(2+x)(x-3)} + \frac{c_1}{(2+x)(x-3)}$$

which simplifies to

$$y = \frac{x^2 + c_1 - x}{(2+x)(x-3)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{c_1}{6}$$

$$c_1 = 6$$

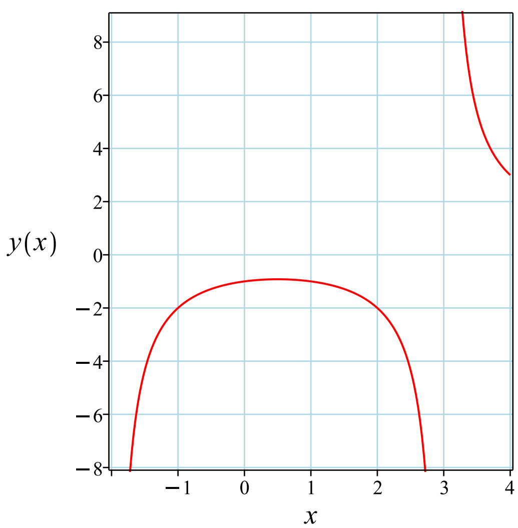
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)}$$

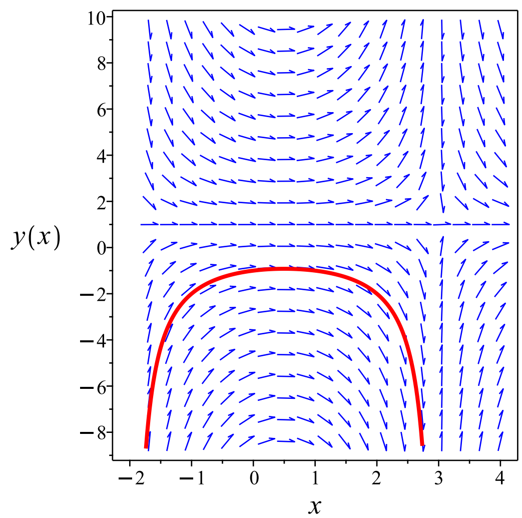
Summary

The solution(s) found are the following

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)}$$

Verified OK.

6.22.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{(2x - 1)(y - 1)}{(2 + x)(x - 3)} \tag{1}$$

Which becomes

$$0 = (-x^2 + x + 6)\, dy + (-(y - 1)(2x - 1))\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2 + x + 6)\, dy + (-(y-1)(2x-1))\, dx = d\big(-(y-1)(x^2 - x) + 6y\big)$$

Hence (2) becomes

$$0 = d\big(-(y-1)(x^2 - x) + 6y\big)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + c_1 - x}{x^2 - x - 6} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{5c_1}{6}$$

$$c_1 = -\frac{6}{5}$$

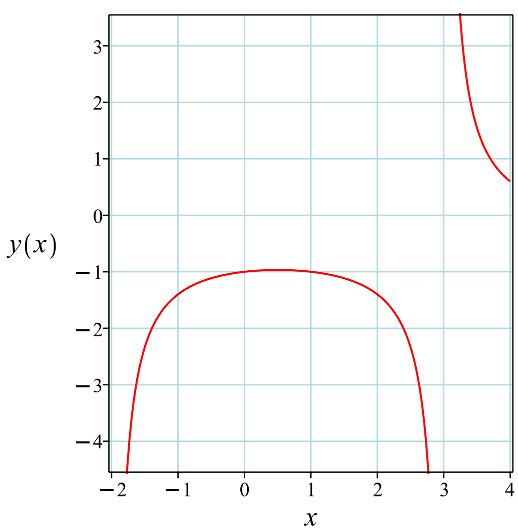
Substituting c_1 found above in the general solution gives

$$y = -\frac{x^2 - x - 30}{5(x^2 - x - 6)}$$

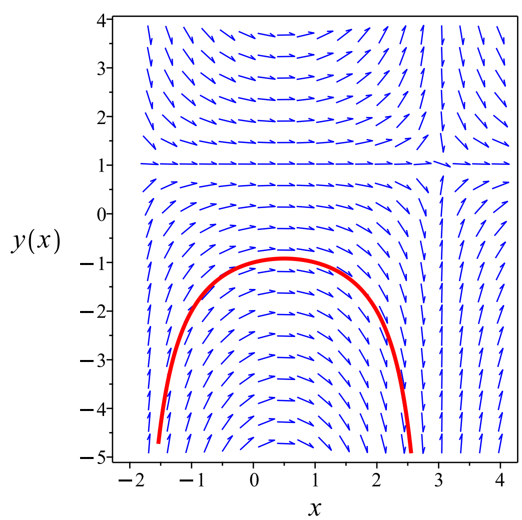
Summary

The solution(s) found are the following

$$y = -\frac{x^2 - x - 30}{5(x^2 - x - 6)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{x^2 - x - 30}{5 (x^2 - x - 6)}$$

Verified OK.

6.22.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(2x - 1) (y - 1)}{(2 + x) (x - 3)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 295: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{(2+x)\,(x-3)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(2+x)(x-3)}} dy \end{aligned}$$

Which results in

$$S = (2 + x)(x - 3) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{(2x - 1)(y - 1)}{(2 + x)(x - 3)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y(2x - 1) \\ S_y &= (2 + x)(x - 3) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 - R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(2+x)(x-3)\,y = x^2 + c_1 - x$$

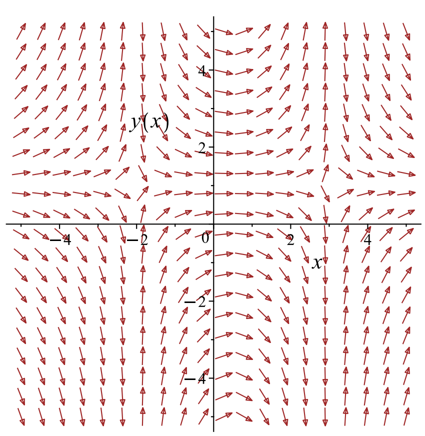
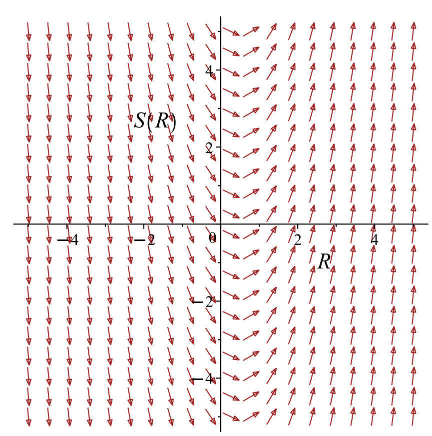
Which simplifies to

$$(2+x)(x-3)\,y = x^2 + c_1 - x$$

Which gives

$$y = \frac{x^2 + c_1 - x}{(2+x)(x-3)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(2x-1)(y-1)}{(2+x)(x-3)}$ 	$R = x$ $S = (2+x)(x-3)\,y$	$\frac{dS}{dR} = 2R - 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{c_1}{6}$$

$$c_1 = 6$$

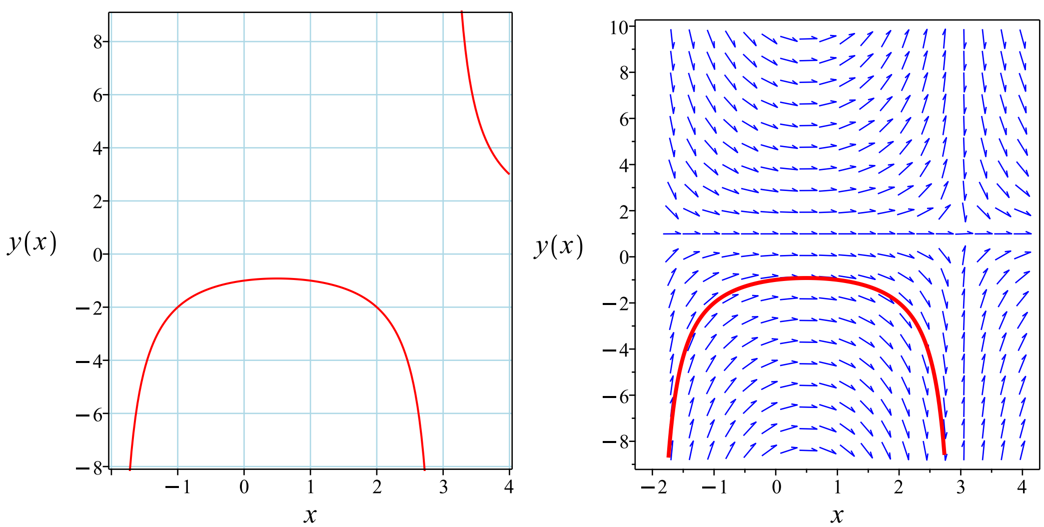
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - x + 6}{(2 + x)(x - 3)}$$

Verified OK.

6.22.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-y+1}\right)\mathrm{d}y=\left(\frac{2x-1}{(2+x)\,(x-3)}\right)\mathrm{d}x$$
$$\left(-\frac{2x-1}{(2+x)\,(x-3)}\right)\mathrm{d}x+\left(\frac{1}{-y+1}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{2x-1}{(2+x)(x-3)} \\ N(x,y) &= \frac{1}{-y+1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x-1}{(2+x)(x-3)}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y+1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{2x-1}{(2+x)(x-3)}\, \mathrm{d}x \end{aligned}$$

$$\phi = -\ln((2+x)(x-3)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y+1}$. Therefore equation (4) becomes

$$\frac{1}{-y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y-1}\right) \mathrm{d}y$$
$$f(y) = -\ln(y-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln((2+x)(x-3)) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln((2+x)(x-3)) - \ln(y-1)$$

The solution becomes

$$y = \frac{(x^2 \mathrm{e}^{c_1} - x\,\mathrm{e}^{c_1} - 6\,\mathrm{e}^{c_1} + 1)\,\mathrm{e}^{-c_1}}{x^2 - x - 6}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 - \frac{\mathrm{e}^{-c_1}}{6}$$

$$c_1 = -\ln(12)$$

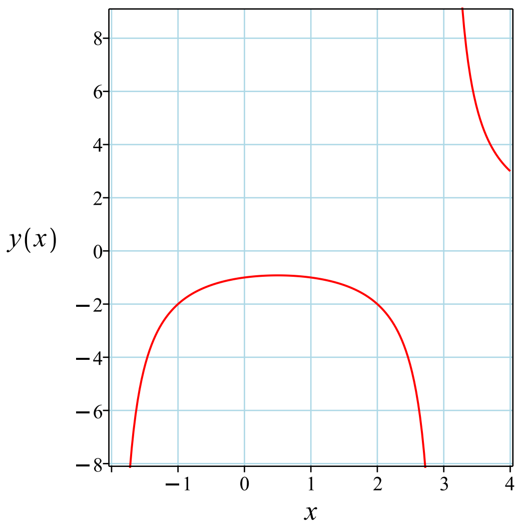
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 - x + 6}{x^2 - x - 6}$$

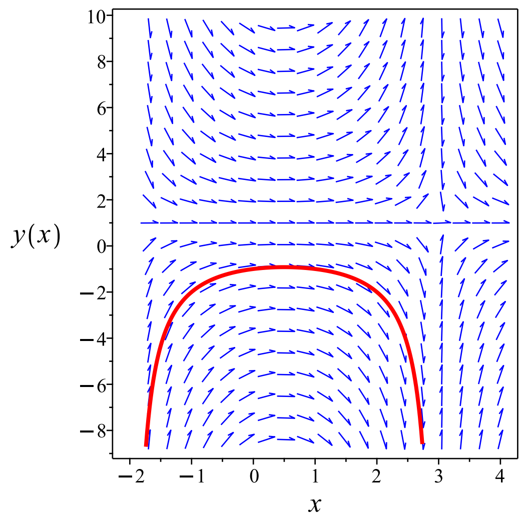
Summary

The solution(s) found are the following

$$y = \frac{x^2 - x + 6}{x^2 - x - 6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 - x + 6}{x^2 - x - 6}$$

Verified OK.

6.22.7 Maple step by step solution

Let's solve

$[(2x-1)(y-1)+(2+x)(x-3)y'=0, y(1)=-1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

$\int ((2x-1)(y-1)+(2+x)(x-3)y')\,dx = \int 0dx + c_1$

- Evaluate integral

  $(x^2-x-6)\,y - x^2 + x = c_1$

- Solve for y

  $y = \frac{x^2+c_1-x}{x^2-x-6}$

- Use initial condition $y(1) = -1$

  $-1 = -\frac{c_1}{6}$

- Solve for c_1

  $c_1 = 6$

- Substitute $c_1 = 6$ into general solution and simplify

  $y = \frac{x^2-x+6}{x^2-x-6}$

- Solution to the IVP

  $y = \frac{x^2-x+6}{x^2-x-6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([((2*x-1)*(y(x)-1))+((x+2)*(x-3))*diff(y(x),x)=0,y(1) = -1],y(x), singsol=all)
```

$$y(x) = \frac{x^2-x+6}{(2+x)(x-3)}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 24

```
DSolve[{((2*x-1)*(y[x]-1))+((x+2)*(x-3))*y'[x]==0,y[1]==-1},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{x^2 - x + 6}{x^2 - x - 6}$$

6.23 problem 23

Internal problem ID [1052]
Internal file name [`OUTPUT/1053_Sunday_June_05_2022_02_01_29_AM_51323999/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{4y + (4x + 3y)\, y' = -7x}$$

6.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$4u(x)\, x + (4x + 3u(x)\, x)\, (u'(x)\, x + u(x)) = -7x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 + 8u + 7}{x\,(3u + 4)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{3u^2+8u+7}{3u+4}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+8u+7}{3u+4}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{3u^2+8u+7}{3u+4}}\,du = \int -\frac{1}{x}\,dx$$
$$\frac{\ln\left(3u^2+8u+7\right)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{3u^2+8u+7} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{3u^2+8u+7} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{3u(x)^2+8u(x)+7} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{3u(x)^2+8u(x)+7} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{3y^2}{x^2}+\frac{8y}{x}+7} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{3y^2+8yx+7x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{3y^2+8yx+7x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

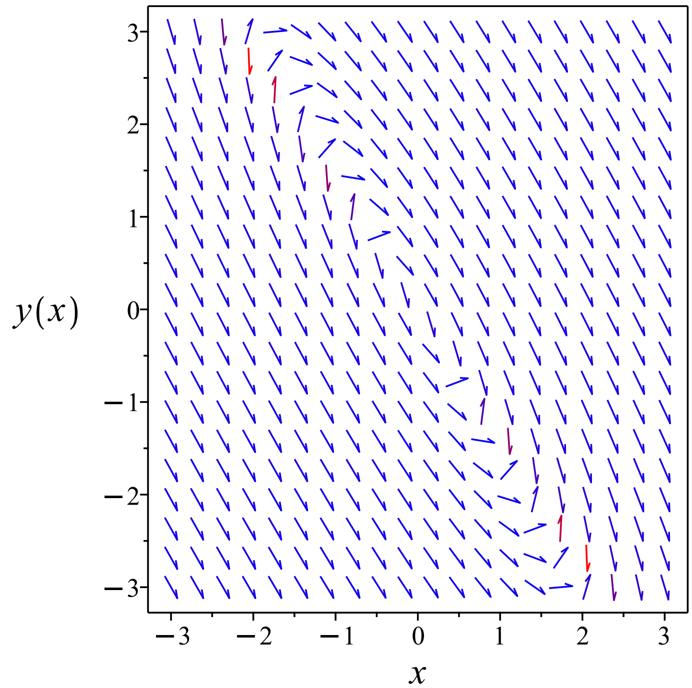


Figure 389: Slope field plot

Verification of solutions

$$\sqrt{\frac{3y^2 + 8yx + 7x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.23.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-7x - 4y}{4x + 3y} \tag{1}$$

Which becomes

$$(3y)\, dy = (-4x)\, dy + (-7x - 4y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-4x)\, dy + (-7x - 4y)\, dx = d\left(-\frac{7}{2}x^2 - 4yx\right)$$

Hence (2) becomes

$$(3y)\, dy = d\left(-\frac{7}{2}x^2 - 4yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{4x}{3} + \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1$$
$$y = -\frac{4x}{3} - \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1$$

Summary
The solution(s) found are the following

$$y = -\frac{4x}{3} + \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1 \tag{1}$$
$$y = -\frac{4x}{3} - \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1 \tag{2}$$

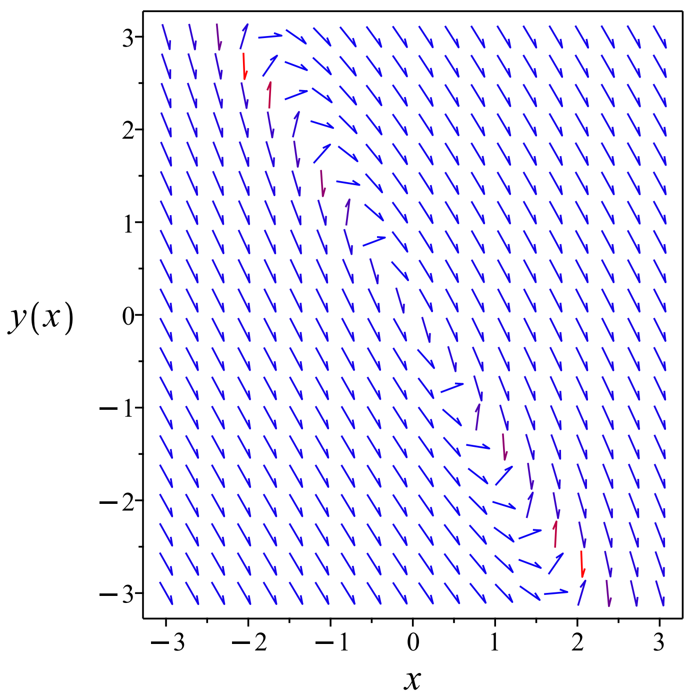


Figure 390: Slope field plot

Verification of solutions

$$y = -\frac{4x}{3} + \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1$$

Verified OK.

$$y = -\frac{4x}{3} - \frac{\sqrt{-5x^2 + 6c_1}}{3} + c_1$$

Verified OK.

6.23.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{7x + 4y}{4x + 3y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(7x + 4y)(b_3 - a_2)}{4x + 3y} - \frac{(7x + 4y)^2 a_3}{(4x + 3y)^2} - \left(-\frac{7}{4x + 3y} + \frac{28x + 16y}{(4x + 3y)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{4}{4x + 3y} + \frac{21x + 12y}{(4x + 3y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{28x^2a_2 - 49x^2a_3 + 11x^2b_2 - 28x^2b_3 + 42xya_2 - 56xya_3 + 24xyb_2 - 42xyb_3 + 12y^2a_2 - 11y^2a_3 + 9y^2b_2 -}{(4x + 3y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$28x^2a_2 - 49x^2a_3 + 11x^2b_2 - 28x^2b_3 + 42xya_2 - 56xya_3 + 24xyb_2 - 42xyb_3 + 12y^2a_2 - 11y^2a_3 + 9y^2b_2 - 12y^2b_3 - 5xb_1 + 5ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$28a_2v_1^2 + 42a_2v_1v_2 + 12a_2v_2^2 - 49a_3v_1^2 - 56a_3v_1v_2 - 11a_3v_2^2 + 11b_2v_1^2 + 24b_2v_1v_2 + 9b_2v_2^2 - 28b_3v_1^2 - 42b_3v_1v_2 - 12b_3v_2^2 + 5a_1v_2 - 5b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(28a_2 - 49a_3 + 11b_2 - 28b_3)\,v_1^2 + (42a_2 - 56a_3 + 24b_2 - 42b_3)\,v_1v_2 - 5b_1v_1 + (12a_2 - 11a_3 + 9b_2 - 12b_3)\,v_2^2 + 5a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
5a_1 &= 0\\
-5b_1 &= 0\\
12a_2 - 11a_3 + 9b_2 - 12b_3 &= 0\\
28a_2 - 49a_3 + 11b_2 - 28b_3 &= 0\\
42a_2 - 56a_3 + 24b_2 - 42b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{8a_3}{3} + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= -\frac{7a_3}{3} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{7x + 4y}{4x + 3y}\right)(x) \\
&= \frac{7x^2 + 8yx + 3y^2}{4x + 3y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{7x^2+8yx+3y^2}{4x+3y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(7x^2+8yx+3y^2)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{7x+4y}{4x+3y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{7x+4y}{7x^2+8yx+3y^2} \\
S_y &= \frac{4x+3y}{7x^2+8yx+3y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

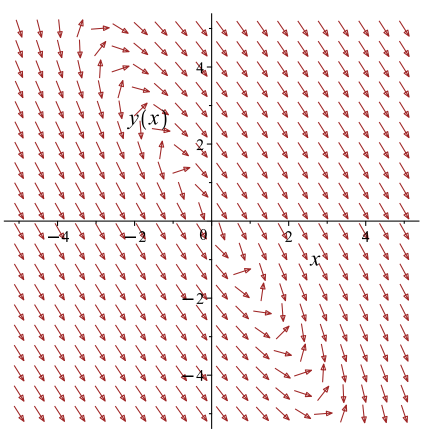
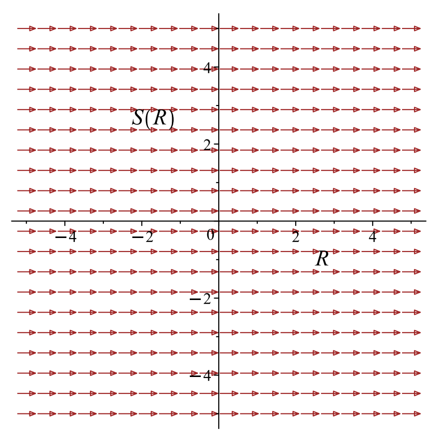
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3y^2 + 8yx + 7x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(3y^2 + 8yx + 7x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{7x+4y}{4x+3y}$ 	$R = x$ $S = \frac{\ln(7x^2 + 8yx + 3y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(3y^2 + 8yx + 7x^2)}{2} = c_1 \tag{1}$$

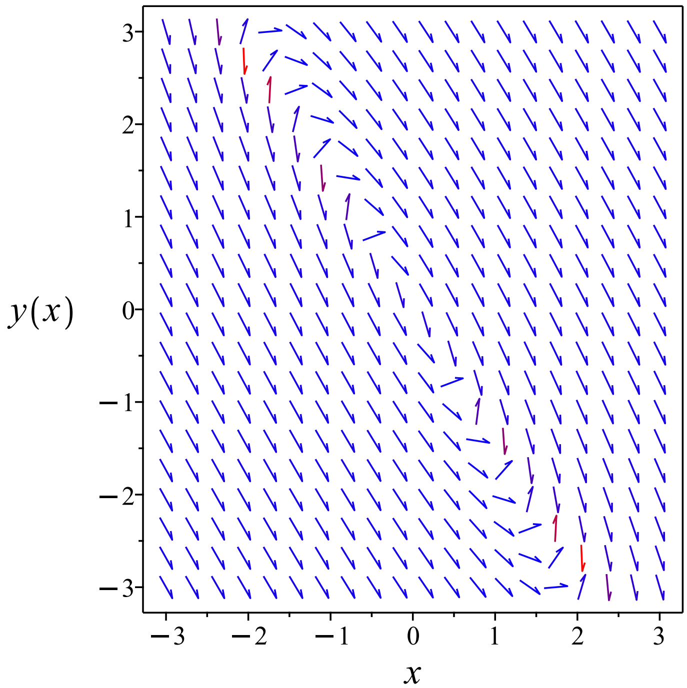


Figure 391: Slope field plot

Verification of solutions

$$\frac{\ln(3y^2 + 8yx + 7x^2)}{2} = c_1$$

Verified OK.

6.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(4x + 3y)\, \mathrm{d}y = (-7x - 4y)\, \mathrm{d}x$$
$$(7x + 4y)\, \mathrm{d}x + (4x + 3y)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 7x + 4y$$
$$N(x, y) = 4x + 3y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(7x + 4y) \\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x + 3y) \\ &= 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int 7x + 4y \, dx$$
$$\phi = \frac{x(7x + 8y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 3y$. Therefore equation (4) becomes

$$4x + 3y = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (3y) \, dy$$
$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(7x + 8y)}{2} + \frac{3y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(7x + 8y)}{2} + \frac{3y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(7x + 8y)}{2} + \frac{3y^2}{2} = c_1 \tag{1}$$

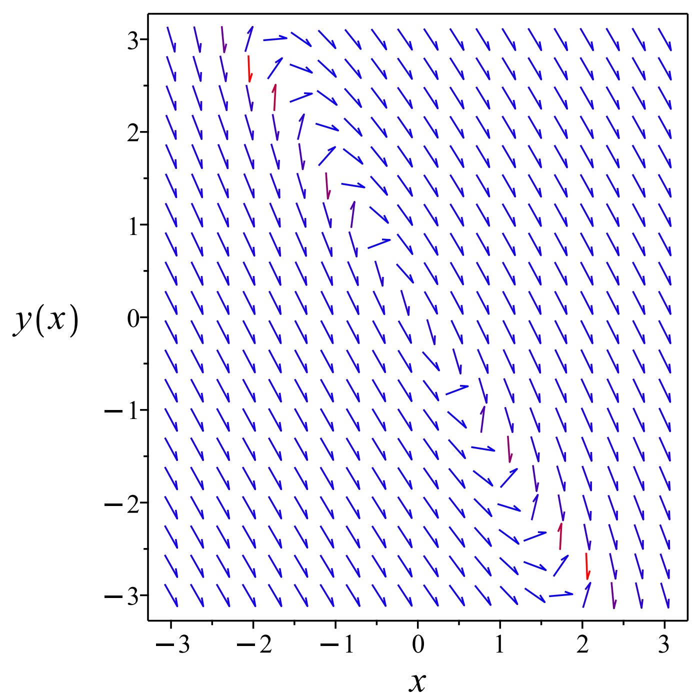


Figure 392: Slope field plot

Verification of solutions

$$\frac{x(7x + 8y)}{2} + \frac{3y^2}{2} = c_1$$

Verified OK.

6.23.5 Maple step by step solution

Let's solve

$4y + (4x + 3y)\, y' = -7x$

- Highest derivative means the order of the ODE is 1

y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - ◦ Evaluate derivatives

    $4 = 4$
  - ◦ Condition met, ODE is exact
- Exact ODE implies solution will be of this form

$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$F(x, y) = \int (7x + 4y)\, dx + f_1(y)$

- Evaluate integral

$F(x, y) = \frac{7x^2}{2} + 4yx + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

$N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

$4x + 3y = 4x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$\frac{d}{dy}f_1(y) = 3y$

- Solve for $f_1(y)$

$f_1(y) = \frac{3y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x, y) = \frac{7}{2}x^2 + 4yx + \frac{3}{2}y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{7}{2}x^2 + 4yx + \frac{3}{2}y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{4x}{3} - \frac{\sqrt{-5x^2+6c_1}}{3}, y = -\frac{4x}{3} + \frac{\sqrt{-5x^2+6c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 53

```
dsolve((7*x+4*y(x))+(4*x+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-4c_1x - \sqrt{-5c_1^2x^2 + 3}}{3c_1}$$
$$y(x) = \frac{-4c_1x + \sqrt{-5c_1^2x^2 + 3}}{3c_1}$$

✓ Solution by Mathematica

Time used: 0.483 (sec). Leaf size: 118

```
DSolve[(7*x+4*y[x])+(4*x+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{3}\left(-4x - \sqrt{-5x^2 + 3e^{2c_1}}\right) \\
y(x) &\to \frac{1}{3}\left(-4x + \sqrt{-5x^2 + 3e^{2c_1}}\right) \\
y(x) &\to \frac{1}{3}\left(-\sqrt{5}\sqrt{-x^2} - 4x\right) \\
y(x) &\to \frac{1}{3}\left(\sqrt{5}\sqrt{-x^2} - 4x\right)
\end{aligned}
$$

6.24 problem 24

Internal problem ID [1053]
Internal file name [`OUTPUT/1054_Sunday_June_05_2022_02_01_30_AM_2174050/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_exact, _Bernoulli]
```

$$\boxed{e^{x}\left(x^{4}y^{2}+4x^{3}y^{2}+1\right)+\left(2x^{4}y\,e^{x}+2y\right)y'=0}$$

6.24.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{e^{x}\left(x^{4}y^{2}+4x^{3}y^{2}+1\right)}{2y\left(e^{x}x^{4}+1\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 299: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y\,(2\,\mathrm{e}^{x} x^4 + 2)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y(2\,\mathrm{e}^{x} x^{4}+2)}} dy \end{aligned}$$

Which results in

$$S = \frac{y^{2}(2\,\mathrm{e}^{x} x^{4} + 2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\mathrm{e}^{x}(x^{4}y^{2} + 4x^{3}y^{2} + 1)}{2y\,(\mathrm{e}^{x} x^{4} + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^{2}\mathrm{e}^{x} x^{3}(x + 4) \\ S_y &= 2y\left(\mathrm{e}^{x} x^{4} + 1\right) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -e^{R} + c_1 \tag{4}$$

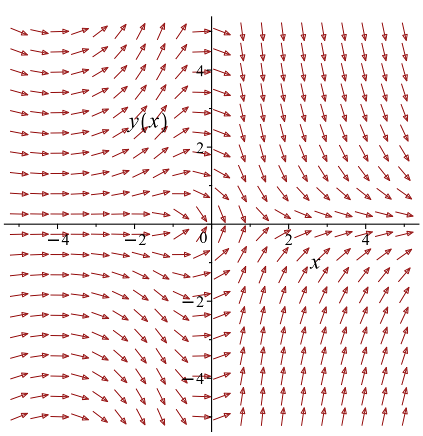
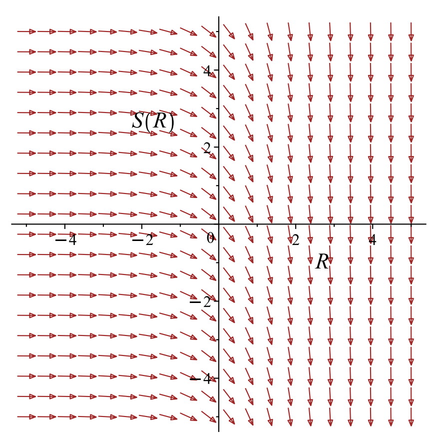
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y^2 \left(e^x x^4 + 1\right) = -e^x + c_1$$

Which simplifies to

$$y^2 \left(e^x x^4 + 1\right) = -e^x + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^x (x^4 y^2 + 4x^3 y^2 + 1)}{2y(e^x x^4 + 1)}$ 	$R = x$ $S = y^2 \left(e^x x^4 + 1\right)$	$\frac{dS}{dR} = -e^{R}$ 

Summary

The solution(s) found are the following

$$y^2 \left(e^x x^4 + 1\right) = -e^x + c_1 \tag{1}$$

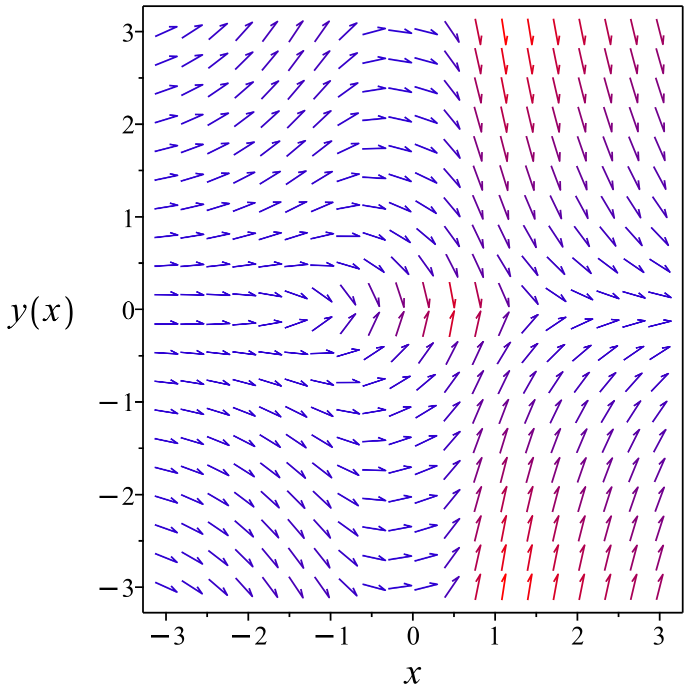


Figure 393: Slope field plot

Verification of solutions

$$y^2(e^x x^4 + 1) = -e^x + c_1$$

Verified OK.

6.24.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{e^x(x^4y^2 + 4x^3y^2 + 1)}{2y(e^x x^4 + 1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{e^x(x^4 + 4x^3)}{2(e^x x^4 + 1)}y - \frac{e^x}{2(e^x x^4 + 1)}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{e^x (x^4 + 4x^3)}{2 (e^x x^4 + 1)} \\ f_1(x) &= -\frac{e^x}{2 (e^x x^4 + 1)} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y' y = -\frac{e^x (x^4 + 4x^3) y^2}{2 (e^x x^4 + 1)} - \frac{e^x}{2 (e^x x^4 + 1)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2 y y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{e^x (x^4 + 4x^3) w(x)}{2 (e^x x^4 + 1)} - \frac{e^x}{2 (e^x x^4 + 1)} \\ w' &= -\frac{e^x (x^4 + 4x^3) w}{e^x x^4 + 1} - \frac{e^x}{e^x x^4 + 1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x) w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{e^{x} x^{3} (x+4)}{e^{x} x^{4}+1} \\
q(x) &= -\frac{e^{x}}{e^{x} x^{4}+1}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{e^{x} x^{3} (x+4)\, w(x)}{e^{x} x^{4}+1} = -\frac{e^{x}}{e^{x} x^{4}+1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{e^{x} x^{3} (x+4)}{e^{x} x^{4}+1} dx} \\
&= e^{x} x^{4}+1
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{e^{x}}{e^{x} x^{4}+1}\right) \\
\frac{d}{dx}\left(\left(e^{x} x^{4}+1\right) w\right) &= \left(e^{x} x^{4}+1\right)\left(-\frac{e^{x}}{e^{x} x^{4}+1}\right) \\
d\left(\left(e^{x} x^{4}+1\right) w\right) &= \left(-e^{x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(e^{x} x^{4}+1\right) w &= \int -e^{x}\, dx \\
\left(e^{x} x^{4}+1\right) w &= -e^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x} x^{4}+1$ results in

$$
w(x) = -\frac{e^{x}}{e^{x} x^{4}+1} + \frac{c_1}{e^{x} x^{4}+1}
$$

which simplifies to

$$
w(x) = \frac{-e^{x}+c_1}{e^{x} x^{4}+1}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = \frac{-e^{x}+c_1}{e^{x} x^{4}+1}
$$

Solving for y gives

$$y(x) = \frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1}$$
$$y(x) = -\frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1} \tag{1}$$
$$y = -\frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1} \tag{2}$$

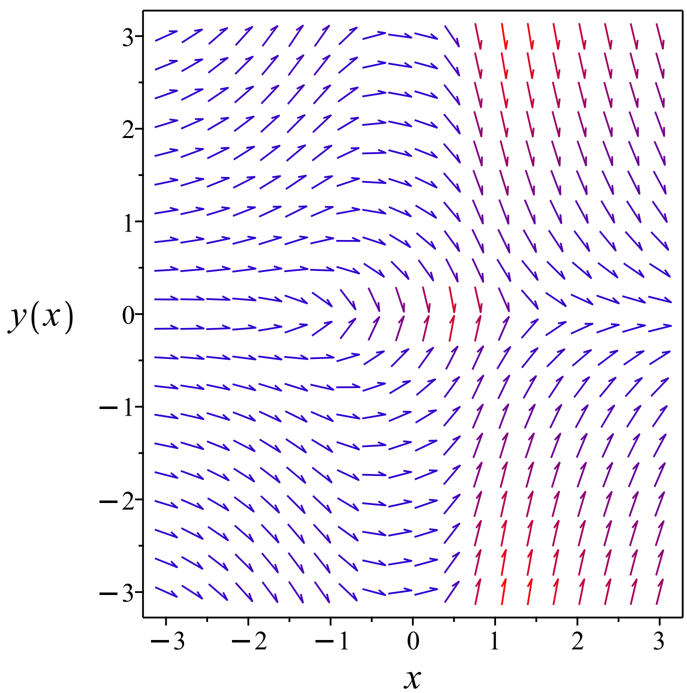


Figure 394: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1}$$

Verified OK.

$$y = -\frac{\sqrt{(e^x x^4 + 1)(-e^x + c_1)}}{e^x x^4 + 1}$$

Verified OK.

6.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2x^4y\,\mathrm{e}^x + 2y\right)\mathrm{d}y &= \left(-\mathrm{e}^x\left(x^4y^2 + 4x^3y^2 + 1\right)\right)\mathrm{d}x\\
\left(\mathrm{e}^x\left(x^4y^2 + 4x^3y^2 + 1\right)\right)\mathrm{d}x + \left(2x^4y\,\mathrm{e}^x + 2y\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \mathrm{e}^x\left(x^4y^2 + 4x^3y^2 + 1\right)\\
N(x,y) &= 2x^4y\,\mathrm{e}^x + 2y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x\left(x^4y^2 + 4x^3y^2 + 1\right)\right)\\
&= 2\,\mathrm{e}^x y\,x^3(x+4)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^4y\,\mathrm{e}^x + 2y\right)\\
&= 2\,\mathrm{e}^x y\,x^3(x+4)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^x\left(x^4y^2 + 4x^3y^2 + 1\right)\mathrm{d}x\\
\phi &= \left(x^4y^2 + 1\right)\mathrm{e}^x + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^4 y\, \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^4 y\, \mathrm{e}^{x} + 2y$. Therefore equation (4) becomes

$$2x^4 y\, \mathrm{e}^{x} + 2y = 2x^4 y\, \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (2y)\, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^4 y^2 + 1\right) \mathrm{e}^{x} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^4 y^2 + 1\right) \mathrm{e}^{x} + y^2$$

Summary
The solution(s) found are the following

$$\left(x^4 y^2 + 1\right) \mathrm{e}^{x} + y^2 = c_1 \tag{1}$$

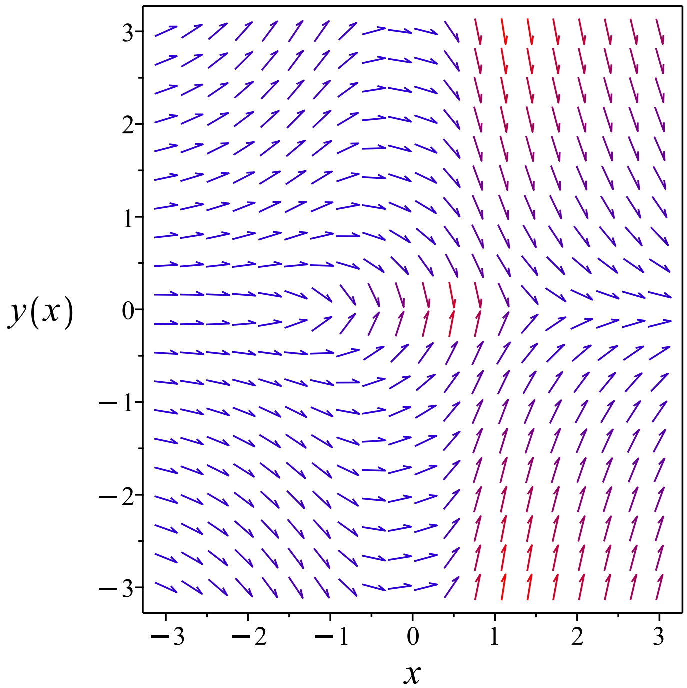


Figure 395: Slope field plot

Verification of solutions

$$\left(x^4y^2+1\right)e^x+y^2=c_1$$

Verified OK.

6.24.4 Maple step by step solution

Let's solve

$e^x(x^4y^2+4x^3y^2+1)+(2x^4y\,e^x+2y)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $e^x (2x^4 y + 8y\,x^3) = 8\,e^x x^3 y + 2x^4 y\,e^x$

- Simplify

  $2\,e^x y\,x^3 (x+4) = 2\,e^x y\,x^3 (x+4)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int e^x (x^4 y^2 + 4x^3 y^2 + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = (x^4 y^2 + 1)\, e^x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2x^4 y\, e^x + 2y = 2x^4 y\, e^x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = (x^4 y^2 + 1)\, e^x + y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $(x^4 y^2 + 1)\, e^x + y^2 = c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{-(e^x x^4 + 1)(-c_1 + e^x)}}{e^x x^4 + 1}, y = -\frac{\sqrt{-(e^x x^4 + 1)(-c_1 + e^x)}}{e^x x^4 + 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve((exp(x)*(x^4*y(x)^2+4*x^3*y(x)^2+1))+(2*x^4*y(x)*exp(x)+2*y(x))*diff(y(x),x)=0,y(x),
```

$$y(x) = \frac{\sqrt{(\mathrm{e}^x x^4 + 1)\,(-\mathrm{e}^x + c_1)}}{\mathrm{e}^x x^4 + 1}$$
$$y(x) = -\frac{\sqrt{(\mathrm{e}^x x^4 + 1)\,(-\mathrm{e}^x + c_1)}}{\mathrm{e}^x x^4 + 1}$$

✓ Solution by Mathematica

Time used: 1.05 (sec). Leaf size: 64

```
DSolve[(Exp[x]*(x^4*y[x]^2+4*x^3*y[x]^2+1))+(2*x^4*y[x]*Exp[x]+2*y[x])*y'[x]==0,y[x],x,Inclu
```

$$y(x) \to -\frac{\sqrt{-2e^x + c_1}}{\sqrt{2e^x x^4 + 2}}$$
$$y(x) \to \frac{\sqrt{-2e^x + c_1}}{\sqrt{2e^x x^4 + 2}}$$

6.25 problem 25

Internal problem ID [1054]
Internal file name [`OUTPUT/1055_Sunday_June_05_2022_02_01_32_AM_69949432/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{x^3y^4 + \left(y^3x^4 + y\right)y' = -x}$$

6.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^3x^4 + y)\,\mathrm{d}y &= (-x^3y^4 - x)\,\mathrm{d}x \\ (x^3y^4 + x)\,\mathrm{d}x + (y^3x^4 + y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^3y^4 + x \\ N(x, y) &= y^3x^4 + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^3y^4 + x) \\ &= 4y^3x^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^3x^4 + y) \\ &= 4y^3x^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int x^3 y^4 + x \, dx$$
$$\phi = \frac{(x^2 y^4 + 1)^2}{4y^4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2(x^2 y^4 + 1)\, x^2}{y} - \frac{(x^2 y^4 + 1)^2}{y^5} + f'(y) \tag{4}$$
$$= \frac{x^4 y^8 - 1}{y^5} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^3 x^4 + y$. Therefore equation (4) becomes

$$y^3 x^4 + y = \frac{x^4 y^8 - 1}{y^5} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^6 + 1}{y^5}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{y^6 + 1}{y^5} \right) dy$$
$$f(y) = \frac{y^2}{2} - \frac{1}{4y^4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 y^4 + 1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2y^4+1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2y^4+1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4} = c_1 \tag{1}$$

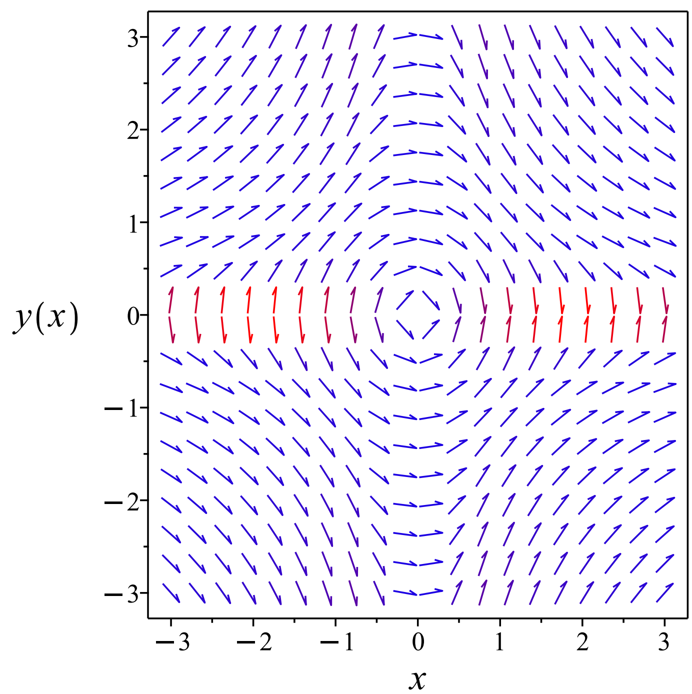


Figure 396: Slope field plot

Verification of solutions

$$\frac{(x^2y^4+1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4} = c_1$$

Verified OK.

6.25.2 Maple step by step solution

Let's solve

$x^3y^4 + (y^3x^4 + y)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $4y^3x^3 = 4y^3x^3$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x^3y^4 + x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{(x^2y^4+1)^2}{4y^4} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y^3x^4 + y = \frac{2(x^2y^4+1)x^2}{y} - \frac{(x^2y^4+1)^2}{y^5} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y^3x^4 + y - \frac{2(x^2y^4+1)x^2}{y} + \frac{(x^2y^4+1)^2}{y^5}$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^2}{2} - \frac{1}{4y^4}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{(x^2y^4+1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4}$
- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{(x^2y^4+1)^2}{4y^4} + \frac{y^2}{2} - \frac{1}{4y^4} = c_1$
- Solve for y

  $\left\{y = \frac{\sqrt{-1-\sqrt{-2x^6+4c_1x^4+1}}}{x^2}, y = \frac{\sqrt{-1+\sqrt{-2x^6+4c_1x^4+1}}}{x^2}, y = -\frac{\sqrt{-1-\sqrt{-2x^6+4c_1x^4+1}}}{x^2}, y = -\frac{\sqrt{-1+\sqrt{-2x^6+4}}}{x^2}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 111

```
dsolve((x^3*y(x)^4+x)+(x^4*y(x)^3+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-1-\sqrt{-2x^6-4c_1x^4+1}}}{x^2}$$
$$y(x) = \frac{\sqrt{-1+\sqrt{-2x^6-4c_1x^4+1}}}{x^2}$$
$$y(x) = -\frac{\sqrt{-1-\sqrt{-2x^6-4c_1x^4+1}}}{x^2}$$
$$y(x) = -\frac{\sqrt{-1+\sqrt{-2x^6-4c_1x^4+1}}}{x^2}$$

✓ Solution by Mathematica

Time used: 11.648 (sec). Leaf size: 135

```
DSolve[(x^3*y[x]^4+x)+(x^4*y[x]^3+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-\frac{1+\sqrt{-2x^6+4c_1x^4+1}}{x^4}}$$
$$y(x) \to \sqrt{-\frac{1+\sqrt{-2x^6+4c_1x^4+1}}{x^4}}$$
$$y(x) \to -\sqrt{\frac{-1+\sqrt{-2x^6+4c_1x^4+1}}{x^4}}$$
$$y(x) \to \sqrt{\frac{-1+\sqrt{-2x^6+4c_1x^4+1}}{x^4}}$$

6.26 problem 26

Internal problem ID [1055]
Internal file name [`OUTPUT/1056_Sunday_June_05_2022_02_01_34_AM_62171422/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{2y + (2y + 2x)\, y' = -3x^2}$$

6.26.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x^2 - 2y}{2y + 2x} \tag{1}$$

Which becomes

$$(2y)\, dy = (-2x)\, dy + \left(-3x^2 - 2y\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\, dy + \left(-3x^2 - 2y\right) dx = d\left(-x^3 - 2yx\right)$$

Hence (2) becomes

$$(2y)\, dy = d\left(-x^3 - 2yx\right)$$

Integrating both sides gives gives these solutions

$$y = -x + \sqrt{-x^3 + x^2 + c_1} + c_1$$
$$y = -x - \sqrt{-x^3 + x^2 + c_1} + c_1$$

Summary
The solution(s) found are the following

$$y = -x + \sqrt{-x^3 + x^2 + c_1} + c_1 \tag{1}$$
$$y = -x - \sqrt{-x^3 + x^2 + c_1} + c_1 \tag{2}$$

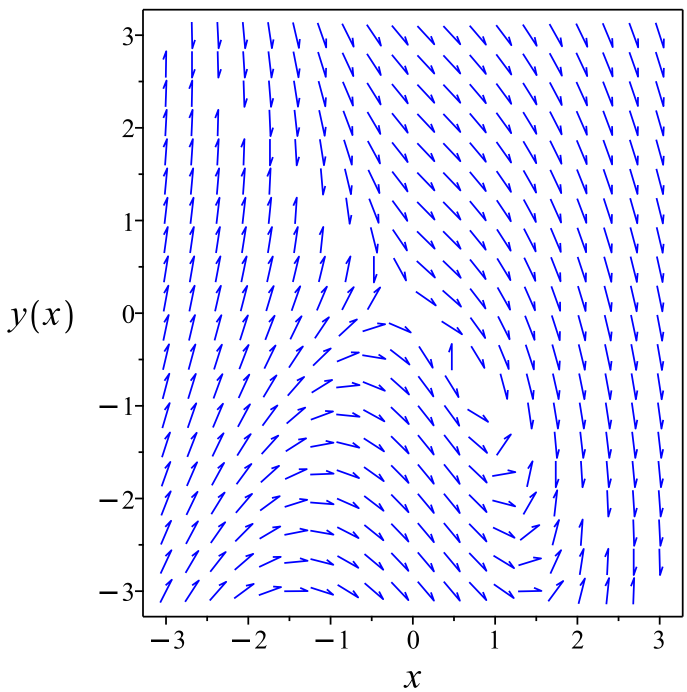


Figure 397: Slope field plot

Verification of solutions

$$y = -x + \sqrt{-x^3 + x^2 + c_1} + c_1$$

Verified OK.

$$y = -x - \sqrt{-x^3 + x^2 + c_1} + c_1$$

Verified OK.

6.26.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2x+2y)\,\mathrm{d}y = \left(-3x^2 - 2y\right)\mathrm{d}x$$
$$\left(3x^2 + 2y\right)\mathrm{d}x + (2x+2y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 3x^2 + 2y$$
$$N(x,y) = 2x + 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 + 2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + 2y) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 3x^2 + 2y \mathrm{d}x\end{aligned}$$

$$\phi = x(x^2 + 2y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + 2y$. Therefore equation (4) becomes

$$2x + 2y = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y) \, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x^2 + 2y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x^2 + 2y) + y^2$$

Summary

The solution(s) found are the following

$$x(x^2 + 2y) + y^2 = c_1 \tag{1}$$

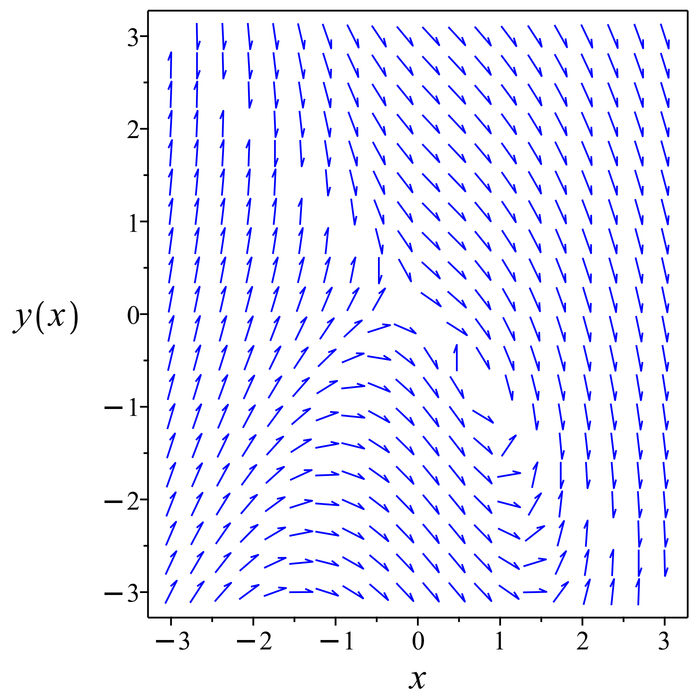


Figure 398: Slope field plot

Verification of solutions

$$x(x^2 + 2y) + y^2 = c_1$$

Verified OK.

6.26.3 Maple step by step solution

Let's solve

$2y + (2y + 2x)\, y' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2 = 2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x^2 + 2y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^3 + 2yx + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2x + 2y = 2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^3 + 2yx + y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^3 + 2yx + y^2 = c_1$

- Solve for y

  $\left\{y = -x - \sqrt{-x^3 + x^2 + c_1}, y = -x + \sqrt{-x^3 + x^2 + c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((3*x^2+2*y(x))+(2*y(x)+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -x - \sqrt{-x^3 + x^2 - c_1}$$
$$y(x) = -x + \sqrt{-x^3 + x^2 - c_1}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 49

```
DSolve[(3*x^2+2*y[x])+(2*y[x]+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \sqrt{-x^3 + x^2 + c_1}$$
$$y(x) \to -x + \sqrt{-x^3 + x^2 + c_1}$$

6.27 problem 27(a)

Internal problem ID [1056]
Internal file name [`OUTPUT/1057_Sunday_June_05_2022_02_01_35_AM_45675956/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 27(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{x^3y^4 + \left(y^3x^4 + 3y\right) y' = -2x}$$

6.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^3x^4 + 3y)\,\mathrm{d}y &= (-x^3y^4 - 2x)\,\mathrm{d}x\\ (x^3y^4 + 2x)\,\mathrm{d}x + (y^3x^4 + 3y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^3y^4 + 2x\\ N(x, y) &= y^3x^4 + 3y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^3y^4 + 2x)\\ &= 4y^3x^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^3x^4 + 3y)\\ &= 4y^3x^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int x^3y^4 + 2x \mathrm{d}x$$
$$\phi = \frac{(x^2y^4+2)^2}{4y^4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2(x^2y^4+2)\,x^2}{y} - \frac{(x^2y^4+2)^2}{y^5} + f'(y) \tag{4}$$
$$= \frac{x^4y^8-4}{y^5} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^3x^4 + 3y$. Therefore equation (4) becomes

$$y^3x^4 + 3y = \frac{x^4y^8-4}{y^5} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3y^6+4}{y^5}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{3y^6+4}{y^5}\right) \mathrm{d}y$$
$$f(y) = \frac{3y^2}{2} - \frac{1}{y^4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4} = c_1 \tag{1}$$

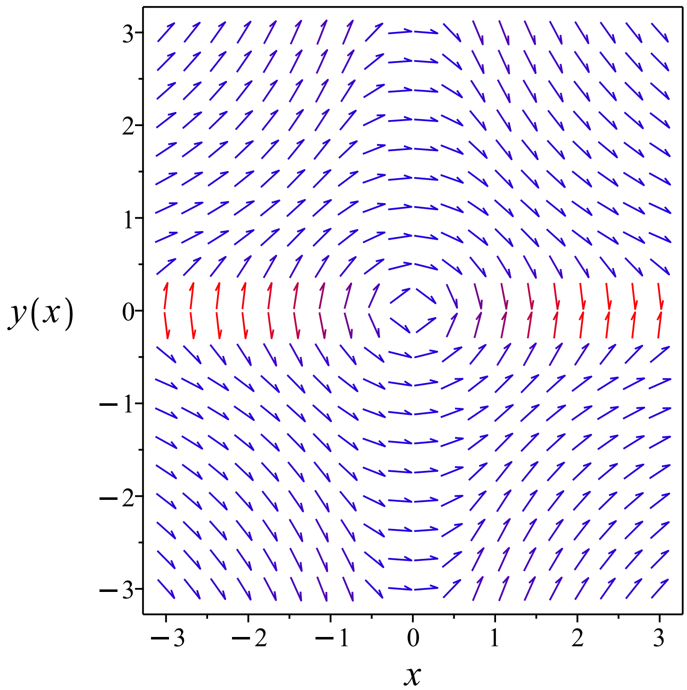


Figure 399: Slope field plot

Verification of solutions

$$\frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4} = c_1$$

Verified OK.

6.27.2 Maple step by step solution

Let's solve

$x^3y^4 + (y^3x^4 + 3y)\, y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $4y^3x^3 = 4y^3x^3$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x^3y^4 + 2x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{(x^2y^4+2)^2}{4y^4} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $y^3x^4 + 3y = \frac{2(x^2y^4+2)x^2}{y} - \frac{(x^2y^4+2)^2}{y^5} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y^3x^4 + 3y - \frac{2(x^2y^4+2)x^2}{y} + \frac{(x^2y^4+2)^2}{y^5}$

- Solve for $f_1(y)$

  $f_1(y) = \frac{3y^2}{2} - \frac{1}{y^4}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{(x^2y^4+2)^2}{4y^4} + \frac{3y^2}{2} - \frac{1}{y^4} = c_1$$

- Solve for y

$$\left\{y = \frac{\sqrt{-3-\sqrt{-4x^6+4c_1x^4+9}}}{x^2}, y = \frac{\sqrt{-3+\sqrt{-4x^6+4c_1x^4+9}}}{x^2}, y = -\frac{\sqrt{-3-\sqrt{-4x^6+4c_1x^4+9}}}{x^2}, y = -\frac{\sqrt{-3+\sqrt{-4x^6+4}}}{x^2}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 111

```
dsolve((x^3*y(x)^4+2*x)+(x^4*y(x)^3+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-3-\sqrt{-4x^6-4c_1x^4+9}}}{x^2}$$
$$y(x) = \frac{\sqrt{-3+\sqrt{-4x^6-4c_1x^4+9}}}{x^2}$$
$$y(x) = -\frac{\sqrt{-3-\sqrt{-4x^6-4c_1x^4+9}}}{x^2}$$
$$y(x) = -\frac{\sqrt{-3+\sqrt{-4x^6-4c_1x^4+9}}}{x^2}$$

✓ Solution by Mathematica

Time used: 11.546 (sec). Leaf size: 135

```
DSolve[(x^3*y[x]^4+2*x)+(x^4*y[x]^3+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\sqrt{-\frac{3+\sqrt{-4x^6+4c_1x^4+9}}{x^4}}$$
$$y(x) \to \sqrt{-\frac{3+\sqrt{-4x^6+4c_1x^4+9}}{x^4}}$$
$$y(x) \to -\sqrt{\frac{-3+\sqrt{-4x^6+4c_1x^4+9}}{x^4}}$$
$$y(x) \to \sqrt{\frac{-3+\sqrt{-4x^6+4c_1x^4+9}}{x^4}}$$

6.28 problem 28(a)

Internal problem ID [1057]
Internal file name [`OUTPUT/1058_Sunday_June_05_2022_02_01_36_AM_99843060/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 28(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{y^2 + 2y'xy = -x^2}$$

6.28.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + 2(u'(x)\,x + u(x))\,x^2u(x) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2+1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{3u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+1}{u}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{3u^2+1}{u}}\,du = \int -\frac{1}{2x}\,dx$$

$$\frac{\ln(3u^2+1)}{6} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\left(3u^2+1\right)^{\frac{1}{6}} = e^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\left(3u^2+1\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

The solution is

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{3y^2}{x^2}+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

$$\left(\frac{x^2+3y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\left(\frac{x^2+3y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}} \tag{1}$$

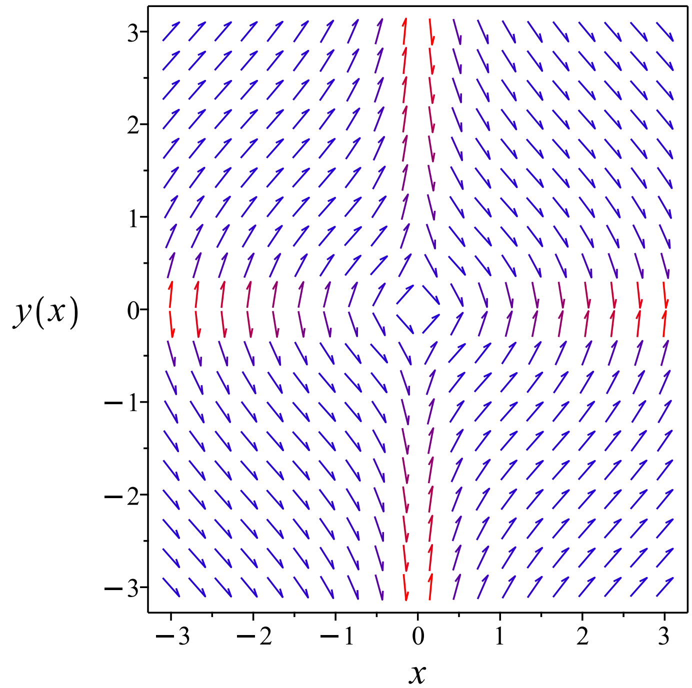


Figure 400: Slope field plot

Verification of solutions

$$\left(\frac{x^2+3y^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Verified OK.

6.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 305: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{yx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx}} dy \end{aligned}$$

Which results in

$$S = \frac{x\, y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= yx \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^2}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{6} + c_1 \tag{4}$$

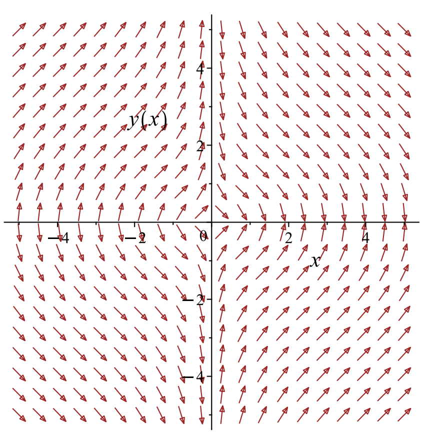
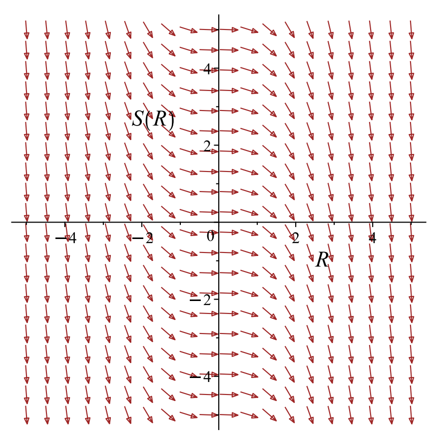
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1$$

Which simplifies to

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{x\,y^2}{2}$	$\frac{dS}{dR} = -\frac{R^2}{2}$ 

Summary
The solution(s) found are the following

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1 \tag{1}$$

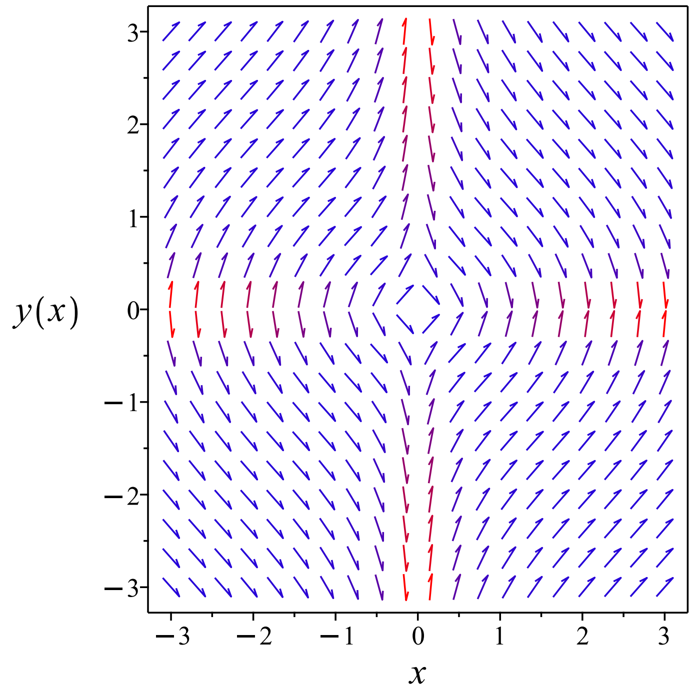


Figure 401: Slope field plot

Verification of solutions

$$\frac{x y^2}{2} = -\frac{x^3}{6} + c_1$$

Verified OK.

6.28.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x} y - \frac{x}{2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x} - \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{x}{2} \\
w' &= -\frac{w}{x} - x
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -x
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}(xw) &= (x)(-x) \\ d(xw) &= (-x^2)\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^2\,dx \\ xw &= -\frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{c_1}{x} - \frac{x^2}{3}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{c_1}{x} - \frac{x^2}{3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \\ y(x) &= -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{2}$$

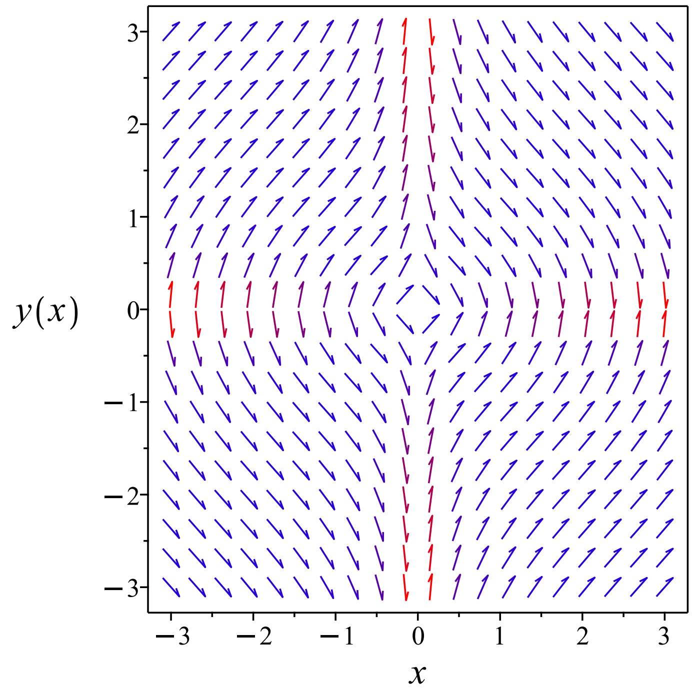


Figure 402: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

6.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(2yx)\,\mathrm{d}y &= \left(-x^2-y^2\right)\mathrm{d}x \\
\left(x^2+y^2\right)\mathrm{d}x + (2yx)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= x^2+y^2 \\
N(x,y) &= 2yx
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2+y^2\right) \\
&= 2y
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int x^2 + y^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{1}{3}x^3 + x\,y^2 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx$. Therefore equation (4) becomes

$$
2yx = 2yx + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{1}{3}x^3 + x\,y^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + x\,y^2$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + xy^2 = c_1 \tag{1}$$

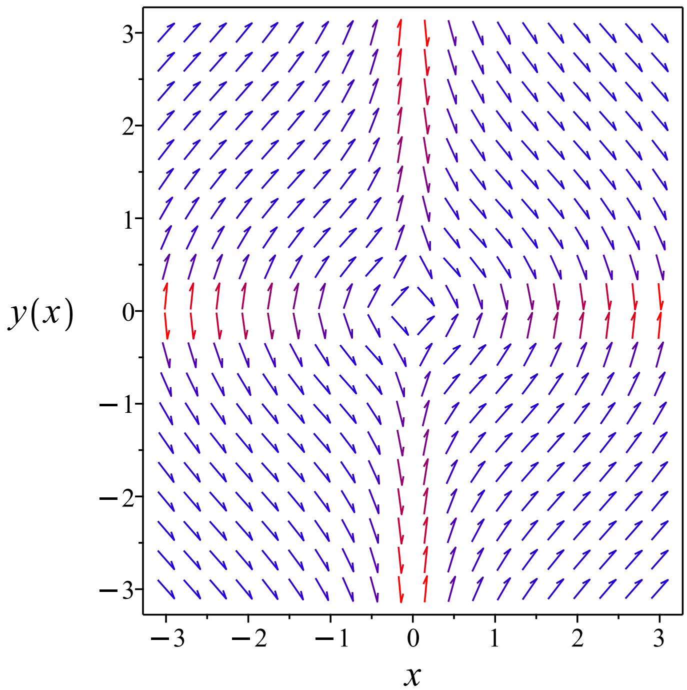


Figure 403: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + xy^2 = c_1$$

Verified OK.

6.28.5 Maple step by step solution

Let's solve

$y^2 + 2y'xy = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2y = 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x^2 + y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^3}{3} + x\, y^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2yx = 2yx + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{1}{3}x^3 + x\,y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{1}{3}x^3 + x\,y^2 = c_1$$

- Solve for y

$$\left\{y = -\frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)+(2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x}$$
$$y(x) = \frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.22 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2)+(2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$

6.29 problem 38

Internal problem ID [1058]
Internal file name [`OUTPUT/1059_Sunday_June_05_2022_02_01_38_AM_16945430/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y' + \frac{2y}{x} + \frac{2xy}{x^2 + 2x^2y + 1} = 0}$$

With initial conditions

$$[y(1) = -2]$$

6.29.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y(2y\,x^2 + 2x^2 + 1)}{x\,(2y\,x^2 + x^2 + 1)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + yx a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + yx b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2x b_4 + y b_5 + b_2 \\
&- \frac{2y(2y\,x^2 + 2x^2 + 1)\left(-2x a_4 + x b_5 - y a_5 + 2y b_6 - a_2 + b_3\right)}{x\left(2y\,x^2 + x^2 + 1\right)} \\
&- \frac{4y^2(2y\,x^2 + 2x^2 + 1)^2\left(x a_5 + 2y a_6 + a_3\right)}{x^2\left(2y\,x^2 + x^2 + 1\right)^2} \\
&- \left(-\frac{2y(4yx + 4x)}{x\left(2y\,x^2 + x^2 + 1\right)} + \frac{2y(2y\,x^2 + 2x^2 + 1)}{x^2\left(2y\,x^2 + x^2 + 1\right)}\right. \\
&\left.+ \frac{2y(2y\,x^2 + 2x^2 + 1)\left(4yx + 2x\right)}{x\left(2y\,x^2 + x^2 + 1\right)^2}\right)\left(x^2 a_4 + yx a_5 + y^2 a_6 + x a_2 + y a_3 + a_1\right) \\
&- \left(-\frac{2(2y\,x^2 + 2x^2 + 1)}{x\left(2y\,x^2 + x^2 + 1\right)} - \frac{4xy}{2y\,x^2 + x^2 + 1}\right. \\
&\left.+ \frac{4y(2y\,x^2 + 2x^2 + 1)\,x}{\left(2y\,x^2 + x^2 + 1\right)^2}\right)\left(x^2 b_4 + yx b_5 + y^2 b_6 + x b_2 + y b_3 + b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{16x^7y^2b_4 + 8x^6y^3a_4 + 4x^6y^3b_5 - 16x^5y^4a_5 - 8x^5y^4b_6 - 40x^4y^5a_6 + 16x^7yb_4 + 12x^6y^2a_4 + 12x^6y^2b_2 - 32x}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&16x^7y^2b_4 + 8x^6y^3a_4 + 4x^6y^3b_5 - 16x^5y^4a_5 - 8x^5y^4b_6 - 40x^4y^5a_6 \\
&+ 16x^7yb_4 + 12x^6y^2a_4 + 12x^6y^2b_2 - 32x^5y^3a_5 - 16x^5y^3b_6 \\
&- 24x^4y^4a_3 - 76x^4y^4a_6 + 6x^7b_4 + 4x^6ya_4 + 12x^6yb_2 + x^6yb_5 \\
&- 16x^5y^2a_5 + 8x^5y^2b_1 - 4x^5y^2b_3 - 4x^5y^2b_6 - 8x^4y^3a_1 - 44x^4y^3a_3 \\
&- 36x^4y^3a_6 + 5x^6b_2 + 8x^5yb_1 + 16x^5yb_4 - 12x^4y^2a_1 - 20x^4y^2a_3 \\
&+ 8x^4y^2a_4 + 4x^4y^2b_5 - 16x^3y^3a_5 - 8x^3y^3b_6 - 40x^2y^4a_6 + 4x^5b_1 \\
&+ 10x^5b_4 - 4x^4ya_1 + 10x^4ya_4 + 12x^4yb_2 + 2x^4yb_5 - 12x^3y^2a_5 \\
&- 6x^3y^2b_6 - 24x^2y^3a_3 - 34x^2y^3a_6 + 8x^4b_2 + 4x^3ya_2 + 8x^3yb_1 \\
&- 8x^2y^2a_1 - 18x^2y^2a_3 + 6x^3b_1 + 4x^3b_4 - 2x^2ya_1 + 2x^2ya_4 + yb_5x^2 \\
&- 4x\,y^2a_5 - 2x\,y^2b_6 - 10y^3a_6 + 3b_2x^2 - 6y^2a_3 + 2xb_1 - 2ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&8a_4v_1^6v_2^3 - 16a_5v_1^5v_2^4 - 40a_6v_1^4v_2^5 + 16b_4v_1^7v_2^2 + 4b_5v_1^6v_2^3 - 8b_6v_1^5v_2^4 \\
&- 24a_3v_1^4v_2^4 + 12a_4v_1^6v_2^2 - 32a_5v_1^5v_2^3 - 76a_6v_1^4v_2^4 + 12b_2v_1^6v_2^2 \\
&+ 16b_4v_1^7v_2 - 16b_6v_1^5v_2^3 - 8a_1v_1^4v_2^3 - 44a_3v_1^4v_2^3 + 4a_4v_1^6v_2 - 16a_5v_1^5v_2^2 \\
&- 36a_6v_1^4v_2^3 + 8b_1v_1^5v_2^2 + 12b_2v_1^6v_2 - 4b_3v_1^5v_2^2 + 6b_4v_1^7 + b_5v_1^6v_2 \\
&- 4b_6v_1^5v_2^2 - 12a_1v_1^4v_2^2 - 20a_3v_1^4v_2^2 + 8a_4v_1^4v_2^2 - 16a_5v_1^3v_2^3 - 40a_6v_1^2v_2^4 \\
&+ 8b_1v_1^5v_2 + 5b_2v_1^6 + 16b_4v_1^5v_2 + 4b_5v_1^4v_2^2 - 8b_6v_1^3v_2^3 - 4a_1v_1^4v_2 \\
&- 24a_3v_1^2v_2^3 + 10a_4v_1^4v_2 - 12a_5v_1^3v_2^2 - 34a_6v_1^2v_2^3 + 4b_1v_1^5 + 12b_2v_1^4v_2 \\
&+ 10b_4v_1^5 + 2b_5v_1^4v_2 - 6b_6v_1^3v_2^2 - 8a_1v_1^2v_2^2 + 4a_2v_1^3v_2 - 18a_3v_1^2v_2^2 \\
&+ 8b_1v_1^3v_2 + 8b_2v_1^4 - 2a_1v_1^2v_2 + 2a_4v_1^2v_2 - 4a_5v_1v_2^2 - 10a_6v_2^3 + 6b_1v_1^3 \\
&+ 4b_4v_1^3 + b_5v_1^2v_2 - 2b_6v_1v_2^2 - 6a_3v_2^2 + 3b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(8a_4 + 4b_5)\, v_1^6 v_2^3 + (12a_4 + 12b_2)\, v_1^6 v_2^2 + (4a_4 + 12b_2 + b_5)\, v_1^6 v_2 \\
&+ (-16a_5 - 8b_6)\, v_1^5 v_2^4 + (-32a_5 - 16b_6)\, v_1^5 v_2^3 \\
&+ (-16a_5 + 8b_1 - 4b_3 - 4b_6)\, v_1^5 v_2^2 + (8b_1 + 16b_4)\, v_1^5 v_2 \\
&+ (-24a_3 - 76a_6)\, v_1^4 v_2^4 + (-8a_1 - 44a_3 - 36a_6)\, v_1^4 v_2^3 \\
&+ (-12a_1 - 20a_3 + 8a_4 + 4b_5)\, v_1^4 v_2^2 + (-4a_1 + 10a_4 + 12b_2 + 2b_5)\, v_1^4 v_2 \\
&+ (-16a_5 - 8b_6)\, v_1^3 v_2^3 + (-12a_5 - 6b_6)\, v_1^3 v_2^2 + (4a_2 + 8b_1)\, v_1^3 v_2 \\
&+ (-24a_3 - 34a_6)\, v_1^2 v_2^3 + (-8a_1 - 18a_3)\, v_1^2 v_2^2 + (-2a_1 + 2a_4 + b_5)\, v_1^2 v_2 \\
&+ (-4a_5 - 2b_6)\, v_1 v_2^2 + (4b_1 + 10b_4)\, v_1^5 + 6b_4 v_1^7 + 5b_2 v_1^6 + 8b_2 v_1^4 \\
&- 10a_6 v_2^3 - 6a_3 v_2^2 + 3b_2 v_1^2 - 2a_1 v_2 + 2b_1 v_1 - 40a_6 v_1^4 v_2^5 \\
&+ 16b_4 v_1^7 v_2^2 + 16b_4 v_1^7 v_2 - 40a_6 v_1^2 v_2^4 + (6b_1 + 4b_4)\, v_1^3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
-6a_3 &= 0 \\
-40a_6 &= 0 \\
-10a_6 &= 0 \\
2b_1 &= 0 \\
3b_2 &= 0 \\
5b_2 &= 0 \\
8b_2 &= 0 \\
6b_4 &= 0 \\
16b_4 &= 0 \\
-8a_1 - 18a_3 &= 0 \\
4a_2 + 8b_1 &= 0 \\
-24a_3 - 76a_6 &= 0 \\
-24a_3 - 34a_6 &= 0 \\
8a_4 + 4b_5 &= 0 \\
12a_4 + 12b_2 &= 0 \\
-32a_5 - 16b_6 &= 0 \\
-16a_5 - 8b_6 &= 0 \\
-12a_5 - 6b_6 &= 0 \\
-4a_5 - 2b_6 &= 0 \\
4b_1 + 10b_4 &= 0 \\
6b_1 + 4b_4 &= 0 \\
8b_1 + 16b_4 &= 0 \\
-8a_1 - 44a_3 - 36a_6 &= 0 \\
-2a_1 + 2a_4 + b_5 &= 0 \\
4a_4 + 12b_2 + b_5 &= 0 \\
-12a_1 - 20a_3 + 8a_4 + 4b_5 &= 0 \\
-4a_1 + 10a_4 + 12b_2 + 2b_5 &= 0 \\
-16a_5 + 8b_1 - 4b_3 - 4b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= -\frac{b_6}{2} \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_6 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{yx}{2} \\
\eta &= y^2 + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y^2 + y - \left(-\frac{2y(2y\,x^2 + 2x^2 + 1)}{x\,(2y\,x^2 + x^2 + 1)}\right)\left(-\frac{yx}{2}\right) \\
&= \frac{x^2y^2 + y\,x^2 + y}{2y\,x^2 + x^2 + 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 y^2 + y\,x^2 + y}{2y\,x^2 + x^2 + 1}} dy \end{aligned}$$

Which results in

$$S = \ln\left(y\left(y\,x^2 + x^2 + 1\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(2y\,x^2 + 2x^2 + 1)}{x\,(2y\,x^2 + x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x(y+1)}{1 + (y+1)\,x^2} \\ S_y &= \frac{1}{y} + \frac{x^2}{1 + (y+1)\,x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

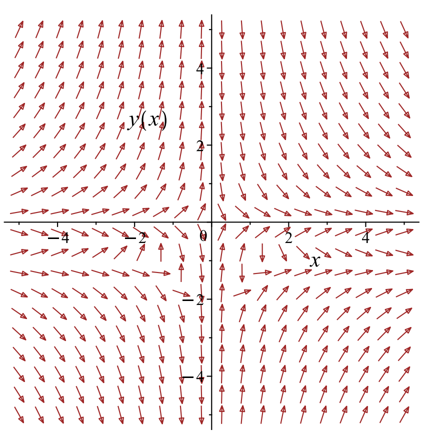
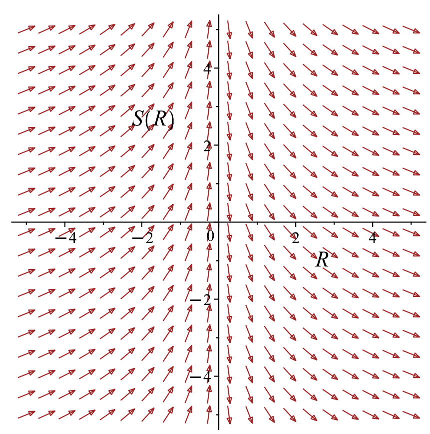
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \ln\left(1 + (1 + y)\, x^2\right) = -2\ln(x) + c_1$$

Which simplifies to

$$\ln(y) + \ln\left(1 + (1 + y)\, x^2\right) = -2\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(2y\,x^2+2x^2+1)}{x(2y\,x^2+x^2+1)}$ 	$R = x$ $S = \ln(y) + \ln(1 + (y +$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-\infty = c_1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

<u>Verification of solutions</u> N/A

6.29.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(2y\,x^2+x^2+1\right)\right)\mathrm{d}y&=\left(-2y\left(2y\,x^2+2x^2+1\right)\right)\mathrm{d}x\\ \left(2y\left(2y\,x^2+2x^2+1\right)\right)\mathrm{d}x+\left(x\left(2y\,x^2+x^2+1\right)\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y\left(2y\,x^2+2x^2+1\right)\\ N(x,y)&=x\left(2y\,x^2+x^2+1\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y\left(2y\,x^{2}+2x^{2}+1\right)\right)\\ &= 2+\left(8y+4\right)x^{2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(2y\,x^{2}+x^{2}+1\right)\right)\\ &= 1+\left(6y+3\right)x^{2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{\left(1+2y\right)x^{3}+x}\left(\left(8y\,x^{2}+4x^{2}+2\right)-\left(2y\,x^{2}+x^{2}+1+x(4yx+2x)\right)\right)\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(2y\left(2y\,x^{2}+2x^{2}+1\right)\right)\\ &= 4\left(\frac{1}{2}+\left(y+1\right)x^{2}\right)xy\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(x\left(2y\,x^2+x^2+1\right)\right) \\
&= (1+2y)\,x^4+x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(4\left(\frac{1}{2}+(y+1)\,x^2\right)xy\right)+\left((1+2y)\,x^4+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int 4\left(\frac{1}{2}+(y+1)\,x^2\right)xy\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=\frac{\left(2y\,x^2+2x^2+1\right)^2 y}{4y+4}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial\phi}{\partial y} &= \frac{4(2y\,x^2+2x^2+1)\,y\,x^2}{4y+4}+\frac{\left(2y\,x^2+2x^2+1\right)^2}{4y+4}-\frac{4\left(2y\,x^2+2x^2+1\right)^2 y}{(4y+4)^2}+f'(y) \qquad (4) \\
&= \frac{1+8\left(\frac{1}{2}+y\right)(y+1)^2\,x^4+4(y+1)^2\,x^2}{4\,(y+1)^2}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=(1+2y)\,x^4+x^2$. Therefore equation (4) becomes

$$
(1+2y)\,x^4+x^2=\frac{1+8\left(\frac{1}{2}+y\right)(y+1)^2\,x^4+4(y+1)^2\,x^2}{4\,(y+1)^2}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{4\left(y+1\right)^{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{4\left(y+1\right)^{2}}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{4y+4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(2y\,x^{2}+2x^{2}+1\right)^{2} y}{4y+4} + \frac{1}{4y+4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(2y\,x^{2}+2x^{2}+1\right)^{2} y}{4y+4} + \frac{1}{4y+4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} = c_1$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{\left(2y\,x^{2}+2x^{2}+1\right)^{2} y}{4y+4} + \frac{1}{4y+4} = \frac{1}{4}$$

Summary

The solution(s) found are the following

$$\frac{1}{4} + \left(y^{2}+y\right)x^{4} + x^{2}y = \frac{1}{4} \tag{1}$$

Verification of solutions

$$\frac{1}{4}+\left(y^2+y\right)x^4+x^2y=\frac{1}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+2/x*y(x)= -(2*x*y(x))/(x^2+2*x^2*y(x)+1),y(1) = -2],y(x), singsol=all)
```

$$y(x)=\frac{-x^2-1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.694 (sec). Leaf size: 38

```
DSolve[{y'[x]+2/x*y[x]== -(2*x*y[x])/(x^2+2*x^2*y[x]+1),y[1]==-2},y[x],x,IncludeSingularSolu
```

$$y(x)\to\frac{1}{2}\left(-\frac{1}{x^2}-\frac{\sqrt{x^3\left(x^2+1\right)^2}}{x^{7/2}}-1\right)$$

6.30 problem 39

Internal problem ID [1059]
Internal file name [OUTPUT/1060_Sunday_June_05_2022_02_01_39_AM_47280605/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y' - \frac{3y}{x} - \frac{2x^4(4x^3 - 3y)}{3x^5 + 3x^3 + 2y} = 0}$$

With initial conditions

$$[y(1) = 1]$$

6.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x\left(3x^5 + 3x^3 + 2y\right)\right)\mathrm{d}y = \left(8x^8 + 3y\,x^5 + 9y\,x^3 + 6y^2\right)\mathrm{d}x$$
$$\left(-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2\right)\mathrm{d}x + \left(x\left(3x^5 + 3x^3 + 2y\right)\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2$$
$$N(x, y) = x\left(3x^5 + 3x^3 + 2y\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2\right)\\ &= -3x^5 - 9x^3 - 12y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(3x^5 + 3x^3 + 2y\right)\right)\\ &= 18x^5 + 12x^3 + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3x^6 + 3x^4 + 2yx}\left(\left(-3x^5 - 9x^3 - 12y\right) - \left(3x^5 + 3x^3 + 2y + x\left(15x^4 + 9x^2\right)\right)\right) \\
&= -\frac{7}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{7}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-7\ln(x)} \\
&= \frac{1}{x^7}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^7}\left(-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2\right) \\
&= \frac{-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2}{x^7}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^7}\left(x\left(3x^5 + 3x^3 + 2y\right)\right) \\
&= \frac{3x^5 + 3x^3 + 2y}{x^6}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2}{x^7}\right) + \left(\frac{3x^5 + 3x^3 + 2y}{x^6}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-8x^8 - 3y\,x^5 - 9y\,x^3 - 6y^2}{x^7} \mathrm{d}x$$
$$\phi = -4x^2 + \frac{3y}{x} + \frac{y^2}{x^6} + \frac{3y}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3}{x} + \frac{2y}{x^6} + \frac{3}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3x^5+3x^3+2y}{x^6}$. Therefore equation (4) becomes

$$\frac{3x^5 + 3x^3 + 2y}{x^6} = \frac{3x^5 + 3x^3 + 2y}{x^6} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4x^2 + \frac{3y}{x} + \frac{y^2}{x^6} + \frac{3y}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4x^2 + \frac{3y}{x} + \frac{y^2}{x^6} + \frac{3y}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$-4x^2 + \frac{3y}{x} + \frac{y^2}{x^6} + \frac{3y}{x^3} = 3$$

The above simplifies to

$$-4x^8 - 3x^6 + 3y\,x^5 + 3y\,x^3 + y^2 = 0$$

Summary
The solution(s) found are the following

$$-4x^8 - 3x^6 + 3yx^5 + 3yx^3 + y^2 = 0 \tag{1}$$

Verification of solutions

$$-4x^8 - 3x^6 + 3yx^5 + 3yx^3 + y^2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 30

```
dsolve([diff(y(x),x)-3/x*y(x)= (2*x^4*(4*x^3-3*y(x)))/(3*x^5+3*x^3+2*y(x)),y(1) = 1],y(x), s
```

$$y(x)=\frac{\left(-3x^{2}+\sqrt{9x^{4}+34x^{2}+21}-3\right)x^{3}}{2}$$

✓ Solution by Mathematica

Time used: 0.734 (sec). Leaf size: 49

```
DSolve[{y'[x]-3/x*y[x]== (2*x^4*(4*x^3-3*y[x]))/(3*x^5+3*x^3+2*y[x]),y[1]==1},y[x],x,Include
```

$$y(x)\to\frac{1}{2}\left(-3x^{5}-3x^{3}+\sqrt{\frac{1}{x^{7}}}\sqrt{x\left(9x^{4}+34x^{2}+21\right)}x^{6}\right)$$

6.31 problem 40

Internal problem ID [1060]
Internal file name [`OUTPUT/1061_Sunday_June_05_2022_02_01_41_AM_99865551/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Section 2.5 Page 79
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_Abel, `2nd type`, `class B`]]
```

$$\boxed{2yx + y' + \frac{e^{-x^2}\left(3x + 2\,e^{x^2} y\right)}{2x + 3\,e^{x^2} y} = 0}$$

With initial conditions

$$[y(0) = -1]$$

6.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{6\,e^{x^2} x\,y^2 + 2y\,e^{-x^2} e^{x^2} + 4y\,x^2 + 3x\,e^{-x^2}}{2x + 3\,e^{x^2} y} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -1$ is

$$\left\{x < \frac{\sqrt{2}}{2\sqrt{-\frac{1}{\mathrm{LambertW}\left(\_Z66, -\frac{9}{2}\right)}}} \vee \frac{\sqrt{2}}{2\sqrt{-\frac{1}{\mathrm{LambertW}\left(\_Z66, -\frac{9}{2}\right)}}} < x\right\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

6.31.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{6\,\mathrm{e}^{x^2} x\, y^2 + 2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} + 4y\, x^2 + 3x\,\mathrm{e}^{-x^2}}{2x + 3\,\mathrm{e}^{x^2} y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$
\xi = x^3 a_7 + y\, x^2 a_8 + x\, y^2 a_9 + y^3 a_{10} + x^2 a_4 + y x a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x^3 b_7 + y\, x^2 b_8 + x\, y^2 b_9 + y^3 b_{10} + x^2 b_4 + y x b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&3x^2b_7+2xyb_8+y^2b_9+2xb_4+yb_5+b_2 \qquad (5E)\\
&-\frac{\left(6\,e^{x^2}x\,y^2+2y\,e^{-x^2}e^{x^2}+4y\,x^2+3x\,e^{-x^2}\right)\left(-3x^2a_7+x^2b_8-2xya_8+2xyb_9-y^2a_9+3y^2b_{10}-2xa_4+x\right.}{2x+3\,e^{x^2}y}\\
&-\frac{\left(6\,e^{x^2}x\,y^2+2y\,e^{-x^2}e^{x^2}+4y\,x^2+3x\,e^{-x^2}\right)^2\left(x^2a_8+2xya_9+3y^2a_{10}+xa_5+2ya_6+a_3\right)}{\left(2x+3\,e^{x^2}y\right)^2}\\
&-\left(-\frac{12\,e^{x^2}x^2y^2+6\,e^{x^2}y^2+8yx+3\,e^{-x^2}-6x^2e^{-x^2}}{2x+3\,e^{x^2}y}\right.\\
&\left.+\frac{\left(6\,e^{x^2}x\,y^2+2y\,e^{-x^2}e^{x^2}+4y\,x^2+3x\,e^{-x^2}\right)\left(2+6x\,e^{x^2}y\right)}{\left(2x+3\,e^{x^2}y\right)^2}\right)\left(x^3a_7\right.\\
&\left.+y\,x^2a_8+x\,y^2a_9+y^3a_{10}+x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1\right)\\
&-\left(-\frac{12x\,e^{x^2}y+2\,e^{x^2}e^{-x^2}+4x^2}{2x+3\,e^{x^2}y}\right.\\
&\left.+\frac{3\left(6\,e^{x^2}x\,y^2+2y\,e^{-x^2}e^{x^2}+4y\,x^2+3x\,e^{-x^2}\right)e^{x^2}}{\left(2x+3\,e^{x^2}y\right)^2}\right)\left(x^3b_7+y\,x^2b_8\right.\\
&\left.+x\,y^2b_9+y^3b_{10}+x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1\right)=0
\end{aligned}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,e^{x^2},e^{-2x^2},e^{-x^2},e^{2x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^{x^2} = v_3, e^{-2x^2} = v_4, e^{-x^2} = v_5, e^{2x^2} = v_6\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0\\
5a_1 &= 0\\
-12a_3 &= 0\\
-9a_3 &= 0\\
-12a_4 &= 0\\
54a_4 &= 0\\
60a_4 &= 0\\
-48a_5 &= 0\\
-36a_5 &= 0\\
-16a_5 &= 0\\
-9a_5 &= 0\\
-132a_6 &= 0\\
-96a_6 &= 0\\
-72a_6 &= 0\\
-60a_6 &= 0\\
-36a_6 &= 0\\
-32a_6 &= 0\\
-24a_6 &= 0\\
-18a_6 &= 0\\
-3a_6 &= 0\\
18a_6 &= 0\\
-12a_7 &= 0\\
-4a_7 &= 0\\
72a_7 &= 0\\
84a_7 &= 0\\
-48a_8 &= 0\\
-36a_8 &= 0\\
-16a_8 &= 0\\
-9a_8 &= 0\\
-96a_9 &= 0\\
-72a_9 &= 0\\
-60a_9 &= 0\\
-32a_9 &= 0\\
-18a_9 &= 0\\
-184a_{10} &= 0\\
-144a_{10} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{b_8}{2} \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{b_8}{2} \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0 \\
b_7 &= 0 \\
b_8 &= b_8 \\
b_9 &= 0 \\
b_{10} &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -\frac{x}{2} \\
\eta &= y\,x^2 - \frac{1}{2}y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y\,x^2 - \frac{1}{2}y - \left(-\frac{6\,\mathrm{e}^{x^2} x\,y^2 + 2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} + 4y\,x^2 + 3x\,\mathrm{e}^{-x^2}}{2x + 3\,\mathrm{e}^{x^2} y}\right)\left(-\frac{x}{2}\right) \\
&= \frac{-3\,\mathrm{e}^{x^2} y^2 - 3x^2\mathrm{e}^{-x^2} - 4yx}{6\,\mathrm{e}^{x^2} y + 4x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-3\,\mathrm{e}^{x^2} y^2 - 3x^2\mathrm{e}^{-x^2} - 4yx}{6\,\mathrm{e}^{x^2} y + 4x}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln\left(3\,\mathrm{e}^{2x^2} y^2 + 4x\,\mathrm{e}^{x^2} y + 3x^2\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{6\,\mathrm{e}^{x^2} x\,y^2 + 2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} + 4y\,x^2 + 3x\,\mathrm{e}^{-x^2}}{2x + 3\,\mathrm{e}^{x^2} y}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-12\,\mathrm{e}^{2x^2} x\, y^2 + (-8y\, x^2 - 4y)\,\mathrm{e}^{x^2} - 6x}{3\,\mathrm{e}^{2x^2} y^2 + 4x\,\mathrm{e}^{x^2} y + 3x^2} \\
S_y &= \frac{-6\,\mathrm{e}^{2x^2} y - 4\,\mathrm{e}^{x^2} x}{3\,\mathrm{e}^{2x^2} y^2 + 4x\,\mathrm{e}^{x^2} y + 3x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

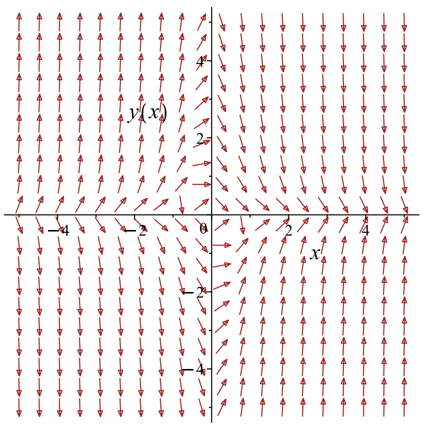
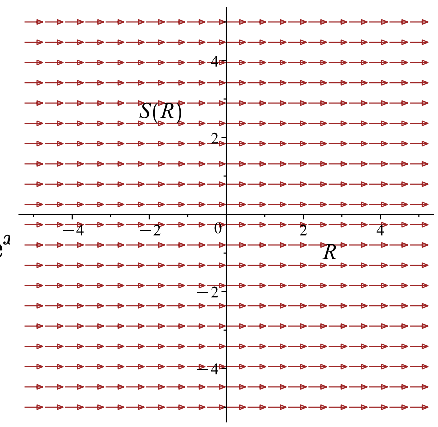
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(3\,\mathrm{e}^{2x^2} y^2 + 4yx\,\mathrm{e}^{x^2} + 3x^2\right) = c_1$$

Which simplifies to

$$-\ln\left(3\,\mathrm{e}^{2x^2} y^2 + 4yx\,\mathrm{e}^{x^2} + 3x^2\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6\,e^{x^2} x\,y^2+2y\,e^{-x^2} e^{x^2}+4y\,x^2+3x\,e^{-x^2}}{2x+3\,e^{x^2} y}$ 	$R = x$ $S = -\ln\left(3\,e^{2x^2} y^2 + 4x\,e^{x}\right.$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(3) = c_1$$

$$c_1 = -\ln(3)$$

Substituting c_1 found above in the general solution gives

$$-\ln\left(3\,e^{2x^2} y^2 + 4x\,e^{x^2} y + 3x^2\right) = -\ln(3)$$

Summary

The solution(s) found are the following

$$-\ln\left(3\,e^{2x^2} y^2 + 4yx\,e^{x^2} + 3x^2\right) = -\ln(3) \tag{1}$$

Verification of solutions

$$-\ln\left(3\,e^{2x^2} y^2 + 4yx\,e^{x^2} + 3x^2\right) = -\ln(3)$$

Verified OK.

6.31.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(2x+3\,\mathrm{e}^{x^2}y\right)\mathrm{d}y=\left(-6\,\mathrm{e}^{x^2}x\,y^2-2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2}-4y\,x^2-3x$$

$$\left(6\,\mathrm{e}^{x^2}x\,y^2+2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2}+4y\,x^2+3x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x+\left(2x+3\,\mathrm{e}^{x^2}y\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=6\,\mathrm{e}^{x^2}x\,y^2+2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2}+4y\,x^2+3x\,\mathrm{e}^{-x^2}\\ N(x,y)&=2x+3\,\mathrm{e}^{x^2}y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(6\,\mathrm{e}^{x^2} x\,y^2 + 2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} + 4y\,x^2 + 3x\,\mathrm{e}^{-x^2}\right)\\ &= 12x\,\mathrm{e}^{x^2} y + 2 + 4x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x + 3\,\mathrm{e}^{x^2} y\right)\\ &= 2 + 6x\,\mathrm{e}^{x^2} y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x + 3\,\mathrm{e}^{x^2} y}\left(\left(12x\,\mathrm{e}^{x^2} y + 2\,\mathrm{e}^{x^2}\mathrm{e}^{-x^2} + 4x^2\right) - \left(2 + 6x\,\mathrm{e}^{x^2} y\right)\right)\\ &= 2x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 2x\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x^2}\\ &= \mathrm{e}^{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x^2}\left(6\,\mathrm{e}^{x^2} x\,y^2 + 2y\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} + 4y\,x^2 + 3x\,\mathrm{e}^{-x^2}\right)\\ &= 6\,\mathrm{e}^{2x^2} x\,y^2 + 2\,\mathrm{e}^{x^2} y\left(2x^2 + 1\right) + 3x\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x^2}\left(2x + 3\,\mathrm{e}^{x^2} y\right) \\
&= 3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(6\,\mathrm{e}^{2x^2} x\,y^2 + 2\,\mathrm{e}^{x^2} y\left(2x^2+1\right) + 3x\right) + \left(3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 6\,\mathrm{e}^{2x^2} x\,y^2 + 2\,\mathrm{e}^{x^2} y\left(2x^2+1\right) + 3x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{3x^2}{2} + 2x\,\mathrm{e}^{x^2} y + \frac{3\,\mathrm{e}^{2x^2} y^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x$. Therefore equation (4) becomes

$$
3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x = 3\,\mathrm{e}^{2x^2} y + 2\,\mathrm{e}^{x^2} x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x^2}{2} + 2x\,e^{x^2} y + \frac{3\,e^{2x^2} y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x^2}{2} + 2x\,e^{x^2} y + \frac{3\,e^{2x^2} y^2}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{3x^2}{2} + 2x\,e^{x^2} y + \frac{3\,e^{2x^2} y^2}{2} = \frac{3}{2}$$

Summary

The solution(s) found are the following

$$\frac{3x^2}{2} + 2yx\,e^{x^2} + \frac{3\,e^{2x^2} y^2}{2} = \frac{3}{2} \tag{1}$$

Verification of solutions

$$\frac{3x^2}{2} + 2yx\,e^{x^2} + \frac{3\,e^{2x^2} y^2}{2} = \frac{3}{2}$$

Verified OK.

The solution

$$-\ln\left(3\,e^{2x^2} y^2 + 4yx\,e^{x^2} + 3x^2\right) = -\ln(3)$$

can be simplified to

$$\ln \left(3\, \mathrm{e}^{2x^2} y^2 + 4yx\, \mathrm{e}^{x^2} + 3x^2\right) = \ln (3)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 36

```
dsolve([diff(y(x),x)+2*x*y(x)= -exp(-x^2)*(3*x+2*y(x)*exp(x^2))/(2*x+3*y(x)*exp(x^2)),y(0) =
```

$$y(x) = -\frac{\left(2x\, \mathrm{e}^{x^2} + \sqrt{\mathrm{e}^{2x^2} \left(-5x^2 + 9\right)}\right) \mathrm{e}^{-2x^2}}{3}$$

✓ Solution by Mathematica

Time used: 33.105 (sec). Leaf size: 44

```
DSolve[{y'[x]+2*x*y[x]== -Exp[-x^2]*(3*x+2*y[x]*Exp[x^2])/(2*x+3*y[x]*Exp[x^2]),y[0]==-1},y[
```

$$y(x) \to -\frac{1}{3} e^{-2x^2} \left(2e^{x^2} x + \sqrt{e^{2x^2} \left(9 - 5x^2\right)}\right)$$

7 Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91

7.1 problem 1(a)

Internal problem ID [1061]
Internal file name [`OUTPUT/1062_Sunday_June_05_2022_02_01_43_AM_33651986/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y + \left(2x + \frac{1}{y}\right) y' = 0}$$

7.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{2yx + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y^2(b_3 - a_2)}{2yx+1} - \frac{y^4 a_3}{(2yx+1)^2} - \frac{2y^3(xa_2 + ya_3 + a_1)}{(2yx+1)^2} - \left(-\frac{2y}{2yx+1} + \frac{2y^2 x}{(2yx+1)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2y^2b_2 - 3y^4a_3 + 2x\,y^2b_1 - 2y^3a_1 + 6xyb_2 + y^2a_2 + y^2b_3 + 2yb_1 + b_2}{(2yx+1)^2} = 0$$

Setting the numerator to zero gives

$$6x^2y^2b_2 - 3y^4a_3 + 2x\,y^2b_1 - 2y^3a_1 + 6xyb_2 + y^2a_2 + y^2b_3 + 2yb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_3v_2^4 + 6b_2v_1^2v_2^2 - 2a_1v_2^3 + 2b_1v_1v_2^2 + a_2v_2^2 + 6b_2v_1v_2 + b_3v_2^2 + 2b_1v_2 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^2v_2^2 + 2b_1v_1v_2^2 + 6b_2v_1v_2 - 3a_3v_2^4 - 2a_1v_2^3 + (a_2 + b_3)\,v_2^2 + 2b_1v_2 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-2a_1 &= 0 \\
-3a_3 &= 0 \\
2b_1 &= 0 \\
6b_2 &= 0 \\
a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y^2}{2yx + 1}\right)(-x) \\
&= \frac{x\,y^2 + y}{2yx + 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} d y \\ &= \int \frac{1}{\frac{x y^{2}+y}{2 y x+1}} d y \end{aligned}$$

Which results in

$$S=\ln (y(y x+1))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{d S}{d R}=\frac{S_{x}+\omega(x, y) S_{y}}{R_{x}+\omega(x, y) R_{y}} \tag{2}$$

Where in the above $R_{x}, R_{y}, S_{x}, S_{y}$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y)=-\frac{y^{2}}{2 y x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_{x} &= 1 \\ R_{y} &= 0 \\ S_{x} &= \frac{y}{y x+1} \\ S_{y} &= \frac{1}{y}+\frac{x}{y x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{d S}{d R}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{d S}{d R}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

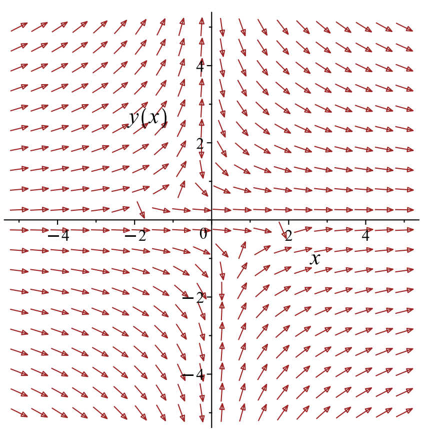
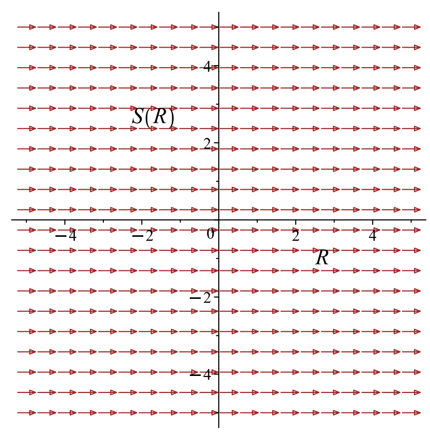
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \ln(yx + 1) = c_1$$

Which simplifies to

$$\ln(y) + \ln(yx + 1) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{2yx+1}$ 	$R = x$ $S = \ln(y) + \ln(yx + 1)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(y) + \ln(yx + 1) = c_1 \tag{1}$$

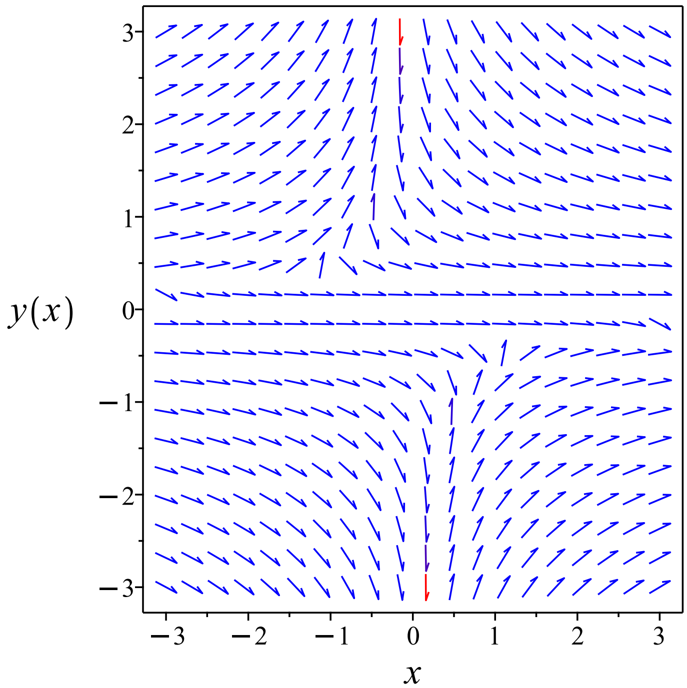


Figure 404: Slope field plot

Verification of solutions

$$\ln(y) + \ln(yx + 1) = c_1$$

Verified OK.

7.1.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2yx + 1)\,\mathrm{d}y &= \left(-y^2\right)\mathrm{d}x\\ \left(y^2\right)\mathrm{d}x + (2yx + 1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2\\ N(x, y) &= 2yx + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx + 1)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int y^2 \,\mathrm{d}x$$
$$\phi = x\,y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx + 1$. Therefore equation (4) becomes

$$2yx + 1 = 2yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (1) \,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^2 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 + y$$

Summary

The solution(s) found are the following

$$y + xy^2 = c_1 \tag{1}$$

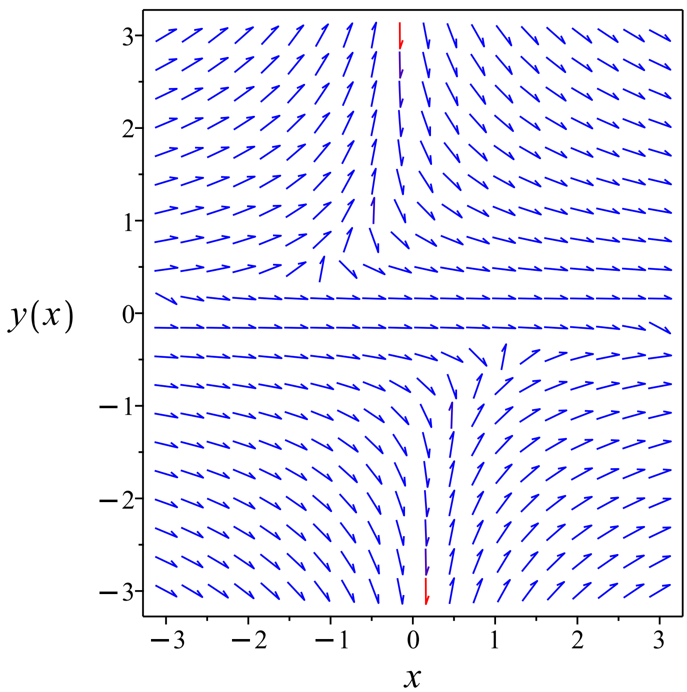


Figure 405: Slope field plot

Verification of solutions

$$y + xy^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(y(x)+(2*x+1/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-1 + \sqrt{4c_1x + 1}}{2x}$$
$$y(x) = \frac{-1 - \sqrt{4c_1x + 1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.321 (sec). Leaf size: 54

```
DSolve[y[x]+(2*x+1/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1 + \sqrt{1 + 4c_1x}}{2x}$$
$$y(x) \to \frac{-1 + \sqrt{1 + 4c_1x}}{2x}$$
$$y(x) \to 0$$

7.2 problem 2(a)

Internal problem ID [1062]
Internal file name [`OUTPUT/1063_Sunday_June_05_2022_02_01_44_AM_71876857/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y^2 + y'x^2 = 0}$$

7.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2}{x^2} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$$
$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx$$
$$-\frac{1}{y} = -\frac{1}{x} + c_1$$

Which results in

$$y = -\frac{x}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - 1} \tag{1}$$

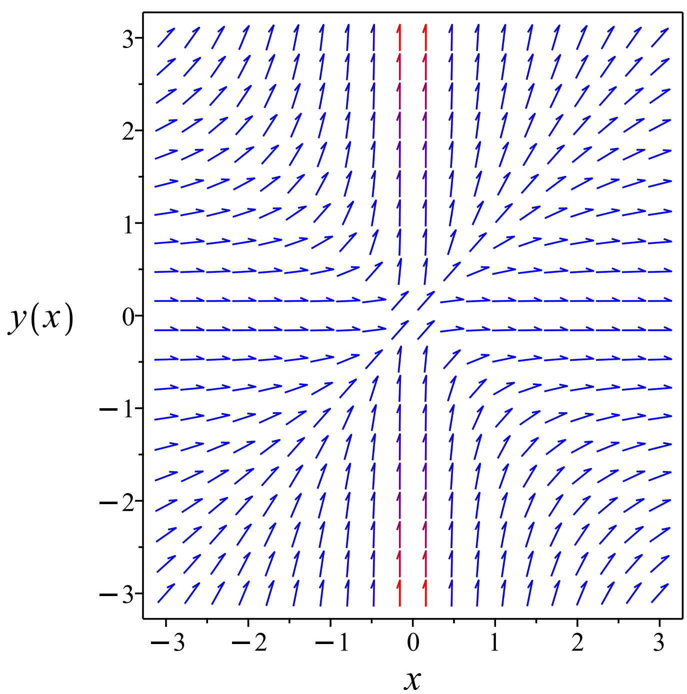


Figure 406: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1 x - 1}$$

Verified OK.

7.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)^2\,x^2 + (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(u-1)}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u\,(u-1)}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u\,(u-1)}\,du &= \int \frac{1}{x}\,dx \\
-\ln\left(u\right) + \ln\left(u-1\right) &= \ln\left(x\right) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u)+\ln(u-1)} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u-1}{u} = c_3 x$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= -\frac{x}{c_3 x - 1}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x}{c_3 x - 1} \tag{1}$$

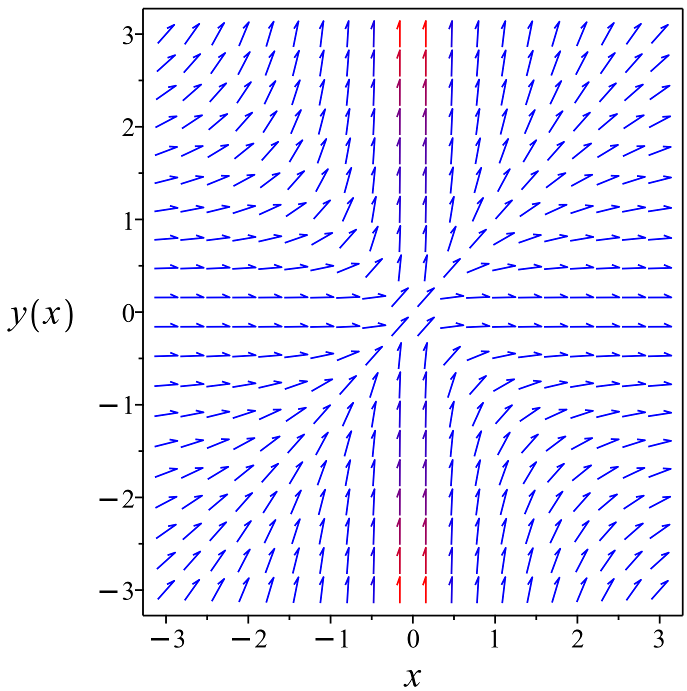


Figure 407: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_3 x - 1}$$

Verified OK.

7.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 308: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = x^2$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

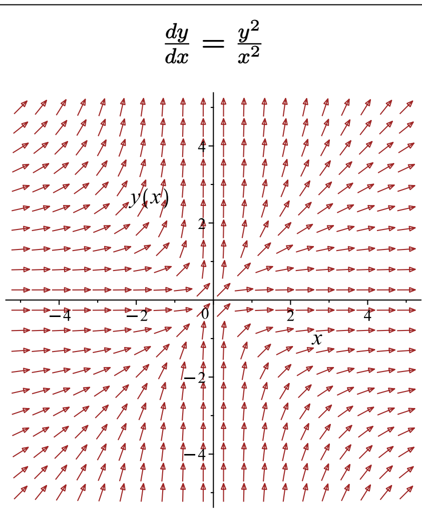
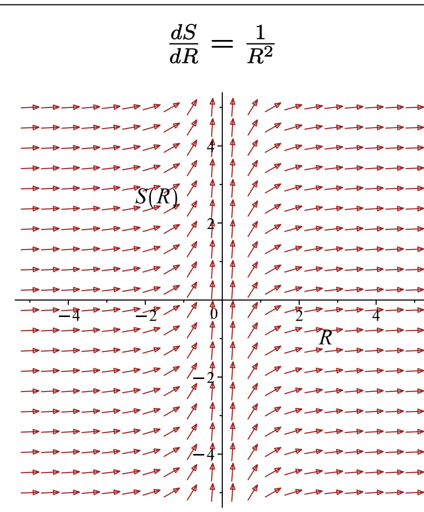
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{c_1 x + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = y$ $S = -\frac{1}{x}$	

Summary
The solution(s) found are the following

$$y = \frac{x}{c_1 x + 1} \tag{1}$$

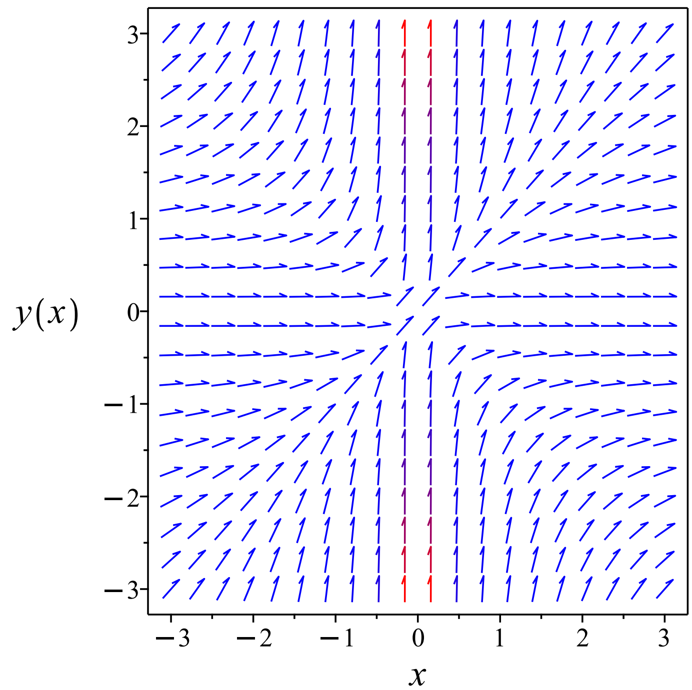


Figure 408: Slope field plot

Verification of solutions

$$y = \frac{x}{c_1 x + 1}$$

Verified OK.

7.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y&=\left(\frac{1}{x^2}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2}\right)\mathrm{d}x+\left(\frac{1}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2}\\ N(x,y)&=\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x^2} \mathrm{d}x\\
\phi &= \frac{1}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^2} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{x}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - 1} \tag{1}$$

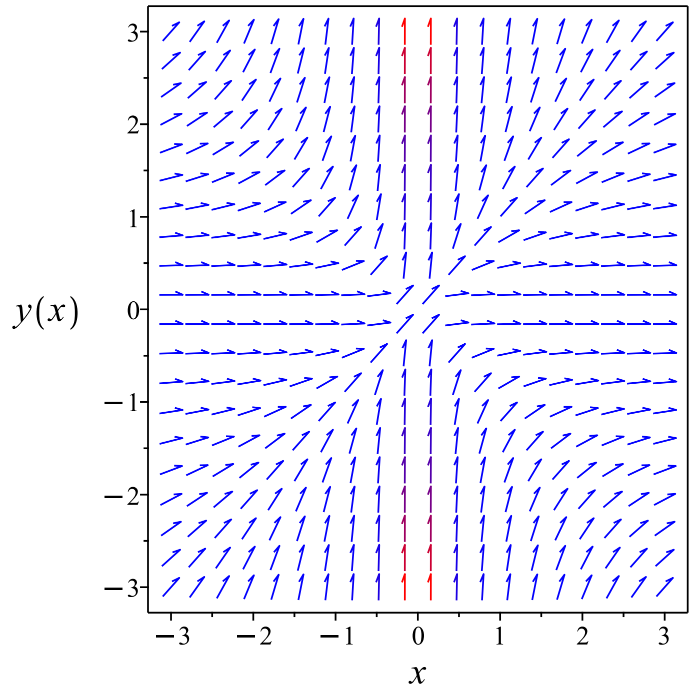


Figure 409: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1 x - 1}$$

Verified OK.

7.2.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{c_1 + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{c_3 + \frac{1}{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_3 + \frac{1}{x}} \tag{1}$$

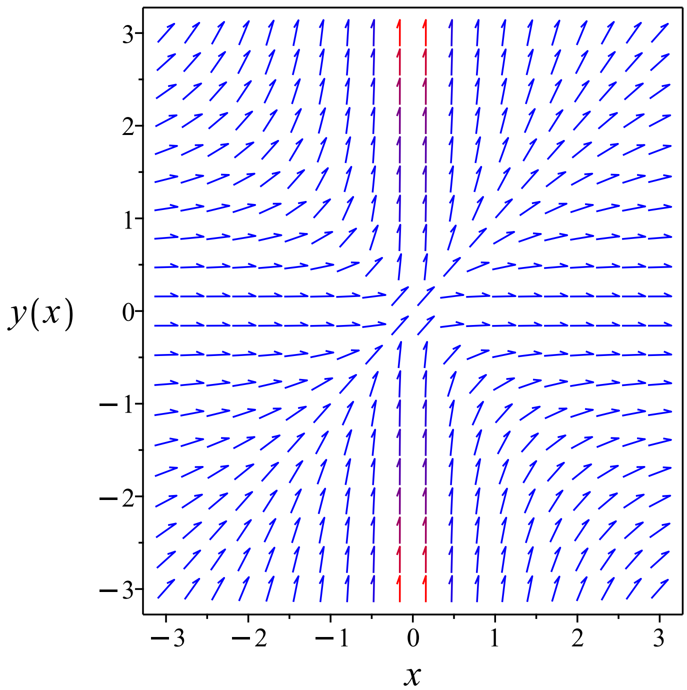


Figure 410: Slope field plot

Verification of solutions

$$y = \frac{1}{c_3 + \frac{1}{x}}$$

Verified OK.

7.2.6 Maple step by step solution

Let's solve

$-y^2 + y' x^2 = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{y^2} = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -\frac{1}{x} + c_1$

- Solve for y

  $y = -\frac{x}{c_1 x - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(-y(x)^2+x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 x + 1}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 21

```
DSolve[-y[x]^2+x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{1 - c_1 x}$$
$$y(x) \to 0$$

7.3 problem 3

Internal problem ID [1063]
Internal file name [`OUTPUT/1064_Sunday_June_05_2022_02_01_45_AM_39758295/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y'x + y = 0}$$

7.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= \mathrm{e}^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

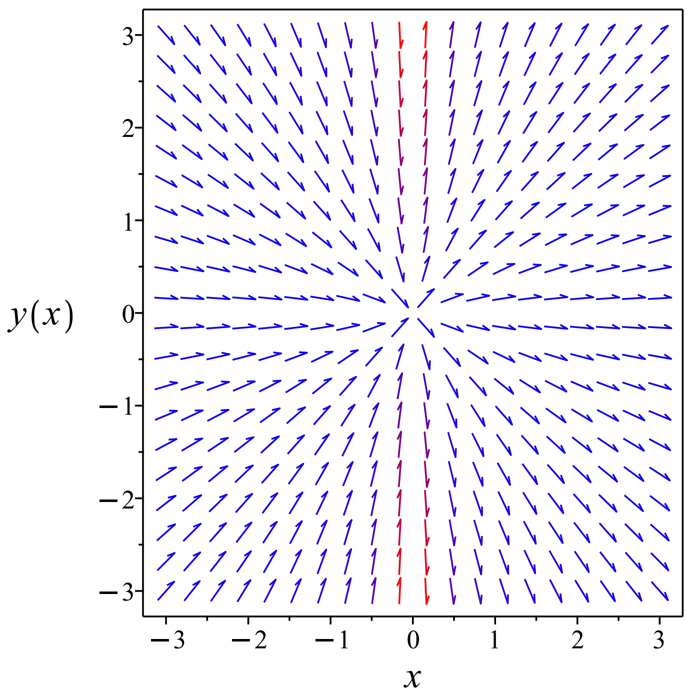


Figure 411: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

7.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

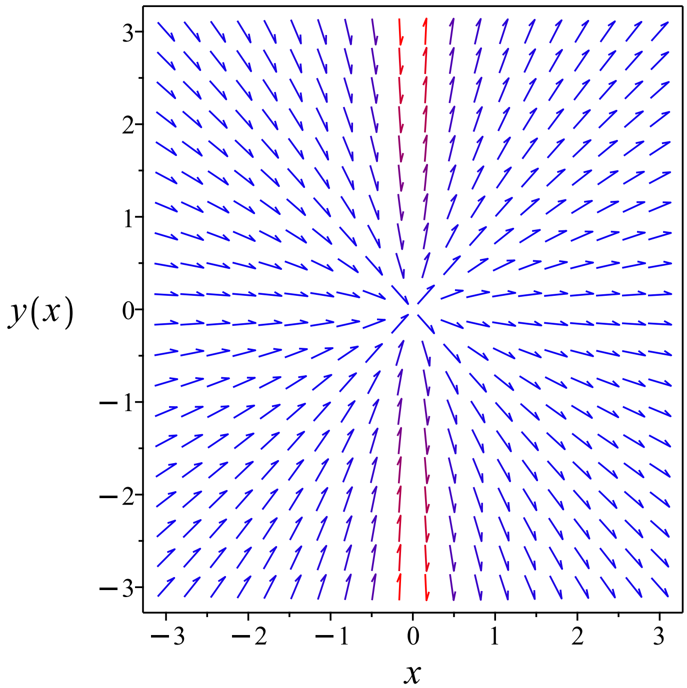


Figure 412: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

7.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-(u'(x)\, x + u(x))\, x + u(x)\, x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \, \mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x \tag{1}$$

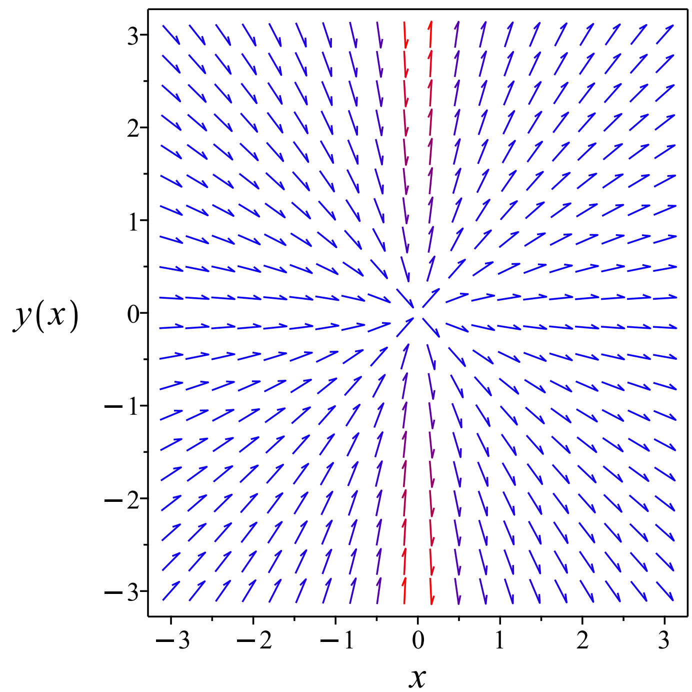


Figure 413: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

7.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 311: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

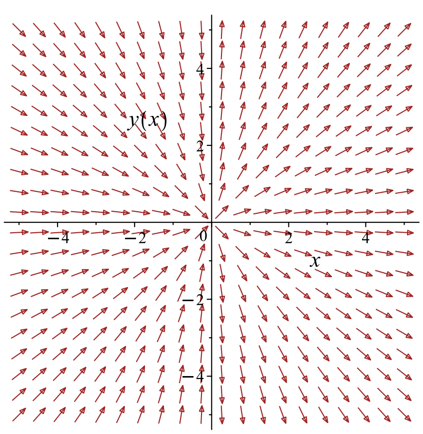
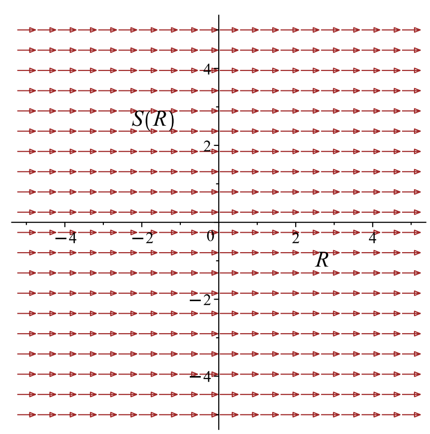
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y}{x}$
</td><td>$R = x$
$S = \frac{y}{x}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

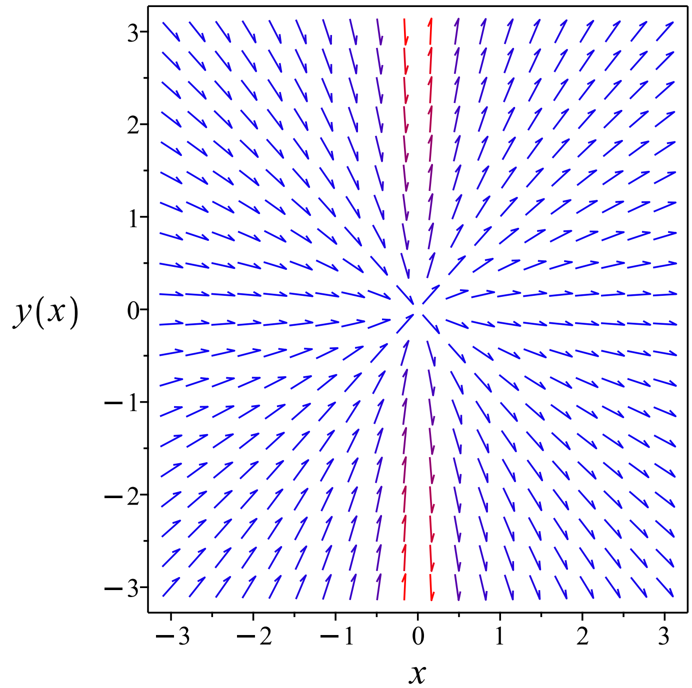


Figure 414: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

7.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) - \ln(x)$$

The solution becomes

$$y = x\,e^{c_1}$$

Summary

The solution(s) found are the following

$$y = x\,e^{c_1} \tag{1}$$

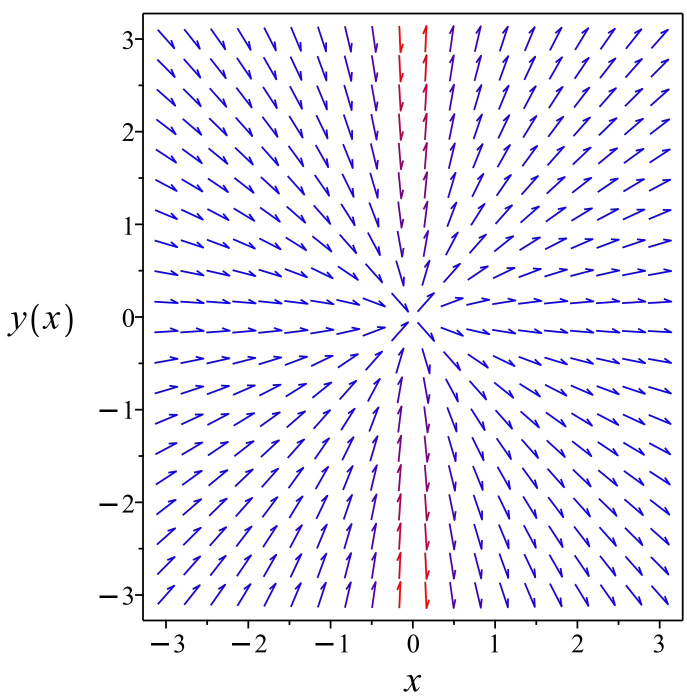


Figure 415: Slope field plot

Verification of solutions

$$y = x\,e^{c_1}$$

Verified OK.

7.3.6 Maple step by step solution

Let's solve

$-y'x + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(y(x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[y[x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

7.4 problem 4

Internal problem ID [1064]
Internal file name [`OUTPUT/1065_Sunday_June_05_2022_02_01_46_AM_94882382/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3x^2y + 2x^3y' = 0}$$

7.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{3y}{2x} \end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{3}{2x}\,dx$$
$$\int \frac{1}{y}\,dy = \int -\frac{3}{2x}\,dx$$
$$\ln(y) = -\frac{3\ln(x)}{2} + c_1$$
$$y = e^{-\frac{3\ln(x)}{2}+c_1}$$
$$= \frac{c_1}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

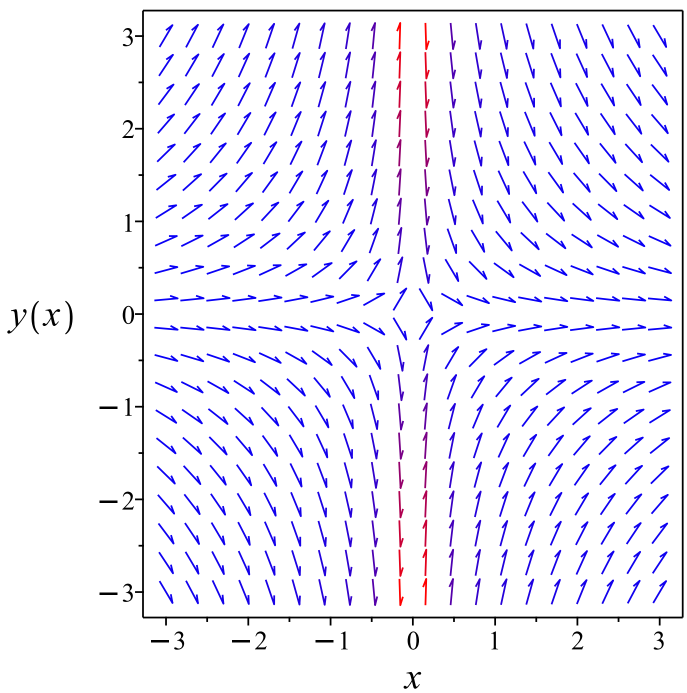


Figure 416: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

7.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{2x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{2x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{3}{2x}dx} \\ &= x^{\frac{3}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(x^{\frac{3}{2}}y\right) &= 0 \end{aligned}$$

Integrating gives

$$x^{\frac{3}{2}}y = c_1$$

Dividing both sides by the integrating factor $\mu = x^{\frac{3}{2}}$ results in

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

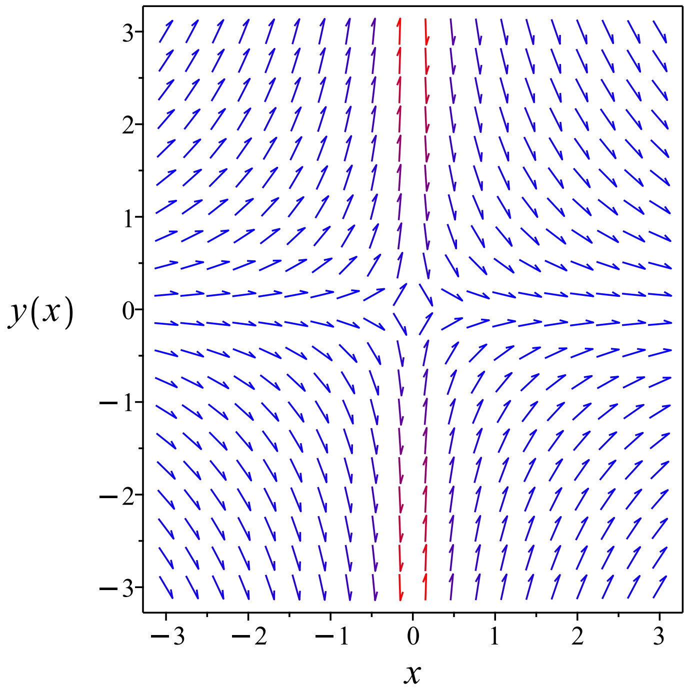


Figure 417: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

7.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$3x^3 u(x) + 2x^3 (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\
\ln(u) &= -\frac{5\ln(x)}{2} + c_2 \\
u &= e^{-\frac{5\ln(x)}{2}+c_2} \\
&= \frac{c_2}{x^{\frac{5}{2}}}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \frac{c_2}{x^{\frac{3}{2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x^{\frac{3}{2}}} \tag{1}
$$

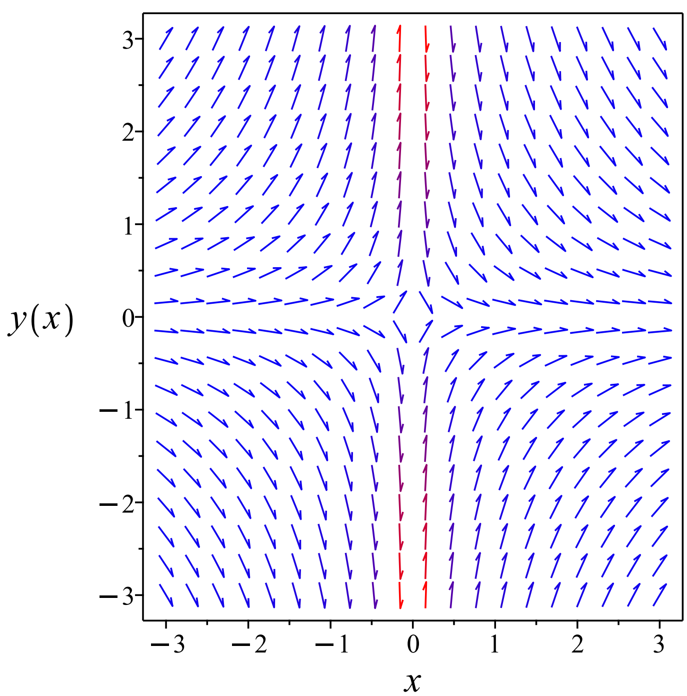


Figure 418: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^{\frac{3}{2}}}$$

Verified OK.

7.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 314: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^{\frac{3}{2}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^{\frac{3}{2}}}} dy\end{aligned}$$

Which results in

$$S = x^{\frac{3}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3y\sqrt{x}}{2} \\ S_y &= x^{\frac{3}{2}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^{\frac{3}{2}} = c_1$$

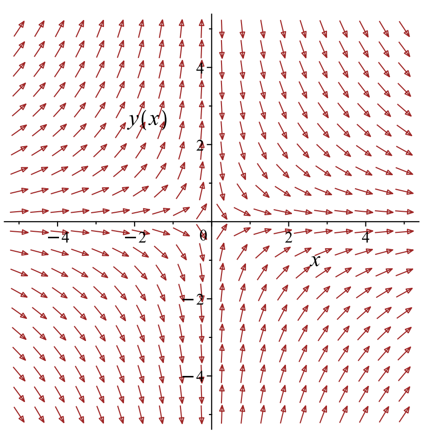
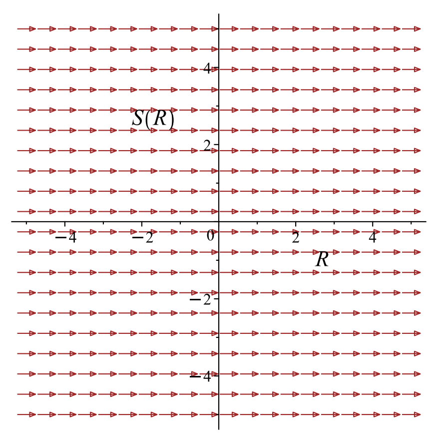
Which simplifies to

$$y x^{\frac{3}{2}} = c_1$$

Which gives

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{3y}{2x}$
</td><td>$R = x$
$S = x^{\frac{3}{2}} y$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} \tag{1}$$

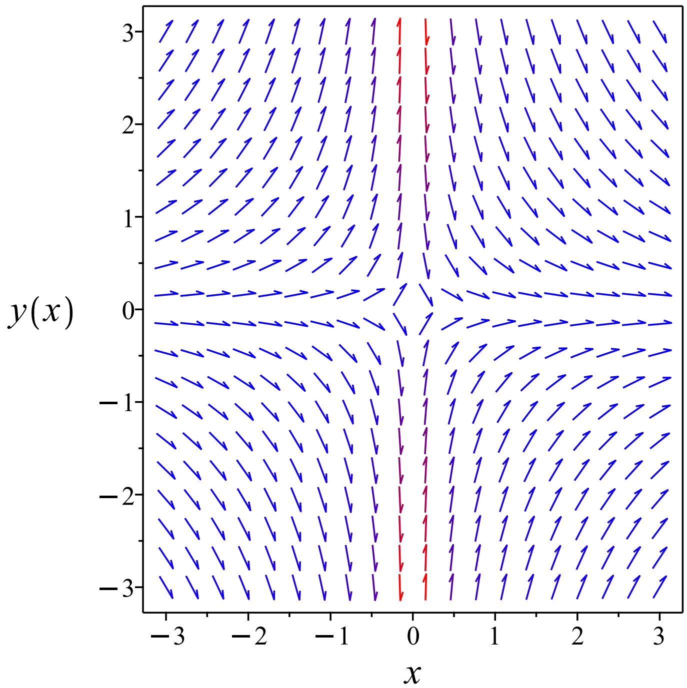


Figure 419: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}}$$

Verified OK.

7.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2}{3y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{2}{3y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= -\frac{2}{3y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2}{3y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2}{3y}$. Therefore equation (4) becomes

$$-\frac{2}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{3y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{2}{3y}\right) \mathrm{d}y$$
$$f(y) = -\frac{2\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{2\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{2\ln(y)}{3}$$

The solution becomes

$$y = e^{-\frac{3\ln(x)}{2} - \frac{3c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{3\ln(x)}{2} - \frac{3c_1}{2}} \tag{1}$$

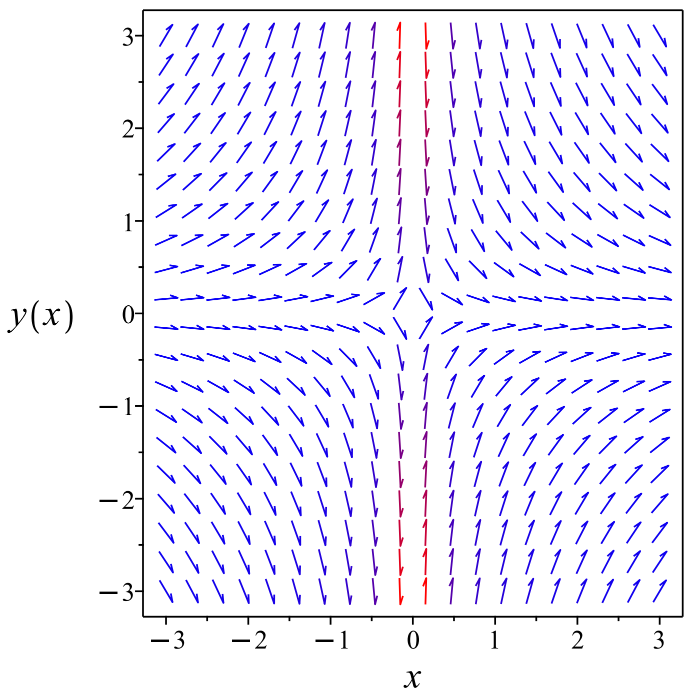


Figure 420: Slope field plot

Verification of solutions

$$y = e^{-\frac{3\ln(x)}{2}-\frac{3c_1}{2}}$$

Verified OK.

7.4.6 Maple step by step solution

Let's solve

$3x^2y + 2x^3y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{3}{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{3}{2x}dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{3\ln(x)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{3\ln(x)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(3*x^2*y(x)+2*x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[3*x^2*y[x]+2*x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^{3/2}}$$
$$y(x) \to 0$$

7.5 problem 5

Internal problem ID [1065]
Internal file name [`OUTPUT/1066_Sunday_June_05_2022_02_01_47_AM_56885570/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{2y^3 + 3y^2 y' = 0}$$

7.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{3}{2y} dy = \int dx$$
$$-\frac{3\ln(y)}{2} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y^{\frac{3}{2}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{y^{\frac{3}{2}}} = c_2 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{1}{(c_2 e^x)^{\frac{2}{3}}} \tag{1}$$

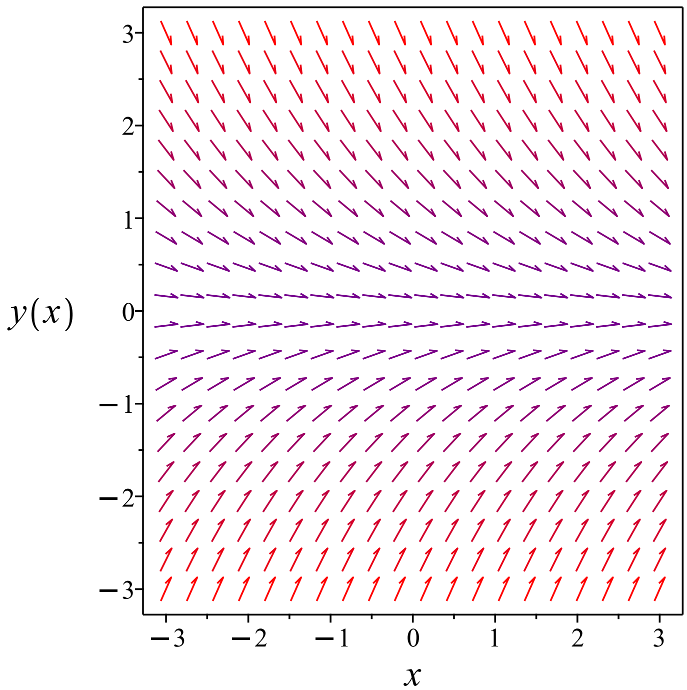


Figure 421: Slope field plot

Verification of solutions

$$y = \frac{1}{(c_2 e^x)^{\frac{2}{3}}}$$

Verified OK.

7.5.2 Maple step by step solution

Let's solve

$2y^3 + 3y^2 y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{y} = -\frac{2}{3}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int -\frac{2}{3} dx + c_1$$

- Evaluate integral

$$\ln(y) = -\frac{2x}{3} + c_1$$

- Solve for y

$$y = e^{-\frac{2x}{3}+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(2*y(x)^3+3*y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1 e^{-\frac{2x}{3}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 25

```
DSolve[2*y[x]^3+3*y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to c_1 e^{-2x/3}$$
$$y(x) \to 0$$

7.6 problem 6

Internal problem ID [1066]
Internal file name [`OUTPUT/1067_Sunday_June_05_2022_02_01_48_AM_58527605/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{5yx + 2y + 2y'x = -5}$$

7.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-5x - 2}{2x}$$
$$q(x) = -\frac{5}{2x}$$

Hence the ode is

$$y' - \frac{(-5x - 2)\,y}{2x} = -\frac{5}{2x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-5x-2}{2x}dx}$$
$$= e^{\frac{5x}{2}+\ln(x)}$$

Which simplifies to

$$\mu = x\,e^{\frac{5x}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{5}{2x}\right)$$
$$\frac{d}{dx}\left(x\,e^{\frac{5x}{2}}y\right) = \left(x\,e^{\frac{5x}{2}}\right)\left(-\frac{5}{2x}\right)$$
$$d\left(x\,e^{\frac{5x}{2}}y\right) = \left(-\frac{5\,e^{\frac{5x}{2}}}{2}\right)dx$$

Integrating gives

$$x\,e^{\frac{5x}{2}}y = \int -\frac{5\,e^{\frac{5x}{2}}}{2}\,dx$$
$$x\,e^{\frac{5x}{2}}y = -e^{\frac{5x}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = x\,e^{\frac{5x}{2}}$ results in

$$y = -\frac{e^{-\frac{5x}{2}}e^{\frac{5x}{2}}}{x} + \frac{c_1 e^{-\frac{5x}{2}}}{x}$$

which simplifies to

$$y = \frac{c_1 e^{-\frac{5x}{2}} - 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{5x}{2}} - 1}{x} \tag{1}$$

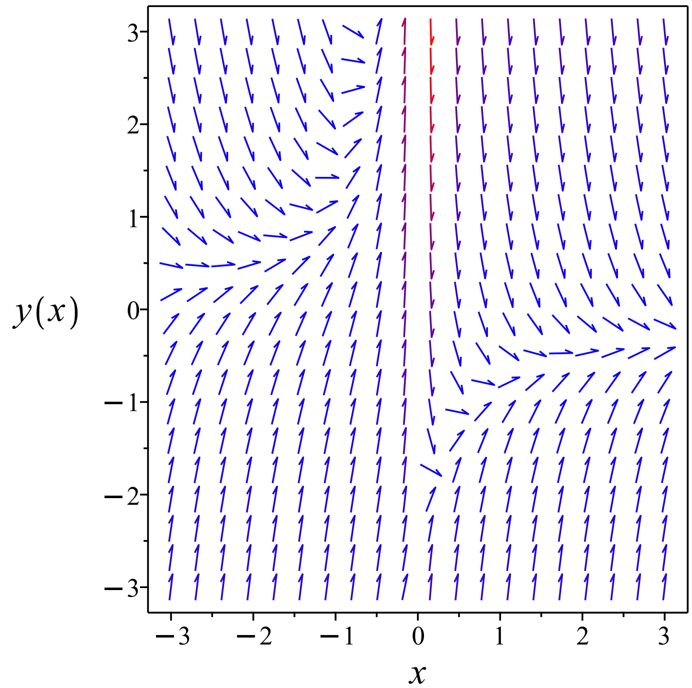


Figure 422: Slope field plot

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{5x}{2}} - 1}{x}$$

Verified OK.

7.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{5yx + 2y + 5}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 318: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-\frac{5x}{2}-\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-\frac{5x}{2}-\ln(x)}} dy
\end{aligned}$$

Which results in

$$S = x\,\mathrm{e}^{\frac{5x}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{5yx + 2y + 5}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \mathrm{e}^{\frac{5x}{2}} y \left(\frac{5x}{2} + 1\right) \\
S_y &= x\,\mathrm{e}^{\frac{5x}{2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{5\,\mathrm{e}^{\frac{5x}{2}}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{5\,\mathrm{e}^{\frac{5R}{2}}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{\frac{5R}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x\,\mathrm{e}^{\frac{5x}{2}} y = -\mathrm{e}^{\frac{5x}{2}} + c_1$$

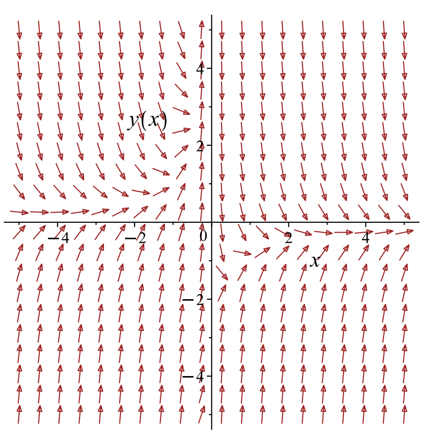
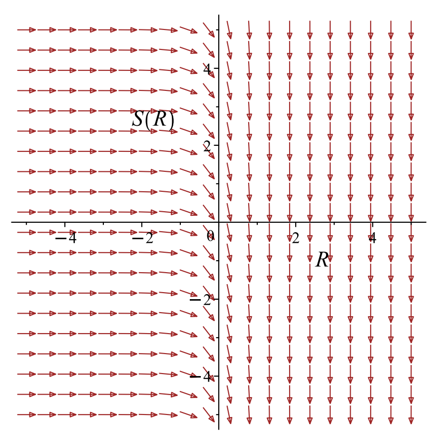
Which simplifies to

$$x\,\mathrm{e}^{\frac{5x}{2}} y = -\mathrm{e}^{\frac{5x}{2}} + c_1$$

Which gives

$$y = -\frac{\left(\mathrm{e}^{\frac{5x}{2}} - c_1\right)\mathrm{e}^{-\frac{5x}{2}}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{5yx+2y+5}{2x}$ 	$R = x$ $S = x\,\mathrm{e}^{\frac{5x}{2}} y$	$\frac{dS}{dR} = -\frac{5\,\mathrm{e}^{\frac{5R}{2}}}{2}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\left(e^{\frac{5x}{2}} - c_1\right) e^{-\frac{5x}{2}}}{x} \tag{1}$$

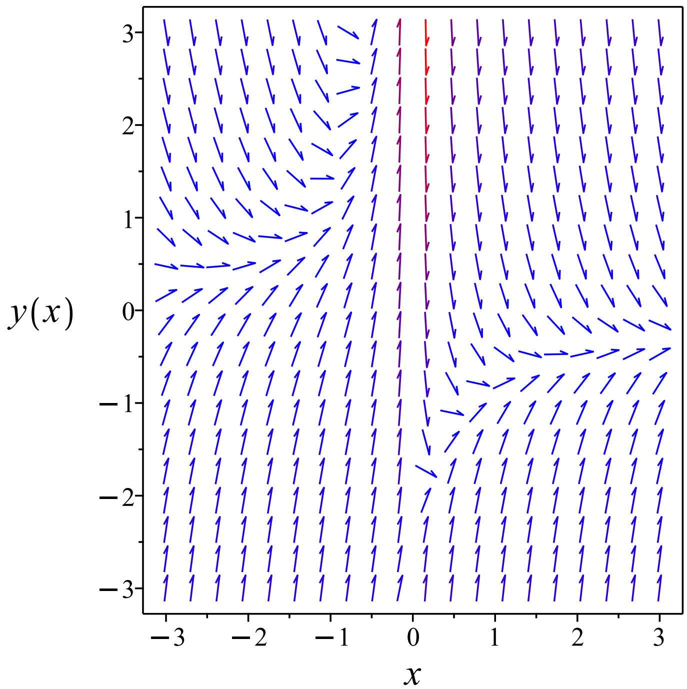


Figure 423: Slope field plot

Verification of solutions

$$y = -\frac{\left(e^{\frac{5x}{2}} - c_1\right) e^{-\frac{5x}{2}}}{x}$$

Verified OK.

7.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x)\,\mathrm{d}y &= (-5yx - 2y - 5)\,\mathrm{d}x \\ (5yx + 2y + 5)\,\mathrm{d}x + (2x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 5yx + 2y + 5 \\ N(x,y) &= 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(5yx+2y+5)\\ &= 5x+2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x}((5x+2)-(2))\\ &= \frac{5}{2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{5}{2}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{5x}{2}}\\ &= \mathrm{e}^{\frac{5x}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\frac{5x}{2}}(5yx+2y+5)\\ &= (5+(5x+2)\,y)\,\mathrm{e}^{\frac{5x}{2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{\frac{5x}{2}}(2x)\\ &= 2x\,\mathrm{e}^{\frac{5x}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\left(5+\left(5x+2\right)y\right)\mathrm{e}^{\frac{5x}{2}}\right) + \left(2x\,\mathrm{e}^{\frac{5x}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \left(5+\left(5x+2\right)y\right)\mathrm{e}^{\frac{5x}{2}}\,\mathrm{d}x$$
$$\phi = 2\,\mathrm{e}^{\frac{5x}{2}}\left(yx+1\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x\,\mathrm{e}^{\frac{5x}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x\,\mathrm{e}^{\frac{5x}{2}}$. Therefore equation (4) becomes

$$2x\,\mathrm{e}^{\frac{5x}{2}} = 2x\,\mathrm{e}^{\frac{5x}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\,\mathrm{e}^{\frac{5x}{2}}\left(yx+1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\,\mathrm{e}^{\frac{5x}{2}}(yx+1)$$

The solution becomes

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}} - c_1\right)\mathrm{e}^{-\frac{5x}{2}}}{2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}} - c_1\right)\mathrm{e}^{-\frac{5x}{2}}}{2x} \tag{1}$$

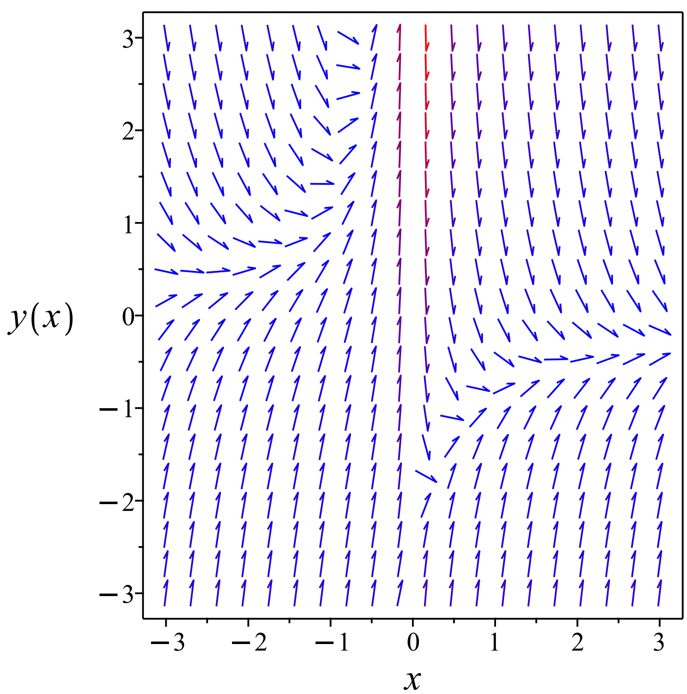


Figure 424: Slope field plot

Verification of solutions

$$y = -\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}} - c_1\right)\mathrm{e}^{-\frac{5x}{2}}}{2x}$$

Verified OK.

7.6.4 Maple step by step solution

Let's solve

$5yx + 2y + 2y'x = -5$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(5x+2)y}{2x} - \frac{5}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(5x+2)y}{2x} = -\frac{5}{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(5x+2)y}{2x}\right) = -\frac{5\mu(x)}{2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{(5x+2)y}{2x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(5x+2)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = x\,\mathrm{e}^{\frac{5x}{2}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{5\mu(x)}{2x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{5\mu(x)}{2x} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{5\mu(x)}{2x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x\mathrm{e}^{\frac{5x}{2}}$

  $y = \frac{\int -\frac{5\,\mathrm{e}^{\frac{5x}{2}}}{2} dx + c_1}{x\,\mathrm{e}^{\frac{5x}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\mathrm{e}^{\frac{5x}{2}} + c_1}{x\,\mathrm{e}^{\frac{5x}{2}}}$

- Simplify

  $$y = \frac{c_1 e^{-\frac{5x}{2}} - 1}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((5*x*y(x)+2*y(x)+5)+(2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-\frac{5x}{2}} c_1 - 1}{x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 21

```
DSolve[(5*x*y[x]+2*y[x]+5)+(2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-1 + c_1 e^{-5x/2}}{x}$$

7.7 problem 7

Internal problem ID [1067]
Internal file name [`OUTPUT/1068_Sunday_June_05_2022_02_01_49_AM_77231025/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{yx + 2y + (x + 1)\, y' = -x - 1}$$

7.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-x - 2}{x + 1}$$
$$q(x) = -1$$

Hence the ode is

$$y' - \frac{(-x - 2)\, y}{x + 1} = -1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-x-2}{x+1}dx}$$
$$= \mathrm{e}^{x+\ln(x+1)}$$

Which simplifies to

$$\mu = (x+1)\,\mathrm{e}^{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\,(-1)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y(x+1)) = ((x+1)\,\mathrm{e}^{x})\,(-1)$$
$$\mathrm{d}(\mathrm{e}^{x} y(x+1)) = (-(x+1)\,\mathrm{e}^{x})\;\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{x} y(x+1) = \int -(x+1)\,\mathrm{e}^{x}\,\mathrm{d}x$$
$$\mathrm{e}^{x} y(x+1) = -x\,\mathrm{e}^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = (x+1)\,\mathrm{e}^{x}$ results in

$$y = -\frac{\mathrm{e}^{-x} x\,\mathrm{e}^{x}}{x+1} + \frac{c_1 \mathrm{e}^{-x}}{x+1}$$

which simplifies to

$$y = \frac{c_1 \mathrm{e}^{-x} - x}{x+1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-x} - x}{x+1} \tag{1}$$

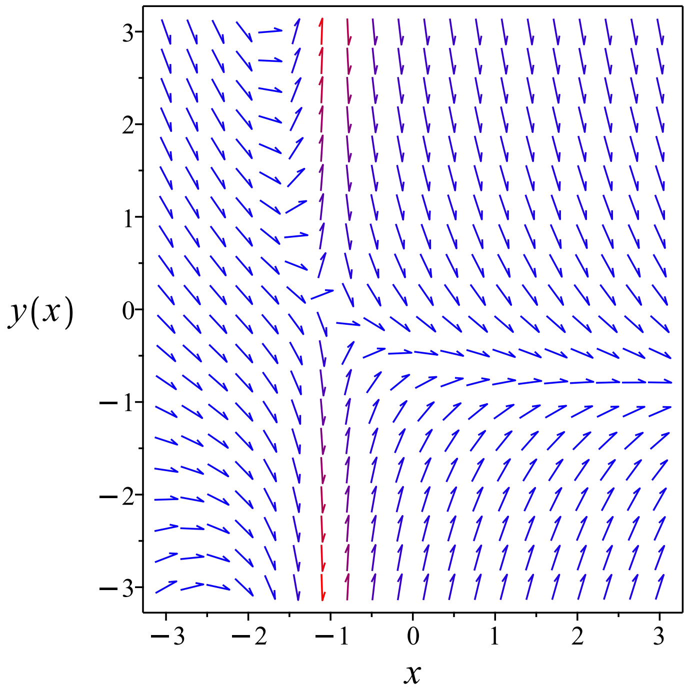


Figure 425: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-x} - x}{x + 1}$$

Verified OK.

7.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{yx + x + 2y + 1}{x + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 321: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-x-\ln(x+1)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x-\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y(x+1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{yx + x + 2y + 1}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y(2+x) \\ S_y &= (x+1)\,\mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{x}(-x-1) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R}(-R-1)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y(x+1) = -x\,\mathrm{e}^{x} + c_1$$

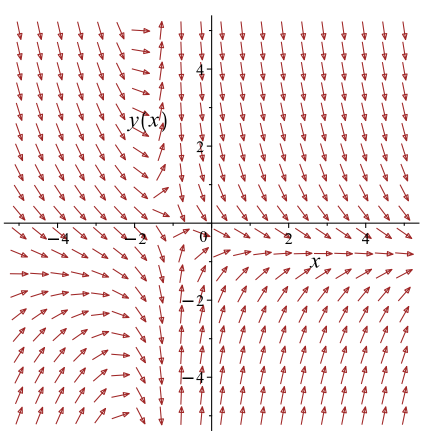
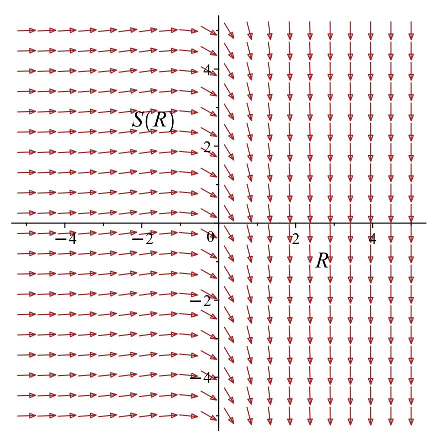
Which simplifies to

$$\mathrm{e}^{x} y(x+1) = -x\,\mathrm{e}^{x} + c_1$$

Which gives

$$y = -\frac{(x\,\mathrm{e}^{x} - c_1)\,\mathrm{e}^{-x}}{x+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx+x+2y+1}{x+1}$ 	$R = x$ $S = \mathrm{e}^{x} y(x+1)$	$\frac{dS}{dR} = \mathrm{e}^{R}(-R-1)$ 

Summary

The solution(s) found are the following

$$y = -\frac{(x\,\mathrm{e}^{x} - c_1)\,\mathrm{e}^{-x}}{x+1} \tag{1}$$

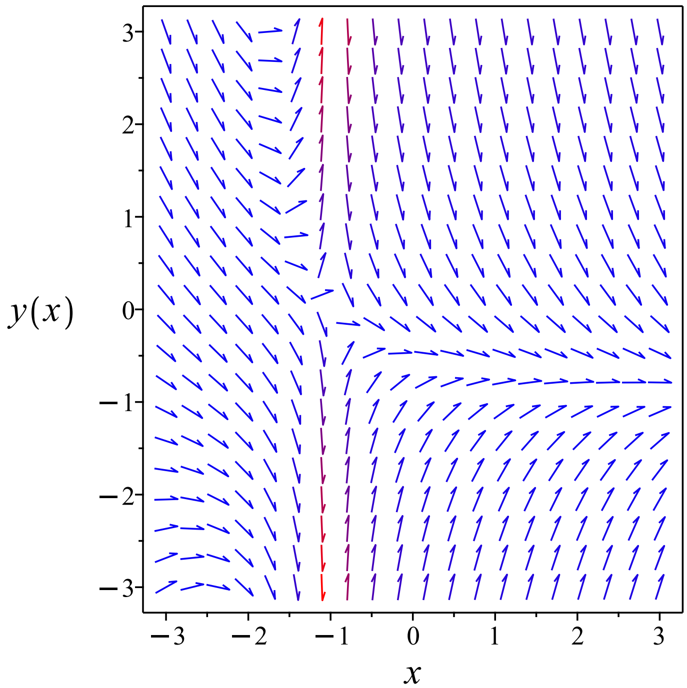


Figure 426: Slope field plot

Verification of solutions

$$y = -\frac{(x\,e^{x} - c_1)\,e^{-x}}{x+1}$$

Verified OK.

7.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x + 1)\, \mathrm{d}y = (-yx - x - 2y - 1)\, \mathrm{d}x$$
$$(yx + x + 2y + 1)\, \mathrm{d}x + (x + 1)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = yx + x + 2y + 1$$
$$N(x, y) = x + 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(yx + x + 2y + 1)$$
$$= 2 + x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x + 1)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x+1}((2+x) - (1)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(yx + x + 2y + 1) \\
&= \mathrm{e}^{x}(yx + x + 2y + 1)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(x+1) \\
&= (x+1)\,\mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\mathrm{e}^{x}(yx + x + 2y + 1)) + ((x+1)\,\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{x}(yx + x + 2y + 1) \, \mathrm{d}x$$
$$\phi = (yx + x + y) \, \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x+1) \, \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x+1) \, \mathrm{e}^{x}$. Therefore equation (4) becomes

$$(x+1) \, \mathrm{e}^{x} = (x+1) \, \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (yx + x + y) \, \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (yx + x + y) \, \mathrm{e}^{x}$$

The solution becomes

$$y = -\frac{(x\,e^{x} - c_1)\,e^{-x}}{x+1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{(x\,e^{x} - c_1)\,e^{-x}}{x+1} \tag{1}$$

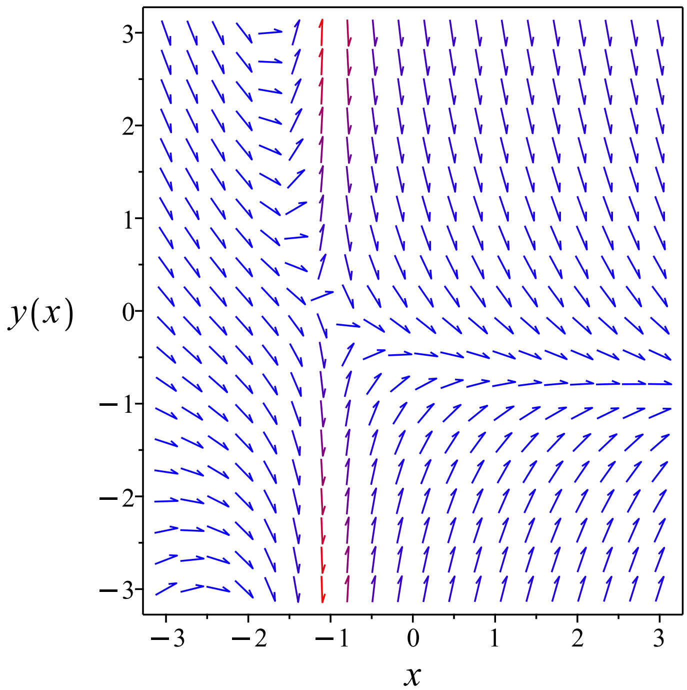


Figure 427: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{(x\,e^{x} - c_1)\,e^{-x}}{x+1}$$

Verified OK.

7.7.4 Maple step by step solution

Let's solve

$yx + 2y + (x+1)\,y' = -x - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -1 - \frac{(2+x)y}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(2+x)y}{x+1} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(2+x)y}{x+1}\right) = -\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{(2+x)y}{x+1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(2+x)}{x+1}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x+\ln(x+1)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x+\ln(x+1)}$

  $y = \frac{\int -\mathrm{e}^{x+\ln(x+1)}dx + c_1}{\mathrm{e}^{x+\ln(x+1)}}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{x\,\mathrm{e}^{x+\ln(x+1)}}{x+1} + c_1}{\mathrm{e}^{x+\ln(x+1)}}$

- Simplify

$y = \frac{c_1 e^{-x} - x}{x+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve((x*y(x)+x+2*y(x)+1)+(x+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-x} c_1 - x}{x+1}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 23

```
DSolve[(x*y[x]+x+2*y[x]+1)+(x+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x - c_1 e^{-x}}{x+1}$$

7.8 problem 8

Internal problem ID [1068]
Internal file name [`OUTPUT/1069_Sunday_June_05_2022_02_01_50_AM_54127861/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{27xy^2 + 8y^3 + \left(18x^2y + 12xy^2\right) y' = 0}$$

7.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$27x^3u(x)^2 + 8u(x)^3\, x^3 + \left(18x^3u(x) + 12x^3u(x)^2\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u(4u+9)}{6x\,(2u+3)} \end{aligned}$$

Where $f(x) = -\frac{5}{6x}$ and $g(u) = \frac{u(4u+9)}{2u+3}$. Integrating both sides gives

$$\frac{1}{\frac{u(4u+9)}{2u+3}} du = -\frac{5}{6x} dx$$

$$\int \frac{1}{\frac{u(4u+9)}{2u+3}} du = \int -\frac{5}{6x} dx$$

$$\frac{\ln (4u+9)}{6} + \frac{\ln (u)}{3} = -\frac{5 \ln (x)}{6} + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(4u+9)}{6} + \frac{\ln(u)}{3}} = e^{-\frac{5\ln(x)}{6} + c_2}$$

Which simplifies to

$$(4u+9)^{\frac{1}{6}} u^{\frac{1}{3}} = \frac{c_3}{x^{\frac{5}{6}}}$$

The solution is

$$(4u(x)+9)^{\frac{1}{6}} u(x)^{\frac{1}{3}} = \frac{c_3}{x^{\frac{5}{6}}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{4y}{x} + 9\right)^{\frac{1}{6}} \left(\frac{y}{x}\right)^{\frac{1}{3}} = \frac{c_3}{x^{\frac{5}{6}}}$$

$$\left(\frac{4y+9x}{x}\right)^{\frac{1}{6}} \left(\frac{y}{x}\right)^{\frac{1}{3}} = \frac{c_3}{x^{\frac{5}{6}}}$$

Summary

The solution(s) found are the following

$$\left(\frac{4y+9x}{x}\right)^{\frac{1}{6}} \left(\frac{y}{x}\right)^{\frac{1}{3}} = \frac{c_3}{x^{\frac{5}{6}}} \tag{1}$$

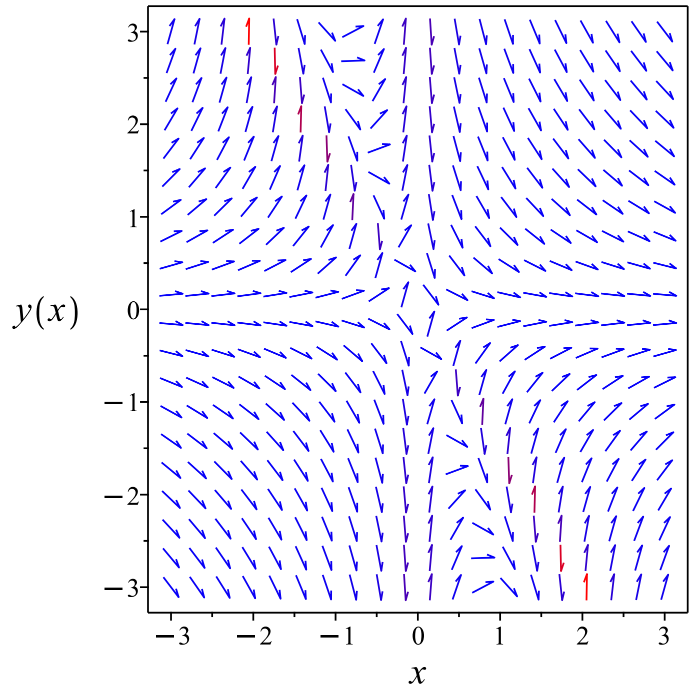


Figure 428: Slope field plot

Verification of solutions

$$\left(\frac{4y+9x}{x}\right)^{\frac{1}{6}}\left(\frac{y}{x}\right)^{\frac{1}{3}}=\frac{c_3}{x^{\frac{5}{6}}}$$

Verified OK.

7.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(27x+8y)}{6x\,(3x+2y)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(27x+8y)\,(b_3-a_2)}{6x\,(3x+2y)} - \frac{y^2(27x+8y)^2\,a_3}{36x^2\,(3x+2y)^2} \\ & - \left(-\frac{9y}{2x\,(3x+2y)} + \frac{y(27x+8y)}{6x^2\,(3x+2y)} + \frac{y(27x+8y)}{2x\,(3x+2y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{27x+8y}{6\,(3x+2y)\,x} - \frac{4y}{3x\,(3x+2y)} + \frac{y(27x+8y)}{3x\,(3x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{810x^4b_2 + 720x^3yb_2 + 180x^2y^2a_2 - 1215x^2y^2a_3 + 240x^2y^2b_2 - 180x^2y^2b_3 - 720x\,y^3a_3 - 160y^4a_3 + 486x^3}{36x^2\,(3x+2y)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 810x^4b_2 + 720x^3yb_2 + 180x^2y^2a_2 - 1215x^2y^2a_3 + 240x^2y^2b_2 \\ & - 180x^2y^2b_3 - 720x\,y^3a_3 - 160y^4a_3 + 486x^3b_1 - 486x^2ya_1 \\ & + 288x^2yb_1 - 288x\,y^2a_1 + 96x\,y^2b_1 - 96y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 180a_2v_1^2v_2^2 - 1215a_3v_1^2v_2^2 - 720a_3v_1v_2^3 - 160a_3v_2^4 + 810b_2v_1^4 \\ & + 720b_2v_1^3v_2 + 240b_2v_1^2v_2^2 - 180b_3v_1^2v_2^2 - 486a_1v_1^2v_2 \\ & - 288a_1v_1v_2^2 - 96a_1v_2^3 + 486b_1v_1^3 + 288b_1v_1^2v_2 + 96b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&810b_2v_1^4 + 720b_2v_1^3v_2 + 486b_1v_1^3 + (180a_2 - 1215a_3 + 240b_2 - 180b_3)\, v_1^2v_2^2 \\ &+ (-486a_1 + 288b_1)\, v_1^2v_2 - 720a_3v_1v_2^3 \\ &+ (-288a_1 + 96b_1)\, v_1v_2^2 - 160a_3v_2^4 - 96a_1v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-96a_1 &= 0\\ -720a_3 &= 0\\ -160a_3 &= 0\\ 486b_1 &= 0\\ 720b_2 &= 0\\ 810b_2 &= 0\\ -486a_1 + 288b_1 &= 0\\ -288a_1 + 96b_1 &= 0\\ 180a_2 - 1215a_3 + 240b_2 - 180b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(27x+8y)}{6x\,(3x+2y)}\right)(x) \\
&= \frac{45yx+20y^2}{18x+12y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{45yx+20y^2}{18x+12y}}dy
\end{aligned}
$$

Which results in

$$
S = \frac{2\ln(y)}{5} + \frac{\ln(9x+4y)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(27x+8y)}{6x\,(3x+2y)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{9}{45x + 20y} \\
S_y &= \frac{18x + 12y}{45yx + 20y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{5x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln(R)}{5} + c_1 \tag{4}$$

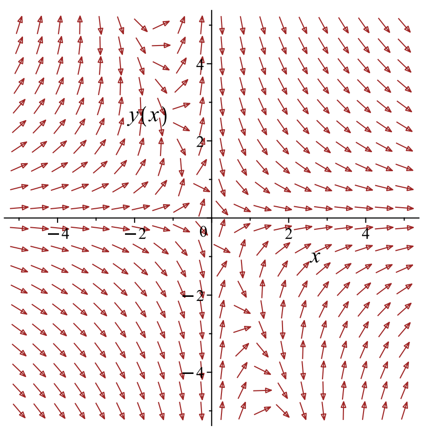
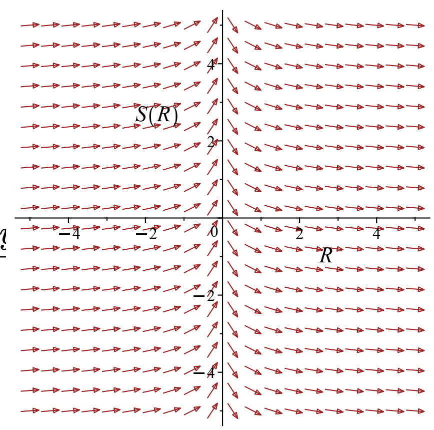
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)}{5} + \frac{\ln(4y + 9x)}{5} = -\frac{2\ln(x)}{5} + c_1$$

Which simplifies to

$$\frac{2\ln(y)}{5} + \frac{\ln(4y + 9x)}{5} = -\frac{2\ln(x)}{5} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(27x+8y)}{6x(3x+2y)}$ 	$R = x$ $S = \frac{2\ln(y)}{5} + \frac{\ln(9x+4y)}{5}$	$\frac{dS}{dR} = -\frac{2}{5R}$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y)}{5} + \frac{\ln(4y+9x)}{5} = -\frac{2\ln(x)}{5} + c_1 \tag{1}$$

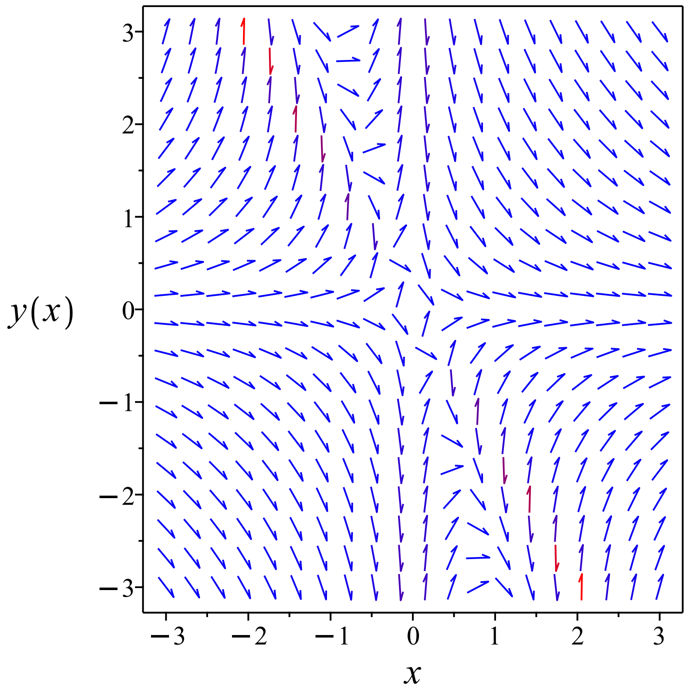


Figure 429: Slope field plot

Verification of solutions

$$\frac{2\ln(y)}{5}+\frac{\ln(4y+9x)}{5}=-\frac{2\ln(x)}{5}+c_1$$

Verified OK.

7.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(18y\,x^2 + 12x\,y^2\right)\mathrm{d}y = \left(-27x\,y^2 - 8y^3\right)\mathrm{d}x$$
$$\left(27x\,y^2 + 8y^3\right)\mathrm{d}x + \left(18y\,x^2 + 12x\,y^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 27x\,y^2 + 8y^3$$
$$N(x,y) = 18y\,x^2 + 12x\,y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(27x\,y^2 + 8y^3\right)$$
$$= 54yx + 24y^2$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(18y\,x^2 + 12x\,y^2\right)$$
$$= 36yx + 12y^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{18y\,x^2 + 12x\,y^2}\left(\left(54yx + 24y^2\right) - \left(36yx + 12y^2\right)\right)\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(27x\,y^2 + 8y^3\right)\\ &= y^2(27x + 8y)\,x\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x\left(18y\,x^2 + 12x\,y^2\right)\\ &= 6y\,x^2(3x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(y^2(27x + 8y)\,x\right) + \left(6y\,x^2(3x + 2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2(27x+8y)\, x \, \mathrm{d}x$$

$$\phi = y^2x^2(9x+4y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\, x^2(9x+4y) + 4x^2y^2 + f'(y) \tag{4}$$

$$= 6y\, x^2(3x+2y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 6y\, x^2(3x+2y)$. Therefore equation (4) becomes

$$6y\, x^2(3x+2y) = 6y\, x^2(3x+2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2x^2(9x+4y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2x^2(9x+4y)$$

Summary

The solution(s) found are the following

$$y^2x^2(4y + 9x) = c_1 \tag{1}$$

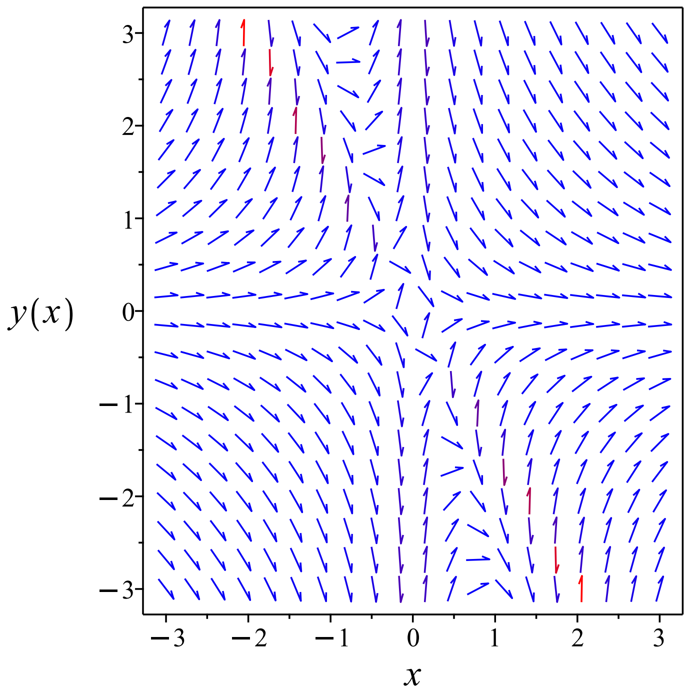


Figure 430: Slope field plot

Verification of solutions

$$y^2x^2(4y + 9x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 1355

```
dsolve((27*x*y(x)^2+8*y(x)^3)+(18*x^2*y(x)+12*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = -\frac{2x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{i\left(-3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}\right)\sqrt{3}+3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = -\frac{2x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{\left(1-i\sqrt{3}\right)\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}+3c_1\left(1+i\sqrt{3}\right)x^5}$$

$$y(x) = -\frac{2x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{i\left(-3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}\right)\sqrt{3}+3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

$$y(x) = -\frac{2x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{\left(1-i\sqrt{3}\right)\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}+3c_1\left(1+i\sqrt{3}\right)x^5}$$

$$y(x) = -\frac{2x\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{1}{3}}}{i\left(-3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}\right)\sqrt{3}+3c_1x^5+\left(2c_1x^5+\sqrt{-27c_1^3x^{15}+4c_1^2x^{10}}\right)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 59.109 (sec). Leaf size: 534

```
DSolve[(27*x*y[x]^2+8*y[x]^3)+(18*x^2*y[x]+12*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to 0$$

$$y(x) \to \frac{1}{4}\left(\frac{9x^2}{\sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}}} + \sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}} - 3x\right)$$

$$y(x) \to \frac{1}{8}\left(-\frac{9\left(1+i\sqrt{3}\right)x^2}{\sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}}} + i\left(\sqrt{3}+i\right)\sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}} - 6x\right)$$

$$y(x) \to \frac{1}{8}\left(\frac{9i\left(\sqrt{3}+i\right)x^2}{\sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}}} - \left(1+i\sqrt{3}\right)\sqrt[3]{\frac{-27x^5+4\sqrt{e^{6c_1}\left(-27x^5+4e^{6c_1}\right)}+8e^{6c_1}}{x^2}} - 6x\right)$$

$$y(x) \to 0$$

$$y(x) \to \frac{3\left(\sqrt[3]{-x^3}+x\right)\left(-2x+\left(1-i\sqrt{3}\right)\sqrt[3]{-x^3}\right)}{8x}$$

$$y(x) \to \frac{3\left(\sqrt[3]{-x^3}+x\right)\left(-2x+\left(1+i\sqrt{3}\right)\sqrt[3]{-x^3}\right)}{8x}$$

$$y(x) \to -\frac{3\left(-\sqrt[3]{-x^3}x+\left(-x^3\right)^{2/3}+x^2\right)}{4x}$$

7.9 problem 9

Internal problem ID [1069]
Internal file name [`OUTPUT/1070_Sunday_June_05_2022_02_01_53_AM_17687407/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

Unable to solve or complete the solution.

$$\boxed{6xy^2 + 2y + (12x^2y + 12xy^2)\, y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 5

```
dsolve((6*x*y(x)^2+2*y(x))+(12*x^2*y(x)+12*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(6*x*y[x]^2+2*y[x])+(12*x^2*y[x]+12*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolution
```

Not solved

7.10 problem 10

Internal problem ID [1070]
Internal file name [OUTPUT/1071_Sunday_June_05_2022_02_01_55_AM_87646166/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y^2 + \left(x y^2 + 6 y x + \frac{1}{y}\right) y' = 0}$$

7.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x\,y^2 + 6yx + \frac{1}{y}\right)\mathrm{d}y = \left(-y^2\right)\mathrm{d}x$$
$$\left(y^2\right)\mathrm{d}x + \left(x\,y^2 + 6yx + \frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y^2$$
$$N(x,y) = x\,y^2 + 6yx + \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right)\\ &= 2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\,y^2+6yx+\frac{1}{y}\right)\\
&= y^2+6y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\
&= \frac{y}{1+y^2\,(y+6)\,x}\left((2y)-\left(y^2+6y\right)\right)\\
&= -\frac{y^2(y+4)}{1+y^2\,(y+6)\,x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\
&= \frac{1}{y^2}\left(\left(y^2+6y\right)-(2y)\right)\\
&= \frac{y+4}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B\,\mathrm{d}y}\\
&= e^{\int \frac{y+4}{y}\,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{y+4\ln(y)}\\
&= y^4\mathrm{e}^y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M\\
&= y^4\mathrm{e}^y\left(y^2\right)\\
&= y^6\mathrm{e}^y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y^4 \mathrm{e}^y \left(x\,y^2 + 6yx + \frac{1}{y} \right) \\
&= \left(x\,y^3 + 6x\,y^2 + 1 \right) y^3 \mathrm{e}^y
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y^6 \mathrm{e}^y \right) + \left(\left(x\,y^3 + 6x\,y^2 + 1 \right) y^3 \mathrm{e}^y \right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int y^6 \mathrm{e}^y \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = y^6 \mathrm{e}^y x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 6y^5 \mathrm{e}^y x + y^6 \mathrm{e}^y x + f'(y) \tag{4}
$$

$$
= \mathrm{e}^y x\,y^5 (y + 6) + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(x\,y^3 + 6x\,y^2 + 1 \right) y^3 \mathrm{e}^y$. Therefore equation (4) becomes

$$
\left(x\,y^3 + 6x\,y^2 + 1 \right) y^3 \mathrm{e}^y = \mathrm{e}^y x\,y^5 (y + 6) + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3 e^y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (y^3 e^y) \, dy$$
$$f(y) = (y^3 - 3y^2 + 6y - 6) e^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^6 e^y x + (y^3 - 3y^2 + 6y - 6) e^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^6 e^y x + (y^3 - 3y^2 + 6y - 6) e^y$$

Summary

The solution(s) found are the following

$$y^6 e^y x + (y^3 - 3y^2 + 6y - 6) e^y = c_1 \tag{1}$$

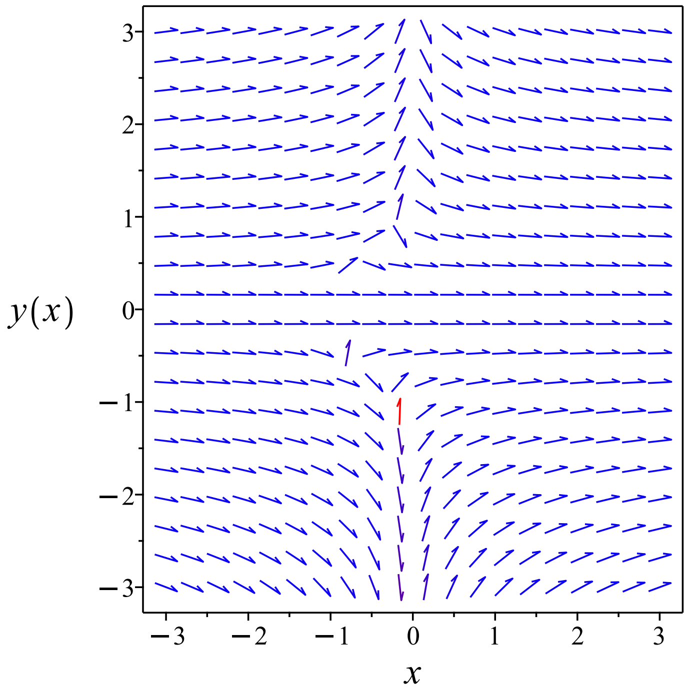


Figure 431: Slope field plot

Verification of solutions

$$y^6 e^y x + (y^3 - 3y^2 + 6y - 6) e^y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve((y(x)^2)+(x*y(x)^2+3*x*y(x)+3*x*y(x)+1/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{xy(x)^{6}+y(x)^{3}-3y(x)^{2}-\mathrm{e}^{-y(x)}c_{1}+6y(x)-6}{y\left(x\right)^{6}}=0$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 41

```
DSolve[(y[x]^2)+(x*y[x]^2+3*x*y[x]+3*x*y[x]+1/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[x=-\frac{y(x)^{3}-3y(x)^{2}+6y(x)-6}{y(x)^{6}}+\frac{c_{1}e^{-y(x)}}{y(x)^{6}},y(x)\right]$$

7.11 problem 11

Internal problem ID [1071]
Internal file name [OUTPUT/1072_Sunday_June_05_2022_02_01_57_AM_89441240/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{12yx^3+24x^2y^2+\left(9x^4+32yx^3+4y\right)y'=0}$$

7.11.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(9x^4+32y\,x^3+4y\right)\mathrm{d}y &= \left(-12y\,x^3-24x^2y^2\right)\mathrm{d}x\\ \left(12y\,x^3+24x^2y^2\right)\mathrm{d}x + \left(9x^4+32y\,x^3+4y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 12y\,x^3+24x^2y^2\\ N(x,y) &= 9x^4+32y\,x^3+4y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(12y\,x^3+24x^2y^2\right)\\ &= 12x^2(x+4y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(9x^4+32y\,x^3+4y\right)\\ &= 36x^3+96y\,x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{9x^4+32y\,x^3+4y}\left(\left(12x^3+48y\,x^2\right)-\left(36x^3+96y\,x^2\right)\right)\\ &= -\frac{24x^2(x+2y)}{9x^4+32y\,x^3+4y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{12y\,x^2\,(x+2y)}\left(\left(36x^3+96y\,x^2\right)-\left(12x^3+48y\,x^2\right)\right) \\
&= \frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B\,\mathrm{d}y} \\
&= e^{\int \frac{2}{y}\,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(y)} \\
&= y^2
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y^2\left(12y\,x^3+24x^2y^2\right) \\
&= 12y^3x^2(x+2y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y^2\left(9x^4+32y\,x^3+4y\right) \\
&= \left(32x^3+4\right)y^3+9x^4y^2
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(12y^3x^2(x+2y)\right)+\left(\left(32x^3+4\right)y^3+9x^4y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 12y^3x^2(x+2y) \, \mathrm{d}x$$

$$\phi = y^3x^3(3x+8y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2x^3(3x+8y) + 8y^3x^3 + f'(y) \tag{4}$$

$$= x^3y^2(9x+32y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (32x^3+4)\,y^3 + 9x^4y^2$. Therefore equation (4) becomes

$$(32x^3+4)\,y^3 + 9x^4y^2 = x^3y^2(9x+32y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (4y^3) \, \mathrm{d}y$$

$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^3x^3(3x+8y) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^3 x^3 (3x + 8y) + y^4$$

Summary
The solution(s) found are the following

$$y^3 x^3 (3x + 8y) + y^4 = c_1 \tag{1}$$

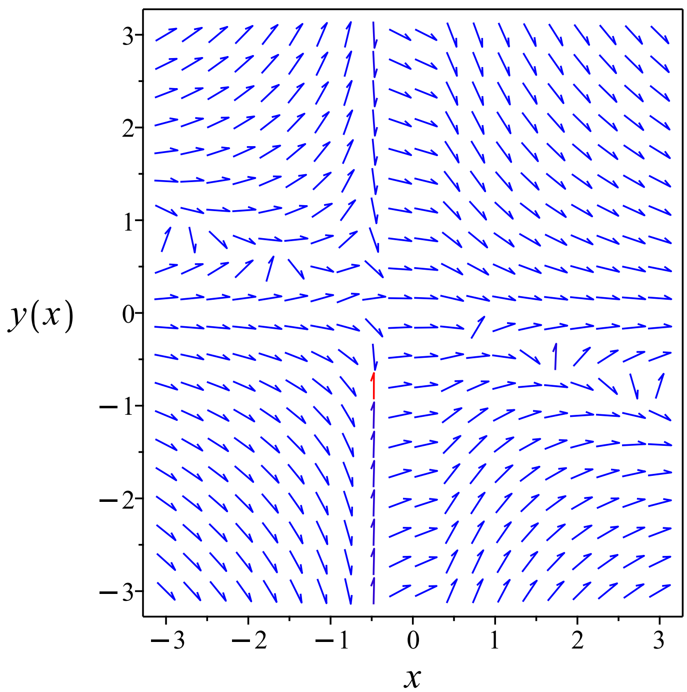


Figure 432: Slope field plot

Verification of solutions

$$y^3 x^3 (3x + 8y) + y^4 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve((12*x^3*y(x)+24*x^2*y(x)^2)+(9*x^4+32*x^3*y(x)+4*y(x))*diff(y(x),x)=0,y(x), singsol=a
```

$$3x^4y(x)^3 + 8x^3y(x)^4 + y(x)^4 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 61.708 (sec). Leaf size: 1733

```
DSolve[(12*x^3*y[x]+24*x^2*y[x]^2)+(9*x^4+32*x^3*y[x]+4*y[x])*y'[x]==0,y[x],x,IncludeSingula
```

$$y(x) \to -\frac{3x^4}{32x^3+4}$$

$$+\frac{1}{2}\sqrt{\frac{9x^8}{4(8x^3+1)^2}+\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}-\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

$$-\frac{1}{2}\sqrt{\frac{9x^8}{2(8x^3+1)^2}-\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}+\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

$$y(x) \to -\frac{3x^4}{32x^3+4}$$

$$+\frac{1}{2}\sqrt{\frac{9x^8}{4(8x^3+1)^2}+\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}-\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

$$+\frac{1}{2}\sqrt{\frac{9x^8}{2(8x^3+1)^2}-\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}+\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

$$y(x) \to -\frac{3x^4}{32x^3+4}$$

$$-\frac{1}{2}\sqrt{\frac{9x^8}{4(8x^3+1)^2}+\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}-\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

$$-\frac{1}{2}\sqrt{\frac{9x^8}{2(8x^3+1)^2}-\frac{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8x^3+1)^3}-243c_1x^8}}{3\sqrt[3]{2}(8x^3+1)}+\frac{4\sqrt[3]{2}c_1}{\sqrt[3]{\sqrt{59049{c_1}^2x^{16}+6912{c_1}^3(8 \cdots}}}}$$

7.12 problem 12

Internal problem ID [1072]
Internal file name [`OUTPUT/1073_Sunday_June_05_2022_02_01_58_AM_30903516/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2y + 4yx + 2y + (x^2 + x)\, y' = 0}$$

7.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(x^2 + 4x + 2)}{x\,(x+1)} \end{aligned}$$

Where $f(x) = -\frac{x^2+4x+2}{x(x+1)}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{x^2+4x+2}{x\,(x+1)}\,dx\\
\int \frac{1}{y}\,dy &= \int -\frac{x^2+4x+2}{x\,(x+1)}\,dx\\
\ln(y) &= -x - 2\ln(x) - \ln(x+1) + c_1\\
y &= \mathrm{e}^{-x-2\ln(x)-\ln(x+1)+c_1}\\
&= c_1\mathrm{e}^{-x-2\ln(x)-\ln(x+1)}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_1\mathrm{e}^{-x}}{x^2\,(x+1)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\mathrm{e}^{-x}}{x^2\,(x+1)} \tag{1}
$$

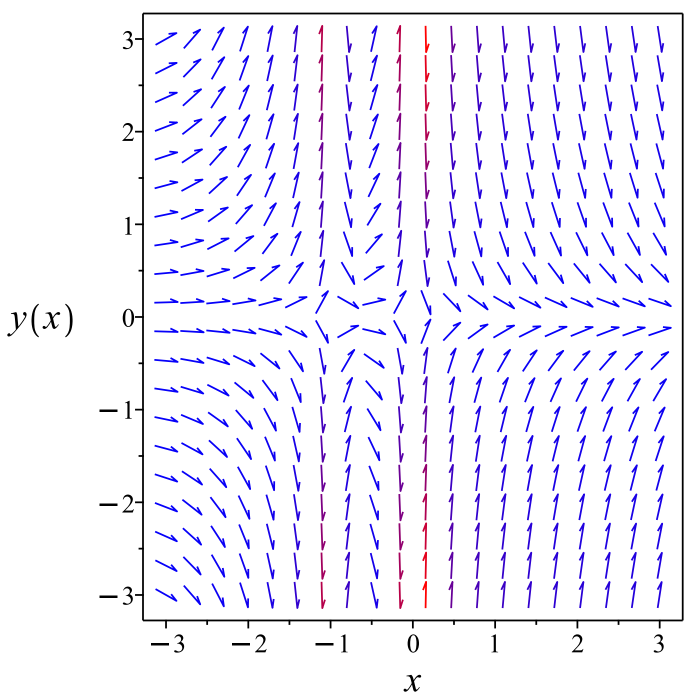


Figure 433: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)}$$

Verified OK.

7.12.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-x^2 - 4x - 2}{x (x+1)}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{(-x^2 - 4x - 2)\, y}{x (x+1)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-x^2-4x-2}{x(x+1)} dx}$$
$$= e^{x+2\ln(x)+\ln(x+1)}$$

Which simplifies to

$$\mu = e^x x^2 (x+1)$$

The ode becomes

$$\frac{d}{dx} \mu y = 0$$
$$\frac{d}{dx} \left(e^x x^2 (x+1)\, y\right) = 0$$

Integrating gives

$$e^x x^2 (x+1)\, y = c_1$$

Dividing both sides by the integrating factor $\mu = e^x x^2 (x+1)$ results in

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)} \tag{1}$$

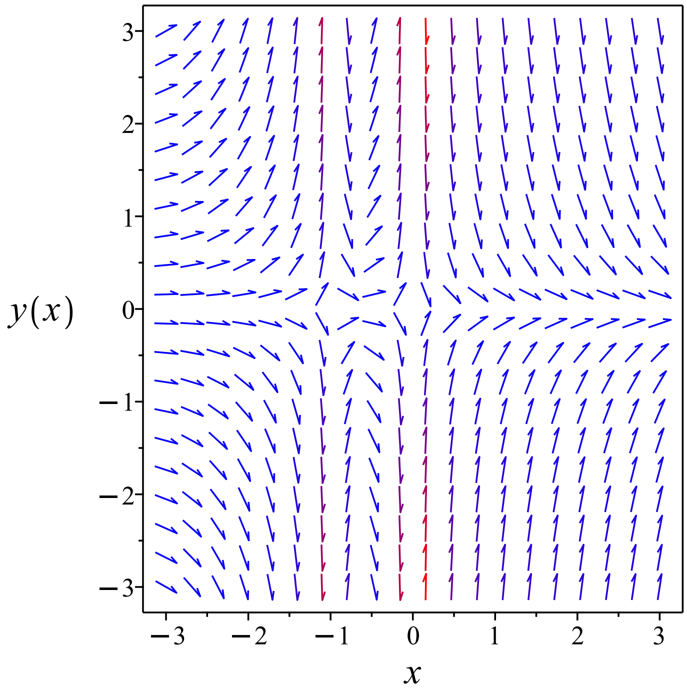


Figure 434: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)}$$

Verified OK.

7.12.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^3 u(x) + 4u(x)\,x^2 + 2u(x)\,x + (x^2 + x)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x^2+5x+3)}{x\left(x+1\right)}
\end{aligned}$$

Where $f(x) = -\frac{x^2+5x+3}{x(x+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^2+5x+3}{x\left(x+1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^2+5x+3}{x\left(x+1\right)}\,dx \\
\ln\left(u\right) &= -x-3\ln\left(x\right)-\ln\left(x+1\right)+c_2 \\
u &= \mathrm{e}^{-x-3\ln(x)-\ln(x+1)+c_2} \\
&= c_2\mathrm{e}^{-x-3\ln(x)-\ln(x+1)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\mathrm{e}^{-x}}{x^3\left(x+1\right)}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2\mathrm{e}^{-x}}{x^2\left(x+1\right)}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2\mathrm{e}^{-x}}{x^2\left(x+1\right)} \tag{1}$$

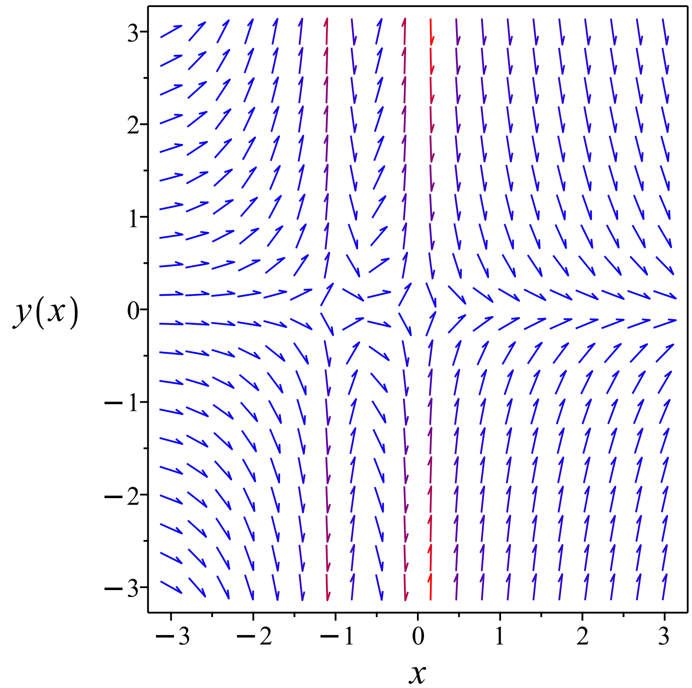


Figure 435: Slope field plot

Verification of solutions

$$y = \frac{c_2 \mathrm{e}^{-x}}{x^2 (x+1)}$$

Verified OK.

7.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(x^2+4x+2)}{x(x+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 324: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-x-2\ln(x)-\ln(x+1)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x-2\ln(x)-\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} x^{2}(x+1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x^2+4x+2)}{x\,(x+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y x\,(x^2+4x+2) \\ S_y &= \mathrm{e}^{x} x^{2}(x+1) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x x^2 (x+1)\, y = c_1$$

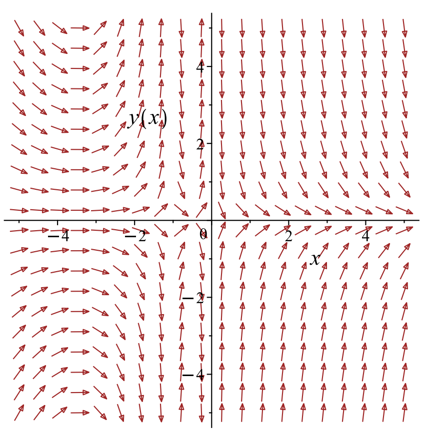
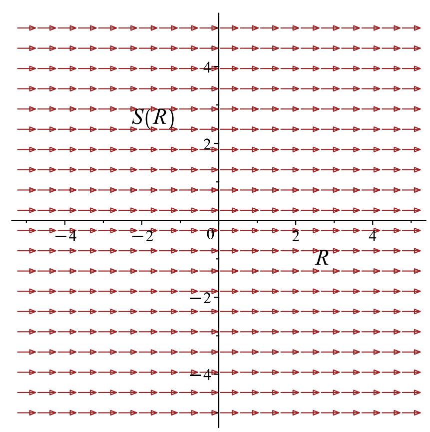
Which simplifies to

$$e^x x^2 (x+1)\, y = c_1$$

Which gives

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^2+4x+2)}{x(x+1)}$ 	$R = x$ $S = e^x x^2 (x+1)\, y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x^2 (x+1)} \tag{1}$$

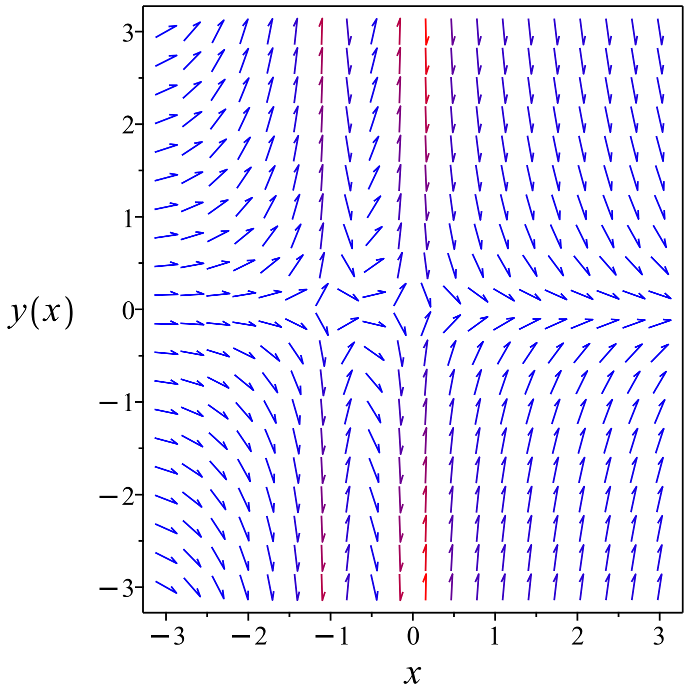


Figure 436: Slope field plot

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-x}}{x^2 (x+1)}$$

Verified OK.

7.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{x^2 + 4x + 2}{x (x + 1)}\right) \mathrm{d}x \\
\left(-\frac{x^2 + 4x + 2}{x (x + 1)}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x^2 + 4x + 2}{x (x + 1)} \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2 + 4x + 2}{x (x + 1)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{x^2 + 4x + 2}{x(x+1)} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x - 2\ln(x) - \ln(x+1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - 2\ln(x) - \ln(x+1) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - 2\ln(x) - \ln(x+1) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-x-c_1}}{(x+1)\,x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x-c_1}}{(x+1)\,x^2} \tag{1}$$

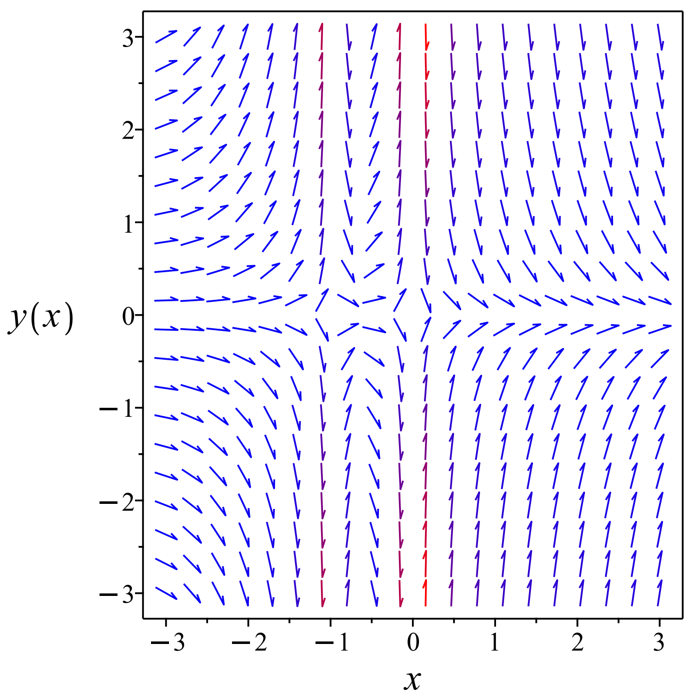


Figure 437: Slope field plot

Verification of solutions

$$y = \frac{e^{-x-c_1}}{(x+1)\,x^2}$$

Verified OK.

7.12.6 Maple step by step solution

Let's solve

$x^2y + 4yx + 2y + (x^2 + x)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x^2+4x+2}{x(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{x^2+4x+2}{x(x+1)}dx + c_1$

- Evaluate integral

  $\ln(y) = -x - 2\ln(x) - \ln(x+1) + c_1$

- Solve for y

  $y = \frac{e^{-x+c_1}}{(x+1)x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((x^2*y(x)+4*x*y(x)+2*y(x))+(x^2+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-x}}{(x+1)\, x^2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 26

```
DSolve[(x^2*y[x]+4*x*y[x]+2*y[x])+(x^2+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^{-x}}{x^2(x+1)}$$
$$y(x) \to 0$$

7.13 problem 13

Internal problem ID [1073]
Internal file name [`OUTPUT/1074_Sunday_June_05_2022_02_01_59_AM_95419976/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y + (x^4 - x)\, y' = 0}$$

7.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x\,(x^3-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(x^3-1)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= \frac{1}{x\,(x^3-1)}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{1}{x\,(x^3-1)}\, dx \\
\ln (y) &= \frac{\ln (x^2+x+1)}{3} - \ln (x) + \frac{\ln (x-1)}{3} + c_1 \\
y &= \mathrm{e}^{\frac{\ln\left(x^2+x+1\right)}{3} - \ln(x) + \frac{\ln(x-1)}{3} + c_1} \\
&= c_1 \mathrm{e}^{\frac{\ln\left(x^2+x+1\right)}{3} - \ln(x) + \frac{\ln(x-1)}{3}}
\end{aligned}$$

Which simplifies to

$$y = \frac{c_1 (x^2+x+1)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2+x+1)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}}{x} \tag{1}$$

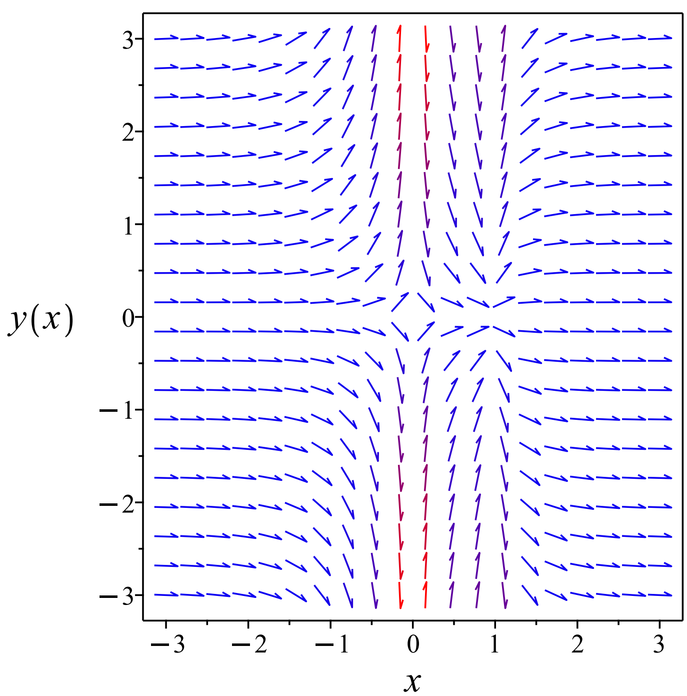


Figure 438: Slope field plot

Verification of solutions

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x}$$

Verified OK.

7.13.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x^4-x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x^4-x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x^4-x}dx}$$
$$= e^{-\frac{\ln(x^2+x+1)}{3}+\ln(x)-\frac{\ln(x-1)}{3}}$$

Which simplifies to

$$\mu = \frac{x}{(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{xy}{(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}\right) = 0$$

Integrating gives

$$\frac{xy}{(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{x}{(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}$ results in

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x} \tag{1}$$

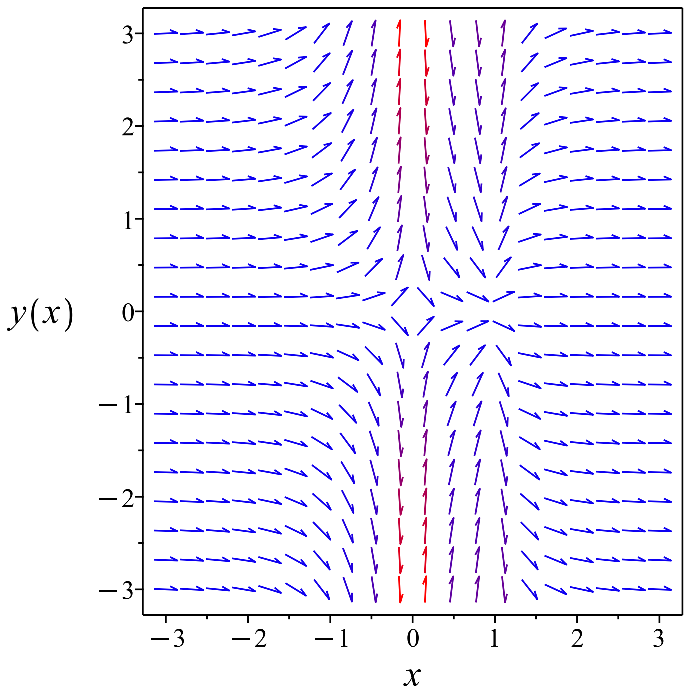


Figure 439: Slope field plot

Verification of solutions

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x}$$

Verified OK.

7.13.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + (x^4 - x)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x^3-2)}{x\,(x^3-1)}
\end{aligned}$$

Where $f(x) = -\frac{x^3-2}{x(x^3-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^3-2}{x\,(x^3-1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^3-2}{x\,(x^3-1)}\,dx \\
\ln(u) &= \frac{\ln(x^2+x+1)}{3} - 2\ln(x) + \frac{\ln(x-1)}{3} + c_2 \\
u &= \mathrm{e}^{\frac{\ln(x^2+x+1)}{3} - 2\ln(x) + \frac{\ln(x-1)}{3} + c_2} \\
&= c_2 \mathrm{e}^{\frac{\ln(x^2+x+1)}{3} - 2\ln(x) + \frac{\ln(x-1)}{3}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2(x^2+x+1)^{\frac{1}{3}}\,(x-1)^{\frac{1}{3}}}{x^2}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2(x^2+x+1)^{\frac{1}{3}}\,(x-1)^{\frac{1}{3}}}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2(x^2+x+1)^{\frac{1}{3}}\,(x-1)^{\frac{1}{3}}}{x} \tag{1}$$

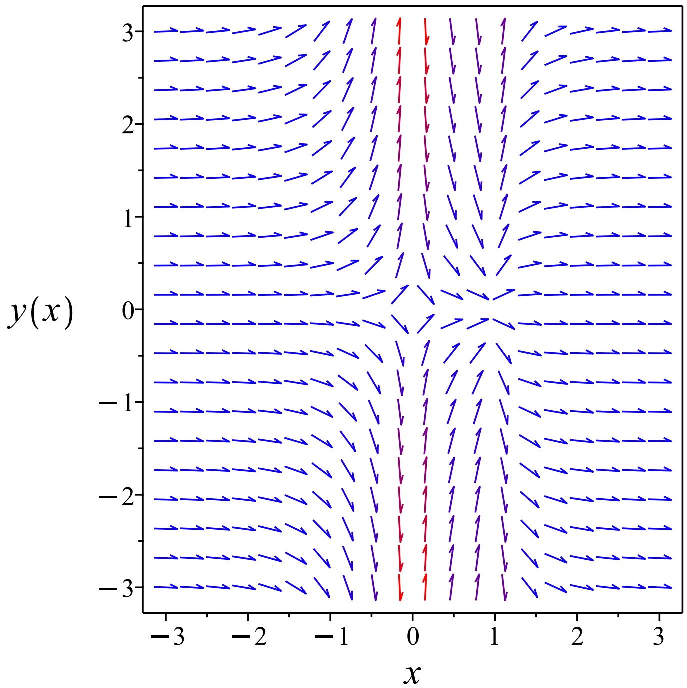


Figure 440: Slope field plot

Verification of solutions

$$y = \frac{c_2(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x}$$

Verified OK.

7.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x(x^3-1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 327: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\ln\left(x^2+x+1\right)}{3} - \ln(x) + \frac{\ln(x-1)}{3}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\ln\left(x^2+x+1\right)}{3} - \ln(x) + \frac{\ln(x-1)}{3}}} dy \end{aligned}$$

Which results in

$$S = x\,\mathrm{e}^{\ln\left(\frac{1}{\left(x^2+x+1\right)^{\frac{1}{3}}}\right) + \ln\left(\frac{1}{(x-1)^{\frac{1}{3}}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x\left(x^3 - 1\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\left(x^2+x+1\right)^{\frac{4}{3}} (x-1)^{\frac{4}{3}}} \\ S_y &= \frac{x}{\left(x^2+x+1\right)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy}{(x^2+x+1)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}} = c_1$$

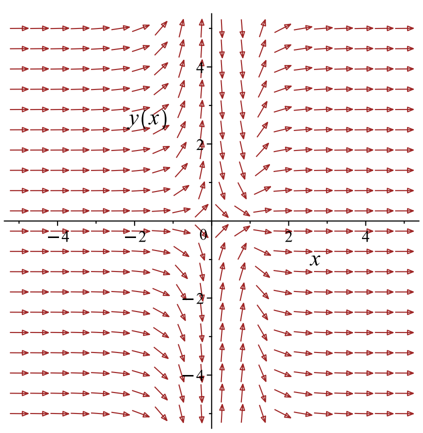
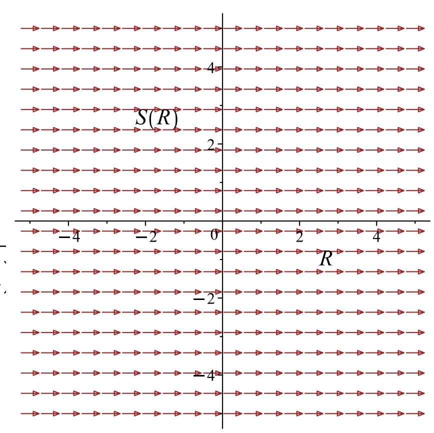
Which simplifies to

$$\frac{xy}{(x^2+x+1)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}} = c_1$$

Which gives

$$y = \frac{c_1 (x^2+x+1)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x(x^3-1)}$ 	$R = x$ $S = \frac{xy}{(x^2+x+1)^{\frac{1}{3}} (x-1}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x} \tag{1}$$

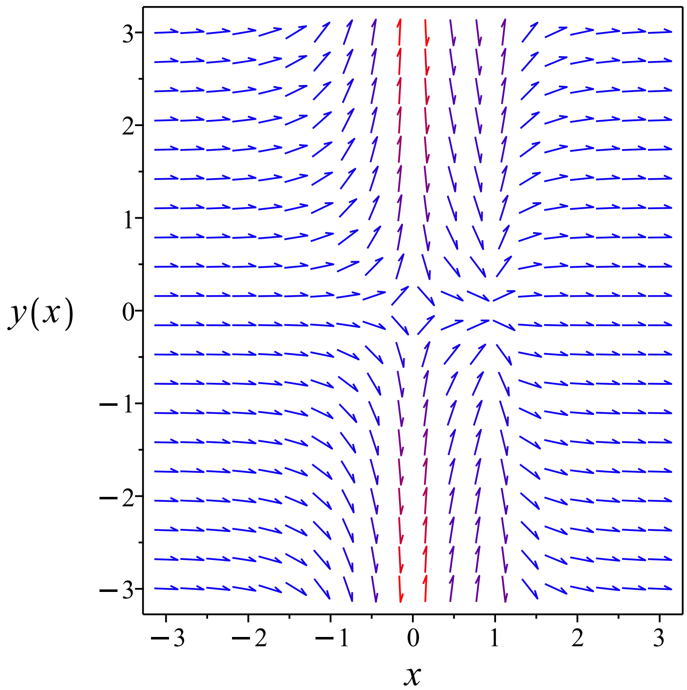


Figure 441: Slope field plot

Verification of solutions

$$y = \frac{c_1(x^2+x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}}{x}$$

Verified OK.

7.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x\,(x^3-1)}\right)\mathrm{d}x\\ \left(-\frac{1}{x\,(x^3-1)}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x\,(x^3-1)}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x\,(x^3-1)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x\,(x^3-1)}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+x+1)}{3} + \ln(x) - \frac{\ln(x-1)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+x+1)}{3} + \ln(x) - \frac{\ln(x-1)}{3} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+x+1)}{3} + \ln(x) - \frac{\ln(x-1)}{3} + \ln(y)$$

The solution becomes

$$y = \frac{\mathrm{e}^{\frac{\ln(x^2+x+1)}{3} + \frac{\ln(x-1)}{3} + c_1}}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{\frac{\ln(x^2+x+1)}{3} + \frac{\ln(x-1)}{3} + c_1}}{x} \tag{1}$$

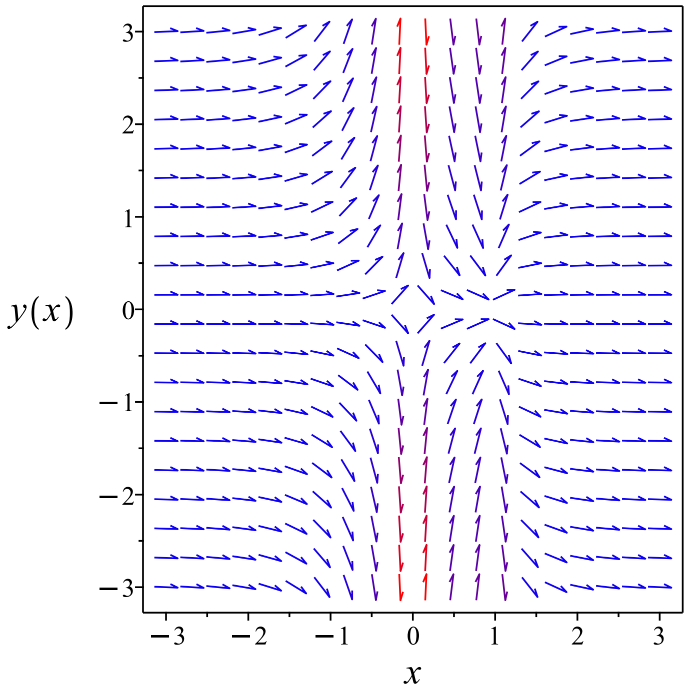


Figure 442: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{\frac{\ln(x^2+x+1)}{3}+\frac{\ln(x-1)}{3}+c_1}}{x}$$

Verified OK.

7.13.6 Maple step by step solution

Let's solve

$-y + (x^4 - x)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x^4-x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x^4-x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x^2+x+1)}{3} - \ln(x) + \frac{\ln(x-1)}{3} + c_1$

- Solve for y

  $y = \frac{e^{\frac{\ln\left(x^2+x+1\right)}{3}+\frac{\ln(x-1)}{3}+c_1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(-y(x)+(x^4-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x-1)^{\frac{1}{3}}\left(x^2+x+1\right)^{\frac{1}{3}}}{x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 27

```
DSolve[-y[x]+(x^4-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1\sqrt[3]{1-x^3}}{x}$$
$$y(x) \to 0$$

7.14 problem 14

Internal problem ID [1074]
Internal file name [`OUTPUT/1075_Sunday_June_05_2022_02_02_00_AM_37850167/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\cos (x) \cos (y)+(\cos (y) \sin (x)-\sin (y) \sin (x)+y) y'=0}$$

7.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y)\sin(x) - \sin(y)\sin(x) + y)\,\mathrm{d}y &= (-\cos(x)\cos(y))\,\mathrm{d}x\\ (\cos(x)\cos(y))\,\mathrm{d}x + (\cos(y)\sin(x) - \sin(y)\sin(x) + y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \cos(x)\cos(y)\\ N(x,y) &= \cos(y)\sin(x) - \sin(y)\sin(x) + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\cos(y))\\ &= -\sin(y)\cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)\sin(x) - \sin(y)\sin(x) + y)\\ &= \cos(x)(-\sin(y) + \cos(y))\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{\sin(x)(-\sin(y) + \cos(y)) + y}((-\sin(y)\cos(x)) - (\cos(x)\cos(y) - \sin(y)\cos(x)))\\ &= -\frac{\cos(x)\cos(y)}{\sin(x)(-\sin(y) + \cos(y)) + y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \sec(y)\sec(x)\left((\cos(x)\cos(y) - \sin(y)\cos(x)) - (-\sin(y)\cos(x))\right) \\
&= 1
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B\,\mathrm{d}y} \\
&= e^{\int 1\,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}(\cos(x)\cos(y)) \\
&= \cos(x)\cos(y)\,\mathrm{e}^{y}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}(\cos(y)\sin(x) - \sin(y)\sin(x) + y) \\
&= (\sin(x)(-\sin(y) + \cos(y)) + y)\,\mathrm{e}^{y}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\cos(x)\cos(y)\,\mathrm{e}^{y}) + ((\sin(x)(-\sin(y) + \cos(y)) + y)\,\mathrm{e}^{y})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \cos(x)\cos(y)\,\mathrm{e}^{y} \mathrm{d}x$$
$$\phi = \sin(x)\cos(y)\,\mathrm{e}^{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\sin(x)\sin(y)\,\mathrm{e}^{y} + \sin(x)\cos(y)\,\mathrm{e}^{y} + f'(y) \tag{4}$$
$$= \mathrm{e}^{y}\sin(x)\left(-\sin(y) + \cos(y)\right) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (\sin(x)(-\sin(y) + \cos(y)) + y)\,\mathrm{e}^{y}$. Therefore equation (4) becomes

$$(\sin(x)(-\sin(y) + \cos(y)) + y)\,\mathrm{e}^{y} = \mathrm{e}^{y}\sin(x)(-\sin(y) + \cos(y)) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{y} y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (\mathrm{e}^{y} y)\,\mathrm{d}y$$
$$f(y) = (y-1)\,\mathrm{e}^{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x)\cos(y)\,\mathrm{e}^{y} + (y-1)\,\mathrm{e}^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x)\cos(y)\,\mathrm{e}^{y} + (y-1)\,\mathrm{e}^{y}$$

Summary
The solution(s) found are the following

$$\sin(x)\cos(y)\,e^{y} + (y-1)\,e^{y} = c_1 \tag{1}$$

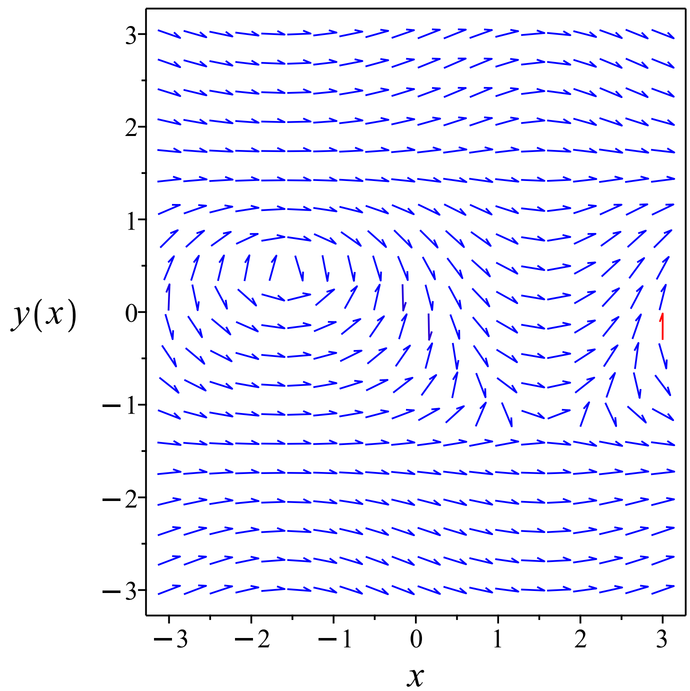


Figure 443: Slope field plot

Verification of solutions

$$\sin(x)\cos(y)\,e^{y} + (y-1)\,e^{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve((cos(x)*cos(y(x)))+(sin(x)*cos(y(x))-sin(x)*sin(y(x))+y(x))*diff(y(x),x)=0,y(x), sing
```

$$(\sin(x)\cos(y(x)) + y(x) - 1)\,\mathrm{e}^{y(x)} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.218 (sec). Leaf size: 28

```
DSolve[(Cos[x]*Cos[y[x]])+(Sin[x]*Cos[y[x]]-Sin[x]*Sin[y[x]]+y[x])*y'[x]==0,y[x],x,IncludeSi
```

$$\text{Solve}\left[-2e^{y(x)}(y(x)-1) - 2e^{y(x)}\sin(x)\cos(y(x)) = c_1, y(x)\right]$$

7.15 problem 15

Internal problem ID [1075]
Internal file name [`OUTPUT/1076_Sunday_June_05_2022_02_02_03_AM_16375660/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{2yx + y^2 + \left(2yx + x^2 - 2xy^2 - 2xy^3\right) y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((2*x*y(x)+y(x)^2)+(2*x*y(x)+x^2-2*x*y(x)^2-2*x*y(x)^3)*diff(y(x),x)=0,y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*x*y[x]+y[x]^2)+(2*x*y[x]+x^2-2*x*y[x]^2-2*x*y[x]^3)*y'[x]==0,y[x],x,IncludeSingula
```

Not solved

7.16 problem 16

Internal problem ID [1076]
Internal file name [`OUTPUT/1077_Sunday_June_05_2022_02_02_05_AM_21785661/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y \sin(y) + x(\sin(y) - y\cos(y))\, y' = 0}$$

7.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y\sin(y)}{x(-\sin(y) + y\cos(y))} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{y\sin(y)}{-\sin(y)+y\cos(y)}$. Integrating both sides gives

$$\frac{1}{\frac{y\sin(y)}{-\sin(y)+y\cos(y)}}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{y\sin(y)}{-\sin(y)+y\cos(y)}}dy = \int \frac{1}{x}\,dx$$

$$-\ln(y)+\ln(\sin(y)) = \ln(x)+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(y)+\ln(\sin(y))} = \mathrm{e}^{\ln(x)+c_1}$$

Which simplifies to

$$\frac{\sin(y)}{y} = c_2 x$$

Summary
The solution(s) found are the following

$$y = \mathrm{RootOf}\left(\_Zc_2x-\sin(\_Z)\right) \tag{1}$$

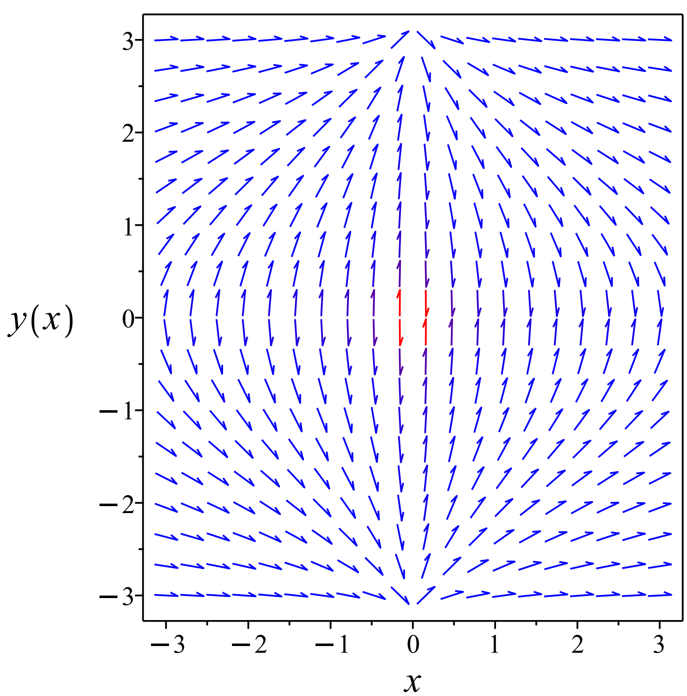


Figure 444: Slope field plot

Verification of solutions

$$y = \mathrm{RootOf}\left(\_Zc_2x-\sin(\_Z)\right)$$

Verified OK.

7.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y \sin(y)}{x(-\sin(y) + y\cos(y))} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 330: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{x} dx\end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\sin(y)}{x(-\sin(y) + y\cos(y))}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-1 + \cot(y)\, y}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-1 + \cot(R)\, R}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + \ln(\sin(R)) + c_1 \tag{4}$$

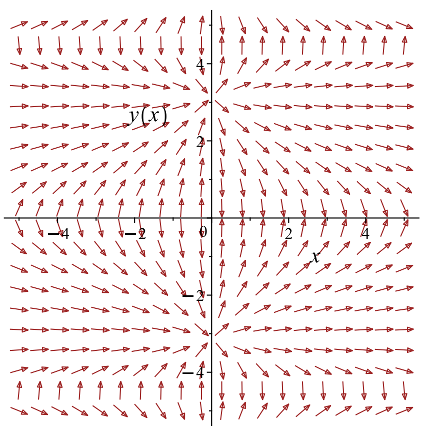
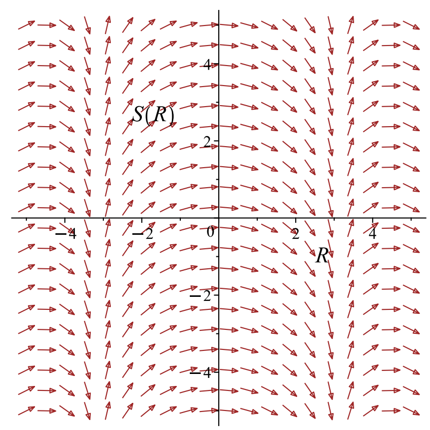
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\ln(y) + \ln(\sin(y)) + c_1$$

Which simplifies to

$$\ln(x) = -\ln(y) + \ln(\sin(y)) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y\sin(y)}{x(-\sin(y)+y\cos(y))}$
</td><td>$R = y$
$S = \ln(x)$</td><td>$\frac{dS}{dR} = \frac{-1+\cot(R)R}{R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\ln(x) = -\ln(y) + \ln(\sin(y)) + c_1 \tag{1}$$

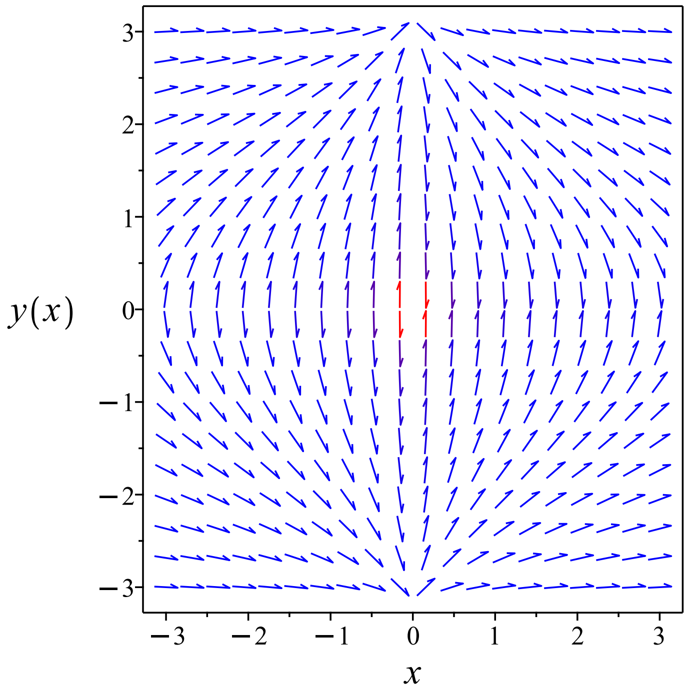


Figure 445: Slope field plot

Verification of solutions

$$\ln(x) = -\ln(y) + \ln(\sin(y)) + c_1$$

Verified OK.

7.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-\sin(y) + y\cos(y)}{y\sin(y)}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{-\sin(y) + y\cos(y)}{y\sin(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{-\sin(y) + y\cos(y)}{y\sin(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-\sin(y) + y\cos(y)}{y\sin(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x \\
\phi &= -\ln(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-\sin(y) + y\cos(y)}{y\sin(y)}$. Therefore equation (4) becomes

$$
\frac{-\sin(y) + y\cos(y)}{y\sin(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\sin(y) - y\cos(y)}{y\sin(y)} \\
&= \frac{-1 + \cot(y)\, y}{y}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-1+\cot(y)\,y}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + \ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y) + \ln(\sin(y))$$

Summary
The solution(s) found are the following

$$-\ln(x) - \ln(y) + \ln(\sin(y)) = c_1 \tag{1}$$

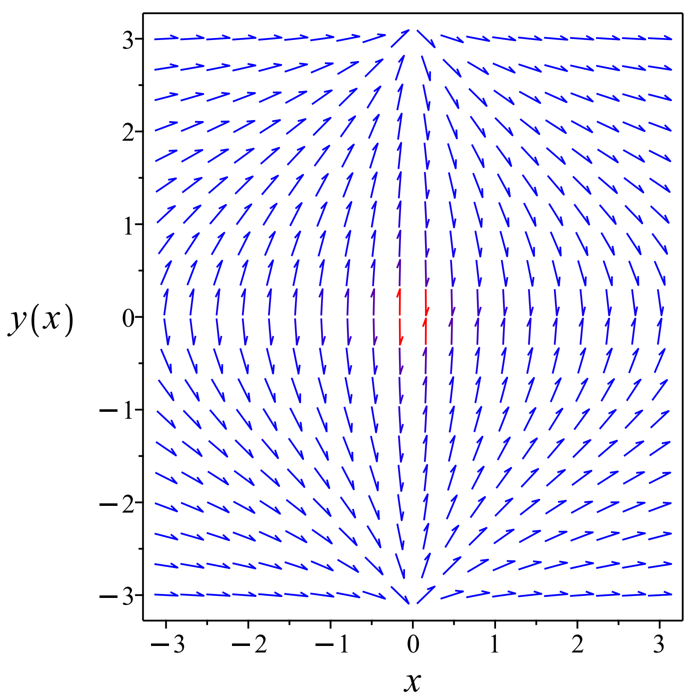


Figure 446: Slope field plot

Verification of solutions

$$-\ln(x)-\ln(y)+\ln(\sin(y))=c_1$$

Verified OK.

7.16.4 Maple step by step solution

Let's solve

$y\sin(y)+x(\sin(y)-y\cos(y))\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(\sin(y)-y\cos(y))}{y\sin(y)}=-\frac{1}{x}$

- Integrate both sides with respect to x

  $\int\frac{y'(\sin(y)-y\cos(y))}{y\sin(y)}dx=\int-\frac{1}{x}dx+c_1$

- Evaluate integral

  $\ln(y)-\ln(\sin(y))=-\ln(x)+c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 16

```
dsolve((y(x)*sin(y(x)))+(x*(sin(y(x))-y(x)*cos(y(x))))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\ln(x)+\ln(y(x))-\ln(\sin(y(x)))+c_1=0$$

✓ Solution by Mathematica

Time used: 0.569 (sec). Leaf size: 27

```
DSolve[(y[x]*Sin[y[x]])+(x*(Sin[y[x]]-y[x]*Cos[y[x]]))*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \text{InverseFunction}[\log(\sin(\#1)) - \log(\#1)\&][\log(x) + c_1]$$
$$y(x) \to 0$$

7.17 problem 18

Internal problem ID [1077]
Internal file name [`OUTPUT/1078_Sunday_June_05_2022_02_02_06_AM_32538710/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{ay + bxy + (cx + dxy)\, y' = 0}$$

7.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(bx+a)}{x\,(dy+c)} \end{aligned}$$

Where $f(x) = -\frac{bx+a}{x}$ and $g(y) = \frac{y}{dy+c}$. Integrating both sides gives

$$\frac{1}{\frac{y}{dy+c}}\, dy = -\frac{bx+a}{x}\, dx$$

$$\int \frac{1}{\frac{y}{dy+c}}\, dy = \int -\frac{bx+a}{x}\, dx$$
$$dy + c\ln(y) = -bx - a\ln(x) + c_1$$

Which results in

$$y = e^{-\frac{c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{bx+a\ln(x)-c_1}{c}}}{c}\right)+a\ln(x)+bx-c_1}{c}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{bx+a\ln(x)-c_1}{c}}}{c}\right)+a\ln(x)+bx-c_1}{c}} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{bx+a\ln(x)-c_1}{c}}}{c}\right)+a\ln(x)+bx-c_1}{c}}$$

Verified OK.

7.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(bx+a)}{x(dy+c)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 333: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x}{bx + a} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{bx+a}} dx \end{aligned}$$

Which results in

$$S = -bx - a \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(bx+a)}{x(dy+c)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{-bx-a}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{dy+c}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{Rd+c}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = Rd + c\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-bx - a\ln(x) = dy + c\ln(y) + c_1$$

Which simplifies to

$$-bx - a\ln(x) = dy + c\ln(y) + c_1$$

Which gives

$$y = e^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,e^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}}$$

Verified OK.

7.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{dy+c}{y}\right)\mathrm{d}y&=\left(\frac{bx+a}{x}\right)\mathrm{d}x\\ \left(-\frac{bx+a}{x}\right)\mathrm{d}x+\left(-\frac{dy+c}{y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{bx+a}{x}\\ N(x,y)&=-\frac{dy+c}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{bx+a}{x}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{dy+c}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{bx+a}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -bx - a\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{dy+c}{y}$. Therefore equation (4) becomes

$$
-\frac{dy+c}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{dy+c}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{-dy-c}{y}\right) \mathrm{d}y \\
f(y) &= -dy - c\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -bx - a\ln(x) - dy - c\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -bx - a\ln(x) - dy - c\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,\mathrm{e}^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,\mathrm{e}^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{a\ln(x)+bx+c\,\mathrm{LambertW}\left(\frac{d\,\mathrm{e}^{-\frac{a\ln(x)+bx+c_1}{c}}}{c}\right)+c_1}{c}}$$

Verified OK.

7.17.4 Maple step by step solution

Let's solve

$ay + bxy + (cx + dxy)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(dy+c)}{y} = -\frac{bx+a}{x}$

- Integrate both sides with respect to x

$$\int \frac{y'(dy+c)}{y}dx = \int -\frac{bx+a}{x}dx + c_1$$

- Evaluate integral

$$dy + c\ln(y) = -bx - a\ln(x) + c_1$$

- Solve for y

$$y = \mathrm{e}^{-\frac{c\mathit{LambertW}\left(\frac{d\,\mathrm{e}^{-\frac{bx+a\ln(x)-c_1}{c}}}{c}\right)+a\ln(x)+bx-c_1}{c}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve((a*y(x)+b*x*y(x))+(c*x+d*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x^{-\frac{a}{c}}\mathrm{e}^{\frac{-bx-c\,\mathrm{LambertW}\left(\frac{d\,x^{-\frac{a}{c}}\,\mathrm{e}^{\frac{-bx-c_1}{c}}}{c}\right)-c_1}{c}}$$

✓ Solution by Mathematica

Time used: 1.133 (sec). Leaf size: 42

```
DSolve[(a*y[x]+b*x*y[x])+(c*x+d*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{cW\left(\frac{dx^{-\frac{a}{c}}e^{\frac{-bx+c_1}{c}}}{c}\right)}{d}$$
$$y(x) \to 0$$

7.18 problem 19

Internal problem ID [1078]
Internal file name [`OUTPUT/1079_Sunday_June_05_2022_02_02_07_AM_94834817/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational, [_Abel, `2nd type`, `class C`]]`

Unable to solve or complete the solution.

$$\boxed{3y^3x^2 - y^2 + y + (-yx + 2x)\, y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
dsolve((3*x^2*y(x)^3-y(x)^2+y(x))+(-x*y(x)+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{4}{\sqrt{x}\,\sqrt{\frac{c_1x+48x^2+4}{x}}+2}$$

$$y(x) = -\frac{4}{\sqrt{x}\,\sqrt{\frac{c_1x+48x^2+4}{x}}-2}$$

✓ Solution by Mathematica

Time used: 0.776 (sec). Leaf size: 80

```
DSolve[(3*x^2*y[x]^3-y[x]^2+y[x])+(-x*y[x]+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{2}{1+\sqrt{-\frac{1}{x^2}}x\sqrt{-12x^2-4c_1x-1}}$$

$$y(x) \to \frac{2x}{x+\frac{\sqrt{-12x^2-4c_1x-1}}{\sqrt{-\frac{1}{x^2}}}}$$

$$y(x) \to 0$$

7.19 problem 20

Internal problem ID [1079]
Internal file name [`OUTPUT/1080_Sunday_June_05_2022_02_02_08_AM_89452171/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y + 3(x^2 + y^3x^2)\, y' = 0}$$

7.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2y}{3x^2\,(y^3+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{3x^2}$ and $g(y) = \frac{y}{y^3+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y^3+1}}\, dy = -\frac{2}{3x^2}\, dx$$

$$\int \frac{1}{\frac{y}{y^3+1}}\, dy = \int -\frac{2}{3x^2}\, dx$$
$$\frac{y^3}{3} + \ln(y) = \frac{2}{3x} + c_1$$

Which results in

$$y = e^{-\frac{x\,\mathrm{LambertW}\left(e^{\frac{3c_1x+2}{x}}\right)-3c_1x-2}{3x}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x\,\mathrm{LambertW}\left(e^{\frac{3c_1x+2}{x}}\right)-3c_1x-2}{3x}} \tag{1}$$

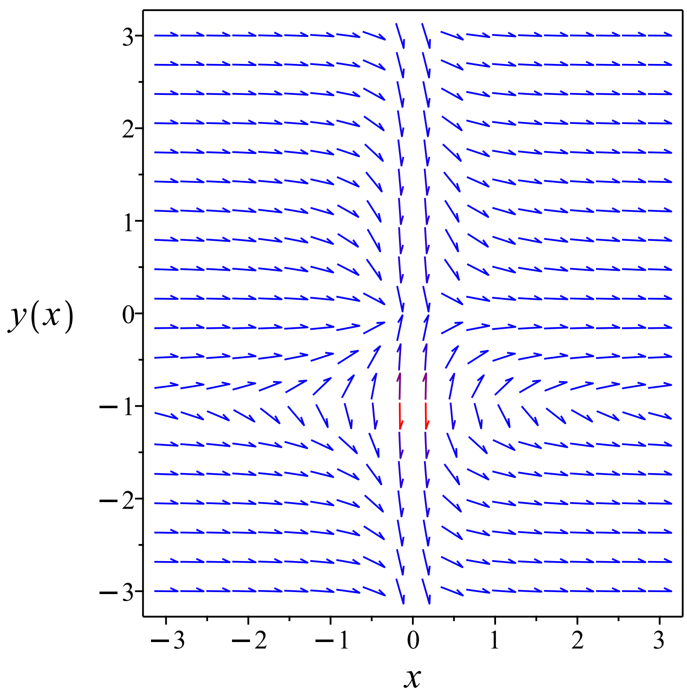


Figure 447: Slope field plot

Verification of solutions

$$y = e^{-\frac{x\,\mathrm{LambertW}\left(e^{\frac{3c_1x+2}{x}}\right)-3c_1x-2}{3x}}$$

Verified OK.

7.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2y}{3x^2(y^3+1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 336: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= -\frac{3x^2}{2} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi}dx \\ &= \int \frac{1}{-\frac{3x^2}{2}}dx\end{aligned}$$

Which results in

$$S = \frac{2}{3x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y}{3x^2(y^3+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{2}{3x^2} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^3+1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^3+1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2}{3x} = \frac{y^3}{3} + \ln(y) + c_1$$

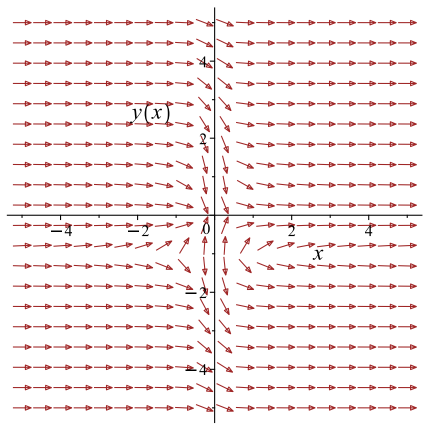
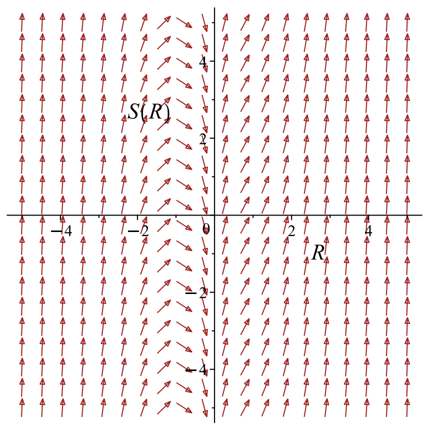
Which simplifies to

$$\frac{2}{3x} = \frac{y^3}{3} + \ln(y) + c_1$$

Which gives

$$y = e^{-\frac{x \operatorname{LambertW}\left(e^{-\frac{3c_1x-2}{x}}\right)+3c_1x-2}{3x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{3x^2(y^3+1)}$ 	$R = y$ $S = \frac{2}{3x}$	$\frac{dS}{dR} = \frac{R^3+1}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{-\frac{x \operatorname{LambertW}\left(e^{-\frac{3c_1 x - 2}{x}}\right) + 3c_1 x - 2}{3x}} \tag{1}$$

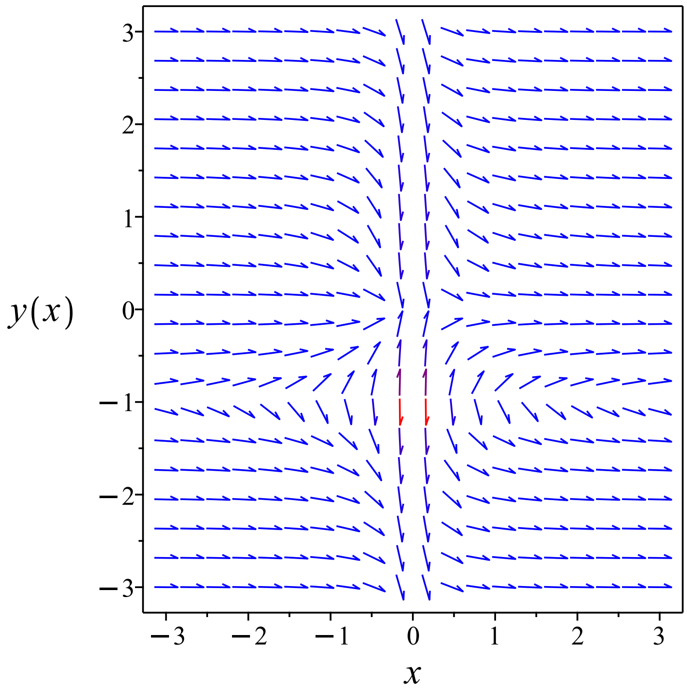


Figure 448: Slope field plot

Verification of solutions

$$y = e^{-\frac{x \operatorname{LambertW}\left(e^{-\frac{3c_1x-2}{x}}\right)+3c_1x-2}{3x}}$$

Verified OK.

7.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\left(-\frac{3(y^3 + 1)}{2y}\right) dy = \left(\frac{1}{x^2}\right) dx$$
$$\left(-\frac{1}{x^2}\right) dx + \left(-\frac{3(y^3 + 1)}{2y}\right) dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2}$$
$$N(x, y) = -\frac{3(y^3 + 1)}{2y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{3(y^3+1)}{2y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3(y^3+1)}{2y}$. Therefore equation (4) becomes

$$
-\frac{3(y^3+1)}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{3(y^3+1)}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-3y^3-3}{2y}\right)\mathrm{d}y$$

$$f(y) = -\frac{y^3}{2} - \frac{3\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{y^3}{2} - \frac{3\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{y^3}{2} - \frac{3\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{-\frac{x\,\mathrm{LambertW}\left(\mathrm{e}^{-\frac{2(c_1x-1)}{x}}\right)+2c_1x-2}{3x}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x\,\mathrm{LambertW}\left(\mathrm{e}^{-\frac{2(c_1x-1)}{x}}\right)+2c_1x-2}{3x}} \tag{1}$$

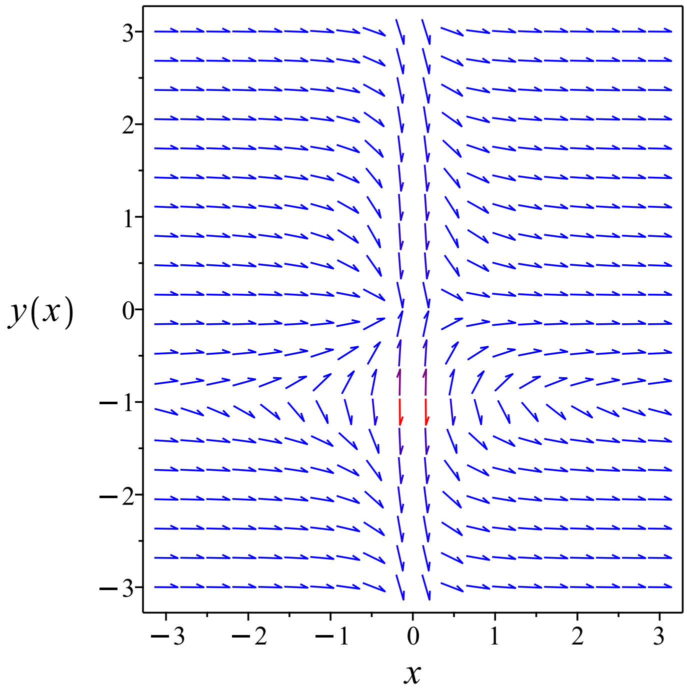


Figure 449: Slope field plot

Verification of solutions

$$y = e^{-\frac{x\,\mathrm{LambertW}\left(e^{-\frac{2(c_1 x-1)}{x}}\right)+2c_1 x-2}{3x}}$$

Verified OK.

7.19.4 Maple step by step solution

Let's solve

$2y + 3(x^2 + y^3x^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y)(y^2-y+1)}{y} = -\frac{2}{3x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y)(y^2-y+1)}{y}dx = \int -\frac{2}{3x^2}dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} + \ln(y) = \frac{2}{3x} + c_1$

- Solve for y

  $y = e^{-\frac{x \mathit{LambertW}\left(e^{\frac{3c_1x+2}{x}}\right) - 3c_1x - 2}{3x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 30

```
dsolve((2*y(x))+3*(x^2+x^2*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x \operatorname{LambertW}\left(e^{\frac{-2c_1x+2}{x}}\right) + 2c_1x - 2}{3x}}$$

✓ Solution by Mathematica

Time used: 4.79 (sec). Leaf size: 82

```
DSolve[(2*y[x])+3*(x^2+x^2*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{W\left(e^{\frac{2}{x}+3c_1}\right)}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{W\left(e^{\frac{2}{x}+3c_1}\right)}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{W\left(e^{\frac{2}{x}+3c_1}\right)}$$
$$y(x) \to 0$$

7.20 problem 21

Internal problem ID [1080]
Internal file name [`OUTPUT/1081_Sunday_June_05_2022_02_02_10_AM_55863699/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{a\cos(x)\,y-\sin(x)\,y^2+(b\cos(x)\,y-\sin(x)\,xy)\,y'=0}$$

7.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{\sin(x)}{b\cos(x)-\sin(x)\,x}$$
$$q(x)=\frac{\cos(x)\,a}{\sin(x)\,x-b\cos(x)}$$

Hence the ode is

$$y'-\frac{\sin(x)\,y}{b\cos(x)-\sin(x)\,x}=\frac{\cos(x)\,a}{\sin(x)\,x-b\cos(x)}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\cos(x)a}{\sin(x)x-b\cos(x)}\right)$$

$$\frac{d}{dx}\left(e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx} y\right) = \left(e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx}\right)\left(\frac{\cos(x)a}{\sin(x)x-b\cos(x)}\right)$$

$$d\left(e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx} y\right) = \left(\frac{\cos(x)a\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)}\right) dx$$

Integrating gives

$$e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx} y = \int \frac{\cos(x)a\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)}\, dx$$

$$e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx} y = \int \frac{\cos(x)a\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)} dx + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{\sin(x)}{b\cos(x)-\sin(x)x}dx}$ results in

$$y = e^{-\left(\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx\right)}\left(\int \frac{\cos(x)a\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)} dx\right) + c_1 e^{-\left(\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx\right)}$$

which simplifies to

$$y = e^{-\left(\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx\right)}\left(a\left(\int \frac{\cos(x)\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)} dx\right) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\left(\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx\right)}\left(a\left(\int \frac{\cos(x)\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)} dx\right) + c_1\right) \quad (1)$$

Verification of solutions

$$y = e^{-\left(\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx\right)}\left(a\left(\int \frac{\cos(x)\,e^{\int \frac{\sin(x)}{\sin(x)x-b\cos(x)}dx}}{\sin(x)x-b\cos(x)} dx\right) + c_1\right)$$

Verified OK.

7.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-\cos(x)\, a + \sin(x)\, y}{\sin(x)\, x - b\cos(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 339: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\ln(-ib+x)-\frac{i\left(\int -\frac{4b}{(ix+b)\left(ix\,\mathrm{e}^{2ix}+b\,\mathrm{e}^{2ix}-ix+b\right)}dx\right)}{2}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\mathrm{e}^{-\ln(-ib+x)-\frac{i\left(\int -\frac{4b}{(ix+b)\left(ix\,\mathrm{e}^{2ix}+b\,\mathrm{e}^{2ix}-ix+b\right)}dx\right)}{2}}}dy
\end{aligned}
$$

7.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx} = 0 \qquad \text{(A)}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \qquad \text{(B)}
$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\sin(x)\,x - b\cos(x))\,\mathrm{d}y = (-\sin(x)\,y + \cos(x)\,a)\,\mathrm{d}x$$
$$(-\cos(x)\,a + \sin(x)\,y)\,\mathrm{d}x + (\sin(x)\,x - b\cos(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\cos(x)\,a + \sin(x)\,y$$
$$N(x, y) = \sin(x)\,x - b\cos(x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)\,a + \sin(x)\,y)\\ &= \sin(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)\,x - b\cos(x))\\ &= x\cos(x) + \sin(x) + b\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\sin(x)\,x - b\cos(x)}\left((\sin(x)) - (x\cos(x) + \sin(x) + b\sin(x))\right) \\
&= \frac{x\cos(x) + b\sin(x)}{b\cos(x) - \sin(x)\,x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int \frac{x\cos(x)+b\sin(x)}{b\cos(x)-\sin(x)x}\,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-ix+\int -\frac{2i(ib+x)}{-ib\,\mathrm{e}^{2ix}+x\,\mathrm{e}^{2ix}-ib-x}dx} \\
&= \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}(-\cos(x)\,a + \sin(x)\,y) \\
&= (-\cos(x)\,a + \sin(x)\,y)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}(\sin(x)\,x - b\cos(x)) \\
&= (\sin(x)\,x - b\cos(x))\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left((-\cos(x)\,a + \sin(x)\,y)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}\right) + \left((\sin(x)\,x - b\cos(x))\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int (-\cos(x)\, a + \sin(x)\, y)\, \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x} dx\right)} \, \mathrm{d}x$$
$$\phi = \int^{x} (-\cos(\_a)\, a + \sin(\_a)\, y)\, \mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{(ib-\_a)\mathrm{e}^{2i\_a}+ib+\_a} d\_a\right)} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int^{x} \sin(\_a)\, \mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{(ib-\_a)\mathrm{e}^{2i\_a}+ib+\_a} d\_a\right)} d\_a + f'(y) \tag{4}$$
$$= \int^{x} \sin(\_a)\, \mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a} d\_a\right)} d\_a + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (\sin(x)\, x - b\cos(x))\, \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x} dx\right)}$. Therefore equation (4) becomes

$$(\sin(x)\, x - b\cos(x))\, \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x} dx\right)} = \int^{x} \sin(\_a)\, \mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a} d\_a\right)} d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \sin(x)\, \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x} dx\right)} x - \cos(x)\, \mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x} dx\right)} b - \left(\int^{x} \sin(\_a)\, \mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a} d\_a\right)} d\_a\right)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\sin(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}x - \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}b - \left(\int^{x} \sin(\_a)\,\mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a}d\_a\right)}d\_a\right)\right)\mathrm{d}y$$

$$f(y) = \int_{0}^{y} \left(\sin(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}x - \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}b - \left(\int^{x} \sin(\_a)\,\mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a}d\_a\right)}d\_a\right)\right)d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} (-\cos(\_a)\,a + \sin(\_a)\,y)\,\mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{(ib-\_a)\mathrm{e}^{2i\_a}+ib+\_a}d\_a\right)}d\_a + \int_{0}^{y} \left(\sin(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}x - \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}b - \left(\int^{x} \sin(\_a)\,\mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a}d\_a\right)}d\_a\right)\right)d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} (-\cos(\_a)\,a + \sin(\_a)\,y)\,\mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{(ib-\_a)\mathrm{e}^{2i\_a}+ib+\_a}d\_a\right)}d\_a + \int_{0}^{y} \left(\sin(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}x - \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}b - \left(\int^{x} \sin(\_a)\,\mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a}d\_a\right)}d\_a\right)\right)d\_a$$

Summary

The solution(s) found are the following

$$\int^{x} (-\cos(\_a)\,a + \sin(\_a)\,y)\,\mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{(ib-\_a)\mathrm{e}^{2i\_a}+ib+\_a}d\_a\right)}d\_a + \int_{0}^{y} \left(\sin(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}x - \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{ix-b}{(ib-x)\mathrm{e}^{2ix}+ib+x}dx\right)}b - \left(\int^{x} \sin(\_a)\,\mathrm{e}^{-i\_a+2\left(\int \frac{ib+\_a}{\mathrm{e}^{2i\_a}(i\_a+b)+b-i\_a}d\_a\right)}d\_a\right)\right)d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^{x} \left(-\cos\left(\_a\right) a+\sin\left(\_a\right) y\right) \mathrm{e}^{-i\_a+2\left(\int \frac{i\_a-b}{\left(i b-\_a\right) \mathrm{e}^{2 i\_a}+i b+\_a}d\_a\right)}d\_a \\ +\int_{0}^{y} \left(\sin\left(x\right) \mathrm{e}^{-i x+2\left(\int \frac{i x-b}{\left(i b-x\right) \mathrm{e}^{2 i x}+i b+x}d x\right)} x-\cos\left(x\right) \mathrm{e}^{-i x+2\left(\int \frac{i x-b}{\left(i b-x\right) \mathrm{e}^{2 i x}+i b+x}d x\right)} b \\ -\left(\int^{x} \sin\left(\_a\right) \mathrm{e}^{-i\_a+2\left(\int \frac{i b+\_a}{\mathrm{e}^{2 i\_a} \left(i\_a+b\right)+b-i\_a}d\_a\right)}d\_a\right)\right) d\_a = c_{1}$$

Warning, solution could not be verified

7.20.4 Maple step by step solution

Let's solve

$a \cos\left(x\right) y-\sin\left(x\right) y^{2}+\left(b \cos\left(x\right) y-\sin\left(x\right) x y\right) y^{\prime}=0$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- Isolate the derivative

  $y^{\prime}=-\frac{\sin\left(x\right) y}{\sin\left(x\right) x-b \cos\left(x\right)}+\frac{\cos\left(x\right) a}{\sin\left(x\right) x-b \cos\left(x\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y^{\prime}+\frac{\sin\left(x\right) y}{\sin\left(x\right) x-b \cos\left(x\right)}=\frac{\cos\left(x\right) a}{\sin\left(x\right) x-b \cos\left(x\right)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y^{\prime}+\frac{\sin\left(x\right) y}{\sin\left(x\right) x-b \cos\left(x\right)}\right)=\frac{\mu(x) \cos\left(x\right) a}{\sin\left(x\right) x-b \cos\left(x\right)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d x}\left(\mu(x) y\right)$

  $\mu(x) \left(y^{\prime}+\frac{\sin\left(x\right) y}{\sin\left(x\right) x-b \cos\left(x\right)}\right)=\mu^{\prime}(x) y+\mu(x) y^{\prime}$

- Isolate $\mu^{\prime}(x)$

  $\mu^{\prime}(x)=\frac{\mu(x) \sin\left(x\right)}{\sin\left(x\right) x-b \cos\left(x\right)}$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{\int \frac{\sin\left(x\right)}{\sin\left(x\right) x-b \cos\left(x\right)}d x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{d x}\left(\mu(x) y\right)\right)d x=\int \frac{\mu(x) \cos\left(x\right) a}{\sin\left(x\right) x-b \cos\left(x\right)}d x+c_{1}$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)\cos(x) a}{\sin(x) x - b\cos(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)\cos(x) a}{\sin(x) x - b\cos(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{\int \frac{\sin(x)}{\sin(x) x - b\cos(x)} dx}$

$$y = \frac{\int \frac{\cos(x) a\, e^{\int \frac{\sin(x)}{\sin(x) x - b\cos(x)} dx}}{\sin(x) x - b\cos(x)} dx + c_1}{e^{\int \frac{\sin(x)}{\sin(x) x - b\cos(x)} dx}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\int \frac{\cos(x) a\, e^{\ln(-\mathrm{I}b + x) - \frac{\mathrm{I}\left(\int \frac{4b}{(\mathrm{I}x + b)\left(\mathrm{I}x\left(e^{\mathrm{I}x}\right)^2 + b\left(e^{\mathrm{I}x}\right)^2 - \mathrm{I}x + b\right)} dx\right)}{2}}}{\sin(x) x - b\cos(x)} dx + c_1}{e^{\ln(-\mathrm{I}b + x) - \frac{\mathrm{I}\left(\int \frac{4b}{(\mathrm{I}x + b)\left(\mathrm{I}x\left(e^{\mathrm{I}x}\right)^2 + b\left(e^{\mathrm{I}x}\right)^2 - \mathrm{I}x + b\right)} dx\right)}{2}}}$$

- Simplify

$$y = \frac{\left(a\left(\int \frac{(-\mathrm{I}b + x)\cos(x) e^{2\,\mathrm{I}b\left(\int \frac{1}{(\mathrm{I}b - x)\left((\mathrm{I}b - x)e^{2\,\mathrm{I}x} + \mathrm{I}b + x\right)} dx\right)}}{b\cos(x) - \sin(x) x} dx\right) - c_1\right) e^{-2\,\mathrm{I}b\left(\int \frac{1}{(\mathrm{I}b - x)\left((\mathrm{I}b - x)e^{2\,\mathrm{I}x} + \mathrm{I}b + x\right)} dx\right)}}{\mathrm{I}b - x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 68

```
dsolve((a*cos(x)*y(x)-y(x)*sin(x)*y(x))+(b*cos(x)*y(x)-x*sin(x)*y(x))*diff(y(x),x)=0,y(x), s
```

$$y(x) = 0$$

$$y(x) = \left(a\left(\int \frac{\cos(x)\, e^{\int \frac{\sin(x)}{x\sin(x) - \cos(x) b} dx}}{x\sin(x) - \cos(x) b} dx\right) + c_1\right) e^{-\left(\int \frac{\sin(x)}{x\sin(x) - \cos(x) b} dx\right)}$$

✓ Solution by Mathematica

Time used: 3.564 (sec). Leaf size: 106

```
DSolve[(a*Cos[x]*y[x]-y[x]*Sin[x]*y[x])+(b*Cos[x]*y[x]-x*Sin[x]*y[x])*y'[x]==0,y[x],x,Includ
```

$$y(x) \to 0$$

$$y(x) \to \exp\left(\int_1^x \frac{\sin(K[1])}{b\cos(K[1]) - K[1]\sin(K[1])} dK[1]\right) \left(\int_1^x -\frac{a\exp\left(-\int_1^{K[2]} \frac{\sin(K[1])}{b\cos(K[1])-K[1]\sin(K[1])} dK[1]\right)\cos(K[2])}{b\cos(K[2]) - K[2]\sin(K[2])} dK[2] + c_1\right)$$

$$y(x) \to 0$$

7.21 problem 22

Internal problem ID [1081]
Internal file name [`OUTPUT/1082_Sunday_June_05_2022_02_02_12_AM_15469839/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^4y^4 + x^5y^3y' = 0}$$

7.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$$
$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$
$$\ln(y) = -\ln(x) + c_1$$
$$y = e^{-\ln(x)+c_1}$$
$$= \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

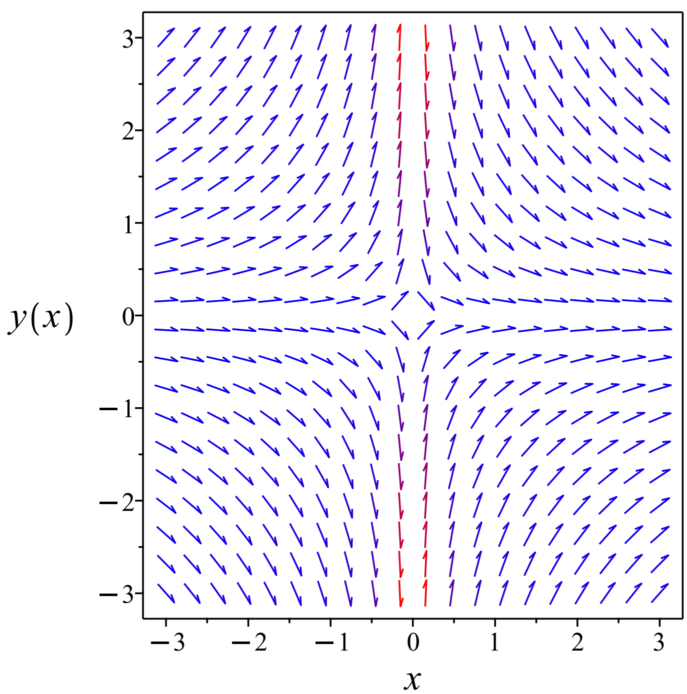


Figure 450: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

7.21.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(yx) &= 0 \end{aligned}$$

Integrating gives

$$yx = c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

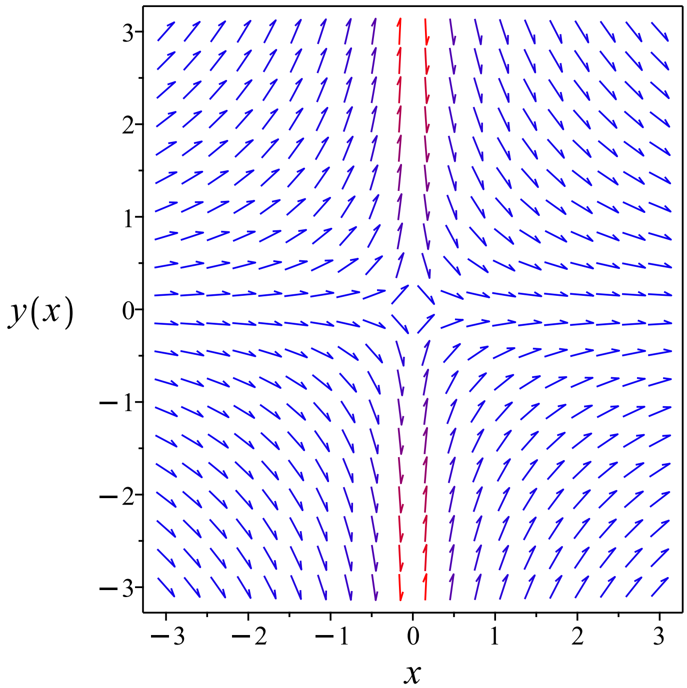


Figure 451: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

7.21.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^8 u(x)^4 + x^8 u(x)^3 \left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_2 \\
u &= e^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= \frac{c_2}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x} \tag{1}
$$

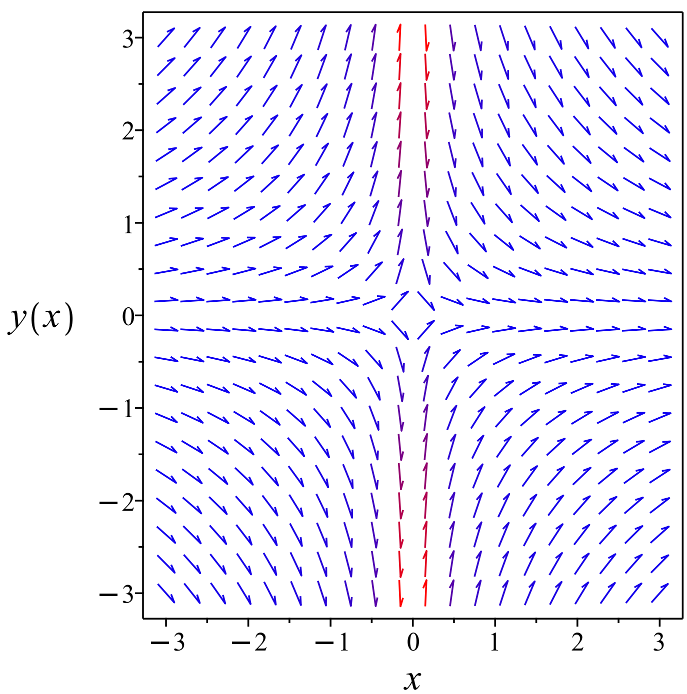


Figure 452: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

7.21.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-yx)$$

Hence (2) becomes

$$0 = d(-yx)$$

Integrating both sides gives gives these solutions

$$y = \frac{c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_1 \tag{1}$$

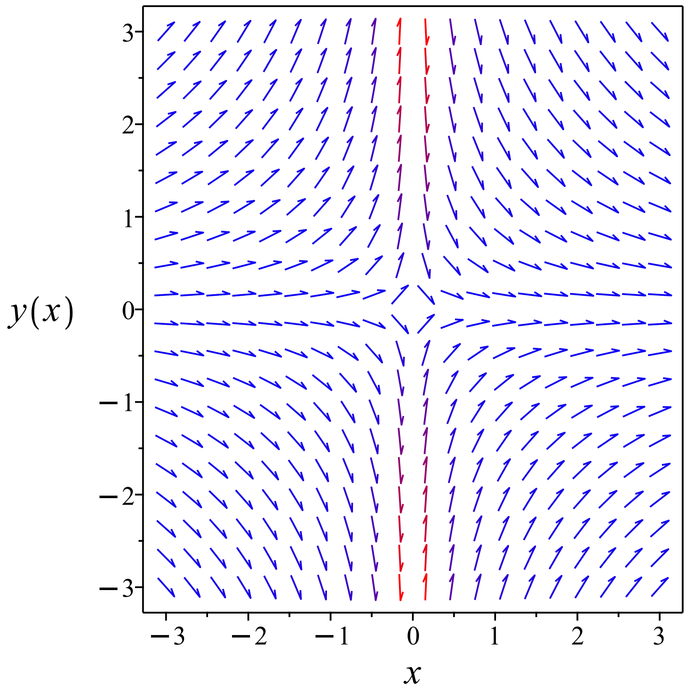


Figure 453: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x} + c_1$$

Verified OK.

7.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 342: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = c_1$$

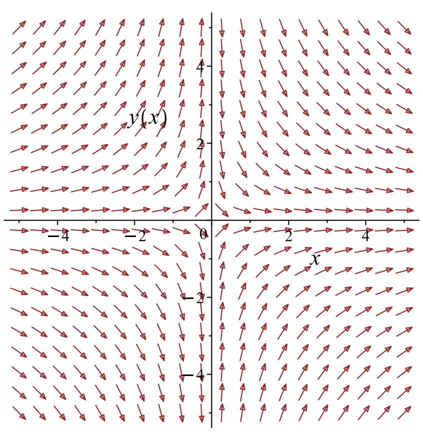
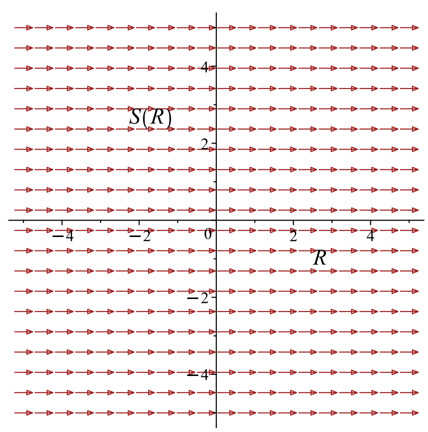
Which simplifies to

$$yx = c_1$$

Which gives

$$y = \frac{c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

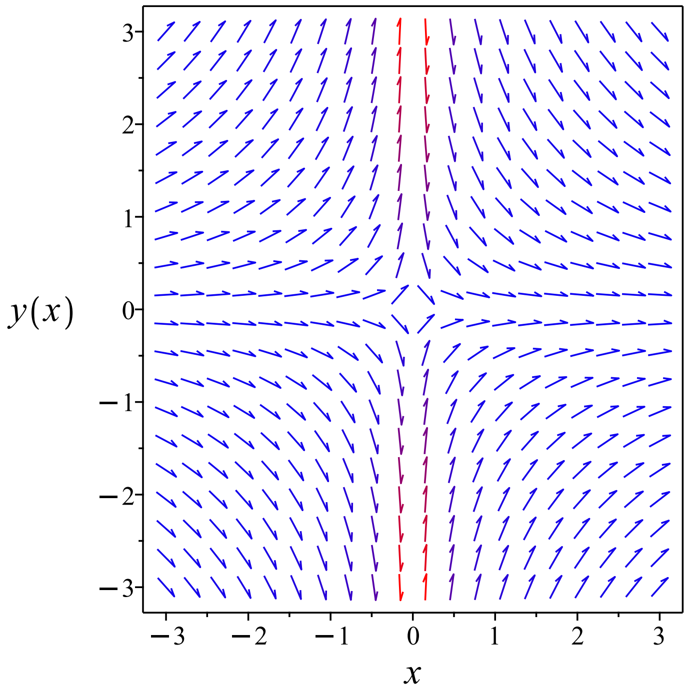


Figure 454: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

7.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-c_1}}{x} \tag{1}$$

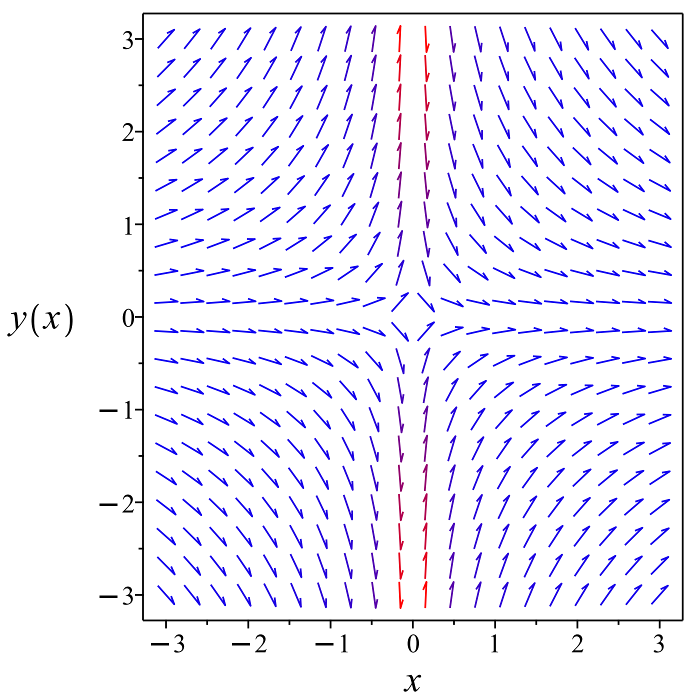


Figure 455: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{x}$$

Verified OK.

7.21.7 Maple step by step solution

Let's solve

$x^4y^4 + x^5y^3y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((x^4*y(x)^4)+(x^5*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 21

```
DSolve[(x^4*y[x]^4)+(x^5*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to 0$$

7.22 problem 23

Internal problem ID [1082]
Internal file name [`OUTPUT/1083_Sunday_June_05_2022_02_02_13_AM_6047940/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y(x\cos(x)+2\sin(x))+x(1+y)\,y'=0}$$

7.22.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{y(x\cos(x)+2\sin(x))}{x\,(y+1)}
\end{aligned}$$

Where $f(x)=-\frac{x\cos(x)+2\sin(x)}{x}$ and $g(y)=\frac{y}{y+1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y+1}}\,dy=-\frac{x\cos(x)+2\sin(x)}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y+1}} dy = \int -\frac{x\cos(x) + 2\sin(x)}{x} dx$$

$$y + \ln(y) = -\sin(x) - 2\,\mathrm{Si}(x) + c_1$$

Which results in

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)+c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)+c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)+c_1} \tag{1}$$

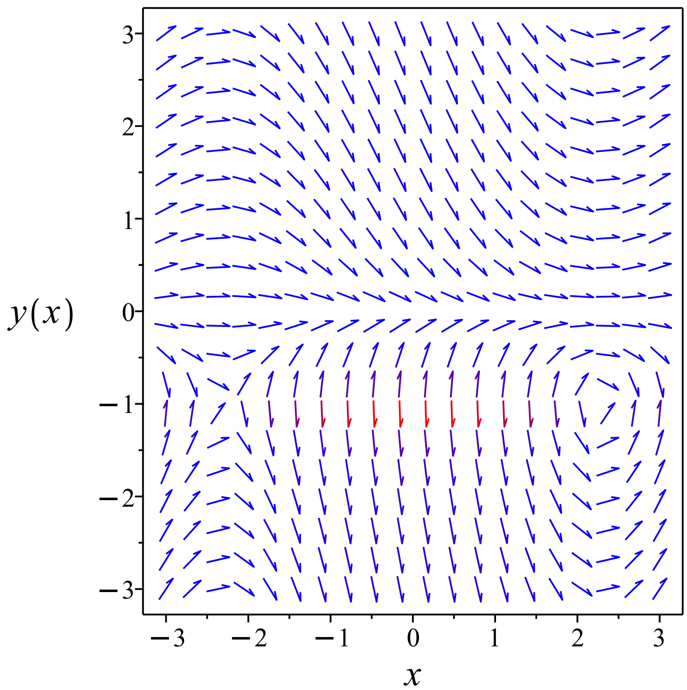


Figure 456: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)+c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)+c_1}$$

Verified OK.

7.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y(x\cos(x) + 2\sin(x))}{x(y+1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 345: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f(\frac{y}{x})$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F(\frac{y}{x})$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{x}{x\cos(x) + 2\sin(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x}{x\cos(x)+2\sin(x)}} dx \end{aligned}$$

Which results in

$$S = -\sin(x) - 2\,\mathrm{Si}(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x\cos(x) + 2\sin(x))}{x(y+1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\cos(x) - \frac{2\sin(x)}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y+1}{y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R+1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R + \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sin(x) - 2\ \mathrm{Si}(x) = y + \ln(y) + c_1
$$

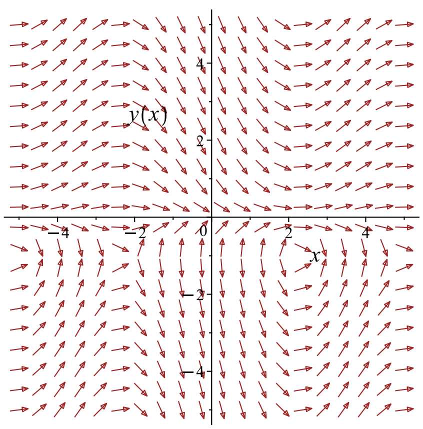
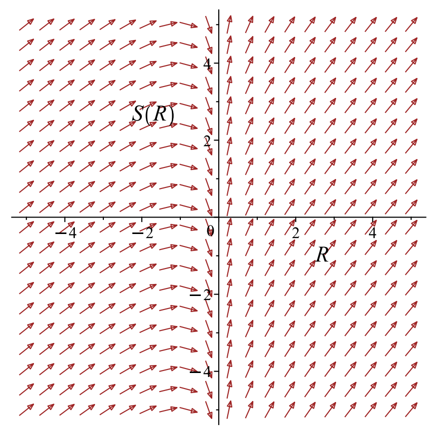
Which simplifies to

$$
-\sin(x) - 2\ \mathrm{Si}(x) = y + \ln(y) + c_1
$$

Which gives

$$
y = \mathrm{e}^{-\operatorname{LambertW}\left(\mathrm{e}^{-\sin(x)-2\ \mathrm{Si}(x)-c_1}\right)-\sin(x)-2\ \mathrm{Si}(x)-c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x\cos(x)+2\sin(x))}{x(y+1)}$ 	$R = y$ $S = -\sin(x) - 2\,\mathrm{Si}(x)$	$\frac{dS}{dR} = \frac{R+1}{R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)-c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)-c_1} \tag{1}$$

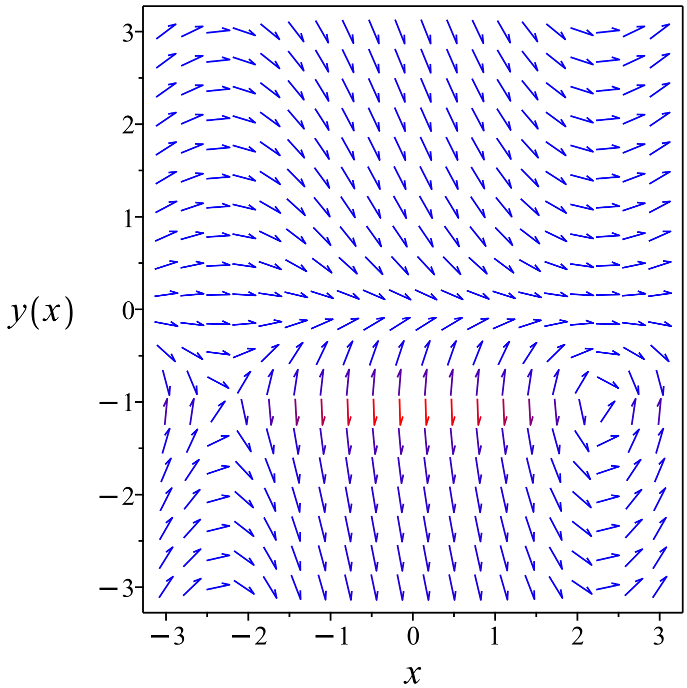


Figure 457: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(e^{-\sin(x)-2\operatorname{Si}(x)-c_1}\right)-\sin(x)-2\operatorname{Si}(x)-c_1}$$

Verified OK.

7.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y+1}{y}\right)\mathrm{d}y = \left(\frac{x\cos(x) + 2\sin(x)}{x}\right)\mathrm{d}x$$
$$\left(-\frac{x\cos(x) + 2\sin(x)}{x}\right)\mathrm{d}x + \left(-\frac{y+1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x\cos(x) + 2\sin(x)}{x}$$
$$N(x, y) = -\frac{y+1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x\cos(x) + 2\sin(x)}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y+1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x\cos(x)+2\sin(x)}{x}\,\mathrm{d}x \\
\phi &= -\sin(x) - 2\ \mathrm{Si}(x) + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y+1}{y}$. Therefore equation (4) becomes

$$
-\frac{y+1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y+1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{-y-1}{y}\right)\mathrm{d}y \\
f(y) &= -y - \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - 2\,\mathrm{Si}(x) - y - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - 2\,\mathrm{Si}(x) - y - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)-c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)-c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)-c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)-c_1} \tag{1}$$

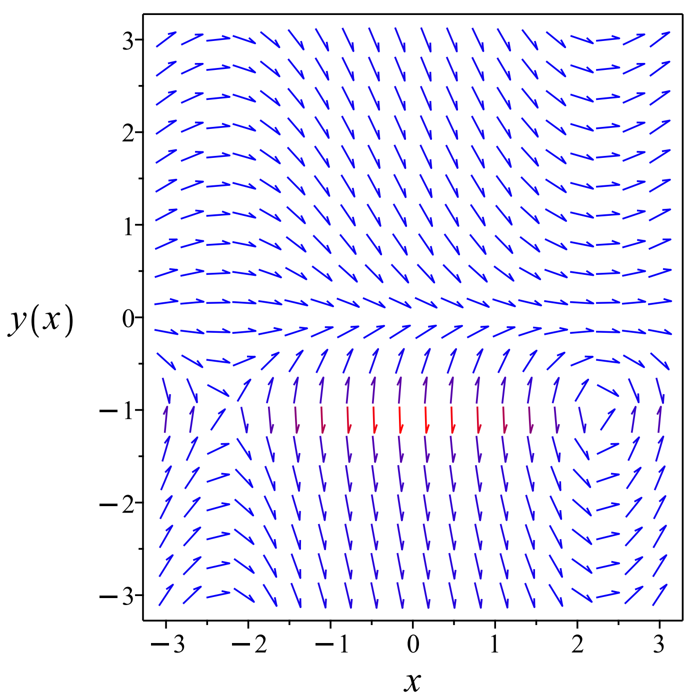


Figure 458: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)-c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)-c_1}$$

Verified OK.

7.22.4 Maple step by step solution

Let's solve

$y(x\cos(x)+2\sin(x))+x(1+y)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y)}{y}=-\frac{x\cos(x)+2\sin(x)}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y)}{y}dx=\int -\frac{x\cos(x)+2\sin(x)}{x}dx+c_1$

- Evaluate integral

  $y+\ln(y)=-\sin(x)-2\,\mathrm{Si}(x)+c_1$

- Solve for y

  $y=\mathrm{e}^{-\mathit{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)+c_1}\right)-\sin(x)-2\,\mathrm{Si}(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 18

```
dsolve((y(x)*(x*cos(x)+2*sin(x)))+(x*(y(x)+1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\mathrm{LambertW}\left(\mathrm{e}^{-\sin(x)-2\,\mathrm{Si}(x)-c_1}\right)$$

✓ Solution by Mathematica

Time used: 2.869 (sec). Leaf size: 24

```
DSolve[(y[x]*(x*Cos[x]+2*Sin[x]))+(x*(y[x]+1))*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to W\left(e^{-2\mathrm{Si}(x)-\sin(x)+c_1}\right)$$
$$y(x) \to 0$$

7.23 problem 24

Internal problem ID [1083]
Internal file name [`OUTPUT/1084_Sunday_June_05_2022_02_02_15_AM_67609577/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y^3x^4 + y + (y^2x^5 - x)\, y' = 0}$$

7.23.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(x^4y^2 + 1)}{x\,(x^4y^2 - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{y(x^4y^2+1)(b_3-a_2)}{x(x^4y^2-1)} - \frac{y^2(x^4y^2+1)^2 a_3}{x^2(x^4y^2-1)^2} \\
& - \left(-\frac{4y^3x^2}{x^4y^2-1} + \frac{y(x^4y^2+1)}{x^2(x^4y^2-1)} + \frac{4y^3(x^4y^2+1)x^2}{(x^4y^2-1)^2}\right)(xa_2+ya_3+a_1) \\
& - \left(-\frac{x^4y^2+1}{x(x^4y^2-1)} - \frac{2y^2x^3}{x^4y^2-1} + \frac{2y^2(x^4y^2+1)x^3}{(x^4y^2-1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$\frac{2x^{10}y^4b_2 - 2x^8y^6a_3 + x^9y^4b_1 - x^8y^5a_1 - 6x^6y^2b_2 - 8x^5y^3a_2 - 4x^5y^3b_3 - 10x^4y^4a_3 - 4x^5y^2b_1 - 8x^4y^3a_1 -}{(x^4y^2-1)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 2x^{10}y^4b_2 - 2x^8y^6a_3 + x^9y^4b_1 - x^8y^5a_1 - 6x^6y^2b_2 - 8x^5y^3a_2 \\
& - 4x^5y^3b_3 - 10x^4y^4a_3 - 4x^5y^2b_1 - 8x^4y^3a_1 - xb_1 + ya_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -2a_3v_1^8v_2^6 + 2b_2v_1^{10}v_2^4 - a_1v_1^8v_2^5 + b_1v_1^9v_2^4 - 8a_2v_1^5v_2^3 - 10a_3v_1^4v_2^4 \\
& - 6b_2v_1^6v_2^2 - 4b_3v_1^5v_2^3 - 8a_1v_1^4v_2^3 - 4b_1v_1^5v_2^2 + a_1v_2 - b_1v_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 2b_2v_1^{10}v_2^4 + b_1v_1^9v_2^4 - 2a_3v_1^8v_2^6 - a_1v_1^8v_2^5 - 6b_2v_1^6v_2^2 + (-8a_2 - 4b_3)\,v_1^5v_2^3 \\ - 4b_1v_1^5v_2^2 - 10a_3v_1^4v_2^4 - 8a_1v_1^4v_2^3 - b_1v_1 + a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -8a_1 &= 0 \\ -a_1 &= 0 \\ -10a_3 &= 0 \\ -2a_3 &= 0 \\ -4b_1 &= 0 \\ -b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ -8a_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2y - \left(-\frac{y(x^4y^2+1)}{x\,(x^4y^2-1)}\right)(x) \\
&= \frac{-y^3x^4+3y}{x^4y^2-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^3x^4+3y}{x^4y^2-1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln\left(y(x^4y^2-3)\right)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(x^4y^2+1)}{x\,(x^4y^2-1)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{4x^3y^2}{3x^4y^2 - 9} \\
S_y &= \frac{-x^4y^2 + 1}{y\left(x^4y^2 - 3\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln\left(R\right)}{3} + c_1 \tag{4}$$

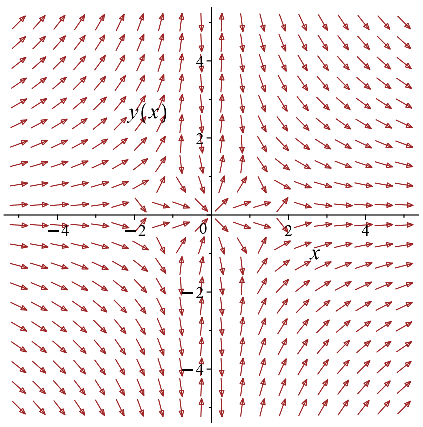
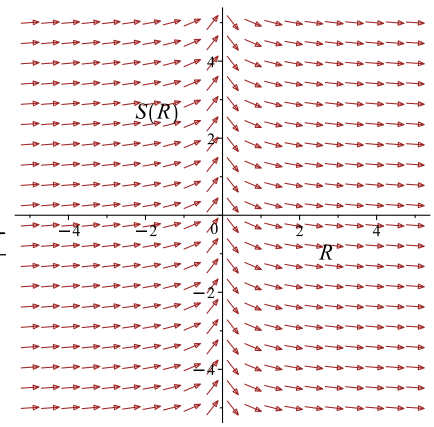
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(y\right)}{3} - \frac{\ln\left(x^4y^2 - 3\right)}{3} = -\frac{\ln\left(x\right)}{3} + c_1$$

Which simplifies to

$$-\frac{\ln\left(y\right)}{3} - \frac{\ln\left(x^4y^2 - 3\right)}{3} = -\frac{\ln\left(x\right)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^4y^2+1)}{x(x^4y^2-1)}$ 	$R = x$ $S = -\frac{\ln (y)}{3} - \frac{\ln (x^4 y^2 -}{3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln (y)}{3} - \frac{\ln (x^4 y^2 - 3)}{3} = -\frac{\ln (x)}{3} + c_1 \tag{1}$$

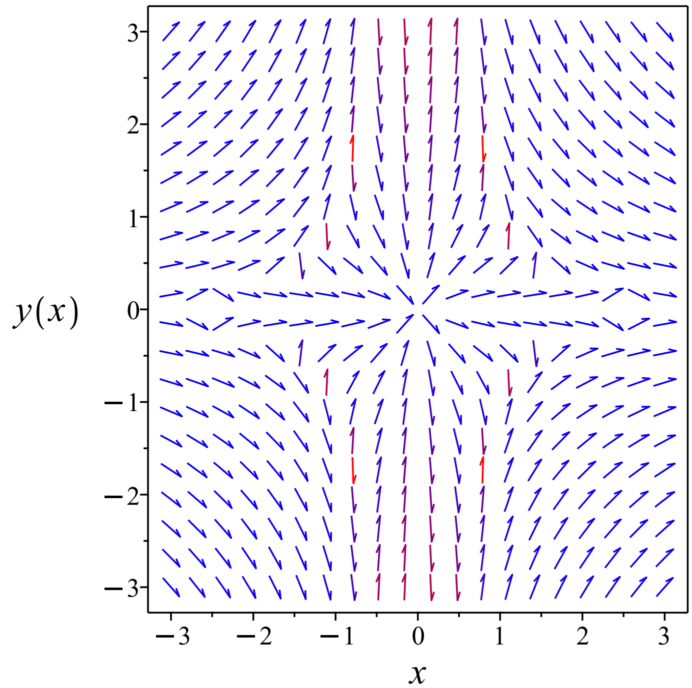


Figure 459: Slope field plot

Verification of solutions

$$-\frac{\ln(y)}{3} - \frac{\ln(x^4y^2-3)}{3} = -\frac{\ln(x)}{3} + c_1$$

Verified OK.

7.23.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y^2 x^5 - x\right) \mathrm{d}y = \left(-y^3 x^4 - y\right) \mathrm{d}x$$
$$\left(y^3 x^4 + y\right) \mathrm{d}x + \left(y^2 x^5 - x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y^3 x^4 + y$$
$$N(x, y) = y^2 x^5 - x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3 x^4 + y\right) \\ &= 3x^4 y^2 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 x^5 - x\right) \\ &= 5x^4 y^2 - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{y^2x^5 - x}\left(\left(3x^4y^2 + 1\right) - \left(5x^4y^2 - 1\right)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(y^3x^4 + y\right) \\
&= \frac{y^3x^4 + y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(y^2x^5 - x\right) \\
&= \frac{x^4y^2 - 1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^3x^4 + y}{x^2}\right) + \left(\frac{x^4y^2 - 1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{y^3x^4 + y}{x^2} \,\mathrm{d}x$$
$$\phi = \frac{y(x^4y^2 - 3)}{3x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^4y^2 - 3}{3x} + \frac{2x^3y^2}{3} + f'(y) \tag{4}$$
$$= \frac{x^4y^2 - 1}{x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^4y^2-1}{x}$. Therefore equation (4) becomes

$$\frac{x^4y^2 - 1}{x} = \frac{x^4y^2 - 1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y(x^4y^2 - 3)}{3x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y(x^4y^2 - 3)}{3x}$$

Summary

The solution(s) found are the following

$$\frac{y(x^4y^2 - 3)}{3x} = c_1 \tag{1}$$

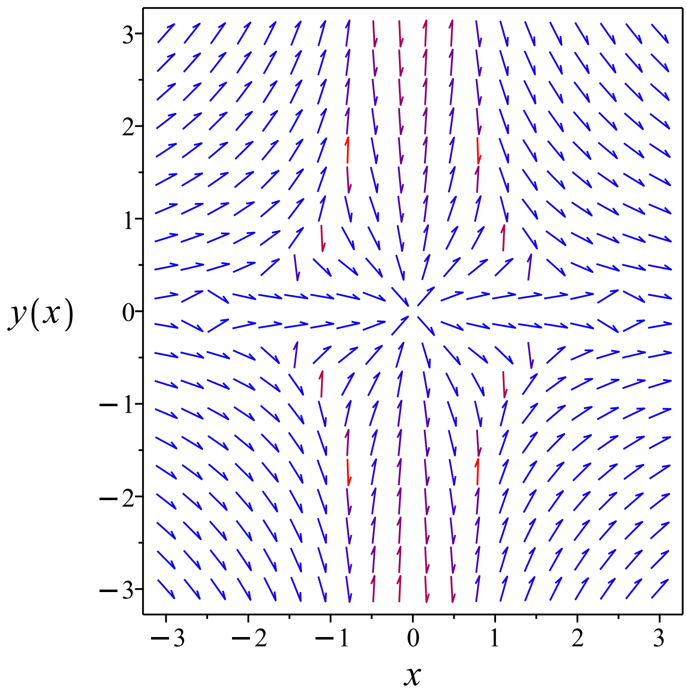


Figure 460: Slope field plot

Verification of solutions

$$\frac{y(x^4y^2 - 3)}{3x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 242

```
dsolve((x^4*y(x)^3+y(x))+(x^5*y(x)^2-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{2}{3}}+4c_1}{2x^2\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

$$y(x) = \frac{-i\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{2}{3}}\sqrt{3}+4i\sqrt{3}\,c_1-\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{2}{3}}-4c_1}{4x^2\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

$$y(x) = -\frac{-i\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{2}{3}}\sqrt{3}+4i\sqrt{3}\,c_1+\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{2}{3}}+4c_1}{4x^2\left(4x^3+4\sqrt{x^6-4c_1^3}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 39.244 (sec). Leaf size: 300

```
DSolve[(x^4*y[x]^3+y[x])+(x^5*y[x]^2-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{2+\frac{\sqrt[3]{2}\left(3c_1x^9+\sqrt{x^{12}(-4+9{c_1}^2x^6)}\right)^{2/3}}{x^4}}{2^{2/3}\sqrt[3]{3c_1x^9+\sqrt{x^{12}\left(-4+9{c_1}^2x^6\right)}}}$$

$$y(x)\to \frac{i\left(\left(\sqrt{3}+i\right)\left(6c_1x^9+2\sqrt{x^{12}\left(-4+9{c_1}^2x^6\right)}\right)^{2/3}-2\sqrt[3]{2}\left(\sqrt{3}-i\right)x^4\right)}{4x^4\sqrt[3]{3c_1x^9+\sqrt{x^{12}\left(-4+9{c_1}^2x^6\right)}}}$$

$$y(x)\to \frac{2i\sqrt[3]{2}\left(\sqrt{3}+i\right)x^4+\left(-1-i\sqrt{3}\right)\left(6c_1x^9+2\sqrt{x^{12}\left(-4+9{c_1}^2x^6\right)}\right)^{2/3}}{4x^4\sqrt[3]{3c_1x^9+\sqrt{x^{12}\left(-4+9{c_1}^2x^6\right)}}}$$

7.24 problem 25

Internal problem ID [1084]
Internal file name [`OUTPUT/1085_Sunday_June_05_2022_02_02_16_AM_22897266/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{3yx + 2y^2 + y + (x^2 + 2yx + x + 2y)\, y' = 0}$$

7.24.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$3u(x)\, x^2 + 2u(x)^2\, x^2 + u(x)\, x + (x^2 + 2u(x)\, x^2 + x + 2u(x)\, x)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(1+2x)\,(u+1)}{x\,(x+1)\,(2u+1)} \end{aligned}$$

Where $f(x) = -\frac{2(1+2x)}{x(x+1)}$ and $g(u) = \frac{u(u+1)}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+1)}{2u+1}}\, du = -\frac{2(1+2x)}{x\,(x+1)}\, dx$$

$$\int \frac{1}{\frac{u(u+1)}{2u+1}}\, du = \int -\frac{2(1+2x)}{x\,(x+1)}\, dx$$

$$\ln\left(u(u+1)\right) = -2\ln\left(x(x+1)\right) + c_2$$

Raising both side to exponential gives

$$u(u+1) = \mathrm{e}^{-2\ln(x(x+1))+c_2}$$

Which simplifies to

$$u(u+1) = \frac{c_3}{x^2\,(x+1)^2}$$

Which simplifies to

$$u(x)\,(u(x)+1) = \frac{c_3 \mathrm{e}^{c_2}}{x^2\,(x+1)^2}$$

The solution is

$$u(x)\,(u(x)+1) = \frac{c_3 \mathrm{e}^{c_2}}{x^2\,(x+1)^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y\left(1+\frac{y}{x}\right)}{x} = \frac{c_3 \mathrm{e}^{c_2}}{x^2\,(x+1)^2}$$

$$\frac{y(x+y)}{x^2} = \frac{c_3 \mathrm{e}^{c_2}}{x^2\,(x+1)^2}$$

Which simplifies to

$$y(x+y) = \frac{c_3 \mathrm{e}^{c_2}}{(x+1)^2}$$

Summary

The solution(s) found are the following

$$y(x+y) = \frac{c_3 \mathrm{e}^{c_2}}{(x+1)^2} \tag{1}$$

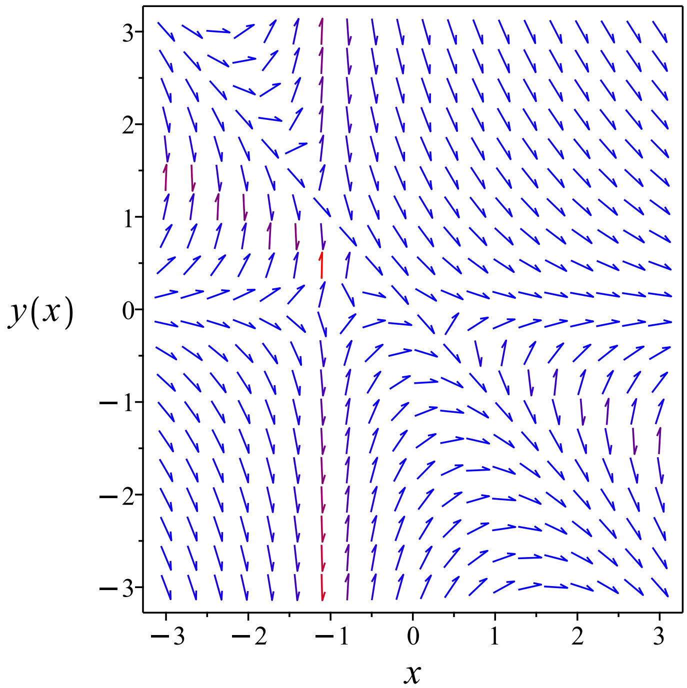


Figure 461: Slope field plot

Verification of solutions

$$y(x+y) = \frac{c_3 e^{c_2}}{(x+1)^2}$$

Verified OK.

7.24.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2 + 2yx + x + 2y\right)\mathrm{d}y &= \left(-3yx - 2y^2 - y\right)\mathrm{d}x\\ \left(3yx + 2y^2 + y\right)\mathrm{d}x + \left(x^2 + 2yx + x + 2y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3yx + 2y^2 + y\\ N(x, y) &= x^2 + 2yx + x + 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3yx + 2y^2 + y\right)\\ &= 3x + 4y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 2yx + x + 2y\right)\\ &= 2x + 2y + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{(x+1)(x+2y)}((3x+4y+1) - (2x+2y+1)) \\ &= \frac{1}{x+1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x+1} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x+1)} \\ &= x+1 \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x+1\left(3yx + 2y^2 + y\right) \\ &= y(3x+2y+1)(x+1) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x+1\left(x^2 + 2yx + x + 2y\right) \\ &= (x+1)^2 (x+2y) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (y(3x+2y+1)(x+1)) + \left((x+1)^2 (x+2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int y(3x + 2y + 1)\,(x + 1)\,\mathrm{d}x$$

$$\phi = \left(x^2 + (2 + y)\,x + 2y + 1\right) xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (2 + x)\,xy + \left(x^2 + (2 + y)\,x + 2y + 1\right) x + f'(y) \tag{4}$$

$$= \left(x^2 + (2y + 2)\,x + 4y + 1\right) x + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x + 1)^2\,(x + 2y)$. Therefore equation (4) becomes

$$(x + 1)^2\,(x + 2y) = \left(x^2 + (2y + 2)\,x + 4y + 1\right) x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (2y) \,\mathrm{d}y$$

$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^2 + (2 + y)\,x + 2y + 1\right) xy + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^2 + (2+y)\,x + 2y + 1\right) xy + y^2$$

<u>Summary</u>
The solution(s) found are the following

$$\left(x^2 + (2+y)\,x + 2y + 1\right) xy + y^2 = c_1 \tag{1}$$

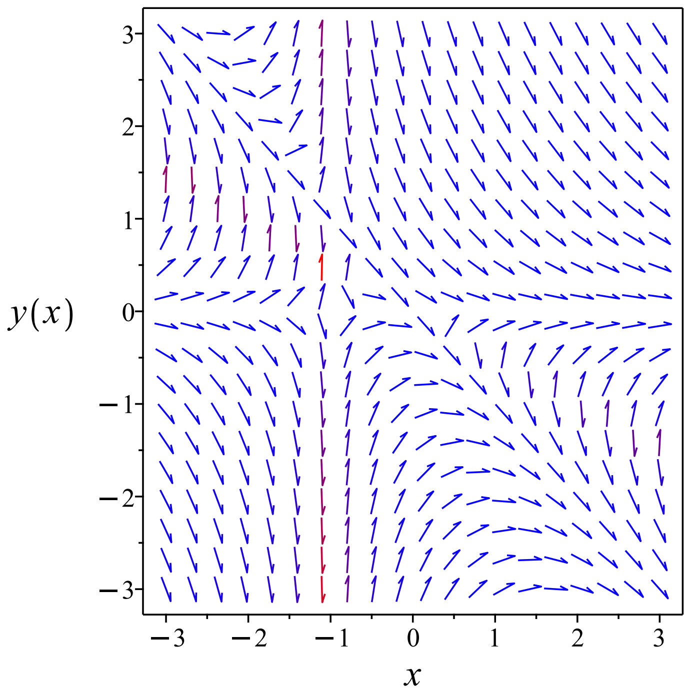


Figure 462: Slope field plot

<u>Verification of solutions</u>

$$\left(x^2 + (2+y)\,x + 2y + 1\right) xy + y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 85

```
dsolve((3*x*y(x)+2*y(x)^2+y(x))+(x^2+2*x*y(x)+x+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{4 + x^2 (x+1)^2 c_1^2} + (-x^2 - x) c_1}{2c_1 (x+1)}$$
$$y(x) = \frac{\sqrt{4 + x^2 (x+1)^2 c_1^2} + (-x^2 - x) c_1}{2c_1 (x+1)}$$

✓ Solution by Mathematica

Time used: 14.424 (sec). Leaf size: 105

```
DSolve[(3*x*y[x]+2*y[x]^2+y[x])+(x^2+2*x*y[x]+x+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{2}\left(-x - \sqrt{x^2 + \frac{4e^{c_1}}{(x+1)^2}}\right)$$
$$y(x) \to \frac{1}{2}\left(-x + \sqrt{x^2 + \frac{4e^{c_1}}{(x+1)^2}}\right)$$
$$y(x) \to \frac{1}{2}\left(-\sqrt{x^2} - x\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{x^2} - x\right)$$

7.25 problem 26

Internal problem ID [1085]
Internal file name [`OUTPUT/1086_Sunday_June_05_2022_02_02_18_AM_91680699/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{12yx + 6y^3 + (9x^2 + 10xy^2)\, y' = 0}$$

7.25.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{6y(y^2 + 2x)}{x\,(10y^2 + 9x)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{6y(y^2+2x)(b_3-a_2)}{x(10y^2+9x)} - \frac{36y^2(y^2+2x)^2 a_3}{x^2(10y^2+9x)^2} - \left(-\frac{12y}{(10y^2+9x)x} + \frac{6y(y^2+2x)}{x^2(10y^2+9x)} + \frac{54y(y^2+2x)}{x(10y^2+9x)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{6(y^2+2x)}{x(10y^2+9x)} - \frac{12y^2}{x(10y^2+9x)} + \frac{120y^2(y^2+2x)}{x(10y^2+9x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{160x^2y^4b_2 - 96y^6a_3 + 222x^3y^2b_2 + 66x^2y^3a_2 - 132x^2y^3b_3 - 252x\,y^4a_3 + 60x\,y^4b_1 - 60y^5a_1 + 189x^4b_2 - 2}{x^2(10y^2+9x)^2} = 0$$

Setting the numerator to zero gives

$$160x^2y^4b_2 - 96y^6a_3 + 222x^3y^2b_2 + 66x^2y^3a_2 - 132x^2y^3b_3 - 252x\,y^4a_3 + 60x\,y^4b_1 - 60y^5a_1 + 189x^4b_2 - 252x^2y^2a_3 + 42x^2y^2b_1 - 108x\,y^3a_1 + 108x^3b_1 - 108x^2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-96a_3v_2^6 + 160b_2v_1^2v_2^4 - 60a_1v_2^5 + 66a_2v_1^2v_2^3 - 252a_3v_1v_2^4 + 60b_1v_1v_2^4 + 222b_2v_1^3v_2^2 - 132b_3v_1^2v_2^3 - 108a_1v_1v_2^3 - 252a_3v_1^2v_2^2 + 42b_1v_1^2v_2^2 + 189b_2v_1^4 - 108a_1v_1^2v_2 + 108b_1v_1^3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &189b_2v_1^4 + 222b_2v_1^3v_2^2 + 108b_1v_1^3 + 160b_2v_1^2v_2^4 \\ &+ (66a_2 - 132b_3)\, v_1^2v_2^3 + (-252a_3 + 42b_1)\, v_1^2v_2^2 - 108a_1v_1^2v_2 \\ &+ (-252a_3 + 60b_1)\, v_1v_2^4 - 108a_1v_1v_2^3 - 96a_3v_2^6 - 60a_1v_2^5 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -108a_1 &= 0 \\ -60a_1 &= 0 \\ -96a_3 &= 0 \\ 108b_1 &= 0 \\ 160b_2 &= 0 \\ 189b_2 &= 0 \\ 222b_2 &= 0 \\ 66a_2 - 132b_3 &= 0 \\ -252a_3 + 42b_1 &= 0 \\ -252a_3 + 60b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{6y(y^2+2x)}{x\,(10y^2+9x)}\right)(2x) \\
&= \frac{22y^3+33yx}{10y^2+9x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{22y^3+33yx}{10y^2+9x}} dy
\end{aligned}$$

Which results in

$$S = \frac{3\ln(y)}{11} + \frac{\ln(2y^2+3x)}{11}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{6y(y^2+2x)}{x\,(10y^2+9x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{22y^2 + 33x} \\
S_y &= \frac{10y^2 + 9x}{22y^3 + 33yx}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{11x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{11R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{3\ln(R)}{11} + c_1 \tag{4}
$$

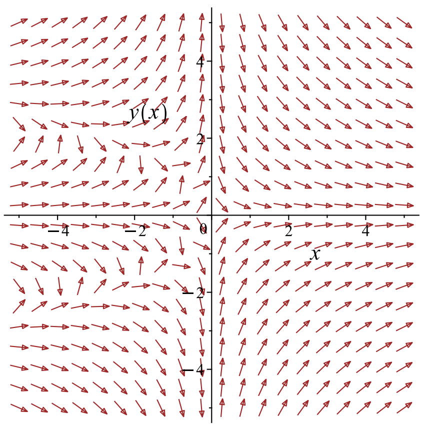
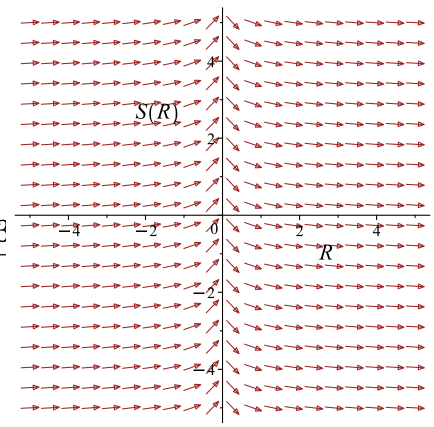
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{3\ln(y)}{11} + \frac{\ln(2y^2 + 3x)}{11} = -\frac{3\ln(x)}{11} + c_1
$$

Which simplifies to

$$
\frac{3\ln(y)}{11} + \frac{\ln(2y^2 + 3x)}{11} = -\frac{3\ln(x)}{11} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6y(y^2+2x)}{x(10y^2+9x)}$  	$R = x$ $S = \frac{3\ln(y)}{11} + \frac{\ln(2y^2 + 3}{11}$	$\frac{dS}{dR} = -\frac{3}{11R}$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(y)}{11} + \frac{\ln(2y^2 + 3x)}{11} = -\frac{3\ln(x)}{11} + c_1 \tag{1}$$

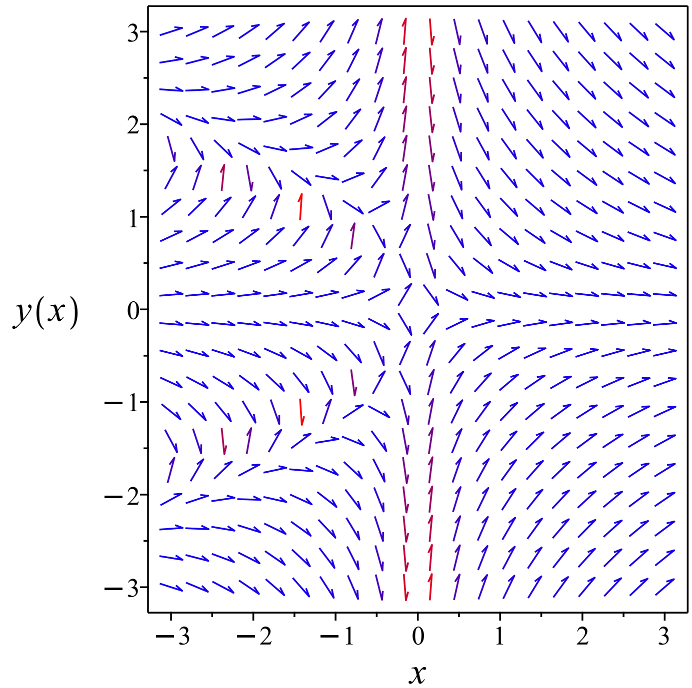


Figure 463: Slope field plot

<u>Verification of solutions</u>

$$\frac{3\ln(y)}{11} + \frac{\ln(2y^2+3x)}{11} = -\frac{3\ln(x)}{11} + c_1$$

Verified OK.

7.25.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(10x\,y^2 + 9x^2\right)\mathrm{d}y = \left(-6y^3 - 12yx\right)\mathrm{d}x$$
$$\left(6y^3 + 12yx\right)\mathrm{d}x + \left(10x\,y^2 + 9x^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 6y^3 + 12yx$$
$$N(x,y) = 10x\,y^2 + 9x^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(6y^3 + 12yx\right)\\ &= 18y^2 + 12x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(10x\,y^2 + 9x^2\right)\\ &= 10y^2 + 18x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{10x\,y^2 + 9x^2}\left(\left(18y^2 + 12x\right) - \left(10y^2 + 18x\right)\right) \\
&= \frac{8y^2 - 6x}{10x\,y^2 + 9x^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{6y^3 + 12yx}\left(\left(10y^2 + 18x\right) - \left(18y^2 + 12x\right)\right) \\
&= \frac{-4y^2 + 3x}{3y^3 + 6yx}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(10y^2 + 18x) - (18y^2 + 12x)}{x\,(6y^3 + 12yx) - y\,(10x\,y^2 + 9x^2)} \\
&= \frac{2}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\mu = e^{2\ln(t)} \\ = t^2$$

Now t is replaced back with xy giving

$$\mu = x^2 y^2$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= x^2 y^2 (6y^3 + 12yx) \\ &= 6x^2 y^5 + 12y^3 x^3 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x^2 y^2 (10x\, y^2 + 9x^2) \\ &= x^3 (10y^2 + 9x)\, y^2 \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (6x^2 y^5 + 12y^3 x^3) + (x^3 (10y^2 + 9x)\, y^2) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 6x^2 y^5 + 12y^3 x^3 \,\mathrm{d}x \end{aligned}$$

$$\phi = y^3 x^3 (2y^2 + 3x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= 3y^2x^3\left(2y^2+3x\right)+4y^4x^3+f'(y) \\ &= x^3\left(10y^2+9x\right)y^2+f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^3(10y^2+9x)\,y^2$. Therefore equation (4) becomes

$$x^3\left(10y^2+9x\right)y^2=x^3\left(10y^2+9x\right)y^2+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=y^3x^3\left(2y^2+3x\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=y^3x^3\left(2y^2+3x\right)$$

Summary

The solution(s) found are the following

$$y^3x^3\left(2y^2+3x\right)=c_1 \tag{1}$$

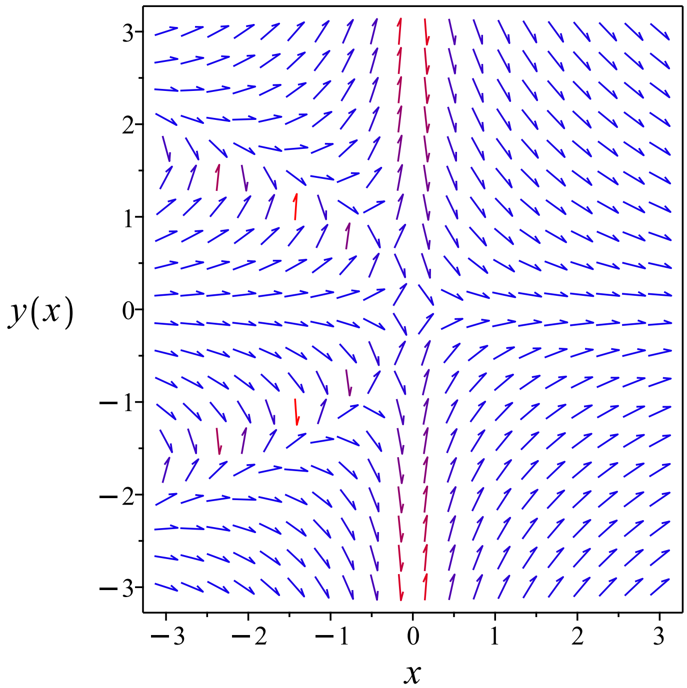


Figure 464: Slope field plot

Verification of solutions

$$y^3x^3(2y^2+3x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 35

```
dsolve((12*x*y(x)+6*y(x)^3)+(9*x^2+10*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\ln(x) - c_1 + \frac{6\ln\left(\frac{y(x)}{\sqrt{x}}\right)}{11} + \frac{2\ln\left(\frac{2y(x)^2+3x}{x}\right)}{11} = 0$$

✓ Solution by Mathematica

Time used: 8.107 (sec). Leaf size: 151

```
DSolve[(12*x*y[x]+6*y[x]^3)+(9*x^2+10*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$\begin{aligned}
y(x) &\to \text{Root}\left[2\#1^5x^3 + 3\#1^3x^4 - c_1\&, 1\right] \\
y(x) &\to \text{Root}\left[2\#1^5x^3 + 3\#1^3x^4 - c_1\&, 2\right] \\
y(x) &\to \text{Root}\left[2\#1^5x^3 + 3\#1^3x^4 - c_1\&, 3\right] \\
y(x) &\to \text{Root}\left[2\#1^5x^3 + 3\#1^3x^4 - c_1\&, 4\right] \\
y(x) &\to \text{Root}\left[2\#1^5x^3 + 3\#1^3x^4 - c_1\&, 5\right]
\end{aligned}$$

7.26 problem 27

Internal problem ID [1086]
Internal file name [`OUTPUT/1087_Sunday_June_05_2022_02_02_20_AM_48374869/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 2, First order equations. Exact equations. Integrating factors. Section 2.6 Page 91
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3x^2y^2 + 2y + 2y'x = 0}$$

7.26.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(3y\,x^2 + 2)}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 348: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\, y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3y\, x^2 + 2)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{1}{x\, y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{3R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = -\frac{3x}{2} + c_1$$

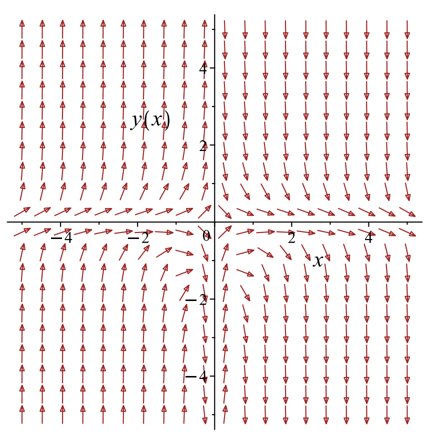
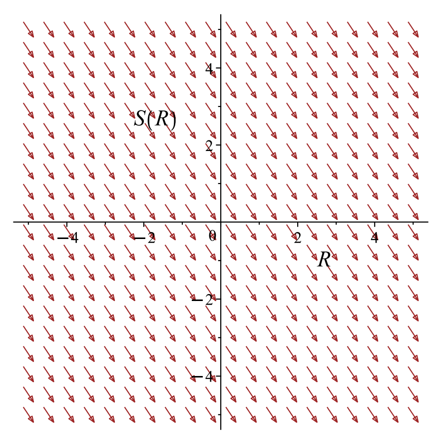
Which simplifies to

$$-\frac{1}{yx} = -\frac{3x}{2} + c_1$$

Which gives

$$y = -\frac{2}{x\left(-3x + 2c_1\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3y\,x^2+2)}{2x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = -\frac{3}{2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2}{x(-3x + 2c_1)} \tag{1}$$

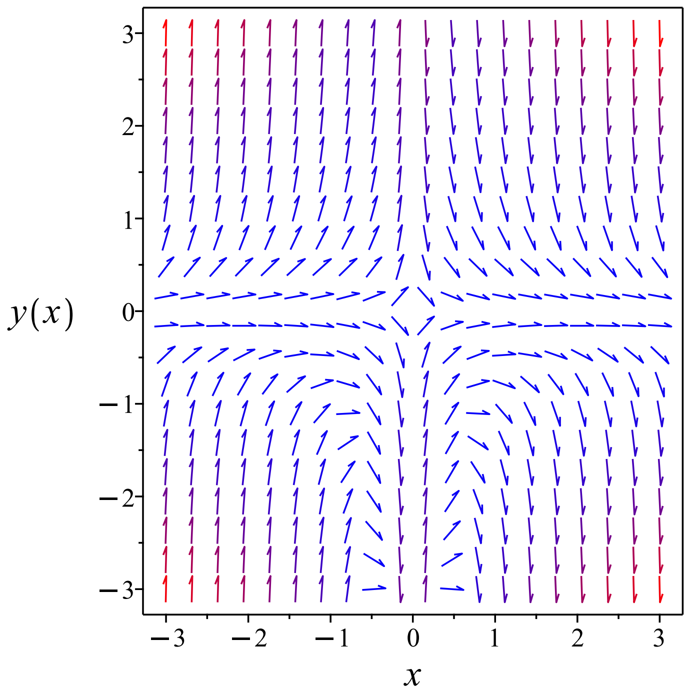


Figure 465: Slope field plot

Verification of solutions

$$y = -\frac{2}{x(-3x + 2c_1)}$$

Verified OK.

7.26.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(3y\,x^2 + 2)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{3x}{2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{3x}{2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} - \frac{3x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} - \frac{3x}{2} \\ w' &= \frac{w}{x} + \frac{3x}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{3x}{2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{3x}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3x}{2}\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{3x}{2}\right) \\ d\left(\frac{w}{x}\right) &= \frac{3}{2}\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x} &= \int \frac{3}{2}\,dx \\ \frac{w}{x} &= \frac{3x}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = \frac{3}{2}x^2 + c_1 x$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{3}{2}x^2 + c_1 x$$

Or

$$y = \frac{1}{\frac{3}{2}x^2 + c_1 x}$$

Which is simplified to

$$y = \frac{2}{x(3x + 2c_1)}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{x(3x + 2c_1)} \tag{1}$$

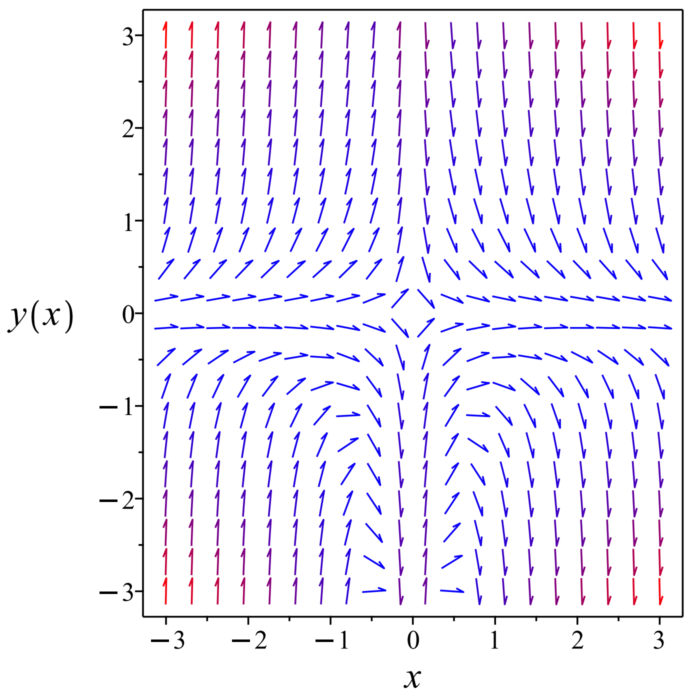


Figure 466: Slope field plot

Verification of solutions

$$y = \frac{2}{x(3x + 2c_1)}$$

Verified OK.

7.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x)\,\mathrm{d}y&=\left(-3x^2y^2-2y\right)\mathrm{d}x\\ \left(3x^2y^2+2y\right)\mathrm{d}x+(2x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2y^2+2y\\ N(x,y)&=2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2y^2 + 2y\right)\\ &= 6y\,x^2 + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x}\left(\left(6y\,x^2 + 2\right) - (2)\right)\\ &= 3yx\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{3x^2y^2 + 2y}\left((2) - \left(6y\,x^2 + 2\right)\right)\\ &= -\frac{6x^2}{3y\,x^2 + 2}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(2) - (6y\,x^2 + 2)}{x\,(3x^2y^2 + 2y) - y\,(2x)} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2y^2}\left(3x^2y^2 + 2y\right) \\ &= \frac{3y\,x^2 + 2}{x^2y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2y^2}(2x) \\ &= \frac{2}{x\,y^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{3y\,x^2+2}{x^2y}\right) + \left(\frac{2}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{3y\,x^2+2}{x^2y}\,\mathrm{d}x$$

$$\phi = \frac{3y\,x^2-2}{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3x}{y} - \frac{3y\,x^2-2}{y^2x} + f'(y) \tag{4}$$

$$= \frac{2}{x\,y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{2}{x\,y^2} = \frac{2}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3y\,x^2 - 2}{yx} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3y\,x^2 - 2}{yx}$$

The solution becomes

$$y = -\frac{2}{(-3x + c_1)\,x}$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{(-3x + c_1)\,x} \tag{1}$$

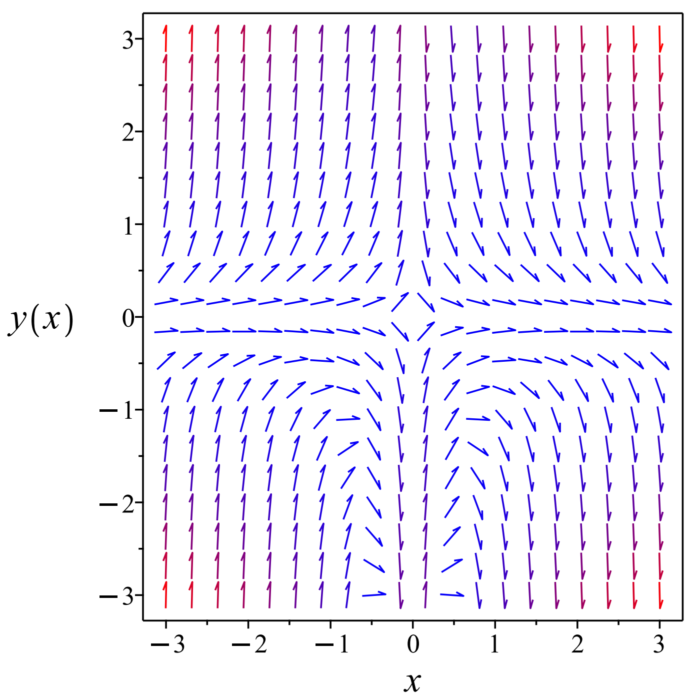


Figure 467: Slope field plot

Verification of solutions

$$y = -\frac{2}{(-3x + c_1)\, x}$$

Verified OK.

7.26.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(3y\, x^2 + 2)}{2x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{3x\, y^2}{2} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{3x}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{3xu}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{3}{2} \\ f_1 f_2 &= \frac{3}{2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{3xu''(x)}{2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = \frac{2c_1}{3x\,(c_1x + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2c_3}{3x\,(c_3x + 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_3}{3x\,(c_3x + 1)} \tag{1}$$

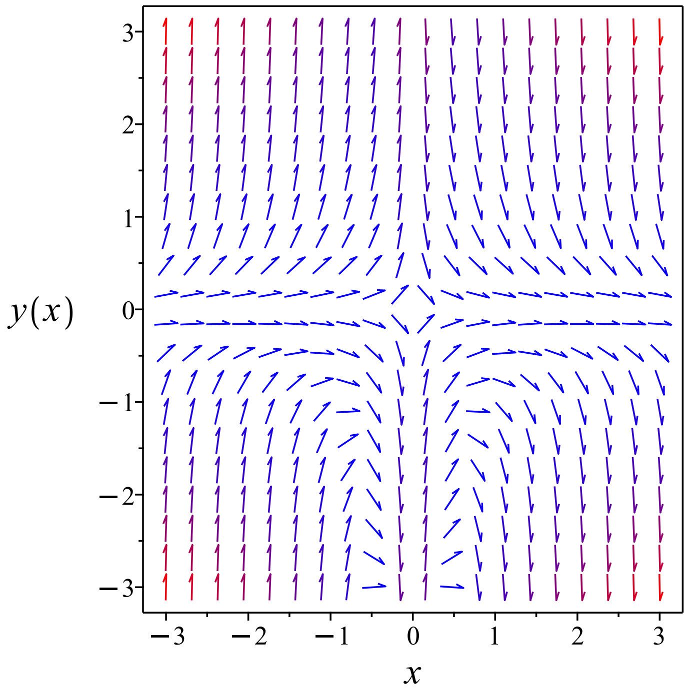


Figure 468: Slope field plot

Verification of solutions

$$y = \frac{2c_3}{3x(c_3x + 1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((3*x^2*y(x)^2+2*y(x))+(2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2}{(3x + 2c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 25

```
DSolve[(3*x^2*y[x]^2+2*y[x])+(2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3x^2 + 2c_1 x}$$

$$y(x) \to 0$$

8 Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203

8.1 problem 1

Internal problem ID [1087]
Internal file name [`OUTPUT/1088_Sunday_June_05_2022_02_02_22_AM_25907932/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 7y' + 10y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 1]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -7 \\ q(x) &= 10 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 7y' + 10y = 0$$

The domain of $p(x) = -7$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -7, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 7\lambda\, \mathrm{e}^{\lambda x} + 10\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -7, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-7^2 - (4)\,(1)\,(10)}\\ &= \frac{7}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{2} + \frac{3}{2}$$

$$\lambda_2 = \frac{7}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 5$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(5)x} + c_2 e^{(2)x}$$

Or

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 \mathrm{e}^{5x} + 2c_2 \mathrm{e}^{2x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 5c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

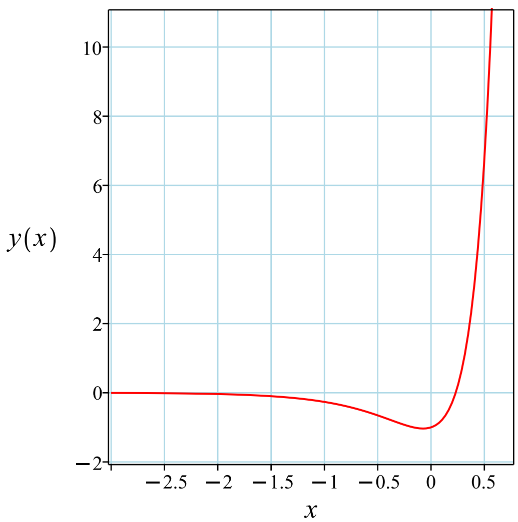
Substituting these values back in above solution results in

$$y = \mathrm{e}^{5x} - 2\,\mathrm{e}^{2x}$$

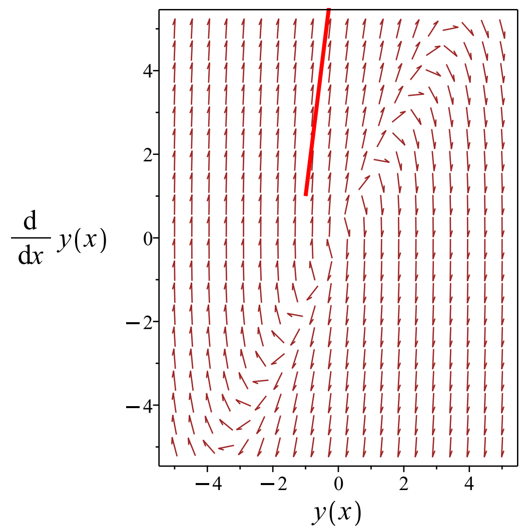
Summary

The solution(s) found are the following

$$y = e^{5x} - 2\,e^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{5x} - 2\,e^{2x}$$

Verified OK.

8.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 350: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{1}\,dx}\\ &= z_1 e^{\frac{7x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{7x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{7x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 \mathrm{e}^{5x}}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$
-1 = c_1 + \frac{c_2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1 \mathrm{e}^{2x} + \frac{5c_2 \mathrm{e}^{5x}}{3}
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = 2c_1 + \frac{5c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 3
\end{aligned}
$$

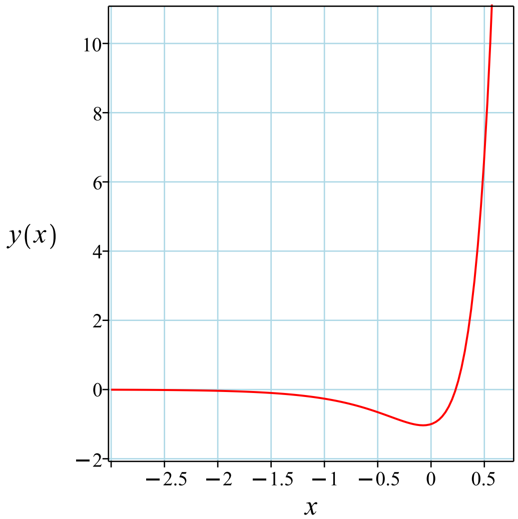
Substituting these values back in above solution results in

$$
y = \mathrm{e}^{5x} - 2\,\mathrm{e}^{2x}
$$

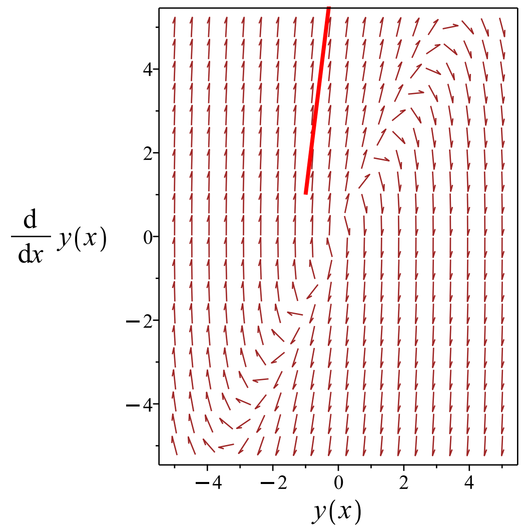
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{5x} - 2\,\mathrm{e}^{2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{5x} - 2\,e^{2x}$$

Verified OK.

8.1.4 Maple step by step solution

Let's solve

$\left[y'' - 7y' + 10y = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-5) = 0$

- Roots of the characteristic polynomial

  $r = (2, 5)$

- 1st solution of the ODE

$y_1(x) = e^{2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} + c_2 e^{5x}$

- □ Check validity of solution $y = c_1 e^{2x} + c_2 e^{5x}$
  - Use initial condition $y(0) = -1$

    $-1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = 2c_1 e^{2x} + 5c_2 e^{5x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = 2c_1 + 5c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = e^{5x} - 2e^{2x}$

- Solution to the IVP

  $y = e^{5x} - 2e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-7*diff(y(x),x)+10*y(x)=0,y(0) = -1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = e^{5x} - 2\,e^{2x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[{y''[x]-7*y'[x]+10*y[x]==0,{y[0]==-1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{2x}\left(e^{3x} - 2\right)$$

8.2 problem 2c

Internal problem ID [1088]
Internal file name [`OUTPUT/1089_Sunday_June_05_2022_02_02_23_AM_99410860/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 2c.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 2y' + 2y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -2]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-\sin(x) c_1 + c_2 \cos(x))$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -5$$

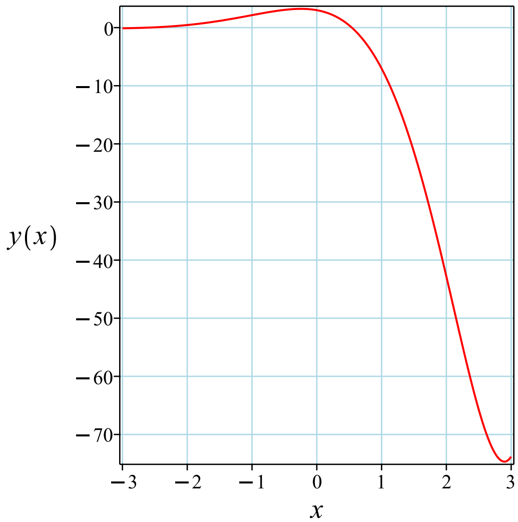
Substituting these values back in above solution results in

$$y = e^{x}(3 \cos(x) - 5 \sin(x))$$

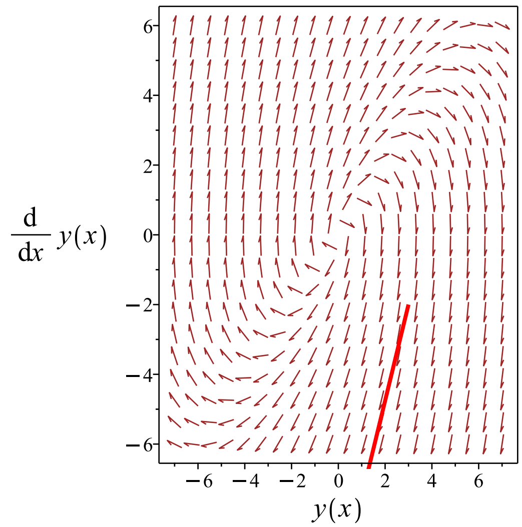
Summary

The solution(s) found are the following

$$y = e^{x}(3\cos(x) - 5\sin(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{x}(3\cos(x) - 5\sin(x))$$

Verified OK.

8.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 352: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos (x) \mathrm{e}^x) + c_2(\cos (x) \mathrm{e}^x (\tan (x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos (x) \mathrm{e}^x + c_2 \sin (x) \mathrm{e}^x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \sin (x) \mathrm{e}^x + c_1 \cos (x) \mathrm{e}^x + c_2 \cos (x) \mathrm{e}^x + c_2 \sin (x) \mathrm{e}^x
$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$
-2 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -5
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3 \cos (x) \mathrm{e}^x - 5 \sin (x) \mathrm{e}^x
$$

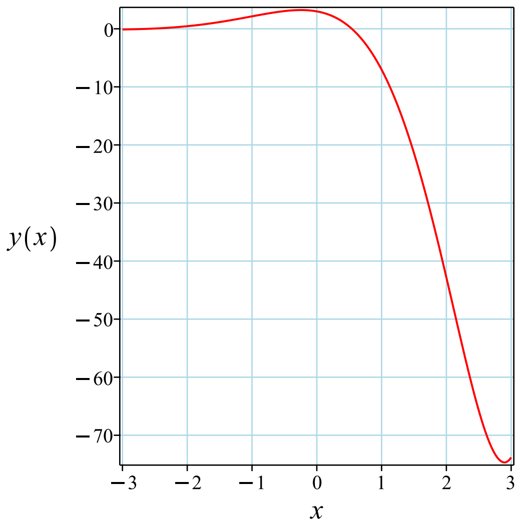
Which simplifies to

$$
y = \mathrm{e}^x (3 \cos (x) - 5 \sin (x))
$$

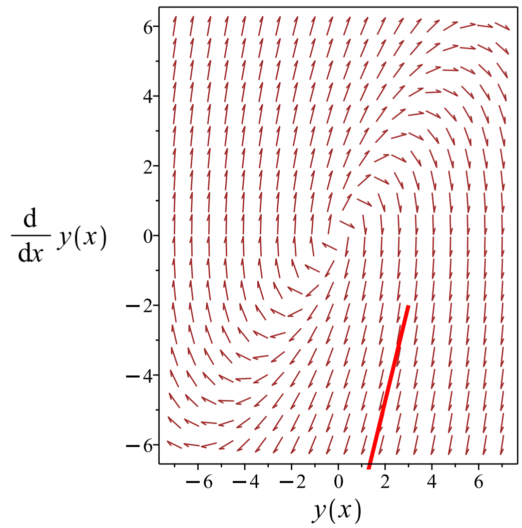
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x (3 \cos (x) - 5 \sin (x)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^x(3\cos(x) - 5\sin(x))$$

Verified OK.

8.2.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - I, 1 + I)$

- 1st solution of the ODE

$y_1(x) = \cos(x)\, e^x$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\, e^x + c_2 \sin(x)\, e^x$

□ Check validity of solution $y = c_1 \cos(x)\, e^x + c_2 \sin(x)\, e^x$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 \sin(x)\, e^x + c_1 \cos(x)\, e^x + c_2 \cos(x)\, e^x + c_2 \sin(x)\, e^x$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -2$

    $-2 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = -5\}$

  - Substitute constant values into general solution and simplify

    $y = e^x(3\cos(x) - 5\sin(x))$

- Solution to the IVP

  $y = e^x(3\cos(x) - 5\sin(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=0,y(0) = 3, D(y)(0) = -2],y(x), singsol=all)
```

$$y(x) = e^x(-5\sin(x) + 3\cos(x))$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==0,{y[0]==3,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^x(3\cos(x) - 5\sin(x))$$

8.3 problem 2d

Internal problem ID [1089]
Internal file name [`OUTPUT/1090_Sunday_June_05_2022_02_02_24_AM_12318990/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 2d.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 2y = 0}$$

With initial conditions

$$[y(0) = k_0, y'(0) = k_1]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Which simplifies to

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 0$ in the above gives

$$k_0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-\sin(x) c_1 + c_2 \cos(x))$$

substituting $y' = k_1$ and $x = 0$ in the above gives

$$k_1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = k_0$$
$$c_2 = k_1 - k_0$$

Substituting these values back in above solution results in

$$y = e^{x}(k_0 \cos(x) - \sin(x) k_0 + \sin(x) k_1)$$

Which simplifies to

$$y = ((k_1 - k_0)\sin(x) + k_0\cos(x))\,\mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = ((k_1 - k_0)\sin(x) + k_0\cos(x))\,\mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = ((k_1 - k_0)\sin(x) + k_0\cos(x))\,\mathrm{e}^{x}$$

Verified OK.

8.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 354: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\, dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)\,\mathrm{e}^{x}) + c_2(\cos(x)\,\mathrm{e}^{x}(\tan(x))) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x)\,\mathrm{e}^{x} + c_2 \sin(x)\,\mathrm{e}^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 0$ in the above gives

$$k_0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) e^x + c_1 \cos(x) e^x + c_2 \cos(x) e^x + c_2 \sin(x) e^x$$

substituting $y' = k_1$ and $x = 0$ in the above gives

$$k_1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= k_0 \\ c_2 &= k_1 - k_0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = k_0 \cos(x) e^x - \sin(x) e^x k_0 + \sin(x) e^x k_1$$

Which simplifies to

$$y = ((k_1 - k_0) \sin(x) + k_0 \cos(x)) e^x$$

Summary

The solution(s) found are the following

$$y = ((k_1 - k_0) \sin(x) + k_0 \cos(x)) e^x \tag{1}$$

Verification of solutions

$$y = ((k_1 - k_0) \sin(x) + k_0 \cos(x)) e^x$$

Verified OK.

8.3.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = 0, y(0) = k_0, y'\Big|_{\{x=0\}} = k_1\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)\,\mathrm{e}^x$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x$

☐ Check validity of solution $y = c_1 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x$

  - Use initial condition $y(0) = k_0$

    $k_0 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 \sin(x)\,\mathrm{e}^x + c_1 \cos(x)\,\mathrm{e}^x + c_2 \cos(x)\,\mathrm{e}^x + c_2 \sin(x)\,\mathrm{e}^x$

  - Use the initial condition $y'\Big|_{\{x=0\}} = k_1$

    $k_1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = k_0, c_2 = k_1 - k_0\}$

  - Substitute constant values into general solution and simplify

    $y = ((k_1 - k_0)\sin(x) + k_0 \cos(x))\,\mathrm{e}^x$

- Solution to the IVP

  $y = ((k_1 - k_0)\sin(x) + k_0 \cos(x))\,\mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=0,y(0) = k__0, D(y)(0) = k__1],y(x), singsol=al
```

$$y(x) = e^x((k_1 - k_0)\sin(x) + k_0\cos(x))$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==0,{y[0]==k0,y'[0]==k1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^x((\text{k1} - \text{k0})\sin(x) + \text{k0}\cos(x))$$

8.4 problem 3c

Internal problem ID [1090]
Internal file name [`OUTPUT/1091_Sunday_June_05_2022_02_02_25_AM_83459772/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 3c.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

With initial conditions

$$[y(0) = 7, y'(0) = 4]$$

8.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$7 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x + c_2 e^x + c_2 x\, e^x$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 7 \\ c_2 &= -3\end{aligned}$$

Substituting these values back in above solution results in

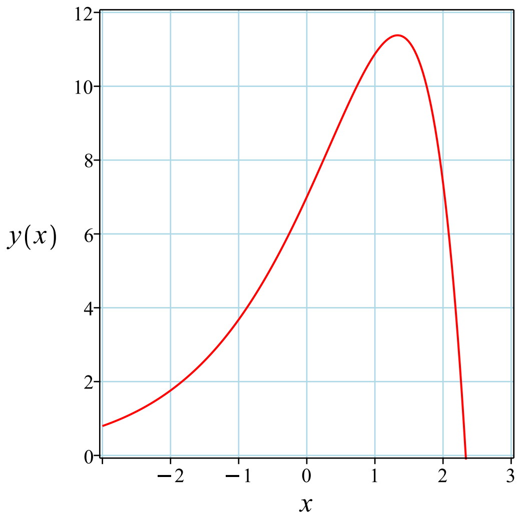
$$y = -3x\, e^x + 7\, e^x$$

Which simplifies to

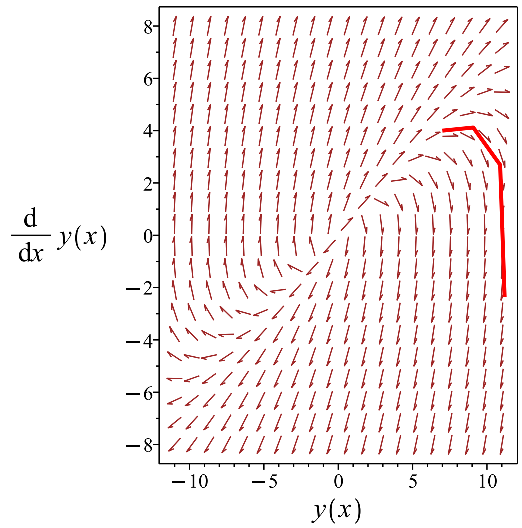
$$y = e^x(7 - 3x)$$

Summary
The solution(s) found are the following

$$y = e^x(7 - 3x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x(7 - 3x)$$

Verified OK.

8.4.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$

$$\left(e^{-x}y\right)'' = 0$$

Integrating once gives

$$\left(e^{-x}y\right)' = c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{-x}}$$

Or

$$y = c_1x\,e^{x} + c_2e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,e^{x} + c_2e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$7 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1e^{x} + c_1x\,e^{x} + c_2e^{x}$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -3 \\ c_2 &= 7 \end{aligned}$$

Substituting these values back in above solution results in

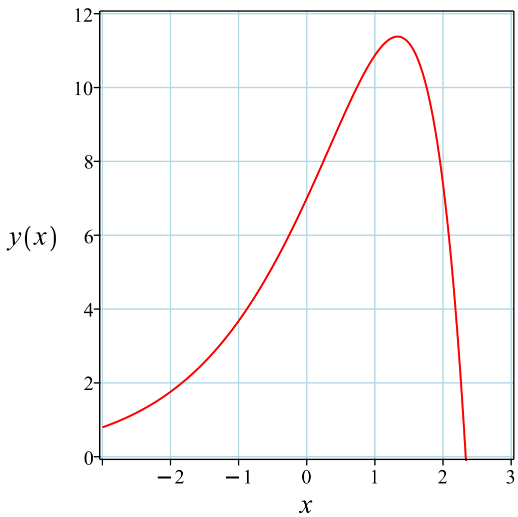
$$y = -3x\,e^{x} + 7\,e^{x}$$

Which simplifies to

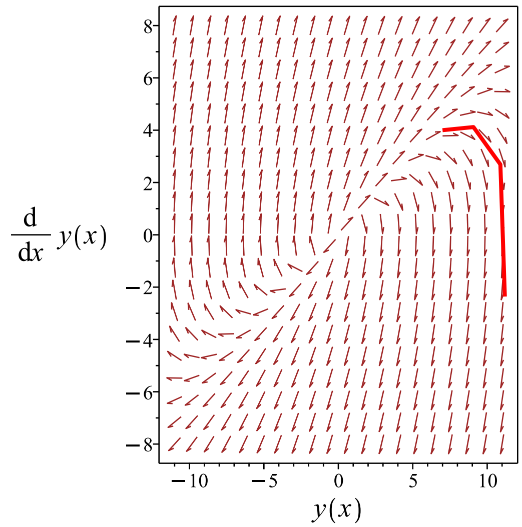
$$y = e^{x}(7 - 3x)$$

Summary
The solution(s) found are the following

$$y = e^{x}(7 - 3x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{x}(7 - 3x)$$

Verified OK.

8.4.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$
7 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x + c_2 x\,\mathrm{e}^x
$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$
4 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 7 \\
c_2 &= -3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -3x\,\mathrm{e}^x + 7\,\mathrm{e}^x
$$

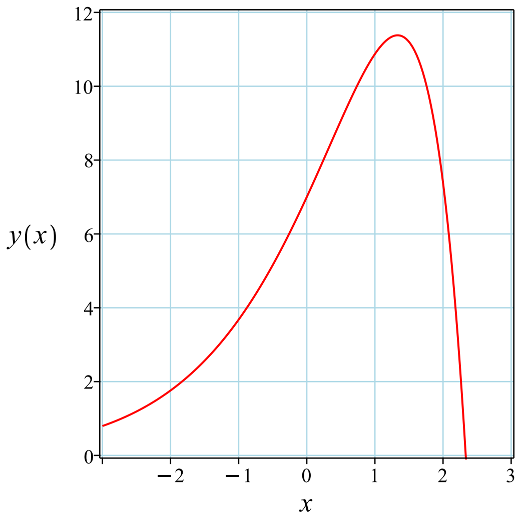
Which simplifies to

$$
y = \mathrm{e}^x(7 - 3x)
$$

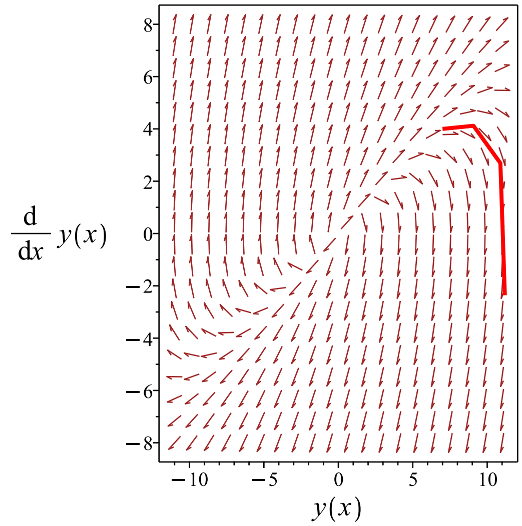
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x(7 - 3x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{x}(7 - 3x)$$

Verified OK.

8.4.5 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + y = 0, y(0) = 7, y'\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

$y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 x\, e^x$

- ☐ Check validity of solution $y = c_1 e^x + c_2 x e^x$
  - Use initial condition $y(0) = 7$

    $7 = c_1$

  - Compute derivative of the solution

    $y' = c_1 e^x + c_2 e^x + c_2 x\, e^x$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 4$

    $4 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 7, c_2 = -3\}$

  - Substitute constant values into general solution and simplify

    $y = e^x(7 - 3x)$

- Solution to the IVP

  $y = e^x(7 - 3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(0) = 7, D(y)(0) = 4],y(x), singsol=all)
```

$$y(x) = e^x(7 - 3x)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 14

```
DSolve[{y''[x]-2*y'[x]+y[x]==0,{y[0]==7,y'[0]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(7 - 3x)$$

8.5 problem 3d

Internal problem ID [1091]
Internal file name [`OUTPUT/1092_Sunday_June_05_2022_02_02_26_AM_3638536/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 3d.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

With initial conditions

$$[y(0) = k_0, y'(0) = k_1]$$

8.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}$$
$$= 1$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 0$ in the above gives

$$k_0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x + c_2 x\,\mathrm{e}^x$$

substituting $y' = k_1$ and $x = 0$ in the above gives

$$k_1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = k_0$$
$$c_2 = k_1 - k_0$$

Substituting these values back in above solution results in

$$y = -x\,\mathrm{e}^x k_0 + x\,\mathrm{e}^x k_1 + k_0 \mathrm{e}^x$$

Which simplifies to

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0)$$

Verified OK.

8.5.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 0$ in the above gives

$$k_0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x + c_1 x\, e^x + c_2 e^x$$

substituting $y' = k_1$ and $x = 0$ in the above gives

$$k_1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= k_1 - k_0 \\ c_2 &= k_0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x\, e^x k_0 + x\, e^x k_1 + k_0 e^x$$

Which simplifies to

$$y = e^x(-k_0 x + k_1 x + k_0)$$

Summary
The solution(s) found are the following

$$y = e^x(-k_0 x + k_1 x + k_0) \tag{1}$$

Verification of solutions

$$y = e^x(-k_0 x + k_1 x + k_0)$$

Verified OK.

8.5.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 0$ in the above gives

$$k_0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}$$

substituting $y' = k_1$ and $x = 0$ in the above gives

$$k_1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= k_0 \\
c_2 &= k_1 - k_0
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -x\,\mathrm{e}^x k_0 + x\,\mathrm{e}^x k_1 + k_0 \mathrm{e}^x$$

Which simplifies to

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x(-k_0 x + k_1 x + k_0)$$

Verified OK.

8.5.5 Maple step by step solution

Let's solve

$\left[y'' - 2y' + y = 0, y(0) = k_0, y'\Big|_{\{x=0\}} = k_1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{x} + c_2 x \mathrm{e}^{x}$

  - Use initial condition $y(0) = k_0$

    $k_0 = c_1$

  - Compute derivative of the solution

    $y' = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = k_1$

    $k_1 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = k_0, c_2 = k_1 - k_0\}$

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{x}(-k_0 x + k_1 x + k_0)$

- Solution to the IVP

  $y = \mathrm{e}^{x}(-k_0 x + k_1 x + k_0)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(0) = k__0, D(y)(0) = k__1],y(x), singsol=all)
```

$$y(x) = -e^{x}((x-1)\,k_{0} - xk_{1})$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[{y''[x]-2*y'[x]+y[x]==0,{y[0]==k0,y'[0]==k1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{x}(\text{k0}(-x) + \text{k0} + \text{k1}x)$$

8.6 problem 4

Internal problem ID [1092]
Internal file name [`OUTPUT/1093_Sunday_June_05_2022_02_02_27_AM_46177798/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2-1)\,y''+4y'x+2y=0}$$

With initial conditions

$$[y(0)=-5, y'(0)=1]$$

8.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{4x}{x^2-1} \\ q(x) &= \frac{2}{x^2-1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{4xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$$

The domain of $p(x) = \frac{4x}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = \frac{4x}{x^2-1}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int \frac{4x}{x^2-1}\,dx} \\ &= x^2-1 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\left(x^2-1\right)y\right)'' = 0$$

Integrating once gives

$$\left(\left(x^2-1\right)y\right)' = c_1$$

Integrating again gives

$$\left(\left(x^2-1\right)y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{x^2-1}$$

Or

$$y = \frac{c_1x}{x^2-1} + \frac{c_2}{x^2-1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1x}{x^2-1} + \frac{c_2}{x^2-1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1x^2}{\left(x^2-1\right)^2} + \frac{c_1}{x^2-1} - \frac{2c_2x}{\left(x^2-1\right)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -\frac{x-5}{x^2-1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x-5}{x^2-1} \tag{1}$$

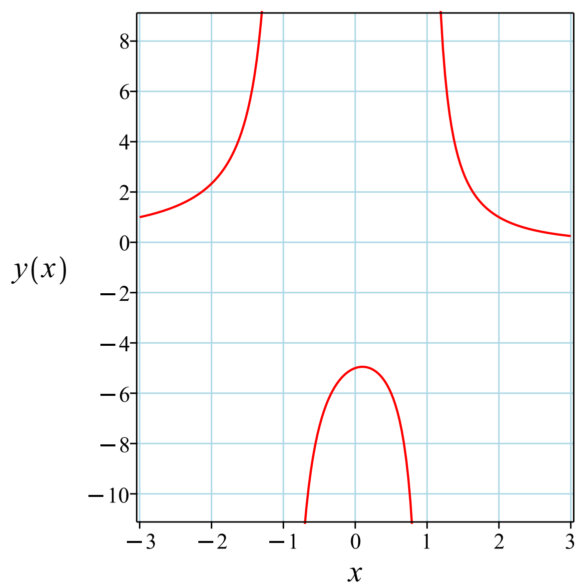


Figure 476: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.3 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{4x}{x^2-1}$$
$$q(x) = \frac{2}{x^2-1}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2}{x^2-1} - \frac{\left(\frac{4x}{x^2-1}\right)'}{2} - \frac{\left(\frac{4x}{x^2-1}\right)^2}{4} \\
&= \frac{2}{x^2-1} - \frac{\left(\frac{4}{x^2-1} - \frac{8x^2}{(x^2-1)^2}\right)}{2} - \frac{\left(\frac{16x^2}{(x^2-1)^2}\right)}{4} \\
&= \frac{2}{x^2-1} - \left(\frac{2}{x^2-1} - \frac{4x^2}{(x^2-1)^2}\right) - \frac{4x^2}{(x^2-1)^2} \\
&= 0
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{4x}{x^2-1}}{2}} \\
&= \frac{1}{x^2-1}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2-1} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\,(z(x))
\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2 - 1}$$

Hence (7) becomes

$$y = \frac{c_1 x + c_2}{x^2 - 1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x + c_2}{x^2 - 1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2 - 1} - \frac{2(c_1 x + c_2) x}{(x^2 - 1)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -\frac{x - 5}{x^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x - 5}{x^2 - 1} \tag{1}$$

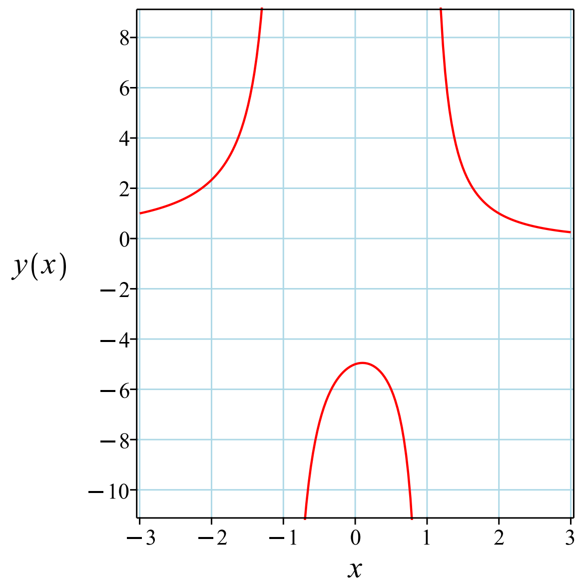


Figure 477: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2-1\right) y'' + 4y'x + 2y\right) dx = 0$$
$$2yx + \left(x^2-1\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2-1}$$
$$q(x) = \frac{c_1}{x^2-1}$$

Hence the ode is

$$y' + \frac{2xy}{x^2 - 1} = \frac{c_1}{x^2 - 1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2-1}dx} \\ &= e^{\ln(x-1)+\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu = x^2 - 1$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2-1}\right) \\ \frac{d}{dx}\left(\left(x^2-1\right)y\right) &= \left(x^2-1\right)\left(\frac{c_1}{x^2-1}\right) \\ d\left(\left(x^2-1\right)y\right) &= c_1\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^2-1\right)y &= \int c_1\, dx \\ \left(x^2-1\right)y &= c_1 x + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$y = \frac{c_1 x}{x^2-1} + \frac{c_2}{x^2-1}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 - 1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x + c_2}{x^2 - 1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2-1} - \frac{2(c_1x+c_2)\,x}{(x^2-1)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -\frac{x-5}{x^2-1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x-5}{x^2-1} \tag{1}$$

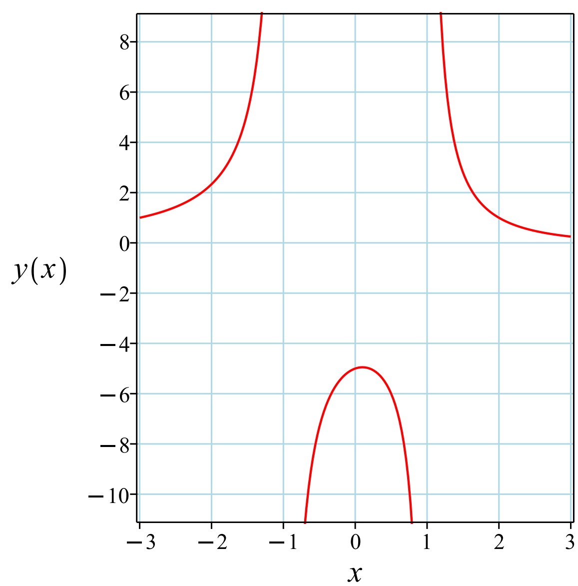


Figure 478: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2-1)\,y''+4y'x+2y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((x^2-1)\,y''+4y'x+2y\right)dx &= 0\\ 2yx+(x^2-1)\,y' &= c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2x}{x^2-1}\\ q(x) &= \frac{c_1}{x^2-1}\end{aligned}$$

Hence the ode is

$$y'+\frac{2xy}{x^2-1}=\frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2x}{x^2-1}dx}\\ &= \mathrm{e}^{\ln(x-1)+\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu=x^2-1$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2-1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-1)\,y\right) &= (x^2-1)\left(\frac{c_1}{x^2-1}\right)\\ \mathrm{d}\left((x^2-1)\,y\right) &= c_1\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\left(x^2-1\right)y = \int c_1\,\mathrm{d}x$$
$$\left(x^2-1\right)y = c_1x + c_2$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$y = \frac{c_1x}{x^2-1} + \frac{c_2}{x^2-1}$$

which simplifies to

$$y = \frac{c_1x + c_2}{x^2-1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1x + c_2}{x^2-1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2-1} - \frac{2(c_1x + c_2)\,x}{\left(x^2-1\right)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -\frac{x-5}{x^2-1}$$

Summary
The solution(s) found are the following

$$y = -\frac{x-5}{x^2-1} \tag{1}$$

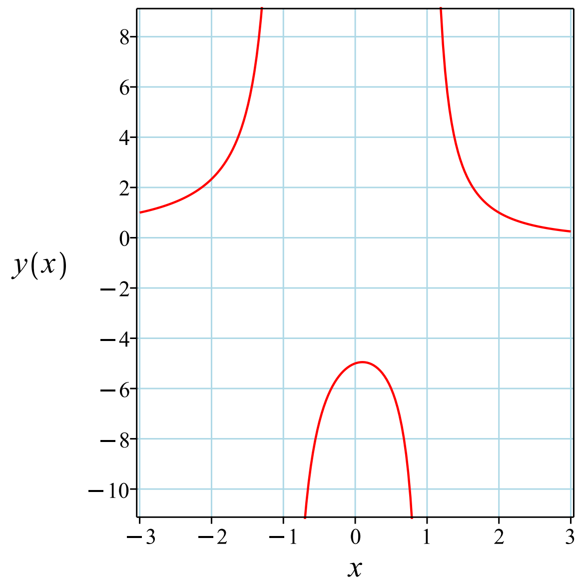


Figure 479: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.6 Solving using Kovacic algorithm

Writing the ode as

$$(x^2-1)\, y'' + 4y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 1 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2-1}\,dx} \\ &= z_1 e^{-\ln(x-1)-\ln(x+1)} \\ &= z_1\left(\frac{1}{x^2-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 - 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x-1)-2\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2 - 1}\right) + c_2 \left(\frac{1}{x^2 - 1}(x)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^2 - 1} + \frac{c_2 x}{x^2 - 1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1 x}{(x^2 - 1)^2} + \frac{c_2}{x^2 - 1} - \frac{2c_2 x^2}{(x^2 - 1)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\frac{x-5}{x^2-1}$$

Summary
The solution(s) found are the following

$$y = -\frac{x-5}{x^2-1} \tag{1}$$

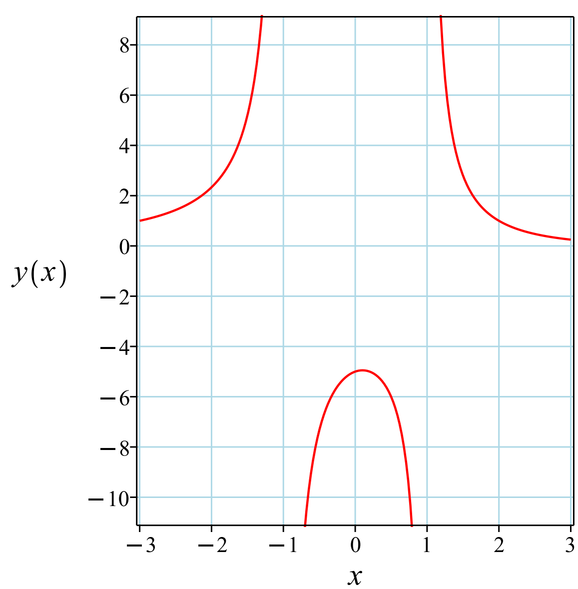


Figure 480: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= x^2 - 1\\ q(x) &= 4x\\ r(x) &= 2\\ s(x) &= 0\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 2\\ q'(x) &= 4\end{aligned}$$

Therefore (1) becomes

$$2 - (4) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$2yx + \left(x^2 - 1\right) y' = c_1$$

We now have a first order ode to solve which is

$$2yx + \left(x^2 - 1\right) y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2-1}$$
$$q(x) = \frac{c_1}{x^2-1}$$

Hence the ode is

$$y' + \frac{2xy}{x^2-1} = \frac{c_1}{x^2-1}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{2x}{x^2-1}dx}$$
$$= \mathrm{e}^{\ln(x-1)+\ln(x+1)}$$

Which simplifies to

$$\mu = x^2 - 1$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x^2-1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-1\right)y\right) = \left(x^2-1\right)\left(\frac{c_1}{x^2-1}\right)$$
$$\mathrm{d}\left(\left(x^2-1\right)y\right) = c_1\,\mathrm{d}x$$

Integrating gives

$$\left(x^2-1\right)y = \int c_1\,\mathrm{d}x$$
$$\left(x^2-1\right)y = c_1x + c_2$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$y = \frac{c_1x}{x^2-1} + \frac{c_2}{x^2-1}$$

which simplifies to

$$y = \frac{c_1x + c_2}{x^2-1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x + c_2}{x^2 - 1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -5$ and $x = 0$ in the above gives

$$-5 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2 - 1} - \frac{2(c_1 x + c_2)\, x}{(x^2 - 1)^2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -\frac{x - 5}{x^2 - 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{x - 5}{x^2 - 1} \tag{1}$$

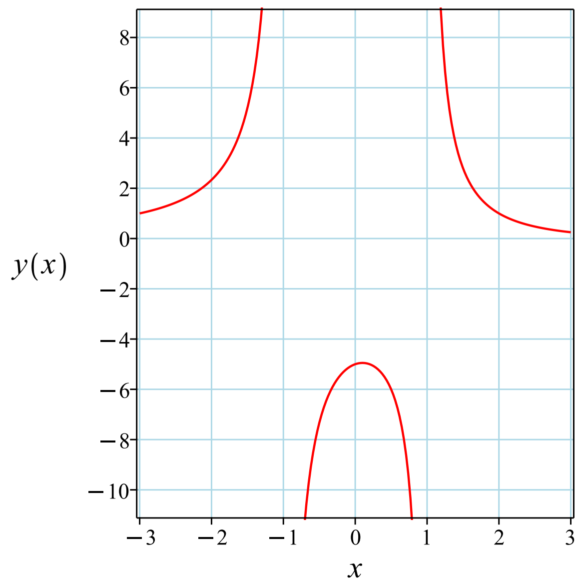


Figure 481: Solution plot

Verification of solutions

$$y = -\frac{x-5}{x^2-1}$$

Verified OK.

8.6.8 Maple step by step solution

Let's solve

$\left[(x^2-1)\,y''+4y'x+2y=0, y(0)=-5, y'\Big|_{\{x=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4xy'}{x^2-1}-\frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{4xy'}{x^2-1}+\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{4x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $$\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 2$$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$(x^2-1)\, y'' + 4y'x + 2y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (4u-4)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$-2a_0r(1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+r+1)\,(k+r+2)+a_k(k+r+2)\,(k+r+1)\right)u^{k+r}\right)=$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\,(k+r+1)\,(-2a_{k+1}+a_k)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{2}$
- Recursion relation for $r=-1$

  $a_{k+1}=\frac{a_k}{2}$
- Solution for $r=-1$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-1}, a_{k+1}=\frac{a_k}{2}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^{k-1}, a_{k+1}=\frac{a_k}{2}\right]$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{2}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k}, a_{k+1}=\frac{a_k}{2}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^{k}, a_{k+1}=\frac{a_k}{2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k}\right), a_{1+k}=\frac{a_k}{2}, b_{1+k}=\frac{b_k}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 17

```
dsolve([(x^2-1)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(0) = -5, D(y)(0) = 1],y(x), sings
```

$$y(x) = \frac{-x+5}{x^2-1}$$

✓ Solution by Mathematica
Time used: 0.024 (sec). Leaf size: 18

```
DSolve[{(x^2-1)*y''[x]+4*x*y'[x]+2*y[x]==0,{y[0]==-5,y'[0]==1}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{5-x}{x^2-1}$$

8.7 problem 10

Internal problem ID [1093]
Internal file name [`OUTPUT/1094_Sunday_June_05_2022_02_02_28_AM_99027787/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 0}$$

8.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} - 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-3)} \\ &= 1 \pm 2\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2$$
$$\lambda_2 = 1 - 2$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x} \tag{1}$$

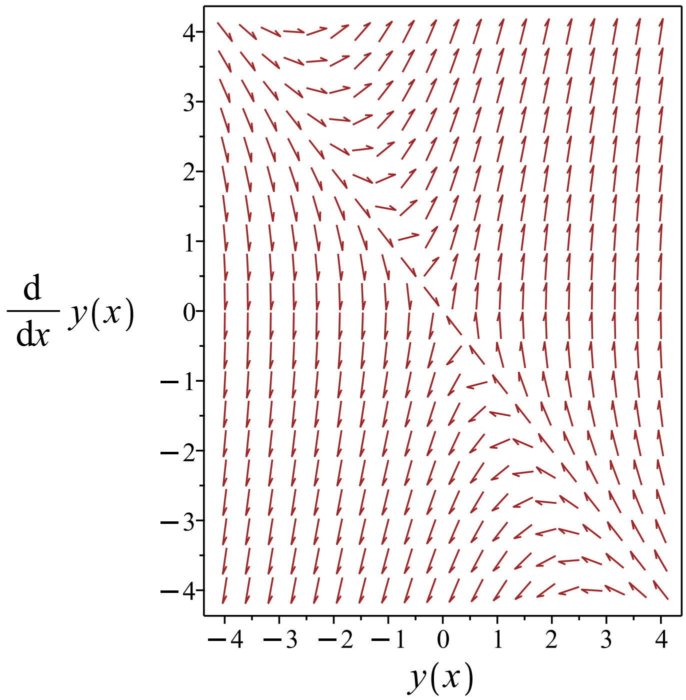


Figure 482: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-x}$$

Verified OK.

8.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} \tag{1}
$$

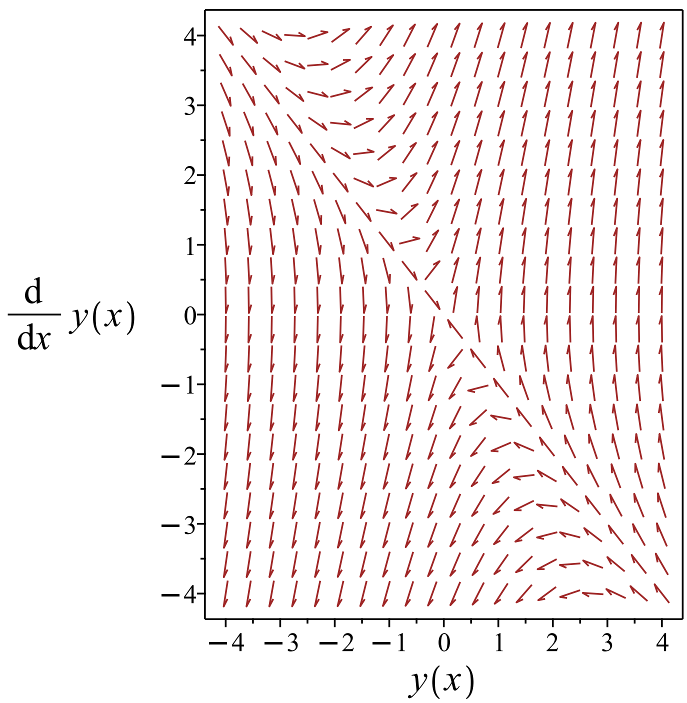


Figure 483: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}
$$

Verified OK.

8.7.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_1 + c_2 \mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[y''[x]-2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2 e^{4x} + c_1\right)$$

8.8 problem 11

Internal problem ID [1094]
Internal file name [`OUTPUT/1095_Sunday_June_05_2022_02_02_29_AM_8236561/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 9y = 0}$$

8.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 x e^{3x} \tag{1}$$

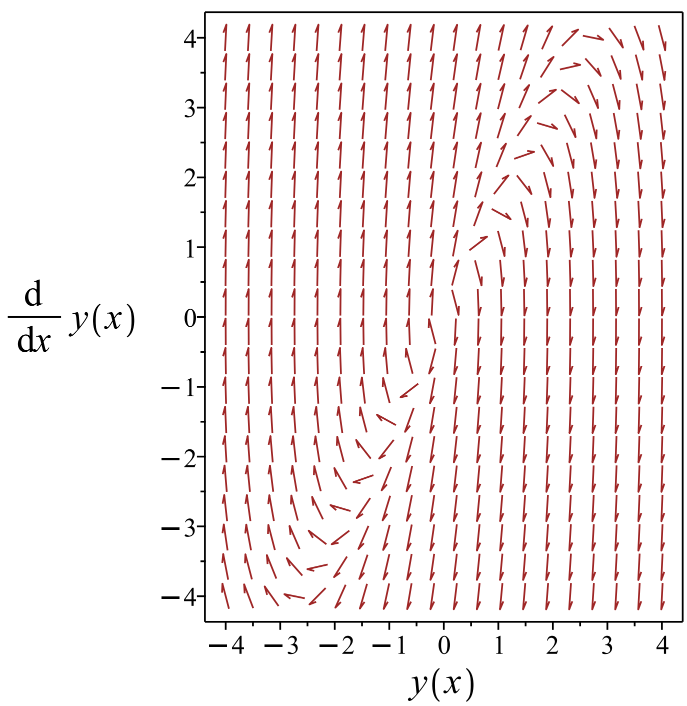


Figure 484: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Verified OK.

8.8.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -6\, dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-3x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} \tag{1}$$

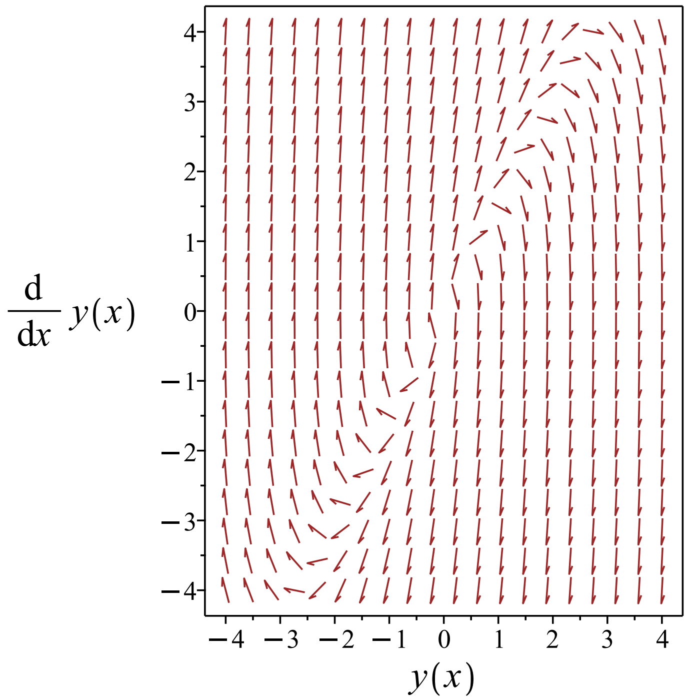


Figure 485: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{3x} + c_2 e^{3x}$$

Verified OK.

8.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{3x}\right) + c_2\left(e^{3x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{3x} + c_2 x\, e^{3x} \tag{1}
$$

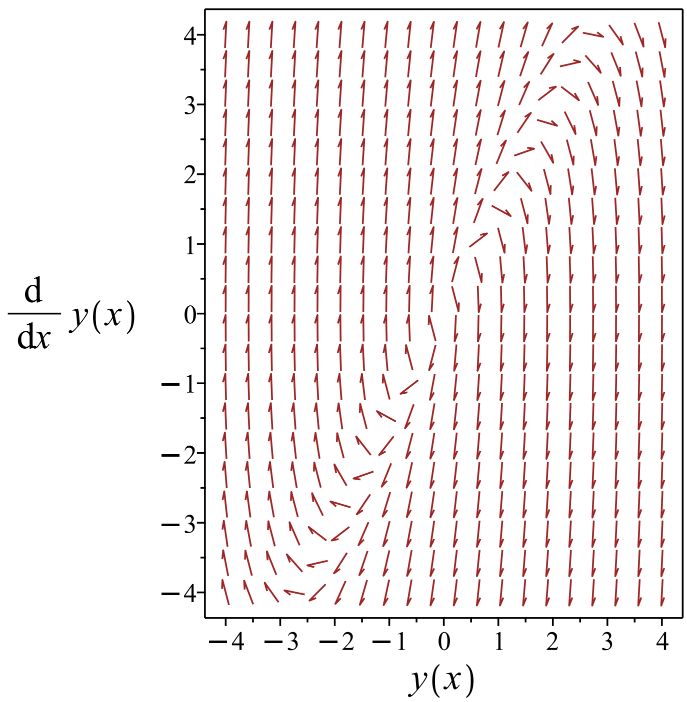


Figure 486: Slope field plot

Verification of solutions

$$
y = c_1 e^{3x} + c_2 x\, e^{3x}
$$

Verified OK.

8.8.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r - 3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{3x} + c_2 x\,e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{3x}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[y''[x]-6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(c_2 x + c_1)$$

8.9 problem 12

Internal problem ID [1095]
Internal file name [`OUTPUT/1096_Sunday_June_05_2022_02_02_30_AM_94023835/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2ay' + a^2y = 0}$$

8.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2a, C = a^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2a\lambda e^{\lambda x} + a^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 - 2a\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2a, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2a)^2 - (4)(1)(a^2)}\\ &= a\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -a$. Therefore the solution is

$$y = c_1 e^{ax} + c_2 x\, e^{ax} \tag{1}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{ax} + c_2 x\, e^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 e^{ax} + c_2 x\, e^{ax}$$

Verified OK.

8.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{(p(x)^2 + p'(x))\, y}{2} = f(x)$$

Where $p(x) = -2a$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int -2a\,dx}\\ &= e^{-ax}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0\\ (e^{-ax}y)'' &= 0\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-ax}y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-ax}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\mathrm{e}^{-ax}}$$

Or

$$y = c_1x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax}$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax} \tag{1}$$

Verification of solutions

$$y = c_1x\,\mathrm{e}^{ax} + c_2\mathrm{e}^{ax}$$

Verified OK.

8.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2ay' + a^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2a \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \qquad (5)$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \qquad (6)$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \qquad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2a}{1}\, dx} \\
&= z_1 e^{ax} \\
&= z_1(\mathrm{e}^{ax})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{ax}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2a}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2ax}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (\mathrm{e}^{ax}) + c_2 (\mathrm{e}^{ax}(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{ax} + c_2 x\,\mathrm{e}^{ax} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{ax} + c_2 x\,\mathrm{e}^{ax}$$

Verified OK.

8.9.4 Maple step by step solution

Let's solve

$y'' - 2ay' + a^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $a^2 - 2ar + r^2 = 0$

- Factor the characteristic polynomial

  $(a - r)^2 = 0$

- Root of the characteristic polynomial

  $r = a$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{ax}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{ax}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

$$y = c_1 e^{ax} + c_2 x \, e^{ax}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-2*a*diff(y(x),x)+a^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{ax}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 18

```
DSolve[y''[x]-2*a*y'[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ax}(c_2 x + c_1)$$

8.10 problem 13

Internal problem ID [1096]
Internal file name [`OUTPUT/1097_Sunday_June_05_2022_02_02_30_AM_77015905/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2 y'' + y' x - y = 0}$$

8.10.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

8.10.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +1 \\ \lambda_2 &= -1 \end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x}$$

Verified OK.

8.10.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}
$$

Verified OK.

8.10.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' + y'x - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x) + c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x
$$

Verified OK.

8.10.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2y'' + y'x - y)\,dx &= 0 \\
y'x^2 - yx &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\
\frac{y}{x} &= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

8.10.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= e^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

8.10.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + y'x - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + y'x - y\right) dx = 0$$
$$y'x^2 - yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)\\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x\\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

8.10.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + y'x - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2\\ B &= x\\ C &= -1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2 x}{2}
$$

Verified OK.

8.10.9 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= x \\
r(x) &= -1 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2-(1)+(-1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y'x^2-yx=c_1
$$

We now have a first order ode to solve which is

$$
y'x^2-yx=c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{y}{x}=\frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)\\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x\\ \frac{y}{x} &= -\frac{c_1}{2x^2}+c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x}+c_2x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{c_1}{2x}+c_2x \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{c_1}{2x}+c_2x$$

Verified OK.

8.10.10 Maple step by step solution

Let's solve

$x^2y''+y'x-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x}+\frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x}-\frac{y}{x^2}=0$

- Multiply by denominators of the ODE

  $x^2y'' + y'x - y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{t}$

- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^{t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x$

- Simplify

  $y = \frac{c_1}{x} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 + c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

8.11 problem 14

Internal problem ID [1097]
Internal file name [`OUTPUT/1098_Sunday_June_05_2022_02_02_31_AM_29375165/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - y'x + y = 0}$$

8.11.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x + \ln(x)\, c_2 x$$

Summary

The solution(s) found are the following

$$y = c_1 x + \ln(x)\, c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x + \ln(x)\, c_2 x$$

Verified OK.

8.11.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\,dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^2}}{x^2} \\ &= \frac{1}{x^4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{1}{x^4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2}$$

Verified OK.

8.11.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau}c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \tag{1}
$$

Verification of solutions

$$
y = c_1 x
$$

Verified OK.

8.11.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2 y'' - y'x + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x \\
&= (c_1 \ln(x) + c_2)\, x
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = (c_1 \ln(x) + c_2)\, x \tag{1}
$$

<u>Verification of solutions</u>

$$
y = (c_1 \ln(x) + c_2)\, x
$$

Verified OK.

8.11.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3 v'' + \left(-x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2 (u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-x)\,(c_1 \ln(x) + c_2) \\
&= -(c_1 \ln(x) + c_2)\, x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -(c_1 \ln(x) + c_2)\, x \tag{1}
$$

Verification of solutions

$$
y = -(c_1 \ln(x) + c_2)\, x
$$

Verified OK.

8.11.6 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' - y'x + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(\ln(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \ln(x) c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x + \ln(x) c_2 x$$

Verified OK.

8.11.7 Maple step by step solution

Let's solve

$x^2 y'' - y' x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - y' x + y = 0$

- Make a change of variables

$t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) + y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$
- Factor the characteristic polynomial

  $(r-1)^2 = 0$
- Root of the characteristic polynomial

  $r = 1$
- 1st solution of the ODE

  $y_1(t) = e^t$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^t$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^t + c_2 t\,e^t$

- Change variables back using $t = \ln(x)$

  $y = c_1 x + \ln(x) c_2 x$
- Simplify

  $y = x(c_1 + c_2 \ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 15

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2 \log(x) + c_1)$$

8.12 problem 15

Internal problem ID [1098]
Internal file name [`OUTPUT/1099_Sunday_June_05_2022_02_02_32_AM_61347599/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - (2a-1)\,xy' + a^2y = 0}$$

8.12.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}(-2a+1)\,xrx^{r-1} + a^2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r(-2a+1)\,r\,x^r + a^2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)\,(-2a+1)\,r + a^2 = 0$$

Or

$$a^2 - 2ra + r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = a$$
$$r_2 = a$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^a + c_2 x^a \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^a + c_2 x^a \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^a + c_2 x^a \ln(x)$$

Verified OK.

8.12.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + (-2ax + x) y' + a^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{-2a + 1}{x}$$
$$q(x) = \frac{a^2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-(\int p(x)dx)} dx \\
&= \int \mathrm{e}^{-(\int \frac{-2a+1}{x} dx)} dx \\
&= \int e^{(2a-1)\ln(x)}\, dx \\
&= \int x^{2a-1} dx \\
&= \frac{x^{2a}}{2a}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a^2}{x^2}}{x^{4a-2}} \\
&= a^2 x^{-4a}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) + a^2 x^{-4a} y(\tau) = 0$$

But in terms of τ

$$a^2 x^{-4a} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\left(c_1 + c_2\ln\left(\frac{x^{2a}}{a}\right) - c_2\ln(2)\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2a}}{a}\right) - c_2 \ln(2)\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2a}}{a}\right) - c_2 \ln(2)\right)}{2}$$

Verified OK.

8.12.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{-2a + 1}{x}$$
$$q(x) = \frac{a^2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-2a + 1)}{x^2} + \frac{a^2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = a \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2a}{x} + \frac{-2a+1}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{u(x)}{x} = 0 \qquad (8)
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u} \, du &= -\frac{1}{x} \, dx \\
\int \frac{1}{u} \, du &= \int -\frac{1}{x} \, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x) + c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x) \, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^a \\
&= (c_1 \ln(x) + c_2)\, x^a
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = (c_1 \ln(x) + c_2)\, x^a \tag{1}
$$

Verification of solutions

$$
y = (c_1 \ln(x) + c_2)\, x^a
$$

Verified OK.

8.12.4 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' + (-2ax + x)\, y' + a^2 y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -2ax + x \\
C &= a^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2ax+x}{x^2} \, dx} \\ &= z_1 e^{\frac{(2a-1)\ln(x)}{2}} \\ &= z_1 \left(x^{a-\frac{1}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^a$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2ax+x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{(2a-1)\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^a) + c_2(x^a(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^a + c_2 x^a \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^a + c_2 x^a \ln(x)$$

Verified OK.

8.12.5 Maple step by step solution

Let's solve

$x^2 y'' + (-2ax + x) y' + a^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{a^2 y}{x^2} + \frac{(2a-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2a-1)y'}{x} + \frac{a^2 y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2a-1}{x}, P_3(x) = \frac{a^2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2a + 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = a^2$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2 y'' - (2a - 1)\, x y' + a^2 y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite DE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} a_k (a-k-r)^2\, x^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k (a-k)^2 = 0$

- Recursion relation that defines series solution to ODE

$a_k = 0$

- Recursion relation for $r = 0$

$a_k = 0$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-(2*a-1)*x*diff(y(x),x)+a^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_2 \ln(x) + c_1) x^a$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-(2*a-1)*x*y'[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^a (ac_2 \log(x) + c_1)$$

8.13 problem 16

Internal problem ID [1099]
Internal file name [`OUTPUT/1100_Sunday_June_05_2022_02_02_33_AM_91454613/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 0}$$

8.13.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{-16x^2 + 3}{4x^2}
\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-16x^2+3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\
&= \frac{-16x^2+3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{-16x^2+3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= -4
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{1}{x}}{2}} \\
&= \sqrt{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\sqrt{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
4x^{\frac{5}{2}}(v''(x) - 4v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - 4 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$v(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}\right)\sqrt{x}$$

Summary
The solution(s) found are the following

$$y = \left(c_1 e^{2x} + c_2 e^{-2x}\right) \sqrt{x} \tag{1}$$

Verification of solutions

$$y = \left(c_1 e^{2x} + c_2 e^{-2x}\right) \sqrt{x}$$

Verified OK.

8.13.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - y'x + \left(-4x^2 + \frac{3}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 1 \\
\beta &= 2i \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \cosh(2x)}{\sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\, \sqrt{ix}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x \cosh(2x)}{\sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\, \sqrt{ix}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x \cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}}$$

Verified OK.

8.13.3 Solving using Kovacic algorithm

Writing the ode as

$$4x^2 y'' - 4y'x + (-16x^2 + 3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}\sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\sqrt{x}\right) + c_2\left(\mathrm{e}^{-2x}\sqrt{x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x}\sqrt{x} + \frac{c_2 \mathrm{e}^{2x}\sqrt{x}}{4} \tag{1}
$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x}\sqrt{x} + \frac{c_2 \mathrm{e}^{2x}\sqrt{x}}{4}$$

Verified OK.

8.13.4 Maple step by step solution

Let's solve

$4x^2 y'' - 4y'x + (-16x^2 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' - 4y'x + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 1\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 16a_{k-2} = 0$$

- Shift index using $k- >k + 2$

  $$4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} - 16a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

8.14 problem 17

Internal problem ID [1100]
Internal file name [`OUTPUT/1101_Sunday_June_05_2022_02_02_34_AM_75601842/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-1)\,y''-y'x+y=0}$$

8.14.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x-1)\,y''-y'x+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=-\frac{x}{x-1}$$
$$q(x)=\frac{1}{x-1}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x-1} + \frac{1}{x-1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 2x + 2)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x\,(x-1)}\,dx \\
\ln(u) &= x - 2\ln(x) + \ln(x-1) + c_1 \\
u &= \mathrm{e}^{x-2\ln(x)+\ln(x-1)+c_1} \\
&= c_1 \mathrm{e}^{x-2\ln(x)+\ln(x-1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{c_1 \mathrm{e}^x}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{c_1 \mathrm{e}^x}{x} + c_2\right)x \\
&= c_1 \mathrm{e}^x + c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{c_1 \mathrm{e}^x}{x} + c_2\right)x \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{c_1 \mathrm{e}^x}{x} + c_2\right)x
$$

Verified OK.

8.14.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x - 1)\,(0) + (-x)\,(-1) + (1)\,(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x(x-1)\,v''+\left(x^2-2x+2\right)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-x^2+x\right)u'(x)+\left(x^2-2x+2\right)u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u(x^2-2x+2)}{x\,(x-1)}
\end{aligned}$$

Where $f(x)=\frac{x^2-2x+2}{x(x-1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x\,(x-1)}\,dx\\
\ln(u) &= x-2\ln(x)+\ln(x-1)+c_1\\
u &= \mathrm{e}^{x-2\ln(x)+\ln(x-1)+c_1}\\
&= c_1\mathrm{e}^{x-2\ln(x)+\ln(x-1)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u\\
&= c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{(x-1)\,c_1\mathrm{e}^x}{x^2}\,\mathrm{d}x\\
&= \frac{c_1\mathrm{e}^x}{x}+c_2
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-x)\left(\frac{c_1 e^x}{x} + c_2\right) \\
&= -c_1 e^x - c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -c_1 e^x - c_2 x \tag{1}
$$

Verification of solutions

$$
y = -c_1 e^x - c_2 x
$$

Verified OK.

8.14.3 Solving using Kovacic algorithm

Writing the ode as

$$
(x-1)\, y'' - y'x + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2\,(x-1)} + \frac{3}{4\,(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(x-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x-1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x-1\right)} + \frac{1}{2} \\
&= \frac{-2+x}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\left(x-1\right)^2}\right) + \left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4\left(x-1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} - c_2 x$$

Verified OK.

8.14.4 Maple step by step solution

Let's solve

$(x-1)\, y'' - y' x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$(x-1)\, y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- > k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1))\,u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{1+k}=\frac{a_k}{1+k}, b_{1+k}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+c_2\mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

8.15 problem 18

Internal problem ID [1101]
Internal file name [`OUTPUT/1102_Sunday_June_05_2022_02_02_36_AM_24070805/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 2y'x + (x^2 + 2)\, y = 0}$$

8.15.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^2 + 2}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 1
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x^3(v''(x) + v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x$$

Summary
The solution(s) found are the following

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x \tag{1}$$

Verification of solutions

$$y = (c_1 \cos(x) + c_2 \sin(x))\, x$$

Verified OK.

8.15.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2y' x + (x^2 + 2)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= \frac{3}{2} \\ \beta &= 1 \\ n &= -\frac{1}{2} \\ \gamma &= 1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Verified OK.

8.15.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2y'x + (x^2 + 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x \cos(x) + c_2 \sin(x) x \tag{1}$$

Verification of solutions

$$y = c_1 x \cos(x) + c_2 \sin(x) x$$

Verified OK.

8.15.4 Maple step by step solution

Let's solve

$x^2 y'' - 2y'x + (x^2+2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2y'x + (x^2+2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(1+k)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

8.16 problem 19

Internal problem ID [1102]
Internal file name [OUTPUT/1103_Sunday_June_05_2022_02_02_37_AM_61910818/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^{2} \sin \left(x \right) y^{\prime \prime}-4 x \left(x \cos \left(x \right)+\sin \left(x \right)\right) y^{\prime}+\left(2 x \cos \left(x \right)+3 \sin \left(x \right)\right) y=0}$$

8.16.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4x^{2} \sin \left(x \right) y^{\prime \prime}-4 x \left(x \cos \left(x \right)+\sin \left(x \right)\right) y^{\prime}+\left(2 x \cos \left(x \right)+3 \sin \left(x \right)\right) y=0 \tag{1}$$

Becomes

$$y^{\prime \prime}+p(x) y^{\prime}+q(x) y=0 \tag{2}$$

Where

$$p(x) = \frac{-1-x \cot \left(x \right)}{x}$$
$$q(x) = \frac{2 x \cot \left(x \right)+3}{4 x^{2}}$$

Applying change of variables on the depndent variable $y = v(x) x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v^{\prime \prime}(x)+\left(\frac{2 n}{x}+p \right) v^{\prime}(x)+\left(\frac{n \left(n -1\right)}{x^{2}}+\frac{n p}{x}+q \right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(-1-x\cot(x))}{x^2}+\frac{2x\cot(x)+3}{4x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=\frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$v''(x)+\left(\frac{1}{x}+\frac{-1-x\cot(x)}{x}\right)v'(x)=0$$
$$v''(x)-\cot(x)\,v'(x)=0 \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)-\cot(x)\,u(x)=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\cot(x)\,u\end{aligned}$$

Where $f(x)=\cot(x)$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du&=\cot(x)\,dx\\\int\frac{1}{u}\,du&=\int\cot(x)\,dx\\\ln(u)&=\ln(\sin(x))+c_1\\u&=e^{\ln(\sin(x))+c_1}\\&=\sin(x)\,c_1\end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -c_1 \cos(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (-c_1 \cos(x) + c_2) \sqrt{x} \\
&= (-c_1 \cos(x) + c_2) \sqrt{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = (-c_1 \cos(x) + c_2) \sqrt{x} \tag{1}
$$

Verification of solutions

$$
y = (-c_1 \cos(x) + c_2) \sqrt{x}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      (2*arccos(t)*t+3*(-t^2+1)^(1/2))*u(t)+(-4*arccos(t)*t^2+4*arccos(t))*diff(u(t),t)+(-4*
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 14

```
dsolve(4*x^2*sin(x)*diff(y(x),x$2)-4*x*(x*cos(x)+sin(x))*diff(y(x),x)+(2*x*cos(x)+3*sin(x))*
```

$$y(x) = \sqrt{x}\,(c_1 + c_2\cos(x))$$

✓ Solution by Mathematica

Time used: 0.356 (sec). Leaf size: 21

```
DSolve[4*x^2*Sin[x]*y''[x]-4*x*(x*Cos[x]+Sin[x])*y'[x]+(2*x*Cos[x]+3*Sin[x])*y[x]==0,y[x],x,
```

$$y(x) \to \sqrt{\arccos(\cos(x))}(c_2\cos(x) + c_1)$$

8.17 problem 20

Internal problem ID [1103]
Internal file name [`OUTPUT/1104_Sunday_June_05_2022_02_02_39_AM_74431363/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(3x-1)\,y''-(3x+2)\,y'+(6x-8)\,y=0$$

8.17.1 Maple step by step solution

Let's solve

$(3x-1)\,y''+(-3x-2)\,y'+(6x-8)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3x-4)y}{3x-1}+\frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(3x+2)y'}{3x-1}+\frac{2(3x-4)y}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{3x+2}{3x-1}, P_3(x) = \frac{2(3x-4)}{3x-1}\right]$

- $\left(x-\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right) \cdot P_2(x)\right)\Big|_{x=\frac{1}{3}} = -1$

- $\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)\right)\Big|_{x=\frac{1}{3}} = 0$

- $x = \frac{1}{3}$is a regular singular point

  Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

  $x_0 = \frac{1}{3}$

- Multiply by denominators

  $(3x-1)\,y'' + (-3x-2)\,y' + (6x-8)\,y = 0$

- Change variables using $x = u + \frac{1}{3}$ so that the regular singular point is at $u = 0$

  $3u\left(\frac{d^2}{du^2}y(u)\right) + (-3u-3)\left(\frac{d}{du}y(u)\right) + (6u-6)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-2+r)\, u^{-1+r} + (3a_1(1+r)(-1+r) - 3a_0(2+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)(k+r-1\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$3a_1(1+r)(-1+r) - 3a_0(2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3a_{k+1}(k+1+r)(k+r-1) + a_k(-3k-3r-6) + 6a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$3a_{k+2}(k+r+2)(k+r) + a_{k+1}(-3k-9-3r) + 6a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{ka_{k+1}+ra_{k+1}-2a_k+3a_{k+1}}{(k+r+2)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{ka_{k+1}-2a_k+3a_{k+1}}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{ka_{k+1}-2a_k+3a_{k+1}}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{ka_{k+1}-2a_k+5a_{k+1}}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{ka_{k+1}-2a_k+5a_{k+1}}{(k+4)(k+2)}, 9a_1 - 12a_0 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x-\frac{1}{3}\right)^{k+2}, a_{k+2}=\frac{k a_{k+1}-2 a_k+5 a_{k+1}}{(k+4)(k+2)}, 9 a_1-12 a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 177

```
dsolve((3*x-1)*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)+(6*x-8)*y(x)=0,y(x), singsol=all)
```

$$y(x)=-\frac{6615\,\mathrm{e}^{-\frac{x\left(i\sqrt{7}-1\right)}{2}}\left(x-\frac{1}{3}\right)^2\left(\left(\left(-\frac{23x}{35}+\frac{296}{735}\right)\sqrt{7}+ix-\frac{248i}{105}\right)c_1\,\mathrm{KummerM}\left(\frac{1}{2}-\frac{5i\sqrt{7}}{14},3,\frac{i\sqrt{7}\,(3x-1)}{3}\right)-\left(\left(-\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 109

```
DSolve[(3*x-1)*y''[x]-(3*x+2)*y'[x]+(6*x-8)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 4e^{\frac{1}{6}\left(1-i\sqrt{7}\right)(3x-1)}(1-3x)^2\left(c_1 \operatorname{HypergeometricU}\left(\frac{3}{2}-\frac{5i}{2\sqrt{7}},3,\frac{1}{3}i\sqrt{7}(3x-1)\right)\right.$$
$$\left.+c_2 L^2_{-\frac{3}{2}+\frac{5i}{2\sqrt{7}}}\left(\frac{1}{3}i\sqrt{7}(3x-1)\right)\right)$$

8.18 problem 21

Internal problem ID [1104]
Internal file name [`OUTPUT/1105_Sunday_June_05_2022_02_02_42_AM_83625445/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2-4)\, y''+4y'x+2y=0}$$

8.18.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = \frac{4x}{x^2-4}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int \frac{4x}{x^2-4}\,dx} \\ &= x^2 - 4 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\left(x^2-4\right)y\right)'' = 0$$

Integrating once gives

$$\left(\left(x^2-4\right)y\right)' = c_1$$

Integrating again gives

$$\left(\left(x^2-4\right)y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{x^2-4}$$

Or

$$y = \frac{c_1x}{x^2-4} + \frac{c_2}{x^2-4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1x}{x^2-4} + \frac{c_2}{x^2-4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1x}{x^2-4} + \frac{c_2}{x^2-4}$$

Verified OK.

8.18.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4x}{x^2 - 4} \\ q(x) &= \frac{2}{x^2 - 4} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2 - 4} - \frac{\left(\frac{4x}{x^2-4}\right)'}{2} - \frac{\left(\frac{4x}{x^2-4}\right)^2}{4} \\ &= \frac{2}{x^2 - 4} - \frac{\left(\frac{4}{x^2-4} - \frac{8x^2}{(x^2-4)^2}\right)}{2} - \frac{\left(\frac{16x^2}{(x^2-4)^2}\right)}{4} \\ &= \frac{2}{x^2 - 4} - \left(\frac{2}{x^2 - 4} - \frac{4x^2}{(x^2 - 4)^2}\right) - \frac{4x^2}{(x^2 - 4)^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{4x}{x^2-4}}{2}} \\ &= \frac{1}{x^2 - 4} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2 - 4} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2 - 4}$$

Hence (7) becomes

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 - 4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Verified OK.

8.18.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{gathered} \int \left(\left(x^2 - 4\right) y'' + 4y'x + 2y\right) dx = 0 \\ 2yx + \left(x^2 - 4\right) y' = c_1 \end{gathered}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2x}{x^2 - 4}$$
$$q(x) = \frac{c_1}{x^2 - 4}$$

Hence the ode is

$$y' + \frac{2xy}{x^2 - 4} = \frac{c_1}{x^2 - 4}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2x}{x^2-4}dx}$$
$$= x^2 - 4$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2 - 4}\right)$$
$$\frac{d}{dx}\left(\left(x^2 - 4\right)y\right) = \left(x^2 - 4\right)\left(\frac{c_1}{x^2 - 4}\right)$$
$$d\left(\left(x^2 - 4\right)y\right) = c_1\, dx$$

Integrating gives

$$\left(x^2 - 4\right)y = \int c_1\, dx$$
$$\left(x^2 - 4\right)y = c_1 x + c_2$$

Dividing both sides by the integrating factor $\mu = x^2 - 4$ results in

$$y = \frac{c_1 x}{x^2 - 4} + \frac{c_2}{x^2 - 4}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 - 4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Verified OK.

8.18.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2-4)\,y''+4y'x+2y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((x^2-4)\,y''+4y'x+2y\right)dx &= 0\\ 2yx+(x^2-4)\,y' &= c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2x}{x^2-4}\\ q(x) &= \frac{c_1}{x^2-4}\end{aligned}$$

Hence the ode is

$$y'+\frac{2xy}{x^2-4}=\frac{c_1}{x^2-4}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2-4}dx}\\ &= x^2-4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2-4}\right)\\ \frac{d}{dx}\left((x^2-4)\,y\right) &= (x^2-4)\left(\frac{c_1}{x^2-4}\right)\\ d\left((x^2-4)\,y\right) &= c_1\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(x^2-4)\,y &= \int c_1\,dx\\ (x^2-4)\,y &= c_1x+c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 - 4$ results in

$$y = \frac{c_1 x}{x^2 - 4} + \frac{c_2}{x^2 - 4}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 - 4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Verified OK.

8.18.5 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 - 4) y'' + 4y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 4 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2-4}\,dx} \\
&= z_1 e^{-\ln(x^2-4)} \\
&= z_1\left(\frac{1}{x^2-4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2-4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2-4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x^2-4)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2 - 4}\right) + c_2 \left(\frac{1}{x^2 - 4}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2 - 4} + \frac{c_2 x}{x^2 - 4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2 - 4} + \frac{c_2 x}{x^2 - 4}
$$

Verified OK.

8.18.6 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 - 4 \\
q(x) &= 4x \\
r(x) &= 2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (4) + (2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)
$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$2yx + \left(x^2 - 4\right) y' = c_1$$

We now have a first order ode to solve which is

$$2yx + \left(x^2 - 4\right) y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2x}{x^2 - 4} \\ q(x) &= \frac{c_1}{x^2 - 4} \end{aligned}$$

Hence the ode is

$$y' + \frac{2xy}{x^2 - 4} = \frac{c_1}{x^2 - 4}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2x}{x^2 - 4} dx} \\ &= x^2 - 4 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{c_1}{x^2 - 4}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2 - 4\right) y\right) &= \left(x^2 - 4\right) \left(\frac{c_1}{x^2 - 4}\right) \\ \mathrm{d}\left(\left(x^2 - 4\right) y\right) &= c_1\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \left(x^2 - 4\right) y &= \int c_1\, \mathrm{d}x \\ \left(x^2 - 4\right) y &= c_1 x + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 - 4$ results in

$$y = \frac{c_1 x}{x^2 - 4} + \frac{c_2}{x^2 - 4}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 - 4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 - 4}$$

Verified OK.

8.18.7 Maple step by step solution

Let's solve

$(x^2 - 4)\, y'' + 4y'x + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4xy'}{x^2-4} - \frac{2y}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4xy'}{x^2-4} + \frac{2y}{x^2-4} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4x}{x^2-4}, P_3(x) = \frac{2}{x^2-4}\right]$

  ◦ $(2+x) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = 2$

  ◦ $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$

$$\left((2+x)^2 \cdot P_3(x)\right)\Bigg|_{x=-2} = 0$$

- $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $(x^2-4)\,y'' + 4y'x + 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right) + (4u-8)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum\limits_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum\limits_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-4a_0r(1+r)\,u^{-1+r} + \left(\sum\limits_{k=0}^{\infty}\left(-4a_{k+1}(k+r+1)\,(k+r+2) + a_k(k+r+2)\,(k+r+1)\right)u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-4r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(k+r+1)(-4a_{k+1}+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{4}$

- Recursion relation for $r=-1$

  $a_{k+1}=\frac{a_k}{4}$

- Solution for $r=-1$

  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1}=\frac{a_k}{4}\right]$$

- Revert the change of variables $u=2+x$

  $$\left[y=\sum_{k=0}^{\infty} a_k(2+x)^{k-1}, a_{k+1}=\frac{a_k}{4}\right]$$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{4}$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{4}\right]$$

- Revert the change of variables $u=2+x$

  $$\left[y=\sum_{k=0}^{\infty} a_k(2+x)^k, a_{k+1}=\frac{a_k}{4}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k(2+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k(2+x)^k\right), a_{1+k}=\frac{a_k}{4}, b_{1+k}=\frac{b_k}{4}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((x^2-4)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x + c_2}{x^2 - 4}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 20

```
DSolve[(x^2-4)*y''[x]+4*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x + c_1}{x^2 - 4}$$

8.19 problem 22

Internal problem ID [1105]
Internal file name [OUTPUT/1106_Sunday_June_05_2022_02_02_43_AM_82361730/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{(1+2x)\,y''-2(2x^2-1)\,y'-4(x+1)\,y=0}$$

8.19.1 Maple step by step solution

Let's solve

$(1+2x)\,y''+(-4x^2+2)\,y'+(-4x-4)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4(x+1)y}{1+2x}+\frac{2(2x^2-1)y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(2x^2-1)y'}{1+2x}-\frac{4(x+1)y}{1+2x}=0$

□ Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2(2x^2-1)}{1+2x}, P_3(x) = -\frac{4(x+1)}{1+2x}\right]$

- $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = \frac{1}{2}$

- $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

- $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(1+2x)\, y'' + (-4x^2+2)\, y' + (-4x-4)\, y = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + (-4u^2+4u+1)\left(\frac{d}{du}y(u)\right) + (-4u-2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r) u^{-1+r} + (a_1(1+r)(1+2r) + 2a_0(-1+2r)) u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(2k+1+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)(1+2r) + 2a_0(-1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + (4a_k - 4a_{k-1}) k + (4a_k - 4a_{k-1}) r - 2a_k = 0$$

- Shift index using $k- >k+1$

$$2(k+2+r)\left(k+\frac{3}{2}+r\right) a_{k+2} + (4a_{k+1} - 4a_k)(k+1) + (4a_{k+1} - 4a_k) r - 2a_{k+1} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2(2a_k k - 2k a_{k+1} + 2a_k r - 2r a_{k+1} + 2a_k - a_{k+1})}{(k+2+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2(2a_k k - 2k a_{k+1} + 2a_k - a_{k+1})}{(k+2)(2k+3)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{2(2a_k k - 2k a_{k+1} + 2a_k - a_{k+1})}{(k+2)(2k+3)}, a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^k, a_{k+2} = \frac{2(2a_k k - 2k a_{k+1} + 2a_k - a_{k+1})}{(k+2)(2k+3)}, a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{2(2a_k k - 2k a_{k+1} + 3a_k - 2a_{k+1})}{\left(k+\frac{5}{2}\right)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2}=\frac{2(2a_k k-2k a_{k+1}+3a_k-2a_{k+1})}{\left(k+\frac{5}{2}\right)(2k+4)}, 3a_1=0\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+\frac{1}{2}}, a_{k+2}=\frac{2(2a_k k-2k a_{k+1}+3a_k-2a_{k+1})}{\left(k+\frac{5}{2}\right)(2k+4)}, 3a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{2}\right)^{k+\frac{1}{2}}\right), a_{k+2}=\frac{2(2k a_k-2k a_{1+k}+2a_k-a_{1+k})}{(k+2)(2k+3)}, a_1-2a_0=0, b_{k+}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 32

```
dsolve((2*x+1)*diff(y(x),x$2)-2*(2*x^2-1)*diff(y(x),x)-4*(x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{HeunB}\left(-\frac{1}{2}, -2, -\frac{1}{2}, 3, x+\frac{1}{2}\right) + c_2 \operatorname{HeunB}\left(\frac{1}{2}, -2, -\frac{1}{2}, 3, x+\frac{1}{2}\right)\sqrt{4x+2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*x+1)*y''[x]-2*(2*x^2-1)*y'[x]-4*(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

Not solved

8.20 problem 23

Internal problem ID [1106]
Internal file name [`OUTPUT/1107_Sunday_June_05_2022_02_02_45_AM_46217271/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.1 Homogeneous linear equations. Page 203
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-2x\right) y''+\left(-x^{2}+2\right) y'+\left(2x-2\right) y=0}$$

8.20.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$\left(x^{2}-2x\right) y''+\left(-x^{2}+2\right) y'+\left(2x-2\right) y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$p(x)=\frac{-x^{2}+2}{x\left(-2+x\right)}$$
$$q(x)=\frac{2x-2}{x\left(-2+x\right)}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-x^2+2)}{x^2\,(-2+x)} + \frac{2x-2}{x\,(-2+x)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4}{x} + \frac{-x^2+2}{x\,(-2+x)}\right) v'(x) &= 0 \\ v''(x) + \frac{(-x^2+4x-6)\, v'(x)}{x\,(-2+x)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-x^2+4x-6)\, u(x)}{x\,(-2+x)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-4x+6)}{x\,(-2+x)} \end{aligned}$$

Where $f(x) = \frac{x^2-4x+6}{x(-2+x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-4x+6}{x(-2+x)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2-4x+6}{x(-2+x)}\,dx \\
\ln(u) &= x - 3\ln(x) + \ln(-2+x) + c_1 \\
u &= \mathrm{e}^{x-3\ln(x)+\ln(-2+x)+c_1} \\
&= c_1 \mathrm{e}^{x-3\ln(x)+\ln(-2+x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(-\frac{2\,\mathrm{e}^x}{x^3} + \frac{\mathrm{e}^x}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{\mathrm{e}^x c_1}{x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{\mathrm{e}^x c_1}{x^2} + c_2\right)x^2 \\
&= c_2 x^2 + c_1 \mathrm{e}^x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{\mathrm{e}^x c_1}{x^2} + c_2\right)x^2 \tag{1}$$

Verification of solutions

$$y = \left(\frac{\mathrm{e}^x c_1}{x^2} + c_2\right)x^2$$

Verified OK.

8.20.2 Solving using Kovacic algorithm

Writing the ode as

$$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-2x\\B&=-x^2+2\\C&=2x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^4-8x^3+24x^2-24x+12}{4\,(x^2-2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^4-8x^3+24x^2-24x+12\\t&=4\big(x^2-2x\big)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^4-8x^3+24x^2-24x+12}{4\,(x^2-2x)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 384: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 4 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4(-2+x)} + \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4(-2+x)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} - \frac{1}{2(-2 + x)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2(-2 + x)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{-4 + 2x} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(-2+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(-2+x)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(-2+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 -}{}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - \frac{1}{2(-2+x)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{-2+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln(-2+x)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{-2+x}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x \sqrt{x}\,\sqrt{-2+x}}{\sqrt{x(-2+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(-2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-x^2 \mathrm{e}^{-x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{-2+x}}{\sqrt{x\,(-2+x)}}\right) + c_2\left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{-2+x}}{\sqrt{x\,(-2+x)}}\left(-x^2\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\mathrm{e}^{x}\sqrt{x}\,\sqrt{-2+x}}{\sqrt{x\,(-2+x)}} - \frac{c_2 x^{\frac{5}{2}}\sqrt{-2+x}}{\sqrt{x\,(-2+x)}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\mathrm{e}^{x}\sqrt{x}\,\sqrt{-2+x}}{\sqrt{x\,(-2+x)}} - \frac{c_2 x^{\frac{5}{2}}\sqrt{-2+x}}{\sqrt{x\,(-2+x)}}
$$

Verified OK.

8.20.3 Maple step by step solution

Let's solve

$(x^2-2x)\,y'' + (-x^2+2)\,y' + (2x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y'}{x(-2+x)} - \frac{2y(x-1)}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(x^2-2)y'}{x(-2+x)} + \frac{2y(x-1)}{x(-2+x)} = 0$$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = -\frac{x^2-2}{x(-2+x)}, P_3(x) = \frac{2(x-1)}{x(-2+x)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x(-2+x) + (-x^2+2)\,y' + (2x-2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\, x^{-1+r} + (-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r))\, x^r + \left(\sum_{k=1}^{\infty} (-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)(k+r-2) - 2k^2 a_{k+1} + (-4r a_{k+1} - a_{k-1})\, k - 2r^2 a_{k+1} - a_{k-1} r + 3a_{k-1} + 2a_{k+1} =$$

- Shift index using $k- > k+1$

$$a_{k+1}(k+2+r)(k+r-1) - 2(k+1)^2 a_{k+2} + (-4r a_{k+2} - a_k)(k+1) - 2r^2 a_{k+2} - r a_k + 3a_k +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_{k+1} + 2kr a_{k+1} + r^2 a_{k+1} - k a_k + k a_{k+1} - r a_k + r a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2kr + r^2 + 2k + 2r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2 + 6k + 8)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2 + 6k + 8)}, -6a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(2*x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + c_2 \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 18

```
DSolve[(x^2-2*x)*y''[x]+(2-x^2)*y'[x]+(2*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_2 x^2 + c_1 e^x$$

9 Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253

9.1 problem 1

Internal problem ID [1107]
Internal file name [`OUTPUT/1108_Sunday_June_05_2022_02_02_47_AM_67231871/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1+2x)\, y''-2y'-(2x+3)\, y=(1+2x)^{2}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{-x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1+2x, B = -2, C = -3-2x, f(x) = 4\left(x+\frac{1}{2}\right)^{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(1+2x)\, y''-2y'+(-3-2x)\, y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^{2}}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2}{1+2x}$$

Therefore

$$\begin{aligned}
y_2(x) &= \mathrm{e}^{-x}\left(\int \mathrm{e}^{-\left(\int -\frac{2}{1+2x}dx\right)}\mathrm{e}^{2x}dx\right)\\
y_2(x) &= \mathrm{e}^{-x}\int \frac{1+2x}{\mathrm{e}^{-2x}},dx\\
y_2(x) &= \mathrm{e}^{-x}\left(\int (1+2x)\,\mathrm{e}^{2x}dx\right)\\
y_2(x) &= \mathrm{e}^{-x}x\,\mathrm{e}^{2x}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1y_1(x) + c_2y_2(x)\\
&= c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-x}x\,\mathrm{e}^{2x}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-x}x\,\mathrm{e}^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x}\\
y_2 &= \mathrm{e}^{-x}x\,\mathrm{e}^{2x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-x} x\, \mathrm{e}^{2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-x} x\, \mathrm{e}^{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-x} x\, \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} x\, \mathrm{e}^{2x} + \mathrm{e}^{-x} \mathrm{e}^{2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\mathrm{e}^{-x} x\, \mathrm{e}^{2x} + \mathrm{e}^{-x} \mathrm{e}^{2x}\right) - \left(\mathrm{e}^{-x} x\, \mathrm{e}^{2x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = 2\,\mathrm{e}^{-2x} x\, \mathrm{e}^{2x} + \mathrm{e}^{-2x} \mathrm{e}^{2x}$$

Which simplifies to

$$W = 1 + 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\,\mathrm{e}^{-x} x\, \mathrm{e}^{2x} \left(x + \frac{1}{2}\right)^2}{\left(1 + 2x\right)^2}\, dx$$

Which simplifies to

$$u_1 = -\int x\, \mathrm{e}^{x} dx$$

Hence

$$u_1 = -(x - 1)\, \mathrm{e}^{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{-x} \left(x + \frac{1}{2}\right)^2}{\left(1+2x\right)^2}\, dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-x} dx$$

Hence

$$u_2 = -\mathrm{e}^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -(x-1)\,\mathrm{e}^{x}\mathrm{e}^{-x} - \mathrm{e}^{-2x} x\,\mathrm{e}^{2x}$$

Which simplifies to

$$y_p(x) = 1 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} x\,\mathrm{e}^{2x}\right) + (1 - 2x) \end{aligned}$$

Which simplifies to

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{x} + 1 - 2x$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{x} + 1 - 2x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{x} + 1 - 2x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([(2*x+1)*diff(y(x),x$2)-2*diff(y(x),x)-(2*x+3)*y(x)=(2*x+1)^2,exp(-x)],singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_2 + x\,\mathrm{e}^{x} c_1 + 1 - 2x$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 33

```
DSolve[(2*x+1)*y''[x]-2*y'[x]-(2*x+3)*y[x]==(2*x+1)^2,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1 e^{-x-\frac{1}{2}} + x\left(-2 + c_2 e^{x+\frac{1}{2}}\right) + 1$$

9.2 problem 2

Internal problem ID [1108]
Internal file name [`OUTPUT/1109_Sunday_June_05_2022_02_02_48_AM_53418085/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + y'x - y = \frac{4}{x^2}}$$

Given that one solution of the ode is

$$y_1 = x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = \frac{4}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x - y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = x\left(\int \frac{\mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x \int \frac{\frac{1}{x}}{x^2},\, dx$$

$$y_2(x) = x\left(\int \frac{1}{x^3}dx\right)$$

$$y_2(x) = -\frac{1}{2x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x - \frac{c_2}{2x} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x - \frac{c_2}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = -\frac{1}{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -\frac{1}{2x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(-\frac{1}{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -\frac{1}{2x} \\ 1 & \frac{1}{2x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{2x^2}\right) - \left(-\frac{1}{2x}\right)(1)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2}{x^3}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{2}{x^4} dx$$

Hence

$$u_1 = -\frac{2}{3x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4}{x}}{x} dx$$

Which simplifies to

$$u_2 = \int \frac{4}{x^2} dx$$

Hence

$$u_2 = -\frac{4}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4}{3x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x - \frac{c_2}{2x}\right) + \left(\frac{4}{3x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x - \frac{c_2}{2x} + \frac{4}{3x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x - \frac{c_2}{2x} + \frac{4}{3x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=4/x^2,x],singsol=all)
```

$$y(x) = c_2x + \frac{4}{3x^2} + \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==4/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{3x^2} + \frac{c_1}{x} + c_2x$$

9.3 problem 3

Internal problem ID [1109]
Internal file name [`OUTPUT/1110_Sunday_June_05_2022_02_02_49_AM_42123006/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + y = x}$$

Given that one solution of the ode is

$$y_1 = x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - y'x + y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{1}{x}$$

Therefore

$$\begin{aligned}
y_2(x) &= x \left(\int \frac{e^{-\left(\int -\frac{1}{x} dx\right)}}{x^2} dx \right) \\
y_2(x) &= x \int \frac{x}{x^2}, dx \\
y_2(x) &= x \left(\int \frac{1}{x} dx \right) \\
y_2(x) &= x \ln(x)
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x + \ln(x)\, c_2 x
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x + \ln(x)\, c_2 x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x \ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}(x \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x \ln(x) \\ 1 & \ln(x) + 1 \end{vmatrix}$$

Therefore

$$W = (x)(\ln(x) + 1) - (x \ln(x))(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x) x^2}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x \ln(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x + \ln(x) c_2 x) + \left(\frac{x \ln(x)^2}{2}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_1 + c_2 \ln(x)) + \frac{x \ln(x)^2}{2}$$

Summary

The solution(s) found are the following

$$y = x(c_1 + c_2 \ln(x)) + \frac{x \ln(x)^2}{2} \tag{1}$$

Verification of solutions

$$y = x(c_1 + c_2 \ln(x)) + \frac{x \ln(x)^2}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x,x],singsol=all)
```

$$y(x) = x\left(c_2 + \ln(x) c_1 + \frac{\ln(x)^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} x\left(\log^2(x) + 2c_2 \log(x) + 2c_1\right)$$

9.4 problem 4

Internal problem ID [1110]
Internal file name [`OUTPUT/1111_Sunday_June_05_2022_02_02_51_AM_38112855/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}}$$

Given that one solution of the ode is

$$y_1 = e^{2x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = \frac{1}{1+e^{-x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -3$$

Therefore

$$\begin{aligned}
y_2(x) &= \mathrm{e}^{2x}\left(\int \mathrm{e}^{-\left(\int(-3)dx\right)}\mathrm{e}^{-4x}dx\right) \\
y_2(x) &= \mathrm{e}^{2x}\int \frac{\mathrm{e}^{3x}}{\mathrm{e}^{4x}}, dx \\
y_2(x) &= \mathrm{e}^{2x}\left(\int \mathrm{e}^{-x}dx\right) \\
y_2(x) &= -\mathrm{e}^{-x}\mathrm{e}^{2x}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\mathrm{e}^{2x} - c_2\mathrm{e}^{-x}\mathrm{e}^{2x}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\mathrm{e}^{2x} - c_2\mathrm{e}^{-x}\mathrm{e}^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= -\mathrm{e}^{-x}\mathrm{e}^{2x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & -e^{-x}e^{2x} \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(-e^{-x}e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & -e^{-x}e^{2x} \\ 2\,e^{2x} & -e^{-x}e^{2x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(-e^{-x}e^{2x}\right) - \left(-e^{-x}e^{2x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}e^{-x}$$

Which simplifies to

$$W = e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{e^{-x}e^{2x}}{1+e^{-x}}}{e^{3x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x}}{1+e^{-x}} dx$$

Hence

$$u_1 = -e^{-x} - \ln\left(e^{x}\right) + \ln\left(1+e^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{1+e^{-x}}}{e^{3x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}}{1+e^{-x}} dx$$

Hence

$$u_2 = -\ln\left(1+e^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-e^{-x} - \ln\left(e^{x}\right) + \ln\left(1+e^{x}\right)\right) e^{2x} + \ln\left(1+e^{-x}\right) e^{-x} e^{2x}$$

Which simplifies to

$$y_p(x) = e^{x}\left(-\ln\left(e^{x}\right) e^{x} + \ln\left(1+e^{x}\right) e^{x} + \ln\left(1+e^{-x}\right) - 1\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} - c_2 e^{-x} e^{2x}\right) + \left(e^{x}\left(-\ln\left(e^{x}\right) e^{x} + \ln\left(1+e^{x}\right) e^{x} + \ln\left(1+e^{-x}\right) - 1\right)\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{2x} - c_2 e^{x} + e^{x}\left(-\ln\left(e^{x}\right) e^{x} + \ln\left(1+e^{x}\right) e^{x} + \ln\left(1+e^{-x}\right) - 1\right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} - c_2 e^{x} + e^{x}\left(-\ln\left(e^{x}\right) e^{x} + \ln\left(1+e^{x}\right) e^{x} + \ln\left(1+e^{-x}\right) - 1\right) \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} - c_2 e^{x} + e^{x}\left(-\ln\left(e^{x}\right) e^{x} + \ln\left(1+e^{x}\right) e^{x} + \ln\left(1+e^{-x}\right) - 1\right)$$

Verified OK.

9.4.1 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{1+e^{-x}}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^x \left(\int \frac{e^{-x}}{1+e^{-x}} dx\right) + e^{2x} \left(\int \frac{e^{-2x}}{1+e^{-x}} dx\right)$$

- Compute integrals

$$y_p(x) = e^x(-\ln(e^x) e^x + \ln(1+e^x) e^x + \ln(1+e^{-x}) - 1)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{2x} + e^x(-\ln(e^x) e^x + \ln(1+e^x) e^x + \ln(1+e^{-x}) - 1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve([diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=1/(1+exp(-x)),exp(2*x)],singsol=all)
```

$$y(x) = e^x(\ln(1+e^x)(1+e^x) + (-1-e^x)\ln(e^x) + e^x c_1 + c_2 - 1)$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 34

```
DSolve[y''[x]-3*y'[x]+2*y[x]==1/(1+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(2(e^x+1)\operatorname{arctanh}(2e^x+1) + c_2 e^x - 1 + c_1)$$

9.5 problem 5

Internal problem ID [1111]
Internal file name [`OUTPUT/1112_Sunday_June_05_2022_02_02_52_AM_61219485/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 7x^{\frac{3}{2}}\mathrm{e}^{x}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 7x^{\frac{3}{2}}\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -2$$

Therefore

$$\begin{aligned} y_2(x) &= e^x \left(\int e^{-\left(\int (-2)dx\right)} e^{-2x} dx \right) \\ y_2(x) &= e^x \int \frac{e^{2x}}{e^{2x}}, dx \\ y_2(x) &= e^x \left(\int 1 dx \right) \\ y_2(x) &= x e^x \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x + c_2 x e^x \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 x e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^x$$
$$y_2 = x\,\mathrm{e}^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \frac{d}{dx}(\mathrm{e}^x) & \frac{d}{dx}(x\,\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^x)(x\,\mathrm{e}^x + \mathrm{e}^x) - (x\,\mathrm{e}^x)(\mathrm{e}^x)$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{7x^{\frac{5}{2}}\mathrm{e}^{2x}}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 7x^{\frac{5}{2}}dx$$

Hence

$$u_1 = -2x^{\frac{7}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{7\,\mathrm{e}^{2x} x^{\frac{3}{2}}}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 7x^{\frac{3}{2}}dx$$

Hence

$$u_2 = \frac{14x^{\frac{5}{2}}}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4x^{\frac{7}{2}}\mathrm{e}^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}) + \left(\frac{4x^{\frac{7}{2}}\mathrm{e}^{x}}{5}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{4x^{\frac{7}{2}}\mathrm{e}^{x}}{5}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{4x^{\frac{7}{2}}\mathrm{e}^{x}}{5} \tag{1}$$

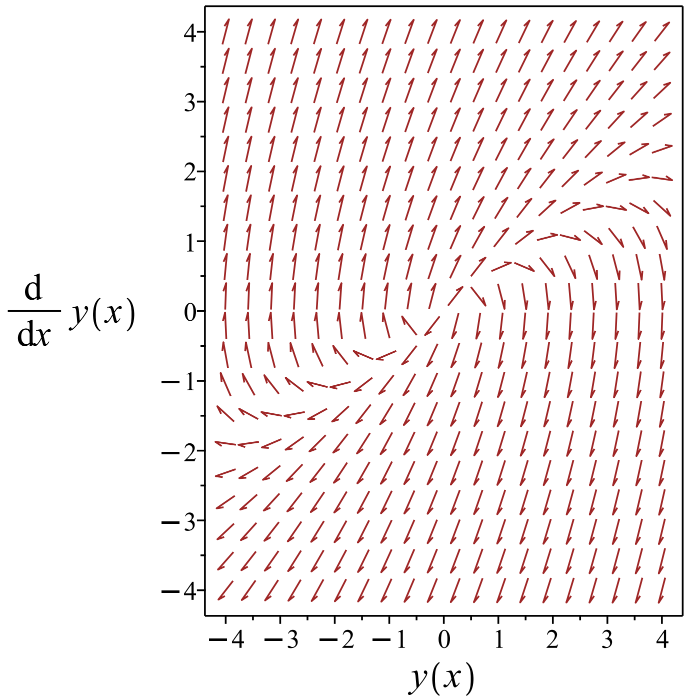


Figure 487: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + \frac{4x^{\frac{7}{2}}e^x}{5}$$

Verified OK.

9.5.1 Maple step by step solution

Let's solve

$y'' - 2y' + y = 7x^{\frac{3}{2}}e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 7x^{\frac{3}{2}}\mathrm{e}^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -7\,\mathrm{e}^x\left(-\left(\int x^{\frac{3}{2}} dx\right) x + \int x^{\frac{5}{2}} dx\right)$

  - Compute integrals

    $y_p(x) = \frac{4x^{\frac{7}{2}}\mathrm{e}^x}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^x + c_1 \mathrm{e}^x + \frac{4x^{\frac{7}{2}}\mathrm{e}^x}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=7*x^(3/2)*exp(x),exp(x)],singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(c_2 + c_1 x + \frac{4x^{\frac{7}{2}}}{5}\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 29

```
DSolve[y''[x]-2*y'[x]+y[x]==7*x^(3/2)*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}e^x\left(4x^{7/2} + 5c_2 x + 5c_1\right)$$

9.6 problem 6

Internal problem ID [1112]
Internal file name [`OUTPUT/1113_Sunday_June_05_2022_02_02_53_AM_88820714/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' + (-8x^2+4x)\, y' + (4x^2-4x-1)\, y = 4\sqrt{x}\, e^x(4x+1)}$$

Given that one solution of the ode is

$$y_1 = \sqrt{x}\, e^x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -8x^2+4x, C = 4x^2-4x-1, f(x) = -(-16x-4)\, e^x\sqrt{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' + (-8x^2+4x)\, y' + (4x^2-4x-1)\, y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-8x^2 + 4x}{4x^2}$$

Therefore

$$y_2(x) = \sqrt{x}\, e^{x} \left(\int \frac{e^{-\left(\int \frac{-8x^2+4x}{4x^2} dx\right)} e^{-2x}}{x} dx\right)$$

$$y_2(x) = \sqrt{x}\, e^{x} \int \frac{e^{2x-\ln(x)}}{x\, e^{2x}}, dx$$

$$y_2(x) = \sqrt{x}\, e^{x} \left(\int \frac{1}{x^2} dx\right)$$

$$y_2(x) = -\frac{e^{x}}{\sqrt{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sqrt{x}\, e^{x} c_1 - \frac{c_2 e^{x}}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \sqrt{x}\, e^{x} c_1 - \frac{c_2 e^{x}}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\, e^{x}$$
$$y_2 = -\frac{e^{x}}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\, e^{x} & -\frac{e^{x}}{\sqrt{x}} \\ \frac{d}{dx}\left(\sqrt{x}\, e^{x}\right) & \frac{d}{dx}\left(-\frac{e^{x}}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\, e^{x} & -\frac{e^{x}}{\sqrt{x}} \\ \frac{e^{x}}{2\sqrt{x}} + \sqrt{x}\, e^{x} & \frac{e^{x}}{2x^{\frac{3}{2}}} - \frac{e^{x}}{\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\, e^{x}\right)\left(\frac{e^{x}}{2x^{\frac{3}{2}}} - \frac{e^{x}}{\sqrt{x}}\right) - \left(-\frac{e^{x}}{\sqrt{x}}\right)\left(\frac{e^{x}}{2\sqrt{x}} + \sqrt{x}\, e^{x}\right)$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x}(-16x-4)}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-4x-1}{x} dx$$

Hence

$$u_1 = 4x + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x\, e^{2x}(-16x-4)}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int (4x+1)\, dx$$

Hence

$$u_2 = 2x^2 + x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (4x + \ln(x))\sqrt{x}\, e^{x} - \frac{(2x^2+x)\, e^{x}}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = e^{x}\sqrt{x}\,(\ln(x) + 2x - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\, e^{x} c_1 - \frac{c_2 e^{x}}{\sqrt{x}}\right) + \left(e^{x}\sqrt{x}\,(\ln(x) + 2x - 1)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{e^{x}(c_1 x - c_2)}{\sqrt{x}} + e^{x}\sqrt{x}\,(\ln(x) + 2x - 1)$$

Summary
The solution(s) found are the following

$$y = \frac{e^{x}(c_1 x - c_2)}{\sqrt{x}} + e^{x}\sqrt{x}\,(\ln(x) + 2x - 1) \tag{1}$$

Verification of solutions

$$y = \frac{e^{x}(c_1 x - c_2)}{\sqrt{x}} + e^{x}\sqrt{x}\,(\ln(x) + 2x - 1)$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.032 (sec). Leaf size: 26

```
dsolve([4*x^2*diff(y(x),x$2)+(4*x-8*x^2)*diff(y(x),x)+(4*x^2-4*x-1)*y(x)=4*x^(1/2)*exp(x)*(1
```

$$y(x) = \frac{(x\ln(x) + 2x^2 + (c_1 - 1)x + c_2)\,e^{x}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==4*x^(1/2)*Exp[x]*(1+4*x),y[x],x,In
```

$$y(x) \to \frac{e^x(2x^2 + x\log(x) + (-1 + c_2)x + c_1)}{\sqrt{x}}$$

9.7 problem 7

Internal problem ID [1113]
Internal file name [`OUTPUT/1114_Sunday_June_05_2022_02_02_54_AM_22826745/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = e^x \sec(x)}$$

Given that one solution of the ode is

$$y_1 = \cos(x) e^x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = e^x \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -2$$

Therefore

$$y_2(x) = \cos(x)\,\mathrm{e}^{x}\left(\int \frac{\mathrm{e}^{-\left(\int(-2)dx\right)}\,\mathrm{e}^{-2x}}{\cos(x)^2}dx\right)$$

$$y_2(x) = \cos(x)\,\mathrm{e}^{x}\int \frac{\mathrm{e}^{2x}}{\cos(x)^2\,\mathrm{e}^{2x}},dx$$

$$y_2(x) = \cos(x)\,\mathrm{e}^{x}\left(\int \sec(x)^2\,dx\right)$$

$$y_2(x) = \cos(x)\,\mathrm{e}^{x}\tan(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= \cos(x)\,\mathrm{e}^{x}c_1 + c_2\cos(x)\,\mathrm{e}^{x}\tan(x) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x)\,\mathrm{e}^{x}c_1 + c_2\cos(x)\,\mathrm{e}^{x}\tan(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x)\,\mathrm{e}^{x} \\ y_2 &= \cos(x)\,\mathrm{e}^{x}\tan(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,e^{x} & \cos(x)\,e^{x}\tan(x) \\ \frac{d}{dx}(\cos(x)\,e^{x}) & \frac{d}{dx}(\cos(x)\,e^{x}\tan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,e^{x} & \cos(x)\,e^{x}\tan(x) \\ -\sin(x)\,e^{x} + \cos(x)\,e^{x} & -\sin(x)\,e^{x}\tan(x) + \cos(x)\,e^{x}\tan(x) + \cos(x)\,e^{x}(1+\tan(x)^2) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & (\cos(x)\,e^{x})\left(-\sin(x)\,e^{x}\tan(x) + \cos(x)\,e^{x}\tan(x) + \cos(x)\,e^{x}(1+\tan(x)^2)\right) \\ & - (\cos(x)\,e^{x}\tan(x))(-\sin(x)\,e^{x} + \cos(x)\,e^{x}) \end{aligned}$$

Which simplifies to

$$W = \tan(x)^2\,e^{2x}\cos(x)^2 + \cos(x)^2\,e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos(x)\,e^{2x}\tan(x)\sec(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\,\mathrm{e}^{2x}\sec(x)}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x)\,\mathrm{e}^x + x\cos(x)\,\mathrm{e}^x\tan(x)$$

Which simplifies to

$$y_p(x) = \mathrm{e}^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,\mathrm{e}^x c_1 + c_2\cos(x)\,\mathrm{e}^x\tan(x)) + (\mathrm{e}^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x(c_1\cos(x) + c_2\sin(x)) + \mathrm{e}^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x(c_1\cos(x) + c_2\sin(x)) + \mathrm{e}^x(\ln(\cos(x))\cos(x) + \sin(x)\,x) \tag{1}$$

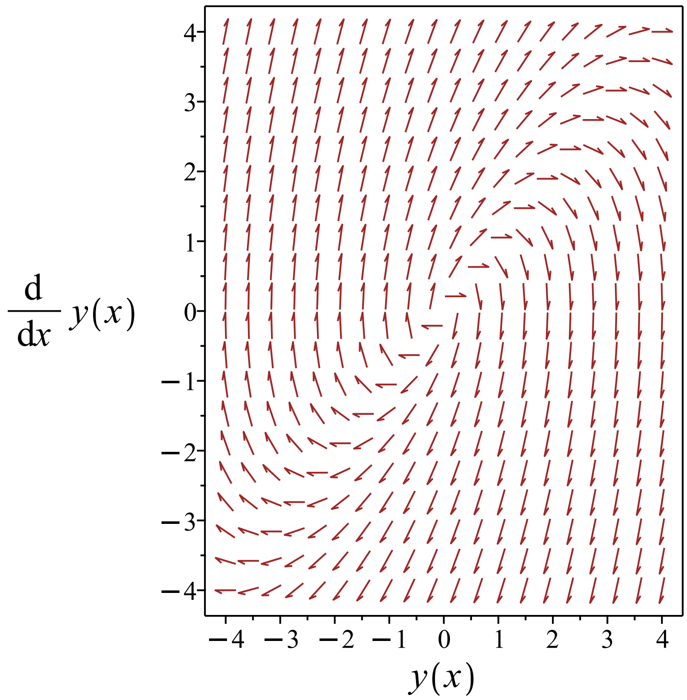


Figure 488: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(\ln(\cos(x)) \cos(x) + \sin(x)\, x)$$

Verified OK.

9.7.1 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = e^{x} \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (1 - I, 1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\, e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\, e^x c_1 + e^x \sin(x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x \sec(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, e^x & \sin(x)\, e^x \\ -\sin(x)\, e^x + \cos(x)\, e^x & \cos(x)\, e^x + \sin(x)\, e^x \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^x\left(\cos(x)\left(\int \tan(x)\, dx\right) - \sin(x)\left(\int 1 dx\right)\right)$

  ◦ Compute integrals

    $y_p(x) = e^x(\ln(\cos(x))\cos(x) + \sin(x)\, x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(x)\, e^x c_1 + e^x \sin(x)\, c_2 + e^x(\ln(\cos(x))\cos(x) + \sin(x)\, x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=exp(x)*sec(x),exp(x)*cos(x)],singsol=all)
```

$$y(x) = (-\cos(x)\ln(\sec(x)) + \cos(x)\,c_1 + \sin(x)\,(c_2 + x))\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 26

```
DSolve[y''[x]-2*y'[x]+2*y[x]==Exp[x]*Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}((x + c_1)\sin(x) + \cos(x)(\log(\cos(x)) + c_2))$$

9.8 problem 8

Internal problem ID [1114]
Internal file name [`OUTPUT/1115_Sunday_June_05_2022_02_02_56_AM_84972689/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y'x + (4x^2 + 2)\, y = 8\,\mathrm{e}^{-x(2+x)}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{-x^2}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4x, C = 4x^2 + 2, f(x) = 8\,\mathrm{e}^{-x(2+x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y'x + (4x^2 + 2)\, y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = 4x$$

Therefore

$$\begin{aligned}
y_2(x) &= \mathrm{e}^{-x^2} \left(\int \mathrm{e}^{-\left(\int 4x dx\right)} \mathrm{e}^{2x^2} dx \right) \\
y_2(x) &= \mathrm{e}^{-x^2} \int \frac{\mathrm{e}^{-2x^2}}{\mathrm{e}^{-2x^2}}, dx \\
y_2(x) &= \mathrm{e}^{-x^2} \left(\int 1 dx \right) \\
y_2(x) &= x\, \mathrm{e}^{-x^2}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \mathrm{e}^{-x^2} + c_2 x\, \mathrm{e}^{-x^2}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x^2} + c_2 x\, \mathrm{e}^{-x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x^2} \\
y_2 &= x\, \mathrm{e}^{-x^2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x^2} & x\,e^{-x^2} \\ \frac{d}{dx}\left(e^{-x^2}\right) & \frac{d}{dx}\left(x\,e^{-x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x^2} & x\,e^{-x^2} \\ -2x\,e^{-x^2} & e^{-x^2} - 2x^2 e^{-x^2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x^2}\right)\left(e^{-x^2} - 2x^2 e^{-x^2}\right) - \left(x\,e^{-x^2}\right)\left(-2x\,e^{-x^2}\right)$$

Which simplifies to

$$W = e^{-2x^2}$$

Which simplifies to

$$W = e^{-2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x\,e^{-x^2} e^{-x(2+x)}}{e^{-2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 8\,e^{-2x} x dx$$

Hence

$$u_1 = 2(1 + 2x)\,\mathrm{e}^{-2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,\mathrm{e}^{-x^2}\mathrm{e}^{-x(2+x)}}{\mathrm{e}^{-2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int 8\,\mathrm{e}^{-2x}dx$$

Hence

$$u_2 = -4\,\mathrm{e}^{-2x}$$

Which simplifies to

$$\begin{aligned} u_1 &= (4x + 2)\,\mathrm{e}^{-2x} \\ u_2 &= -4\,\mathrm{e}^{-2x} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (4x + 2)\,\mathrm{e}^{-2x}\mathrm{e}^{-x^2} - 4\,\mathrm{e}^{-2x}x\,\mathrm{e}^{-x^2}$$

Which simplifies to

$$y_p(x) = 2\,\mathrm{e}^{-x(2+x)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x^2} + c_2x\,\mathrm{e}^{-x^2}\right) + \left(2\,\mathrm{e}^{-x(2+x)}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x^2}(c_2x + c_1) + 2\,\mathrm{e}^{-x(2+x)}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x^2}(c_2x + c_1) + 2\,\mathrm{e}^{-x(2+x)} \tag{1}$$

Verification of solutions

$$y = e^{-x^2}(c_2x + c_1) + 2\,e^{-x(2+x)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=8*exp(-x*(x+2)),exp(-x^2)],singsol=al
```

$$y(x) = (c_1x + c_2)\,e^{-x^2} + 2\,e^{-x(2+x)}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 29

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+2)*y[x]==8*Exp[-x*(x+2)],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-x(x+2)}\left(2 + e^{2x}(c_2x + c_1)\right)$$

9.9 problem 9

Internal problem ID [1115]
Internal file name [`OUTPUT/1116_Sunday_June_05_2022_02_02_57_AM_23492223/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x - 4y = -6x - 4}$$

Given that one solution of the ode is

$$y_1 = x^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = -6x - 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x - 4y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1}{x}$$

Therefore

$$y_2(x) = x^2\left(\int \frac{e^{-\left(\int \frac{1}{x}dx\right)}}{x^4}dx\right)$$

$$y_2(x) = x^2 \int \frac{\frac{1}{x}}{x^4}, dx$$

$$y_2(x) = x^2\left(\int \frac{1}{x^5}dx\right)$$

$$y_2(x) = -\frac{1}{4x^2}$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^2 - \frac{c_2}{4x^2} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1x^2 - \frac{c_2}{4x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$

$$y_2 = -\frac{1}{4x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & -\frac{1}{4x^2} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(-\frac{1}{4x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & -\frac{1}{4x^2} \\ 2x & \frac{1}{2x^3} \end{vmatrix}$$

Therefore

$$W = (x^2)\left(\frac{1}{2x^3}\right) - \left(-\frac{1}{4x^2}\right)(2x)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{-6x-4}{4x^2}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{3x+2}{2x^3}dx$$

Hence

$$u_1 = \frac{3}{2x} + \frac{1}{2x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(-6x-4)}{x}\,dx$$

Which simplifies to

$$u_2 = \int \left(-6x^2 - 4x\right) dx$$

Hence

$$u_2 = -2x^3 - 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{3}{2x} + \frac{1}{2x^2}\right) x^2 - \frac{-2x^3 - 2x^2}{4x^2}$$

Which simplifies to

$$y_p(x) = 1 + 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 - \frac{c_2}{4x^2}\right) + (1 + 2x) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x^2 - \frac{c_2}{4x^2} + 1 + 2x \tag{1}$$

Verification of solutions

$$y = c_1x^2 - \frac{c_2}{4x^2} + 1 + 2x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=-6*x-4,x^2],singsol=all)
```

$$y(x) = \frac{c_2}{x^2} + c_1x^2 + 2x + 1$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]-4*y[x]==-6*x-4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x^2 + \frac{c_1}{x^2} + 2x + 1$$

9.10 problem 10

Internal problem ID [1116]
Internal file name [`OUTPUT/1117_Sunday_June_05_2022_02_02_58_AM_90097776/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 2x(x-1)\,y' + \left(x^2-2x+2\right)y = \mathrm{e}^{2x}x^3}$$

Given that one solution of the ode is

$$y_1 = x\,\mathrm{e}^{-x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x^2 - 2x, C = x^2 - 2x + 2, f(x) = \mathrm{e}^{2x}x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(2x^2-2x\right)y' + \left(x^2-2x+2\right)y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2x^2 - 2x}{x^2}$$

Therefore

$$\begin{aligned}
y_2(x) &= x\,\mathrm{e}^{-x} \left(\int \frac{\mathrm{e}^{-\left(\int \frac{2x^2-2x}{x^2} dx\right)}\mathrm{e}^{2x}}{x^2} dx \right) \\
y_2(x) &= x\,\mathrm{e}^{-x} \int \frac{\mathrm{e}^{-2x+2\ln(x)}}{x^2\mathrm{e}^{-2x}}, dx \\
y_2(x) &= x\,\mathrm{e}^{-x} \left(\int 1 dx \right) \\
y_2(x) &= x^2\mathrm{e}^{-x}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= x\,\mathrm{e}^{-x} c_1 + c_2 x^2\mathrm{e}^{-x}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = x\,\mathrm{e}^{-x} c_1 + c_2 x^2\mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x\,\mathrm{e}^{-x} \\
y_2 &= x^2\mathrm{e}^{-x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{-x} & x^2\mathrm{e}^{-x} \\ \frac{d}{dx}(x\,\mathrm{e}^{-x}) & \frac{d}{dx}(x^2\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{-x} & x^2\mathrm{e}^{-x} \\ \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} & 2x\,\mathrm{e}^{-x} - x^2\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = \left(x\,\mathrm{e}^{-x}\right)\left(2x\,\mathrm{e}^{-x} - x^2\mathrm{e}^{-x}\right) - \left(x^2\mathrm{e}^{-x}\right)\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = x^2\mathrm{e}^{-2x}$$

Which simplifies to

$$W = x^2\mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5\mathrm{e}^{-x}\mathrm{e}^{2x}}{x^4\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x\,\mathrm{e}^{3x}dx$$

Hence

$$u_1 = -\frac{(3x-1)\,\mathrm{e}^{3x}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \mathrm{e}^{-x} \mathrm{e}^{2x}}{x^4 \mathrm{e}^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{3x} dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{3x}}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(3x-1)\, \mathrm{e}^{3x} x\, \mathrm{e}^{-x}}{9} + \frac{\mathrm{e}^{3x} x^2 \mathrm{e}^{-x}}{3}$$

Which simplifies to

$$y_p(x) = \frac{x\, \mathrm{e}^{2x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x\, \mathrm{e}^{-x} c_1 + c_2 x^2 \mathrm{e}^{-x}\right) + \left(\frac{x\, \mathrm{e}^{2x}}{9}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x} x (c_2 x + c_1) + \frac{x\, \mathrm{e}^{2x}}{9}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x} x (c_2 x + c_1) + \frac{x\, \mathrm{e}^{2x}}{9} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x} x (c_2 x + c_1) + \frac{x\, \mathrm{e}^{2x}}{9}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 27

```
dsolve([x^2*diff(y(x),x$2)+2*x*(x-1)*diff(y(x),x)+(x^2-2*x+2)*y(x)=x^3*exp(2*x),x*exp(-x)],s
```

$$y(x) = \frac{x(e^{2x} + 9c_1 x e^{-x} + 9 e^{-x} c_2)}{9}$$

✓ Solution by Mathematica
Time used: 0.054 (sec). Leaf size: 30

```
DSolve[x^2*y''[x]+2*x*(x-1)*y'[x]+(x^2-2*x+2)*y[x]==x^3*Exp[2*x],y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{9}e^{-x}x\left(e^{3x} + 9(c_2 x + c_1)\right)$$

9.11 problem 11

Internal problem ID [1117]
Internal file name [`OUTPUT/1118_Sunday_June_05_2022_02_03_00_AM_69233257/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - x(2x-1)\,y' + \left(x^2 - x - 1\right)y = x^2\mathrm{e}^x}$$

Given that one solution of the ode is

$$y_1 = x\,\mathrm{e}^x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x^2 + x, C = x^2 - x - 1, f(x) = x^2\mathrm{e}^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(-2x^2 + x\right)y' + \left(x^2 - x - 1\right)y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2x^2 + x}{x^2}$$

Therefore

$$\begin{aligned}
y_2(x) &= x\,\mathrm{e}^x \left(\int \frac{\mathrm{e}^{-\left(\int \frac{-2x^2+x}{x^2}dx\right)}\mathrm{e}^{-2x}}{x^2} dx\right) \\
y_2(x) &= x\,\mathrm{e}^x \int \frac{\mathrm{e}^{2x-\ln(x)}}{x^2 \mathrm{e}^{2x}}, dx \\
y_2(x) &= x\,\mathrm{e}^x \left(\int \frac{1}{x^3} dx\right) \\
y_2(x) &= -\frac{\mathrm{e}^x}{2x}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\,\mathrm{e}^x - \frac{c_2 \mathrm{e}^x}{2x}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x\,\mathrm{e}^x - \frac{c_2 \mathrm{e}^x}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\,e^x$$
$$y_2 = -\frac{e^x}{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,e^x & -\frac{e^x}{2x} \\ \frac{d}{dx}(x\,e^x) & \frac{d}{dx}\left(-\frac{e^x}{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,e^x & -\frac{e^x}{2x} \\ x\,e^x + e^x & -\frac{e^x}{2x} + \frac{e^x}{2x^2} \end{vmatrix}$$

Therefore

$$W = (x\,e^x)\left(-\frac{e^x}{2x} + \frac{e^x}{2x^2}\right) - \left(-\frac{e^x}{2x}\right)(x\,e^x + e^x)$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{x\,e^{2x}}{2}}{x\,e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{2} dx$$

Hence

$$u_1 = \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x^3}{x\, e^{2x}} dx$$

Which simplifies to

$$u_2 = \int x^2 dx$$

Hence

$$u_2 = \frac{x^3}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2 e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x\, e^x - \frac{c_2 e^x}{2x}\right) + \left(\frac{x^2 e^x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, e^x - \frac{c_2 e^x}{2x} + \frac{x^2 e^x}{3} \tag{1}$$

Verification of solutions

$$y = c_1 x\, e^x - \frac{c_2 e^x}{2x} + \frac{x^2 e^x}{3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 24

```
dsolve([x^2*diff(y(x),x$2)-x*(2*x-1)*diff(y(x),x)+(x^2-x-1)*y(x)=x^2*exp(x),x*exp(x)],singso
```

$$y(x) = \frac{e^x(3c_1x^2 + x^3 + 3c_2)}{3x}$$

✓ Solution by Mathematica
Time used: 0.038 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]-x*(2*x-1)*y'[x]+(x^2-x-1)*y[x]==x^2*Exp[x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^x(2x^3 + 3c_2x^2 + 6c_1)}{6x}$$

9.12 problem 12

Internal problem ID [1118]
Internal file name [`OUTPUT/1119_Sunday_June_05_2022_02_03_01_AM_77209729/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{(1-2x)\,y''+2y'+(-3+2x)\,y=\left(4x^2-4x+1\right)\mathrm{e}^x}$$

Given that one solution of the ode is

$$y_1=\mathrm{e}^x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1-2x, B=2, C=-3+2x, f(x)=(2x-1)^2\,\mathrm{e}^x$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(1-2x)\,y''+2y'+(-3+2x)\,y=0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int\frac{\mathrm{e}^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2}{1 - 2x}$$

Therefore

$$\begin{aligned}
y_2(x) &= \mathrm{e}^{x} \left(\int \mathrm{e}^{-\left(\int \frac{2}{1-2x}dx\right)} \mathrm{e}^{-2x} dx\right) \\
y_2(x) &= \mathrm{e}^{x} \int \frac{1 - 2x}{\mathrm{e}^{2x}}, dx \\
y_2(x) &= \mathrm{e}^{x} \left(\int (1 - 2x)\, \mathrm{e}^{-2x} dx\right) \\
y_2(x) &= \mathrm{e}^{x} \mathrm{e}^{-2x} x
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \mathrm{e}^{-2x} x
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \mathrm{e}^{-2x} x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= \mathrm{e}^{x} \mathrm{e}^{-2x} x
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^x e^{-2x} x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^x e^{-2x} x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^x e^{-2x} x \\ e^x & -e^x e^{-2x} x + e^{-2x} e^x \end{vmatrix}$$

Therefore

$$W = (e^x)\left(-e^x e^{-2x} x + e^{-2x} e^x\right) - \left(e^x e^{-2x} x\right)(e^x)$$

Which simplifies to

$$W = -2\, e^{-2x} x\, e^{2x} + e^{2x} e^{-2x}$$

Which simplifies to

$$W = 1 - 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} e^{-2x} x (2x-1)^2}{(1-2x)^2}\, dx$$

Which simplifies to

$$u_1 = -\int x dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x}(2x-1)^2}{(1-2x)^2}\, dx$$

Which simplifies to

$$u_2 = \int e^{2x} dx$$

Hence

$$u_2 = \frac{e^{2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2 e^x}{2} + \frac{e^{2x} e^x e^{-2x} x}{2}$$

Which simplifies to

$$y_p(x) = -\frac{e^x x(x-1)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^x e^{-2x} x\right) + \left(-\frac{e^x x(x-1)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^x + x\, e^{-x} c_2 - \frac{e^x x(x-1)}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + x\, e^{-x} c_2 - \frac{e^x x(x-1)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^x + x\,e^{-x} c_2 - \frac{e^x x(x-1)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve([(1-2*x)*diff(y(x),x$2)+2*diff(y(x),x)+(2*x-3)*y(x)=(1-4*x+4*x^2)*exp(x),exp(x)],sing
```

$$y(x) = c_1 x\,e^{-x} - \frac{e^x(x^2 - 2c_2 - x)}{2}$$

✓ Solution by Mathematica

Time used: 0.254 (sec). Leaf size: 77

```
DSolve[(1-2*x)*y''[x]+2*y'[x]+(2*x-3)*y[x]==(1-4*x+4*x^2)*Exp[x],y[x],x,IncludeSingularSolut
```

$$y(x) \to -\frac{1}{2}e^x(x-1)x - \frac{c_2 e^{\frac{1}{2}-x}\sqrt{1-2x}x}{\sqrt{2x-1}} + \frac{c_1 e^{x-\frac{1}{2}}\sqrt{1-2x}}{\sqrt{2x-1}}$$

9.13 problem 13

Internal problem ID [1119]
Internal file name [`OUTPUT/1120_Sunday_June_05_2022_02_03_02_AM_15823335/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + 4y = 4x^4}$$

Given that one solution of the ode is

$$y_1 = x^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 4x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{3}{x}$$

Therefore

$$\begin{aligned}
y_2(x) &= x^2 \left(\int \frac{e^{-\left(\int -\frac{3}{x} dx\right)}}{x^4} dx \right) \\
y_2(x) &= x^2 \int \frac{x^3}{x^4}, dx \\
y_2(x) &= x^2 \left(\int \frac{1}{x} dx \right) \\
y_2(x) &= \ln(x) x^2
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 + c_2 \ln(x) x^2
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x^2 + c_2 \ln(x) x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2x\ln(x)) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\ln(x)\, x^6}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int 4x \ln(x)\, dx$$

Hence

$$u_1 = -2\ln(x)\, x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^6}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Which simplifies to

$$\begin{aligned} u_1 &= x^2(1 - 2\ln(x)) \\ u_2 &= 2x^2 \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^4(1 - 2\ln(x)) + 2x^4 \ln(x)$$

Which simplifies to

$$y_p(x) = x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 \ln(x)\, x^2\right) + \left(x^4\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + x^4$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + x^4 \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + x^4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=4*x^4,x^2],singsol=all)
```

$$y(x) = x^2(\ln(x)\, c_1 + x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==4*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x^2 + 2c_2 \log(x) + c_1)$$

9.14 problem 14

Internal problem ID [1120]
Internal file name [`OUTPUT/1121_Sunday_June_05_2022_02_03_03_AM_94218790/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{2xy'' + (4x+1)\,y' + (1+2x)\,y = 3\sqrt{x}\,\mathrm{e}^{-x}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{-x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x, B = 4x+1, C = 1+2x, f(x) = 3\sqrt{x}\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2xy'' + (4x+1)\,y' + (1+2x)\,y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{4x+1}{2x}$$

Therefore

$$y_2(x) = \mathrm{e}^{-x} \left(\int \mathrm{e}^{-\left(\int \frac{4x+1}{2x}dx\right)} \mathrm{e}^{2x} dx \right)$$

$$y_2(x) = \mathrm{e}^{-x} \int \frac{\mathrm{e}^{-2x-\frac{\ln(x)}{2}}}{\mathrm{e}^{-2x}}, dx$$

$$y_2(x) = \mathrm{e}^{-x} \left(\int \frac{1}{\sqrt{x}} dx \right)$$

$$y_2(x) = 2\sqrt{x}\, \mathrm{e}^{-x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \mathrm{e}^{-x} + 2c_2 \sqrt{x}\, \mathrm{e}^{-x} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + 2c_2 \sqrt{x}\, \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= 2\sqrt{x}\, \mathrm{e}^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & 2\sqrt{x}\,\mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(2\sqrt{x}\,\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & 2\sqrt{x}\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}}{\sqrt{x}} - 2\sqrt{x}\,\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}} - 2\sqrt{x}\,\mathrm{e}^{-x}\right) - (2\sqrt{x}\,\mathrm{e}^{-x})(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

Which simplifies to

$$W = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\,\mathrm{e}^{-2x} x}{2\,\mathrm{e}^{-2x}\sqrt{x}}\,dx$$

Which simplifies to

$$u_1 = -\int 3\sqrt{x}dx$$

Hence

$$u_1 = -2x^{\frac{3}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\,\mathrm{e}^{-2x}\sqrt{x}}{2\,\mathrm{e}^{-2x}\sqrt{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{3}{2}dx$$

Hence

$$u_2 = \frac{3x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^{\frac{3}{2}}\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + 2c_2\sqrt{x}\,\mathrm{e}^{-x}\right) + \left(x^{\frac{3}{2}}\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}\left(c_1 + 2\sqrt{x}\,c_2\right) + x^{\frac{3}{2}}\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_1 + 2\sqrt{x}\,c_2\right) + x^{\frac{3}{2}}\mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}\left(c_1 + 2\sqrt{x}\,c_2\right) + x^{\frac{3}{2}}\mathrm{e}^{-x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 19

```
dsolve([2*x*diff(y(x),x$2)+(4*x+1)*diff(y(x),x)+(2*x+1)*y(x)=3*x^(1/2)*exp(-x),exp(-x)],sing
```

$$y(x) = \mathrm{e}^{-x}\left(c_2 + c_1\sqrt{x} + x^{\frac{3}{2}}\right)$$

✓ Solution by Mathematica
Time used: 0.049 (sec). Leaf size: 28

```
DSolve[2*x*y''[x]+(4*x+1)*y'[x]+(2*x+1)*y[x]==3*x^(1/2)*Exp[-x],y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{-x}\left(x^{3/2} + 2c_2\sqrt{x} + c_1\right)$$

9.15 problem 15

Internal problem ID [1121]
Internal file name [`OUTPUT/1122_Sunday_June_05_2022_02_03_05_AM_27284816/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - (1+2x)\,y' + (x+1)\,y = -\mathrm{e}^{-x}}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1-2x, C = x+1, f(x) = -\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + (-1-2x)\,y' + (x+1)\,y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-1 - 2x}{x}$$

Therefore

$$y_2(x) = \mathrm{e}^{x}\left(\int \mathrm{e}^{-\left(\int \frac{-1-2x}{x}dx\right)}\mathrm{e}^{-2x}dx\right)$$

$$y_2(x) = \mathrm{e}^{x}\int \frac{\mathrm{e}^{2x+\ln(x)}}{\mathrm{e}^{2x}}, dx$$

$$y_2(x) = \mathrm{e}^{x}\left(\int x dx\right)$$

$$y_2(x) = \frac{x^{2}\mathrm{e}^{x}}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\mathrm{e}^{x} + \frac{c_2 x^{2}\mathrm{e}^{x}}{2} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\mathrm{e}^{x} + \frac{c_2 x^{2}\mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{x}$$

$$y_2 = \frac{x^{2}\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{x^2 e^x}{2} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{x^2 e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{x^2 e^x}{2} \\ e^x & x\,e^x + \frac{x^2 e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(x\,e^x + \frac{x^2 e^x}{2}\right) - \left(\frac{x^2 e^x}{2}\right)(e^x)$$

Which simplifies to

$$W = x\,e^{2x}$$

Which simplifies to

$$W = x\,e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{x^2 e^x e^{-x}}{2}}{x^2 e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x}}{2}dx$$

Hence

$$u_1 = -\frac{e^{-2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-e^{-x}e^{x}}{x^2 e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{-2x}}{x^2} dx$$

Hence

$$u_2 = \frac{e^{-2x}}{x} - 2\ \mathrm{expIntegral}_1\left(2x\right)$$

Which simplifies to

$$u_1 = -\frac{e^{-2x}}{4}$$
$$u_2 = \frac{-2\ \mathrm{expIntegral}_1\left(2x\right) x + e^{-2x}}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-2x}e^{x}}{4} + \frac{\left(-2\ \mathrm{expIntegral}_1\left(2x\right) x + e^{-2x}\right) x\, e^{x}}{2}$$

Which simplifies to

$$y_p(x) = -e^{x}\ \mathrm{expIntegral}_1\left(2x\right) x^2 + \frac{e^{-x}(2x-1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + \frac{c_2 x^2 e^{x}}{2}\right) + \left(-e^{x}\ \mathrm{expIntegral}_1\left(2x\right) x^2 + \frac{e^{-x}(2x-1)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{x}\left(c_1 + \frac{c_2 x^2}{2}\right) - e^{x}\ \mathrm{expIntegral}_1\left(2x\right) x^2 + \frac{e^{-x}(2x-1)}{4}$$

Summary
The solution(s) found are the following

$$y = e^{x}\left(c_1 + \frac{c_2 x^2}{2}\right) - e^{x} \operatorname{expIntegral}_1(2x)\, x^2 + \frac{e^{-x}(2x-1)}{4} \tag{1}$$

Verification of solutions

$$y = e^{x}\left(c_1 + \frac{c_2 x^2}{2}\right) - e^{x} \operatorname{expIntegral}_1(2x)\, x^2 + \frac{e^{-x}(2x-1)}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve([x*diff(y(x),x$2)-(2*x+1)*diff(y(x),x)+(x+1)*y(x)=-exp(-x),exp(x)],singsol=all)
```

$$y(x) = -\operatorname{expIntegral}_1(2x)\, x^2 e^{x} + \frac{e^{-x}(2x-1)}{4} + e^{x}\left(c_1 x^2 + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 52

```
DSolve[x*y''[x]-(2*x+1)*y'[x]+(x+1)*y[x]==-Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x x^2 \operatorname{ExpIntegralEi}(-2x) + \frac{1}{4} e^{-x} \left(2c_2 e^{2x} x^2 + 2x + 4c_1 e^{2x} - 1\right)$$

9.16 problem 16

Internal problem ID [1122]
Internal file name [`OUTPUT/1123_Sunday_June_05_2022_02_03_06_AM_43908000/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 4x^{\frac{5}{2}}\mathrm{e}^{2x}}$$

Given that one solution of the ode is

$$y_1 = \sqrt{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -4x^2 - 4x, C = 2x + 3, f(x) = 4x^{\frac{5}{2}}\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-4x^2 - 4x}{4x^2}$$

Therefore

$$y_2(x) = \sqrt{x}\left(\int \frac{\mathrm{e}^{-\left(\int \frac{-4x^2-4x}{4x^2}dx\right)}}{x}dx\right)$$

$$y_2(x) = \sqrt{x}\int \frac{\mathrm{e}^{x+\ln(x)}}{x}, dx$$

$$y_2(x) = \sqrt{x}\left(\int \mathrm{e}^x dx\right)$$

$$y_2(x) = \sqrt{x}\,\mathrm{e}^x$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sqrt{x}\,c_1 + c_2\sqrt{x}\,\mathrm{e}^x \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \sqrt{x}\,c_1 + c_2\sqrt{x}\,\mathrm{e}^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{x} \\ y_2 &= \sqrt{x}\,\mathrm{e}^x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\mathrm{e}^{x} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{e}^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\mathrm{e}^{x} \\ \frac{1}{2\sqrt{x}} & \frac{\mathrm{e}^{x}}{2\sqrt{x}} + \sqrt{x}\,\mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)\left(\frac{\mathrm{e}^{x}}{2\sqrt{x}} + \sqrt{x}\,\mathrm{e}^{x}\right) - \left(\sqrt{x}\,\mathrm{e}^{x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\,\mathrm{e}^{x}$$

Which simplifies to

$$W = x\,\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x^3 \mathrm{e}^{x} \mathrm{e}^{2x}}{4\,\mathrm{e}^{x} x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \mathrm{e}^{2x} dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{2x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,e^{2x} x^3}{4\,e^{x} x^3}\, dx$$

Which simplifies to

$$u_2 = \int e^{x} dx$$

Hence

$$u_2 = e^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{2x}\sqrt{x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^{2x}\sqrt{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\, c_1 + c_2\sqrt{x}\, e^{x}\right) + \left(\frac{e^{2x}\sqrt{x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = (c_1 + c_2 e^{x})\sqrt{x} + \frac{e^{2x}\sqrt{x}}{2}$$

Summary

The solution(s) found are the following

$$y = (c_1 + c_2 e^{x})\sqrt{x} + \frac{e^{2x}\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = (c_1 + c_2 e^{x})\sqrt{x} + \frac{e^{2x}\sqrt{x}}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve([4*x^2*diff(y(x),x$2)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x)=4*x^(5/2)*exp(2*x),x^(1/2)]
```

$$y(x) = \left(c_2 + e^x c_1 + \frac{e^{2x}}{2}\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 31

```
DSolve[4*x^2*y''[x]-4*x*(x+1)*y'[x]+(2*x+3)*y[x]==4*x^(5/2)*Exp[2*x],y[x],x,IncludeSingularS
```

$$y(x) \to \frac{1}{2}\sqrt{x}\left(e^{2x} + 2c_2e^x + 2c_1\right)$$

9.17 problem 17

Internal problem ID [1123]
Internal file name [`OUTPUT/1124_Sunday_June_05_2022_02_03_08_AM_68602730/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 5y'x + 8y = 4x^2}$$

Given that one solution of the ode is

$$y_1 = x^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 5y'x + 8y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{5}{x}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{e^{-(\int -\frac{5}{x} dx)}}{x^4} dx \right)$$

$$y_2(x) = x^2 \int \frac{x^5}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int x dx \right)$$

$$y_2(x) = \frac{x^4}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 + \frac{1}{2} c_2 x^4 \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x^2 + \frac{1}{2} c_2 x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$

$$y_2 = \frac{x^4}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(\frac{x^4}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ 2x & 2x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(2x^3) - \left(\frac{x^4}{2}\right)(2x)$$

Which simplifies to

$$W = x^5$$

Which simplifies to

$$W = x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^6}{x^7}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{x} dx$$

Hence

$$u_1 = -2\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^4}{x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{4}{x^3} dx$$

Hence

$$u_2 = -\frac{2}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2\ln(x)\, x^2 - x^2$$

Which simplifies to

$$y_p(x) = x^2(-1 - 2\ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + \frac{1}{2} c_2 x^4\right) + \left(x^2(-1 - 2\ln(x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + \frac{c_2 x^4}{2} + x^2(-1 - 2\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + \frac{c_2 x^4}{2} + x^2(-1 - 2\ln(x))$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+8*y(x)=4*x^2,x^2],singsol=all)
```

$$y(x) = x^2\left(c_2x^2 - 2\ln(x) + c_1 - 1\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]-5*x*y'[x]+8*y[x]==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2\left(c_2x^2 - 2\log(x) - 1 + c_1\right)$$

9.18 problem 18

Internal problem ID [1124]
Internal file name [`OUTPUT/1125_Sunday_June_05_2022_02_03_09_AM_23081873/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-2x + 2)\, y' + (-2 + x)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = e^x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2x + 2}{x}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int \frac{-2x+2}{x} dx\right)} e^{-2x} dx \right)$$

$$y_2(x) = e^x \int \frac{e^{2x-2\ln(x)}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int \frac{1}{x^2} dx \right)$$

$$y_2(x) = -\frac{e^x}{x}$$

Hence the solution is

$$y = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 e^x - \frac{c_2 e^x}{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x - \frac{c_2 e^x}{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^x - \frac{c_2 e^x}{x}$$

Verified OK.

9.18.1 Maple step by step solution

Let's solve

$y''x + (-2x + 2) y' + (-2 + x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-2+x)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x-1)y'}{x} + \frac{(-2+x)y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{-2+x}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x + (-2x+2)\,y' + (-2+x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r) x^{-1+r} + (a_1(1+r)(2+r) - 2a_0(1+r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+2+r) - 2a_k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)(2+r) - 2a_0(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2+r)(k+3+r) - 2a_{k+1}(k+1) - 2r a_{k+1} - 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + 4a_{k+1}}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2k a_{1+k} - a_k + 2a_{1+k}}{(1+k)(k+2)}, 0 = 0, b_{k+2} = \frac{2k b_{1+k} - b_k + 4b_{1+k}}{(k+2)(k+3)}, 2b_1 - 2b\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([x*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = \frac{e^x(c_1 x + c_2)}{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{x}$$

9.19 problem 19

Internal problem ID [1125]
Internal file name [`OUTPUT/1126_Sunday_June_05_2022_02_03_10_AM_52368277/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

Given that one solution of the ode is

$$y_1 = x^2$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{4}{x}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{e^{-(\int -\frac{4}{x} dx)}}{x^4} dx \right)$$

$$y_2(x) = x^2 \int \frac{x^4}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int 1 dx \right)$$

$$y_2(x) = x^3$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_2 x^3 + c_1 x^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^3 + c_1 x^2$$

Verified OK.

9.19.1 Maple step by step solution

Let's solve

$x^2 y'' - 4y'x + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 4y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

$$y_2(t) = e^{3t}$$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 e^{2t} + c_2 e^{3t}$$

- Change variables back using $t = \ln(x)$

$$y = c_2 x^3 + c_1 x^2$$

- Simplify

$$y = x^2(c_2 x + c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,x^2],singsol=all)
```

$$y(x) = x^2(c_1 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(c_2 x + c_1)$$

9.20 problem 20

Internal problem ID [1126]
Internal file name [`OUTPUT/1127_Sunday_June_05_2022_02_03_11_AM_7309673/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 \ln(x)^2 y'' - 2x \ln(x) y' + (2 + \ln(x)) y = 0}$$

Given that one solution of the ode is

$$y_1 = \ln(x)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2}{x \ln(x)}$$

Therefore

$$y_2(x) = \ln(x)\left(\int \frac{e^{-\left(\int -\frac{2}{x\ln(x)}dx\right)}}{\ln(x)^2}dx\right)$$

$$y_2(x) = \ln(x)\int \frac{\ln(x)^2}{\ln(x)^2}, dx$$

$$y_2(x) = \ln(x)\left(\int 1dx\right)$$

$$y_2(x) = x\ln(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\ln(x) + \ln(x)\,c_2x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\ln(x) + \ln(x)\,c_2x \tag{1}$$

Verification of solutions

$$y = c_1\ln(x) + \ln(x)\,c_2x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([x^2*(ln(x))^2*diff(y(x),x$2)-2*x*ln(x)*diff(y(x),x)+(2+ln(x))*y(x)=0,ln(x)],singsol=
```

$$y(x) = \ln(x)(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 15

```
DSolve[x^2*Log[x]^2*y''[x]-2*x*Log[x]*y'[x]+(2+Log[x])*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to (c_2x + c_1)\log(x)$$

9.21 problem 21

Internal problem ID [1127]
Internal file name [`OUTPUT/1128_Sunday_June_05_2022_02_03_11_AM_90055498/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{4xy''+2y'+y=0}$$

Given that one solution of the ode is

$$y_1=\sin\left(\sqrt{x}\right)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int\frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=\frac{1}{2x}$$

Therefore

$$y_2(x) = \sin\left(\sqrt{x}\right)\left(\int \frac{e^{-\left(\int \frac{1}{2x}dx\right)}}{\sin\left(\sqrt{x}\right)^2}dx\right)$$

$$y_2(x) = \sin\left(\sqrt{x}\right)\int \frac{\frac{1}{\sqrt{x}}}{\sin\left(\sqrt{x}\right)^2}, dx$$

$$y_2(x) = \sin\left(\sqrt{x}\right)\left(\int \frac{\csc\left(\sqrt{x}\right)^2}{\sqrt{x}}dx\right)$$

$$y_2(x) = -2\sin\left(\sqrt{x}\right)\cot\left(\sqrt{x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= \sin\left(\sqrt{x}\right)c_1 - 2c_2\sin\left(\sqrt{x}\right)\cot\left(\sqrt{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin\left(\sqrt{x}\right)c_1 - 2c_2\sin\left(\sqrt{x}\right)\cot\left(\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = \sin\left(\sqrt{x}\right)c_1 - 2c_2\sin\left(\sqrt{x}\right)\cot\left(\sqrt{x}\right)$$

Verified OK.

9.21.1 Maple step by step solution

Let's solve

$4y''x + 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{4x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{4x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4y''x + 2y' + y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

    Rewrite ODE with series expansions

$$2a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+1+2r)+a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2(2k+1+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{2(2k+1)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(2k+1)(k+1)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{a_k}{2(2k+1)(1+k)}, b_{1+k} = -\frac{b_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([4*x*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,sin(sqrt(x))],singsol=all)
```

$$y(x) = c_1 \sin\left(\sqrt{x}\right) + c_2 \cos\left(\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 24

```
DSolve[4*x*y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

9.22 problem 22

Internal problem ID [1128]
Internal file name [`OUTPUT/1129_Sunday_June_05_2022_02_03_12_AM_21078271/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (2+2x)\,y' + (2+x)\,y = 0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2x-2}{x}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int \frac{-2x-2}{x} dx\right)} e^{-2x} dx \right)$$

$$y_2(x) = e^x \int \frac{e^{2x+2\ln(x)}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int x^2 dx \right)$$

$$y_2(x) = \frac{e^x x^3}{3}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x + \frac{c_2 e^x x^3}{3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 e^x x^3}{3} \tag{1}$$

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^x x^3}{3}$$

Verified OK.

9.22.1 Maple step by step solution

Let's solve

$y''x + (-2x-2)\, y' + (2+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2+x)y}{x} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x+1)y'}{x} + \frac{(2+x)y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{2+x}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x + (-2x-2)\, y' + (2+x)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r) x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2+r)(k+r-1) - 2a_{k+1}(k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2k a_{k+1} - a_k}{(k+2)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = \frac{2k a_{k+1} - a_k}{(k+2)(k-1)}$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2k a_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x$2)-(2*x+2)*diff(y(x),x)+(x+2)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = e^x (c_2 x^3 + c_1)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} e^x (c_2 x^3 + 3c_1)$$

9.23 problem 23

Internal problem ID [1129]
Internal file name [`OUTPUT/1130_Sunday_June_05_2022_02_03_13_AM_79199182/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - (2a-1)\,xy' + a^2y = 0}$$

Given that one solution of the ode is

$$y_1 = x^a$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2a+1}{x}$$

Therefore

$$y_2(x) = x^a \left(\int e^{-\left(\int \frac{-2a+1}{x} dx\right)} x^{-2a} dx \right)$$

$$y_2(x) = x^a \int \frac{e^{-(-2a+1)\ln(x)}}{x^{2a}}, dx$$

$$y_2(x) = x^a \left(\int \frac{1}{x} dx \right)$$

$$y_2(x) = x^a \ln(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= x^a c_1 + c_2 x^a \ln(x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^a c_1 + c_2 x^a \ln(x) \tag{1}$$

Verification of solutions

$$y = x^a c_1 + c_2 x^a \ln(x)$$

Verified OK.

9.23.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2ax + x) y' + a^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{a^2 y}{x^2} + \frac{(2a-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2a-1)y'}{x} + \frac{a^2 y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2a-1}{x}, P_3(x) = \frac{a^2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2a + 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = a^2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - (2a - 1)\,xy' + a^2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite DE with series expansions

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k(a - k - r)^2\, x^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(a - k)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_k = 0$

- Recursion relation for $r = 0$

$a_k = 0$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([x^2*diff(y(x),x$2)-(2*a-1)*x*diff(y(x),x)+a^2*y(x)=0,x^a],singsol=all)
```

$$y(x) = (c_2 \ln (x) + c_1) x^a$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-(2*a-1)*x*y'[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^a (a c_2 \log(x) + c_1)$$

9.24 problem 24

Internal problem ID [1130]
Internal file name [`OUTPUT/1131_Sunday_June_05_2022_02_03_14_AM_34255493/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2y'x + (x^2 + 2)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = \sin(x)\, x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2}{x}$$

Therefore

$$y_2(x) = \sin(x)\,x\left(\int \frac{\mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)}}{\sin(x)^2\,x^2}dx\right)$$

$$y_2(x) = \sin(x)\,x\int \frac{x^2}{x^2\sin(x)^2},dx$$

$$y_2(x) = \sin(x)\,x\left(\int \csc(x)^2\,dx\right)$$

$$y_2(x) = -\sin(x)\cot(x)\,x$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= \sin(x)\,xc_1 - c_2\sin(x)\cot(x)\,x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin(x)\,xc_1 - c_2\sin(x)\cot(x)\,x \tag{1}$$

Verification of solutions

$$y = \sin(x)\,xc_1 - c_2\sin(x)\cot(x)\,x$$

Verified OK.

9.24.1 Maple step by step solution

Let's solve

$x^2y'' - 2y'x + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' - 2y'x + (x^2 + 2) y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)(-2+r)x^r + a_1 r(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-1)(k+r-2) + a_{k-2} = 0$

- Shift index using $k- > k+2$

$a_{k+2}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(1+k)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,x*sin(x)],singsol=all)
```

$$y(x) = x(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

9.25 problem 25

Internal problem ID [1131]
Internal file name [`OUTPUT/1132_Sunday_June_05_2022_02_03_15_AM_56049452/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (4x+1)\, y' + (4x+2)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = e^{2x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-4x-1}{x}$$

Therefore

$$y_2(x) = e^{2x}\left(\int e^{-\left(\int \frac{-4x-1}{x}dx\right)} e^{-4x} dx\right)$$

$$y_2(x) = e^{2x}\int \frac{e^{4x+\ln(x)}}{e^{4x}}, dx$$

$$y_2(x) = e^{2x}\left(\int x dx\right)$$

$$y_2(x) = \frac{x^2 e^{2x}}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{2x} + \frac{c_2 x^2 e^{2x}}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + \frac{c_2 x^2 e^{2x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + \frac{c_2 x^2 e^{2x}}{2}$$

Verified OK.

9.25.1 Maple step by step solution

Let's solve

$y''x + (-4x-1)\,y' + (4x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(1+2x)y}{x} + \frac{(4x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4x+1)y'}{x} + \frac{2(1+2x)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{4x+1}{x}, P_3(x) = \frac{2(1+2x)}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (-4x - 1)\, y' + (4x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) x^{-1+r} + (a_1(1+r)(-1+r) - 2a_0(-1+2r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$a_1(1+r)(-1+r) - 2a_0(-1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2(2ka_{k+1} + 2ra_{k+1} - 2a_k + a_{k+1})}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + a_{k+1})}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + a_{k+1})}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + 5a_{k+1})}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + 5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([x*diff(y(x),x$2)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x)=0,exp(2*x)],singsol=all)
```

$$y(x) = e^{2x}(c_2x^2 + c_1)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(4*x+1)*y'[x]+(4*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{2x}(c_2x^2 + 2c_1)$$

9.26 problem 26

Internal problem ID [1132]
Internal file name [`OUTPUT/1133_Sunday_June_05_2022_02_03_16_AM_33275409/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$4x^2 \sin(x) y'' - 4x(x\cos(x) + \sin(x)) y' + (2x\cos(x) + 3\sin(x)) y = 0$$

Given that one solution of the ode is

$$y_1 = \sqrt{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-4x^2\cos(x) - 4\sin(x) x}{4\sin(x) x^2}$$

Therefore

$$y_2(x) = \sqrt{x}\left(\int \frac{e^{-\left(\int \frac{-4x^2\cos(x)-4\sin(x)x}{4\sin(x)x^2}dx\right)}}{x}dx\right)$$

$$y_2(x) = \sqrt{x}\int \frac{e^{\ln(\sin(x))+\ln(x)}}{x}, dx$$

$$y_2(x) = \sqrt{x}\left(\int \sin(x)\,dx\right)$$

$$y_2(x) = -\cos(x)\sqrt{x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sqrt{x}\,c_1 - c_2\cos(x)\sqrt{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x}\,c_1 - c_2\cos(x)\sqrt{x} \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\,c_1 - c_2\cos(x)\sqrt{x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      (2*arccos(t)*t+3*(-t^2+1)^(1/2))*u(t)+(-4*arccos(t)*t^2+4*arccos(t))*diff(u(t),t)+(-4*
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 14

```
dsolve([4*x^2*sin(x)*diff(y(x),x$2)-4*x*(x*cos(x)+sin(x))*diff(y(x),x)+(2*x*cos(x)+3*sin(x))
```

$$y(x) = \sqrt{x}\,(c_1 + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.322 (sec). Leaf size: 21

```
DSolve[4*x^2*Sin[x]*y''[x]-4*x*(x*Cos[x]+Sin[x])*y'[x]+(2*x*Cos[x]+3*Sin[x])*y[x]==0,y[x],x,
```

$$y(x) \to \sqrt{\arccos(\cos(x))}(c_2 \cos(x) + c_1)$$

9.27 problem 27

Internal problem ID [1133]
Internal file name [`OUTPUT/1134_Sunday_June_05_2022_02_03_17_AM_32416935/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 0}$$

Given that one solution of the ode is

$$y_1 = e^{2x}\sqrt{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{1}{x}$$

Therefore

$$y_2(x) = e^{2x}\sqrt{x}\left(\int \frac{e^{-\left(\int -\frac{1}{x}dx\right)} e^{-4x}}{x} dx\right)$$

$$y_2(x) = e^{2x}\sqrt{x}\int \frac{x}{x\, e^{4x}}, dx$$

$$y_2(x) = e^{2x}\sqrt{x}\left(\int e^{-4x} dx\right)$$

$$y_2(x) = -\frac{e^{2x}\sqrt{x}\, e^{-4x}}{4}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= e^{2x}\sqrt{x}\, c_1 - \frac{c_2 e^{2x}\sqrt{x}\, e^{-4x}}{4} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x}\sqrt{x}\, c_1 - \frac{c_2 e^{2x}\sqrt{x}\, e^{-4x}}{4} \tag{1}$$

Verification of solutions

$$y = e^{2x}\sqrt{x}\, c_1 - \frac{c_2 e^{2x}\sqrt{x}\, e^{-4x}}{4}$$

Verified OK.

9.27.1 Maple step by step solution

Let's solve

$4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)(-3+2r)x^r + a_1(1+2r)(-1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)(2k+2r-3) - 1\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k - 16a_{k-2} = 0$
- Shift index using $k- > k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right)a_{k+2} - 16a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,sqrt(x)*exp(2*x)],singsol=al
```

$$y(x) = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\big(c_2 e^{4x} + 4c_1\big)$$

9.28 problem 28

Internal problem ID [1134]
Internal file name [OUTPUT/1135_Sunday_June_05_2022_02_03_18_AM_50290424/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+2x)\,xy''-2(2x^2-1)\,y'-4(x+1)\,y=0}$$

Given that one solution of the ode is

$$y_1=\frac{1}{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int\frac{e^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=\frac{-4x^2+2}{2x^2+x}$$

Therefore

$$
\begin{aligned}
y_2(x) &= \frac{\int e^{-\left(\int \frac{-4x^2+2}{2x^2+x}dx\right)} x^2 dx}{x} \\
y_2(x) &= \frac{1}{x} \int \frac{e^{2x-2\ln(x)+\ln(1+2x)}}{\frac{1}{x^2}}, dx \\
y_2(x) &= \frac{\int (1+2x)\, e^{2x} dx}{x} \\
y_2(x) &= e^{2x}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1}{x} + c_2 e^{2x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + c_2 e^{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + c_2 e^{2x}
$$

Verified OK.

9.28.1 Maple step by step solution

Let's solve

$(2x^2 + x)\, y'' + (-4x^2 + 2)\, y' + (-4x - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4(x+1)y}{(1+2x)x} + \frac{2(2x^2-1)y'}{(1+2x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(2x^2-1)y'}{(1+2x)x} - \frac{4(x+1)y}{(1+2x)x} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(2x^2-1)}{(1+2x)x}, P_3(x) = -\frac{4(x+1)}{(1+2x)x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $(1+2x)\, xy'' + (-4x^2+2)\, y' + (-4x-4)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)(2+r) + 2a_0(1+r)(-2+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k+2+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)(2+r) + 2a_0(1+r)(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)(k+2+r) + 2a_k(k+r+1)(k+r-2) - 4a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)(k+3+r) + 2a_{k+1}(k+2+r)(k+r-1) - 4a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2(k^2 a_{k+1} + 2kra_{k+1} + r^2 a_{k+1} - 2ka_k + ka_{k+1} - 2ra_k + ra_{k+1} - 2a_k - 2a_{k+1})}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2(k^2 a_{k+1} - 2ka_k - ka_{k+1} - 2a_{k+1})}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2(k^2 a_{k+1} - 2ka_k - ka_{k+1} - 2a_{k+1})}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2(k^2 a_{k+1} - 2ka_k + ka_{k+1} - 2a_k - 2a_{k+1})}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2(k^2 a_{k+1} - 2ka_k + ka_{k+1} - 2a_k - 2a_{k+1})}{(k+2)(k+3)}, 2a_1 - 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{2(k^2 a_{1+k} - 2k a_k - k a_{1+k} - 2a_{1+k})}{(1+k)(k+2)}, 0 = 0, b_{k+2} = -\frac{2(k^2 b_{1+k} - 2k$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([(2*x+1)*x*diff(y(x),x$2)-2*(2*x^2-1)*diff(y(x),x)-4*(x+1)*y(x)=0,1/x],singsol=all)
```

$$y(x) = \frac{c_2 e^{2x} x + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 28

```
DSolve[(2*x+1)*x*y''[x]-2*(2*x^2-1)*y'[x]-4*(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_2 e^{2x+1} x + c_1}{\sqrt{e} x}$$

9.29 problem 29

Internal problem ID [1135]
Internal file name [OUTPUT/1136_Sunday_June_05_2022_02_03_18_AM_35042066/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0}$$

Given that one solution of the ode is

$$y_1=\mathrm{e}^x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int\frac{\mathrm{e}^{-(\int pdx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=\frac{-x^2+2}{x^2-2x}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int \frac{-x^2+2}{x^2-2x}dx\right)} e^{-2x} dx\right)$$

$$y_2(x) = e^x \int \frac{e^{x+\ln(x)+\ln(-2+x)}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int x(-2+x)\, e^{-x} dx\right)$$

$$y_2(x) = -e^x x^2 e^{-x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x - c_2 e^x x^2 e^{-x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x - c_2 e^x x^2 e^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 e^x x^2 e^{-x}$$

Verified OK.

9.29.1 Maple step by step solution

Let's solve

$(x^2-2x)\, y'' + (-x^2+2)\, y' + (2x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y'}{x(-2+x)} - \frac{2y(x-1)}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-2)y'}{x(-2+x)} + \frac{2y(x-1)}{x(-2+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = -\frac{x^2-2}{x(-2+x)}, P_3(x) = \frac{2(x-1)}{x(-2+x)}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x(-2+x) + (-x^2+2)\, y' + (2x-2)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - ◦ Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\, x^{-1+r} + (-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r))\, x^r + \left(\sum_{k=1}^{\infty} (-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) - 2k^2 a_{k+1} + (-4ra_{k+1} - a_{k-1})\, k - 2r^2 a_{k+1} - a_{k-1} r + 3a_{k-1} + 2a_{k+1} =$

- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)(k+r-1) - 2(k+1)^2 a_{k+2} + (-4ra_{k+2} - a_k)(k+1) - 2r^2 a_{k+2} - ra_k + 3a_k +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{k^2 a_{k+1} + 2kra_{k+1} + r^2 a_{k+1} - ka_k + ka_{k+1} - ra_k + ra_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{k^2 a_{k+1} - ka_k + ka_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2k)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{k^2 a_{k+1} - ka_k + ka_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2k)}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{k^2 a_{k+1} - ka_k + 5ka_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_{k+1} - ka_k + 5ka_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}, -6a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([(x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(2*x-2)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = c_1x^2 + c_2\mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 18

```
DSolve[(x^2-2*x)*y''[x]+(2-x^2)*y'[x]+(2*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_2x^2 + c_1e^x$$

9.30 problem 30

Internal problem ID [1136]
Internal file name [`OUTPUT/1137_Sunday_June_05_2022_02_03_19_AM_79319869/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (4x+1)\,y' + (4x+2)\,y = 0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{2x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-4x-1}{x}$$

Therefore

$$y_2(x) = \mathrm{e}^{2x} \left(\int \mathrm{e}^{-\left(\int \frac{-4x-1}{x} dx\right)} \mathrm{e}^{-4x} dx \right)$$

$$y_2(x) = \mathrm{e}^{2x} \int \frac{\mathrm{e}^{4x+\ln(x)}}{\mathrm{e}^{4x}}, dx$$

$$y_2(x) = \mathrm{e}^{2x} \left(\int x dx \right)$$

$$y_2(x) = \frac{x^2 \mathrm{e}^{2x}}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2}$$

Verified OK.

9.30.1 Maple step by step solution

Let's solve

$y''x + (-4x - 1) y' + (4x + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(1+2x)y}{x} + \frac{(4x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4x+1)y'}{x} + \frac{2(1+2x)y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $$\left[P_2(x) = -\frac{4x+1}{x}, P_3(x) = \frac{2(1+2x)}{x}\right]$$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $$x_0 = 0$$
- • Multiply by denominators

  $$y''x + (-4x - 1)\, y' + (4x + 2)\, y = 0$$
- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - ◦ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$
  - ◦ Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$
  - ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) x^{-1+r} + (a_1(1+r)(-1+r) - 2a_0(-1+2r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$a_1(1+r)(-1+r) - 2a_0(-1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2+r)(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2(2ka_{k+1} + 2ra_{k+1} - 2a_k + a_{k+1})}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + a_{k+1})}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + a_{k+1})}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + 5a_{k+1})}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1} - 2a_k + 5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve([x*diff(y(x),x$2)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x)=0,exp(2*x)],singsol=all)
```

$$y(x) = e^{2x}(c_2x^2 + c_1)$$

✓ Solution by Mathematica
Time used: 0.025 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(4*x+1)*y'[x]+(4*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{2x}(c_2x^2 + 2c_1)$$

9.31 problem 31

Internal problem ID [1137]
Internal file name [`OUTPUT/1138_Sunday_June_05_2022_02_03_20_AM_21604343/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + 4y = 4x^4}$$

Given that one solution of the ode is

$$y_1 = x^2$$

With initial conditions

$$[y(-1) = 7, y'(-1) = -8]$$

9.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \\ F &= 4x^2 \end{aligned}$$

Hence the ode is

$$y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 4x^2$$

The domain of $p(x) = -\frac{3}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{4}{x^2}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = -1$ is also inside this domain. The domain of $F = 4x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 4x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{3}{x}$$

Therefore

$$
\begin{aligned}
y_2(x) &= x^2 \left(\int \frac{e^{-\left(\int -\frac{3}{x} dx\right)}}{x^4} dx \right) \\
y_2(x) &= x^2 \int \frac{x^3}{x^4}, dx \\
y_2(x) &= x^2 \left(\int \frac{1}{x} dx \right) \\
y_2(x) &= \ln(x)\, x^2
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 + c_2 \ln(x)\, x^2
\end{aligned}
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1 x^2 + c_2 \ln(x)\, x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\ln(x)\, x^6}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int 4x\ln(x)\, dx$$

Hence

$$u_1 = -2\ln(x)\, x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^6}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Which simplifies to

$$\begin{aligned} u_1 &= x^2(1 - 2\ln(x)) \\ u_2 &= 2x^2 \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^4(1 - 2\ln(x)) + 2x^4 \ln(x)$$

Which simplifies to

$$y_p(x) = x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 \ln(x) x^2\right) + \left(x^4\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + x^4$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^2(c_1 + c_2 \ln(x)) + x^4 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = -1$ in the above gives

$$7 = \pi c_2 i + c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2x(c_1 + c_2 \ln(x)) + c_2 x + 4x^3$$

substituting $y' = -8$ and $x = -1$ in the above gives

$$-8 = -2\pi c_2 i - 2c_1 - c_2 - 4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 8i\pi + 6$$
$$c_2 = -8$$

Substituting these values back in above solution results in

$$y = 8ix^2\pi + x^4 - 8\ln(x)\,x^2 + 6x^2$$

Summary

The solution(s) found are the following

$$y = \left(-8\ln(x) + 8i\pi + x^2 + 6\right)x^2 \tag{1}$$

Verification of solutions

$$y = \left(-8\ln(x) + 8i\pi + x^2 + 6\right)x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 21

```
dsolve([x^2*diff(diff(y(x),x),x)-3*x*diff(y(x),x)+4*y(x) = 4*x^4, x^2, y(-1) = 7, D(y)(-1) =
```

$$y(x) = \left(8i\pi + x^2 - 8\ln(x) + 6\right)x^2$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==4*x^2,{y[-1]==7,y'[-1]==8},y[x],x,IncludeSingularSolutio
```

$$y(x) \to x^2\left(2\log^2(x) + (-22 - 4i\pi)\log(x) - 2\pi^2 + 22i\pi + 7\right)$$

9.32 problem 32

Internal problem ID [1138]
Internal file name [`OUTPUT/1139_Sunday_June_05_2022_02_03_21_AM_93815312/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{2x}$$

With initial conditions

$$[y(0)=2, y'(0)=3]$$

9.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{-3x-2}{3x-1} \\ q(x) &= \frac{-6x+8}{3x-1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-3x-2)\, y'}{3x-1} + \frac{(-6x+8)\, y}{3x-1} = 0$$

The domain of $p(x) = \frac{-3x-2}{3x-1}$ is

$$\left\{x < \frac{1}{3} \vee \frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{-6x+8}{3x-1}$ is

$$\left\{x < \frac{1}{3} \vee \frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-3x-2}{3x-1}$$

Therefore

$$y_2(x) = \mathrm{e}^{2x} \left(\int \mathrm{e}^{-\left(\int \frac{-3x-2}{3x-1} dx\right)} \mathrm{e}^{-4x} dx \right)$$

$$y_2(x) = \mathrm{e}^{2x} \int \frac{\mathrm{e}^{x+\ln(3x-1)}}{\mathrm{e}^{4x}}, dx$$

$$y_2(x) = \mathrm{e}^{2x} \left(\int (3x-1)\, \mathrm{e}^{-3x} dx \right)$$

$$y_2(x) = -\mathrm{e}^{2x} x\, \mathrm{e}^{-3x}$$

Hence the solution is

$$
\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \mathrm{e}^{2x} - c_2 \mathrm{e}^{2x} x \,\mathrm{e}^{-3x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} - c_2 \mathrm{e}^{2x} x \,\mathrm{e}^{-3x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x \,\mathrm{e}^{-3x} - c_2 \mathrm{e}^{2x} \mathrm{e}^{-3x}
$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$
3 = 2c_1 - c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\mathrm{e}^{2x} x \,\mathrm{e}^{-3x} + 2\,\mathrm{e}^{2x}
$$

Which simplifies to

$$
y = -x \,\mathrm{e}^{-x} + 2\,\mathrm{e}^{2x}
$$

Summary

The solution(s) found are the following

$$
y = -x \,\mathrm{e}^{-x} + 2\,\mathrm{e}^{2x} \tag{1}
$$

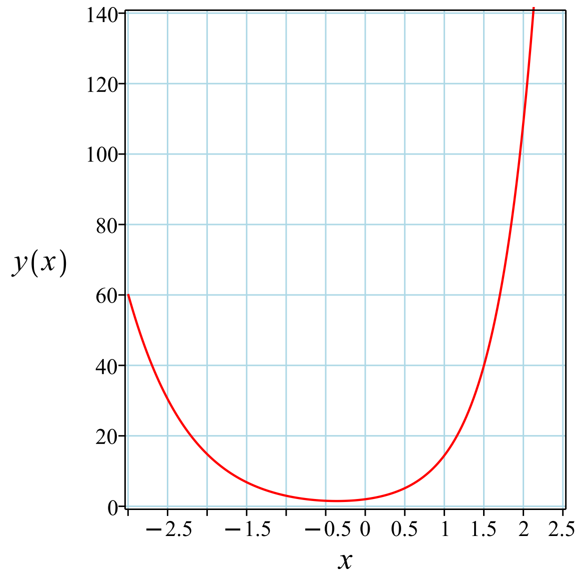


Figure 489: Solution plot

Verification of solutions

$$y = -x\,e^{-x} + 2\,e^{2x}$$

Verified OK.

9.32.2 Maple step by step solution

Let's solve

$$\left[(3x-1)\,y'' + (-3x-2)\,y' + (-6x+8)\,y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(3x-4)y}{3x-1} + \frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3x+2)y'}{3x-1} - \frac{2(3x-4)y}{3x-1} = 0$

□ Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{3x+2}{3x-1}, P_3(x) = -\frac{2(3x-4)}{3x-1}\right]$

- $\left(x - \frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x - \frac{1}{3}\right) \cdot P_2(x)\right) \bigg|_{x=\frac{1}{3}} = -1$

- $\left(x - \frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x - \frac{1}{3}\right)^2 \cdot P_3(x)\right) \bigg|_{x=\frac{1}{3}} = 0$

- $x = \frac{1}{3}$is a regular singular point

  Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

  $x_0 = \frac{1}{3}$

- Multiply by denominators

  $(3x-1)\, y'' + (-3x-2)\, y' + (-6x+8)\, y = 0$

- Change variables using $x = u + \frac{1}{3}$ so that the regular singular point is at $u = 0$

  $3u\left(\frac{d^2}{du^2}y(u)\right) + (-3u-3)\left(\frac{d}{du}y(u)\right) + (-6u+6)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

◦ Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(-2+r)\, u^{-1+r} + (3a_1(1+r)(-1+r) - 3a_0(-2+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $3a_1(1+r)(-1+r) - 3a_0(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$
- Shift index using $k- > k+1$

  $3a_{k+2}(k+2+r)(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{k a_{k+1} + r a_{k+1} + 2a_k - a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$
- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x-\frac{1}{3}\right)^{k+2}, a_{k+2}=\frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}, 9 a_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.032 (sec). Leaf size: 18

```
dsolve([(3*x-1)*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x) = 0, exp(2*x), y(0) =
```

$$y(x)=2\,\mathrm{e}^{2x}-x\,\mathrm{e}^{-x}$$

✓ Solution by Mathematica
Time used: 0.199 (sec). Leaf size: 21

```
DSolve[(3*x-1)*y''[x]-(3*x+2)*y'[x]-(6*x-8)*y[x]==0,{y[0]==2,y'[0]==3},y[x],x,IncludeSingula
```

$$y(x)\to 2e^{2x}-e^{-x}x$$

9.33 problem 33

Internal problem ID [1139]
Internal file name [`OUTPUT/1140_Sunday_June_05_2022_02_03_23_AM_90044175/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)^2 y'' - 2(x+1) y' - (x^2+2x-1) y = (x+1)^3 e^x}$$

Given that one solution of the ode is

$$y_1 = (x+1) e^x$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

9.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{-2x-2}{(x+1)^2} \\ q(x) &= \frac{-x^2-2x+1}{(x+1)^2} \\ F &= (x+1) e^x \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-2x-2)\,y'}{(x+1)^2} + \frac{(-x^2-2x+1)\,y}{(x+1)^2} = (x+1)\,e^x$$

The domain of $p(x) = \frac{-2x-2}{(x+1)^2}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{-x^2-2x+1}{(x+1)^2}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = (x+1)\,e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 + 2x + 1, B = -2x - 2, C = -x^2 - 2x + 1, f(x) = (x+1)^3\,e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 2x + 1)\,y'' + (-2x-2)\,y' + (-x^2 - 2x + 1)\,y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2x - 2}{x^2 + 2x + 1}$$

Therefore

$$y_2(x) = (x+1)\,\mathrm{e}^{x} \left(\int \frac{\mathrm{e}^{-\left(\int \frac{-2x-2}{x^2+2x+1}dx\right)}\mathrm{e}^{-2x}}{(x+1)^2}dx\right)$$

$$y_2(x) = (x+1)\,\mathrm{e}^{x} \int \frac{(x+1)^2}{(x+1)^2\,\mathrm{e}^{2x}}, dx$$

$$y_2(x) = (x+1)\,\mathrm{e}^{x} \left(\int \mathrm{e}^{-2x}dx\right)$$

$$y_2(x) = -\frac{(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= (x+1)\,\mathrm{e}^{x} c_1 - \frac{c_2 (x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = (x+1)\,\mathrm{e}^{x} c_1 - \frac{c_2 (x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x+1)\,\mathrm{e}^{x}$$

$$y_2 = -\frac{(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x+1)\,\mathrm{e}^{x} & -\frac{(x+1)\mathrm{e}^{x}\mathrm{e}^{-2x}}{2} \\ \frac{d}{dx}\left((x+1)\,\mathrm{e}^{x}\right) & \frac{d}{dx}\left(-\frac{(x+1)\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x+1)\,\mathrm{e}^{x} & -\frac{(x+1)\mathrm{e}^{x}\mathrm{e}^{-2x}}{2} \\ \mathrm{e}^{x} + (x+1)\,\mathrm{e}^{x} & -\frac{\mathrm{e}^{-2x}\mathrm{e}^{x}}{2} + \frac{(x+1)\mathrm{e}^{x}\mathrm{e}^{-2x}}{2} \end{vmatrix}$$

Therefore

$$W = \left((x+1)\,\mathrm{e}^{x}\right)\left(-\frac{\mathrm{e}^{-2x}\mathrm{e}^{x}}{2} + \frac{(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}\right) - \left(-\frac{(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}\right)\left(\mathrm{e}^{x} + (x+1)\,\mathrm{e}^{x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x}\mathrm{e}^{2x}x^{2} + 2\,\mathrm{e}^{-2x}x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = (x+1)^{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{(x+1)^{4}\mathrm{e}^{2x}\mathrm{e}^{-2x}}{2}}{(x^{2}+2x+1)\,(x+1)^{2}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{2}dx$$

Hence

$$u_1 = \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x+1)^4 \mathrm{e}^{2x}}{(x^2+2x+1)(x+1)^2}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{2x} dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x(x+1)\,\mathrm{e}^{x}}{2} - \frac{\mathrm{e}^{2x}(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x}(2x^2+x-1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((x+1)\,\mathrm{e}^{x} c_1 - \frac{c_2(x+1)\,\mathrm{e}^{x}\mathrm{e}^{-2x}}{2}\right) + \left(\frac{\mathrm{e}^{x}(2x^2+x-1)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{(x+1)\,(2c_1\mathrm{e}^{x} - c_2\mathrm{e}^{-x})}{2} + \frac{\mathrm{e}^{x}(2x^2+x-1)}{4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(x+1)\,(2c_1\mathrm{e}^{x} - c_2\mathrm{e}^{-x})}{2} + \frac{\mathrm{e}^{x}(2x^2+x-1)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = -\frac{1}{4} + c_1 - \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^x - \frac{c_2 e^{-x}}{2} + \frac{(x+1)(2c_1 e^x + c_2 e^{-x})}{2} + \frac{e^x(2x^2 + x - 1)}{4} + \frac{e^x(4x+1)}{4}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_2 = -\frac{7}{2}$$

Substituting these values back in above solution results in

$$y = \frac{7x\, e^{-x}}{4} - \frac{x\, e^x}{4} + \frac{7\, e^{-x}}{4} - \frac{3\, e^x}{4} + \frac{x^2 e^x}{2}$$

Which simplifies to

$$y = \frac{(2x\, e^x - 3\, e^x + 7\, e^{-x})(x+1)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x\, e^x - 3\, e^x + 7\, e^{-x})(x+1)}{4} \tag{1}$$

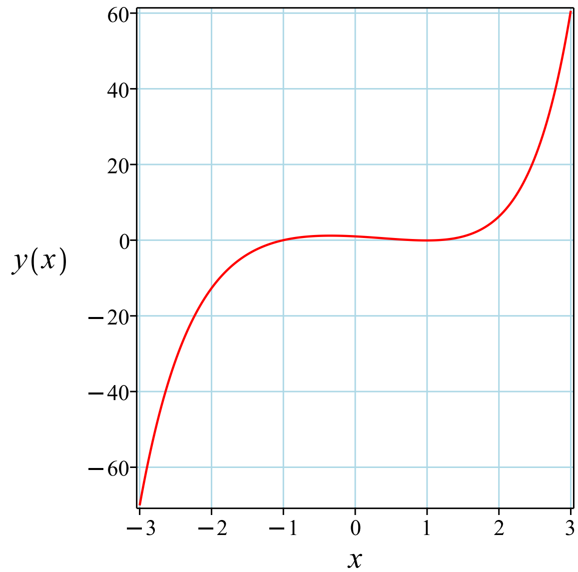


Figure 490: Solution plot

Verification of solutions

$$y = \frac{(2x\,e^{x} - 3\,e^{x} + 7\,e^{-x})\,(x+1)}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve([(1+x)^2*diff(diff(y(x),x),x)-2*(1+x)*diff(y(x),x)-(x^2+2*x-1)*y(x) = (1+x)^3*exp(x),
```

$$y(x) = \frac{(x+1)\left(x\,\mathrm{e}^{x}-5\sinh\left(x\right)+2\cosh\left(x\right)\right)}{2}$$

✓ Solution by Mathematica

Time used: 21.262 (sec). Leaf size: 5749

```
DSolve[(x+1)^2*y''[x]-2*(x+1)*x*y'[x]-(x^2+2*x-1)*y[x]==(x+1)^3*Exp[x],{y[0]==1,y'[0]==-1},y
```

Too large to display

9.34 problem 34

Internal problem ID [1140]
Internal file name [`OUTPUT/1141_Sunday_June_05_2022_02_03_24_AM_67262954/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + 2y'x - 2y = x^2}$$

Given that one solution of the ode is

$$y_1 = x$$

With initial conditions

$$\left[y(1) = \frac{5}{4}, y'(1) = \frac{3}{2}\right]$$

9.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{2}{x^2}$$
$$F = 1$$

Hence the ode is

$$y'' + \frac{2y'}{x} - \frac{2y}{x^2} = 1$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x, C = -2, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 2y'x - 2y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2}{x}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int \frac{2}{x}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{\frac{1}{x^2}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{1}{x^4}dx\right)$$

$$y_2(x) = -\frac{1}{3x^2}$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x - \frac{c_2}{3x^2} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1x - \frac{c_2}{3x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = -\frac{1}{3x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -\frac{1}{3x^2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(-\frac{1}{3x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -\frac{1}{3x^2} \\ 1 & \frac{2}{3x^3} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{2}{3x^3}\right) - \left(-\frac{1}{3x^2}\right)(1)$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{1}{3}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{3}dx$$

Hence

$$u_1 = \frac{x}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int x^3 dx$$

Hence

$$u_2 = \frac{x^4}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x - \frac{c_2}{3x^2}\right) + \left(\frac{x^2}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x - \frac{c_2}{3x^2} + \frac{x^2}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{5}{4}$ and $x = 1$ in the above gives

$$\frac{5}{4} = c_1 - \frac{c_2}{3} + \frac{1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + \frac{2c_2}{3x^3} + \frac{x}{2}$$

substituting $y' = \frac{3}{2}$ and $x = 1$ in the above gives

$$\frac{3}{2} = c_1 + \frac{2c_2}{3} + \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x(x+4)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{x(x+4)}{4} \tag{1}$$

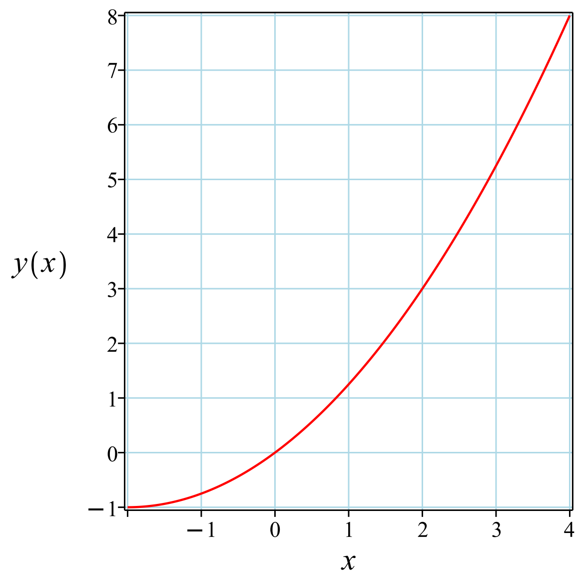


Figure 491: Solution plot

Verification of solutions

$$y = \frac{x(x+4)}{4}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 11

```
dsolve([x^2*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = x^2, x, y(1) = 5/4, D(y)(1) = 3/2
```

$$y(x) = x + \frac{1}{4}x^2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 13

```
DSolve[x^2*y''[x]+2*x*y'[x]-2*y[x]==x^2,{y[1]==5/4,y'[1]==3/2},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{1}{4}x(x+4)$$

9.35 problem 35

Internal problem ID [1141]
Internal file name [`OUTPUT/1142_Sunday_June_05_2022_02_03_26_AM_72372577/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x^2-4)\,y''+4y'x+2y=2+x}$$

Given that one solution of the ode is

$$y_1=\frac{1}{-2+x}$$

With initial conditions

$$\left[y(0)=-\frac{1}{3},y'(0)=-1\right]$$

9.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$
\begin{aligned}
p(x) &= \frac{4x}{x^2-4} \\
q(x) &= \frac{2}{x^2-4} \\
F &= \frac{2+x}{x^2-4}
\end{aligned}
$$

Hence the ode is

$$
y'' + \frac{4xy'}{x^2-4} + \frac{2y}{x^2-4} = \frac{2+x}{x^2-4}
$$

The domain of $p(x) = \frac{4x}{x^2-4}$ is

$$
\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{x^2-4}$ is

$$
\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}
$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{2+x}{x^2-4}$ is

$$
\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

This is second order nonhomogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2 - 4, B = 4x, C = 2, f(x) = 2 + x$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x^2 - 4)\, y'' + 4y'x + 2y = 0
$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$
y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx \right)
$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{4x}{x^2 - 4}$$

Therefore

$$\begin{aligned}
y_2(x) &= \frac{\int \mathrm{e}^{-\left(\int \frac{4x}{x^2-4}dx\right)} (-2+x)^2\, dx}{-2+x} \\
y_2(x) &= \frac{1}{-2+x} \int \frac{\frac{1}{(x^2-4)^2}}{\frac{1}{(-2+x)^2}},\, dx \\
y_2(x) &= \frac{\int \frac{1}{(2+x)^2} dx}{-2+x} \\
y_2(x) &= -\frac{1}{(-2+x)\,(2+x)}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1}{-2+x} - \frac{c_2}{(-2+x)\,(2+x)}
\end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1}{-2+x} - \frac{c_2}{(-2+x)\,(2+x)}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{-2+x}$$

$$y_2 = -\frac{1}{(-2+x)(2+x)}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{-2+x} & -\frac{1}{(-2+x)(2+x)} \\ \frac{d}{dx}\left(\frac{1}{-2+x}\right) & \frac{d}{dx}\left(-\frac{1}{(-2+x)(2+x)}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{-2+x} & -\frac{1}{(-2+x)(2+x)} \\ -\frac{1}{(-2+x)^2} & \frac{1}{(-2+x)^2(2+x)} + \frac{1}{(-2+x)(2+x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{-2+x}\right)\left(\frac{1}{(-2+x)^2(2+x)} + \frac{1}{(-2+x)(2+x)^2}\right) - \left(-\frac{1}{(-2+x)(2+x)}\right)\left(-\frac{1}{(-2+x)^2}\right)$$

Which simplifies to

$$W = \frac{1}{(-2+x)^2(2+x)^2}$$

Which simplifies to

$$W = \frac{1}{(-2+x)^2(2+x)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{1}{-2+x}}{\frac{x^2-4}{(-2+x)^2(2+x)^2}} \, dx$$

Which simplifies to

$$u_1 = -\int (-x-2) \, dx$$

Hence

$$u_1 = \frac{1}{2}x^2 + 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2+x}{-2+x}}{\frac{x^2-4}{(-2+x)^2(2+x)^2}} \, dx$$

Which simplifies to

$$u_2 = \int (2+x)^2 \, dx$$

Hence

$$u_2 = \frac{(2+x)^3}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{2}x^2 + 2x}{-2+x} - \frac{(2+x)^2}{3(-2+x)}$$

Which simplifies to

$$y_p(x) = \frac{x^2 + 4x - 8}{-12 + 6x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{-2+x} - \frac{c_2}{(-2+x)(2+x)}\right) + \left(\frac{x^2 + 4x - 8}{-12 + 6x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1(2+x)-c_2}{x^2-4}+\frac{x^2+4x-8}{-12+6x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1(2+x)-c_2}{x^2-4}+\frac{x^2+4x-8}{-12+6x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=-\frac{1}{3}$ and $x=0$ in the above gives

$$-\frac{1}{3} = \frac{c_2}{4}+\frac{2}{3}-\frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^2-4}-\frac{2(c_1(2+x)-c_2)\,x}{(x^2-4)^2}+\frac{2x+4}{-12+6x}-\frac{6(x^2+4x-8)}{(-12+6x)^2}$$

substituting $y'=-1$ and $x=0$ in the above gives

$$-1 = -\frac{c_1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 4$$

Substituting these values back in above solution results in

$$y = \frac{x^3+6x^2+24x+8}{6\,(-2+x)\,(2+x)}$$

Which simplifies to

$$y = \frac{x^3+6x^2+24x+8}{6x^2-24}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3+6x^2+24x+8}{6x^2-24} \tag{1}$$

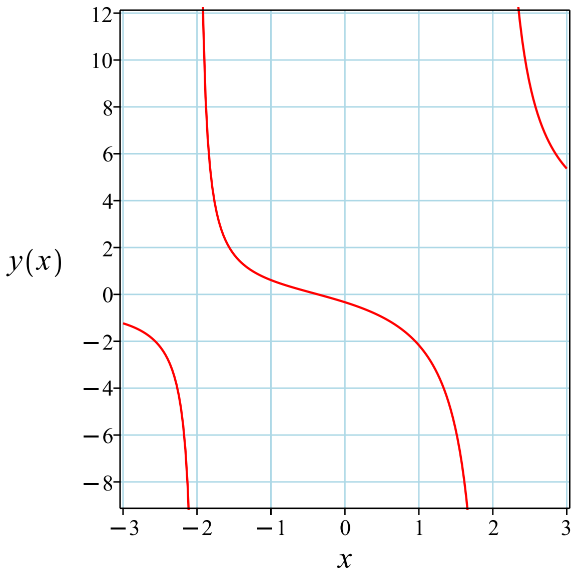


Figure 492: Solution plot

Verification of solutions

$$y = \frac{x^3 + 6x^2 + 24x + 8}{6x^2 - 24}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([(x^2-4)*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+2*y(x) = 2+x, 1/(-2+x), y(0) = -1/3, D
```

$$y(x) = \frac{x^3 + 6x^2 + 24x + 8}{6x^2 - 24}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 30

```
DSolve[(x^2-4)*y''[x]+4*x*y'[x]+2*y[x]==x+2,{y[1]==5/4,y'[1]==3/2},y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{-2x^3 - 12x^2 + 54x + 5}{48 - 12x^2}$$

9.36 problem 38 part (a)

Internal problem ID [1142]
Internal file name [`OUTPUT/1143_Sunday_June_05_2022_02_03_27_AM_95477654/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = -k^2}$$

9.36.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-k^2 - y^2} dy = x + c_1$$
$$-\frac{\arctan\left(\frac{y}{k}\right)}{k} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\tan(c_1 k + kx) k$$

Summary
The solution(s) found are the following

$$y = -\tan(c_1 k + kx) k \tag{1}$$

Verification of solutions

$$y = -\tan(c_1 k + kx) k$$

Verified OK.

9.36.2 Maple step by step solution

Let's solve

$y' + y^2 = -k^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-k^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-k^2}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{\arctan\left(\frac{y}{k}\right)}{k} = x + c_1$

- Solve for y

  $y = -\tan\left(c_1k + kx\right)k$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+y(x)^2+k^2=0,y(x), singsol=all)
```

$$y(x) = -\tan(k(c_1 + x)) k$$

✓ Solution by Mathematica

Time used: 4.112 (sec). Leaf size: 35

```
DSolve[y'[x]+y[x]^2+k^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -k \tan(k(x - c_1))$$
$$y(x) \to -ik$$
$$y(x) \to ik$$

9.37 problem 38 part (b)

Internal problem ID [1143]
Internal file name [`OUTPUT/1144_Sunday_June_05_2022_02_03_28_AM_82082756/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 - 3y = -2}$$

9.37.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + 3y - 2} dy = \int dx$$
$$\ln(y-1) - \ln(y-2) = x + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y-2)} = e^{x+c_1}$$

Which simplifies to

$$\frac{y-1}{y-2} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{2c_2 e^x - 1}{-1 + c_2 e^x} \tag{1}$$

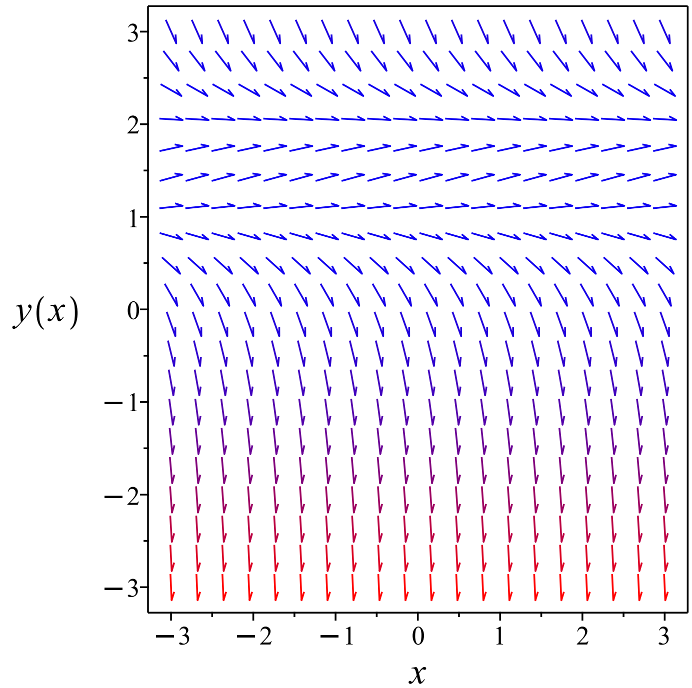


Figure 493: Slope field plot

Verification of solutions

$$y = \frac{2c_2 e^x - 1}{-1 + c_2 e^x}$$

Verified OK.

9.37.2 Maple step by step solution

Let's solve

$y' + y^2 - 3y = -2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2+3y-2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2+3y-2}dx = \int 1dx + c_1$

- Evaluate integral

$$\ln(y-1)-\ln(y-2)=x+c_1$$

- Solve for y

$$y=\frac{2\,\mathrm{e}^{x+c_1}-1}{\mathrm{e}^{x+c_1}-1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)+y(x)^2-3*y(x)+2=0,y(x), singsol=all)
```

$$y(x)=\frac{2\,\mathrm{e}^{x}c_1-1}{\mathrm{e}^{x}c_1-1}$$

✓ Solution by Mathematica

Time used: 0.93 (sec). Leaf size: 40

```
DSolve[y'[x]+y[x]^2-3*y[x]+2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{2e^x-e^{c_1}}{e^x-e^{c_1}}$$
$$y(x)\to 1$$
$$y(x)\to 2$$

9.38 problem 38 part (c)

Internal problem ID [1144]
Internal file name [`OUTPUT/1145_Sunday_June_05_2022_02_03_29_AM_30770682/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + y^2 + 5y = 6}$$

9.38.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 - 5y + 6} dy = \int dx$$
$$\frac{\ln (y+6)}{7} - \frac{\ln (y-1)}{7} = x + c_1$$

The above can be written as

$$\left(\frac{1}{7}\right) (\ln (y+6) - \ln (y-1)) = x + c_1$$
$$\ln (y+6) - \ln (y-1) = (7) (x + c_1)$$
$$= 7x + 7c_1$$

Raising both side to exponential gives

$$e^{\ln(y+6)-\ln(y-1)} = 7c_1 e^{7x}$$

Which simplifies to

$$\frac{y+6}{y-1} = c_2 e^{7x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 e^{7x} + 6}{-1 + c_2 e^{7x}} \tag{1}$$

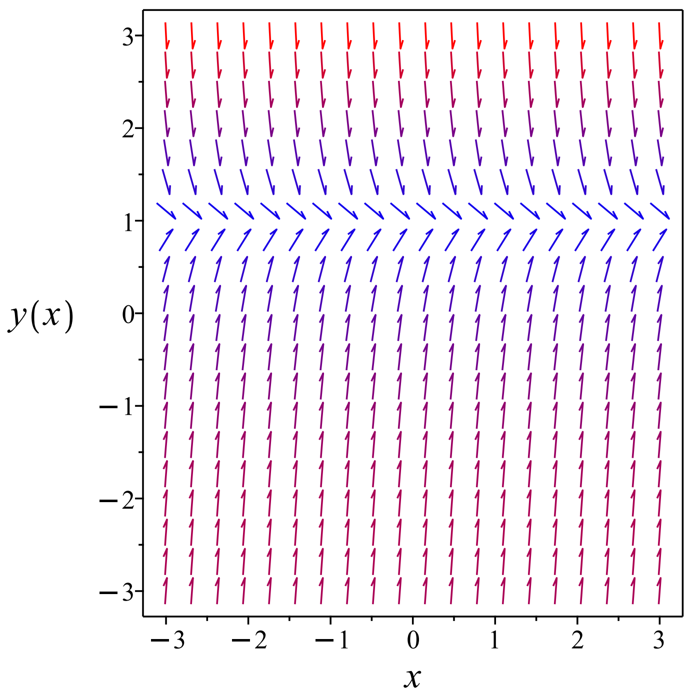


Figure 494: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^{7x} + 6}{-1 + c_2 e^{7x}}$$

Verified OK.

9.38.2 Maple step by step solution

Let's solve

$y' + y^2 + 5y = 6$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-5y+6} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-5y+6}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(y+6)}{7} - \frac{\ln(y-1)}{7} = x + c_1$

- Solve for y

  $y = \frac{6+\mathrm{e}^{7x+7c_1}}{\mathrm{e}^{7x+7c_1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+y(x)^2+5*y(x)-6=0,y(x), singsol=all)
```

$$y(x) = \frac{6 + c_1 e^{7x}}{c_1 e^{7x} - 1}$$

✓ Solution by Mathematica

Time used: 0.71 (sec). Leaf size: 46

```
DSolve[y'[x]+y[x]^2+5*y[x]-6==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{7x} + 6e^{7c_1}}{e^{7x} - e^{7c_1}}$$
$$y(x) \to -6$$
$$y(x) \to 1$$

9.39 problem 38 part (d)

Internal problem ID [1145]
Internal file name [`OUTPUT/1146_Sunday_June_05_2022_02_03_31_AM_37228273/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 + 8y = -7}$$

9.39.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 - 8y - 7} dy = \int dx$$
$$-\frac{\ln(y+1)}{6} + \frac{\ln(7+y)}{6} = x + c_1$$

The above can be written as

$$\left(-\frac{1}{6}\right)(\ln(y+1) - \ln(7+y)) = x + c_1$$
$$\ln(y+1) - \ln(7+y) = (-6)(x + c_1)$$
$$= -6x - 6c_1$$

Raising both side to exponential gives

$$e^{\ln(y+1)-\ln(7+y)} = -6c_1 e^{-6x}$$

Which simplifies to

$$\frac{y+1}{7+y} = c_2 e^{-6x}$$

Summary

The solution(s) found are the following

$$y = -\frac{7c_2 e^{-6x} - 1}{-1 + c_2 e^{-6x}} \quad (1)$$

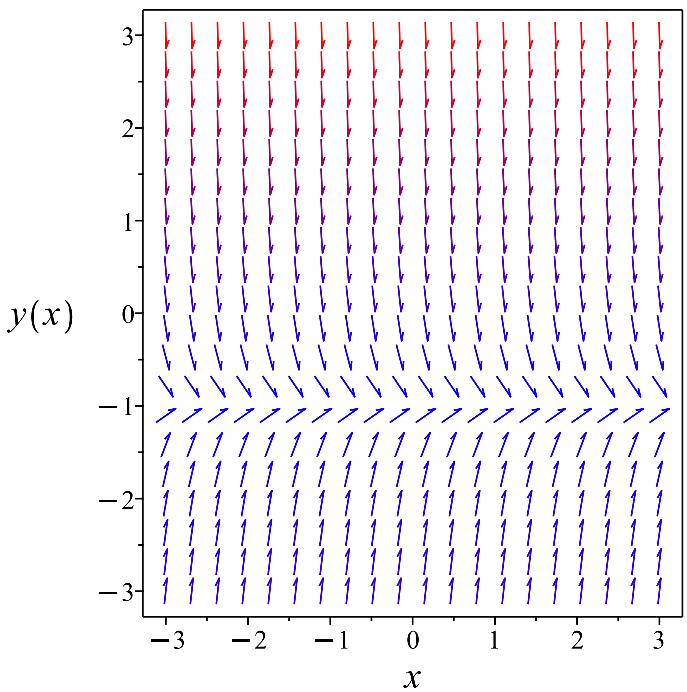


Figure 495: Slope field plot

Verification of solutions

$$y = -\frac{7c_2 e^{-6x} - 1}{-1 + c_2 e^{-6x}}$$

Verified OK.

9.39.2 Maple step by step solution

Let's solve

$y' + y^2 + 8y = -7$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-8y-7} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-8y-7}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{\ln(1+y)}{6} + \frac{\ln(7+y)}{6} = x + c_1$

- Solve for y

  $y = -\frac{-7+e^{6x+6c_1}}{e^{6x+6c_1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)+y(x)^2+8*y(x)+7=0,y(x), singsol=all)
```

$$y(x) = \frac{7 - c_1 e^{6x}}{c_1 e^{6x} - 1}$$

✓ Solution by Mathematica

Time used: 0.655 (sec). Leaf size: 47

```
DSolve[y'[x]+y[x]^2+8*y[x]+7==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{6x} - 7e^{6c_1}}{e^{6x} - e^{6c_1}}$$
$$y(x) \to -7$$
$$y(x) \to -1$$

9.40 problem 38 part (e)

Internal problem ID [1146]
Internal file name [`OUTPUT/1147_Sunday_June_05_2022_02_03_32_AM_85019389/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 + 14y = -50}$$

9.40.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 - 14y - 50} dy = x + c_1$$
$$-\arctan(7 + y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = -7 - \tan(x + c_1)$$

Summary
The solution(s) found are the following

$$y = -7 - \tan(x + c_1) \tag{1}$$

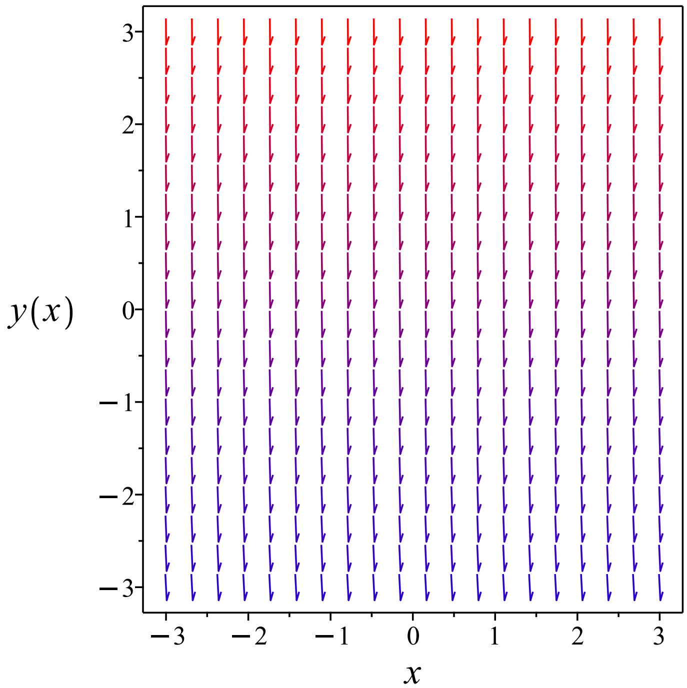


Figure 496: Slope field plot

Verification of solutions

$$y = -7 - \tan(x + c_1)$$

Verified OK.

9.40.2 Maple step by step solution

Let's solve

$y' + y^2 + 14y = -50$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-14y-50} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-14y-50}dx = \int 1dx + c_1$

- Evaluate integral

$$-\arctan(7+y) = x + c_1$$

- Solve for y

$$y = -7 - \tan(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)+y(x)^2+14*y(x)+50=0,y(x), singsol=all)
```

$$y(x) = -7 - \tan(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.553 (sec). Leaf size: 30

```
DSolve[y'[x]+y[x]^2+14*y[x]+50==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -7 - \tan(x - c_1)$$
$$y(x) \to -7 - i$$
$$y(x) \to -7 + i$$

9.41 problem 38 part (f)

Internal problem ID [1147]
Internal file name [`OUTPUT/1148_Sunday_June_05_2022_02_03_33_AM_27793055/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{6y' + 6y^2 - y = 1}$$

9.41.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + \frac{1}{6}y + \frac{1}{6}} dy = \int dx$$
$$-\frac{6\ln(-1+2y)}{5} + \frac{6\ln(3y+1)}{5} = x + c_1$$

The above can be written as

$$\left(-\frac{6}{5}\right)(\ln(-1+2y) - \ln(3y+1)) = x + c_1$$
$$\ln(-1+2y) - \ln(3y+1) = \left(-\frac{5}{6}\right)(x + c_1)$$
$$= -\frac{5x}{6} - \frac{5c_1}{6}$$

Raising both side to exponential gives

$$e^{\ln(-1+2y) - \ln(3y+1)} = -\frac{5c_1 e^{-\frac{5x}{6}}}{6}$$

Which simplifies to

$$\frac{-1+2y}{3y+1} = c_2 e^{-\frac{5x}{6}}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_2 e^{-\frac{5x}{6}}+1}{-2+3c_2 e^{-\frac{5x}{6}}} \tag{1}$$

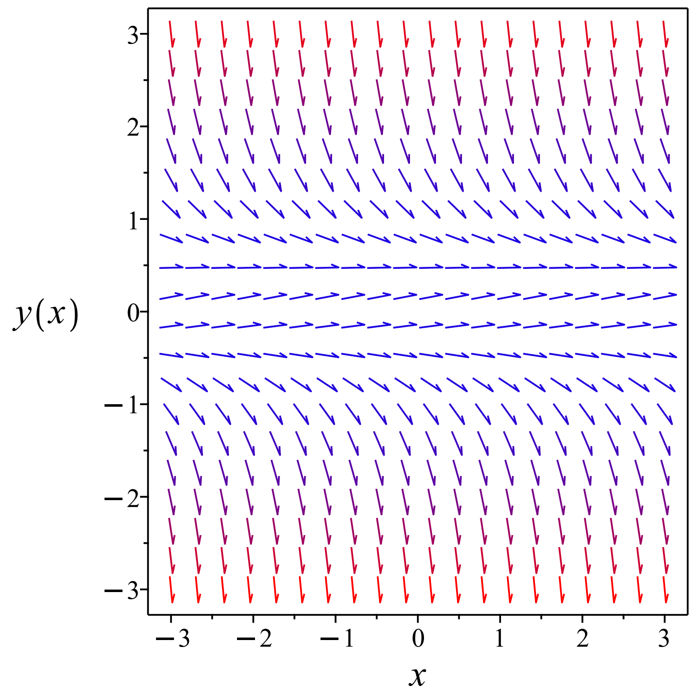


Figure 497: Slope field plot

Verification of solutions

$$y = -\frac{c_2 e^{-\frac{5x}{6}}+1}{-2+3c_2 e^{-\frac{5x}{6}}}$$

Verified OK.

9.41.2 Maple step by step solution

Let's solve

$6y' + 6y^2 - y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2+\frac{y}{6}+\frac{1}{6}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2+\frac{y}{6}+\frac{1}{6}}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{6\ln(2y-1)}{5} + \frac{6\ln(3y+1)}{5} = x + c_1$

- Solve for y

  $y = \frac{1+\mathrm{e}^{\frac{5x}{6}+\frac{5c_1}{6}}}{2\,\mathrm{e}^{\frac{5x}{6}+\frac{5c_1}{6}}-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(6*diff(y(x),x)+6*y(x)^2-y(x)-1=0,y(x), singsol=all)
```

$$y(x) = \frac{1 + c_1 e^{\frac{5x}{6}}}{2c_1 e^{\frac{5x}{6}} - 3}$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 56

```
DSolve[6*y'[x]+6*y[x]^2-y[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{5x/6} - e^{5c_1}}{2e^{5x/6} + 3e^{5c_1}}$$
$$y(x) \to -\frac{1}{3}$$
$$y(x) \to \frac{1}{2}$$

9.42 problem 38 part (g)

Internal problem ID [1148]
Internal file name [`OUTPUT/1149_Sunday_June_05_2022_02_03_34_AM_91257170/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 38 part (g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{36y' + 36y^2 - 12y = -1}$$

9.42.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2 + \frac{1}{3}y - \frac{1}{36}} dy = x + c_1$$
$$\frac{6}{6y - 1} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{c_1 + x + 6}{6x + 6c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 + x + 6}{6x + 6c_1} \tag{1}$$

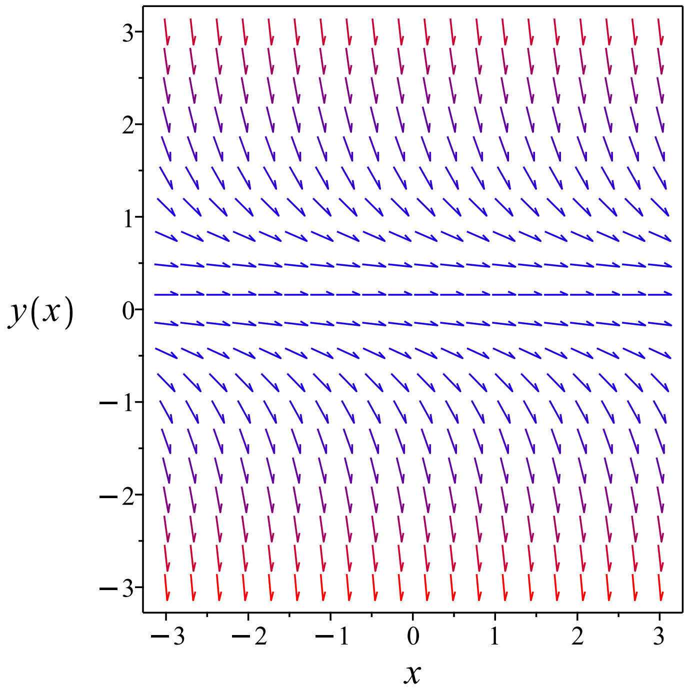


Figure 498: Slope field plot

Verification of solutions

$$y = \frac{c_1 + x + 6}{6x + 6c_1}$$

Verified OK.

9.42.2 Maple step by step solution

Let's solve

$36y' + 36y^2 - 12y = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2 + \frac{y}{3} - \frac{1}{36}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2 + \frac{y}{3} - \frac{1}{36}} dx = \int 1 dx + c_1$

- Evaluate integral

$$\frac{6}{6y-1} = x + c_1$$

- Solve for y

$$y = \frac{c_1+x+6}{6(x+c_1)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(36*diff(y(x),x)+36*y(x)^2-12*y(x)+1=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + x + 6}{6x + 6c_1}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 30

```
DSolve[36*y'[x]+36*y[x]^2-12*y[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + 6 - 36c_1}{6x - 216c_1}$$
$$y(x) \to \frac{1}{6}$$

9.43 problem 39 part(a)

Internal problem ID [1149]
Internal file name [`OUTPUT/1150_Sunday_June_05_2022_02_03_35_AM_37748633/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
   _Riccati]
```

$$\boxed{x^2(y'+y^2)-x(2+x)\,y=-x-2}$$

9.43.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x^2y^2-y\,x^2-2yx+x+2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y'=-y^2+y+\frac{2y}{x}-\frac{1}{x}-\frac{2}{x^2}$$

With Riccati ODE standard form

$$y'=f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\frac{2+x}{x^2}$, $f_1(x)=-\frac{-x^2-2x}{x^2}$ and $f_2(x)=-1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{-u} \qquad (1) \end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \frac{-x^2-2x}{x^2} \\ f_2^2f_0 &= -\frac{2+x}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) - \frac{(-x^2-2x)\,u'(x)}{x^2} - \frac{(2+x)\,u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(c_1 + c_2\mathrm{e}^x)$$

The above shows that

$$u'(x) = (x+1)\,c_2\mathrm{e}^x + c_1$$

Using the above in (1) gives the solution

$$y = \frac{(x+1)\,c_2\mathrm{e}^x + c_1}{x\,(c_1 + c_2\mathrm{e}^x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x\,\mathrm{e}^x + \mathrm{e}^x + c_3}{x\,(c_3 + \mathrm{e}^x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x\,\mathrm{e}^x + \mathrm{e}^x + c_3}{x\,(c_3 + \mathrm{e}^x)} \tag{1}$$

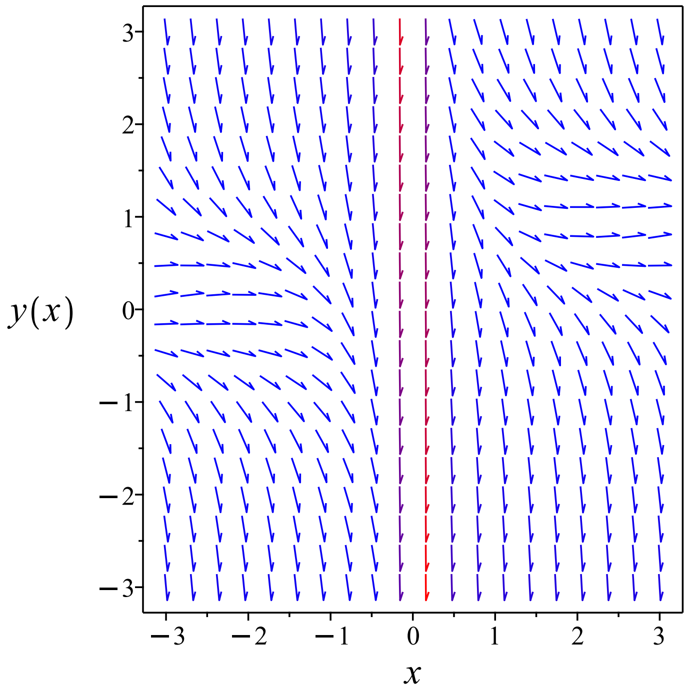


Figure 499: Slope field plot

Verification of solutions

$$y = \frac{x\,\mathrm{e}^{x} + \mathrm{e}^{x} + c_3}{x\,(c_3 + \mathrm{e}^{x})}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^2*(diff(y(x),x)+y(x)^2)-x*(x+2)*y(x)+x+2=0,y(x), singsol=all)
```

$$y(x) = \frac{x\,\mathrm{e}^{x} + \mathrm{e}^{x} - c_1}{(-c_1 + \mathrm{e}^{x})\,x}$$

✓ Solution by Mathematica

Time used: 0.179 (sec). Leaf size: 49

```
DSolve[x^2*(y'[x]+y[x])-x*(x+2)+y[x]+x+2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{1}{x}-x}\left(\int_1^x \frac{e^{K[1]-\frac{1}{K[1]}}\left(K[1]^2+K[1]-2\right)}{K[1]^2}dK[1]+c_1\right)$$

9.44 problem 39 part(b)

Internal problem ID [1150]
Internal file name [OUTPUT/1151_Sunday_June_05_2022_02_03_37_AM_78391930/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' + y^2 + 4yx = -4x^2 - 2}$$

9.44.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -4x^2 - 4yx - y^2 - 2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (-4x^2 - 4yx - y^2 - 2)(b_3 - a_2) - (-4x^2 - 4yx - y^2 - 2)^2 a_3 - (-8x - 4y)(xa_2 + ya_3 + a_1) - (-4x - 2y)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-16x^4a_3 - 32x^3ya_3 - 24x^2y^2a_3 - 8x\,y^3a_3 - y^4a_3 + 12x^2a_2 - 16x^2a_3 + 4x^2b_2 - 4x^2b_3 + 8xya_2 - 8xya_3 + 2xyb_2 + y^2a_2 + y^2b_3 + 8xa_1 + 4xb_1 + 4ya_1 + 2yb_1 + 2a_2 - 4a_3 + b_2 - 2b_3 = 0$$

Setting the numerator to zero gives

$$-16x^4a_3 - 32x^3ya_3 - 24x^2y^2a_3 - 8x\,y^3a_3 - y^4a_3 + 12x^2a_2 - 16x^2a_3 + 4x^2b_2 - 4x^2b_3 + 8xya_2 - 8xya_3 + 2xyb_2 + y^2a_2 + y^2b_3 + 8xa_1 + 4xb_1 + 4ya_1 + 2yb_1 + 2a_2 - 4a_3 + b_2 - 2b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-16a_3v_1^4 - 32a_3v_1^3v_2 - 24a_3v_1^2v_2^2 - 8a_3v_1v_2^3 - a_3v_2^4 + 12a_2v_1^2 + 8a_2v_1v_2 + a_2v_2^2 - 16a_3v_1^2 - 8a_3v_1v_2 + 4b_2v_1^2 + 2b_2v_1v_2 - 4b_3v_1^2 + b_3v_2^2 + 8a_1v_1 + 4a_1v_2 + 4b_1v_1 + 2b_1v_2 + 2a_2 - 4a_3 + b_2 - 2b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
-16a_3v_1^4 - 32a_3v_1^3v_2 - 24a_3v_1^2v_2^2 + (12a_2 - 16a_3 + 4b_2 - 4b_3)\, v_1^2 \\
- 8a_3v_1v_2^3 + (8a_2 - 8a_3 + 2b_2)\, v_1v_2 + (8a_1 + 4b_1)\, v_1 - a_3v_2^4 \\
+ (a_2 + b_3)\, v_2^2 + (4a_1 + 2b_1)\, v_2 + 2a_2 - 4a_3 + b_2 - 2b_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-32a_3 &= 0 \\
-24a_3 &= 0 \\
-16a_3 &= 0 \\
-8a_3 &= 0 \\
-a_3 &= 0 \\
4a_1 + 2b_1 &= 0 \\
8a_1 + 4b_1 &= 0 \\
a_2 + b_3 &= 0 \\
8a_2 - 8a_3 + 2b_2 &= 0 \\
2a_2 - 4a_3 + b_2 - 2b_3 &= 0 \\
12a_2 - 16a_3 + 4b_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= -2a_1 \\
b_2 &= 4b_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= -2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2 - \left(-4x^2 - 4yx - y^2 - 2\right)(1) \\
&= 4x^2 + 4yx + y^2 \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{4x^2 + 4yx + y^2} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{2x + y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -4x^2 - 4yx - y^2 - 2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{(2x+y)^2} \\
S_y &= \frac{1}{(2x+y)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{1}{2x+y} = -x + c_1
$$

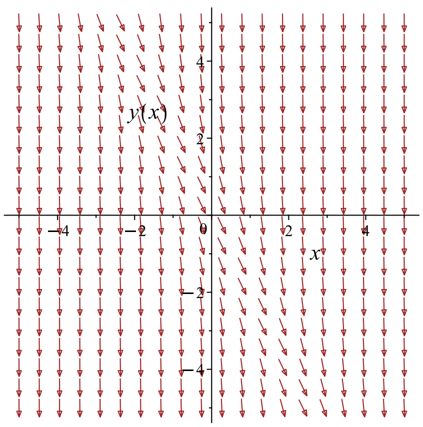
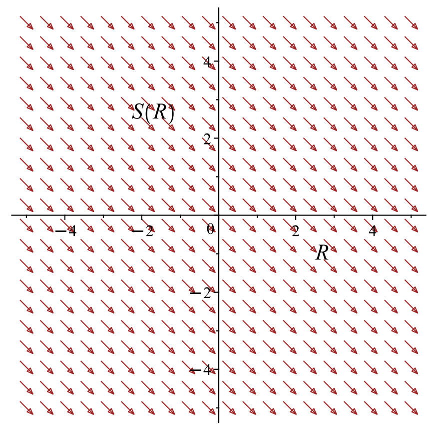
Which simplifies to

$$
-\frac{1}{2x+y} = -x + c_1
$$

Which gives

$$
y = -\frac{2c_1 x - 2x^2 + 1}{-x + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -4x^2 - 4yx - y^2 - 2$ 	$R = x$ $S = -\frac{1}{2x+y}$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = -\frac{2c_1x - 2x^2 + 1}{-x + c_1} \tag{1}$$

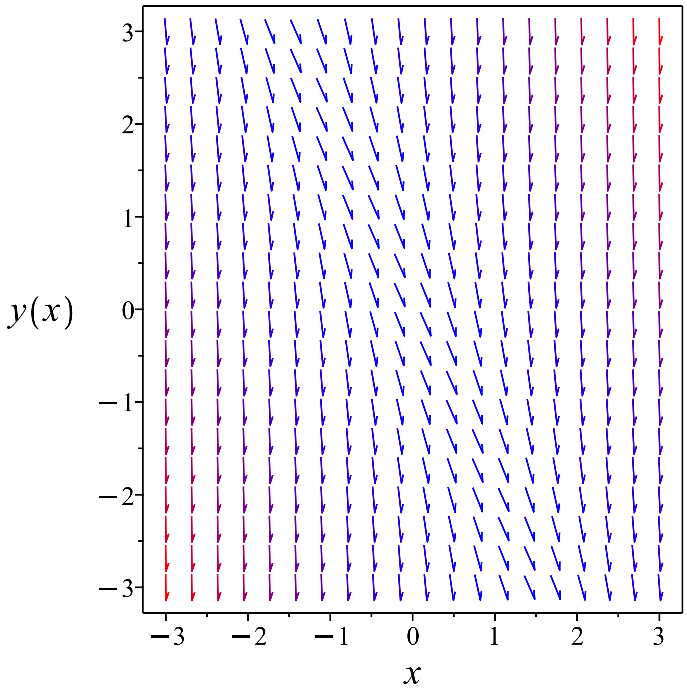


Figure 500: Slope field plot

Verification of solutions

$$y = -\frac{2c_1x - 2x^2 + 1}{-x + c_1}$$

Verified OK.

9.44.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -4x^2 - 4yx - y^2 - 2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -4x^2 - 4yx - y^2 - 2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -4x^2 - 2$, $f_1(x) = -4x$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 4x \\ f_2^2 f_0 &= -4x^2 - 2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) - 4xu'(x) + (-4x^2 - 2) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{-x^2}(c_2 x + c_1)$$

The above shows that

$$u'(x) = e^{-x^2}(-2c_2 x^2 - 2c_1 x + c_2)$$

Using the above in (1) gives the solution

$$y = \frac{-2c_2 x^2 - 2c_1 x + c_2}{c_2 x + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-2c_3 x - 2x^2 + 1}{c_3 + x}$$

Summary

The solution(s) found are the following

$$y = \frac{-2c_3x - 2x^2 + 1}{c_3 + x} \tag{1}$$

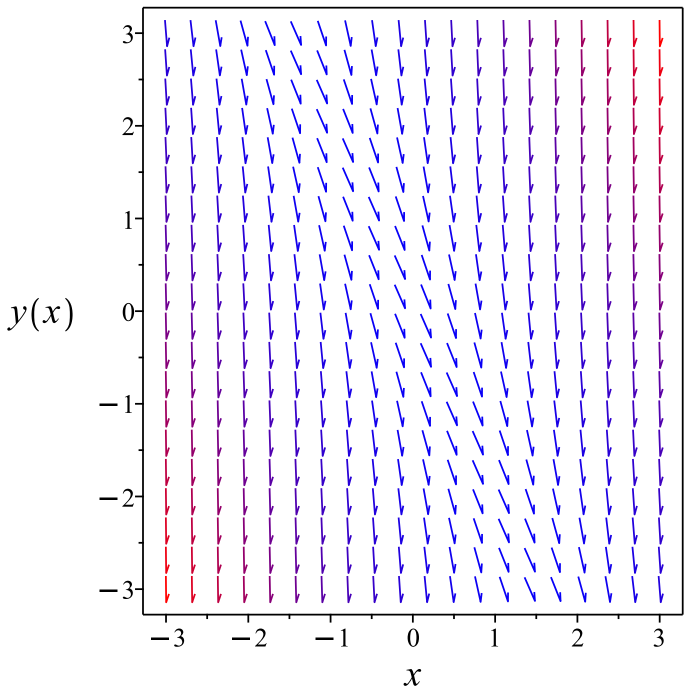


Figure 501: Slope field plot

Verification of solutions

$$y = \frac{-2c_3x - 2x^2 + 1}{c_3 + x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+y(x)^2+4*x*y(x)+4*x^2+2=0,y(x), singsol=all)
```

$$y(x) = \frac{-2c_1x + 2x^2 - 1}{c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 22

```
DSolve[y'[x]+y[x]^2+4*x*y[x]+4*x^2+2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + \frac{1}{x + c_1}$$
$$y(x) \to -2x$$

9.45 problem 39 part(c)

Internal problem ID [1151]
Internal file name [OUTPUT/1152_Sunday_June_05_2022_02_03_38_AM_44514293/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(1+2x)\left(y'+y^2\right)-2y=2x+3}$$

9.45.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x\,y^2+y^2-2x-2y-3}{1+2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7+y\,x^2a_8+x\,y^2a_9+y^3a_{10}+x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = x^3b_7+y\,x^2b_8+x\,y^2b_9+y^3b_{10}+x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\ &- \frac{(2x\,y^2 + y^2 - 2x - 2y - 3)\,(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6}{1+2x} \\ &- \frac{(2x\,y^2 + y^2 - 2x - 2y - 3)^2\,(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{(1+2x)^2} \\ &- \left(-\frac{2y^2-2}{1+2x} + \frac{4x\,y^2 + 2y^2 - 4x - 4y - 6}{(1+2x)^2}\right)(x^3a_7 + y\,x^2a_8 \\ &+ x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\ &+ \frac{(4yx + 2y - 2)\,(x^3b_7 + y\,x^2b_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1)}{1+2x} \\ &= 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

Expression too large to display (7E)

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(4b_5+16a_8+12a_7+4b_8+8a_4+8a_5)\,v_2^2v_1^3\\
&+(12a_5+16a_6+32a_9+12a_8)\,v_2^3v_1^2\\
&+(-a_5+16a_6+48a_{10}+12a_9-4b_{10}-4a_3)\,v_2^4v_1\\
&+(8b_4-8a_9+2b_7+8b_8-8a_7-16a_8+8b_9+8b_2)\,v_2v_1^3\\
&+(8a_4+16a_5+4b_5-12a_{10}+8b_9+3a_7-2a_8\\
&-20a_9+b_8+12b_{10}+4a_2+8a_3+4b_3)\,v_2^2v_1^2\\
&+(8a_5+32a_6-24a_{10}+2a_8+4a_9+8b_{10}+8a_3)\,v_2^3v_1\\
&+(-4a_4-12a_5-8a_6+2b_4+4b_5+8b_6-24a_8\\
&+8b_8-6a_7+16b_9-24a_9+8b_1+8b_2)\,v_2v_1^2\\
&+(2a_4+2a_5-16a_6+b_5+4b_6-28a_9+6b_9\\
&-4a_8+24b_{10}-36a_{10}+4a_2+16a_3+4b_3)\,v_2^2v_1\\
&+(-4a_4-24a_6+16b_6-16a_5+4b_5-6a_8\\
&+6b_9-18a_9+2b_8-8a_3+8b_1+2b_2)\,v_2v_1\\
&+(-4a_8-4a_5)\,v_2^4v_1^3+(-8a_9-8a_6)\,v_2^5v_1^2\\
&+(8a_8+12a_7+4b_8)\,v_2^2v_1^4+(16a_9+16a_8)\,v_2^3v_1^3\\
&+(-a_8+24a_{10}+20a_9-4b_{10}-4a_5-4a_3)\,v_2^4v_1^2\\
&+(-2a_9+24a_{10}-8a_6)\,v_2^5v_1+(8b_7+8b_4)\,v_2v_1^4\\
&+4a_1-3a_2-9a_3-2b_1+b_2+3b_3\\
&+(-8a_4-4a_5+8b_8-12a_8+4b_4+4b_5-20a_7+10b_7)\,v_1^3\\
&+(3b_7-4a_2-4a_3+4b_3-12a_4+6b_4-9a_7+3b_8-9a_8\\
&-12a_5+8b_5)\,v_1^2+(b_9+a_2+6a_3+b_3-20a_6+2b_6-3a_9\\
&+9b_{10}-27a_{10}-2a_5)\,v_2^2+(8b_7-12a_7+4b_8-4a_8)\,v_1^4\\
&+(4a_3-2a_9+a_5+8a_6-32a_{10}+4b_{10})\,v_2^3+(-4a_2\\
&-12a_3-6a_4-9a_5-4b_1+2b_2+8b_3+2b_4+3b_5)\,v_1\\
&+(4a_1-2a_2-8a_3-3a_5-18a_6+2b_1+b_5+6b_6)\,v_2\\
&+(-2a_6+12a_{10})\,v_2^5-3v_2^6a_{10}-4v_1^4v_2^4a_8\\
&-8v_1^3v_2^5a_9-12v_1^2v_2^6a_{10}-12v_1v_2^6a_{10}+8v_1^5v_2b_7\\
&+(8a_6-a_3+a_9-b_{10}+10a_{10})\,v_2^4=0
\end{aligned}\tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_8 &= 0 \\
-8a_9 &= 0 \\
-12a_{10} &= 0 \\
-3a_{10} &= 0 \\
8b_7 &= 0 \\
-2a_6 + 12a_{10} &= 0 \\
-4a_8 - 4a_5 &= 0 \\
-8a_9 - 8a_6 &= 0 \\
16a_9 + 16a_8 &= 0 \\
8b_7 + 8b_4 &= 0 \\
8a_8 + 12a_7 + 4b_8 &= 0 \\
-2a_9 + 24a_{10} - 8a_6 &= 0 \\
12a_5 + 16a_6 + 32a_9 + 12a_8 &= 0 \\
8b_7 - 12a_7 + 4b_8 - 4a_8 &= 0 \\
8a_6 - a_3 + a_9 - b_{10} + 10a_{10} &= 0 \\
4a_1 - 3a_2 - 9a_3 - 2b_1 + b_2 + 3b_3 &= 0 \\
4a_3 - 2a_9 + a_5 + 8a_6 - 32a_{10} + 4b_{10} &= 0 \\
-a_5 + 16a_6 + 48a_{10} + 12a_9 - 4b_{10} - 4a_3 &= 0 \\
-a_8 + 24a_{10} + 20a_9 - 4b_{10} - 4a_5 - 4a_3 &= 0 \\
4b_5 + 16a_8 + 12a_7 + 4b_8 + 8a_4 + 8a_5 &= 0 \\
8a_5 + 32a_6 - 24a_{10} + 2a_8 + 4a_9 + 8b_{10} + 8a_3 &= 0 \\
4a_1 - 2a_2 - 8a_3 - 3a_5 - 18a_6 + 2b_1 + b_5 + 6b_6 &= 0 \\
-8a_4 - 4a_5 + 8b_8 - 12a_8 + 4b_4 + 4b_5 - 20a_7 + 10b_7 &= 0 \\
8b_4 - 8a_9 + 2b_7 + 8b_8 - 8a_7 - 16a_8 + 8b_9 + 8b_2 &= 0 \\
-4a_2 - 12a_3 - 6a_4 - 9a_5 - 4b_1 + 2b_2 + 8b_3 + 2b_4 + 3b_5 &= 0 \\
b_9 + a_2 + 6a_3 + b_3 - 20a_6 + 2b_6 - 3a_9 + 9b_{10} - 27a_{10} - 2a_5 &= 0 \\
3b_7 - 4a_2 - 4a_3 + 4b_3 - 12a_4 + 6b_4 - 9a_7 + 3b_8 - 9a_8 - 12a_5 + 8b_5 &= 0 \\
-4a_4 - 24a_6 + 16b_6 - 16a_5 + 4b_5 - 6a_8 + 6b_9 - 18a_9 + 2b_8 - 8a_3 + 8b_1 + 2b_2 &= 0 \\
-4a_4 - 12a_5 - 8a_6 + 2b_4 + 4b_5 + 8b_6 - 24a_8 + 8b_8 - 6a_7 + 16b_9 - 24a_9 + 8b_1 + 8b_2 &= 0 \\
2a_4 + 2a_5 - 16a_6 + b_5 + 4b_6 - 28a_9 + 6b_9 - 4a_8 + 24b_{10} - 36a_{10} + 4a_2 + 16a_3 + 4b_3 &= 0 \\
8a_4 + 16a_5 + 4b_5 - 12a_{10} + 8b_9 + 3a_7 - 2a_8 - 20a_9 + b_8 + 12b_{10} + 4a_2 + 8a_3 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{b_9}{2} - b_{10} \\
a_2 &= -b_9 \\
a_3 &= -b_{10} \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= -\frac{3b_9}{2} + b_{10} \\
b_2 &= -b_9 \\
b_3 &= -b_9 - b_{10} \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= \frac{b_9}{2} - b_{10} \\
b_7 &= 0 \\
b_8 &= 0 \\
b_9 &= b_9 \\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y - 1 \\
\eta &= y^3 - y^2 - y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y^3 - y^2 - y + 1 - \left(-\frac{2x\,y^2 + y^2 - 2x - 2y - 3}{1+2x}\right)(-y-1) \\
&= \frac{-4x\,y^2 + 4x + 4y + 4}{1+2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-4x\,y^2+4x+4y+4}{1+2x}} dy
\end{aligned}
$$

Which results in

$$
S = \left(\frac{1}{4} + \frac{x}{2}\right)\left(\frac{\ln(y+1)}{1+2x} - \frac{\ln(yx - x - 1)}{1+2x}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2x\,y^2 + y^2 - 2x - 2y - 3}{1+2x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-y+1}{-4+(4y-4)\,x} \\
S_y &= \frac{-1-2x}{4\,(-1+(y-1)\,x)\,(y+1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(1+y)}{4} - \frac{\ln(-1+(y-1)\,x)}{4} = \frac{x}{2} + c_1
$$

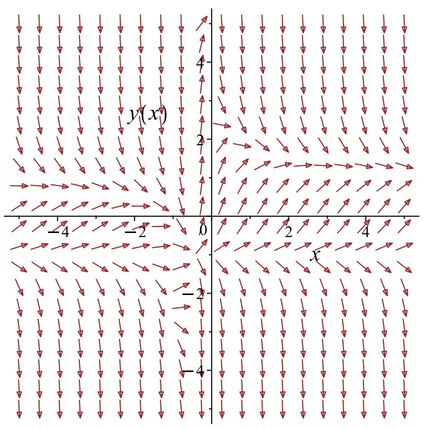
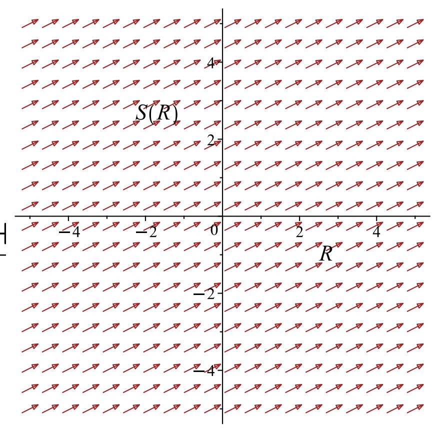
Which simplifies to

$$
\frac{\ln(1+y)}{4} - \frac{\ln(-1+(y-1)\,x)}{4} = \frac{x}{2} + c_1
$$

Which gives

$$
y = \frac{x\,\mathrm{e}^{2x+4c_1} + \mathrm{e}^{2x+4c_1} + 1}{-1 + x\,\mathrm{e}^{2x+4c_1}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x\,y^2+y^2-2x-2y-3}{1+2x}$ 	$R = x$ $S = \frac{\ln(y+1)}{4} - \ln(-1+$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x\,\mathrm{e}^{2x+4c_1} + \mathrm{e}^{2x+4c_1} + 1}{-1 + x\,\mathrm{e}^{2x+4c_1}} \tag{1}$$

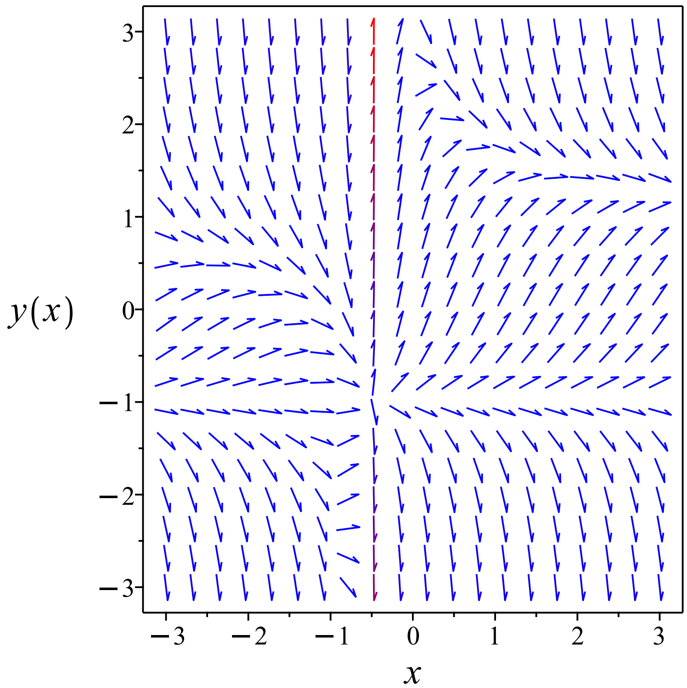


Figure 502: Slope field plot

Verification of solutions

$$y = \frac{x\,\mathrm{e}^{2x+4c_1} + \mathrm{e}^{2x+4c_1} + 1}{-1 + x\,\mathrm{e}^{2x+4c_1}}$$

Verified OK.

9.45.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2x\,y^2 + y^2 - 2x - 2y - 3}{1+2x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2x\,y^2}{1+2x} - \frac{y^2}{1+2x} + \frac{2x}{1+2x} + \frac{2y}{1+2x} + \frac{3}{1+2x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{-3-2x}{1+2x}$, $f_1(x) = \frac{2}{1+2x}$ and $f_2(x) = -1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= -\frac{2}{1+2x} \\
f_2^2 f_0 &= -\frac{-3-2x}{1+2x}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-u''(x) + \frac{2u'(x)}{1+2x} - \frac{(-3-2x)\, u(x)}{1+2x} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{x}
$$

The above shows that

$$
u'(x) = -c_1 \mathrm{e}^{-x} + (x+1)\, c_2 \mathrm{e}^{x}
$$

Using the above in (1) gives the solution

$$
y = \frac{-c_1 \mathrm{e}^{-x} + (x+1)\, c_2 \mathrm{e}^{x}}{c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{x}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{x\, \mathrm{e}^{2x} + \mathrm{e}^{2x} - c_3}{x\, \mathrm{e}^{2x} + c_3}
$$

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{2x} + e^{2x} - c_3}{x\,e^{2x} + c_3} \tag{1}$$

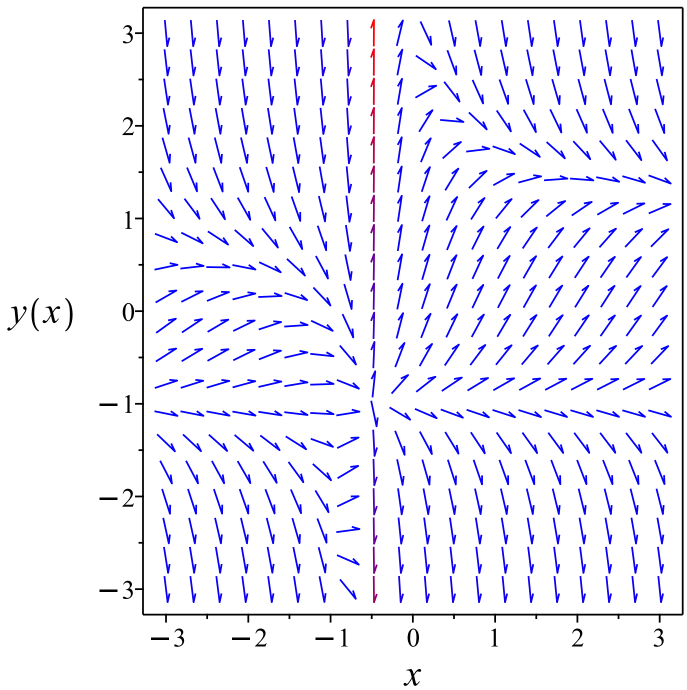


Figure 503: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{2x} + e^{2x} - c_3}{x\,e^{2x} + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (b) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve((2*x+1)*(diff(y(x),x)+y(x)^2)-2*y(x)-(2*x+3)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{2x}x + e^{2x} - c_1}{e^{2x}x + c_1}$$

✓ Solution by Mathematica

Time used: 0.403 (sec). Leaf size: 41

```
DSolve[(2*x+1)*(y'[x]+y[x]^2)-2*y[x]-(2*x+3)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x+1}(x+1) - c_1}{e^{2x+1}x + c_1}$$
$$y(x) \to -1$$

9.46 problem 39 part(d)

Internal problem ID [1152]
Internal file name [`OUTPUT/1153_Sunday_June_05_2022_02_03_39_AM_15976872/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{(3x-1)\left(y'+y^2\right)-y(3x+2)=6x-8}$$

9.46.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3x\,y^2-3yx-y^2-6x-2y+8}{3x-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7+y\,x^2a_8+x\,y^2a_9+y^3a_{10}+x^2a_4+yxa_5+y^2a_6+xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = x^3b_7+y\,x^2b_8+x\,y^2b_9+y^3b_{10}+x^2b_4+yxb_5+y^2b_6+xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 \\
&- \frac{(3x\,y^2 - 3yx - y^2 - 6x - 2y + 8)\,(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5}{3x-1} \\
&- \frac{(3x\,y^2 - 3yx - y^2 - 6x - 2y + 8)^2\,(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3)}{(3x-1)^2} \\
&- \left(-\frac{3y^2 - 3y - 6}{3x-1} + \frac{9x\,y^2 - 9yx - 3y^2 - 18x - 6y + 24}{(3x-1)^2}\right)\big(x^3a_7 \\
&+ y\,x^2a_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + yxa_5 + y^2a_6 + xa_2 + ya_3 + a_1\big) \\
&+ \frac{(6yx - 3x - 2y - 2)\,(x^3b_7 + y\,x^2b_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + yxb_5 + y^2b_6 + xb_2 + yb_3 + b_1)}{3x-1} \\
&= 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(24a_8 + 54a_9 + 18a_5)\, v_2^3 v_1^3 + (21a_9 - 9b_{10} - a_8 + 81a_{10} - 9a_3 + 6a_5 + 36a_6)\, v_2^4 v_1^2 \\
&+ (6a_8 + 36a_9 - 9a_5)\, v_2^4 v_1^3 + (12a_9 + 54a_{10} - 18a_6)\, v_2^5 v_1^2 \\
&+ (18a_{10} - 2a_9 + 12a_6)\, v_2^5 v_1 + (-36a_8 - 12b_7 - 27a_7 + 18b_4)\, v_2 v_1^4 \\
&+ (-72a_9 - 18a_7 - 90a_8 - 6b_8 + 9b_9 + 18a_4 + 27a_5 + 9b_5)\, v_2^2 v_1^3 \\
&+ (-108a_{10} - 16a_8 - 153a_9 + 18b_{10} + 18a_3 + 15a_5 + 54a_6)\, v_2^3 v_1^2 \\
&+ (-216a_{10} - a_5 + 12a_6 - 14a_9 + 6b_{10} + 6a_3)\, v_2^4 v_1 \\
&+ (-72a_9 + 2b_7 - 18a_4 - 36a_5 - 12b_4 + 18b_8 - 12a_8 + 36b_9 + 18b_2)\, v_2 v_1^3 + (-108a_{10} - 12a_4 \\
&- 81a_5 - 72a_6 - 6b_5 + 9b_6 + 9a_2 + 27a_3 + 9b_3 + 12b_9 + 3a_7 + 15a_8 + 30a_9 + b_8 + 54b_{10})\, v_2^2 v_1^2 \\
&+ (72a_{10} - 10a_5 - 144a_6 + 6a_3 + 2a_8 + 30a_9 + 6b_{10})\, v_2^3 v_1 + (192a_9 + 3a_4 - 9a_2 - 36a_3 \\
&+ 18b_1 - 12b_2 + 6a_7 - 60b_9 + 6a_5 - 72a_6 + 2b_4 + 9b_5 + 36b_6 + 74a_8 - 12b_8)\, v_2 v_1^2 \\
&+ (288a_{10} - 6a_2 - 72a_3 - 6b_3 + 4a_8 - 90b_{10} + 2a_4 + 18a_5 + 48a_6 + b_5 + 3b_6 + 76a_9 - 8b_9)\, v_2^2 v_1 \\
&+ (-128a_9 + 6a_2 + 24a_3 - 12b_1 + 2b_2 + 2b_8 - 16a_8 + 16b_9 + 4a_4 + 192a_6 \\
&- 60b_6 + 44a_5 - 6b_5)\, v_2 v_1 + (27a_7 + 9b_8 + 27a_8)\, v_2^2 v_1^4 - 18a_1 - 8a_2 - 64a_3 \\
&+ 2b_1 + b_2 + 8b_3 + 18v_1^5 v_2 b_7 - 9v_1^4 v_2^4 a_8 - 18v_1^3 v_2^5 a_9 - 27v_1^2 v_2^6 a_{10} + 18v_1^4 v_2^3 a_8 \\
&+ 18v_1 v_2^6 a_{10} - 3v_2^6 a_{10} - 9v_1^5 b_7 + (24b_7 - 54a_7 + 18b_8 - 36a_8 - 9b_4)\, v_1^4 \\
&+ (-30b_8 + 96a_8 + 72a_7 - 16b_7 - 9b_2 - 36a_4 - 36a_5 + 15b_4 + 18b_5)\, v_1^3 \\
&+ (b_9 - 8a_9 + 24b_{10} - 192a_{10} + a_2 + 21a_3 + b_3 + 2a_5 + 46a_6 - 2b_6)\, v_2^2 \\
&+ (3b_7 - 24a_7 + 8b_8 - 64a_8 - 18a_2 - 36a_3 - 9b_1 + 6b_2 + 18b_3 + 96a_5 - 30b_5 + 42a_4 - 10b_4)\, v_1^2 \\
&+ (2a_9 + 78a_{10} - 4b_{10} - 4a_3 + a_5 + 33a_6)\, v_2^3 + (a_9 - b_{10} + 45a_{10} - a_3 - 8a_6)\, v_2^4 \\
&+ (12a_2 + 96a_3 - 16a_4 - 64a_5 - 3b_1 - 4b_2 - 30b_3 + 2b_4 + 8b_5)\, v_1 \\
&+ (9a_1 + 2a_2 + 14a_3 - 8a_5 - 128a_6 + 2b_1 + b_5 + 16b_6)\, v_2 + (-12a_{10} - 2a_6)\, v_2^5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-9a_8 =$$
$$18a_8 =$$
$$-18a_9 =$$
$$-27a_{10} =$$
$$-3a_{10} =$$
$$18a_{10} =$$
$$-9b_7 =$$
$$18b_7 =$$
$$-12a_{10} - 2a_6 =$$
$$27a_7 + 9b_8 + 27a_8 =$$
$$6a_8 + 36a_9 - 9a_5 =$$
$$24a_8 + 54a_9 + 18a_5 =$$
$$12a_9 + 54a_{10} - 18a_6 =$$
$$18a_{10} - 2a_9 + 12a_6 =$$
$$-36a_8 - 12b_7 - 27a_7 + 18b_4 =$$
$$a_9 - b_{10} + 45a_{10} - a_3 - 8a_6 =$$
$$24b_7 - 54a_7 + 18b_8 - 36a_8 - 9b_4 =$$
$$-18a_1 - 8a_2 - 64a_3 + 2b_1 + b_2 + 8b_3 =$$
$$2a_9 + 78a_{10} - 4b_{10} - 4a_3 + a_5 + 33a_6 =$$
$$-216a_{10} - a_5 + 12a_6 - 14a_9 + 6b_{10} + 6a_3 =$$
$$21a_9 - 9b_{10} - a_8 + 81a_{10} - 9a_3 + 6a_5 + 36a_6 =$$
$$-108a_{10} - 16a_8 - 153a_9 + 18b_{10} + 18a_3 + 15a_5 + 54a_6 =$$
$$72a_{10} - 10a_5 - 144a_6 + 6a_3 + 2a_8 + 30a_9 + 6b_{10} =$$
$$9a_1 + 2a_2 + 14a_3 - 8a_5 - 128a_6 + 2b_1 + b_5 + 16b_6 =$$
$$-72a_9 - 18a_7 - 90a_8 - 6b_8 + 9b_9 + 18a_4 + 27a_5 + 9b_5 =$$
$$12a_2 + 96a_3 - 16a_4 - 64a_5 - 3b_1 - 4b_2 - 30b_3 + 2b_4 + 8b_5 =$$
$$-72a_9 + 2b_7 - 18a_4 - 36a_5 - 12b_4 + 18b_8 - 12a_8 + 36b_9 + 18b_2 =$$
$$-30b_8 + 96a_8 + 72a_7 - 16b_7 - 9b_2 - 36a_4 - 36a_5 + 15b_4 + 18b_5 =$$
$$b_9 - 8a_9 + 24b_{10} - 192a_{10} + a_2 + 21a_3 + b_3 + 2a_5 + 46a_6 - 2b_6 =$$
$$-128a_9 + 6a_2 + 24a_3 - 12b_1 + 2b_2 + 2b_8 - 16a_8 + 16b_9 + 4a_4 + 192a_6 - 60b_6 + 44a_5 - 6b_5 =$$
$$288a_{10} - 6a_2 - 72a_3 - 6b_3 + 4a_8 - 90b_{10} + 2a_4 + 18a_5 + 48a_6 + b_5 + 3b_6 + 76a_9 - 8b_9 =$$
$$3b_7 - 24a_7 + 8b_8 - 64a_8 - 18a_2 - 36a_3 - 9b_1 + 6b_2 + 18b_3 + 96a_5 - 30b_5 + 42a_4 - 10b_4 =$$
$$192a_9 + 3a_4 - 9a_2 - 36a_3 + 18b_1 - 12b_2 + 6a_7 - 60b_9 + 6a_5 - 72a_6 + 2b_4 + 9b_5 + 36b_6 + 74a_8 - 12b_8 =$$
$$-108a_{10} - 12a_4 - 81a_5 - 72a_6 - 6b_5 + 9b_6 + 9a_2 + 27a_3 + 9b_3 + 12b_9 + 3a_7 + 15a_8 + 30a_9 + b_8 + 54b_{10} =$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{b_9}{3} + 2b_{10} \\
a_2 &= -b_9 \\
a_3 &= -b_{10} \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
a_7 &= 0 \\
a_8 &= 0 \\
a_9 &= 0 \\
a_{10} &= 0 \\
b_1 &= \frac{8b_9}{3} - 2b_{10} \\
b_2 &= -2b_9 \\
b_3 &= -\frac{2b_9}{3} - 3b_{10} \\
b_4 &= 0 \\
b_5 &= -b_9 \\
b_6 &= -\frac{b_9}{3} \\
b_7 &= 0 \\
b_8 &= 0 \\
b_9 &= b_9 \\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y + 2 \\
\eta &= y^3 - 3y - 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y^3 - 3y - 2 - \left(-\frac{3x\,y^2 - 3yx - y^2 - 6x - 2y + 8}{3x - 1}\right)(-y+2) \\
&= \frac{9x\,y^2 - 9yx - 18x - 9y + 18}{3x - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{9x\,y^2 - 9yx - 18x - 9y + 18}{3x - 1}} dy
\end{aligned}$$

Which results in

$$S = \left(\frac{x}{3} - \frac{1}{9}\right)\left(-\frac{\ln(yx + x - 1)}{3x - 1} + \frac{\ln(y - 2)}{3x - 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x\,y^2 - 3yx - y^2 - 6x - 2y + 8}{3x - 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{-y-1}{-9+(9y+9)\,x}\\
S_y &= \frac{3x-1}{9\,(y-2)\,(yx+x-1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{R}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(yx+x-1)}{9} + \frac{\ln(y-2)}{9} = -\frac{x}{3} + c_1
$$

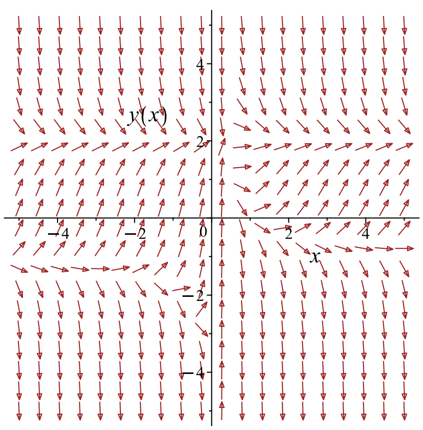
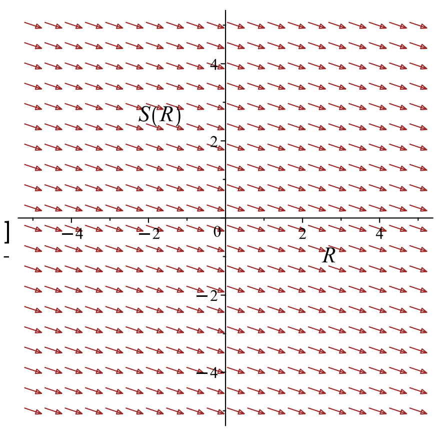
Which simplifies to

$$
-\frac{\ln(yx+x-1)}{9} + \frac{\ln(y-2)}{9} = -\frac{x}{3} + c_1
$$

Which gives

$$
y = -\frac{x\,\mathrm{e}^{-3x+9c_1} - \mathrm{e}^{-3x+9c_1} + 2}{-1 + x\,\mathrm{e}^{-3x+9c_1}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x\,y^2-3yx-y^2-6x-2y+8}{3x-1}$ 	$R = x$ $S = -\frac{\ln(yx+x-1)}{9} +$	$\frac{dS}{dR} = -\frac{1}{3}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x\,\mathrm{e}^{-3x+9c_1} - \mathrm{e}^{-3x+9c_1} + 2}{-1 + x\,\mathrm{e}^{-3x+9c_1}} \tag{1}$$

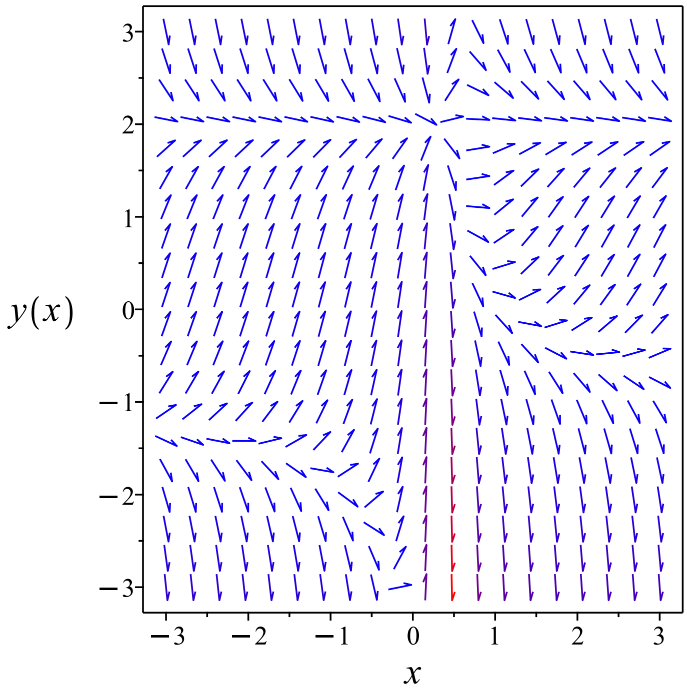


Figure 504: Slope field plot

Verification of solutions

$$y = -\frac{x\,\mathrm{e}^{-3x+9c_1} - \mathrm{e}^{-3x+9c_1} + 2}{-1 + x\,\mathrm{e}^{-3x+9c_1}}$$

Verified OK.

9.46.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{3x\,y^2 - 3yx - y^2 - 6x - 2y + 8}{3x - 1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{3x\,y^2}{3x-1} + \frac{3yx}{3x-1} + \frac{y^2}{3x-1} + \frac{6x}{3x-1} + \frac{2y}{3x-1} - \frac{8}{3x-1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{-6x+8}{3x-1}$, $f_1(x) = -\frac{-3x-2}{3x-1}$ and $f_2(x) = -1$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= \frac{-3x-2}{3x-1} \\
f_2^2 f_0 &= -\frac{-6x+8}{3x-1}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-u''(x) - \frac{(-3x-2) u'(x)}{3x-1} - \frac{(-6x+8) u(x)}{3x-1} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{-x} c_2
$$

The above shows that

$$
u'(x) = -c_2(x-1)\,\mathrm{e}^{-x} + 2c_1 \mathrm{e}^{2x}
$$

Using the above in (1) gives the solution

$$
y = \frac{-c_2(x-1)\,\mathrm{e}^{-x} + 2c_1 \mathrm{e}^{2x}}{c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{-x} c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{2c_3 \mathrm{e}^{3x} - x + 1}{c_3 \mathrm{e}^{3x} + x}
$$

Summary

The solution(s) found are the following

$$y = \frac{2c_3 e^{3x} - x + 1}{c_3 e^{3x} + x} \tag{1}$$

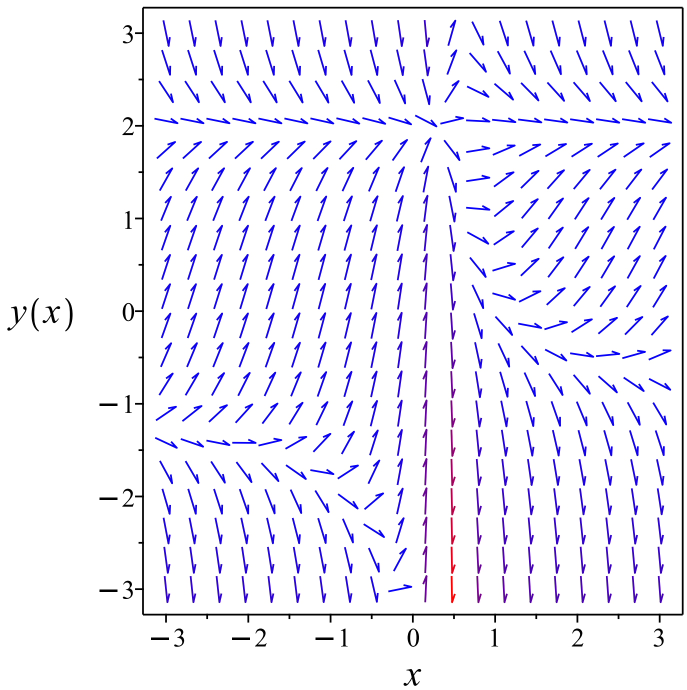


Figure 505: Slope field plot

Verification of solutions

$$y = \frac{2c_3 e^{3x} - x + 1}{c_3 e^{3x} + x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
   <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 31

```
dsolve((3*x-1)*(diff(y(x),x)+y(x)^2)-(3*x+2)*y(x)-6*x+8=0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x + 2\,e^{3x-1} + c_1}{c_1 x + e^{3x-1}}$$

✓ Solution by Mathematica

Time used: 0.557 (sec). Leaf size: 41

```
DSolve[(3*x-1)*(y'[x]+y[x]^2)-(3*x+2)*y[x]-6*x+8==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2(-ex + c_1 e^{3x} + e)}{2ex + c_1 e^{3x}}$$
$$y(x) \to 2$$

9.47 problem 39 part(e)

Internal problem ID [1153]
Internal file name [`OUTPUT/1154_Sunday_June_05_2022_02_03_40_AM_94871224/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
   _Riccati]
```

$$\boxed{x^2(y'+y^2)+yx=-x^2+\frac{1}{4}}$$

9.47.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{4x^2y^2+4x^2+4yx-1}{4x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2-1-\frac{y}{x}+\frac{1}{4x^2}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = -\frac{4x^2-1}{4x^2}$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= \frac{1}{x} \\ f_2^2f_0 &= -\frac{4x^2-1}{4x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) - \frac{u'(x)}{x} - \frac{(4x^2-1)\,u(x)}{4x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\sin(x)\,c_1 + c_2\cos(x)}{\sqrt{x}}$$

The above shows that

$$u'(x) = \frac{\left(c_1x - \frac{c_2}{2}\right)\cos(x) - \left(c_2x + \frac{c_1}{2}\right)\sin(x)}{x^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(c_1x - \frac{c_2}{2}\right)\cos(x) - \left(c_2x + \frac{c_1}{2}\right)\sin(x)}{x\,(\sin(x)\,c_1 + c_2\cos(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(2c_3x - 1)\cos(x) - 2\left(x + \frac{c_3}{2}\right)\sin(x)}{2\,(c_3\sin(x) + \cos(x))\,x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2c_3x - 1)\cos(x) - 2\left(x + \frac{c_3}{2}\right)\sin(x)}{2\,(c_3\sin(x) + \cos(x))\,x} \tag{1}$$

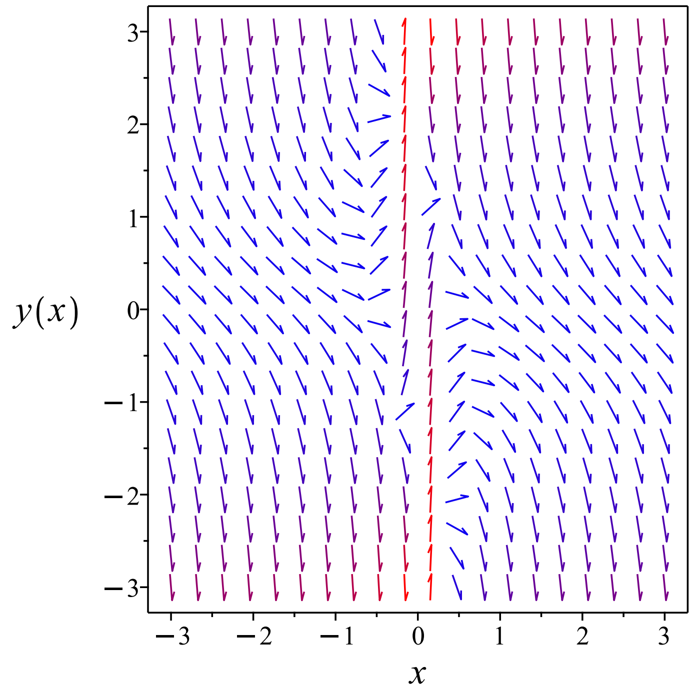


Figure 506: Slope field plot

Verification of solutions

$$y = \frac{(2c_3x - 1)\cos(x) - 2\left(x + \frac{c_3}{2}\right)\sin(x)}{2\left(c_3 \sin(x) + \cos(x)\right)x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 46

```
dsolve(x^2*(diff(y(x),x)+y(x)^2)+x*y(x)+x^2-1/4=0,y(x), singsol=all)
```

$$y(x) = \frac{-4c_1x - e^{-2ix} - 2ie^{-2ix}x - 2ic_1}{2x\left(e^{-2ix} + 2ic_1\right)}$$

✓ Solution by Mathematica

Time used: 0.377 (sec). Leaf size: 22

```
DSolve[x^2*(y'[x]+y[x]^2)+x*y[x]+x^2-1/4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2x} - \tan(x - c_1)$$

9.48 problem 39 part(f)

Internal problem ID [1154]
Internal file name [`OUTPUT/1155_Sunday_June_05_2022_02_03_42_AM_21891696/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.6 Reduction or order. Page 253
Problem number: 39 part(f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{x^2(y'+y^2)-7yx=-7}$$

9.48.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x^2y^2-7yx+7}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x^2y^2 - 7yx + 7)(b_3 - a_2)}{x^2} - \frac{(x^2y^2 - 7yx + 7)^2 a_3}{x^4} - \left(-\frac{2x\,y^2 - 7y}{x^2} + \frac{2x^2y^2 - 14yx + 14}{x^3}\right)(xa_2 + ya_3 + a_1) + \frac{(2y\,x^2 - 7x)(xb_2 + yb_3 + b_1)}{x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{-x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 + 14x^3y^3a_3 + 2x^4yb_1 - 6b_2x^4 - 56x^2y^2a_3 - 7x^3b_1 + 7x^2ya_1 - 7x^2a_2}{x^4} = 0$$

Setting the numerator to zero gives

$$-x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 + 14x^3y^3a_3 + 2x^4yb_1 - 6b_2x^4 - 56x^2y^2a_3 - 7x^3b_1 + 7x^2ya_1 - 7x^2a_2 - 7x^2b_3 + 84xya_3 - 14xa_1 - 49a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^4v_2^4 + a_2v_1^4v_2^2 + 14a_3v_1^3v_2^3 + 2b_2v_1^5v_2 + b_3v_1^4v_2^2 + 2b_1v_1^4v_2 - 56a_3v_1^2v_2^2 - 6b_2v_1^4 + 7a_1v_1^2v_2 - 7b_1v_1^3 - 7a_2v_1^2 + 84a_3v_1v_2 - 7b_3v_1^2 - 14a_1v_1 - 49a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 2b_2v_1^5v_2 - a_3v_1^4v_2^4 + (a_2 + b_3)\, v_1^4v_2^2 + 2b_1v_1^4v_2 - 6b_2v_1^4 + 14a_3v_1^3v_2^3 - 7b_1v_1^3 \\ - 56a_3v_1^2v_2^2 + 7a_1v_1^2v_2 + (-7a_2 - 7b_3)\, v_1^2 + 84a_3v_1v_2 - 14a_1v_1 - 49a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -14a_1 &= 0 \\ 7a_1 &= 0 \\ -56a_3 &= 0 \\ -49a_3 &= 0 \\ -a_3 &= 0 \\ 14a_3 &= 0 \\ 84a_3 &= 0 \\ -7b_1 &= 0 \\ 2b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ -7a_2 - 7b_3 &= 0 \\ a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x^2y^2 - 7yx + 7}{x^2}\right)(-x) \\
&= \frac{-x^2y^2 + 8yx - 7}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2y^2+8yx-7}{x}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln(yx-7)}{6} + \frac{\ln(yx-1)}{6}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x^2y^2 - 7yx + 7}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{(yx-1)(yx-7)} \\
S_y &= -\frac{x}{(yx-1)(yx-7)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(yx-7)}{6} + \frac{\ln(yx-1)}{6} = \ln(x) + c_1
$$

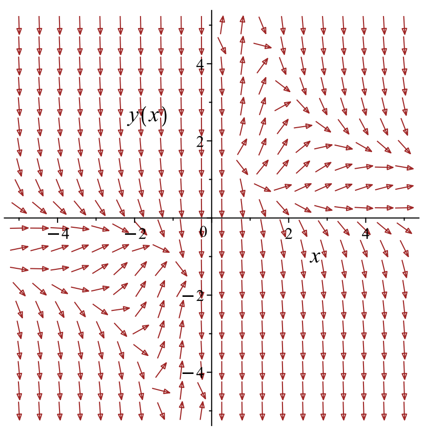
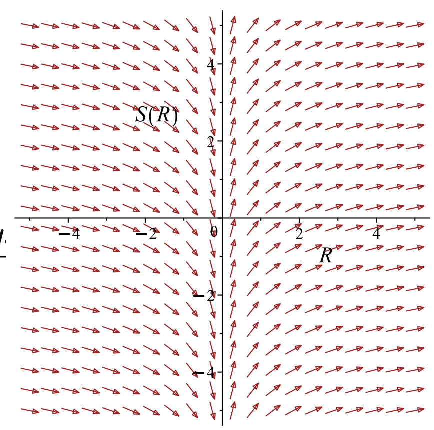
Which simplifies to

$$
-\frac{\ln(yx-7)}{6} + \frac{\ln(yx-1)}{6} = \ln(x) + c_1
$$

Which gives

$$
y = \frac{7x^6 e^{6c_1} - 1}{x\left(x^6 e^{6c_1} - 1\right)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2y^2-7yx+7}{x^2}$ 	$R = x$ $S = -\frac{\ln(yx-7)}{6} + \frac{\ln(y}{}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{7x^6 e^{6c_1} - 1}{x\left(x^6 e^{6c_1} - 1\right)} \tag{1}$$

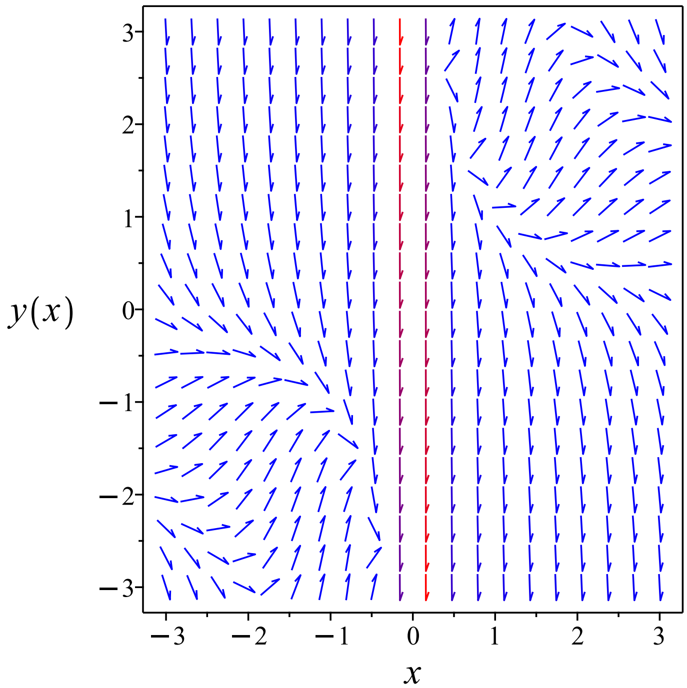


Figure 507: Slope field plot

Verification of solutions

$$y = \frac{7x^6 e^{6c_1} - 1}{x\left(x^6 e^{6c_1} - 1\right)}$$

Verified OK.

9.48.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x^2) \, \mathrm{d}y &= (-x^2y^2 + 7yx - 7) \, \mathrm{d}x \\ (x^2y^2 - 7yx + 7) \, \mathrm{d}x + (x^2) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x^2y^2 - 7yx + 7$$
$$N(x, y) = x^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2y^2 - 7yx + 7) \\ &= 2y\,x^2 - 7x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\ &= 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2}\left(\left(2y\,x^2 - 7x\right) - (2x)\right) \\
&= \frac{2yx - 9}{x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{x^2y^2 - 7yx + 7}\left((2x) - \left(2y\,x^2 - 7x\right)\right) \\
&= \frac{-2y\,x^2 + 9x}{x^2y^2 - 7yx + 7}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2x) - \left(2y\,x^2 - 7x\right)}{x\left(x^2y^2 - 7yx + 7\right) - y\left(x^2\right)} \\
&= \frac{-2yx + 9}{x^2y^2 - 8yx + 7}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t + 9}{t^2 - 8t + 7}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2t+9}{t^2-8t+7}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{7\ln(t-1)}{6}-\frac{5\ln(t-7)}{6}}\\ &= \frac{1}{(t-1)^{\frac{7}{6}}(t-7)^{\frac{5}{6}}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\left(x^2y^2-7yx+7\right)\\ &= \frac{x^2y^2-7yx+7}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\left(x^2\right)\\ &= \frac{x^2}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x^2y^2-7yx+7}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\right)+\left(\frac{x^2}{(yx-1)^{\frac{7}{6}}(yx-7)^{\frac{5}{6}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x^2y^2 - 7yx + 7}{(yx-1)^{\frac{7}{6}} (yx-7)^{\frac{5}{6}}} \mathrm{d}x$$

$$\phi = \frac{(yx-7)^{\frac{1}{6}} x}{(yx-1)^{\frac{1}{6}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{(yx-7)^{\frac{1}{6}} x^2}{6(yx-1)^{\frac{7}{6}}} + \frac{x^2}{6(yx-1)^{\frac{1}{6}} (yx-7)^{\frac{5}{6}}} + f'(y) \tag{4}$$

$$= \frac{x^2}{(yx-1)^{\frac{7}{6}} (yx-7)^{\frac{5}{6}}} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2}{(yx-1)^{\frac{7}{6}} (yx-7)^{\frac{5}{6}}}$. Therefore equation (4) becomes

$$\frac{x^2}{(yx-1)^{\frac{7}{6}} (yx-7)^{\frac{5}{6}}} = \frac{x^2}{(yx-1)^{\frac{7}{6}} (yx-7)^{\frac{5}{6}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(yx-7)^{\frac{1}{6}} x}{(yx-1)^{\frac{1}{6}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(yx-7)^{\frac{1}{6}} x}{(yx-1)^{\frac{1}{6}}}$$

The solution becomes

$$y = \frac{c_1^6 - 7x^6}{(c_1^6 - x^6)\, x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1^6 - 7x^6}{(c_1^6 - x^6)\, x} \tag{1}$$

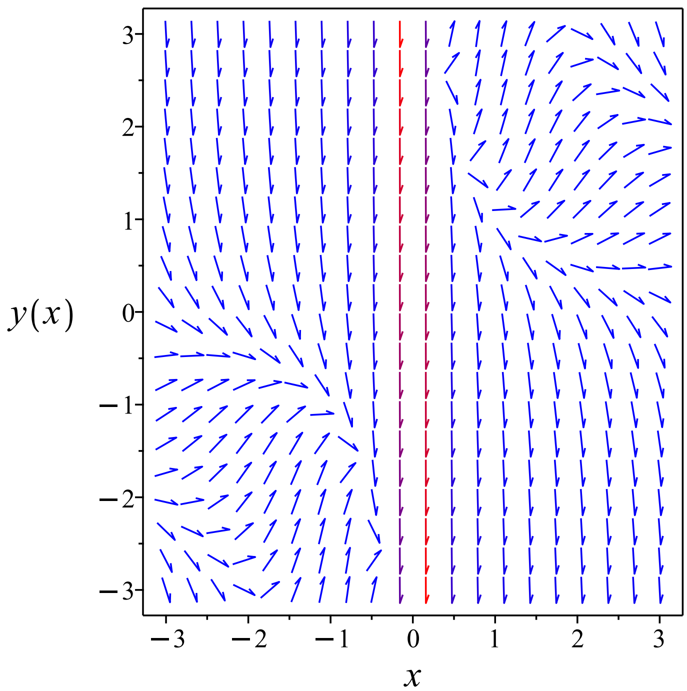


Figure 508: Slope field plot

Verification of solutions

$$y = \frac{c_1^6 - 7x^6}{(c_1^6 - x^6)\, x}$$

Verified OK.

9.48.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x^2y^2-7yx+7}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2+\frac{7y}{x}-\frac{7}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x) = -\frac{7}{x^2}$, $f_1(x) = \frac{7}{x}$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-(f_2'+f_1f_2)\,u'(x)+f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -\frac{7}{x} \\ f_2^2f_0 &= -\frac{7}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x)+\frac{7u'(x)}{x}-\frac{7u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(c_1x^6+c_2)$$

The above shows that

$$u'(x) = 7c_1x^6 + c_2$$

Using the above in (1) gives the solution

$$y = \frac{7c_1x^6 + c_2}{x\,(c_1x^6 + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{7c_3x^6 + 1}{x\,(c_3x^6 + 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{7c_3x^6 + 1}{x\,(c_3x^6 + 1)} \tag{1}$$

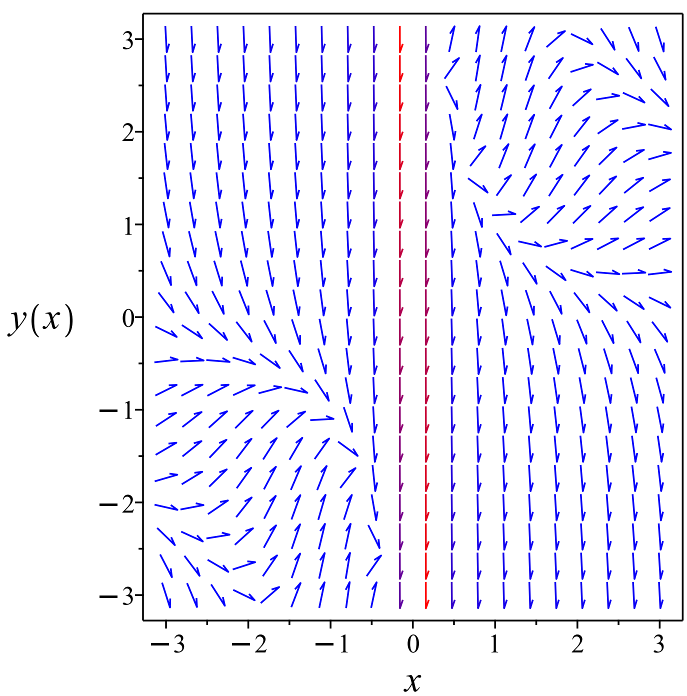


Figure 509: Slope field plot

Verification of solutions

$$y = \frac{7c_3x^6 + 1}{x(c_3x^6 + 1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 24

```
dsolve(x^2*(diff(y(x),x)+y(x)^2)-7*x*y(x)+7=0,y(x), singsol=all)
```

$$y(x) = \frac{-7x^6 + c_1}{x(-x^6 + c_1)}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 34

```
DSolve[x^2*(y'[x]+y[x]^2)-7*x*y[x]+7==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{7x^6 - 6c_1}{x^7 - 6c_1x}$$

$$y(x) \to \frac{1}{x}$$

10 Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262

10.1 problem 1

Internal problem ID [1155]
Internal file name [`OUTPUT/1156_Sunday_June_05_2022_02_03_44_AM_59119923/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \tan(3x)}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \tan(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \sin(3x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(3\cos(3x)) - (\sin(3x))(-3\sin(3x))$$

Which simplifies to

$$W = 3\cos(3x)^2 + 3\sin(3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\sin(3x)\tan(3x)}{3}dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{\sin(3x)\tan(3x)}{3}dx\right)$$

Hence

$$u_1 = \frac{\sin(3x)}{9} - \frac{\ln(\sec(3x) + \tan(3x))}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\tan(3x)}{3}dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(3x)}{3}dx$$

Hence

$$u_2 = -\frac{\cos(3x)}{9}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(3x)}{9} - \frac{\ln(\sec(3x) + \tan(3x))}{9}\right)\cos(3x) - \frac{\cos(3x)\sin(3x)}{9}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(3x)\ln(\sec(3x) + \tan(3x))}{9}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(-\frac{\cos(3x)\ln(\sec(3x) + \tan(3x))}{9}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\cos(3x)\ln(\sec(3x) + \tan(3x))}{9} \tag{1}
$$

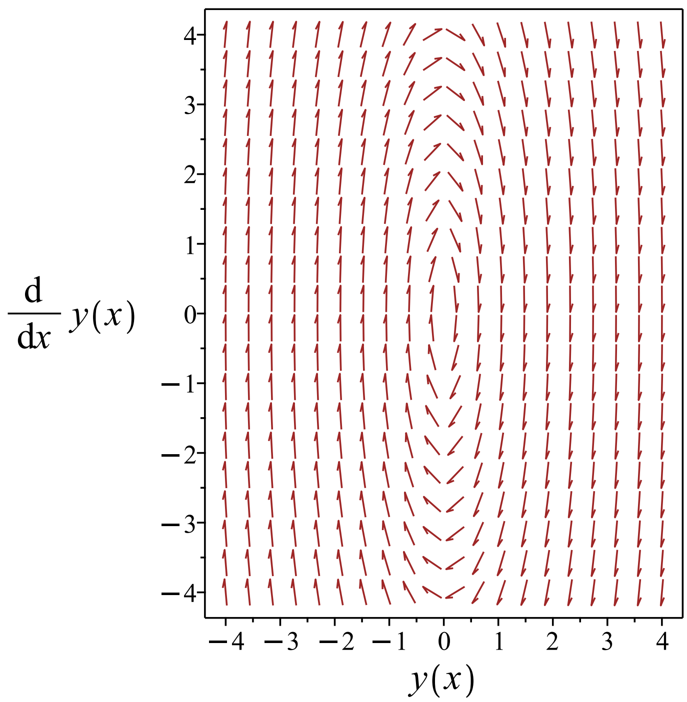


Figure 510: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\cos(3x)\ln(\sec(3x) + \tan(3x))}{9}
$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(3x)\tan(3x)}{3}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(3x)\tan(3x)}{3}dx$$

Hence

$$u_1 = \frac{\sin(3x)}{9} - \frac{\ln(\sec(3x) + \tan(3x))}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\tan(3x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(3x)\,dx$$

Hence

$$u_2 = -\frac{\cos(3x)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(3x)}{9} - \frac{\ln(\sec(3x) + \tan(3x))}{9}\right)\cos(3x) - \frac{\cos(3x)\sin(3x)}{9}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(3x) + \frac{c_2\sin(3x)}{3}\right) + \left(-\frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(3x) + \frac{c_2\sin(3x)}{3} - \frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9} \tag{1}$$

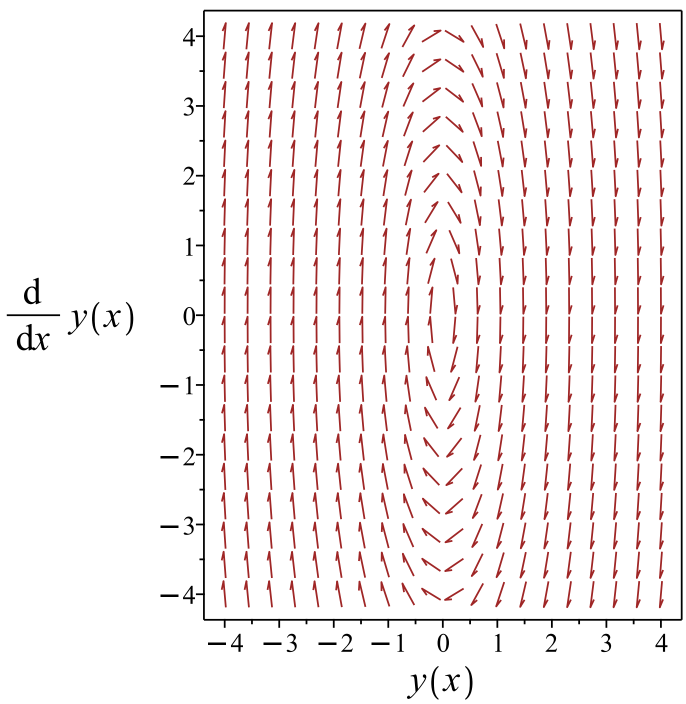


Figure 511: Slope field plot

Verification of solutions

$$y = c_1\cos(3x) + \frac{c_2\sin(3x)}{3} - \frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$y'' + 9y = \tan(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(3x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)\tan(3x)dx\right)}{3} + \frac{\sin(3x)\left(\int \sin(3x)dx\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(3x) + c_2\sin(3x) - \frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+9*y(x)=tan(3*x),y(x), singsol=all)
```

$$y(x) = \sin(3x)\,c_2 + \cos(3x)\,c_1 - \frac{\cos(3x)\ln(\sec(3x)+\tan(3x))}{9}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 33

```
DSolve[y''[x]+9*y[x]==Tan[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{9}\cos(3x)\text{arctanh}(\sin(3x)) + c_1\cos(3x) + c_2\sin(3x)$$

10.2 problem 2

Internal problem ID [1156]
Internal file name [`OUTPUT/1157_Sunday_June_05_2022_02_03_45_AM_2739520/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(2x)\sec(2x)^2}$$

10.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \tan(2x)\sec(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\tan(2x)\sec(2x)}{2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\tan(2x)^2}{2}dx$$

Hence

$$u_1 = -\frac{\tan(2x)}{4} + \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\tan(2x)\sec(2x)}{2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\tan(2x)}{2}dx$$

Hence

$$u_2 = \frac{\ln\left(1+\tan(2x)^2\right)}{8}$$

Which simplifies to

$$u_1 = -\frac{\tan(2x)}{4} + \frac{x}{2}$$

$$u_2 = \frac{\ln\left(\sec(2x)^2\right)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\tan(2x)}{4} + \frac{x}{2}\right)\cos(2x) + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8}$$

Which simplifies to

$$y_p(x) = \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(\frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2} \tag{1}$$

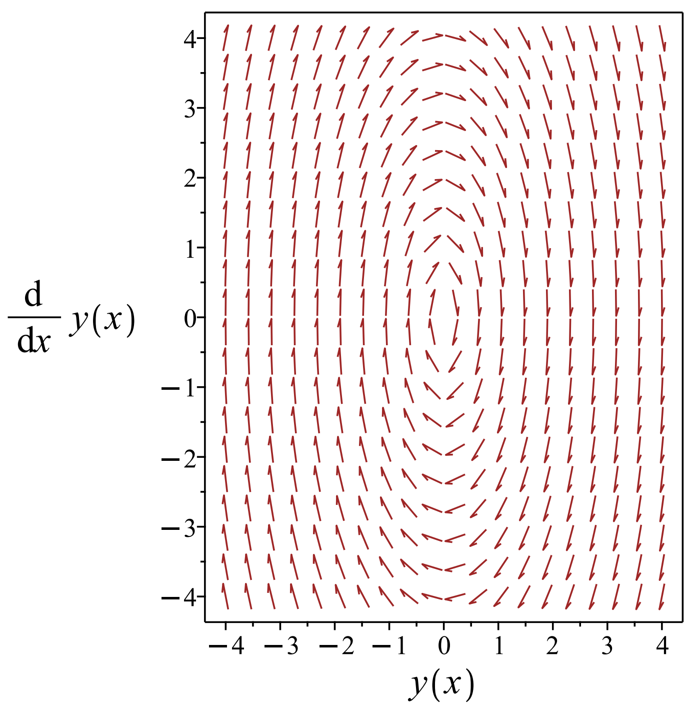


Figure 512: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

Verified OK.

10.2.2 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' + 4y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= 0 \tag{3} \\ C &= 4 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= -4 \\ t &= 1 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\tan(2x)\sec(2x)}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\tan(2x)^2}{2} dx$$

Hence

$$u_1 = -\frac{\tan(2x)}{4} + \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\tan(2x)\sec(2x)}{1} dx$$

Which simplifies to

$$u_2 = \int \tan(2x)\, dx$$

Hence

$$u_2 = \frac{\ln\left(1 + \tan(2x)^2\right)}{4}$$

Which simplifies to

$$u_1 = -\frac{\tan(2x)}{4} + \frac{x}{2}$$
$$u_2 = \frac{\ln\left(\sec(2x)^2\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\tan(2x)}{4} + \frac{x}{2}\right)\cos(2x) + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8}$$

Which simplifies to

$$y_p(x) = \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2} \tag{1}$$

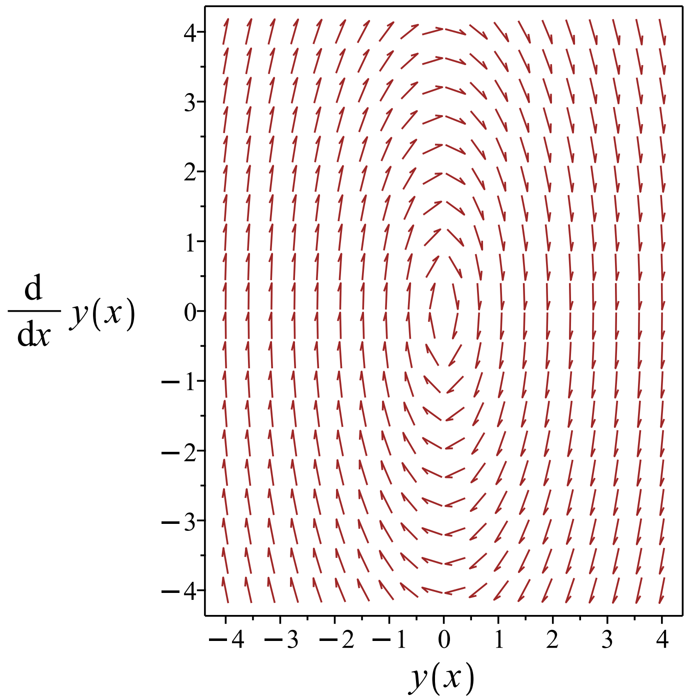


Figure 513: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\ln(\sec(2x)^2) \sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x \cos(2x)}{2}$$

Verified OK.

10.2.3 Maple step by step solution

Let's solve

$y'' + 4y = \tan(2x) \sec(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(2x)\sec(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $$y_p(x) = -\frac{\cos(2x)\left(\int \tan(2x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \tan(2x) dx\right)}{2}$$

  - Compute integrals

    $$y_p(x) = \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\ln\left(\sec(2x)^2\right)\sin(2x)}{8} - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(2*x)*sec(2*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\ln(\sec(2x))\sin(2x)}{4} + \frac{(4c_2 - 1)\sin(2x)}{4} + \frac{\cos(2x)(2c_1 + x)}{2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sin[2*x]*Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + c_1)\cos(2x) + \sin(x)\cos(x)(2\log(\cos(x)) - 1 + 2c_2)$$

10.3 problem 3

Internal problem ID [1157]
Internal file name [`OUTPUT/1158_Sunday_June_05_2022_02_03_46_AM_98592241/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \frac{4}{1 + \mathrm{e}^{-x}}}$$

10.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = \frac{4}{1+\mathrm{e}^{-x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\ &= \frac{3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2\,e^{2x} & e^{x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(e^{x}\right) - \left(e^{x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -e^x e^{2x}$$

Which simplifies to

$$W = -e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^x}{1+e^{-x}}}{-e^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{4\,e^{-2x}}{1+e^{-x}}dx$$

Hence

$$u_1 = -4\,e^{-x} - 4\ln\left(e^x\right) + 4\ln\left(1+e^x\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}}{1+e^{-x}}}{-e^{3x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{4\,e^{-x}}{1+e^{-x}}dx$$

Hence

$$u_2 = 4\ln\left(1+e^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-4\,e^{-x} - 4\ln\left(e^x\right) + 4\ln\left(1+e^x\right)\right)e^{2x} + 4\ln\left(1+e^{-x}\right)e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^x\right) + \left(\left(-4\,e^{-x} - 4\ln\left(e^x\right) + 4\ln\left(1+e^x\right)\right)e^{2x} + 4\ln\left(1+e^{-x}\right)e^x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{x} + \left(-4\,e^{-x} - 4\ln\left(e^{x}\right) + 4\ln\left(1+e^{x}\right)\right) e^{2x} + 4\ln\left(1+e^{-x}\right) e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + \left(-4\,e^{-x} - 4\ln\left(e^{x}\right) + 4\ln\left(1+e^{x}\right)\right) e^{2x} + 4\ln\left(1+e^{-x}\right) e^{x}$$

Verified OK.

10.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x}))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + c_2 e^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2\,e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{x})\left(2\,e^{2x}\right) - \left(e^{2x}\right)(e^{x})$$

Which simplifies to

$$W = e^{x} e^{2x}$$

Which simplifies to

$$W = e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{2x}}{1+e^{-x}}}{e^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{4\,e^{-x}}{1+e^{-x}} dx$$

Hence

$$u_1 = 4\ln\left(1+e^{-x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{x}}{1+e^{-x}}}{e^{3x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4\,e^{-2x}}{1+e^{-x}} dx$$

Hence

$$u_2 = -4\,e^{-x} - 4\ln(e^{x}) + 4\ln(1+e^{x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-4\,e^{-x} - 4\ln(e^{x}) + 4\ln(1+e^{x})\right)e^{2x} + 4\ln\left(1+e^{-x}\right)e^{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x}\right) + \left(\left(-4\,\mathrm{e}^{-x} - 4\ln\left(\mathrm{e}^x\right) + 4\ln\left(1+\mathrm{e}^x\right)\right)\mathrm{e}^{2x} + 4\ln\left(1+\mathrm{e}^{-x}\right)\mathrm{e}^x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} + \left(-4\,\mathrm{e}^{-x} - 4\ln\left(\mathrm{e}^x\right) + 4\ln\left(1+\mathrm{e}^x\right)\right)\mathrm{e}^{2x} + 4\ln\left(1+\mathrm{e}^{-x}\right)\mathrm{e}^x \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} + \left(-4\,\mathrm{e}^{-x} - 4\ln\left(\mathrm{e}^x\right) + 4\ln\left(1+\mathrm{e}^x\right)\right)\mathrm{e}^{2x} + 4\ln\left(1+\mathrm{e}^{-x}\right)\mathrm{e}^x
$$

Verified OK.

10.3.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \frac{4}{1+\mathrm{e}^{-x}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{2x} + y_p(x)$$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{4}{1+e^{-x}}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = e^{3x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -4e^x\left(\int \frac{e^{-x}}{1+e^{-x}}dx\right) + 4e^{2x}\left(\int \frac{e^{-2x}}{1+e^{-x}}dx\right)$$

- Compute integrals

  $$y_p(x) = -4e^x(\ln(e^x)e^x - \ln(1+e^x)e^x - \ln(1+e^{-x}) + 1)$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^x + c_2 e^{2x} - 4e^x(\ln(e^x)e^x - \ln(1+e^x)e^x - \ln(1+e^{-x}) + 1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=4/(1+exp(-x)),y(x), singsol=all)
```

$$y(x) = e^x((4e^x + 4)\ln(1+e^x) + (-4e^x - 4)\ln(e^x) + e^x c_1 + c_2 - 4)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 34

```
DSolve[y''[x]-3*y'[x]+2*y[x]==4/(1+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(8(e^x + 1)\operatorname{arctanh}(2e^x + 1) + c_2e^x - 4 + c_1)$$

10.4 problem 4

Internal problem ID [1158]
Internal file name [`OUTPUT/1159_Sunday_June_05_2022_02_03_48_AM_99963137/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = 3\,\mathrm{e}^{x} \sec(x)}$$

10.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = 3\,\mathrm{e}^{x} \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos{(x)} + c_2 \sin{(x)})$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos{(x)} + c_2 \sin{(x)})$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$
$$y_2 = \sin(x)\,\mathrm{e}^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{x} & \sin(x)\,\mathrm{e}^{x} \\ \frac{d}{dx}(\cos(x)\,\mathrm{e}^{x}) & \frac{d}{dx}(\sin(x)\,\mathrm{e}^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{x} & \sin(x)\,\mathrm{e}^{x} \\ -\sin(x)\,\mathrm{e}^{x} + \cos(x)\,\mathrm{e}^{x} & \cos(x)\,\mathrm{e}^{x} + \sin(x)\,\mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (\cos(x)\,\mathrm{e}^{x})(\cos(x)\,\mathrm{e}^{x} + \sin(x)\,\mathrm{e}^{x}) - (\sin(x)\,\mathrm{e}^{x})(-\sin(x)\,\mathrm{e}^{x} + \cos(x)\,\mathrm{e}^{x})$$

Which simplifies to

$$W = \mathrm{e}^{2x}\cos(x)^2 + \mathrm{e}^{2x}\sin(x)^2$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\sin(x)\,e^{2x}\sec(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 3\tan(x)\,dx$$

Hence

$$u_1 = 3\ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\cos(x)\,e^{2x}\sec(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 3dx$$

Hence

$$u_2 = 3x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3\ln(\cos(x))\cos(x)\,e^{x} + 3\,e^{x}\sin(x)\,x$$

Which simplifies to

$$y_p(x) = 3\,e^{x}(\ln(\cos(x))\cos(x) + \sin(x)\,x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x}(c_1\cos(x) + c_2\sin(x))) + (3\,e^{x}(\ln(\cos(x))\cos(x) + \sin(x)\,x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + 3\,e^{x}(\ln(\cos(x)) \cos(x) + \sin(x)\, x) \tag{1}$$

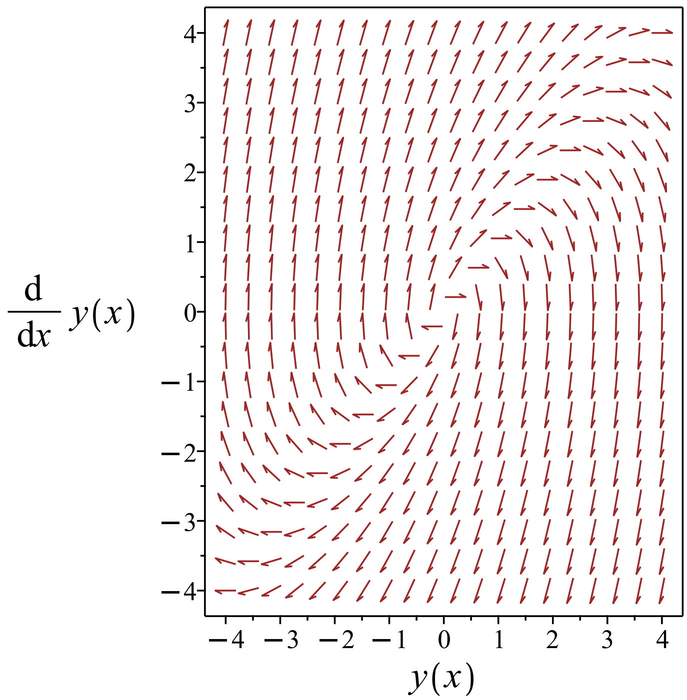


Figure 514: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + 3\,e^{x}(\ln(\cos(x)) \cos(x) + \sin(x)\, x)$$

Verified OK.

10.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\, e^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)\, e^{x}) + c_2(\cos(x)\, e^{x}(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{x} \cos(x)\, c_1 + e^{x} \sin(x)\, c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,e^x$$
$$y_2 = \sin(x)\,e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,e^x & \sin(x)\,e^x \\ \frac{d}{dx}(\cos(x)\,e^x) & \frac{d}{dx}(\sin(x)\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,e^x & \sin(x)\,e^x \\ -\sin(x)\,e^x + \cos(x)\,e^x & \cos(x)\,e^x + \sin(x)\,e^x \end{vmatrix}$$

Therefore

$$W = (\cos(x)\,e^x)(\cos(x)\,e^x + \sin(x)\,e^x) - (\sin(x)\,e^x)(-\sin(x)\,e^x + \cos(x)\,e^x)$$

Which simplifies to

$$W = e^{2x}\cos(x)^2 + e^{2x}\sin(x)^2$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\sin(x)\,e^{2x}\sec(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 3\tan(x)\,dx$$

Hence

$$u_1 = 3\ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\cos(x)\,e^{2x}\sec(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 3dx$$

Hence

$$u_2 = 3x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3\ln(\cos(x))\cos(x)\,e^x + 3\,e^x\sin(x)\,x$$

Which simplifies to

$$y_p(x) = 3\,e^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x\cos(x)\,c_1 + e^x\sin(x)\,c_2) + (3\,e^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)) \end{aligned}$$

Which simplifies to

$$y = e^x(c_1\cos(x) + c_2\sin(x)) + 3\,e^x(\ln(\cos(x))\cos(x) + \sin(x)\,x)$$

Summary

The solution(s) found are the following

$$y = e^x(c_1\cos(x) + c_2\sin(x)) + 3\,e^x(\ln(\cos(x))\cos(x) + \sin(x)\,x) \tag{1}$$

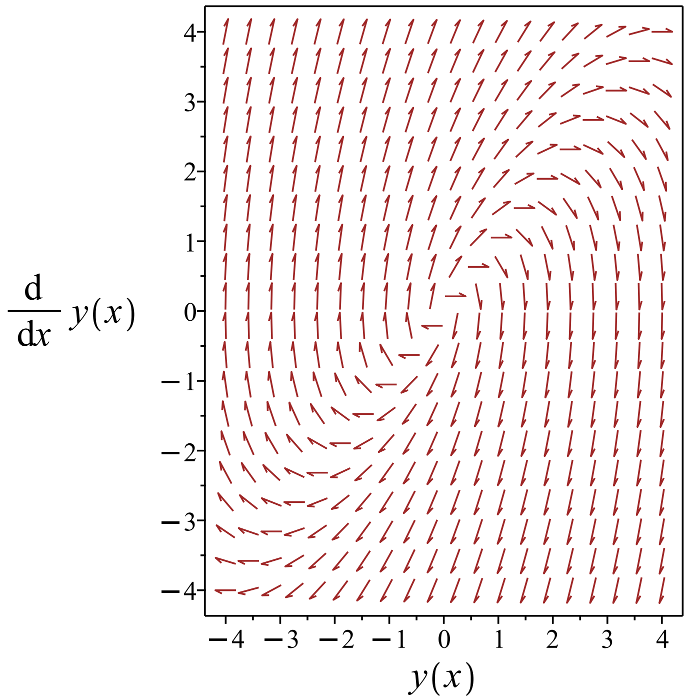


Figure 515: Slope field plot

Verification of solutions

$$y = e^x(c_1\cos(x) + c_2\sin(x)) + 3e^x(\ln(\cos(x))\cos(x) + \sin(x)x)$$

Verified OK.

10.4.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = 3e^x\sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\, \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\, \mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^x \cos(x)\, c_1 + \mathrm{e}^x \sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\, \mathrm{e}^x \sec(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, \mathrm{e}^x & \sin(x)\, \mathrm{e}^x \\ -\sin(x)\, \mathrm{e}^x + \cos(x)\, \mathrm{e}^x & \cos(x)\, \mathrm{e}^x + \sin(x)\, \mathrm{e}^x \end{bmatrix}$$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -3\, \mathrm{e}^x \left(\cos(x) \left(\int \tan(x)\, dx\right) - \sin(x) \left(\int 1 dx\right)\right)$

  ◦ Compute integrals

    $y_p(x) = 3\, \mathrm{e}^x (\ln(\cos(x)) \cos(x) + \sin(x)\, x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x \cos(x)\, c_1 + \mathrm{e}^x \sin(x)\, c_2 + 3\, \mathrm{e}^x (\ln(\cos(x)) \cos(x) + \sin(x)\, x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=3*exp(x)*sec(x),y(x), singsol=all)
```

$$y(x) = e^{x}(\sin(x) c_2 + \cos(x) c_1 + 3x \sin(x) - 3\cos(x) \ln(\sec(x)))$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 30

```
DSolve[y''[x]-2*y'[x]+2*y[x]==3*Exp[x]*Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}((3x + c_1)\sin(x) + \cos(x)(3\log(\cos(x)) + c_2))$$

10.5 problem 5

Internal problem ID [1159]
Internal file name [`OUTPUT/1160_Sunday_June_05_2022_02_03_49_AM_43898604/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 14x^{\frac{3}{2}}\mathrm{e}^{x}}$$

10.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 14x^{\frac{3}{2}}\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 x\, e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^x\\ y_2 &= x\, e^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{x} & x\,e^{x} \\ \frac{d}{dx}(e^{x}) & \frac{d}{dx}(x\,e^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{x} & x\,e^{x} \\ e^{x} & x\,e^{x} + e^{x} \end{vmatrix}$$

Therefore

$$W = (e^{x})(x\,e^{x} + e^{x}) - (x\,e^{x})(e^{x})$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{14x^{\frac{5}{2}} e^{2x}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 14x^{\frac{5}{2}}\,dx$$

Hence

$$u_1 = -4x^{\frac{7}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{14\,\mathrm{e}^{2x} x^{\frac{3}{2}}}{\mathrm{e}^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int 14 x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \frac{28 x^{\frac{5}{2}}}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8 x^{\frac{7}{2}} \mathrm{e}^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}) + \left(\frac{8 x^{\frac{7}{2}} \mathrm{e}^{x}}{5}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x} (c_2 x + c_1) + \frac{8 x^{\frac{7}{2}} \mathrm{e}^{x}}{5}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x} (c_2 x + c_1) + \frac{8 x^{\frac{7}{2}} \mathrm{e}^{x}}{5} \tag{1}$$

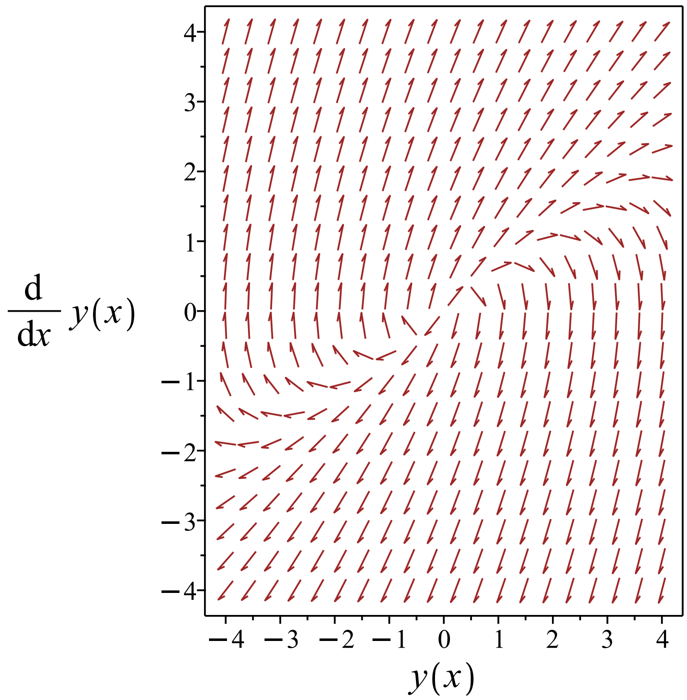


Figure 516: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{8x^{\frac{7}{2}} e^{x}}{5}$$

Verified OK.

10.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 14\,\mathrm{e}^{-x} x^{\frac{3}{2}} \mathrm{e}^{x} \\
\left(\mathrm{e}^{-x} y\right)'' &= 14\,\mathrm{e}^{-x} x^{\frac{3}{2}} \mathrm{e}^{x}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-x} y\right)' = \frac{28x^{\frac{5}{2}}}{5} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-x} y\right) = c_1 x + \frac{8x^{\frac{7}{2}}}{5} + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + \frac{8x^{\frac{7}{2}}}{5} + c_2}{\mathrm{e}^{-x}}
$$

Or

$$
y = \frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5} + c_1 x\,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5} + c_1 x\,\mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}
$$

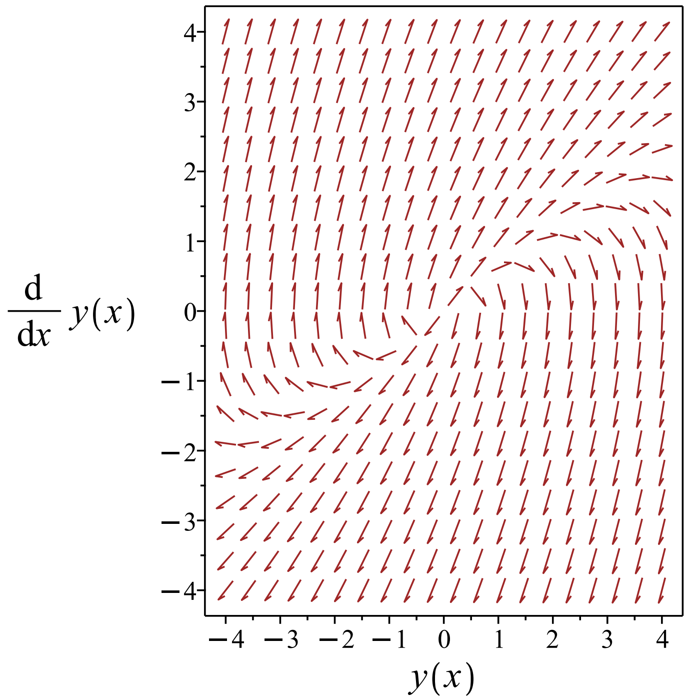


Figure 517: Slope field plot

Verification of solutions

$$y = \frac{8x^{\frac{7}{2}} e^{x}}{5} + c_1 x \,e^{x} + c_2 e^{x}$$

Verified OK.

10.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= x \,\mathrm{e}^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{14x^{\frac{5}{2}}\,e^{2x}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 14x^{\frac{5}{2}}\,dx$$

Hence

$$u_1 = -4x^{\frac{7}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{14\,e^{2x}x^{\frac{3}{2}}}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int 14x^{\frac{3}{2}}\,dx$$

Hence

$$u_2 = \frac{28x^{\frac{5}{2}}}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{x} + c_2 x \, \mathrm{e}^{x}) + \left(\frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \frac{8x^{\frac{7}{2}} \mathrm{e}^{x}}{5} \tag{1}$$

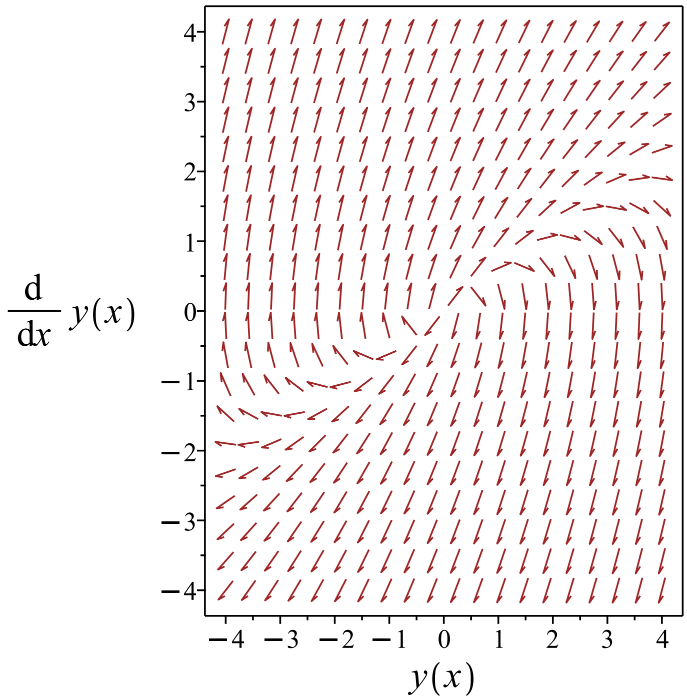


Figure 518: Slope field plot

Verification of solutions

$$y = e^{x}(c_2x + c_1) + \frac{8x^{\frac{7}{2}}e^{x}}{5}$$

Verified OK.

10.5.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 14x^{\frac{3}{2}}e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 14x^{\frac{3}{2}}\mathrm{e}^x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -14\,\mathrm{e}^x\left(-\left(\int x^{\frac{3}{2}}dx\right)x + \int x^{\frac{5}{2}}dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{8x^{\frac{7}{2}}\mathrm{e}^x}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^x + c_1 \mathrm{e}^x + \frac{8x^{\frac{7}{2}}\mathrm{e}^x}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=14*x^(3/2)*exp(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(c_2 + c_1 x + \frac{8x^{\frac{7}{2}}}{5}\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 29

```
DSolve[y''[x]-2*y'[x]+y[x]==14*x^(3/2)*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}e^x\left(8x^{7/2} + 5c_2 x + 5c_1\right)$$

10.6 problem 6

Internal problem ID [1160]
Internal file name [`OUTPUT/1161_Sunday_June_05_2022_02_03_50_AM_15784279/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \frac{4\,\mathrm{e}^{-x}}{1 - \mathrm{e}^{-2x}}}$$

10.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = -\frac{4\,\mathrm{e}^{-x}}{-1+\mathrm{e}^{-2x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(-e^{-x}\right) - \left(e^{-x}\right)(e^x)$$

Which simplifies to

$$W = -2\,e^{-x} e^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{4\,e^{-2x}}{-1+e^{-2x}}}{-2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\,e^{-2x}}{-1+e^{-2x}}dx$$

Hence

$$u_1 = \ln\left(-1+e^{-2x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{4\,e^{x}e^{-x}}{-1+e^{-2x}}}{-2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{-1+e^{-2x}}dx$$

Hence

$$u_2 = \ln\left(e^{-2x}\right) - \ln\left(-1+e^{-2x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln\left(-1+e^{-2x}\right)e^{x} + \left(\ln\left(e^{-2x}\right) - \ln\left(-1+e^{-2x}\right)\right)e^{-x}$$

Which simplifies to

$$y_p(x) = \left(e^{x}-e^{-x}\right)\ln\left(-1+e^{-2x}\right) + e^{-x}\ln\left(e^{-2x}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + \left(\left(e^{x}-e^{-x}\right)\ln\left(-1+e^{-2x}\right) + e^{-x}\ln\left(e^{-2x}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \left(e^x - e^{-x}\right) \ln\left(-1 + e^{-2x}\right) + e^{-x} \ln\left(e^{-2x}\right) \tag{1}$$

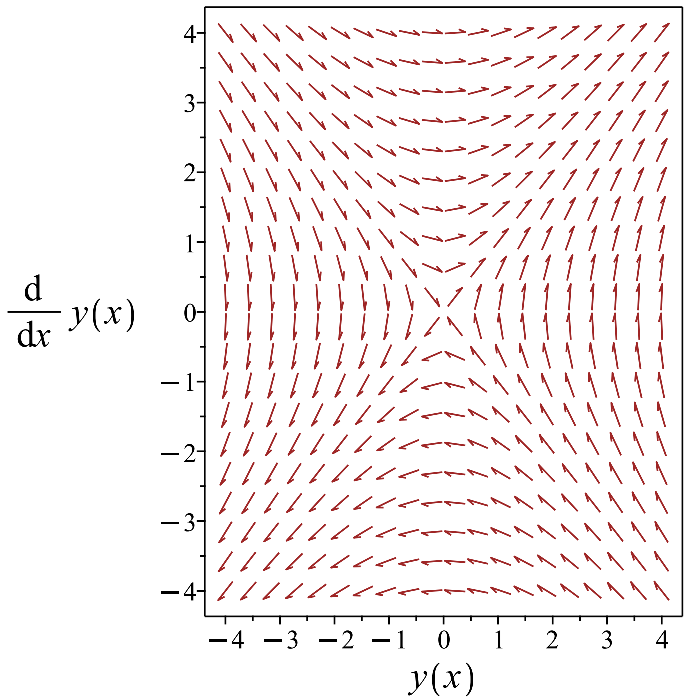


Figure 519: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \left(e^x - e^{-x}\right) \ln\left(-1 + e^{-2x}\right) + e^{-x} \ln\left(e^{-2x}\right)$$

Verified OK.

10.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2\,e^{x} e^{-x}}{-1+e^{-2x}}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2}{-1+e^{-2x}}dx$$

Hence

$$u_1 = \ln\left(e^{-2x}\right) - \ln\left(-1+e^{-2x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{4\,e^{-2x}}{-1+e^{-2x}}}{1}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{4\,e^{-2x}}{-1+e^{-2x}}dx$$

Hence

$$u_2 = 2\ln\left(-1+e^{-2x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln\left(-1+e^{-2x}\right)e^{x} + \left(\ln\left(e^{-2x}\right) - \ln\left(-1+e^{-2x}\right)\right)e^{-x}$$

Which simplifies to

$$y_p(x) = \left(e^{x} - e^{-x}\right)\ln\left(-1+e^{-2x}\right) + e^{-x}\ln\left(e^{-2x}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + \left(\left(e^{x} - e^{-x}\right)\ln\left(-1+e^{-2x}\right) + e^{-x}\ln\left(e^{-2x}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \left(e^{x} - e^{-x}\right) \ln \left(-1 + e^{-2x}\right) + e^{-x} \ln \left(e^{-2x}\right) \tag{1}$$

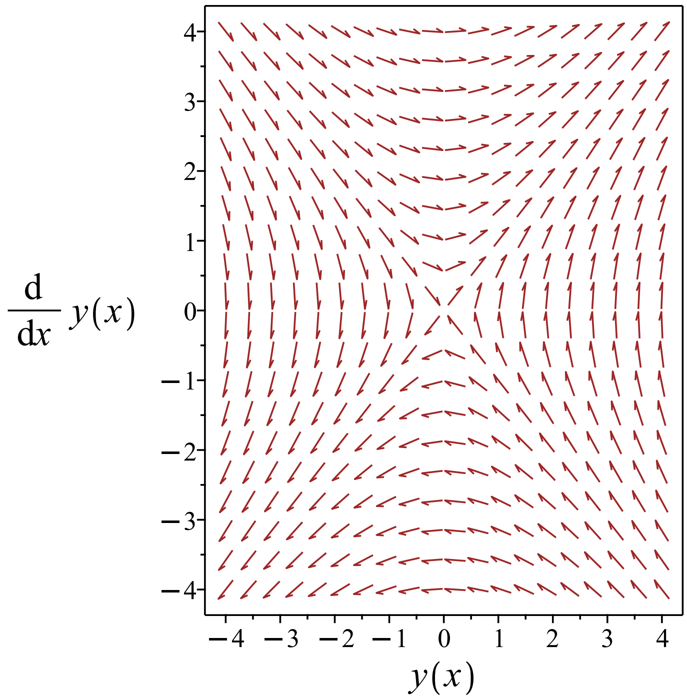


Figure 520: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \left(e^{x} - e^{-x}\right) \ln \left(-1 + e^{-2x}\right) + e^{-x} \ln \left(e^{-2x}\right)$$

Verified OK.

10.6.3 Maple step by step solution

Let's solve

$y'' - y = -\frac{4 e^{-x}}{-1+e^{-2x}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -\frac{4\,\mathrm{e}^{-x}}{-1+\mathrm{e}^{-2x}}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,\mathrm{e}^{-x}\left(\int \frac{1}{-1+\mathrm{e}^{-2x}} dx\right) - 2\,\mathrm{e}^{x}\left(\int \frac{\mathrm{e}^{-2x}}{-1+\mathrm{e}^{-2x}} dx\right)$

  ◦ Compute integrals

  $y_p(x) = (\mathrm{e}^{x} - \mathrm{e}^{-x}) \ln(-1 + \mathrm{e}^{-2x}) + \mathrm{e}^{-x} \ln(\mathrm{e}^{-2x})$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + (\mathrm{e}^{x} - \mathrm{e}^{-x}) \ln(-1 + \mathrm{e}^{-2x}) + \mathrm{e}^{-x} \ln(\mathrm{e}^{-2x})$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)-y(x)=4*exp(-x)/(1-exp(-2*x)),y(x), singsol=all)
```

$$y(x) = \ln\left(1 - e^{-2x}\right) e^{x} + e^{-x} \ln\left(e^{-2x}\right) - e^{-x} \ln\left(-1 + e^{-2x}\right) + c_2 e^{x} + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 67

```
DSolve[y''[x]-y[x]==4*Exp[-x]/(1-Exp[-2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(-2e^{2x} \log\left(e^{x}\right) + \left(e^{2x} - 1\right) \log\left(e^{x} - 1\right) + e^{2x} \log\left(e^{x} + 1\right) - \log\left(e^{x} + 1\right) + c_1 e^{2x} + c_2\right)$$

10.7 problem 7

Internal problem ID [1161]
Internal file name [`OUTPUT/1162_Sunday_June_05_2022_02_03_52_AM_43594281/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + y'x - y = 2x^2 + 2}$$

10.7.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 2x^2 + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + y'x - y = 2x^2 + 2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x(2x^2+2)}{2x}\,dx$$

Which simplifies to

$$u_1 = -\int \left(x^2+1\right) dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2x^2+2}{x}}{2x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2+1}{x^2}dx$$

Hence

$$u_2 = x - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \left(x - \frac{1}{x}\right)x$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{2x^2}{3} - 2 + \frac{c_1}{x} + c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2x^2}{3} - 2 + \frac{c_1}{x} + c_2 x \tag{1}
$$

Verification of solutions

$$
y = \frac{2x^2}{3} - 2 + \frac{c_1}{x} + c_2 x
$$

Verified OK.

10.7.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + y'x - y = 0
$$

In normal form the ode

$$
x^2y'' + y'x - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln\left(x\right)\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'\left(x\right)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= -1\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}\\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^2+2}{x}}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2-1}{x^2} dx$$

Hence

$$u_1 = x - \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^2+2)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int \left(-x^2 - 1\right) dx$$

Hence

$$u_2 = -\frac{1}{3}x^3 - x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \left(x - \frac{1}{x}\right)x$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1x^2 + c_2}{x}\right) + \left(\frac{2x^2}{3} - 2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1x^2 + c_2}{x} + \frac{2x^2}{3} - 2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1x^2 + c_2}{x} + \frac{2x^2}{3} - 2$$

Verified OK.

10.7.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 2x^2 + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{1}{x}\\
q(x) &= -\frac{1}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q}\\
&= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\
&= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2}\\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1) x^2 - ic_2 + c_1}{2x}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + y' x - y = 2x^2 + 2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^2+2}{x}}{-2x} dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2 - 1}{x^2} dx$$

Hence

$$u_1 = x - \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^2+2)}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int (-x^2 - 1)\,dx$$

Hence

$$u_2 = -\frac{1}{3}x^3 - x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \left(x - \frac{1}{x}\right)x$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}\right) + \left(\frac{2x^2}{3} - 2\right) \\ &= \frac{2x^2}{3} - 2 + \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x} \end{aligned}$$

Which simplifies to

$$y = \frac{3c_2x^2i + 3c_1x^2 + 4x^3 - 3ic_2 + 3c_1 - 12x}{6x}$$

Summary
The solution(s) found are the following

$$y = \frac{3c_2x^2i + 3c_1x^2 + 4x^3 - 3ic_2 + 3c_1 - 12x}{6x} \tag{1}$$

Verification of solutions

$$y = \frac{3c_2x^2i + 3c_1x^2 + 4x^3 - 3ic_2 + 3c_1 - 12x}{6x}$$

Verified OK.

10.7.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 2x^2 + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n}{x^2}-\frac{1}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x)+\frac{3v'(x)}{x}=0$$

$$v''(x)+\frac{3v'(x)}{x}=0 \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{3u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x)=-\frac{3}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x)+c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x - y = 2x^2 + 2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^2+2}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2-1}{x^2}dx$$

Hence

$$u_1 = x - \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^2+2)}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int \left(-x^2 - 1\right) dx$$

Hence

$$u_2 = -\frac{1}{3}x^3 - x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \left(x - \frac{1}{x}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{2x^2}{3} - 2\right) \\ &= \frac{2x^2}{3} - 2 + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = -\frac{-6c_2x^2 - 4x^3 + 3c_1 + 12x}{6x}$$

Summary

The solution(s) found are the following

$$y = -\frac{-6c_2x^2 - 4x^3 + 3c_1 + 12x}{6x} \tag{1}$$

Verification of solutions

$$y = -\frac{-6c_2x^2 - 4x^3 + 3c_1 + 12x}{6x}$$

Verified OK.

10.7.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + y'x - y)\,dx = \int (2x^2 + 2)\,dx$$
$$y'x^2 - yx = \frac{2}{3}x^3 + 2x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{2x^3 + 3c_1 + 6x}{3x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{2x^3 + 3c_1 + 6x}{3x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{2x^3 + 3c_1 + 6x}{3x^3}\right)\,dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{2x^3 + 3c_1 + 6x}{3x^3}\,dx$$
$$\frac{y}{x} = \frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x \tag{1}$$

Verification of solutions

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Verified OK.

10.7.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= 2x^2 + 2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3 v'' + \left(3x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2 (u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x) + c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^2+2}{x}}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2-1}{x^2} dx$$

Hence

$$u_1 = x - \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(2x^2+2)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int \left(-x^2 - 1\right) dx$$

Hence

$$u_2 = -\frac{1}{3}x^3 - x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \left(x - \frac{1}{x}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{2x^2}{3} - 2\right) \\ &= \frac{6c_2x^2 + 4x^3 - 3c_1 - 12x}{6x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{6c_2x^2 + 4x^3 - 3c_1 - 12x}{6x} \tag{1}$$

Verification of solutions

$$y = \frac{6c_2x^2 + 4x^3 - 3c_1 - 12x}{6x}$$

Verified OK.

10.7.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + y'x - y = 2x^2 + 2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + y'x - y)\, dx = \int (2x^2 + 2)\, dx$$

$$y'x^2 - yx = \frac{2}{3}x^3 + 2x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$

$$q(x) = \frac{2x^3 + 3c_1 + 6x}{3x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{2x^3 + 3c_1 + 6x}{3x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$

$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right)$$

$$d\left(\frac{y}{x}\right) = \left(\frac{2x^3 + 3c_1 + 6x}{3x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{2x^3 + 3c_1 + 6x}{3x^3} \, dx$$
$$\frac{y}{x} = \frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x \tag{1}$$

Verification of solutions

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Verified OK.

10.7.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + y'x - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0\\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= e^{\int -\frac{1}{2x} dx}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2} \, dx}\\ &= z_1 e^{-\frac{\ln(x)}{2}}\\ &= z_1 \left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + y'x - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x} + \frac{c_2 x}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x(2x^2+2)}{2}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int (x^2+1)\, dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2x^2+2}{x}}{x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x^2+2}{x^2} dx$$

Hence

$$u_2 = 2x - \frac{2}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - x}{x} + \frac{\left(2x - \frac{2}{x}\right)x}{2}$$

Which simplifies to

$$y_p(x) = \frac{2x^2}{3} - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + \left(\frac{2x^2}{3} - 2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{2x^2}{3} - 2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{2x^2}{3} - 2$$

Verified OK.

10.7.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= 2x^2 + 2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y'x^2 - yx = \int 2x^2 + 2\,dx$$

We now have a first order ode to solve which is

$$y'x^2 - yx = \frac{2}{3}x^3 + 2x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{2x^3 + 3c_1 + 6x}{3x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{2x^3 + 3c_1 + 6x}{3x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{2x^3 + 3c_1 + 6x}{3x^2}\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{2x^3 + 3c_1 + 6x}{3x^3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{2x^3 + 3c_1 + 6x}{3x^3}\,dx \\ \frac{y}{x} &= \frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x \tag{1}$$

Verification of solutions

$$y = x\left(\frac{2x}{3} - \frac{2}{x} - \frac{c_1}{2x^2}\right) + c_2 x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=2*x^2+2,y(x), singsol=all)
```

$$y(x) = c_2 x + \frac{2x^2}{3} + \frac{c_1}{x} - 2$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 24

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==2*x^2+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^2}{3} + c_2 x + \frac{c_1}{x} - 2$$

10.8 problem 8

Internal problem ID [1162]
Internal file name [OUTPUT/1163_Sunday_June_05_2022_02_03_53_AM_77332042/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$x^2 y'' + (-2x + 2)\, y' + (-2 + x)\, y = e^{2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(x^2*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=exp(2*x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^2*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True
```

Not solved

10.9 problem 9

Internal problem ID [1163]
Internal file name [`OUTPUT/1164_Sunday_June_05_2022_02_03_55_AM_6679980/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 4\sqrt{x}\,\mathrm{e}^x}$$

10.9.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{1-2x}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int \frac{1-2x}{x}\,dx} \\ &= \sqrt{x}\,\mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{e^{-x}e^{x}}{x} \\
\left(\sqrt{x}\,e^{-x}y\right)'' &= \frac{e^{-x}e^{x}}{x}
\end{aligned}
$$

Integrating once gives

$$
\left(\sqrt{x}\,e^{-x}y\right)' = \ln(x) + c_1
$$

Integrating again gives

$$
\left(\sqrt{x}\,e^{-x}y\right) = x(\ln(x) + c_1 - 1) + c_2
$$

Hence the solution is

$$
y = \frac{x(\ln(x) + c_1 - 1) + c_2}{\sqrt{x}\,e^{-x}}
$$

Or

$$
y = c_1\sqrt{x}\,e^{x} + \sqrt{x}\,e^{x}\ln(x) + \frac{c_2 e^{x}}{\sqrt{x}} - \sqrt{x}\,e^{x}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\,e^{x} + \sqrt{x}\,e^{x}\ln(x) + \frac{c_2 e^{x}}{\sqrt{x}} - \sqrt{x}\,e^{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,e^{x} + \sqrt{x}\,e^{x}\ln(x) + \frac{c_2 e^{x}}{\sqrt{x}} - \sqrt{x}\,e^{x}
$$

Verified OK.

10.9.2 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + (-8x^2 + 4x)\,y' + (4x^2 - 4x - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2} \frac{-8x^2+4x}{4x^2}\, dx} \\
&= z_1 e^{x-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^x}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) + c_2 \left(\frac{\mathrm{e}^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^x}{\sqrt{x}}$$

$$y_2 = \sqrt{x}\, e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\, e^x \\ \frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) & \frac{d}{dx}\left(\sqrt{x}\, e^x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\, e^x \\ -\frac{e^x}{2x^{\frac{3}{2}}} + \frac{e^x}{\sqrt{x}} & \frac{e^x}{2\sqrt{x}} + \sqrt{x}\, e^x \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{x}}{\sqrt{x}}\right)\left(\frac{e^{x}}{2\sqrt{x}} + \sqrt{x}\, e^{x}\right) - \left(\sqrt{x}\, e^{x}\right)\left(-\frac{e^{x}}{2x^{\frac{3}{2}}} + \frac{e^{x}}{\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\, e^{2x}}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\, e^{2x}}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln (x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \sqrt{x}\, e^{x} \ln (x) - \sqrt{x}\, e^{x}$$

Which simplifies to

$$y_p(x) = \sqrt{x}\, e^{x} (\ln(x) - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 e^{x}}{\sqrt{x}} + c_2 \sqrt{x}\, e^{x}\right) + \left(\sqrt{x}\, e^{x} (\ln(x) - 1)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{e^{x} (c_2 x + c_1)}{\sqrt{x}} + \sqrt{x}\, e^{x} (\ln(x) - 1)$$

Summary

The solution(s) found are the following

$$y = \frac{e^{x} (c_2 x + c_1)}{\sqrt{x}} + \sqrt{x}\, e^{x} (\ln(x) - 1) \tag{1}$$

Verification of solutions

$$y = \frac{e^{x} (c_2 x + c_1)}{\sqrt{x}} + \sqrt{x}\, e^{x} (\ln(x) - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(4*x^2*diff(y(x),x$2)+(4*x-8*x^2)*diff(y(x),x)+(4*x^2-4*x-1)*y(x)=4*x^(1/2)*exp(x),y(x
```

$$y(x) = \frac{e^{x}(x\ln(x) + (c_1 - 1)x + c_2)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 27

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==4*x^(1/2)*Exp[x],y[x],x,IncludeSin
```

$$y(x) \to \frac{e^x(x\log(x) + (-1 + c_2)x + c_1)}{\sqrt{x}}$$

10.10 problem 10

Internal problem ID [1164]
Internal file name [`OUTPUT/1165_Sunday_June_05_2022_02_03_56_AM_13240388/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''+4y'x+\left(4x^2+2\right)y=4\,\mathrm{e}^{-x(2+x)}}$$

10.10.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y''+p(x)\,y'+\frac{\left(p(x)^2+p'(x)\right)y}{2}=f(x)$$

Where $p(x)=4x$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x)&=e^{\frac{1}{2}\int p\,dx}\\&=e^{\int 4x\,dx}\\&=\mathrm{e}^{x^2}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\,\mathrm{e}^{x^2}\mathrm{e}^{-x(2+x)} \\
\left(\mathrm{e}^{x^2}y\right)'' &= 4\,\mathrm{e}^{x^2}\mathrm{e}^{-x(2+x)}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{x^2}y\right)' = -2\,\mathrm{e}^{-2x} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{x^2}y\right) = c_1x + \mathrm{e}^{-2x} + c_2
$$

Hence the solution is

$$
y = \frac{c_1x + \mathrm{e}^{-2x} + c_2}{\mathrm{e}^{x^2}}
$$

Or

$$
y = c_1x\,\mathrm{e}^{-x^2} + c_2\mathrm{e}^{-x^2} + \mathrm{e}^{-x^2}\mathrm{e}^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,\mathrm{e}^{-x^2} + c_2\mathrm{e}^{-x^2} + \mathrm{e}^{-x^2}\mathrm{e}^{-2x} \tag{1}
$$

Verification of solutions

$$
y = c_1x\,\mathrm{e}^{-x^2} + c_2\mathrm{e}^{-x^2} + \mathrm{e}^{-x^2}\mathrm{e}^{-2x}
$$

Verified OK.

10.10.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + 4y'x + \left(4x^2+2\right)y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4x \\
C &= 4x^2+2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y'x + \left(4x^2 + 2\right) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x^2} + c_2 x\, \mathrm{e}^{-x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x^2}$$
$$y_2 = x\,\mathrm{e}^{-x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & x\,\mathrm{e}^{-x^2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & x\,\mathrm{e}^{-x^2} \\ -2x\,\mathrm{e}^{-x^2} & \mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x^2}\right)\left(\mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2}\right) - \left(x\,\mathrm{e}^{-x^2}\right)\left(-2x\,\mathrm{e}^{-x^2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x^2}$$

Which simplifies to

$$W = e^{-2x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\, e^{-x^2} e^{-x(2+x)}}{e^{-2x^2}} \, dx$$

Which simplifies to

$$u_1 = -\int 4\, e^{-2x} x dx$$

Hence

$$u_1 = (1 + 2x)\, e^{-2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\, e^{-x^2} e^{-x(2+x)}}{e^{-2x^2}} \, dx$$

Which simplifies to

$$u_2 = \int 4\, e^{-2x} dx$$

Hence

$$u_2 = -2\, e^{-2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (1 + 2x)\, e^{-2x} e^{-x^2} - 2\, e^{-2x} x\, e^{-x^2}$$

Which simplifies to

$$y_p(x) = e^{-x(2+x)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x^2} + c_2 x\, e^{-x^2}\right) + \left(e^{-x(2+x)}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x^2}(c_2 x + c_1) + e^{-x(2+x)}$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}(c_2 x + c_1) + e^{-x(2+x)} \tag{1}$$

Verification of solutions

$$y = e^{-x^2}(c_2 x + c_1) + e^{-x(2+x)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=4*exp(-x*(x+2)),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2) e^{-x^2} + e^{-x(2+x)}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 27

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==4*x^(1/2)*Exp[x],y[x],x,IncludeSin
```

$$y(x) \to \frac{e^x(x\log(x) + (-1 + c_2)x + c_1)}{\sqrt{x}}$$

10.11 problem 11

Internal problem ID [1165]
Internal file name [`OUTPUT/1166_Sunday_June_05_2022_02_03_57_AM_75415380/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4y'x + 6y = x^{\frac{5}{2}}}$$

10.11.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^{\frac{5}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 4y'x + 6y = x^{\frac{5}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{11}{2}}}{x^6}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{\sqrt{x}}dx$$

Hence

$$u_1 = -2\sqrt{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{9}{2}}}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^{\frac{3}{2}}}dx$$

Hence

$$u_2 = -\frac{2}{\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -4x^{\frac{5}{2}} + c_2x^3 + c_1x^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -4x^{\frac{5}{2}} + c_2x^3 + c_1x^2 \tag{1}$$

Verification of solutions

$$y = -4x^{\frac{5}{2}} + c_2x^3 + c_1x^2$$

Verified OK.

10.11.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= \frac{1}{x^{\frac{3}{2}}} \\
\left(\frac{y}{x^2}\right)'' &= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = -\frac{2}{\sqrt{x}} + c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x - 4\sqrt{x} + c_2$$

Hence the solution is

$$y = \frac{c_1x - 4\sqrt{x} + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2 - 4x^{\frac{5}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1x^3 + c_2x^2 - 4x^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = c_1x^3 + c_2x^2 - 4x^{\frac{5}{2}}$$

Verified OK.

10.11.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}\,dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}\,dx \\
&= \int e^{4\ln(x)}\,dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{2}{5} \\ r_2 &= \frac{3}{5} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{5}} + c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^{\frac{5}{2}}}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_1 = -\frac{2(x^5)^{\frac{3}{5}}}{x^{\frac{5}{2}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^{\frac{5}{2}}}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = -\frac{2(x^5)^{\frac{2}{5}}}{x^{\frac{5}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}\right) + \left(-4x^{\frac{5}{2}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - 4x^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - 4x^{\frac{5}{2}}$$

Verified OK.

10.11.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^{\frac{5}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c\sqrt{6}}{6}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Now the particular solution to this ODE is found

$$x^2 y'' - 4y'x + 6y = x^{\frac{5}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^5)^{\frac{2}{5}}$$
$$y_2 = (x^5)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{2}{5}}\right)\left(\frac{3x^4}{(x^5)^{\frac{2}{5}}}\right) - \left((x^5)^{\frac{3}{5}}\right)\left(\frac{2x^4}{(x^5)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}\, x^{\frac{5}{2}}}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_1 = -\frac{2(x^5)^{\frac{3}{5}}}{x^{\frac{5}{2}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}\, x^{\frac{5}{2}}}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = -\frac{2(x^5)^{\frac{2}{5}}}{x^{\frac{5}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(-4x^{\frac{5}{2}}\right) \\ &= -4x^{\frac{5}{2}} + x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = x^{\frac{5}{2}}\left(-4 + c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = x^{\frac{5}{2}}\left(-4 + c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = x^{\frac{5}{2}}\left(-4 + c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Verified OK.

10.11.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^{\frac{3}{2}} v''(x) = 1$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{1}{x^{\frac{3}{2}}}$$

Integrating once gives

$$v'(x) = -\frac{2}{\sqrt{x}} + c_1$$

Integrating again gives

$$v(x) = -4\sqrt{x} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 x - 4\sqrt{x} + c_2\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = \left(c_1 x - 4\sqrt{x} + c_2\right) x^2$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 x - 4\sqrt{x} + c_2\right) x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{2}{5}}\right)\left(\frac{3x^4}{(x^5)^{\frac{2}{5}}}\right) - \left((x^5)^{\frac{3}{5}}\right)\left(\frac{2x^4}{(x^5)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}\, x^{\frac{5}{2}}}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_1 = -\frac{2(x^5)^{\frac{3}{5}}}{x^{\frac{5}{2}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^{\frac{5}{2}}}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = -\frac{2(x^5)^{\frac{2}{5}}}{x^{\frac{5}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 x - 4\sqrt{x} + c_2\right) x^2\right) + \left(-4x^{\frac{5}{2}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(c_1 x - 4\sqrt{x} + c_2\right) x^2 - 4x^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = \left(c_1 x - 4\sqrt{x} + c_2\right) x^2 - 4x^{\frac{5}{2}}$$

Verified OK.

10.11.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^{\frac{5}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^3 \\
&= (c_2 x - c_1)\, x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 4y'x + 6y = x^{\frac{5}{2}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{11}{2}}}{x^6}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{\sqrt{x}}dx$$

Hence

$$u_1 = -2\sqrt{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{9}{2}}}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^{\frac{3}{2}}}dx$$

Hence

$$u_2 = -\frac{2}{\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^3\right) + \left(-4x^{\frac{5}{2}}\right) \\ &= -4x^{\frac{5}{2}} + \left(-\frac{c_1}{x} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = -4x^{\frac{5}{2}} + c_2 x^3 - c_1 x^2$$

Summary
The solution(s) found are the following

$$y = -4x^{\frac{5}{2}} + c_2 x^3 - c_1 x^2 \tag{1}$$

Verification of solutions

$$y = -4x^{\frac{5}{2}} + c_2 x^3 - c_1 x^2$$

Verified OK.

10.11.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2\right) + c_2 \left(x^2(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 4y'x + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x^3 + c_1 x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{11}{2}}}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{\sqrt{x}} dx$$

Hence

$$u_1 = -2\sqrt{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{9}{2}}}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = -\frac{2}{\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4x^{\frac{5}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x^3 + c_1 x^2) + \left(-4x^{\frac{5}{2}}\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_2 x + c_1) - 4x^{\frac{5}{2}}$$

Summary

The solution(s) found are the following

$$y = x^2(c_2 x + c_1) - 4x^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = x^2(c_2 x + c_1) - 4x^{\frac{5}{2}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=x^(5/2),y(x), singsol=all)
```

$$y(x) = c_2 x^3 + c_1 x^2 - 4x^{\frac{5}{2}}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==x^(5/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2\left(-4\sqrt{x} + c_2 x + c_1\right)$$

10.12 problem 12

Internal problem ID [1166]
Internal file name [`OUTPUT/1167_Sunday_June_05_2022_02_03_58_AM_19511484/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + 3y = 2\sin(x)\, x^4}$$

10.12.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2\sin(x)\, x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r - 3r\, x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3y'x + 3y = 2\sin(x)\, x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^2 dx$$

Hence

$$u_1 = x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x\right) x - x^3 \cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x)\, x + \cos(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x^2 - 2\sin(x)\, x - 2\cos(x) + c_1\right) x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = (c_2x^2 - 2\sin(x)\,x - 2\cos(x) + c_1)\,x \tag{1}$$

Verification of solutions

$$y = (c_2x^2 - 2\sin(x)\,x - 2\cos(x) + c_1)\,x$$

Verified OK.

10.12.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 3y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{3}{x} dx\right)} dx \\
&= \int e^{3\ln(x)}\, dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{x^2}}{x^6} \\
&= \frac{3}{x^8} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^4\right)^{\frac{1}{4}}$$
$$y_2 = \left(x^4\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^4\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^4\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{x^3}{\left(x^4\right)^{\frac{3}{4}}} & \frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^4\right)^{\frac{1}{4}}\right)\left(\frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}}\right) - \left(\left(x^4\right)^{\frac{3}{4}}\right)\left(\frac{x^3}{\left(x^4\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x^4)^{\frac{3}{4}} \sin(x)\, x^4}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}} \sin(x)}{x} dx$$

Hence

$$u_1 = \frac{(x^4)^{\frac{3}{4}} (x^2 - 2) \cos(x)}{x^3} - \frac{2(x^4)^{\frac{3}{4}} \sin(x)}{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(x^4)^{\frac{1}{4}} \sin(x)\, x^4}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} \sin(x)}{x} dx$$

Hence

$$u_2 = -\frac{(x^4)^{\frac{1}{4}} \cos(x)}{x}$$

Which simplifies to

$$u_1 = \frac{(x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x) (x^4)^{\frac{3}{4}}}{x^3}$$

$$u_2 = -\frac{(x^4)^{\frac{1}{4}} \cos(x)}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x)\, x - x^3 \cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x)\,x + \cos(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}\right) + (-2x(\sin(x)\,x + \cos(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4} - 2x(\sin(x)\,x + \cos(x)) \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4} - 2x(\sin(x)\,x + \cos(x))$$

Verified OK.

10.12.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2\sin(x)\,x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 3y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{3}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{4c\sqrt{3}}{3} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau} y(\tau)\right)}{3} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}} \left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + i c_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right) \right)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c} \sqrt{q}\, dx \\ &= \frac{\int \sqrt{3}\sqrt{\frac{1}{x^2}} dx}{c} \\ &= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{\left(\left(i c_2 + c_1\right) x^2 - i c_2 + c_1\right) x}{2}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3y'x + 3y = 2\sin(x)\, x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \left(x^4\right)^{\frac{1}{4}} \\ y_2 &= \left(x^4\right)^{\frac{3}{4}} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{d}{dx}\left((x^4)^{\frac{1}{4}}\right) & \frac{d}{dx}\left((x^4)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{x^3}{(x^4)^{\frac{3}{4}}} & \frac{3x^3}{(x^4)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((x^4)^{\frac{1}{4}}\right)\left(\frac{3x^3}{(x^4)^{\frac{1}{4}}}\right) - \left((x^4)^{\frac{3}{4}}\right)\left(\frac{x^3}{(x^4)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x^4)^{\frac{3}{4}} \sin(x)\, x^4}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}} \sin(x)}{x} dx$$

Hence

$$u_1 = \frac{(x^4)^{\frac{3}{4}} (x^2 - 2) \cos(x)}{x^3} - \frac{2(x^4)^{\frac{3}{4}} \sin(x)}{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(x^4)^{\frac{1}{4}} \sin(x) x^4}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} \sin(x)}{x} dx$$

Hence

$$u_2 = -\frac{(x^4)^{\frac{1}{4}} \cos(x)}{x}$$

Which simplifies to

$$u_1 = \frac{(x^2 \cos(x) - 2\cos(x) - 2\sin(x) x) (x^4)^{\frac{3}{4}}}{x^3}$$

$$u_2 = -\frac{(x^4)^{\frac{1}{4}} \cos(x)}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (x^2 \cos(x) - 2\cos(x) - 2\sin(x) x) x - x^3 \cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x) x + \cos(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{((ic_2 + c_1) x^2 - ic_2 + c_1) x}{2}\right) + (-2x(\sin(x) x + \cos(x))) \\ &= -2x(\sin(x) x + \cos(x)) + \frac{((ic_2 + c_1) x^2 - ic_2 + c_1) x}{2} \end{aligned}$$

Which simplifies to

$$y = \frac{(-4\cos(x) - 4\sin(x)x + (ic_2 + c_1)x^2 - ic_2 + c_1)x}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(-4\cos(x) - 4\sin(x)x + (ic_2 + c_1)x^2 - ic_2 + c_1)x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(-4\cos(x) - 4\sin(x)x + (ic_2 + c_1)x^2 - ic_2 + c_1)x}{2}$$

Verified OK.

10.12.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = 2\sin(x)x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 3y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right)x^3 \\
&= c_2x^3 - \frac{1}{2}c_1x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 3y'x + 3y = 2\sin(x)\,x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= x^3
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^2 dx$$

Hence

$$u_1 = x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x\right) x - x^3 \cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x)\, x + \cos(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x^3\right) + (-2x(\sin(x)\, x + \cos(x))) \\ &= -2x(\sin(x)\, x + \cos(x)) + \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = x\left(-2\sin(x)\, x - 2\cos(x) - \frac{c_1}{2} + c_2 x^2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(-2\sin(x)\, x - 2\cos(x) - \frac{c_1}{2} + c_2 x^2\right) \tag{1}$$

Verification of solutions

$$y = x\left(-2\sin(x)\, x - 2\cos(x) - \frac{c_1}{2} + c_2 x^2\right)$$

Verified OK.

10.12.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \\ F &= 2\sin(x)\,x^4 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-3x)(-3) + (3)(-3x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-3x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-3x^2(u'(x)\, x - u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{1}{x}\, dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int c_1 x \,\mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-3x)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{3x(c_1 x^2 + 2c_2)}{2}
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^2 dx$$

Hence

$$u_1 = x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5 \sin(x)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x\right) x - x^3 \cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x)\, x + \cos(x))$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{3x(c_1 x^2 + 2c_2)}{2}\right) + (-2x(\sin(x)\, x + \cos(x))) \\ &= x\left(-\frac{3c_1 x^2}{2} - 3c_2 - 2\sin(x)\, x - 2\cos(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\left(-\frac{3c_1x^2}{2} - 3c_2 - 2\sin(x)\,x - 2\cos(x)\right) \tag{1}$$

Verification of solutions

$$y = x\left(-\frac{3c_1x^2}{2} - 3c_2 - 2\sin(x)\,x - 2\cos(x)\right)$$

Verified OK.

10.12.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2}} \\ &= z_1 \left(x^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 3y'x + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + \frac{1}{2} c_2 x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{x^3}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ 1 & \frac{3x^2}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{3x^2}{2}\right) - \left(\frac{x^3}{2}\right)(1)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7 \sin(x)}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^2 dx$$

Hence

$$u_1 = x^2 \cos(x) - 2\cos(x) - 2\sin(x)\, x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5 \sin(x)}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 2\sin(x)\,dx$$

Hence

$$u_2 = -2\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (x^2\cos(x) - 2\cos(x) - 2\sin(x)\,x)\,x - x^3\cos(x)$$

Which simplifies to

$$y_p(x) = -2x(\sin(x)\,x + \cos(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1x + \frac{1}{2}c_2x^3\right) + (-2x(\sin(x)\,x + \cos(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + \frac{c_2x^3}{2} - 2x(\sin(x)\,x + \cos(x)) \tag{1}$$

Verification of solutions

$$y = c_1x + \frac{c_2x^3}{2} - 2x(\sin(x)\,x + \cos(x))$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+3*y(x)=2*x^4*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(c_1 x^2 - 4x\sin(x) - 4\cos(x) + 2c_2)\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-3*x*y'[x]+3*y[x]==2*x^4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(c_2 x^2 - 2x\sin(x) - 2\cos(x) + c_1\right)$$

10.13 problem 13

Internal problem ID [1167]
Internal file name [OUTPUT/1168_Sunday_June_05_2022_02_03_59_AM_11180758/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(1+2x)\,y''-2y'-(2x+3)\,y=(1+2x)^2\,\mathrm{e}^{-x}}$$

10.13.1 Solving using Kovacic algorithm

Writing the ode as

$$(1+2x)\,y''-2y'+(-3-2x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1+2x \\ B &= -2 \\ C &= -3-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (1 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(1 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x + \frac{1}{2}\right)^2} + \frac{1}{x + \frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} + (-)\,(1) \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{1+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2 + 8x + 6}{(1+2x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{1+2x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1+2x}\, dx} \\
&= z_1 e^{\frac{\ln(1+2x)}{2}} \\
&= z_1\left(\sqrt{1+2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1+2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x\, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(1 + 2x)\, y'' - 2y' + (-3 - 2x)\, y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{x} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{x} \\ -\mathrm{e}^{-x} & x\,\mathrm{e}^{x} + \mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(x\,\mathrm{e}^{x} + \mathrm{e}^{x}\right) - \left(x\,\mathrm{e}^{x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = 2x\,\mathrm{e}^{x}\mathrm{e}^{-x} + \mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$W = 1 + 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,\mathrm{e}^{x}(1+2x)^2\,\mathrm{e}^{-x}}{(1+2x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int x dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-2x}(1+2x)^2}{(1+2x)^2}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-2x} dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{-2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2\mathrm{e}^{-x}}{2} - \frac{\mathrm{e}^{-2x}x\,\mathrm{e}^{x}}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^{-x}x(x+1)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2x\,\mathrm{e}^{x}\right) + \left(-\frac{\mathrm{e}^{-x}x(x+1)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 x e^{x} - \frac{e^{-x} x(x+1)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 x e^{x} - \frac{e^{-x} x(x+1)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve((2*x+1)*diff(y(x),x$2)-2*diff(y(x),x)-(2*x+3)*y(x)=(2*x+1)^2*exp(-x),y(x), singsol=al
```

$$y(x) = \frac{(-x^2 + 2c_2 - x) e^{-x}}{2} + x e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 42

```
DSolve[(2*x+1)*y''[x]-2*y'[x]-(2*x+3)*y[x]==(2*x+1)^2*Exp[-x],y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{1}{2}e^{-x}x(x+1) + c_1e^{-x-\frac{1}{2}} + c_2e^{x+\frac{1}{2}}x$$

10.14 problem 14

Internal problem ID [1168]
Internal file name [`OUTPUT/1169_Sunday_June_05_2022_02_04_02_AM_60845432/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{2xy'' + 2y' + 2y = \sin\left(\sqrt{x}\right)}$$

10.14.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + yx = \frac{x \sin\left(\sqrt{x}\right)}{2} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 0 \\ \beta &= 2 \\ n &= 0 \\ \gamma &= \frac{1}{2} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) \\ y_2 &= \operatorname{BesselY}\left(0, 2\sqrt{x}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \text{BesselJ}\left(0, 2\sqrt{x}\right) & \text{BesselY}\left(0, 2\sqrt{x}\right) \\ \frac{d}{dx}\left(\text{BesselJ}\left(0, 2\sqrt{x}\right)\right) & \frac{d}{dx}\left(\text{BesselY}\left(0, 2\sqrt{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \text{BesselJ}\left(0, 2\sqrt{x}\right) & \text{BesselY}\left(0, 2\sqrt{x}\right) \\ -\frac{\text{BesselJ}\left(1, 2\sqrt{x}\right)}{\sqrt{x}} & -\frac{\text{BesselY}\left(1, 2\sqrt{x}\right)}{\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\text{BesselJ}\left(0, 2\sqrt{x}\right)\right)\left(-\frac{\text{BesselY}\left(1, 2\sqrt{x}\right)}{\sqrt{x}}\right) - \left(\text{BesselY}\left(0, 2\sqrt{x}\right)\right)\left(-\frac{\text{BesselJ}\left(1, 2\sqrt{x}\right)}{\sqrt{x}}\right)$$

Which simplifies to

$$W = -\frac{\text{BesselJ}\left(0, 2\sqrt{x}\right)\text{BesselY}\left(1, 2\sqrt{x}\right) - \text{BesselY}\left(0, 2\sqrt{x}\right)\text{BesselJ}\left(1, 2\sqrt{x}\right)}{\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{x\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\text{BesselY}\left(0, 2\sqrt{x}\right) x \sin\left(\sqrt{x}\right)}{2}}{\frac{x}{\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\text{BesselY}\left(0, 2\sqrt{x}\right)\sin\left(\sqrt{x}\right)\pi}{2} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\text{BesselY}\left(0, 2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)\pi}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\text{BesselJ}\left(0,2\sqrt{x}\right)x\sin\left(\sqrt{x}\right)}{2}}{\frac{x}{\pi}}dx$$

Which simplifies to

$$u_2 = \int \frac{\text{BesselJ}\left(0,2\sqrt{x}\right)\sin\left(\sqrt{x}\right)\pi}{2}dx$$

Hence

$$u_2 = \int_0^x \frac{\text{BesselJ}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)\pi}{2}d\alpha$$

Which simplifies to

$$u_1 = -\frac{\pi\left(\int_0^x \text{BesselY}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)}{2}$$
$$u_2 = \frac{\pi\left(\int_0^x \text{BesselJ}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\pi\left(\int_0^x \text{BesselY}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{BesselJ}\left(0,2\sqrt{x}\right)}{2} + \frac{\pi\left(\int_0^x \text{BesselJ}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{BesselY}\left(0,2\sqrt{x}\right)}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\pi\left(\left(\int_0^x \text{BesselY}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{BesselJ}\left(0,2\sqrt{x}\right) - \left(\int_0^x \text{BesselJ}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{BesselY}\right.}{2}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1\,\text{BesselJ}\left(0,2\sqrt{x}\right) + c_2\,\text{BesselY}\left(0,2\sqrt{x}\right)\right) + \left(-\frac{\pi\left(\left(\int_0^x \text{BesselY}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{BesselJ}\left(0,2\sqrt{x}\right) - \left(\int_0^x \text{BesselJ}\left(0,2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\text{Be}\right.}{2}\right.$$

Summary

The solution(s) found are the following

$$y = c_1 \,\mathrm{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \,\mathrm{BesselY}\left(0, 2\sqrt{x}\right) - \frac{\pi\left(\left(\int_0^x \mathrm{BesselY}\left(0, 2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\mathrm{BesselJ}\left(0, 2\sqrt{x}\right) - \left(\int_0^x \mathrm{BesselJ}\left(0, 2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\mathrm{BesselY}\right.}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \,\mathrm{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \,\mathrm{BesselY}\left(0, 2\sqrt{x}\right) - \frac{\pi\left(\left(\int_0^x \mathrm{BesselY}\left(0, 2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\mathrm{BesselJ}\left(0, 2\sqrt{x}\right) - \left(\int_0^x \mathrm{BesselJ}\left(0, 2\sqrt{\alpha}\right)\sin\left(\sqrt{\alpha}\right)d\alpha\right)\mathrm{BesselY}\right.}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
dsolve(2*x*diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=sin(sqrt(x)),y(x), singsol=all)
```

$$y(x) = \text{BesselJ}\left(0, 2\sqrt{x}\right) c_2 + \text{BesselY}\left(0, 2\sqrt{x}\right) c_1 + \frac{\pi\left(\int \text{BesselJ}\left(0, 2\sqrt{x}\right) \sin\left(\sqrt{x}\right) dx\right) \text{BesselY}\left(0, 2\sqrt{x}\right)}{2} - \frac{\pi\left(\int \text{BesselY}\left(0, 2\sqrt{x}\right) \sin\left(\sqrt{x}\right) dx\right) \text{BesselJ}\left(0, 2\sqrt{x}\right)}{2}$$

✓ Solution by Mathematica

Time used: 11.396 (sec). Leaf size: 110

```
DSolve[2*x*y''[x]+2*y'[x]+2*y[x]==Sin[Sqrt[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{BesselJ}\left(0, 2\sqrt{x}\right) \int_1^x -\frac{1}{2}\pi \text{BesselY}\left(0, 2\sqrt{K[1]}\right) \sin\left(\sqrt{K[1]}\right) dK[1] + 2\,\text{BesselY}\left(0, 2\sqrt{x}\right) \int_1^x \frac{1}{4}\pi \text{BesselJ}\left(0, 2\sqrt{K[2]}\right) \sin\left(\sqrt{K[2]}\right) dK[2] + c_1 \text{BesselJ}\left(0, 2\sqrt{x}\right) + 2c_2 \text{BesselY}\left(0, 2\sqrt{x}\right)$$

10.15 problem 15

Internal problem ID [1169]
Internal file name [OUTPUT/1170_Sunday_June_05_2022_02_04_03_AM_35419903/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - (2+2x)\, y' + (2+x)\, y = 6\,\mathrm{e}^{x} x^{3}}$$

10.15.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x-2)\, y' + (2+x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, e^x) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' + (-2x - 2)\, y' + (2 + x)\, y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + \frac{c_2 \mathrm{e}^x x^3}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= \frac{\mathrm{e}^x x^3}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{e^x x^3}{3} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{e^x x^3}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{e^x x^3}{3} \\ e^x & \frac{e^x x^3}{3} + x^2 e^x \end{vmatrix}$$

Therefore

$$W = (e^x)\left(\frac{e^x x^3}{3} + x^2 e^x\right) - \left(\frac{e^x x^3}{3}\right)(e^x)$$

Which simplifies to

$$W = x^2 e^{2x}$$

Which simplifies to

$$W = x^2 e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2 e^{2x} x^6}{e^{2x} x^3}\, dx$$

Which simplifies to

$$u_1 = -\int 2x^3 dx$$

Hence

$$u_1 = -\frac{x^4}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6 e^{2x} x^3}{e^{2x} x^3}\, dx$$

Which simplifies to

$$u_2 = \int 6dx$$

Hence

$$u_2 = 6x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{3\,e^x x^4}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^x x^3}{3}\right) + \left(\frac{3\,e^x x^4}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^x\left(c_1 + \frac{c_2 x^3}{3}\right) + \frac{3\,e^x x^4}{2}$$

Summary

The solution(s) found are the following

$$y = e^x\left(c_1 + \frac{c_2 x^3}{3}\right) + \frac{3\,e^x x^4}{2} \tag{1}$$

Verification of solutions

$$y = e^x\left(c_1 + \frac{c_2 x^3}{3}\right) + \frac{3\,e^x x^4}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)-(2*x+2)*diff(y(x),x)+(x+2)*y(x)=6*x^3*exp(x),y(x), singsol=all)
```

$$y(x) = e^x \left(c_2 + c_1 x^3 + \frac{3}{2} x^4\right)$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 29

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(x+2)*y[x]==6*x^3*Exp[x],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{6} e^x \left(9x^4 + 2c_2 x^3 + 6c_1\right)$$

10.16 problem 16

Internal problem ID [1170]
Internal file name [`OUTPUT/1171_Sunday_June_05_2022_02_04_04_AM_62651681/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - (2a-1)\,xy' + a^2y = x^{1+a}}$$

10.16.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = (-2a+1)\,x, C = a^2, f(x) = x^{1+a}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + (-2a+1)\,y'x + a^2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}(-2a+1)\, xrx^{r-1} + a^2x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r(-2a+1)\, r\, x^r + a^2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)\,(-2a+1)\, r + a^2 = 0$$

Or

$$a^2 - 2ra + r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= a \\ r_2 &= a \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^a + c_2x^a \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' + (-2a+1)\, y'x + a^2y = x^{1+a}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^a \\ y_2 &= x^a \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{d}{dx}(x^a) & \frac{d}{dx}(x^a \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{x^a a}{x} & \frac{x^a a \ln(x)}{x} + \frac{x^a}{x} \end{vmatrix}$$

Therefore

$$W = (x^a)\left(\frac{x^a a \ln(x)}{x} + \frac{x^a}{x}\right) - (x^a \ln(x))\left(\frac{x^a a}{x}\right)$$

Which simplifies to

$$W = \frac{x^{2a}}{x}$$

Which simplifies to

$$W = x^{2a-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^a \ln(x)\, x^{1+a}}{x^2 x^{2a-1}}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, dx$$

Hence

$$u_1 = -x \ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^a x^{1+a}}{x^2 x^{2a-1}}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x \ln(x) + x)\, x^a + x\, x^a \ln(x)$$

Which simplifies to

$$y_p(x) = x^{1+a}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^a(x + c_1 + c_2 \ln(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^a(x + c_1 + c_2 \ln(x)) \tag{1}$$

Verification of solutions

$$y = x^a(x + c_1 + c_2 \ln(x))$$

Verified OK.

10.16.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0$$

In normal form the ode

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2a+1}{x} \\ q(x) &= \frac{a^2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{-2a+1}{x} dx\right)} dx \\
&= \int e^{(2a-1)\ln(x)} dx \\
&= \int x^{2a-1} dx \\
&= \frac{x^{2a}}{2a}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{a^2}{x^2}}{x^{4a-2}} \\
&= a^2 x^{-4a}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + a^2 x^{-4a} y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
a^2 x^{-4a} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2} y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2} y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\left(c_1 + c_2\ln\left(\frac{x^{2a}}{a}\right) - c_2\ln(2)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\left(c_1 + c_2\ln\left(\frac{x^{2a}}{a}\right) - c_2\ln(2)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{\frac{x^{2a}}{a}}$$

$$y_2 = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln(2)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{\frac{x^{2a}}{a}} & \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln(2)}{2} \\ \frac{d}{dx}\left(\sqrt{\frac{x^{2a}}{a}}\right) & \frac{d}{dx}\left(\frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{\frac{x^{2a}}{a}} & \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\ln(2)}{2} \\ \frac{x^{2a}}{\sqrt{\frac{x^{2a}}{a}}\,x} & \frac{\sqrt{2}\ln\left(\frac{x^{2a}}{a}\right)x^{2a}}{2\sqrt{\frac{x^{2a}}{a}}\,x} + \frac{\sqrt{2}\sqrt{\frac{x^{2a}}{a}}\,a}{x} - \frac{\sqrt{2}\ln(2)x^{2a}}{2\sqrt{\frac{x^{2a}}{a}}\,x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{\frac{x^{2a}}{a}}\right)\left(\frac{\sqrt{2}\,\ln\left(\frac{x^{2a}}{a}\right)x^{2a}}{2\sqrt{\frac{x^{2a}}{a}}\,x} + \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,a}{x} - \frac{\sqrt{2}\,\ln(2)\,x^{2a}}{2\sqrt{\frac{x^{2a}}{a}}\,x}\right) - \left(\frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln(2)}{2}\right)\left(\frac{x^{2a}}{\sqrt{\frac{x^{2a}}{a}}\,x}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}\, x^{2a}}{x}$$

Which simplifies to

$$W = x^{2a-1}\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln\left(\frac{x^{2a}}{a}\right)}{2} - \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln(2)}{2}\right) x^{1+a}}{x^2 x^{2a-1}\sqrt{2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{-a}\sqrt{\frac{x^{2a}}{a}}\left(-\ln\left(\frac{x^{2a}}{a}\right) + \ln(2)\right)}{2} dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{\alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right) + \ln(2)\right)}{2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{\frac{x^{2a}}{a}}\, x^{1+a}}{x^2 x^{2a-1}\sqrt{2}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\, x^{-a}\sqrt{\frac{x^{2a}}{a}}}{2} dx$$

Hence

$$u_2 = \frac{\sqrt{2}\, x^{-a}\sqrt{\frac{x^{2a}}{a}}\, x}{2}$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right)+\ln(2)\right)d\alpha\right)}{2}$$

$$u_2 = \frac{x^{1-a}\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right)+\ln(2)\right)d\alpha\right)\sqrt{\frac{x^{2a}}{a}}}{2} + \frac{x^{1-a}\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\left(\frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln\left(\frac{x^{2a}}{a}\right)}{2}-\frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\ln(2)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right)+\ln(2)\right)d\alpha\right)\sqrt{\frac{x^{2a}}{a}}\,a-\left(-\ln\left(\frac{x^{2a}}{a}\right)+\ln(2)\right)x^{1+a}}{2a}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\left(c_1+c_2\ln\left(\frac{x^{2a}}{a}\right)-c_2\ln(2)\right)}{2}\right) \\ &\quad + \left(\frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right)+\ln(2)\right)d\alpha\right)\sqrt{\frac{x^{2a}}{a}}\,a-\left(-\ln\left(\frac{x^{2a}}{a}\right)+\ln(2)\right)x^{1+a}}{2a}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2a}}{a}\right) - c_2 \ln(2)\right)}{2} + \frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right) + \ln(2)\right) d\alpha\right)\sqrt{\frac{x^{2a}}{a}}\, a - \left(-\ln\left(\frac{x^{2a}}{a}\right) + \ln(2)\right) x^{1+a}}{2a} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{\frac{x^{2a}}{a}}\,\left(c_1 + c_2 \ln\left(\frac{x^{2a}}{a}\right) - c_2 \ln(2)\right)}{2} + \frac{\left(\int_0^x \alpha^{-a}\sqrt{\frac{\alpha^{2a}}{a}}\left(-\ln\left(\frac{\alpha^{2a}}{a}\right) + \ln(2)\right) d\alpha\right)\sqrt{\frac{x^{2a}}{a}}\, a - \left(-\ln\left(\frac{x^{2a}}{a}\right) + \ln(2)\right) x^{1+a}}{2a}$$

Verified OK.

10.16.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2ax + x, C = a^2, f(x) = x^{1+a}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0$$

In normal form the ode

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2a+1}{x} \\
q(x) &= \frac{a^2}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n(-2a+1)}{x^2} + \frac{a^2}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = a \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2a}{x} + \frac{-2a+1}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= (c_1 \ln(x) + c_2)\,x^a \\ &= (c_1 \ln(x) + c_2)\,x^a \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + (-2ax + x)\,y' + a^2 y = x^{1+a}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^a$$
$$y_2 = x^a \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{d}{dx}(x^a) & \frac{d}{dx}(x^a \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{x^a a}{x} & \frac{x^a a \ln(x)}{x} + \frac{x^a}{x} \end{vmatrix}$$

Therefore

$$W = (x^a)\left(\frac{x^a a \ln(x)}{x} + \frac{x^a}{x}\right) - (x^a \ln(x))\left(\frac{x^a a}{x}\right)$$

Which simplifies to

$$W = \frac{x^{2a}}{x}$$

Which simplifies to

$$W = x^{2a-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^a \ln(x)\, x^{1+a}}{x^2 x^{2a-1}}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\,dx$$

Hence

$$u_1 = -x\ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^a x^{1+a}}{x^2 x^{2a-1}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x\ln(x) + x)\,x^a + x\,x^a\ln(x)$$

Which simplifies to

$$y_p(x) = x^{1+a}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= ((c_1\ln(x) + c_2)\,x^a) + (x^{1+a}) \\ &= x^{1+a} + (c_1\ln(x) + c_2)\,x^a \end{aligned}$$

Which simplifies to

$$y = x^a(x + c_1\ln(x) + c_2)$$

Summary

The solution(s) found are the following

$$y = x^a(x + c_1\ln(x) + c_2) \tag{1}$$

Verification of solutions

$$y = x^a(x + c_1\ln(x) + c_2)$$

Verified OK.

10.16.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + (-2ax + x)\,y' + a^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2ax + x \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2ax+x}{x^2}\,dx} \\
&= z_1 e^{\frac{(2a-1)\ln(x)}{2}} \\
&= z_1\left(x^{a-\frac{1}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^a
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2ax+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{(2a-1)\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^a) + c_2(x^a(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + (-2ax + x)\,y' + a^2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^a + c_2 x^a \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^a \\ y_2 &= x^a \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{d}{dx}(x^a) & \frac{d}{dx}(x^a \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^a & x^a \ln(x) \\ \frac{x^a a}{x} & \frac{x^a a \ln(x)}{x} + \frac{x^a}{x} \end{vmatrix}$$

Therefore

$$W = (x^a) \left(\frac{x^a a \ln(x)}{x} + \frac{x^a}{x} \right) - (x^a \ln(x)) \left(\frac{x^a a}{x} \right)$$

Which simplifies to

$$W = \frac{x^{2a}}{x}$$

Which simplifies to

$$W = x^{2a-1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^a \ln(x)\, x^{1+a}}{x^2 x^{2a-1}}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, dx$$

Hence

$$u_1 = -x \ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^a x^{1+a}}{x^2 x^{2a-1}}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x \ln(x) + x)\, x^a + x\, x^a \ln(x)$$

Which simplifies to

$$y_p(x) = x^{1+a}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 x^a + c_2 x^a \ln(x)) + (x^{1+a})
\end{aligned}
$$

Which simplifies to

$$
y = (c_1 + c_2 \ln(x))\, x^a + x^{1+a}
$$

Summary

The solution(s) found are the following

$$
y = (c_1 + c_2 \ln(x))\, x^a + x^{1+a} \tag{1}
$$

Verification of solutions

$$
y = (c_1 + c_2 \ln(x))\, x^a + x^{1+a}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-(2*a-1)*x*diff(y(x),x)+a^2*y(x)=x^(a+1),y(x), singsol=all)
```

$$
y(x) = x^a(c_2 + \ln(x)\, c_1 + x)
$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]-(2*a-1)*x*y'[x]+a^2*y[x]==x^(a+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^a(ac_2 \log(x) + x + c_1)$$

10.17 problem 17

Internal problem ID [1171]
Internal file name [`OUTPUT/1172_Sunday_June_05_2022_02_04_06_AM_52712814/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 2y'x + (x^2+2)\,y = x^3\cos(x)}$$

10.17.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2y'x + (x^2+2)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{x^2+2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v(x) + v''(x) = \cos(x)
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x)$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(x), \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x \cos(x) + A_2 \sin(x) x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{\sin(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\sin(x)\, x}{2}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right) x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right) x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x \cos(x)$$
$$y_2 = \sin(x) x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x \cos(x) & \sin(x) x \\ \frac{d}{dx}(x \cos(x)) & \frac{d}{dx}(\sin(x) x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x \cos(x) & \sin(x) x \\ \cos(x) - \sin(x) x & x \cos(x) + \sin(x) \end{vmatrix}$$

Therefore

$$W = (x \cos(x)) (x \cos(x) + \sin(x)) - (\sin(x) x) (\cos(x) - \sin(x) x)$$

Which simplifies to

$$W = \cos(x)^2 x^2 + x^2 \sin(x)^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) x^4 \cos(x)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)}{2} dx$$

Hence

$$u_1 = \frac{\cos(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \cos(x)^2}{x^4} dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2 \, dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)}{4}$$
$$u_2 = \frac{\sin(2x)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)\, x \cos(x)}{4} + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right) \sin(x)\, x$$

Which simplifies to

$$y_p(x) = \frac{\sin(x)\, x^2}{2} + \frac{x \cos(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right) x\right) + \left(\frac{\sin(x)\, x^2}{2} + \frac{x \cos(x)}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right) x + \frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4} \tag{1}$$

Verification of solutions

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2}\right) x + \frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4}$$

Verified OK.

10.17.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 2y'x + (x^2 + 2)\, y = x^3 \cos(x) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{3}{2}$$
$$\beta = 1$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x \sqrt{2}\, \sin(x)}{\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x \cos(x)$$
$$y_2 = \sin(x)\, x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x \cos(x) & \sin(x)\, x \\ \frac{d}{dx}(x \cos(x)) & \frac{d}{dx}(\sin(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\cos(x) & \sin(x)\, x \\ \cos(x) - \sin(x)\, x & x\cos(x) + \sin(x) \end{vmatrix}$$

Therefore

$$W = (x\cos(x))\,(x\cos(x) + \sin(x)) - (\sin(x)\, x)\,(\cos(x) - \sin(x)\, x)$$

Which simplifies to

$$W = \cos(x)^2\, x^2 + x^2 \sin(x)^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\, x^4 \cos(x)}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)}{2} dx$$

Hence

$$u_1 = \frac{\cos(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \cos(x)^2}{x^4}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2\, dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)}{4}$$
$$u_2 = \frac{\sin(2x)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)\, x \cos(x)}{4} + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right)\sin(x)\, x$$

Which simplifies to

$$y_p(x) = \frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2 x\sqrt{2}\,\sin(x)}{\sqrt{\pi}}\right) + \left(\frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2 x\sqrt{2}\,\sin(x)}{\sqrt{\pi}} + \frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2 x\sqrt{2}\,\sin(x)}{\sqrt{\pi}} + \frac{\sin(x)\, x^2}{2} + \frac{x\cos(x)}{4}$$

Verified OK.

10.17.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x \cos(x)) + c_2(x \cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 2y'x + (x^2 + 2)\, y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x \cos(x) + c_2 \sin(x)\, x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\cos(x)$$
$$y_2 = \sin(x)\,x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\cos(x) & \sin(x)\,x \\ \frac{d}{dx}(x\cos(x)) & \frac{d}{dx}(\sin(x)\,x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\cos(x) & \sin(x)\,x \\ \cos(x) - \sin(x)\,x & x\cos(x) + \sin(x) \end{vmatrix}$$

Therefore

$$W = (x\cos(x))\,(x\cos(x) + \sin(x)) - (\sin(x)\,x)\,(\cos(x) - \sin(x)\,x)$$

Which simplifies to

$$W = \cos(x)^2\,x^2 + x^2\sin(x)^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\, x^4 \cos(x)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)}{2} dx$$

Hence

$$u_1 = \frac{\cos(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \cos(x)^2}{x^4} dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2 \, dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)}{4}$$
$$u_2 = \frac{\sin(2x)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)\, x \cos(x)}{4} + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right) \sin(x)\, x$$

Which simplifies to

$$y_p(x) = \frac{\sin(x)\,x^2}{2} + \frac{x\cos(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x\cos(x) + c_2\sin(x)\,x) + \left(\frac{\sin(x)\,x^2}{2} + \frac{x\cos(x)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\cos(x) + c_2\sin(x)\,x + \frac{\sin(x)\,x^2}{2} + \frac{x\cos(x)}{4} \tag{1}$$

Verification of solutions

$$y = c_1 x\cos(x) + c_2\sin(x)\,x + \frac{\sin(x)\,x^2}{2} + \frac{x\cos(x)}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=x^3*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{((x + 2c_2)\sin(x) + 2\cos(x)c_1)x}{2}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 49

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==x^3*Cos[x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{8}e^{-ix}x\left(2ix + e^{2ix}(-2ix + 1 - 4ic_2) + 1 + 8c_1\right)$$

10.18 problem 18

Internal problem ID [1172]
Internal file name [`OUTPUT/1173_Sunday_June_05_2022_02_04_07_AM_2082207/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' - y' - 4yx^3 = 8x^5}$$

10.18.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' - 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' - 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)} dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-4x^2}{x^2} \\ &= -4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\lambda_1 = +2$$
$$\lambda_2 = -2$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau}$$
$$y(\tau) = c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}$$

Or

$$y(\tau) = c_1 e^{2\tau} + c_2 e^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 e^{x^2} + c_2 e^{-x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x^2} + c_2 e^{-x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x^2}$$
$$y_2 = e^{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x^2} & e^{x^2} \\ \frac{d}{dx}\left(e^{-x^2}\right) & \frac{d}{dx}\left(e^{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x^2} & e^{x^2} \\ -2x\,e^{-x^2} & 2\,e^{x^2}x \end{vmatrix}$$

Therefore

$$W = \left(e^{-x^2}\right)\left(2\,e^{x^2}x\right) - \left(e^{x^2}\right)\left(-2x\,e^{-x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\,e^{x^2}x^5}{4x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,e^{x^2}x^3 dx$$

Hence

$$u_1 = -\left(x^2 - 1\right)e^{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,e^{-x^2}x^5}{4x^2}\,dx$$

Which simplifies to

$$u_2 = \int 2x^3 e^{-x^2} dx$$

Hence

$$u_2 = -\left(x^2 + 1\right)e^{-x^2}$$

Which simplifies to

$$u_1 = \left(-x^2+1\right) e^{x^2}$$
$$u_2 = -\left(x^2+1\right) e^{-x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2+1-\left(x^2+1\right) e^{-x^2} e^{x^2}$$

Which simplifies to

$$y_p(x) = -2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x^2} + c_2 e^{-x^2}\right) + \left(-2x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - 2x^2 \tag{1}$$

Verification of solutions

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - 2x^2$$

Verified OK.

10.18.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x, B = -1, C = -4x^3, f(x) = 8x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$xy'' - y' - 4yx^3 = 0$$

In normal form the ode

$$xy'' - y' - 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-x^2}}{c} \\ \tau'' &= -\frac{2x}{c\sqrt{-x^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2x}{c\sqrt{-x^2}} - \frac{1}{x}\frac{2\sqrt{-x^2}}{c}}{\left(\frac{2\sqrt{-x^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{-x^2} dx}{c} \\
&= \frac{x\sqrt{-x^2}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right)
$$

Now the particular solution to this ODE is found

$$
x y'' - y' - 4 y x^3 = 8x^5
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x^2} \\
y_2 &= \mathrm{e}^{x^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x^2} & e^{x^2} \\ \frac{d}{dx}\left(e^{-x^2}\right) & \frac{d}{dx}\left(e^{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x^2} & e^{x^2} \\ -2x\,e^{-x^2} & 2\,e^{x^2} x \end{vmatrix}$$

Therefore

$$W = \left(e^{-x^2}\right)\left(2\,e^{x^2} x\right) - \left(e^{x^2}\right)\left(-2x\,e^{-x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\,e^{x^2} x^5}{4x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,e^{x^2} x^3 dx$$

Hence

$$u_1 = -\left(x^2 - 1\right) e^{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,\mathrm{e}^{-x^2} x^5}{4x^2}\, dx$$

Which simplifies to

$$u_2 = \int 2x^3 \mathrm{e}^{-x^2} dx$$

Hence

$$u_2 = -\left(x^2+1\right)\mathrm{e}^{-x^2}$$

Which simplifies to

$$u_1 = \left(-x^2+1\right)\mathrm{e}^{x^2}$$
$$u_2 = -\left(x^2+1\right)\mathrm{e}^{-x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2 + 1 - \left(x^2+1\right)\mathrm{e}^{-x^2}\mathrm{e}^{x^2}$$

Which simplifies to

$$y_p(x) = -2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right)\right) + \left(-2x^2\right) \\ &= -2x^2 + c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y = -2x^2 + c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right)$$

Summary

The solution(s) found are the following

$$y = -2x^2 + c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right) \tag{1}$$

Verification of solutions

$$y = -2x^2 + c_1 \cosh\left(x^2\right) + i c_2 \sinh\left(x^2\right)$$

Verified OK.

10.18.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - y'x - 4yx^4 = 8x^6 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\, xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= i \\ n &= \frac{1}{2} \\ \gamma &= 2 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{ic_1x\sqrt{2}\, \sinh(x^2)}{\sqrt{\pi}\, \sqrt{ix^2}} - \frac{c_2x\sqrt{2}\, \cosh(x^2)}{\sqrt{\pi}\, \sqrt{ix^2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{ic_1x\sqrt{2}\, \sinh(x^2)}{\sqrt{\pi}\, \sqrt{ix^2}} - \frac{c_2x\sqrt{2}\, \cosh(x^2)}{\sqrt{\pi}\, \sqrt{ix^2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x^2}$$
$$y_2 = \mathrm{e}^{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & \mathrm{e}^{x^2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x^2}\right) & \frac{d}{dx}\left(\mathrm{e}^{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x^2} & \mathrm{e}^{x^2} \\ -2x\,\mathrm{e}^{-x^2} & 2\,\mathrm{e}^{x^2} x \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x^2}\right)\left(2\,\mathrm{e}^{x^2} x\right) - \left(\mathrm{e}^{x^2}\right)\left(-2x\,\mathrm{e}^{-x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\,\mathrm{e}^{x^2} x^6}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,\mathrm{e}^{x^2} x^3 dx$$

Hence

$$u_1 = -\left(x^2 - 1\right) \mathrm{e}^{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,\mathrm{e}^{-x^2} x^6}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int 2x^3 \mathrm{e}^{-x^2} dx$$

Hence

$$u_2 = -\left(x^2 + 1\right) \mathrm{e}^{-x^2}$$

Which simplifies to

$$u_1 = \left(-x^2 + 1\right) \mathrm{e}^{x^2}$$
$$u_2 = -\left(x^2 + 1\right) \mathrm{e}^{-x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2 + 1 - \left(x^2 + 1\right) \mathrm{e}^{-x^2} \mathrm{e}^{x^2}$$

Which simplifies to

$$y_p(x) = -2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{i c_1 x \sqrt{2}\, \sinh\left(x^2\right)}{\sqrt{\pi}\, \sqrt{i x^2}} - \frac{c_2 x \sqrt{2}\, \cosh\left(x^2\right)}{\sqrt{\pi}\, \sqrt{i x^2}}\right) + \left(-2x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{i c_1 x \sqrt{2}\, \sinh(x^2)}{\sqrt{\pi}\, \sqrt{i x^2}} - \frac{c_2 x \sqrt{2}\, \cosh(x^2)}{\sqrt{\pi}\, \sqrt{i x^2}} - 2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{i c_1 x \sqrt{2}\, \sinh(x^2)}{\sqrt{\pi}\, \sqrt{i x^2}} - \frac{c_2 x \sqrt{2}\, \cosh(x^2)}{\sqrt{\pi}\, \sqrt{i x^2}} - 2x^2$$

Verified OK.

10.18.4 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' - 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= -4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 16x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2x + \frac{3}{16x^3} - \frac{9}{1024x^7} + \frac{27}{32768x^{11}} - \frac{405}{4194304x^{15}} + \frac{1701}{134217728x^{19}} - \frac{15309}{8589934592x^{23}} + \frac{72171}{274877906944x^{27}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (4x^2) + \left(\frac{3}{4x^2}\right) \\ &= 4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2x$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(2x) \\
&= -\frac{1}{2x} - 2x \\
&= -\frac{1}{2x} - 2x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2x\right)(0) + \left(\left(\frac{1}{2x^2} - 2\right) + \left(-\frac{1}{2x} - 2x\right)^2 - \left(\frac{16x^4 + 3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2x\right) dx} \\ &= \frac{e^{-x^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{2x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x^2}\right) + c_2 \left(e^{-x^2} \left(\frac{e^{2x^2}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' - 4yx^3 = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-x^2} + \frac{c_2 e^{x^2}}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x^2}$$

$$y_2 = \frac{e^{x^2}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x^2} & \frac{e^{x^2}}{4} \\ \frac{d}{dx}\left(e^{-x^2}\right) & \frac{d}{dx}\left(\frac{e^{x^2}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x^2} & \frac{e^{x^2}}{4} \\ -2x\,e^{-x^2} & \frac{e^{x^2}x}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x^2}\right)\left(\frac{e^{x^2}x}{2}\right) - \left(\frac{e^{x^2}}{4}\right)\left(-2x\,e^{-x^2}\right)$$

Which simplifies to

$$W = e^{-x^2} e^{x^2} x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{x^2} x^5}{x^2}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,\mathrm{e}^{x^2} x^3 dx$$

Hence

$$u_1 = -\left(x^2-1\right)\mathrm{e}^{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,\mathrm{e}^{-x^2} x^5}{x^2}\,dx$$

Which simplifies to

$$u_2 = \int 8x^3 \mathrm{e}^{-x^2} dx$$

Hence

$$u_2 = -4\left(x^2+1\right)\mathrm{e}^{-x^2}$$

Which simplifies to

$$u_1 = \left(-x^2+1\right)\mathrm{e}^{x^2}$$
$$u_2 = -4\left(x^2+1\right)\mathrm{e}^{-x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-x^2+1\right)\mathrm{e}^{x^2}\mathrm{e}^{-x^2} - \left(x^2+1\right)\mathrm{e}^{-x^2}\mathrm{e}^{x^2}$$

Which simplifies to

$$y_p(x) = -2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x^2} + \frac{c_2\mathrm{e}^{x^2}}{4}\right) + \left(-2x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x^2} + \frac{c_2 e^{x^2}}{4} - 2x^2 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x^2} + \frac{c_2 e^{x^2}}{4} - 2x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)-4*x^3*y(x)=8*x^5,y(x), singsol=all)
```

$$y(x) = \sinh\left(x^2\right) c_2 + \cosh\left(x^2\right) c_1 - 2x^2$$

✓ Solution by Mathematica
Time used: 0.06 (sec). Leaf size: 28

```
DSolve[x*y''[x]-y'[x]-4*x^3*y[x]==8*x^5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x^2 + c_1 \cosh\left(x^2\right) + ic_2 \sinh\left(x^2\right)$$

10.19 problem 19

Internal problem ID [1173]
Internal file name [`OUTPUT/1174_Sunday_June_05_2022_02_04_09_AM_30580542/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{\sin(x) y'' + (2\sin(x) - \cos(x)) y' + (-\cos(x) + \sin(x)) y = e^{-x}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
      <- Kovacics algorithm successful
      Change of variables used:
         [x = arccos(t)]
      Linear ODE actually solved:
         ((-t^2+1)^(1/2)-t)*u(t)+(2*t^2-2)*diff(u(t),t)+(-t^2+1)^(3/2)*diff(diff(u(t),t),t)
   <- change of variables successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 58

```
dsolve(sin(x)*diff(y(x),x$2)+(2*sin(x)-cos(x))*diff(y(x),x)+(sin(x)-cos(x))*y(x)=exp(-x),y(x
```

$$y(x) = -e^{\arcsin(\cos(x))}\left(\left(\left(\int \csc(x)^2 e^{-\arcsin(\cos(x))-x}dx\right)\cos(x) - \cos(x) c_1 - \left(\int \cot(x)\csc(x) e^{-\arcsin(\cos(x))-x}dx\right) - c_2\right)\right)$$

✓ Solution by Mathematica

Time used: 0.809 (sec). Leaf size: 121

```
DSolve[Sin[x]*y''[x]+(2*Sin[x]-Cos[x])*y'[x]+(Sin[x]-Cos[x])*y[x]==Exp[-x],y[x],x,IncludeSin
```

$$y(x) \to -\sqrt{\sin^2(x)} \exp\left(-\arccos(\cos(x)) - 4\arctan\left(\frac{\sqrt{\sin^2(x)}}{\cos(x)+1}\right) + 4\cot^{-1}\left(\frac{\cos(x)+1}{\sqrt{\sin^2(x)}}\right)\right) + c_2\cos(x)\exp\left(2\left(\cot^{-1}\left(\frac{\cos(x)+1}{\sqrt{\sin^2(x)}}\right) - 2\arctan\left(\frac{\sqrt{\sin^2(x)}}{\cos(x)+1}\right)\right)\right) + c_1 e^{-2\cot^{-1}\left(\frac{\cos(x)+1}{\sqrt{\sin^2(x)}}\right)}$$

10.20 problem 20

Internal problem ID [1174]
Internal file name [`OUTPUT/1175_Sunday_June_05_2022_02_04_13_AM_47362518/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 8x^{\frac{5}{2}}}$$

10.20.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' - 4y'x + (-16x^2 + 3)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{-16x^2+3}{4x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-16x^2+3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\
&= \frac{-16x^2+3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{-16x^2+3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= -4
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{1}{x}}{2}} \\
&= \sqrt{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\sqrt{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) - 4v(x) = 2
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = -4, f(x) = 2$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - 4v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$v(x) = c_1 e^{2x} + c_2 e^{-2x}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -\frac{1}{2}$$

Therefore the general solution is

$$
\begin{aligned}
v &= v_h + v_p \\
&= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}\right) + \left(-\frac{1}{2}\right)
\end{aligned}
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} - \frac{1}{2}\right) (z(x))
\end{aligned}
\tag{7}
$$

But from (5)

$$
z(x) = \sqrt{x}
$$

Hence (7) becomes

$$
y = \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} - \frac{1}{2}\right) \sqrt{x}
$$

Therefore the homogeneous solution y_h is

$$
y_h = \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} - \frac{1}{2}\right) \sqrt{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \sqrt{x} \\
y_2 &= \mathrm{e}^{-2x} \sqrt{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x}\sqrt{x} & e^{-2x}\sqrt{x} \\ \frac{d}{dx}\left(e^{2x}\sqrt{x}\right) & \frac{d}{dx}\left(e^{-2x}\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x}\sqrt{x} & e^{-2x}\sqrt{x} \\ 2\,e^{2x}\sqrt{x} + \frac{e^{2x}}{2\sqrt{x}} & -2\,e^{-2x}\sqrt{x} + \frac{e^{-2x}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\sqrt{x}\right)\left(-2\,e^{-2x}\sqrt{x} + \frac{e^{-2x}}{2\sqrt{x}}\right) - \left(e^{-2x}\sqrt{x}\right)\left(2\,e^{2x}\sqrt{x} + \frac{e^{2x}}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = -4\,e^{2x} x\,e^{-2x}$$

Which simplifies to

$$W = -4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8\,e^{-2x} x^3}{-16x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x}}{2}dx$$

Hence

$$u_1 = -\frac{e^{-2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,e^{2x} x^3}{-16x^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{2x}}{2} dx$$

Hence

$$u_2 = -\frac{e^{2x}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-2x} e^{2x} \sqrt{x}}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}\right)\sqrt{x}\right) + \left(-\frac{\sqrt{x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}\right)\sqrt{x} - \frac{\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = \left(c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}\right)\sqrt{x} - \frac{\sqrt{x}}{2}$$

Verified OK.

10.20.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - y'x + \left(-4x^2 + \frac{3}{4}\right)y = 2x^{\frac{5}{2}} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\,y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\mathrm{BesselJ}\,(n, \beta\, x^{\gamma}) + c_2\,\mathrm{BesselY}\,(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 1\\
\beta &= 2i\\
n &= -\frac{1}{2}\\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1x\cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2x\sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1x\cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{ic_2x\sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{2x}\sqrt{x} \\ y_2 &= e^{-2x}\sqrt{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x}\sqrt{x} & e^{-2x}\sqrt{x} \\ \frac{d}{dx}\left(e^{2x}\sqrt{x}\right) & \frac{d}{dx}\left(e^{-2x}\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x}\sqrt{x} & e^{-2x}\sqrt{x} \\ 2\,e^{2x}\sqrt{x} + \frac{e^{2x}}{2\sqrt{x}} & -2\,e^{-2x}\sqrt{x} + \frac{e^{-2x}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\sqrt{x}\right)\left(-2\,e^{-2x}\sqrt{x} + \frac{e^{-2x}}{2\sqrt{x}}\right) - \left(e^{-2x}\sqrt{x}\right)\left(2\,e^{2x}\sqrt{x} + \frac{e^{2x}}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = -4\,e^{2x} x\,e^{-2x}$$

Which simplifies to

$$W = -4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{-2x} x^3}{-4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-2x}}{2} dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{2x} x^3}{-4x^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{2x}}{2} dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{2x}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\mathrm{e}^{-2x}\mathrm{e}^{2x}\sqrt{x}}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x \cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}}\right) + \left(-\frac{\sqrt{x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x \cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}} - \frac{\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x \cosh(2x)}{\sqrt{\pi}\,\sqrt{ix}} + \frac{i c_2 x \sinh(2x)}{\sqrt{\pi}\,\sqrt{ix}} - \frac{\sqrt{x}}{2}$$

Verified OK.

10.20.3 Solving using Kovacic algorithm

Writing the ode as

$$4x^2 y'' - 4y'x + (-16x^2 + 3)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\sqrt{x}\right) + c_2\left(\mathrm{e}^{-2x}\sqrt{x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$4x^2y''-4y'x+\left(-16x^2+3\right)y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x}\sqrt{x}+\frac{c_2 \mathrm{e}^{2x}\sqrt{x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-2x}\sqrt{x} \\ y_2 &= \frac{\mathrm{e}^{2x}\sqrt{x}}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \mathrm{e}^{-2x}\sqrt{x} & \frac{\mathrm{e}^{2x}\sqrt{x}}{4} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\sqrt{x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}\sqrt{x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x}\sqrt{x} & \frac{e^{2x}\sqrt{x}}{4} \\ -2\,e^{-2x}\sqrt{x}+\frac{e^{-2x}}{2\sqrt{x}} & \frac{e^{2x}\sqrt{x}}{2}+\frac{e^{2x}}{8\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\sqrt{x}\right)\left(\frac{e^{2x}\sqrt{x}}{2}+\frac{e^{2x}}{8\sqrt{x}}\right)-\left(\frac{e^{2x}\sqrt{x}}{4}\right)\left(-2\,e^{-2x}\sqrt{x}+\frac{e^{-2x}}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = e^{2x}x\,e^{-2x}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{2x}x^3}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}}{2}dx$$

Hence

$$u_1 = -\frac{e^{2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,e^{-2x}x^3}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int 2\,e^{-2x}dx$$

Hence

$$u_2 = -e^{-2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-2x} e^{2x} \sqrt{x}}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} \sqrt{x} + \frac{c_2 e^{2x} \sqrt{x}}{4}\right) + \left(-\frac{\sqrt{x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} \sqrt{x} + \frac{c_2 e^{2x} \sqrt{x}}{4} - \frac{\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} \sqrt{x} + \frac{c_2 e^{2x} \sqrt{x}}{4} - \frac{\sqrt{x}}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=8*x^(5/2),y(x), singsol=all)
```

$$y(x) = \sqrt{x}\left(-\frac{1}{2}+\sinh(2x)\, c_2+\cosh(2x)\, c_1\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==8*x^(5/2),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(-2e^{2x}+c_2e^{4x}+4c_1\right)$$

10.21 problem 21

Internal problem ID [1175]
Internal file name [`OUTPUT/1176_Sunday_June_05_2022_02_04_14_AM_97933508/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' - 4y'x + (4x^2 + 3)\, y = x^{\frac{7}{2}}}$$

10.21.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' - 4y'x + (4x^2 + 3)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{4x^2+3}{4x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{4x^2+3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\
&= \frac{4x^2+3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{4x^2+3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\
&= e^{-\int \frac{-\frac{1}{x}}{2}} \\
&= \sqrt{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\sqrt{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
4v''(x) + 4v(x) = x
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 4, B = 0, C = 4, f(x) = x$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$4v''(x) + 4v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 4, B = 0, C = 4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(4)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 4A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x}{4}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 \cos(x) + c_2 \sin(x) + \frac{x}{4}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}$$

Hence (7) becomes

$$y = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{x}{4}\right) \sqrt{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \cos(x) + c_2 \sin(x) + \frac{x}{4}\right) \sqrt{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \sqrt{x} \\ y_2 &= \sin(x) \sqrt{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ \frac{d}{dx}\left(\cos(x)\sqrt{x}\right) & \frac{d}{dx}\left(\sin(x)\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ -\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}} & \cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\sqrt{x}\right)\left(\cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}}\right) - \left(\sin(x)\sqrt{x}\right)\left(-\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\left(\cos(x)^2 + \sin(x)^2\right)$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\, x^4}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)\, x}{4} dx$$

Hence

$$u_1 = -\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \cos(x)}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x\cos(x)}{4}dx$$

Hence

$$u_2 = \frac{\cos(x)}{4} + \frac{\sin(x)\,x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}\right)\cos(x)\sqrt{x} + \left(\frac{\cos(x)}{4} + \frac{\sin(x)\,x}{4}\right)\sin(x)\sqrt{x}$$

Which simplifies to

$$y_p(x) = \frac{x^{\frac{3}{2}}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\cos(x) + c_2\sin(x) + \frac{x}{4}\right)\sqrt{x}\right) + \left(\frac{x^{\frac{3}{2}}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(c_1\cos(x) + c_2\sin(x) + \frac{x}{4}\right)\sqrt{x} + \frac{x^{\frac{3}{2}}}{4} \tag{1}$$

Verification of solutions

$$y = \left(c_1\cos(x) + c_2\sin(x) + \frac{x}{4}\right)\sqrt{x} + \frac{x^{\frac{3}{2}}}{4}$$

Verified OK.

10.21.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - y'x + \left(x^2 + \frac{3}{4}\right)y = \frac{x^{\frac{7}{2}}}{4} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= 1\\ n &= -\frac{1}{2}\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sin(x)}{\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{x}\,\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sin(x)}{\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\sqrt{x}$$
$$y_2 = \sin(x)\sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ \frac{d}{dx}\left(\cos(x)\sqrt{x}\right) & \frac{d}{dx}\left(\sin(x)\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ -\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}} & \cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\sqrt{x}\right)\left(\cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}}\right) - \left(\sin(x)\sqrt{x}\right)\left(-\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\left(\cos(x)^2 + \sin(x)^2\right)$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)x^4}{4}}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)\, x}{4} dx$$

Hence

$$u_1 = -\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^4\cos(x)}{4}}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{x\cos(x)}{4} dx$$

Hence

$$u_2 = \frac{\cos(x)}{4} + \frac{\sin(x)\, x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}\right)\cos(x)\sqrt{x} + \left(\frac{\cos(x)}{4} + \frac{\sin(x)\, x}{4}\right)\sin(x)\sqrt{x}$$

Which simplifies to

$$y_p(x) = \frac{x^{\frac{3}{2}}}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1\sqrt{x}\,\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sin(x)}{\sqrt{\pi}}\right) + \left(\frac{x^{\frac{3}{2}}}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sin(x)}{\sqrt{\pi}} + \frac{x^{\frac{3}{2}}}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}\,\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2\sqrt{x}\,\sqrt{2}\,\sin(x)}{\sqrt{\pi}} + \frac{x^{\frac{3}{2}}}{4}
$$

Verified OK.

10.21.3 Solving using Kovacic algorithm

Writing the ode as

$$
4x^2y'' - 4y'x + (4x^2+3)\,y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4x^2 \\
B &= -4x \\
C &= 4x^2 + 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos(x)\sqrt{x}\right) + c_2\left(\cos(x)\sqrt{x}(\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' - 4y'x + (4x^2 + 3)\, y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \sqrt{x}\, \cos(x)\, c_1 + \sqrt{x}\, \sin(x)\, c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\sqrt{x}$$
$$y_2 = \sin(x)\sqrt{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ \frac{d}{dx}\left(\cos(x)\sqrt{x}\right) & \frac{d}{dx}\left(\sin(x)\sqrt{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\sqrt{x} & \sin(x)\sqrt{x} \\ -\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}} & \cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\sqrt{x}\right)\left(\cos(x)\sqrt{x} + \frac{\sin(x)}{2\sqrt{x}}\right) - \left(\sin(x)\sqrt{x}\right)\left(-\sin(x)\sqrt{x} + \frac{\cos(x)}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\left(\cos(x)^2 + \sin(x)^2\right)$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\,x^4}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)\,x}{4}dx$$

Hence

$$u_1 = -\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4\cos(x)}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x\cos(x)}{4}dx$$

Hence

$$u_2 = \frac{\cos(x)}{4} + \frac{\sin(x)\,x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\sin(x)}{4} + \frac{x\cos(x)}{4}\right)\cos(x)\sqrt{x} + \left(\frac{\cos(x)}{4} + \frac{\sin(x)\,x}{4}\right)\sin(x)\sqrt{x}$$

Which simplifies to

$$y_p(x) = \frac{x^{\frac{3}{2}}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\,\cos(x)\,c_1 + \sqrt{x}\,\sin(x)\,c_2\right) + \left(\frac{x^{\frac{3}{2}}}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \sqrt{x}\,(c_1\cos(x) + c_2\sin(x)) + \frac{x^{\frac{3}{2}}}{4}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x}\,(c_1\cos(x) + c_2\sin(x)) + \frac{x^{\frac{3}{2}}}{4} \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\,(c_1\cos(x) + c_2\sin(x)) + \frac{x^{\frac{3}{2}}}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2+3)*y(x)=x^(7/2),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x}\,(x + 4\sin(x)\, c_2 + 4\cos(x)\, c_1)}{4}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 48

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(4*x^2+3)*y[x]==x^(7/2),y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{4} e^{-ix} \sqrt{x} \left(e^{ix} x - 2ic_2 e^{2ix} + 4c_1\right)$$

10.22 problem 22

Internal problem ID [1176]
Internal file name [`OUTPUT/1177_Sunday_June_05_2022_02_04_15_AM_70243093/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 2y'x - (x^2 - 2)\, y = 3x^4}$$

10.22.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2y'x + (-x^2 + 2)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{-x^2+2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{-x^2+2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{-x^2+2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{-x^2+2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= -1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
-v(x) + v''(x) = 3x
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = -1, f(x) = 3x$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$-v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ v(x) &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$v(x) = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 x - A_1 = 3x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -3]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -3x$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + (-3x) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} - 3x\right) (z(x)) \end{aligned}\tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} - 3x\right) x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} - 3x\right) x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x\,\mathrm{e}^{x}\\ y_2 &= x\,\mathrm{e}^{-x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x\,\mathrm{e}^{-x} \\ \frac{d}{dx}\left(x\,\mathrm{e}^{x}\right) & \frac{d}{dx}\left(x\,\mathrm{e}^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x\,\mathrm{e}^{-x} \\ x\,\mathrm{e}^{x} + \mathrm{e}^{x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (x\,\mathrm{e}^{x})\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) - \left(x\,\mathrm{e}^{-x}\right)(x\,\mathrm{e}^{x} + \mathrm{e}^{x})$$

Which simplifies to

$$W = -2\,\mathrm{e}^{x}\mathrm{e}^{-x}x^{2}$$

Which simplifies to

$$W = -2x^{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3x^{5}\mathrm{e}^{-x}}{-2x^{4}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{3x\,\mathrm{e}^{-x}}{2} dx$$

Hence

$$u_1 = -\frac{3(x+1)\,\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\,\mathrm{e}^{x}x^{5}}{-2x^{4}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3x\,\mathrm{e}^{x}}{2} dx$$

Hence

$$u_2 = -\frac{3(x-1)\,\mathrm{e}^{x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3(x+1)\,\mathrm{e}^{-x} x\,\mathrm{e}^{x}}{2} - \frac{3(x-1)\,\mathrm{e}^{x} x\,\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 3x\right)x\right) + \left(-3x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 3x\right)x - 3x^2 \tag{1}$$

Verification of solutions

$$y = \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} - 3x\right)x - 3x^2$$

Verified OK.

10.22.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - 2y'x + \left(-x^2 + 2\right)y = 3x^4 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}\left(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)\right) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{3}{2} \\
\beta &= i \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cosh\left(x\right)}{\sqrt{\pi}\, \sqrt{i x}} + \frac{i c_2 x^{\frac{3}{2}} \sqrt{2}\, \sinh\left(x\right)}{\sqrt{\pi}\, \sqrt{i x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cosh\left(x\right)}{\sqrt{\pi}\, \sqrt{i x}} + \frac{i c_2 x^{\frac{3}{2}} \sqrt{2}\, \sinh\left(x\right)}{\sqrt{\pi}\, \sqrt{i x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x\,\mathrm{e}^{x} \\
y_2 &= x\,\mathrm{e}^{-x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x\,\mathrm{e}^{-x} \\ \frac{d}{dx}(x\,\mathrm{e}^{x}) & \frac{d}{dx}(x\,\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x\,\mathrm{e}^{-x} \\ x\,\mathrm{e}^{x} + \mathrm{e}^{x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (x\,\mathrm{e}^{x})\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right) - \left(x\,\mathrm{e}^{-x}\right)(x\,\mathrm{e}^{x} + \mathrm{e}^{x})$$

Which simplifies to

$$W = -2\,\mathrm{e}^{x}\mathrm{e}^{-x}x^{2}$$

Which simplifies to

$$W = -2x^{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3x^{5}\mathrm{e}^{-x}}{-2x^{4}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{3x\,\mathrm{e}^{-x}}{2}dx$$

Hence

$$u_1 = -\frac{3(x+1)\,\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\,\mathrm{e}^{x}x^{5}}{-2x^{4}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{3x\,\mathrm{e}^{x}}{2} dx$$

Hence

$$u_2 = -\frac{3(x-1)\,\mathrm{e}^{x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3(x+1)\,\mathrm{e}^{-x} x\,\mathrm{e}^{x}}{2} - \frac{3(x-1)\,\mathrm{e}^{x} x\,\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cosh(x)}{\sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 x^{\frac{3}{2}} \sqrt{2}\, \sinh(x)}{\sqrt{\pi}\, \sqrt{ix}}\right) + \left(-3x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cosh(x)}{\sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 x^{\frac{3}{2}} \sqrt{2}\, \sinh(x)}{\sqrt{\pi}\, \sqrt{ix}} - 3x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{3}{2}} \sqrt{2}\, \cosh(x)}{\sqrt{\pi}\, \sqrt{ix}} + \frac{i c_2 x^{\frac{3}{2}} \sqrt{2}\, \sinh(x)}{\sqrt{\pi}\, \sqrt{ix}} - 3x^2$$

Verified OK.

10.22.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x + (-x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \,\mathrm{e}^{-x}\right) + c_2 \left(x \,\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 2y'x + \left(-x^2 + 2\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = x \,\mathrm{e}^{-x} c_1 + \frac{c_2 x \,\mathrm{e}^{x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\,\mathrm{e}^{-x}$$
$$y_2 = \frac{x\,\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{-x} & \frac{x\,\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(x\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{x\,\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{-x} & \frac{x\,\mathrm{e}^{x}}{2} \\ \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} & \frac{x\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(x\,\mathrm{e}^{-x}\right)\left(\frac{x\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{x\,\mathrm{e}^{x}}{2}\right)\left(\mathrm{e}^{-x} - x\,\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3\,e^{x} x^5}{2}}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{3x\,e^{x}}{2} dx$$

Hence

$$u_1 = -\frac{3(x-1)\,e^{x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^5 e^{-x}}{x^4} dx$$

Which simplifies to

$$u_2 = \int 3x\,e^{-x} dx$$

Hence

$$u_2 = -3(x+1)\,e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3(x+1)\,e^{-x} x\,e^{x}}{2} - \frac{3(x-1)\,e^{x} x\,e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x\,e^{-x} c_1 + \frac{c_2 x\,e^{x}}{2}\right) + \left(-3x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\,\mathrm{e}^{-x}c_1 + \frac{c_2 x\,\mathrm{e}^{x}}{2} - 3x^2 \tag{1}$$

Verification of solutions

$$y = x\,\mathrm{e}^{-x}c_1 + \frac{c_2 x\,\mathrm{e}^{x}}{2} - 3x^2$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)-(x^2-2)*y(x)=3*x^4,y(x), singsol=all)
```

$$y(x) = \sinh(x)\,xc_2 + \cosh(x)\,xc_1 - 3x^2$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 29

```
DSolve[x^2*y''[x]-2*x*y'[x]-(x^2-2)*y[x]==3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x\left(-6x + 2c_1e^{-x} + c_2e^x\right)$$

10.23 problem 23

Internal problem ID [1177]
Internal file name [`OUTPUT/1178_Sunday_June_05_2022_02_04_17_AM_99613705/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 2x(x+1)\,y' + \left(x^2+2x+2\right)y = \mathrm{e}^x x^3}$$

10.23.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(-2x^2-2x\right)y' + \left(x^2+2x+2\right)y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2x^2 - 2x}{x^2} \\
q(x) &= \frac{x^2 + 2x + 2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2 + 2x + 2}{x^2} - \frac{\left(\frac{-2x^2-2x}{x^2}\right)'}{2} - \frac{\left(\frac{-2x^2-2x}{x^2}\right)^2}{4} \\
&= \frac{x^2 + 2x + 2}{x^2} - \frac{\left(\frac{-2-4x}{x^2} - \frac{2(-2x^2-2x)}{x^3}\right)}{2} - \frac{\left(\frac{(-2x^2-2x)^2}{x^4}\right)}{4} \\
&= \frac{x^2 + 2x + 2}{x^2} - \left(\frac{-2 - 4x}{2x^2} - \frac{-2x^2 - 2x}{x^3}\right) - \frac{(-2x^2 - 2x)^2}{4x^4} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-2x^2-2x}{x^2}}{2}} \\
&= x\, \mathrm{e}^{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x\, \mathrm{e}^{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) = 1
$$

Which is now solved for $v(x)$ The ODE can be written as

$$
v''(x) = 1
$$

Integrating once gives

$$v'(x) = x + c_1$$

Integrating again gives

$$v(x) = \frac{x^2}{2} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(\frac{1}{2}x^2 + c_1 x + c_2\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x\,\mathrm{e}^{x}$$

Hence (7) becomes

$$y = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) x\,\mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) x\,\mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x\,\mathrm{e}^{x} \\ y_2 &= x^2 \mathrm{e}^{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x^2\mathrm{e}^{x} \\ \frac{d}{dx}(x\,\mathrm{e}^{x}) & \frac{d}{dx}(x^2\mathrm{e}^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x^2\mathrm{e}^{x} \\ x\,\mathrm{e}^{x} + \mathrm{e}^{x} & 2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (x\,\mathrm{e}^{x})\left(2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x}\right) - \left(x^2\mathrm{e}^{x}\right)(x\,\mathrm{e}^{x} + \mathrm{e}^{x})$$

Which simplifies to

$$W = x^2\mathrm{e}^{2x}$$

Which simplifies to

$$W = x^2\mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5\mathrm{e}^{2x}}{\mathrm{e}^{2x}x^4}\,dx$$

Which simplifies to

$$u_1 = -\int x dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{2x}x^4}{\mathrm{e}^{2x}x^4}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x x^3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{1}{2}x^2 + c_1 x + c_2\right) x\, e^x\right) + \left(\frac{e^x x^3}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) x\, e^x + \frac{e^x x^3}{2} \tag{1}$$

Verification of solutions

$$y = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) x\, e^x + \frac{e^x x^3}{2}$$

Verified OK.

10.23.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + \left(-2x^2 - 2x\right) y' + \left(x^2 + 2x + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2 - 2x \\ C &= x^2 + 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\,dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\,\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,\mathrm{e}^{x}) + c_2(x\,\mathrm{e}^{x}(x))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(-2x^2 - 2x\right)y' + \left(x^2 + 2x + 2\right)y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x\,\mathrm{e}^{x} + c_2 x^2 \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\,\mathrm{e}^{x}$$
$$y_2 = x^2 \mathrm{e}^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x^2\mathrm{e}^{x} \\ \frac{d}{dx}(x\,\mathrm{e}^{x}) & \frac{d}{dx}(x^2\mathrm{e}^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & x^2\mathrm{e}^{x} \\ x\,\mathrm{e}^{x} + \mathrm{e}^{x} & 2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (x\,\mathrm{e}^{x})\left(2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x}\right) - \left(x^2\mathrm{e}^{x}\right)(x\,\mathrm{e}^{x} + \mathrm{e}^{x})$$

Which simplifies to

$$W = x^2\mathrm{e}^{2x}$$

Which simplifies to

$$W = x^2\mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 e^{2x}}{e^{2x} x^4} \, dx$$

Which simplifies to

$$u_1 = -\int x dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x^4}{e^{2x} x^4} \, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x x^3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x \, e^x + c_2 x^2 e^x\right) + \left(\frac{e^x x^3}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^x x (c_2 x + c_1) + \frac{e^x x^3}{2}$$

Summary
The solution(s) found are the following

$$y = e^x x(c_2 x + c_1) + \frac{e^x x^3}{2} \tag{1}$$

Verification of solutions

$$y = e^x x(c_2 x + c_1) + \frac{e^x x^3}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x)=x^3*exp(x),y(x), singsol=a
```

$$y(x) = \frac{e^x x(2c_1 x + x^2 + 2c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.867 (sec). Leaf size: 227

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2*x+2)*y[x]==x^3*Exp[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{ix}x\Bigg(\text{HypergeometricU}(-i,0,-2ix)\int_1^x$$

$$-\frac{ie^{(1-i)K[1]}L_i^{-1}(-2iK[1])}{4\,\text{Hypergeometric1F1}(1-i,2,-2iK[1])\,\text{HypergeometricU}(1-i,1,-2iK[1])K[1]-2\,\text{Hypergeomet}}$$

$$+L_i^{-1}(-2ix)\int_1^x\frac{ie^{(1-i)K[2]}\,\text{HypergeometricU}(-i,}{4\,\text{Hypergeometric1F1}(1-i,2,-2iK[2])\,\text{HypergeometricU}(1-i,1,-2iK[2])K[2]-2}$$

$$+c_1\,\text{HypergeometricU}(-i,0,-2ix)+c_2L_i^{-1}(-2ix)\Bigg)$$

10.24 problem 24

Internal problem ID [1178]
Internal file name [`OUTPUT/1179_Sunday_June_05_2022_02_04_18_AM_69533080/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - y'x - 3y = x^{\frac{3}{2}}}$$

10.24.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = -3, f(x) = x^{\frac{3}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - y'x - 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 3 = 0$$

Or

$$r^2 - 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x^3$$

Next, we find the particular solution to the ODE

$$x^2y'' - y'x - 3y = x^{\frac{3}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(3x^2) - (x^3)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{9}{2}}}{4x^3} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{3}{2}}}{4} dx$$

Hence

$$u_1 = -\frac{x^{\frac{5}{2}}}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}}{4x^3} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{1}{6x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4x^{\frac{3}{2}}}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{4x^{\frac{3}{2}}}{15} + \frac{c_1}{x} + c_2 x^3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{4x^{\frac{3}{2}}}{15} + \frac{c_1}{x} + c_2 x^3 \tag{1}$$

Verification of solutions

$$y = -\frac{4x^{\frac{3}{2}}}{15} + \frac{c_1}{x} + c_2 x^3$$

Verified OK.

10.24.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - y'x - 3y = 0$$

In normal form the ode

$$x^2y'' - y'x - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x} dx\right)} dx \\
&= \int e^{\ln(x)}\, dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{x^2}}{x^2} \\
&= -\frac{3}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{3y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{3}{x^4} = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(3x^2) - (x^3)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{9}{2}}}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{3}{2}}}{4}dx$$

Hence

$$u_1 = -\frac{x^{\frac{5}{2}}}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^{\frac{5}{2}}}dx$$

Hence

$$u_2 = -\frac{1}{6x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4x^{\frac{3}{2}}}{15}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}\right) + \left(-\frac{4x^{\frac{3}{2}}}{15}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x} - \frac{4x^{\frac{3}{2}}}{15} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x} - \frac{4x^{\frac{3}{2}}}{15}
$$

Verified OK.

10.24.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -x, C = -3, f(x) = x^{\frac{3}{2}}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - y'x - 3y = 0
$$

In normal form the ode

$$
x^2y'' - y'x - 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{3}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} - \frac{n}{x^2} - \frac{3}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 3 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{5v'(x)}{x} &= 0 \\
v''(x) + \frac{5v'(x)}{x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{5u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= e^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{4x^4} + c_2\right)x^3 \\ &= \frac{4c_2x^4 - c_1}{4x} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - y'x - 3y = x^{\frac{3}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(3x^2) - (x^3)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{9}{2}}}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{3}{2}}}{4} dx$$

Hence

$$u_1 = -\frac{x^{\frac{5}{2}}}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}}{4x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{1}{6x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4x^{\frac{3}{2}}}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right) x^3\right) + \left(-\frac{4x^{\frac{3}{2}}}{15}\right) \\ &= -\frac{4x^{\frac{3}{2}}}{15} + \left(-\frac{c_1}{4x^4} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = -\frac{-60c_2x^4 + 16x^{\frac{5}{2}} + 15c_1}{60x}$$

Summary
The solution(s) found are the following

$$y = -\frac{-60c_2x^4 + 16x^{\frac{5}{2}} + 15c_1}{60x} \tag{1}$$

Verification of solutions

$$y = -\frac{-60c_2x^4 + 16x^{\frac{5}{2}} + 15c_1}{60x}$$

Verified OK.

10.24.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' - y'x - 3y\right) dx = \int x^{\frac{3}{2}} dx$$
$$y'x^2 - 3yx = \frac{2x^{\frac{5}{2}}}{5} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x}dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right)$$
$$d\left(\frac{y}{x^3}\right) = \left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^5}\right) dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{2x^{\frac{5}{2}} + 5c_1}{5x^5} dx$$
$$\frac{y}{x^3} = -\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4}\right) + c_2 x^3$$

which simplifies to

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Verified OK.

10.24.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - y'x - 3y = x^{\frac{3}{2}}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - y'x - 3y)\,dx = \int x^{\frac{3}{2}}\,dx$$
$$y'x^2 - 3yx = \frac{2x^{\frac{5}{2}}}{5} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x}dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right)$$
$$d\left(\frac{y}{x^3}\right) = \left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^5}\right)dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{2x^{\frac{5}{2}} + 5c_1}{5x^5} \mathrm{d}x$$
$$\frac{y}{x^3} = -\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4}\right) + c_2 x^3$$

which simplifies to

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Summary
The solution(s) found are the following

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Verified OK.

10.24.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - y'x - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 434: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= e^{\int -\frac{3}{2x} dx}\\ &= \frac{1}{x^{\frac{3}{2}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\, dx}\\ &= z_1 e^{\frac{\ln(x)}{2}}\\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - y' x - 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x} + \frac{c_2 x^3}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x^3}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^3}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ -\frac{1}{x^2} & \frac{3x^2}{4} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{3x^2}{4}\right) - \left(\frac{x^3}{4}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^{\frac{9}{2}}}{4}}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{3}{2}}}{4} dx$$

Hence

$$u_1 = -\frac{x^{\frac{5}{2}}}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{2}{3x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4x^{\frac{3}{2}}}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x^3}{4}\right) + \left(-\frac{4x^{\frac{3}{2}}}{15}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} - \frac{4x^{\frac{3}{2}}}{15} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} - \frac{4x^{\frac{3}{2}}}{15}$$

Verified OK.

10.24.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= -x \\ r(x) &= -3 \\ s(x) &= x^{\frac{3}{2}} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= -1 \end{aligned}$$

Therefore (1) becomes

$$2 - (-1) + (-3) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y'x^2 - 3yx = \int x^{\frac{3}{2}}\,dx$$

We now have a first order ode to solve which is

$$y'x^2 - 3yx = \frac{2x^{\frac{5}{2}}}{5} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}
\end{aligned}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^2}\right) \\
d\left(\frac{y}{x^3}\right) &= \left(\frac{2x^{\frac{5}{2}} + 5c_1}{5x^5}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x^3} &= \int \frac{2x^{\frac{5}{2}} + 5c_1}{5x^5} dx \\
\frac{y}{x^3} &= -\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4}\right) + c_2x^3$$

which simplifies to

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^3\left(-\frac{4}{15x^{\frac{3}{2}}} - \frac{c_1}{4x^4} + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=x^(3/2),y(x), singsol=all)
```

$$y(x) = \frac{15c_2x^4 - 4x^{\frac{5}{2}} + 15c_1}{15x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]-x*y'[x]-3*y[x]==x^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{4x^{3/2}}{15}+c_2x^3+\frac{c_1}{x}$$

10.25 problem 25

Internal problem ID [1179]
Internal file name [`OUTPUT/1180_Sunday_June_05_2022_02_04_19_AM_23808311/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - x(x+4)\,y' + 2y(x+3) = \mathrm{e}^x x^4}$$

10.25.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(-x^2 - 4x\right) y' + (2x+6)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-x^2-4x}{x^2} \\
q(x) &= \frac{2x+6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2x+6}{x^2} - \frac{\left(\frac{-x^2-4x}{x^2}\right)'}{2} - \frac{\left(\frac{-x^2-4x}{x^2}\right)^2}{4} \\
&= \frac{2x+6}{x^2} - \frac{\left(\frac{-2x-4}{x^2} - \frac{2(-x^2-4x)}{x^3}\right)}{2} - \frac{\left(\frac{(-x^2-4x)^2}{x^4}\right)}{4} \\
&= \frac{2x+6}{x^2} - \left(\frac{-2x-4}{2x^2} - \frac{-x^2-4x}{x^3}\right) - \frac{(-x^2-4x)^2}{4x^4} \\
&= -\frac{1}{4}
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-x^2-4x}{x^2}}{2}} \\
&= x^2 \mathrm{e}^{\frac{x}{2}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x^2 \mathrm{e}^{\frac{x}{2}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\mathrm{e}^{\frac{x}{2}} (4v''(x) - v(x)) = 4\, \mathrm{e}^{x}
$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$
v''(x) - \frac{v(x)}{4} = \mathrm{e}^{\frac{x}{2}}
$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -\frac{1}{4}, f(x) = \mathrm{e}^{\frac{x}{2}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - \frac{v(x)}{4} = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \frac{\mathrm{e}^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\left(-\frac{1}{4}\right)}\\ &= \pm\frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{\frac{x}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{\frac{x}{2}}\right\}$$

Since $\mathrm{e}^{\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{\mathrm{e}^{\frac{x}{2}} x\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 \mathrm{e}^{\frac{x}{2}} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{\frac{x}{2}} = e^{\frac{x}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = e^{\frac{x}{2}} x$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) + \left(e^{\frac{x}{2}} x\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + e^{\frac{x}{2}} x\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2 e^{\frac{x}{2}}$$

Hence (7) becomes

$$y = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + e^{\frac{x}{2}} x\right) x^2 e^{\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + e^{\frac{x}{2}} x\right) x^2 e^{\frac{x}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2 e^x$$
$$y_2 = e^{-\frac{x}{2}} x^2 e^{\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 e^x & e^{-\frac{x}{2}} x^2 e^{\frac{x}{2}} \\ \frac{d}{dx}(x^2 e^x) & \frac{d}{dx}\left(e^{-\frac{x}{2}} x^2 e^{\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 e^x & e^{-\frac{x}{2}} x^2 e^{\frac{x}{2}} \\ 2x\, e^x + x^2 e^x & 2\, e^{-\frac{x}{2}} x\, e^{\frac{x}{2}} \end{vmatrix}$$

Therefore

$$W = \left(x^2 e^x\right)\left(2\, e^{-\frac{x}{2}} x\, e^{\frac{x}{2}}\right) - \left(e^{-\frac{x}{2}} x^2 e^{\frac{x}{2}}\right)\left(2x\, e^x + x^2 e^x\right)$$

Which simplifies to

$$W = -e^{\frac{3x}{2}} e^{-\frac{x}{2}} x^4$$

Which simplifies to

$$W = -e^x x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\frac{x}{2}} x^6 e^{\frac{x}{2}} e^x}{-x^6 e^x} \, dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{(\mathrm{e}^x)^2\, x^6}{-x^6 \mathrm{e}^x}\,dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^x dx$$

Hence

$$u_2 = -\mathrm{e}^x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \mathrm{e}^x x^3 - \mathrm{e}^x \mathrm{e}^{-\frac{x}{2}} x^2 \mathrm{e}^{\frac{x}{2}}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^x x^2(x-1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{x}{2}} x\right) x^2 \mathrm{e}^{\frac{x}{2}}\right) + \left(\mathrm{e}^x x^2 (x-1)\right) \end{aligned}$$

Which simplifies to

$$y = ((x + c_1)\,\mathrm{e}^x + c_2)\,x^2 + \mathrm{e}^x x^2 (x-1)$$

Summary

The solution(s) found are the following

$$y = ((x + c_1)\,\mathrm{e}^x + c_2)\,x^2 + \mathrm{e}^x x^2 (x-1) \tag{1}$$

Verification of solutions

$$y = ((x + c_1)\,\mathrm{e}^x + c_2)\,x^2 + \mathrm{e}^x x^2 (x-1)$$

Verified OK.

10.25.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x^2 - 4x, C = 2x + 6, f(x) = e^x x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + (-x^2 - 4x) y' + (2x + 6) y = 0$$

In normal form the ode

$$x^2y'' + (-x^2 - 4x) y' + (2x + 6) y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-x - 4}{x} \\ q(x) &= \frac{2x + 6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-x-4)}{x^2} + \frac{2x+6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4}{x} + \frac{-x-4}{x}\right) v'(x) &= 0 \\ v''(x) - v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) - u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. Integrating both sides gives

$$\begin{aligned} \int \frac{1}{u} du &= x + c_1 \\ \ln(u) &= x + c_1 \\ u &= e^{x+c_1} \\ u &= c_1 e^x \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 e^x + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (c_1 e^x + c_2)\, x^2 \\ &= (c_1 e^x + c_2)\, x^2 \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + (-x^2 - 4x)\, y' + (2x + 6)\, y = e^x x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^2 e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2 e^x \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^2 e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2 e^x \\ 2x & 2x\, e^x + x^2 e^x \end{vmatrix}$$

Therefore

$$W = (x^2)\left(2x\, e^x + x^2 e^x\right) - \left(x^2 e^x\right)(2x)$$

Which simplifies to

$$W = \mathrm{e}^{x} x^{4}$$

Which simplifies to

$$W = \mathrm{e}^{x} x^{4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{2x} x^{6}}{x^{6} \mathrm{e}^{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \mathrm{e}^{x} dx$$

Hence

$$u_1 = -\mathrm{e}^{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{6} \mathrm{e}^{x}}{x^{6} \mathrm{e}^{x}}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \mathrm{e}^{x} x^{3} - x^{2} \mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{x} x^{2} (x - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 e^x + c_2\right) x^2\right) + \left(e^x x^2 (x-1)\right) \\ &= e^x x^2 (x-1) + \left(c_1 e^x + c_2\right) x^2 \end{aligned}$$

Which simplifies to

$$y = \left((x + c_1 - 1) e^x + c_2\right) x^2$$

Summary

The solution(s) found are the following

$$y = \left((x + c_1 - 1) e^x + c_2\right) x^2 \tag{1}$$

Verification of solutions

$$y = \left((x + c_1 - 1) e^x + c_2\right) x^2$$

Verified OK.

10.25.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + \left(-x^2 - 4x\right) y' + (2x + 6) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 4x \\ C &= 2x + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-4x}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+2\ln(x)} \\ &= z_1\left(x^2 \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\mathrm{e}^x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + \left(-x^2 - 4x\right)y' + (2x+6)\,y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1x^2 + c_2x^2\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^2\mathrm{e}^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2 e^x \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^2 e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2 e^x \\ 2x & 2x\, e^x + x^2 e^x \end{vmatrix}$$

Therefore

$$W = (x^2)\left(2x\, e^x + x^2 e^x\right) - \left(x^2 e^x\right)(2x)$$

Which simplifies to

$$W = e^x x^4$$

Which simplifies to

$$W = e^x x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} x^6}{x^6 e^x}\, dx$$

Which simplifies to

$$u_1 = -\int e^x dx$$

Hence

$$u_1 = -e^x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6 e^x}{x^6 e^x}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = e^x x^3 - x^2 e^x$$

Which simplifies to

$$y_p(x) = e^x x^2 (x - 1)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2 + c_2 x^2 e^x) + (e^x x^2 (x - 1)) \end{aligned}$$

Which simplifies to

$$y = x^2 (c_1 + c_2 e^x) + e^x x^2 (x - 1)$$

Summary

The solution(s) found are the following

$$y = x^2 (c_1 + c_2 e^x) + e^x x^2 (x - 1) \tag{1}$$

Verification of solutions

$$y = x^2 (c_1 + c_2 e^x) + e^x x^2 (x - 1)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x$2)-x*(x+4)*diff(y(x),x)+2*(x+3)*y(x)=x^4*exp(x),y(x), singsol=all)
```

$$y(x) = ((c_2 + x)\,\mathrm{e}^{x} + c_1)\,x^2$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]-x*(x+4)*y'[x]+2*(x+3)*y[x]==x^4*Exp[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to x^2(e^x(x - 1 + c_2) + c_1)$$

10.26 problem 26

Internal problem ID [1180]
Internal file name [`OUTPUT/1181_Sunday_June_05_2022_02_04_21_AM_43398011/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' - 2x(2+x)\,y' + \left(x^2+4x+6\right)y = 2x\,\mathrm{e}^x}$$

10.26.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\,dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1\left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\mathrm{e}^x\right) + c_2\left(x^2\mathrm{e}^x(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + \left(-2x^2 - 4x\right)y' + \left(x^2 + 4x + 6\right)y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = x^2\mathrm{e}^x c_1 + c_2\mathrm{e}^x x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2\mathrm{e}^x \\
y_2 &= \mathrm{e}^x x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 e^x & e^x x^3 \\ \frac{d}{dx}(x^2 e^x) & \frac{d}{dx}(e^x x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 e^x & e^x x^3 \\ 2x\, e^x + x^2 e^x & e^x x^3 + 3x^2 e^x \end{vmatrix}$$

Therefore

$$W = (x^2 e^x)(e^x x^3 + 3x^2 e^x) - (e^x x^3)(2x\, e^x + x^2 e^x)$$

Which simplifies to

$$W = e^{2x} x^4$$

Which simplifies to

$$W = e^{2x} x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\, e^{2x} x^4}{e^{2x} x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{x^2} dx$$

Hence

$$u_1 = \frac{2}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\, e^{2x} x^3}{e^{2x} x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,\mathrm{e}^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^2\mathrm{e}^x c_1 + c_2\mathrm{e}^x x^3\right) + \left(x\,\mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x x^2(c_2 x + c_1) + x\,\mathrm{e}^x$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^x x^2(c_2 x + c_1) + x\,\mathrm{e}^x \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x x^2(c_2 x + c_1) + x\,\mathrm{e}^x$$

Verified OK.

10.26.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$x^2y'' + \left(-2x^2 - 4x\right)y' + \left(x^2 + 4x + 6\right)y = 2x\,\mathrm{e}^x \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2x - 4}{x} \\ q(x) &= \frac{x^2 + 4x + 6}{x^2} \\ r(x) &= \frac{2\,\mathrm{e}^x}{x} \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(-2x-4)\,\xi(x)}{x}\right)' + \left(\frac{(x^2+4x+6)\,\xi(x)}{x^2}\right) &= 0 \\
\xi''(x) - \frac{(-2x-4)\,\xi'(x)}{x} + \left(\frac{2}{x} + \frac{-2x-4}{x^2} + \frac{x^2+4x+6}{x^2}\right)\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$
\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{2x^2+4x}{x^2} \\
q(x) &= \frac{x^2+4x+2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+4x+2}{x^2} - \frac{\left(\frac{2x^2+4x}{x^2}\right)'}{2} - \frac{\left(\frac{2x^2+4x}{x^2}\right)^2}{4} \\
&= \frac{x^2+4x+2}{x^2} - \frac{\left(\frac{4x+4}{x^2} - \frac{2(2x^2+4x)}{x^3}\right)}{2} - \frac{\left(\frac{(2x^2+4x)^2}{x^4}\right)}{4} \\
&= \frac{x^2+4x+2}{x^2} - \left(\frac{4x+4}{2x^2} - \frac{2x^2+4x}{x^3}\right) - \frac{(2x^2+4x)^2}{4x^4} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
\xi(x) = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2x^2+4x}{x^2}}{2}} \\
&= \frac{\mathrm{e}^{-x}}{x^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$\xi(x) = \frac{v(x)\,\mathrm{e}^{-x}}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x)\,\mathrm{e}^{-x} = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned}\xi(x) &= v(x)\,z(x)\\ &= (c_1 x + c_2)\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{\mathrm{e}^{-x}}{x^2}$$

Hence (7) becomes

$$\xi(x) = \frac{(c_1 x + c_2)\,\mathrm{e}^{-x}}{x^2}$$

The original ode (2) now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}\\ y' + y\left(\frac{-2x-4}{x} - \frac{\left(\frac{c_3\mathrm{e}^{-x}}{x^2} - \frac{2(c_3x+c_2)\mathrm{e}^{-x}}{x^3} - \frac{(c_3x+c_2)\mathrm{e}^{-x}}{x^2}\right)x^2\mathrm{e}^{x}}{c_3x+c_2}\right) &= -\frac{\mathrm{e}^{x}(2c_3x+c_2)}{c_3x+c_2}\end{aligned}$$

Which is now a first order ode. This is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{c_3x^2 + (c_2 + 3c_3)\,x + 2c_2}{x\,(c_3x + c_2)}\\ q(x) &= \frac{(-2c_3x - c_2)\,\mathrm{e}^{x}}{c_3x + c_2}\end{aligned}$$

Hence the ode is

$$y' - \frac{(c_3x^2 + (c_2+3c_3)\,x + 2c_2)\,y}{x\,(c_3x+c_2)} = \frac{(-2c_3x - c_2)\,\mathrm{e}^x}{c_3x+c_2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{c_3x^2+(c_2+3c_3)x+2c_2}{x(c_3x+c_2)}dx}\\ &= \mathrm{e}^{-x-2\ln(x)-\ln(c_3x+c_2)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{-x}}{x^2\,(c_3x+c_2)}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-2c_3x-c_2)\,\mathrm{e}^x}{c_3x+c_2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{-x}y}{x^2\,(c_3x+c_2)}\right) &= \left(\frac{\mathrm{e}^{-x}}{x^2\,(c_3x+c_2)}\right)\left(\frac{(-2c_3x-c_2)\,\mathrm{e}^x}{c_3x+c_2}\right)\\ \mathrm{d}\left(\frac{\mathrm{e}^{-x}y}{x^2\,(c_3x+c_2)}\right) &= \left(\frac{-2c_3x-c_2}{x^2\,(c_3x+c_2)^2}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\mathrm{e}^{-x}y}{x^2\,(c_3x+c_2)} &= \int \frac{-2c_3x-c_2}{x^2\,(c_3x+c_2)^2}\,\mathrm{d}x\\ \frac{\mathrm{e}^{-x}y}{x^2\,(c_3x+c_2)} &= -\frac{c_3}{c_2\,(c_3x+c_2)} + \frac{1}{c_2x} + c_3\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{-x}}{x^2(c_3x+c_2)}$ results in

$$y = x^2(c_3x+c_2)\,\mathrm{e}^x\left(-\frac{c_3}{c_2\,(c_3x+c_2)} + \frac{1}{c_2x}\right) + c_3x^2(c_3x+c_2)\,\mathrm{e}^x$$

which simplifies to

$$y = \mathrm{e}^x x\left(c_3^2x^2 + c_2c_3x + 1\right)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \mathrm{e}^x c_2c_3x^2 + x\,\mathrm{e}^x\left(c_3^2x^2 + 1\right)$$

Summary
The solution(s) found are the following

$$y = e^{x} c_2 c_3 x^2 + x e^{x} (c_3^2 x^2 + 1) \tag{1}$$

Verification of solutions

$$y = e^{x} c_2 c_3 x^2 + x e^{x} (c_3^2 x^2 + 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x)=2*x*exp(x),y(x), singsol=a
```

$$y(x) = e^{x} x (c_1 x^2 + c_2 x + 1)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-2*x*(x+2)*y'[x]+(x^2+4*x+6)*y[x]==2*x*Exp[x],y[x],x,IncludeSingularSolutio
```

$$y(x) \to e^x x\left(c_2 x^2 + c_1 x + 1\right)$$

10.27 problem 27

Internal problem ID [1181]
Internal file name [`OUTPUT/1182_Sunday_June_05_2022_02_04_22_AM_92007663/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 4y'x + (x^2 + 6)\, y = x^4}$$

10.27.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + (x^2 + 6)\, y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{x^2+6}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2+6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{x^2+6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{x^2+6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x^2 \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + v(x) = 1
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 1, f(x) = 1$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = 1$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x) + 1)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 \cos(x) + c_2 \sin(x) + 1)\, x^2$$

Therefore the homogeneous solution y_h is

$$y_h = (c_1 \cos(x) + c_2 \sin(x) + 1)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \cos(x) \\ y_2 &= \sin(x)\, x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x) x^2 \\ \frac{d}{dx}(x^2 \cos(x)) & \frac{d}{dx}(\sin(x) x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x) x^2 \\ 2x \cos(x) - \sin(x) x^2 & x^2 \cos(x) + 2\sin(x) x \end{vmatrix}$$

Therefore

$$W = \left(x^2 \cos(x)\right)\left(x^2 \cos(x) + 2\sin(x) x\right) - \left(\sin(x) x^2\right)\left(2x \cos(x) - \sin(x) x^2\right)$$

Which simplifies to

$$W = \cos(x)^2 x^4 + \sin(x)^2 x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) x^6}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, dx$$

Hence

$$u_1 = \cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6 \cos(x)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos(x)^2 x^2 + x^2 \sin(x)^2$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \cos(x) + c_2 \sin(x) + 1)\, x^2\right) + \left(x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \cos(x) + c_2 \sin(x) + 1)\, x^2 + x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \cos(x) + c_2 \sin(x) + 1)\, x^2 + x^2$$

Verified OK.

10.27.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - 4y'x + (x^2 + 6)\,y = x^4 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\,y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2)\,y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1\,\text{BesselJ}\,(n, \beta\,x^{\gamma}) + c_2\,\text{BesselY}\,(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{5}{2} \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1x^2\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2x^2\sqrt{2}\,\sin(x)}{\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1x^2\sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2x^2\sqrt{2}\,\sin(x)}{\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2 \cos(x)$$
$$y_2 = \sin(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x)\, x^2 \\ \frac{d}{dx}\left(x^2 \cos(x)\right) & \frac{d}{dx}\left(\sin(x)\, x^2\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x)\, x^2 \\ 2x \cos(x) - \sin(x)\, x^2 & x^2 \cos(x) + 2\sin(x)\, x \end{vmatrix}$$

Therefore

$$W = \left(x^2 \cos(x)\right)\left(x^2 \cos(x) + 2\sin(x)\, x\right) - \left(\sin(x)\, x^2\right)\left(2x \cos(x) - \sin(x)\, x^2\right)$$

Which simplifies to

$$W = \cos(x)^2\, x^4 + \sin(x)^2\, x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\,x^6}{x^6}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\,dx$$

Hence

$$u_1 = \cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6 \cos(x)}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \cos(x)\,dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos(x)^2 x^2 + x^2 \sin(x)^2$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2 \sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\,\sin(x)}{\sqrt{\pi}}\right) + (x^2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 \sqrt{2}\,\cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\,\sin(x)}{\sqrt{\pi}} + x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 \sqrt{2}\, \cos(x)}{\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\, \sin(x)}{\sqrt{\pi}} + x^2$$

Verified OK.

10.27.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4y'x + (x^2 + 6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\cos(x)\right) + c_2\left(x^2\cos(x)\,(\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y'' - 4y'x + \left(x^2+6\right)y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, x^2 c_1 + \sin(x)\, x^2 c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2 \cos(x)$$
$$y_2 = \sin(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x)\, x^2 \\ \frac{d}{dx}\left(x^2 \cos(x)\right) & \frac{d}{dx}\left(\sin(x)\, x^2\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 \cos(x) & \sin(x)\, x^2 \\ 2x \cos(x) - \sin(x)\, x^2 & x^2 \cos(x) + 2\sin(x)\, x \end{vmatrix}$$

Therefore

$$W = \left(x^2 \cos(x)\right)\left(x^2 \cos(x) + 2\sin(x)\, x\right) - \left(\sin(x)\, x^2\right)\left(2x \cos(x) - \sin(x)\, x^2\right)$$

Which simplifies to

$$W = \cos(x)^2 x^4 + \sin(x)^2 x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) x^6}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \sin(x) dx$$

Hence

$$u_1 = \cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6 \cos(x)}{x^6} dx$$

Which simplifies to

$$u_2 = \int \cos(x) dx$$

Hence

$$u_2 = \sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos(x)^2 x^2 + x^2 \sin(x)^2$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\cos(x)\, x^2 c_1 + \sin(x)\, x^2 c_2\right) + \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y = x^2(c_1 \cos(x) + c_2 \sin(x)) + x^2
$$

Summary

The solution(s) found are the following

$$
y = x^2(c_1 \cos(x) + c_2 \sin(x)) + x^2 \tag{1}
$$

Verification of solutions

$$
y = x^2(c_1 \cos(x) + c_2 \sin(x)) + x^2
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(x^2+6)*y(x)=x^4,y(x), singsol=all)
```

$$y(x) = x^2(1 + \sin(x)\, c_2 + \cos(x)\, c_1)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 38

```
DSolve[x^2*y''[x]-4*x*y'[x]+(x^2+6)*y[x]==x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x^2\left(2c_1e^{-ix} - ic_2e^{ix} + 2\right)$$

10.28 problem 28

Internal problem ID [1182]
Internal file name [`OUTPUT/1183_Sunday_June_05_2022_02_04_23_AM_76507277/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-y'x+y=2(x-1)^2\,\mathrm{e}^{x}}$$

10.28.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x-1, B=-x, C=1, f(x)=2(x-1)^2\,\mathrm{e}^{x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(x-1)\,y''-y'x+y=0$$

In normal form the ode

$$(x-1)\,y''-y'x+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x)&=-\frac{x}{x-1}\\ q(x)&=\frac{1}{x-1} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{n}{x-1}+\frac{1}{x-1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}-\frac{x}{x-1}\right)v'(x)&=0\\ v''(x)+\left(\frac{2}{x}-\frac{x}{x-1}\right)v'(x)&=0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-2x+2)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x\,(x-1)}\,dx \\ \ln(u) &= x - 2\ln(x) + \ln(x-1) + c_1 \\ u &= \mathrm{e}^{x-2\ln(x)+\ln(x-1)+c_1} \\ &= c_1 \mathrm{e}^{x-2\ln(x)+\ln(x-1)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \frac{c_1 \mathrm{e}^x}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\frac{c_1 \mathrm{e}^x}{x} + c_2\right) x \\ &= c_1 \mathrm{e}^x + c_2 x \end{aligned}$$

Now the particular solution to this ODE is found

$$(x-1)\,y''-y'x+y=2(x-1)^2\,\mathrm{e}^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=x$$
$$y_2=\mathrm{e}^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}x & \mathrm{e}^x\\ \frac{d}{dx}(x) & \frac{d}{dx}(\mathrm{e}^x)\end{vmatrix}$$

Which gives

$$W=\begin{vmatrix}x & \mathrm{e}^x\\ 1 & \mathrm{e}^x\end{vmatrix}$$

Therefore

$$W=(x)\,(\mathrm{e}^x)-(\mathrm{e}^x)\,(1)$$

Which simplifies to

$$W = x\,e^{x} - e^{x}$$

Which simplifies to

$$W = (x-1)\,e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x-1)^2\,e^{2x}}{(x-1)^2\,e^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,e^{x} dx$$

Hence

$$u_1 = -2\,e^{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x(x-1)^2\,e^{x}}{(x-1)^2\,e^{x}}\,dx$$

Which simplifies to

$$u_2 = \int 2x dx$$

Hence

$$u_2 = x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x\,e^{x} + x^2 e^{x}$$

Which simplifies to

$$y_p(x) = x(-2+x)\,e^{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{c_1 e^x}{x} + c_2\right) x\right) + (x(-2+x)\, e^x) \\
&= x(-2+x)\, e^x + \left(\frac{c_1 e^x}{x} + c_2\right) x
\end{aligned}
$$

Which simplifies to

$$
y = \left(x^2 + c_1 - 2x\right) e^x + c_2 x
$$

Summary

The solution(s) found are the following

$$
y = \left(x^2 + c_1 - 2x\right) e^x + c_2 x \tag{1}
$$

Verification of solutions

$$
y = \left(x^2 + c_1 - 2x\right) e^x + c_2 x
$$

Verified OK.

10.28.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1 \\
F &= 2(x - 1)^2 \, \mathrm{e}^x
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x - 1)\,(0) + (-x)\,(-1) + (1)\,(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x(x - 1)\,v'' + \left(x^2 - 2x + 2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(-x^2 + x\right) u'(x) + \left(x^2 - 2x + 2\right) u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u(x^2 - 2x + 2)}{x\,(x - 1)}
\end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x\,(x-1)}\,dx \\
\ln(u) &= x - 2\ln(x) + \ln(x-1) + c_1 \\
u &= \mathrm{e}^{x-2\ln(x)+\ln(x-1)+c_1} \\
&= c_1 \mathrm{e}^{x-2\ln(x)+\ln(x-1)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{(x-1)\,c_1\mathrm{e}^x}{x^2}\,\mathrm{d}x \\
&= \frac{c_1\mathrm{e}^x}{x} + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-x)\left(\frac{c_1\mathrm{e}^x}{x} + c_2\right) \\
&= -c_1\mathrm{e}^x - c_2 x
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}$$

Therefore

$$W = (x)(e^x) - (e^x)(1)$$

Which simplifies to

$$W = x e^x - e^x$$

Which simplifies to

$$W = (x-1)e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x-1)^2 e^{2x}}{(x-1)^2 e^x} dx$$

Which simplifies to

$$u_1 = -\int 2\,\mathrm{e}^x dx$$

Hence

$$u_1 = -2\,\mathrm{e}^x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x(x-1)^2\,\mathrm{e}^x}{(x-1)^2\,\mathrm{e}^x}\,dx$$

Which simplifies to

$$u_2 = \int 2x dx$$

Hence

$$u_2 = x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x\,\mathrm{e}^x + x^2\mathrm{e}^x$$

Which simplifies to

$$y_p(x) = x(-2+x)\,\mathrm{e}^x$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-c_1\mathrm{e}^x - c_2 x) + (x(-2+x)\,\mathrm{e}^x) \\ &= \left(x^2 - c_1 - 2x\right)\mathrm{e}^x - c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(x^2 - c_1 - 2x\right)\mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = \left(x^2 - c_1 - 2x\right)\mathrm{e}^x - c_2 x$$

Verified OK.

10.28.3 Solving using Kovacic algorithm

Writing the ode as

$$(x-1)\,y''-y'x+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{x^2-4x+6}{4\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=x^2-4x+6\\t&=4(x-1)^2\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 438: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-4x+6}{4x^2-8x+4} \\ &= Q + \frac{R}{4x^2-8x+4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x+5}{4x^2-8x+4}\right) \\ &= \frac{1}{4} + \frac{-2x+5}{4x^2-8x+4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2-4x+6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\,(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2\,(x-1)} + \frac{1}{2} \\ &= \frac{-2+x}{2x-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x-1)\,y'' - y'x + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x - c_2 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= -x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & -x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(-x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & -x \\ e^x & -1 \end{vmatrix}$$

Therefore

$$W = (e^x)(-1) - (-x)(e^x)$$

Which simplifies to

$$W = x e^x - e^x$$

Which simplifies to

$$W = (x-1)e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2x(x-1)^2 e^x}{(x-1)^2 e^x}\,dx$$

Which simplifies to

$$u_1 = -\int -2x dx$$

Hence

$$u_1 = x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(x-1)^2 e^{2x}}{(x-1)^2 e^x}\,dx$$

Which simplifies to

$$u_2 = \int 2\,\mathrm{e}^x dx$$

Hence

$$u_2 = 2\,\mathrm{e}^x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x\,\mathrm{e}^x + x^2\mathrm{e}^x$$

Which simplifies to

$$y_p(x) = x(-2+x)\,\mathrm{e}^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\mathrm{e}^x - c_2x) + (x(-2+x)\,\mathrm{e}^x) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\mathrm{e}^x - c_2x + x(-2+x)\,\mathrm{e}^x \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\mathrm{e}^x - c_2x + x(-2+x)\,\mathrm{e}^x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=2*(x-1)^2*exp(x),y(x), singsol=all)
```

$$y(x) = (x^2 + c_1 - 2x)\,e^x + c_2 x$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 24

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==2*(x-1)^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (x^2 - 2x + c_1) - c_2 x$$

10.29 problem 29

Internal problem ID [1183]
Internal file name [`OUTPUT/1184_Sunday_June_05_2022_02_04_25_AM_71535634/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = x^{\frac{5}{2}}\mathrm{e}^x}$$

10.29.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{-4x^2 - 4x}{4x^2}$$
$$q(x) = \frac{2x + 3}{4x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2x+3}{4x^2} - \frac{\left(\frac{-4x^2-4x}{4x^2}\right)'}{2} - \frac{\left(\frac{-4x^2-4x}{4x^2}\right)^2}{4} \\
&= \frac{2x+3}{4x^2} - \frac{\left(\frac{-8x-4}{4x^2} - \frac{-4x^2-4x}{2x^3}\right)}{2} - \frac{\left(\frac{(-4x^2-4x)^2}{16x^4}\right)}{4} \\
&= \frac{2x+3}{4x^2} - \left(\frac{-8x-4}{8x^2} - \frac{-4x^2-4x}{4x^3}\right) - \frac{(-4x^2-4x)^2}{64x^4} \\
&= -\frac{1}{4}
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= e^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-4x^2-4x}{4x^2}}{2}} \\
&= \sqrt{x}\, e^{\frac{x}{2}}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\sqrt{x}\, e^{\frac{x}{2}} \tag{4}$$

Applying this change of variable to the original ode results in

$$e^{\frac{x}{2}} (4v''(x) - v(x)) = e^x$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) - \frac{v(x)}{4} = \frac{e^{\frac{x}{2}}}{4}$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -\frac{1}{4}, f(x) = \frac{e^{\frac{x}{2}}}{4}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - \frac{v(x)}{4} = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \frac{e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(-\frac{1}{4}\right)} \\ &= \pm\frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{\mathrm{e}^{\frac{x}{2}}}{4}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{\frac{x}{2}}\right\}$$

Since $\mathrm{e}^{\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{\mathrm{e}^{\frac{x}{2}} x\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 \mathrm{e}^{\frac{x}{2}} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{\frac{x}{2}} = \frac{e^{\frac{x}{2}}}{4}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = \frac{e^{\frac{x}{2}} x}{4}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) + \left(\frac{e^{\frac{x}{2}} x}{4}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{e^{\frac{x}{2}} x}{4}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}\, e^{\frac{x}{2}}$$

Hence (7) becomes

$$y = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{e^{\frac{x}{2}} x}{4}\right) \sqrt{x}\, e^{\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{e^{\frac{x}{2}} x}{4}\right) \sqrt{x}\, e^{\frac{x}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\, e^{x}$$
$$y_2 = e^{-\frac{x}{2}} \sqrt{x}\, e^{\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\, e^{x} & e^{-\frac{x}{2}} \sqrt{x}\, e^{\frac{x}{2}} \\ \frac{d}{dx}\left(\sqrt{x}\, e^{x}\right) & \frac{d}{dx}\left(e^{-\frac{x}{2}} \sqrt{x}\, e^{\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\, e^{x} & e^{-\frac{x}{2}} \sqrt{x}\, e^{\frac{x}{2}} \\ \sqrt{x}\, e^{x} + \frac{e^{x}}{2\sqrt{x}} & \frac{e^{-\frac{x}{2}} e^{\frac{x}{2}}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\, e^{x}\right) \left(\frac{e^{-\frac{x}{2}} e^{\frac{x}{2}}}{2\sqrt{x}}\right) - \left(e^{-\frac{x}{2}} \sqrt{x}\, e^{\frac{x}{2}}\right) \left(\sqrt{x}\, e^{x} + \frac{e^{x}}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = -e^{\frac{3x}{2}} e^{-\frac{x}{2}} x$$

Which simplifies to

$$W = -x\,\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-\frac{x}{2}} x^3 \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{x}}{-4\,\mathrm{e}^{x} x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{4} dx$$

Hence

$$u_1 = \frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(\mathrm{e}^{x})^2\, x^3}{-4\,\mathrm{e}^{x} x^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{x}}{4} dx$$

Hence

$$u_2 = -\frac{\mathrm{e}^{x}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{\frac{3}{2}} \mathrm{e}^{x}}{4} - \frac{\mathrm{e}^{x} \mathrm{e}^{-\frac{x}{2}} \sqrt{x}\,\mathrm{e}^{\frac{x}{2}}}{4}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x} \sqrt{x}\,(x-1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} + \frac{\mathrm{e}^{\frac{x}{2}} x}{4}\right) \sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right) + \left(\frac{\mathrm{e}^{x} \sqrt{x}\,(x-1)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{((x + 4c_1)\,e^x + 4c_2)\,\sqrt{x}}{4} + \frac{e^x \sqrt{x}\,(x - 1)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{((x + 4c_1)\,e^x + 4c_2)\,\sqrt{x}}{4} + \frac{e^x \sqrt{x}\,(x - 1)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{((x + 4c_1)\,e^x + 4c_2)\,\sqrt{x}}{4} + \frac{e^x \sqrt{x}\,(x - 1)}{4}$$

Verified OK.

10.29.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -4x^2 - 4x, C = 2x + 3, f(x) = x^{\frac{5}{2}} e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$4x^2y'' + (-4x^2 - 4x)\,y' + (2x + 3)\,y = 0$$

In normal form the ode

$$4x^2y'' + (-4x^2 - 4x)\,y' + (2x + 3)\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-x-1}{x} \\
q(x) &= \frac{2x+3}{4x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n(-x-1)}{x^2} + \frac{2x+3}{4x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = \frac{1}{2} \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{1}{x} + \frac{-x-1}{x}\right) v'(x) &= 0 \\
v''(x) - v'(x) &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) - u(x) = 0 \tag{8}
$$

The above is now solved for $u(x)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{u} du &= x + c_1 \\
\ln(u) &= x + c_1 \\
u &= e^{x+c_1} \\
u &= c_1 e^x
\end{aligned}
$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \mathrm{e}^{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^{n} \\ &= (c_1 \mathrm{e}^{x} + c_2)\,\sqrt{x} \\ &= (c_1 \mathrm{e}^{x} + c_2)\,\sqrt{x} \end{aligned}$$

Now the particular solution to this ODE is found

$$4x^{2}y'' + \left(-4x^{2} - 4x\right) y' + (2x+3)\,y = x^{\frac{5}{2}} \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{x} \\ y_2 &= \sqrt{x}\,\mathrm{e}^{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,e^{x} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(\sqrt{x}\,e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,e^{x} \\ \frac{1}{2\sqrt{x}} & \sqrt{x}\,e^{x} + \frac{e^{x}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)\left(\sqrt{x}\,e^{x} + \frac{e^{x}}{2\sqrt{x}}\right) - \left(\sqrt{x}\,e^{x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\,e^{x}$$

Which simplifies to

$$W = x\,e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} x^3}{4\,e^{x} x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{x}}{4} dx$$

Hence

$$u_1 = -\frac{e^{x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} x^3}{4\,e^{x} x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4} dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{\frac{3}{2}} \mathrm{e}^x}{4} - \frac{\sqrt{x}\, \mathrm{e}^x}{4}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^x \sqrt{x}\, (x - 1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \mathrm{e}^x + c_2) \sqrt{x}\right) + \left(\frac{\mathrm{e}^x \sqrt{x}\, (x - 1)}{4}\right) \\ &= \frac{\mathrm{e}^x \sqrt{x}\, (x - 1)}{4} + (c_1 \mathrm{e}^x + c_2) \sqrt{x} \end{aligned}$$

Which simplifies to

$$y = \frac{\sqrt{x}\, ((x + 4c_1 - 1)\, \mathrm{e}^x + 4c_2)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x}\, ((x + 4c_1 - 1)\, \mathrm{e}^x + 4c_2)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{x}\, ((x + 4c_1 - 1)\, \mathrm{e}^x + 4c_2)}{4}$$

Verified OK.

10.29.3 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + (-4x^2 - 4x)\,y' + (2x+3)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2-4x}{4x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4x^2y'' + \left(-4x^2 - 4x\right) y' + (2x + 3)\, y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \sqrt{x}\, c_1 + c_2\sqrt{x}\, \mathrm{e}^{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}$$
$$y_2 = \sqrt{x}\,\mathrm{e}^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\mathrm{e}^{x} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{e}^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\mathrm{e}^{x} \\ \frac{1}{2\sqrt{x}} & \sqrt{x}\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)\left(\sqrt{x}\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}}{2\sqrt{x}}\right) - \left(\sqrt{x}\,\mathrm{e}^{x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x\,\mathrm{e}^{x}$$

Which simplifies to

$$W = x\,\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{2x} x^{3}}{4\,\mathrm{e}^{x} x^{3}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{x}}{4}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{x} x^{3}}{4\,\mathrm{e}^{x} x^{3}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4}dx$$

Hence

$$u_2 = \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{\frac{3}{2}}\mathrm{e}^{x}}{4} - \frac{\sqrt{x}\,\mathrm{e}^{x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x}\sqrt{x}\,(x-1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\,c_1 + c_2\sqrt{x}\,\mathrm{e}^{x}\right) + \left(\frac{\mathrm{e}^{x}\sqrt{x}\,(x-1)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = (c_1 + c_2 e^x)\sqrt{x} + \frac{e^x\sqrt{x}\,(x-1)}{4}$$

Summary

The solution(s) found are the following

$$y = (c_1 + c_2 e^x)\sqrt{x} + \frac{e^x\sqrt{x}\,(x-1)}{4} \tag{1}$$

Verification of solutions

$$y = (c_1 + c_2 e^x)\sqrt{x} + \frac{e^x\sqrt{x}\,(x-1)}{4}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x)=x^(5/2)*exp(x),y(x), singsol
```

$$y(x) = \frac{\sqrt{x}\left(\left(x+4c_1\right)\mathrm{e}^x+4c_2\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 30

```
DSolve[4*x^2*y''[x]-4*x*(x+1)*y'[x]+(2*x+3)*y[x]==x^(5/2)*Exp[x],y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{4}\sqrt{x}(e^x(x-1+4c_2)+4c_1)$$

10.30 problem 30

Internal problem ID [1184]
Internal file name [`OUTPUT/1185_Sunday_June_05_2022_02_04_27_AM_59910487/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=(3x-1)^2\,\mathrm{e}^{2x}}$$

With initial conditions

$$[y(0)=1, y'(0)=2]$$

10.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{-3x-2}{3x-1}\\q(x)&=\frac{-6x+8}{3x-1}\\F&=(3x-1)\,\mathrm{e}^{2x}\end{aligned}$$

Hence the ode is

$$y'' + \frac{(-3x-2)\,y'}{3x-1} + \frac{(-6x+8)\,y}{3x-1} = (3x-1)\,\mathrm{e}^{2x}$$

The domain of $p(x) = \frac{-3x-2}{3x-1}$ is

$$\left\{x < \frac{1}{3} \lor \frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{-6x+8}{3x-1}$ is

$$\left\{x < \frac{1}{3} \lor \frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = (3x-1)\,\mathrm{e}^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.30.2 Solving using Kovacic algorithm

Writing the ode as

$$(3x-1)\,y'' + (-3x-2)\,y' + (-6x+8)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2 - 108x + 54 \\ t &= 4(3x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4(3x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 440: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4(3x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x - 6}{6x - 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x - 1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\sqrt{3x-1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-x\,\mathrm{e}^{-3x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-x\,\mathrm{e}^{-3x}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(3x-1)\,y'' + (-3x-2)\,y' + (-6x+8)\,y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} - x\,e^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{2x} \\ y_2 &= -x\,e^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & -x\,e^{-x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(-x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & -x\,e^{-x} \\ 2\,e^{2x} & -e^{-x} + x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(-e^{-x} + x\,e^{-x}\right) - \left(-x\,e^{-x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = 3x\,\mathrm{e}^{-x}\mathrm{e}^{2x} - \mathrm{e}^{2x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = (3x-1)\,\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-x\,\mathrm{e}^{-x}(3x-1)^2\,\mathrm{e}^{2x}}{(3x-1)^2\,\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int -x dx$$

Hence

$$u_1 = \frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{4x}(3x-1)^2}{(3x-1)^2\,\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{3x} dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{3x}}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{3x}x\,\mathrm{e}^{-x}}{3}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{2x}x(3x-2)}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{2x} - x\,\mathrm{e}^{-x} c_2\right) + \left(\frac{\mathrm{e}^{2x} x(3x-2)}{6}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} - x\,\mathrm{e}^{-x} c_2 + \frac{\mathrm{e}^{2x} x(3x-2)}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1 \mathrm{e}^{2x} - c_2 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{\mathrm{e}^{2x} x(3x-2)}{3} + \frac{\mathrm{e}^{2x}(3x-2)}{6} + \frac{x\,\mathrm{e}^{2x}}{2}
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = 2c_1 - c_2 - \frac{1}{3} \tag{2A}
$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -\frac{1}{3}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{2x} + \frac{x\,\mathrm{e}^{-x}}{3} + \frac{x^2 \mathrm{e}^{2x}}{2} - \frac{x\,\mathrm{e}^{2x}}{3}
$$

Which simplifies to

$$
y = \frac{(3x^2 - 2x + 6)\,\mathrm{e}^{2x}}{6} + \frac{x\,\mathrm{e}^{-x}}{3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(3x^2 - 2x + 6)\,\mathrm{e}^{2x}}{6} + \frac{x\,\mathrm{e}^{-x}}{3} \tag{1}
$$

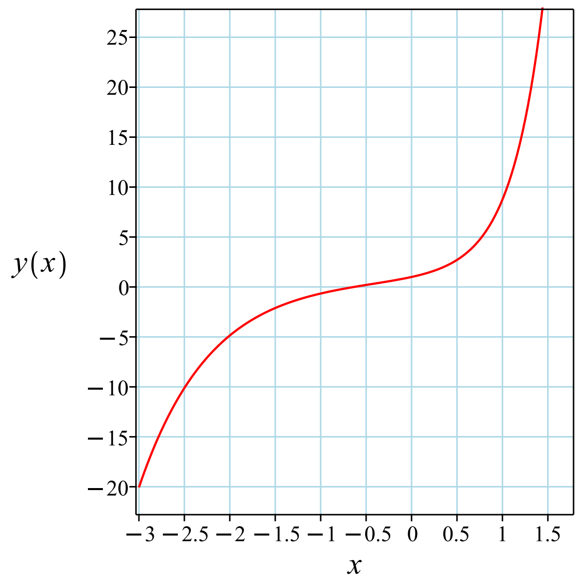


Figure 521: Solution plot

Verification of solutions

$$y = \frac{(3x^2 - 2x + 6)\,e^{2x}}{6} + \frac{x\,e^{-x}}{3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve([(3*x-1)*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x)=(3*x-1)^2*exp(2*x),y(0) = 1
```

$$y(x) = \frac{(3x^2 - 2x + 6)\,e^{2x}}{6} + \frac{x\,e^{-x}}{3}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 34

```
DSolve[{(3*x-1)*y''[x]-(3*x+2)*y'[x]-(6*x-8)*y[x]==(3*x-1)^2*Exp[2*x],{y[0]==1,y'[0]==2}},y[
```

$$y(x) \to \frac{1}{6}e^{-x}\left(e^{3x}\left(3x^2 - 2x + 6\right) + 2x\right)$$

10.31 problem 31

Internal problem ID [1185]
Internal file name [`OUTPUT/1186_Sunday_June_05_2022_02_04_29_AM_25254914/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)^2 y''-2(x-1)\,y'+2y=(x-1)^2}$$

With initial conditions

$$[y(0)=3, y'(0)=-6]$$

10.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{-2x+2}{(x-1)^2} \\ q(x) &= \frac{2}{(x-1)^2} \\ F &= 1 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-2x+2)\,y'}{(x-1)^2} + \frac{2y}{(x-1)^2} = 1$$

The domain of $p(x) = \frac{-2x+2}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.31.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -\frac{2}{x-1}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{x-1}\,dx} \\ &= \frac{1}{x-1} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{1}{x-1} \\ \left(\frac{y}{x-1}\right)'' &= \frac{1}{x-1} \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x-1}\right)' = \ln(x-1) + c_1$$

Integrating again gives

$$\left(\frac{y}{x-1}\right) = \ln(x-1)(x-1) + 1 + (c_1 - 1)x + c_2$$

Hence the solution is

$$y = \frac{\ln(x-1)(x-1) + 1 + (c_1 - 1)x + c_2}{\frac{1}{x-1}}$$

Or

$$y = x^2 \ln(x-1) - x^2 - 2x \ln(x-1) + (x^2 - x) c_1 + (x-1) c_2 + 2x + \ln(x-1) - 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^2 \ln(x-1) - x^2 - 2x \ln(x-1) + (x^2 - x) c_1 + (x-1) c_2 + 2x + \ln(x-1) - 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = i\pi - c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2x\ln(x-1) + \frac{x^2}{x-1} - 2x - 2\ln(x-1) - \frac{2x}{x-1} + (2x-1)c_1 + c_2 + 2 + \frac{1}{x-1}$$

substituting $y' = -6$ and $x = 0$ in the above gives

$$-6 = -2i\pi - c_1 + c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -i\pi + 3 \\ c_2 &= i\pi - 4 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2\ln(x-1) + 2x^2 - 2x\ln(x-1) - 5x - ix^2\pi + 2ix\pi - i\pi + 3 + \ln(x-1)$$

Summary
The solution(s) found are the following

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1) \tag{1}$$

Verification of solutions

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1)$$

Verified OK.

10.31.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x-1)^2 y'' + (-2x+2)y' + 2y = 0$$

In normal form the ode

$$(x-1)^2 y'' + (-2x+2) y' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{1-x} \\ q(x) &= \frac{2}{(x-1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x) \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x) dx)} dx \\ &= \int e^{-\left(\int \frac{2}{1-x} dx\right)} dx \\ &= \int e^{2\ln(1-x)} dx \\ &= \int (x-1)^2 dx \\ &= \frac{(x-1)^3}{3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{(x-1)^2}}{(x-1)^4} \\
&= \frac{2}{(x-1)^6}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{(x-1)^6} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{2}{(x-1)^6} = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} \left((x-1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x-1)^3\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left((x-1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x-1)^3\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3 - 3x^2 + 3x - 1\right)^{\frac{1}{3}}$$
$$y_2 = \left(x^3 - 3x^2 + 3x - 1\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3-3x^2+3x-1)^{\frac{1}{3}} & (x^3-3x^2+3x-1)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3-3x^2+3x-1)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3-3x^2+3x-1)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3-3x^2+3x-1)^{\frac{1}{3}} & (x^3-3x^2+3x-1)^{\frac{2}{3}} \\ \frac{3x^2-6x+3}{3(x^3-3x^2+3x-1)^{\frac{2}{3}}} & \frac{2x^2-4x+2}{(x^3-3x^2+3x-1)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3-3x^2+3x-1)^{\frac{1}{3}}\right)\left(\frac{2x^2-4x+2}{(x^3-3x^2+3x-1)^{\frac{1}{3}}}\right) - \left((x^3-3x^2+3x-1)^{\frac{2}{3}}\right)\left(\frac{3x^2-6x+3}{3(x^3-3x^2+3x-1)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2 - 2x + 1$$

Which simplifies to

$$W = (x-1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3-3x^2+3x-1)^{\frac{2}{3}}(x-1)^2}{(x-1)^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\left((x-1)^3\right)^{\frac{2}{3}}}{(x-1)^2}dx$$

Hence

$$u_1 = -\frac{\left((x-1)^3\right)^{\frac{2}{3}} x}{(x-1)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3 - 3x^2 + 3x - 1)^{\frac{1}{3}} (x - 1)^2}{(x - 1)^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\left((x - 1)^3\right)^{\frac{1}{3}}}{(x - 1)^2} dx$$

Hence

$$u_2 = \frac{\left((x - 1)^3\right)^{\frac{1}{3}} \ln (x - 1)}{x - 1}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) \\ &= -\frac{\left((x - 1)^3\right)^{\frac{2}{3}} x(x^3 - 3x^2 + 3x - 1)^{\frac{1}{3}}}{(x - 1)^2} + \frac{\left((x - 1)^3\right)^{\frac{1}{3}} \ln (x - 1) (x^3 - 3x^2 + 3x - 1)^{\frac{2}{3}}}{x - 1} \end{aligned}$$

Which simplifies to

$$y_p(x) = (x - 1) (\ln (x - 1) (x - 1) - x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left((x - 1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x - 1)^3\right)^{\frac{2}{3}}}{3}\right) + ((x - 1) (\ln (x - 1) (x - 1) - x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 3^{\frac{2}{3}} \left((x - 1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x - 1)^3\right)^{\frac{2}{3}}}{3} + (x - 1) (\ln (x - 1) (x - 1) - x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = \frac{3^{\frac{2}{3}} c_1}{6} + \frac{i 3^{\frac{1}{6}} c_1}{2} - \frac{3^{\frac{1}{3}} c_2}{6} + \frac{i 3^{\frac{5}{6}} c_2}{6} + i\pi \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 3^{\frac{2}{3}} (x-1)^2}{3 \left((x-1)^3\right)^{\frac{2}{3}}} + \frac{2 c_2 3^{\frac{1}{3}} (x-1)^2}{3 \left((x-1)^3\right)^{\frac{1}{3}}} + 2 \ln(x-1)(x-1) - x$$

substituting $y' = -6$ and $x = 0$ in the above gives

$$-6 = -\frac{3^{\frac{2}{3}} c_1}{6} - \frac{i 3^{\frac{1}{6}} c_1}{2} + \frac{3^{\frac{1}{3}} c_2}{3} - \frac{i 3^{\frac{5}{6}} c_2}{3} - 2i\pi \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -\frac{6(3i + \pi)}{3^{\frac{5}{6}} + i 3^{\frac{1}{3}}}$$

Substituting these values back in above solution results in

$$y = \frac{3^{\frac{5}{6}} \ln(x-1) x^2 - 2\, 3^{\frac{5}{6}} \ln(x-1) x - 3^{\frac{5}{6}} x^2 + 3^{\frac{5}{6}} \ln(x-1) + 3^{\frac{5}{6}} x + i 3^{\frac{1}{3}} \ln(x-1) x^2 - 2 i 3^{\frac{1}{3}} \ln(x-1)}{3^{\frac{5}{6}} + i 3^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{-6\left(i + \frac{\pi}{3}\right) 3^{\frac{1}{3}} \left((x-1)^3\right)^{\frac{2}{3}} + (\ln(x-1)(x-1) - x)\left(3^{\frac{5}{6}} + i 3^{\frac{1}{3}}\right)(x-1)}{3^{\frac{5}{6}} + i 3^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{-6\left(i + \frac{\pi}{3}\right) 3^{\frac{1}{3}} \left((x-1)^3\right)^{\frac{2}{3}} + (\ln(x-1)(x-1) - x)\left(3^{\frac{5}{6}} + i 3^{\frac{1}{3}}\right)(x-1)}{3^{\frac{5}{6}} + i 3^{\frac{1}{3}}}$$

Verified OK.

10.31.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= (x-1)^2 \\ B &= -2x + 2 \\ C &= 2 \\ F &= (x-1)^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left((x-1)^2\right)(0) + (-2x+2)(-2) + (2)(-2x+2) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2(x-1)^3\, v'' + (0)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2(x-1)^3\, u'(x) = 0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 \,\mathrm{d}x \\ &= c_1 x + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-2x+2)\,(c_1 x + c_2) \\ &= -2(x-1)\,(c_1 x + c_2) \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= -2x + 2 \\ y_2 &= -2x^2 + 2x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -2x+2 & -2x^2+2x \\ \frac{d}{dx}(-2x+2) & \frac{d}{dx}(-2x^2+2x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -2x+2 & -2x^2+2x \\ -2 & -4x+2 \end{vmatrix}$$

Therefore

$$W = (-2x+2)(-4x+2) - (-2x^2+2x)(-2)$$

Which simplifies to

$$W = 4x^2 - 8x + 4$$

Which simplifies to

$$W = 4(x-1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-2x^2+2x)(x-1)^2}{4(x-1)^4}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x}{2x-2}\,dx$$

Hence

$$u_1 = \frac{x}{2} + \frac{\ln(x-1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-1)^2(-2x+2)}{4(x-1)^4}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{2x-2} dx$$

Hence

$$u_2 = -\frac{\ln(x-1)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{2} + \frac{\ln(x-1)}{2}\right)(-2x+2) - \frac{\ln(x-1)(-2x^2+2x)}{2}$$

Which simplifies to

$$y_p(x) = (x-1)(\ln(x-1)(x-1) - x)$$

Hence the complete solution is

$$\begin{aligned}y(x) &= y_h + y_p \\ &= (-2(x-1)(c_1x + c_2)) + ((x-1)(\ln(x-1)(x-1) - x)) \\ &= (x-1)^2 \ln(x-1) - (x-1)(2c_1x + 2c_2 + x)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (x-1)^2 \ln(x-1) - (x-1)(2c_1x + 2c_2 + x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = i\pi + 2c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\ln(x-1)(x-1) - 1 - 2c_1x - 2c_2 - (x-1)(2c_1+1)$$

substituting $y' = -6$ and $x = 0$ in the above gives

$$-6 = -2i\pi + 2c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{3}{2} + \frac{i\pi}{2} \\ c_2 &= \frac{3}{2} - \frac{i\pi}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2 \ln(x-1) + 2x^2 - 2x\ln(x-1) - 5x - ix^2\pi + 2ix\pi - i\pi + 3 + \ln(x-1)$$

Summary

The solution(s) found are the following

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1) \tag{1}$$

Verification of solutions

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1)$$

Verified OK.

10.31.5 Solving using Kovacic algorithm

Writing the ode as

$$(x-1)^2 y'' + (-2x+2)y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-1)^2 \\ B &= -2x+2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x+2}{(x-1)^2}\,dx} \\
&= z_1 e^{\ln(x-1)} \\
&= z_1(x-1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+2}{(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2(x - 1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
(x-1)^2 y'' + (-2x+2)\, y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(x-1) + c_2 x(x-1)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x - 1 \\
y_2 &= x(x-1)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x-1 & x(x-1) \\ \frac{d}{dx}(x-1) & \frac{d}{dx}(x(x-1)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x-1 & x(x-1) \\ 1 & 2x-1 \end{vmatrix}$$

Therefore

$$W = (x-1)(2x-1) - (x(x-1))(1)$$

Which simplifies to

$$W = x^2 - 2x + 1$$

Which simplifies to

$$W = (x-1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x(x-1)^3}{(x-1)^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x-1}dx$$

Hence

$$u_1 = -x - \ln(x-1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-1)^3}{(x-1)^4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x-1}dx$$

Hence

$$u_2 = \ln(x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x - \ln(x-1))(x-1) + \ln(x-1)\,x(x-1)$$

Which simplifies to

$$y_p(x) = (x-1)(\ln(x-1)(x-1) - x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1(x-1) + c_2 x(x-1)) + ((x-1)(\ln(x-1)(x-1) - x)) \end{aligned}$$

Which simplifies to

$$y = (x-1)(c_2 x + c_1) + (x-1)(\ln(x-1)(x-1) - x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (x-1)(c_2 x + c_1) + (x-1)(\ln(x-1)(x-1) - x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = i\pi - c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 x + c_1 + (x-1)c_2 + 2\ln(x-1)(x-1) - x$$

substituting $y' = -6$ and $x = 0$ in the above gives

$$-6 = -2i\pi + c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = i\pi - 3$$
$$c_2 = -i\pi + 3$$

Substituting these values back in above solution results in

$$y = x^2 \ln(x-1) + 2x^2 - 2x \ln(x-1) - 5x - ix^2\pi + 2ix\pi - i\pi + 3 + \ln(x-1)$$

Summary

The solution(s) found are the following

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1) \tag{1}$$

Verification of solutions

$$y = -((1-x)\ln(x-1) + ix\pi - i\pi - 2x + 3)(x-1)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 33

```
dsolve([(x-1)^2*diff(y(x),x$2)-2*(x-1)*diff(y(x),x)+2*y(x)=(x-1)^2,y(0) = 3, D(y)(0) = -6],y
```

$$y(x) = (-i\pi x + i\pi + \ln(x-1)x - \ln(x-1) + 2x - 3)(x-1)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 30

```
DSolve[{(x-1)^2*y''[x]-2*(x-1)*y'[x]+2*y[x]==(x-1)^2,{y[0]==3,y'[0]==-6}},y[x],x,IncludeSing
```

$$y(x) \to (x-1)(-i\pi(x-1)+2x+(x-1)\log(x-1)-3)$$

10.32 problem 32

Internal problem ID [1186]
Internal file name [`OUTPUT/1187_Sunday_June_05_2022_02_04_30_AM_23388380/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{(x-1)^2 y''-\left(x^2-1\right) y'+(x-1)^3 y=(x-1)^3 \mathrm{e}^x}$$

With initial conditions

$$[y(0)=4, y'(0)=-6]$$

10.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{-x^2+1}{(x-1)^2} \\ q(x) &= x-1 \\ F &= (x-1)\,\mathrm{e}^x \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-x^2+1)\, y'}{(x-1)^2} + y(x-1) = (x-1)\, e^x$$

The domain of $p(x) = \frac{-x^2+1}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x - 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = (x-1)\, e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
```

✗ Solution by Maple

```
dsolve([(x-1)^2*diff(y(x),x$2)-(x^2-1)*diff(y(x),x)+(x-1)^3*y(x)=(x-1)^3*exp(x),y(0) = 4, D(
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x-1)^2*y''[x]-(x^2-1)*y'[x]+(x-1)^3*y[x]==(x-1)^3*Exp[x],{y[0]==4,y'[0]==-6}},y[x],
```

Not solved

10.33 problem 33

Internal problem ID [1187]
Internal file name [`OUTPUT/1188_Sunday_June_05_2022_02_04_33_AM_77966869/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x-1)^2 y'' + 4y'x + 2y = 2x}$$

With initial conditions

$$[y(0) = 0, y'(0) = -2]$$

10.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= \frac{4x}{(x-1)^2}\\ q(x) &= \frac{2}{(x-1)^2}\\ F &= \frac{2x}{(x-1)^2}\end{aligned}$$

Hence the ode is

$$y'' + \frac{4xy'}{(x-1)^2} + \frac{2y}{(x-1)^2} = \frac{2x}{(x-1)^2}$$

The domain of $p(x) = \frac{4x}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{2x}{(x-1)^2}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.33.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((x-1)^2 y'' + 4y'x + 2y\right) dx &= \int 2x dx\\ (2+2x)\, y + \left(x^2 - 2x + 1\right) y' &= x^2 + c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x-2}{(x-1)^2}$$
$$q(x) = \frac{x^2+c_1}{(x-1)^2}$$

Hence the ode is

$$y' - \frac{(-2x-2)\,y}{(x-1)^2} = \frac{x^2+c_1}{(x-1)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2x-2}{(x-1)^2}dx}$$
$$= e^{2\ln(x-1)-\frac{4}{x-1}}$$

Which simplifies to

$$\mu = (x-1)^2\,e^{-\frac{4}{x-1}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2+c_1}{(x-1)^2}\right)$$
$$\frac{d}{dx}\left((x-1)^2\,e^{-\frac{4}{x-1}}y\right) = \left((x-1)^2\,e^{-\frac{4}{x-1}}\right)\left(\frac{x^2+c_1}{(x-1)^2}\right)$$
$$d\left((x-1)^2\,e^{-\frac{4}{x-1}}y\right) = \left((x^2+c_1)\,e^{-\frac{4}{x-1}}\right)\,dx$$

Integrating gives

$$(x-1)^2\,e^{-\frac{4}{x-1}}y = \int (x^2+c_1)\,e^{-\frac{4}{x-1}}\,dx$$
$$(x-1)^2\,e^{-\frac{4}{x-1}}y = -\frac{e^{-\frac{4}{x-1}}(x-1)}{3} + \frac{4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + 4c_1\left(\frac{e^{-\frac{4}{x-1}}(x-1)}{4} - \text{expIntegral}_1\left(\frac{4}{x-1}\right)\right.$$

Dividing both sides by the integrating factor $\mu = (x-1)^2\,e^{-\frac{4}{x-1}}$ results in

$$y = \frac{e^{\frac{4}{x-1}}\left(-\frac{e^{-\frac{4}{x-1}}(x-1)}{3} + \frac{4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + 4c_1\left(\frac{e^{-\frac{4}{x-1}}(x-1)}{4} - \text{expIntegral}_1\left(\frac{4}{x-1}\right)\right) + \frac{(x-1)^2 e^{-\frac{4}{x-1}}}{3} + \frac{(x-1)^3}{\ldots}}{(x-1)^2}$$

which simplifies to

$$y = \frac{(-12c_1+4)\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)+3c_2\mathrm{e}^{\frac{4}{x-1}}+(x-1)\,(x^2+3c_1-x-1)}{3\,(x-1)^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(-12c_1+4)\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)+3c_2\mathrm{e}^{\frac{4}{x-1}}+(x-1)\,(x^2+3c_1-x-1)}{3\,(x-1)^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $x=0$ in the above gives

$$0 = \frac{1}{3}+\frac{4(-3c_1+1)\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\,(-4)}{3}+c_2\mathrm{e}^{-4}-c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-\frac{4(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{(x-1)^2}+\frac{(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}}\mathrm{e}^{-\frac{4}{x-1}}}{x-1}-\frac{12c_2\mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2}+x^2+3c_1-x-1+(x-1)\,(2x-1)}{3\,(x-1)^2}$$

substituting $y'=-2$ and $x=0$ in the above gives

$$-2 = -\frac{2}{3}+\frac{8(3c_1-1)\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\,(-4)}{3}-2c_2\mathrm{e}^{-4}+3c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= -\frac{28\,\mathrm{expIntegral}_1\,(-4)}{3}-\frac{7\,\mathrm{e}^4}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^3+28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}}-28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\,(-4)-7\,\mathrm{e}^{\frac{4x}{x-1}}-2x^2-6x+7}{3x^2-6x+3}$$

Which simplifies to

$$y = \frac{x^3+28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}}-28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\,(-4)-7\,\mathrm{e}^{\frac{4x}{x-1}}-2x^2-6x+7}{3\,(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} - 28\, e^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\, e^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3(x-1)^2} \tag{1}$$

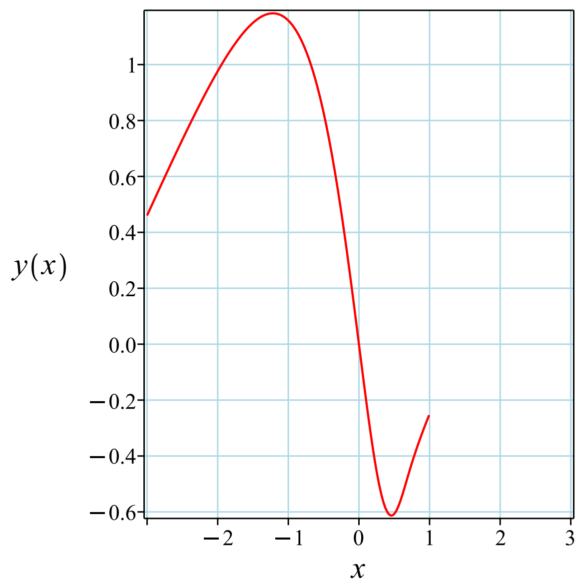


Figure 522: Solution plot

Verification of solutions

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} - 28\, e^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\, e^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3(x-1)^2}$$

Verified OK.

10.33.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x-1)^2 y'' + 4y'x + 2y = 2x$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x-1\right)^{2} y''+4y'x+2y\right)dx = \int 2x dx$$
$$\left(2+2x\right) y+\left(x^{2}-2x+1\right) y' = x^{2}+c_{1}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x-2}{\left(x-1\right)^{2}}$$
$$q(x) = \frac{x^{2}+c_{1}}{\left(x-1\right)^{2}}$$

Hence the ode is

$$y'-\frac{\left(-2x-2\right) y}{\left(x-1\right)^{2}} = \frac{x^{2}+c_{1}}{\left(x-1\right)^{2}}$$

The integrating factor μ is

$$\mu = {\mathrm e}^{\int -\frac{-2x-2}{\left(x-1\right)^{2}}d x}$$
$$= {\mathrm e}^{2\ln\left(x-1\right)-\frac{4}{x-1}}$$

Which simplifies to

$$\mu = \left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}}$$

The ode becomes

$$\frac{\mathop{d}}{dx}(\mu y) = (\mu)\left(\frac{x^{2}+c_{1}}{\left(x-1\right)^{2}}\right)$$
$$\frac{\mathop{d}}{dx}\left(\left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}} y\right) = \left(\left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}}\right)\left(\frac{x^{2}+c_{1}}{\left(x-1\right)^{2}}\right)$$
$$\mathrm{d}\left(\left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}} y\right) = \left(\left(x^{2}+c_{1}\right) {\mathrm e}^{-\frac{4}{x-1}}\right)\mathrm{d}x$$

Integrating gives

$$\left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}} y = \int \left(x^{2}+c_{1}\right) {\mathrm e}^{-\frac{4}{x-1}}\,\mathrm{d}x$$
$$\left(x-1\right)^{2} {\mathrm e}^{-\frac{4}{x-1}} y = -\frac{{\mathrm e}^{-\frac{4}{x-1}} \left(x-1\right)}{3}+\frac{4\,\mathrm{expIntegral}_{1}\left(\frac{4}{x-1}\right)}{3}+4c_{1} \left(\frac{{\mathrm e}^{-\frac{4}{x-1}} \left(x-1\right)}{4}-\mathrm{expIntegral}_{1}\left(\frac{4}{x-1}\right)\right)$$

Dividing both sides by the integrating factor $\mu = (x-1)^2 \mathrm{e}^{-\frac{4}{x-1}}$ results in

$$y = \frac{\mathrm{e}^{\frac{4}{x-1}} \left(-\frac{\mathrm{e}^{-\frac{4}{x-1}}(x-1)}{3} + \frac{4 \,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + 4c_1 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}}(x-1)}{4} - \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\right) + \frac{(x-1)^2 \mathrm{e}^{-\frac{4}{x-1}}}{3} + \frac{(x-1)^3}{\ldots}}{(x-1)^2}$$

which simplifies to

$$y = \frac{(-12c_1 + 4)\,\mathrm{e}^{\frac{4}{x-1}} \,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) + 3c_2 \mathrm{e}^{\frac{4}{x-1}} + (x-1)\left(x^2 + 3c_1 - x - 1\right)}{3(x-1)^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(-12c_1 + 4)\,\mathrm{e}^{\frac{4}{x-1}} \,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) + 3c_2 \mathrm{e}^{\frac{4}{x-1}} + (x-1)\left(x^2 + 3c_1 - x - 1\right)}{3(x-1)^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{1}{3} + \frac{4(-3c_1 + 1)\,\mathrm{e}^{-4} \,\mathrm{expIntegral}_1(-4)}{3} + c_2 \mathrm{e}^{-4} - c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-\frac{4(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}} \,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{(x-1)^2} + \frac{(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}} \mathrm{e}^{-\frac{4}{x-1}}}{x-1} - \frac{12c_2 \mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2} + x^2 + 3c_1 - x - 1 + (x-1)(2x-1)}{3(x-1)^2}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -\frac{2}{3} + \frac{8(3c_1 - 1)\,\mathrm{e}^{-4} \,\mathrm{expIntegral}_1(-4)}{3} - 2c_2 \mathrm{e}^{-4} + 3c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = -\frac{28 \,\mathrm{expIntegral}_1(-4)}{3} - \frac{7\,\mathrm{e}^4}{3}$$

Substituting these values back in above solution results in

$$y = \frac{x^3 + 28 \,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}} \,\mathrm{expIntegral}_1(-4) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3x^2 - 6x + 3}$$

Which simplifies to

$$y = \frac{x^3 + 28 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} - 28 e^{\frac{4}{x-1}} \operatorname{expIntegral}_1(-4) - 7 e^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 28 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} - 28 e^{\frac{4}{x-1}} \operatorname{expIntegral}_1(-4) - 7 e^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3(x-1)^2} \tag{1}$$

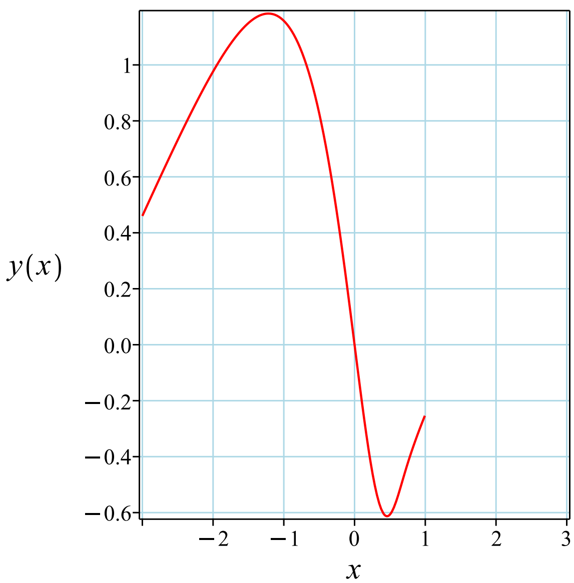


Figure 523: Solution plot

Verification of solutions

$$y = \frac{x^3 + 28 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} - 28 e^{\frac{4}{x-1}} \operatorname{expIntegral}_1(-4) - 7 e^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3(x-1)^2}$$

Verified OK.

10.33.4 Solving using Kovacic algorithm

Writing the ode as

$$(x-1)^2 y'' + 4y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-1)^2 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x}{(x-1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x \\ t &= (x-1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x}{(x-1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 442: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x-1)^4$. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = \frac{4}{(x-1)^3} + \frac{4}{(x-1)^4}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} + \frac{1}{x-1} - \frac{3}{8} + \frac{x}{8} - \frac{5(x-1)^2}{64} + \frac{7(x-1)^3}{128} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 4. Therefore

$$\begin{aligned} b &= (4) - (0) \\ &= 4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{4}{2} + 2\right) = 2 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{4}{2} + 2\right) = 0 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x}{(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	4	$\frac{2}{(x-1)^2}$	2	0

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{(x-1)^2} + (0) \\
&= -\frac{2}{(x-1)^2} \\
&= -\frac{2}{(x-1)^2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{(x-1)^2}\right)(0) + \left(\left(\frac{4}{(x-1)^3}\right) + \left(-\frac{2}{(x-1)^2}\right)^2 - \left(\frac{4x}{(x-1)^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{2}{(x-1)^2} dx} \\
&= \mathrm{e}^{\frac{2}{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{(x-1)^2} \, dx} \\
&= z_1 e^{-2\ln(x-1) + \frac{2}{x-1}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{2}{x-1}}}{(x-1)^2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{\frac{4}{x-1}}}{(x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)+\frac{4}{x-1}}}{(y_1)^2}\,dx \\
&= y_1 \left(e^{-\frac{4}{x-1}} (x-1) - 4 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\frac{4}{x-1}}}{(x-1)^2}\right) + c_2 \left(\frac{e^{\frac{4}{x-1}}}{(x-1)^2} \left(e^{-\frac{4}{x-1}} (x-1) - 4 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right)\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x-1)^2 y'' + 4y'x + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 e^{\frac{4}{x-1}}}{(x-1)^2} + \frac{c_2 \left(-4 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1\right)}{(x-1)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^{\frac{4}{x-1}}}{(x-1)^2}$$

$$y_2 = \frac{-4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1}{(x-1)^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^{\frac{4}{x-1}}}{(x-1)^2} & \frac{-4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1}{(x-1)^2} \\ \frac{d}{dx}\left(\frac{e^{\frac{4}{x-1}}}{(x-1)^2}\right) & \frac{d}{dx}\left(\frac{-4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1}{(x-1)^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^{\frac{4}{x-1}}}{(x-1)^2} & \frac{-4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1}{(x-1)^2} \\ -\frac{2 e^{\frac{4}{x-1}}}{(x-1)^3} - \frac{4 e^{\frac{4}{x-1}}}{(x-1)^4} & \frac{-\frac{4 e^{-\frac{4}{x-1}} e^{\frac{4}{x-1}}}{x-1} + \frac{16\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}}}{(x-1)^2} + 1}{(x-1)^2} - \frac{2\left(-4\,\text{expIntegral}_1\left(\frac{4}{x-1}\right) e^{\frac{4}{x-1}} + x - 1\right)}{(x-1)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{\frac{4}{x-1}}}{(x-1)^2}\right)\left(\frac{-\frac{4\,e^{-\frac{4}{x-1}}\,e^{\frac{4}{x-1}}}{x-1}+\frac{16\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)e^{\frac{4}{x-1}}}{(x-1)^2}+1}{(x-1)^2} - \frac{2\left(-4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)e^{\frac{4}{x-1}}+x-1\right)}{(x-1)^3}\right) - \left(\frac{-4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)e^{\frac{4}{x-1}}+x-1}{(x-1)^2}\right)\left(-\frac{2\,e^{\frac{4}{x-1}}}{(x-1)^3}-\frac{4\,e^{\frac{4}{x-1}}}{(x-1)^4}\right)$$

Which simplifies to

$$W = -\frac{e^{\frac{4}{x-1}}\left(4\,e^{-\frac{4}{x-1}}\,e^{\frac{4}{x-1}}-x-3\right)}{(x-1)^5}$$

Which simplifies to

$$W = \frac{e^{\frac{4}{x-1}}}{(x-1)^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\left(-4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)e^{\frac{4}{x-1}}+x-1\right)x}{(x-1)^2}}{\frac{e^{\frac{4}{x-1}}}{(x-1)^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 2\left(e^{-\frac{4}{x-1}}(x-1)-4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\right)x\,dx$$

Hence

$$u_1 = -\frac{2\,e^{-\frac{4}{x-1}}x^3}{3}+\frac{e^{-\frac{4}{x-1}}x^2}{3}+4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)x^2+\frac{e^{-\frac{4}{x-1}}}{3}+\frac{4\ \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\,\mathrm{e}^{\frac{4}{x-1}} x}{(x-1)^2}}{\frac{\mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2}}\,dx$$

Which simplifies to

$$u_2 = \int 2x dx$$

Hence

$$u_2 = x^2$$

Which simplifies to

$$u_1 = \frac{4(3x^2+1)\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + \frac{(-2x^3+x^2+1)\,\mathrm{e}^{-\frac{4}{x-1}}}{3}$$
$$u_2 = x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\frac{4(3x^2+1)\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + \frac{(-2x^3+x^2+1)\,\mathrm{e}^{-\frac{4}{x-1}}}{3}\right)\mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2} + \frac{x^2\left(-4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} + x - 1\right)}{(x-1)^2}$$

Which simplifies to

$$y_p(x) = \frac{x^3 - 2x^2 + 4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} + 1}{3\,(x-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2} + \frac{c_2\left(-4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} + x - 1\right)}{(x-1)^2}\right) \\ &\quad + \left(\frac{x^3 - 2x^2 + 4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} + 1}{3\,(x-1)^2}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{-4\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) c_2 + \mathrm{e}^{\frac{4}{x-1}} c_1 + (x-1)\,c_2}{(x-1)^2} + \frac{x^3 - 2x^2 + 4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} + 1}{3\,(x-1)^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{-4\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) c_2 + \mathrm{e}^{\frac{4}{x-1}} c_1 + (x-1)\,c_2}{(x-1)^2} + \frac{x^3 - 2x^2 + 4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} + 1}{3\,(x-1)^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{1}{3} + \frac{4(-3c_2+1)\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1(-4)}{3} + \mathrm{e}^{-4} c_1 - c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\frac{16\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) c_2}{(x-1)^2} - \frac{4\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{e}^{-\frac{4}{x-1}} c_2}{x-1} - \frac{4c_1\,\mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2} + c_2}{(x-1)^2} - \frac{2\left(-4\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) c_2 + \mathrm{e}^{\frac{4}{x-1}} c_1 + \right.}{(x-1)^3}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -\frac{2}{3} + \frac{8(3c_2-1)\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1(-4)}{3} - 2\,\mathrm{e}^{-4} c_1 + 3c_2 \tag{2A}$$

Equations $\{1\mathrm{A}, 2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{28\,\mathrm{expIntegral}_1(-4)}{3} - \frac{7\,\mathrm{e}^4}{3}$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3x^2 - 6x + 3}$$

Which simplifies to

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\,(x-1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\,(x-1)^2} \tag{1}$$

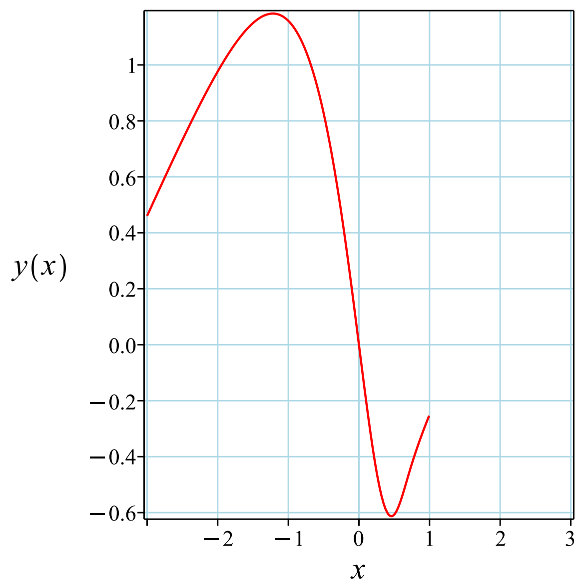


Figure 524: Solution plot

Verification of solutions

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1(-4) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\,(x-1)^2}$$

Verified OK.

10.33.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= (x-1)^2 \\
q(x) &= 4x \\
r(x) &= 2 \\
s(x) &= 2x
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}$$

Therefore (1) becomes

$$2 - (4) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$(x-1)^2\,y' + (2+2x)\,y = \int 2x\,dx$$

We now have a first order ode to solve which is

$$(x-1)^2\,y' + (2+2x)\,y = x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x-2}{(x-1)^2}$$
$$q(x) = \frac{x^2+c_1}{(x-1)^2}$$

Hence the ode is

$$y' - \frac{(-2x-2)\,y}{(x-1)^2} = \frac{x^2+c_1}{(x-1)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2x-2}{(x-1)^2}dx}$$
$$= e^{2\ln(x-1)-\frac{4}{x-1}}$$

Which simplifies to

$$\mu = (x-1)^2 e^{-\frac{4}{x-1}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2+c_1}{(x-1)^2}\right)$$
$$\frac{d}{dx}\left((x-1)^2 e^{-\frac{4}{x-1}}y\right) = \left((x-1)^2 e^{-\frac{4}{x-1}}\right)\left(\frac{x^2+c_1}{(x-1)^2}\right)$$
$$d\left((x-1)^2 e^{-\frac{4}{x-1}}y\right) = \left((x^2+c_1)\,e^{-\frac{4}{x-1}}\right)dx$$

Integrating gives

$$(x-1)^2 e^{-\frac{4}{x-1}}y = \int (x^2+c_1)\,e^{-\frac{4}{x-1}}\,dx$$
$$(x-1)^2 e^{-\frac{4}{x-1}}y = -\frac{e^{-\frac{4}{x-1}}(x-1)}{3} + \frac{4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + 4c_1\left(\frac{e^{-\frac{4}{x-1}}(x-1)}{4} - \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\right.$$

Dividing both sides by the integrating factor $\mu = (x-1)^2 e^{-\frac{4}{x-1}}$ results in

$$y = \frac{e^{\frac{4}{x-1}}\left(-\frac{e^{-\frac{4}{x-1}}(x-1)}{3} + \frac{4\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)}{3} + 4c_1\left(\frac{e^{-\frac{4}{x-1}}(x-1)}{4} - \mathrm{expIntegral}_1\left(\frac{4}{x-1}\right)\right) + \frac{(x-1)^2 e^{-\frac{4}{x-1}}}{3} + \frac{(x-1)^3}{\cdots}\right.}{(x-1)^2}$$

which simplifies to

$$y = \frac{(-12c_1+4)\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) + 3c_2\mathrm{e}^{\frac{4}{x-1}} + (x-1)\left(x^2+3c_1-x-1\right)}{3\left(x-1\right)^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(-12c_1+4)\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) + 3c_2\mathrm{e}^{\frac{4}{x-1}} + (x-1)\left(x^2+3c_1-x-1\right)}{3\left(x-1\right)^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{1}{3} + \frac{4(-3c_1+1)\,\mathrm{e}^{-4} \operatorname{expIntegral}_1\left(-4\right)}{3} + c_2\mathrm{e}^{-4} - c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-\frac{4(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right)}{(x-1)^2} + \frac{(-12c_1+4)\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{e}^{-\frac{4}{x-1}}}{x-1} - \frac{12c_2\mathrm{e}^{\frac{4}{x-1}}}{(x-1)^2} + x^2+3c_1-x-1+(x-1)(2x-1)}{3\left(x-1\right)^2}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -\frac{2}{3} + \frac{8(3c_1-1)\,\mathrm{e}^{-4} \operatorname{expIntegral}_1\left(-4\right)}{3} - 2c_2\mathrm{e}^{-4} + 3c_1 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= -\frac{28 \operatorname{expIntegral}_1\left(-4\right)}{3} - \frac{7\,\mathrm{e}^4}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^3 + 28 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(-4\right) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3x^2-6x+3}$$

Which simplifies to

$$y = \frac{x^3 + 28 \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right)\mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(-4\right) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\left(x-1\right)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(-4\right) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\left(x-1\right)^2} \tag{1}$$

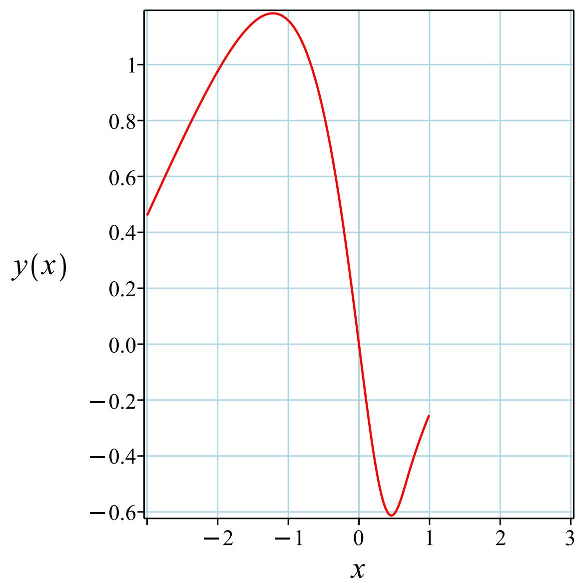


Figure 525: Solution plot

Verification of solutions

$$y = \frac{x^3 + 28\,\mathrm{expIntegral}_1\left(\frac{4}{x-1}\right) \mathrm{e}^{\frac{4}{x-1}} - 28\,\mathrm{e}^{\frac{4}{x-1}}\,\mathrm{expIntegral}_1\left(-4\right) - 7\,\mathrm{e}^{\frac{4x}{x-1}} - 2x^2 - 6x + 7}{3\left(x-1\right)^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 67

```
dsolve([(x-1)^2*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=2*x,y(0) = 0, D(y)(0) = -2],y(x), sin
```

$$y(x) = \frac{x^3 - 28\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(-4\right) - 7\,\mathrm{e}^{\frac{4x}{x-1}} + 28\,\mathrm{e}^{\frac{4}{x-1}} \operatorname{expIntegral}_1\left(\frac{4}{x-1}\right) - 2x^2 - 6x + 7}{3\left(x-1\right)^2}$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 71

```
DSolve[{(x-1)^2*y''[x]+4*x*y'[x]+2*y[x]==2*x,{y[0]==0,y'[0]==-2}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{-28e^{\frac{4}{x-1}} \operatorname{ExpIntegralEi}\left(-\frac{4}{x-1}\right) + 28 \operatorname{ExpIntegralEi}(4)e^{\frac{4}{x-1}} + x^3 - 2x^2 - 6x - 7e^{\frac{4x}{x-1}} + 7}{3(x-1)^2}$$

10.34 problem 34

Internal problem ID [1188]
Internal file name [`OUTPUT/1189_Sunday_June_05_2022_02_04_39_AM_5592745/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2y'x - 2y = -2x^2}$$

With initial conditions

$$[y(1) = 1, y'(1) = -1]$$

10.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= \frac{2}{x}\\ q(x) &= -\frac{2}{x^2}\\ F &= -2\end{aligned}$$

Hence the ode is

$$y'' + \frac{2y'}{x} - \frac{2y}{x^2} = -2$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

10.34.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x, C = -2, f(x) = -2x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 2y'x - 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r - 2 = 0$$

Or

$$r^2 + r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x$$

Next, we find the particular solution to the ODE

$$x^2 y'' + 2y'x - 2y = -2x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{x^2} \\ y_2 &= x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x \\ -\frac{2}{x^3} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(1) - (x)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{3}{x^2}$$

Which simplifies to

$$W = \frac{3}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2x^3}{3} \, dx$$

Which simplifies to

$$u_1 = -\int -\frac{2x^3}{3} dx$$

Hence

$$u_1 = \frac{x^4}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2}{3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{2}{3}dx$$

Hence

$$u_2 = -\frac{2x}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{x^2}{2} + \frac{c_1}{x^2} + c_2 x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2}{2} + \frac{c_1}{x^2} + c_2 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -x - \frac{2c_1}{x^3} + c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -2c_1 + c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2} \tag{1}$$

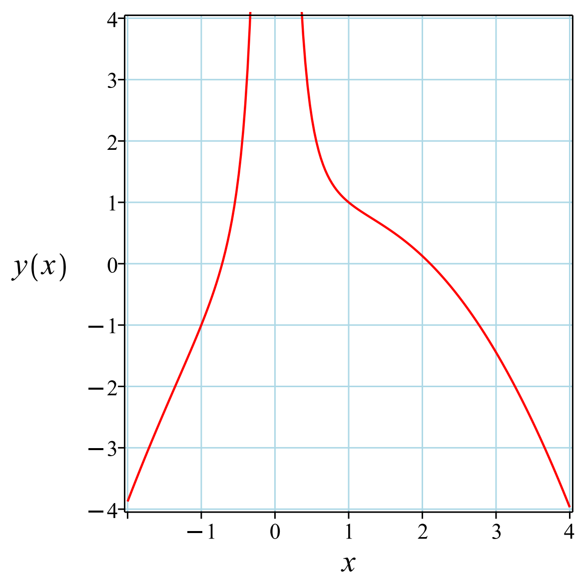


Figure 526: Solution plot

Verification of solutions

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Verified OK.

10.34.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 2y'x - 2y = 0$$

In normal form the ode

$$x^2y'' + 2y'x - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\, dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{x^2}}{\frac{1}{x^4}} \\
&= -2x^2
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2x^2 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
-2x^2 = -\frac{2}{\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0
$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-c_1 x^3 + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{-c_1 x^3 + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x^2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x^2} \\ 1 & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)(1)$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2}{-3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{3}dx$$

Hence

$$u_1 = -\frac{2x}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x^3}{-3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2x^3}{3}dx$$

Hence

$$u_2 = \frac{x^4}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-c_1x^3 + c_2}{x^2}\right) + \left(-\frac{x^2}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{-c_1x^3 + c_2}{x^2} - \frac{x^2}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -c_1 + c_2 - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 - \frac{2(-c_1x^3 + c_2)}{x^3} - x$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 - 2c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{1}{2}$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2} \tag{1}$$

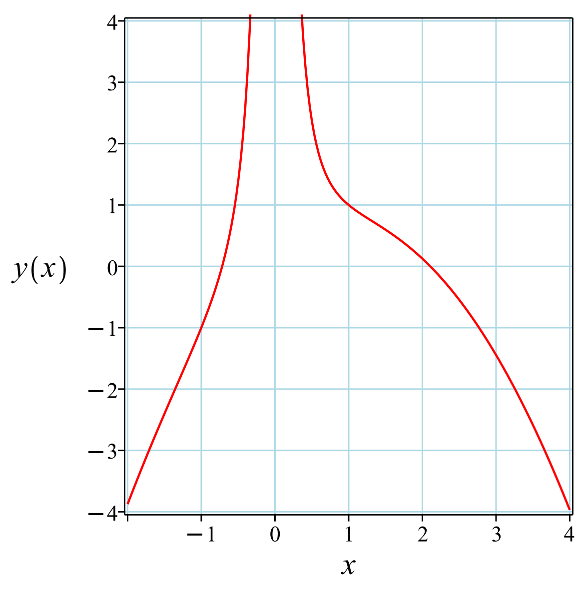


Figure 527: Solution plot

Verification of solutions

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Verified OK.

10.34.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x, C = -2, f(x) = -2x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 2y'x - 2y = 0$$

In normal form the ode

$$x^2y'' + 2y'x - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{4v'(x)}{x} &= 0 \\ v''(x) + \frac{4v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{4u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{4}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{4}{x}\,dx \\ \ln(u) &= -4\ln(x) + c_1 \\ u &= e^{-4\ln(x)+c_1} \\ &= \frac{c_1}{x^4} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{3x^3} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 2y'x - 2y = -2x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & \frac{1}{x^2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x^2} \\ 1 & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)(1)$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2}{-3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{3}dx$$

Hence

$$u_1 = -\frac{2x}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x^3}{-3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2x^3}{3}dx$$

Hence

$$u_2 = \frac{x^4}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{3x^3} + c_2\right) x\right) + \left(-\frac{x^2}{2}\right) \\
&= -\frac{x^2}{2} + \left(-\frac{c_1}{3x^3} + c_2\right) x
\end{aligned}$$

Which simplifies to

$$y = -\frac{-6c_2x^3 + 3x^4 + 2c_1}{6x^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{-6c_2x^3 + 3x^4 + 2c_1}{6x^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{3} + c_2 - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-6c_2x^3 + 3x^4 + 2c_1}{3x^3} - \frac{-18c_2x^2 + 12x^3}{6x^2}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{2c_1}{3} + c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= -\frac{3}{2} \\
c_2 &= 1
\end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2} \tag{1}$$

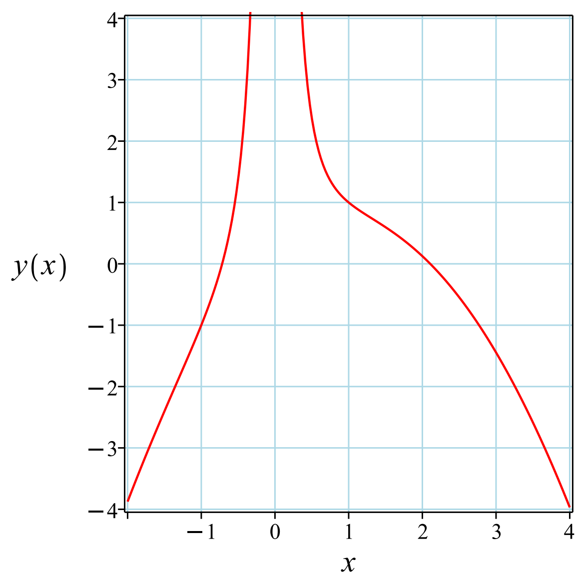


Figure 528: Solution plot

Verification of solutions

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Verified OK.

10.34.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right)v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right)u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 2x \\
C &= -2 \\
F &= -2x^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (2x)(2) + (-2)(2x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
2x^3v'' + \left(8x^2\right)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
2x^2(u'(x)\,x + 4u(x)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{4u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{4}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{4}{x}\, dx \\
\ln(u) &= -4\ln(x) + c_1 \\
u &= \mathrm{e}^{-4\ln(x)+c_1} \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^4}\, \mathrm{d}x \\
&= -\frac{c_1}{3x^3} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (2x)\left(-\frac{c_1}{3x^3} + c_2\right) \\
&= \frac{6c_2x^3 - 2c_1}{3x^2}
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \frac{1}{x^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x^2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x^2} \\ 1 & -\frac{2}{x^3} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{2}{x^3}\right) - \left(\frac{1}{x^2}\right)(1)$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Which simplifies to

$$W = -\frac{3}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2}{-3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{3} dx$$

Hence

$$u_1 = -\frac{2x}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x^3}{-3} dx$$

Which simplifies to

$$u_2 = \int \frac{2x^3}{3} dx$$

Hence

$$u_2 = \frac{x^4}{6}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\frac{6c_2x^3 - 2c_1}{3x^2}\right) + \left(-\frac{x^2}{2}\right) \\ &= \frac{12c_2x^3 - 3x^4 - 4c_1}{6x^2} \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{12c_2x^3 - 3x^4 - 4c_1}{6x^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{2c_1}{3} + 2c_2 - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{36c_2x^2 - 12x^3}{6x^2} - \frac{12c_2x^3 - 3x^4 - 4c_1}{3x^3}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{4c_1}{3} + 2c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{3}{4}$$
$$c_2 = \frac{1}{2}$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2} \tag{1}$$

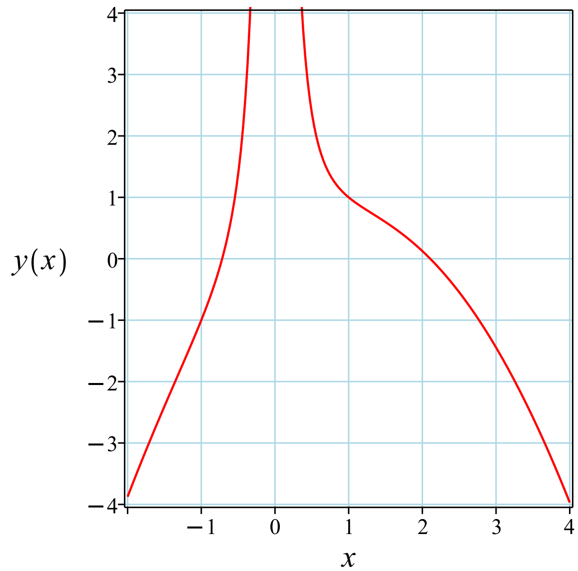


Figure 529: Solution plot

<u>Verification of solutions</u>

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Verified OK.

10.34.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2y'x - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2} \left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + 2y'x - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^2} + \frac{c_2 x}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{x}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x}{3} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x}{3} \\ -\frac{2}{x^3} & \frac{1}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{1}{3}\right) - \left(\frac{x}{3}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2x^3}{3}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2x^3}{3}dx$$

Hence

$$u_1 = \frac{x^4}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2}{1}\,dx$$

Which simplifies to

$$u_2 = \int (-2)\, dx$$

Hence

$$u_2 = -2x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2 x}{3}\right) + \left(-\frac{x^2}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^2} + \frac{c_2 x}{3} - \frac{x^2}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + \frac{c_2}{3} - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1}{x^3} + \frac{c_2}{3} - x$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -2c_1 + \frac{c_2}{3} - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{2} \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2} \tag{1}$$

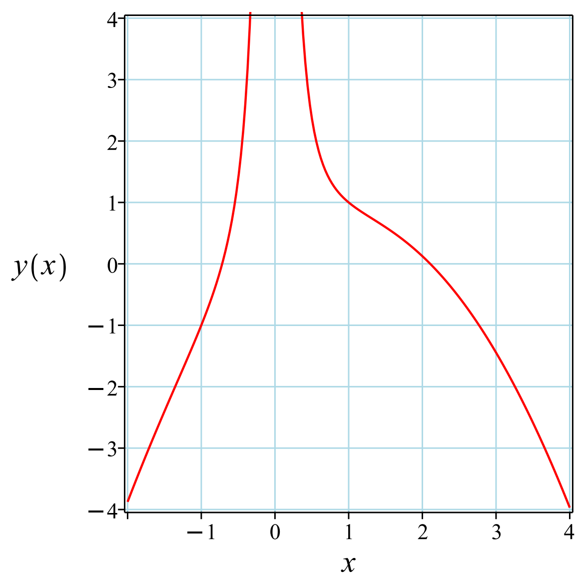


Figure 530: Solution plot

Verification of solutions

$$y = -\frac{x^4 - 2x^3 - 1}{2x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=-2*x^2,y(1) = 1, D(y)(1) = -1],y(x), sing
```

$$y(x) = \frac{1}{2x^2} + x - \frac{x^2}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 21

```
DSolve[{x^2*y''[x]+2*x*y'[x]-2*y[x]==-2*x^2,{y[1]==1,y'[1]==-1}},y[x],x,IncludeSingularSolut
```

$$y(x) \to -\frac{x^2}{2} + \frac{1}{2x^2} + x$$

10.35 problem 35

Internal problem ID [1189]
Internal file name [`OUTPUT/1190_Sunday_June_05_2022_02_04_40_AM_39778088/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 5 linear second order equations. Section 5.7 Variation of Parameters. Page 262
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)(2x+3)y''+2(2+x)y'-2y=(2x+3)^2}$$

With initial conditions

$$[y(0)=0, y'(0)=0]$$

10.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}
p(x) &= \frac{2x + 4}{2x^2 + 5x + 3} \\
q(x) &= -\frac{2}{2x^2 + 5x + 3} \\
F &= \frac{4\left(x + \frac{3}{2}\right)^2}{2x^2 + 5x + 3}
\end{aligned}$$

Hence the ode is

$$y'' + \frac{(2x + 4)\, y'}{2x^2 + 5x + 3} - \frac{2y}{2x^2 + 5x + 3} = \frac{4\left(x + \frac{3}{2}\right)^2}{2x^2 + 5x + 3}$$

The domain of $p(x) = \frac{2x+4}{2x^2+5x+3}$ is

$$\left\{-\infty \le x < -1, -1 < x < -\frac{3}{2}, -\frac{3}{2} < x \le \infty\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{2}{2x^2+5x+3}$ is

$$\left\{-\infty \le x < -1, -1 < x < -\frac{3}{2}, -\frac{3}{2} < x \le \infty\right\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{4\left(x+\frac{3}{2}\right)^2}{2x^2+5x+3}$ is

$$\left\{-\infty \le x < -1, -1 < x < -\frac{3}{2}, -\frac{3}{2} < x \le \infty\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.35.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''(2x^2+5x+3)+(2x+4)\,y'-2y)\,dx = \int 4\left(x+\frac{3}{2}\right)^2 dx$$

$$(-1-2x)\,y+(2x^2+5x+3)\,y' = \frac{4(x+\frac{3}{2})^3}{3}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1+2x}{2x^2+5x+3} \\ q(x) &= \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18} \end{aligned}$$

Hence the ode is

$$y'-\frac{(1+2x)\,y}{2x^2+5x+3} = \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1+2x}{2x^2+5x+3}dx} \\ &= \mathrm{e}^{-2\ln(2x+3)+\ln(x+1)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{x+1}{(2x+3)^2}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)\,y}{(2x+3)^2}\right) &= \left(\frac{x+1}{(2x+3)^2}\right)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right) \\ \mathrm{d}\left(\frac{(x+1)\,y}{(2x+3)^2}\right) &= \left(\frac{8x^3+36x^2+6c_1+54x+27}{6\,(2x+3)^3}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{(x+1)\,y}{(2x+3)^2} = \int \frac{8x^3+36x^2+6c_1+54x+27}{6\,(2x+3)^3}\,\mathrm{d}x$$
$$\frac{(x+1)\,y}{(2x+3)^2} = \frac{x}{6}-\frac{c_1}{4\,(2x+3)^2}+c_2$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+3)^2}$ results in

$$y = \frac{(2x+3)^2\left(\frac{x}{6}-\frac{c_1}{4(2x+3)^2}\right)}{x+1}+\frac{c_2(2x+3)^2}{x+1}$$

which simplifies to

$$y = \frac{8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2}{12x+12}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2}{12x+12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_1}{4}+9c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{24x^2+2(48c_2+24)\,x+144c_2+18}{12x+12}-\frac{12(8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2)}{(12x+12)^2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{4}+3c_2+\frac{3}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{9}{2}$$
$$c_2 = -\frac{1}{8}$$

Substituting these values back in above solution results in

$$y = \frac{x^2(4x+9)}{6x+6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2(4x+9)}{6x+6} \tag{1}$$

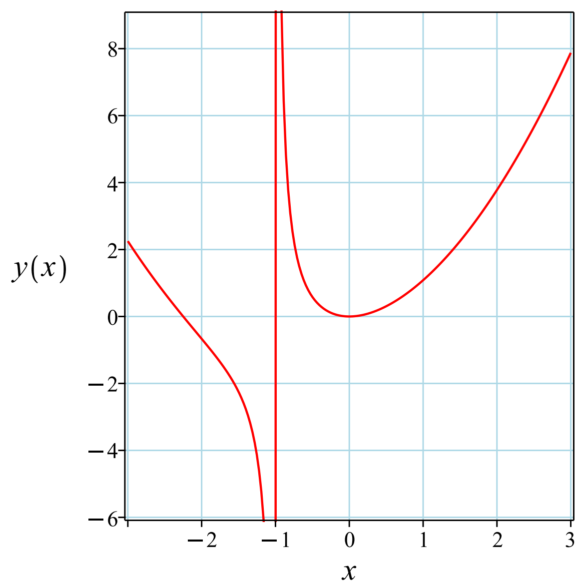


Figure 531: Solution plot

Verification of solutions

$$y = \frac{x^2(4x+9)}{6x+6}$$

Verified OK.

10.35.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 2x^2 + 5x + 3 \\
B &= 2x + 4 \\
C &= -2 \\
F &= 4\left(x + \frac{3}{2}\right)^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(2x^2 + 5x + 3\right)(0) + (2x + 4)(2) + (-2)(2x + 4) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
4x^3 + 18x^2 + 26x + 12v'' + \left(12x^2 + 36x + 28\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(4x^3 + 18x^2 + 26x + 12\right) u'(x) + 12\left(x^2 + 3x + \frac{7}{3}\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u(3x^2+9x+7)}{2x^3+9x^2+13x+6}
\end{aligned}$$

Where $f(x) = -\frac{2(3x^2+9x+7)}{2x^3+9x^2+13x+6}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2(3x^2+9x+7)}{2x^3+9x^2+13x+6}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2(3x^2+9x+7)}{2x^3+9x^2+13x+6}\,dx \\
\ln(u) &= \ln(2x+3) - 2\ln(x+1) - 2\ln(2+x) + c_1 \\
u &= e^{\ln(2x+3)-2\ln(x+1)-2\ln(2+x)+c_1} \\
&= c_1 e^{\ln(2x+3)-2\ln(x+1)-2\ln(2+x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{2x}{(x+1)^2(2+x)^2} + \frac{3}{(x+1)^2(2+x)^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\left(\frac{2x}{(x+1)^2(2+x)^2} + \frac{3}{(x+1)^2(2+x)^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1(2x+3)}{(x+1)^2(2+x)^2}\,\mathrm{d}x \\
&= c_1\left(\frac{1}{2+x} - \frac{1}{x+1}\right) + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (2x+4)\left(c_1\left(\frac{1}{2+x} - \frac{1}{x+1}\right) + c_2\right) \\
&= \frac{(2x^2+6x+4)c_2 - 2c_1}{x+1}
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$

$$y_2 = \frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1} \\ -\frac{1}{(x+1)^2} & \frac{4x}{x+1} - \frac{2x^2}{(x+1)^2} - \frac{6x}{(x+1)^2} + \frac{6}{x+1} - \frac{4}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(\frac{4x}{x+1} - \frac{2x^2}{(x+1)^2} - \frac{6x}{(x+1)^2} + \frac{6}{x+1} - \frac{4}{(x+1)^2}\right) - \left(\frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{4x+6}{(x+1)^2}$$

Which simplifies to

$$W = \frac{4x+6}{(x+1)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(\frac{2x^2}{x+1} + \frac{6x}{x+1} + \frac{4}{x+1}\right)\left(x+\frac{3}{2}\right)^2}{\frac{(2x^2+5x+3)(4x+6)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_1 = -\int (2+x)(x+1)\,dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - \frac{3}{2}x^2 - 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\left(x+\frac{3}{2}\right)^2}{x+1}}{\frac{(2x^2+5x+3)(4x+6)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2}dx$$

Hence

$$u_2 = \frac{x}{2}$$

Which simplifies to

$$u_1 = -\frac{\left(x^2+\frac{9}{2}x+6\right)x}{3}$$
$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(x^2+\frac{9}{2}x+6\right)x}{3\left(x+1\right)} + \frac{x\left(\frac{2x^2}{x+1}+\frac{6x}{x+1}+\frac{4}{x+1}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x^2(4x+9)}{6x+6}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\frac{\left(2x^2+6x+4\right)c_2 - 2c_1}{x+1}\right) + \left(\frac{x^2(4x+9)}{6x+6}\right) \\
&= \frac{4x^3+\left(12c_2+9\right)x^2+36c_2x-12c_1+24c_2}{6x+6}
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{4x^3+\left(12c_2+9\right)x^2+36c_2x-12c_1+24c_2}{6x+6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -2c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{12x^2+2(12c_2+9)\,x+36c_2}{6x+6} - \frac{6(4x^3+\left(12c_2+9\right)x^2+36c_2x-12c_1+24c_2)}{\left(6x+6\right)^2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^2(4x+9)}{6x+6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2(4x+9)}{6x+6} \tag{1}$$

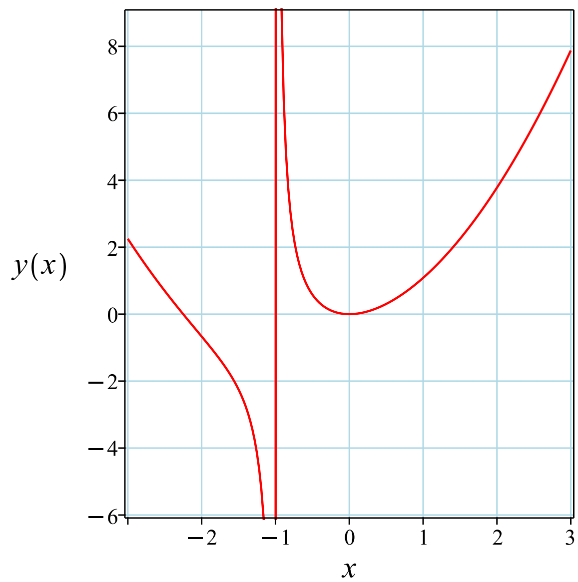


Figure 532: Solution plot

Verification of solutions

$$y = \frac{x^2(4x+9)}{6x+6}$$

Verified OK.

10.35.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''(2x^2+5x+3)+(2x+4)\,y'-2y = 4\left(x+\frac{3}{2}\right)^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''\left(2x^2+5x+3\right)+\left(2x+4\right)y'-2y\right)dx = \int 4\left(x+\frac{3}{2}\right)^2 dx$$
$$\left(-1-2x\right)y+\left(2x^2+5x+3\right)y' = \frac{4\left(x+\frac{3}{2}\right)^3}{3}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1+2x}{2x^2+5x+3}$$
$$q(x) = \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}$$

Hence the ode is

$$y'-\frac{\left(1+2x\right)y}{2x^2+5x+3} = \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1+2x}{2x^2+5x+3}dx}$$
$$= e^{-2\ln(2x+3)+\ln(x+1)}$$

Which simplifies to

$$\mu = \frac{x+1}{\left(2x+3\right)^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right)$$
$$\frac{d}{dx}\left(\frac{\left(x+1\right)y}{\left(2x+3\right)^2}\right) = \left(\frac{x+1}{\left(2x+3\right)^2}\right)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right)$$
$$d\left(\frac{\left(x+1\right)y}{\left(2x+3\right)^2}\right) = \left(\frac{8x^3+36x^2+6c_1+54x+27}{6\left(2x+3\right)^3}\right)dx$$

Integrating gives

$$
\begin{aligned}
\frac{(x+1)\,y}{(2x+3)^2} &= \int \frac{8x^3+36x^2+6c_1+54x+27}{6\,(2x+3)^3}\, \mathrm{d}x \\
\frac{(x+1)\,y}{(2x+3)^2} &= \frac{x}{6}-\frac{c_1}{4\,(2x+3)^2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+3)^2}$ results in

$$
y = \frac{(2x+3)^2\left(\frac{x}{6}-\frac{c_1}{4(2x+3)^2}\right)}{x+1}+\frac{c_2(2x+3)^2}{x+1}
$$

which simplifies to

$$
y = \frac{8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2}{12x+12}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2}{12x+12} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $x=0$ in the above gives

$$
0 = -\frac{c_1}{4}+9c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{24x^2+2(48c_2+24)\,x+144c_2+18}{12x+12}-\frac{12(8x^3+(48c_2+24)\,x^2+(144c_2+18)\,x-3c_1+108c_2)}{(12x+12)^2}
$$

substituting $y'=0$ and $x=0$ in the above gives

$$
0 = \frac{c_1}{4}+3c_2+\frac{3}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{9}{2} \\
c_2 &= -\frac{1}{8}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{x^2(4x+9)}{6x+6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2(4x+9)}{6x+6} \tag{1}$$

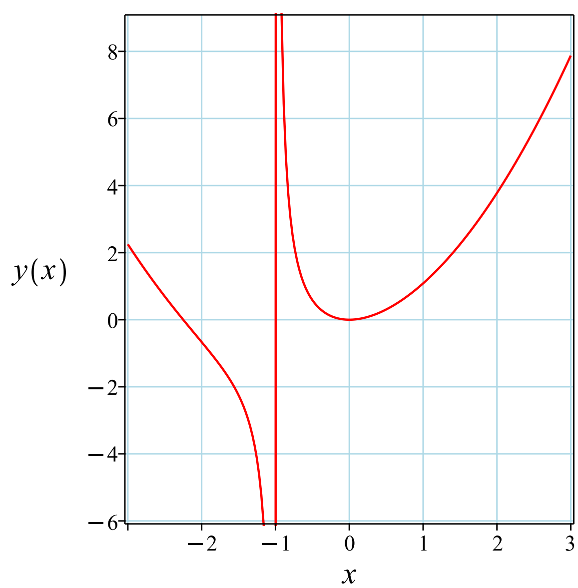


Figure 533: Solution plot

Verification of solutions

$$y = \frac{x^2(4x+9)}{6x+6}$$

Verified OK.

10.35.5 Solving using Kovacic algorithm

Writing the ode as

$$y''(2x^2+5x+3) + (2x+4)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 2x^2 + 5x + 3 \\
B &= 2x + 4 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{(2x + 3)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= (2x + 3)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{(2x + 3)^2} \right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 444: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x + 3)^2$. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x + \frac{3}{2}\right)^2}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{(2x+3)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(2x+3)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{3}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x + \frac{3}{2}\right)} + (-)(0) \\
&= -\frac{1}{2\left(x + \frac{3}{2}\right)} \\
&= -\frac{1}{2x + 3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x + \frac{3}{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(x + \frac{3}{2}\right)^2} \right) + \left(-\frac{1}{2\left(x + \frac{3}{2}\right)} \right)^2 - \left(\frac{3}{(2x + 3)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2\left(x + \frac{3}{2}\right)} dx} \\
&= \frac{1}{\sqrt{2x + 3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+4}{2x^2+5x+3}\,dx} \\
&= z_1 e^{\frac{\ln(2x+3)}{2}-\ln(x+1)} \\
&= z_1 \left(\frac{\sqrt{2x+3}}{x+1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+4}{2x^2+5x+3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x+3)-2\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1(x(x+3))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x+1}\right) + c_2\left(\frac{1}{x+1}(x(x+3))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''\left(2x^2+5x+3\right)+(2x+4)\,y'-2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x+1}+\frac{c_2x(x+3)}{x+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{x+1}$$

$$y_2=\frac{x(x+3)}{x+1}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{x+1} & \frac{x(x+3)}{x+1}\\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{x(x+3)}{x+1}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x(x+3)}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{x(x+3)}{(x+1)^2} + \frac{x+3}{x+1} + \frac{x}{x+1} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(-\frac{x(x+3)}{(x+1)^2} + \frac{x+3}{x+1} + \frac{x}{x+1}\right) - \left(\frac{x(x+3)}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{2x+3}{(x+1)^2}$$

Which simplifies to

$$W = \frac{2x+3}{(x+1)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4x(x+3)\left(x+\frac{3}{2}\right)^2}{x+1}}{\frac{(2x^2+5x+3)(2x+3)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_1 = -\int x(x+3)\,dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - \frac{3}{2}x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\left(x+\frac{3}{2}\right)^2}{x+1}}{\frac{(2x^2+5x+3)(2x+3)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - \frac{3}{2}x^2}{x+1} + \frac{x^2(x+3)}{x+1}$$

Which simplifies to

$$y_p(x) = \frac{x^2(4x+9)}{6x+6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1} + \frac{c_2 x(x+3)}{x+1}\right) + \left(\frac{x^2(4x+9)}{6x+6}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 x^2 + 3c_2 x + c_1}{x+1} + \frac{x^2(4x+9)}{6x+6}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_2 x^2 + 3c_2 x + c_1}{x+1} + \frac{x^2(4x+9)}{6x+6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_2 x + 3c_2}{x+1} - \frac{c_2 x^2 + 3c_2 x + c_1}{(x+1)^2} + \frac{2x(4x+9)}{6x+6} + \frac{4x^2}{6x+6} - \frac{6x^2(4x+9)}{(6x+6)^2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{x^2(4x+9)}{6x+6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2(4x+9)}{6x+6} \tag{1}$$

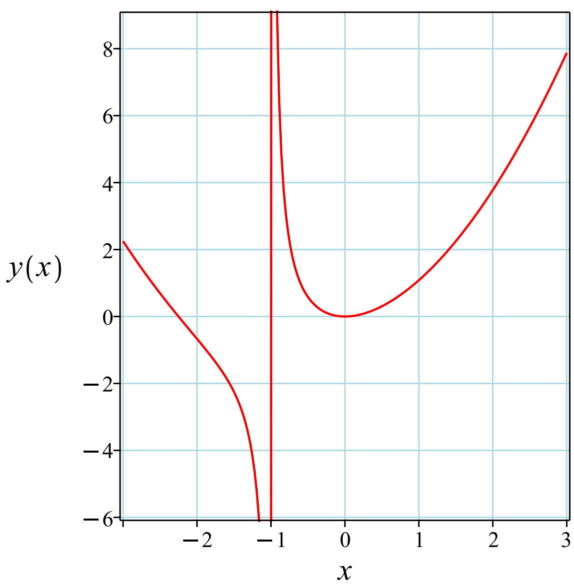


Figure 534: Solution plot

Verification of solutions

$$y = \frac{x^2(4x+9)}{6x+6}$$

Verified OK.

10.35.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 2x^2 + 5x + 3 \\ q(x) &= 2x + 4 \\ r(x) &= -2 \\ s(x) &= 4\left(x + \frac{3}{2}\right)^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 4 \\ q'(x) &= 2 \end{aligned}$$

Therefore (1) becomes

$$4 - (2) + (-2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$(-1 - 2x)\,y + \left(2x^2 + 5x + 3\right)y' = \int 4\left(x + \frac{3}{2}\right)^2 dx$$

We now have a first order ode to solve which is

$$(-1 - 2x)\,y + \left(2x^2 + 5x + 3\right)y' = \frac{4\left(x + \frac{3}{2}\right)^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1+2x}{2x^2+5x+3} \\
q(x) &= \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}
\end{aligned}$$

Hence the ode is

$$y' - \frac{(1+2x)\, y}{2x^2+5x+3} = \frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1+2x}{2x^2+5x+3}dx} \\
&= \mathrm{e}^{-2\ln(2x+3)+\ln(x+1)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{x+1}{(2x+3)^2}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)\, y}{(2x+3)^2}\right) &= \left(\frac{x+1}{(2x+3)^2}\right)\left(\frac{8x^3+36x^2+6c_1+54x+27}{12x^2+30x+18}\right) \\
\mathrm{d}\left(\frac{(x+1)\, y}{(2x+3)^2}\right) &= \left(\frac{8x^3+36x^2+6c_1+54x+27}{6\,(2x+3)^3}\right)\, \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{(x+1)\, y}{(2x+3)^2} &= \int \frac{8x^3+36x^2+6c_1+54x+27}{6\,(2x+3)^3}\, \mathrm{d}x \\
\frac{(x+1)\, y}{(2x+3)^2} &= \frac{x}{6} - \frac{c_1}{4\,(2x+3)^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+3)^2}$ results in

$$y = \frac{(2x+3)^2\left(\frac{x}{6} - \frac{c_1}{4(2x+3)^2}\right)}{x+1} + \frac{c_2(2x+3)^2}{x+1}$$

which simplifies to

$$y = \frac{8x^3 + (48c_2 + 24)\,x^2 + (144c_2 + 18)\,x - 3c_1 + 108c_2}{12x + 12}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{8x^3 + (48c_2 + 24)\,x^2 + (144c_2 + 18)\,x - 3c_1 + 108c_2}{12x + 12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_1}{4} + 9c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{24x^2 + 2(48c_2 + 24)\,x + 144c_2 + 18}{12x + 12} - \frac{12(8x^3 + (48c_2 + 24)\,x^2 + (144c_2 + 18)\,x - 3c_1 + 108c_2)}{(12x + 12)^2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_1}{4} + 3c_2 + \frac{3}{2} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{9}{2}$$
$$c_2 = -\frac{1}{8}$$

Substituting these values back in above solution results in

$$y = \frac{x^2(4x + 9)}{6x + 6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2(4x + 9)}{6x + 6} \tag{1}$$

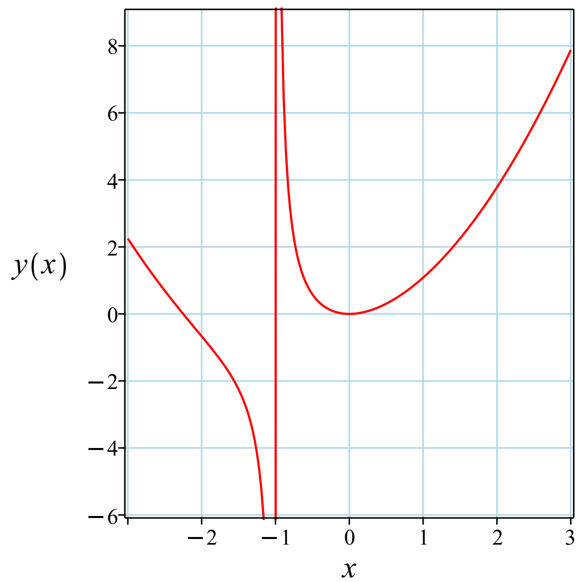


Figure 535: Solution plot

Verification of solutions

$$y = \frac{x^2(4x+9)}{6x+6}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve([(x+1)*(2*x+3)*diff(y(x),x$2)+2*(x+2)*diff(y(x),x)-2*y(x)=(2*x+3)^2,y(0) = 0, D(y)(0)
```

$$y(x) = \frac{x^2(4x+9)}{6x+6}$$

✓ Solution by Mathematica

Time used: 0.694 (sec). Leaf size: 22

```
DSolve[{(x+1)*(2*x+3)*y''[x]+2*(x+2)*y'[x]-2*y[x]==(2*x+3)^2,{y[0]==0,y'[0]==0}},y[x],x,Incl
```

$$y(x) \to \frac{x^2(4x+9)}{6(x+1)}$$

11 Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318

11.1 problem 11

Internal problem ID [1190]
Internal file name [`OUTPUT/1191_Sunday_June_05_2022_02_04_42_AM_34872227/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2+x)\, y''+y'x+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (785)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (786)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y + y'x}{2 + x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 - 3x - 8)\, y' + 3(x + 1)\, y}{(2 + x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^2 + 8x + 8)\, y' - 3y(x - 4)}{(2 + x)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^2 - 13x + 12)\, y' + 3y(x - 9)}{(2 + x)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^3 + 16x^2 - 16x - 104)\, y' - 3y(x^2 - 12x - 8)}{(2 + x)^3}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2} \\
F_1 &= \frac{3y(0)}{4} - 2y'(0) \\
F_2 &= 3y(0) + 2y'(0) \\
F_3 &= -\frac{27y(0)}{4} + 3y'(0) \\
F_4 &= 3y(0) - 13y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{3}{4}x^2+\frac{1}{8}x^3+\frac{1}{8}x^4-\frac{9}{160}x^5+\frac{1}{240}x^6\right)y(0) \\ +\left(x-\frac{1}{3}x^3+\frac{1}{12}x^4+\frac{1}{40}x^5-\frac{13}{720}x^6\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2+x)\,y''+y'x+3y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2+x)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)x+3\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty} 3a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,x^{n-1}a_n(n-1)=\sum_{n=1}^{\infty}(n+1)\,a_{n+1}n\,x^n$$
$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right) + \left(\sum_{n=0}^{\infty} 3 a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 2(n+2)\, a_{n+2}(n+1) + n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_n + n a_{n+1} + 3a_n}{2\,(n+2)\,(n+1)} \\ &= -\frac{(n+3)\, a_n}{2\,(n+2)\,(n+1)} - \frac{(n^2+n)\, a_{n+1}}{2\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 12a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{8} - \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 24a_4 + 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{9a_0}{160} + \frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{240} - \frac{13a_1}{720}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_0}{3360} + \frac{41a_1}{10080}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{3a_0x^2}{4} + \left(\frac{a_0}{8} - \frac{a_1}{3}\right)x^3 + \left(\frac{a_0}{8} + \frac{a_1}{12}\right)x^4 + \left(-\frac{9a_0}{160} + \frac{a_1}{40}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5\right)a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5\right)c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5\right)c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5 + \frac{1}{240}x^6\right)y(0) \\ &+ \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{13}{720}x^6\right)y'(0) + O(x^6) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5\right)c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5\right)c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$\begin{aligned} y = &\left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5 + \frac{1}{240}x^6\right)y(0) \\ &+ \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{13}{720}x^6\right)y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 - \frac{9}{160}x^5\right)c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{40}x^5\right)c_2 + O(x^6)$$

Verified OK.

11.1.1 Maple step by step solution

Let's solve

$(2 + x)\,y'' + y'x + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{3y}{2+x} - \frac{xy'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{xy'}{2+x} + \frac{3y}{2+x} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

- Define functions

$\left[P_2(x) = \frac{x}{2+x}, P_3(x) = \frac{3}{2+x}\right]$

- $(2+x) \cdot P_2(x)$ is analytic at $x = -2$

$((2+x) \cdot P_2(x))\Big|_{x=-2} = -2$

- $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$

$((2+x)^2 \cdot P_3(x))\Big|_{x=-2} = 0$

- $x = -2$is a regular singular point

Check to see if $x_0 = -2$ is a regular singular point

$x_0 = -2$

- Multiply by denominators

$(2+x)\, y'' + y'x + 3y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (u-2)\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-2+r) + a_k(k+r+3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k-2+r) + a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(2+x)^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve((2+x)*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{3}{4}x^2+\frac{1}{8}x^3+\frac{1}{8}x^4-\frac{9}{160}x^5\right)y(0)+\left(x-\frac{1}{3}x^3+\frac{1}{12}x^4+\frac{1}{40}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(2+x)*y''[x]+x*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{40}+\frac{x^4}{12}-\frac{x^3}{3}+x\right)+c_1\left(-\frac{9x^5}{160}+\frac{x^4}{8}+\frac{x^3}{8}-\frac{3x^2}{4}+1\right)$$

11.2 problem 12

Internal problem ID [1191]
Internal file name [`OUTPUT/1192_Sunday_June_05_2022_02_04_43_AM_4731487/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2+1)\,y''+3y'x^2-2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{788}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{789}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3y'x^2 - 2y}{3x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(9x^4 + 6x^2 - 6x + 2)\, y' - 6x(2+x)\, y}{(3x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-27x^6 - 36x^4 - 18x^3 + 42x^2 - 24x - 6)\, y' + 18\left(x^4 + 2x^3 + \frac{20}{3}x^2 - \frac{4}{3}x - \frac{4}{9}\right) y}{(3x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(81x^8 + 162x^6 + 216x^5 + 414x^4 - 612x^3 + 420x^2 + 168x - 32)\, y' - 54\left(x^6 + 2x^5 + \frac{22}{3}x^4 + 26x^3 - \frac{92}{9}\right.}{(3x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-243x^{10} - 648x^8 - 1296x^7 - 4428x^6 - 6480x^5 + 10260x^4 - 6804x^3 - 4824x^2 + 1944x + 132)\, y' +}{(3x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 2y'(0) \\
F_2 &= -8y(0) - 6y'(0) \\
F_3 &= -36y(0) - 32y'(0) \\
F_4 &= 272y(0) + 132y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + x^2 - \frac{1}{3}x^4 - \frac{3}{10}x^5 + \frac{17}{45}x^6\right) y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{4}{15}x^5 + \frac{11}{60}x^6\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(3x^2 + 1\right) y'' + 3y' x^2 - 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(3x^2 + 1\right) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 3\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$
$$\sum_{n=1}^{\infty} 3n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 3(n-1) a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \left(\sum_{n=2}^{\infty} 3(n-1) a_{n-1} x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$6a_3 - 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$3na_n(n-1) + (n+2) a_{n+2}(1+n) + 3(n-1) a_{n-1} - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3n^2 a_n - 3na_n + 3na_{n-1} - 2a_n - 3a_{n-1}}{(n+2)(1+n)} \\ &= -\frac{(3n^2 - 3n - 2) a_n}{(n+2)(1+n)} - \frac{(3n-3) a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 12a_4 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3} - \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$16a_3 + 20a_5 + 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{4a_1}{15} - \frac{3a_0}{10}$$

For $n = 4$ the recurrence equation gives

$$34a_4 + 30a_6 + 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{17a_0}{45} + \frac{11a_1}{60}$$

For $n = 5$ the recurrence equation gives

$$58a_5 + 42a_7 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{277a_1}{630} + \frac{107a_0}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + a_0x^2 + \frac{a_1x^3}{3} + \left(-\frac{a_0}{3} - \frac{a_1}{4}\right)x^4 + \left(-\frac{4a_1}{15} - \frac{3a_0}{10}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}\right) a_{0}+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}\right) a_{1}+O\left(x^{6}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}\right) c_{1}+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}+\frac{17}{45}x^{6}\right) y(0) \\ &+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}+\frac{11}{60}x^{6}\right) y'(0)+O\left(x^{6}\right) \end{aligned} \tag{1}$$

$$y = \left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}\right) c_{1}+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = \left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}+\frac{17}{45}x^{6}\right) y(0)+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}+\frac{11}{60}x^{6}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1+x^{2}-\frac{1}{3}x^{4}-\frac{3}{10}x^{5}\right) c_{1}+\left(x+\frac{1}{3}x^{3}-\frac{1}{4}x^{4}-\frac{4}{15}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
Order:=6;
dsolve((1+3*x^2)*diff(y(x),x$2)+3*x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 - \frac{1}{3}x^4 - \frac{3}{10}x^5\right) y(0) + \left(x + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{4}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 52

```
AsymptoticDSolveValue[(1+3*x^2)*y''[x]+3*x^2*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{4x^5}{15} - \frac{x^4}{4} + \frac{x^3}{3} + x\right) + c_1\left(-\frac{3x^5}{10} - \frac{x^4}{3} + x^2 + 1\right)$$

11.3 problem 13

Internal problem ID [1192]
Internal file name [`OUTPUT/1193_Sunday_June_05_2022_02_04_45_AM_35783344/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\,y''+(-3x+2)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\ &= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y_0'} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{791}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{792}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3y'x-2y'-4y}{2x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{(-5x^2-4x+3)\,y'+4(2+x)\,y}{(2x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(13x^3+38x^2-13x-2)\,y'-4y(x^2+12x+2)}{(2x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(-47x^4-312x^3+34x^2+72x-17)\,y'-20\left(x^3-\frac{82}{5}x^2-7x+2\right)y}{(2x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(195x^5+2934x^4+390x^3-1548x^2+285x+66)\,y'+388y\left(x^4-\frac{672}{97}x^3-\frac{539}{97}x^2+\frac{252}{97}x+\frac{52}{97}\right)}{(2x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0)-2y'(0) \\
F_1 &= 8y(0)+3y'(0) \\
F_2 &= -8y(0)-2y'(0) \\
F_3 &= -40y(0)-17y'(0) \\
F_4 &= 208y(0)+66y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-2x^{2}+\frac{4}{3}x^{3}-\frac{1}{3}x^{4}-\frac{1}{3}x^{5}+\frac{13}{45}x^{6}\right) y(0) \\
&+\left(x-x^{2}+\frac{1}{2}x^{3}-\frac{1}{12}x^{4}-\frac{17}{120}x^{5}+\frac{11}{120}x^{6}\right) y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(2x^{2}+1\right) y''+\left(-3x+2\right) y'+4y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(2x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(-3x+2\right)\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+4\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right) \\
&+\sum_{n=1}^{\infty}\left(-3n a_{n} x^{n}\right)+\left(\sum_{n=1}^{\infty} 2n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 4a_{n} x^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-3n a_n x^n) + \left(\sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 4a_0 = 0$$

$$a_2 = -2a_0 - a_1$$

$n = 1$ gives

$$6a_3 + a_1 + 4a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_0}{3} + \frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2) a_{n+2}(n+1) - 3na_n + 2(n+1) a_{n+1} + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_n - 5na_n + 2na_{n+1} + 4a_n + 2a_{n+1}}{(n+2)(n+1)} \\ &= -\frac{(2n^2 - 5n + 4) a_n}{(n+2)(n+1)} - \frac{(2n+2) a_{n+1}}{(n+2)(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 12a_4 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 20a_5 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{3} - \frac{17a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$16a_4 + 30a_6 + 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{13a_0}{45} + \frac{11a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$29a_5 + 42a_7 + 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{31a_0}{210} + \frac{361a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + (-2a_0 - a_1) x^2 + \left(\frac{4a_0}{3} + \frac{a_1}{2}\right) x^3 + \left(-\frac{a_0}{3} - \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{3} - \frac{17a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5\right) a_0 + \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5\right) c_1 + \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5 + \frac{13}{45}x^6\right) y(0) \\ &+ \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5 + \frac{11}{120}x^6\right) y'(0) + O(x^6) \end{aligned} \quad (1)$$

$$y = \left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5\right) c_1 + \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$\begin{aligned} y = &\left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5 + \frac{13}{45}x^6\right) y(0) \\ &+ \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5 + \frac{11}{120}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5\right) c_1 + \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((1+2*x^2)*diff(y(x),x$2)+(2-3*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - 2x^2 + \frac{4}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5\right) y(0) + \left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4 - \frac{17}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 66

```
AsymptoticDSolveValue[(1+2*x^2)*y''[x]+(2-3*x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^5}{3} - \frac{x^4}{3} + \frac{4x^3}{3} - 2x^2 + 1\right) + c_2 \left(-\frac{17x^5}{120} - \frac{x^4}{12} + \frac{x^3}{2} - x^2 + x\right)$$

11.4 problem 14

Internal problem ID [1193]
Internal file name [`OUTPUT/1194_Sunday_June_05_2022_02_04_46_AM_81043293/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''+(2-x)\, y'+3y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{794}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{795}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x-2y'-3y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{-3y'x^2+3yx+2y'+6y}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(6x^3+12x^2-9x+2)\,y'+(-24x-3)\,y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(-12x^4-72x^3+27x^2+8x-16)\,y'-18\left(x^3-\frac{14}{3}x^2-\frac{5}{2}x+\frac{5}{3}\right)y}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(18x^5+396x^4-12x^3-264x^2+195x+10)\,y'+126\left(x^4-\frac{16}{7}x^3-\frac{25}{7}x^2+\frac{64}{21}x+\frac{31}{42}\right)y}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0)-2y'(0) \\
F_1 &= 6y(0)+2y'(0) \\
F_2 &= -3y(0)+2y'(0) \\
F_3 &= -30y(0)-16y'(0) \\
F_4 &= 93y(0)+10y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{3}{2}x^{2}+x^{3}-\frac{1}{8}x^{4}-\frac{1}{4}x^{5}+\frac{31}{240}x^{6}\right) y(0) \\
&+\left(x-x^{2}+\frac{1}{3}x^{3}+\frac{1}{12}x^{4}-\frac{2}{15}x^{5}+\frac{1}{72}x^{6}\right) y'(0)+O(x^{6})
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x^{2}+1)\, y''+(2-x)\, y'+3y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(x^{2}+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+(2-x)\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+3\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \quad (1)
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right) \\
&+\left(\sum_{n=1}^{\infty} 2n a_{n} x^{n-1}\right)+\sum_{n=1}^{\infty} (-n a_{n} x^{n})+\left(\sum_{n=0}^{\infty} 3a_{n} x^{n}\right) = 0
\end{aligned} \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2} - a_1$$

$n = 1$ gives

$$6a_3 + 4a_2 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_0 + \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) + 2(n+1)\, a_{n+1} - n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n - 2n a_n + 2n a_{n+1} + 3a_n + 2a_{n+1}}{(n+2)\,(n+1)} \\ &= -\frac{(n^2 - 2n + 3)\, a_n}{(n+2)\,(n+1)} - \frac{(2n+2)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 + 12a_4 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$6a_3 + 20a_5 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{4} - \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$11a_4 + 30a_6 + 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{31a_0}{240} + \frac{a_1}{72}$$

For $n = 5$ the recurrence equation gives

$$18a_5 + 42a_7 + 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{59a_0}{840} + \frac{67a_1}{1260}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{3a_0}{2} - a_1\right) x^2 + \left(a_0 + \frac{a_1}{3}\right) x^3 + \left(-\frac{a_0}{8} + \frac{a_1}{12}\right) x^4 + \left(-\frac{a_0}{4} - \frac{2a_1}{15}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5\right) a_0 + \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5\right) c_1 + \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5 + \frac{31}{240}x^6\right) y(0) \\ &+ \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5 + \frac{1}{72}x^6\right) y'(0) + O(x^6) \end{aligned} \quad (1)$$

$$y = \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5\right) c_1 + \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$\begin{aligned} y &= \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5 + \frac{31}{240}x^6\right) y(0) \\ &+ \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5 + \frac{1}{72}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 + x^3 - \frac{1}{8}x^4 - \frac{1}{4}x^5\right) c_1 + \left(x - x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+(2-x)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\frac{3}{2}x^2+x^3-\frac{1}{8}x^4-\frac{1}{4}x^5\right)y(0)+\left(x-x^2+\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{2}{15}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 64

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+(2-x)*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{4}-\frac{x^4}{8}+x^3-\frac{3x^2}{2}+1\right)+c_2\left(-\frac{2x^5}{15}+\frac{x^4}{12}+\frac{x^3}{3}-x^2+x\right)$$

11.5 problem 15

Internal problem ID [1194]
Internal file name [`OUTPUT/1195_Sunday_June_05_2022_02_04_47_AM_50425243/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2+1)\,y''-2y'x+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{797}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{798}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y'x - 4y}{3x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-14y'x^2 + 16yx - 2y'}{\left(3x^2 + 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{104x^3y' - 88x^2y + 8y'x + 24y}{\left(3x^2 + 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{32(-31x^4 + 6x^2 + 1)\, y' + 640(x^3 - x)\, y}{\left(3x^2 + 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(11840x^5 - 8320x^3 - 960x\right) y' + \left(-5632x^4 + 14592x^2 - 768\right) y}{\left(3x^2 + 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 32y'(0) \\
F_4 &= -768y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - 2x^2 + x^4 - \frac{16}{15}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(3x^2 + 1)\, y'' - 2y'x + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(3x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 4a_0 = 0$$

$$a_2 = -2a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$3na_n(n-1) + (n+2)\, a_{n+2}(n+1) - 2na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(3n^2 - 5n + 4)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$16a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$32a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{16a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$54a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{12a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - 2a_0x^2 - \frac{1}{3}a_1x^3 + a_0x^4 + \frac{4}{15}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^4 - 2x^2 + 1\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{4}{15}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(x^4 - 2x^2 + 1\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O\left(x^6\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 - 2x^2 + x^4 - \frac{16}{15}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{4}{15}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(x^4 - 2x^2 + 1\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{4}{15}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1-2x^2+x^4-\frac{16}{15}x^6\right) y(0)+\left(x-\frac{1}{3}x^3+\frac{4}{15}x^5\right) y'(0)+O(x^6)$$

Verified OK.

$$y = (x^4-2x^2+1)\, c_1+\left(x-\frac{1}{3}x^3+\frac{4}{15}x^5\right) c_2+O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve((1+3*x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4-2x^2+1)\, y(0)+\left(x-\frac{1}{3}x^3+\frac{4}{15}x^5\right) D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(1+3*x^2)*y''[x]-2*x*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{4x^5}{15} - \frac{x^3}{3} + x\right) + c_1\left(x^4 - 2x^2 + 1\right)$$

11.6 problem 16

Internal problem ID [1195]
Internal file name [`OUTPUT/1196_Sunday_June_05_2022_02_04_49_AM_56123841/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (2x+4)\,y' + (2+x)\,y = 0}$$

With the expansion point for the power series method at $x = -1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(-1+t) + (2t+2)\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{800}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{801}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx} \left(\frac{df}{dx} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)\,t + 2t\left(\frac{d}{dt}y(t)\right) + y(t) + 2\frac{d}{dt}y(t)}{-1+t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(3t^2+8t+9\right)\left(\frac{d}{dt}y(t)\right) + 2y(t)\left(t^2+2t+2\right)}{\left(-1+t\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-4t^3-20t^2-48t-48\right)\left(\frac{d}{dt}y(t)\right) - 3\left(t^3+\frac{11}{3}t^2+\frac{25}{3}t+7\right)y(t)}{\left(-1+t\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(5t^4+40t^3+154t^2+332t+309\right)\left(\frac{d}{dt}y(t)\right) + 4y(t)\left(t^4+6t^3+22t^2+42t+34\right)}{\left(-1+t\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-6t^5-70t^4-384t^3-1320t^2-2618t-2322\right)\left(\frac{d}{dt}y(t)\right) - 5y(t)\left(t^5+9t^4+\frac{234}{5}t^3+\frac{734}{5}t^2+\frac{1321}{5}t+\right.}{\left(-1+t\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) + 2y'(0) \\
F_1 &= 4y(0) + 9y'(0) \\
F_2 &= 21y(0) + 48y'(0) \\
F_3 &= 136y(0) + 309y'(0) \\
F_4 &= 1021y(0) + 2322y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1+\frac{1}{2}t^2+\frac{2}{3}t^3+\frac{7}{8}t^4+\frac{17}{15}t^5+\frac{1021}{720}t^6\right)y(0) + \left(t+t^2+\frac{3}{2}t^3+2t^4+\frac{103}{40}t^5+\frac{129}{40}t^6\right)y'(0)+O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(-1+t)+(2t+2)\left(\frac{d}{dt}y(t)\right)+(t+1)\,y(t)=0$$

Let the solution be represented as power series of the form

$$y(t)=\sum_{n=0}^{\infty}a_n t^n$$

Then

$$\frac{d}{dt}y(t)=\sum_{n=1}^{\infty}n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t)=\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)(-1+t)+(2t+2)\left(\sum_{n=1}^{\infty}n a_n t^{n-1}\right)+(t+1)\left(\sum_{n=0}^{\infty}a_n t^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}n\,t^{n-1}a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1)\,a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty}2n a_n t^n\right) + \left(\sum_{n=1}^{\infty}2n a_n t^{n-1}\right)+\left(\sum_{n=0}^{\infty}t^{1+n}a_n\right)+\left(\sum_{n=0}^{\infty}a_n t^n\right)=0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (1+n)\, t^n\right)$$

$$\sum_{n=1}^{\infty} 2n a_n t^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (1+n)\, t^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 2a_1 + a_0 = 0$$

$$a_2 = \frac{a_0}{2} + a_1$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n - (n+2)\, a_{n+2} (1+n) + 2n a_n + 2(1+n)\, a_{1+n} + a_{n-1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{1+n} + 2n a_n + 3n a_{1+n} + a_n + 2a_{1+n} + a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{(2n+1)\, a_n}{(n+2)\,(1+n)} + \frac{(n^2+3n+2)\, a_{1+n}}{(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 - 6a_3 + 3a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_0}{3} + \frac{3a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_3 - 12a_4 + 5a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{8} + 2a_1$$

For $n = 3$ the recurrence equation gives

$$20a_4 - 20a_5 + 7a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{17a_0}{15} + \frac{103a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_5 - 30a_6 + 9a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1021a_0}{720} + \frac{129a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$42a_6 - 42a_7 + 11a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{243a_0}{140} + \frac{6631a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t &+ \left(\frac{a_0}{2} + a_1\right) t^2 + \left(\frac{2a_0}{3} + \frac{3a_1}{2}\right) t^3 + \left(\frac{7a_0}{8} + 2a_1\right) t^4 + \left(\frac{17a_0}{15} + \frac{103a_1}{40}\right) t^5 \\ &+ \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \frac{7}{8}t^4 + \frac{17}{15}t^5\right) a_0 + \left(t + t^2 + \frac{3}{2}t^3 + 2t^4 + \frac{103}{40}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \frac{7}{8}t^4 + \frac{17}{15}t^5\right) c_1 + \left(t + t^2 + \frac{3}{2}t^3 + 2t^4 + \frac{103}{40}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x+1$ results in

$$\begin{aligned} y = &\left(1 + \frac{(x+1)^2}{2} + \frac{2(x+1)^3}{3} + \frac{7(x+1)^4}{8} + \frac{17(x+1)^5}{15} + \frac{1021(x+1)^6}{720}\right) y(-1) \\ &+ \left(x + 1 + (x+1)^2 + \frac{3(x+1)^3}{2} + 2(x+1)^4 + \frac{103(x+1)^5}{40} + \frac{129(x+1)^6}{40}\right) y'(-1) \\ &+ O\left((x+1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(1+\frac{(x+1)^{2}}{2}+\frac{2(x+1)^{3}}{3}+\frac{7(x+1)^{4}}{8}+\frac{17(x+1)^{5}}{15}+\frac{1021(x+1)^{6}}{720}\right)y(-1) + \left(x+1+(x+1)^{2}+\frac{3(x+1)^{3}}{2}+2(x+1)^{4}+\frac{103(x+1)^{5}}{40}+\frac{129(x+1)^{6}}{40}\right)y'(-1) + O\left((x+1)^{6}\right) \tag{1}$$

Verification of solutions

$$y = \left(1+\frac{(x+1)^{2}}{2}+\frac{2(x+1)^{3}}{3}+\frac{7(x+1)^{4}}{8}+\frac{17(x+1)^{5}}{15}+\frac{1021(x+1)^{6}}{720}\right)y(-1) + \left(x+1+(x+1)^{2}+\frac{3(x+1)^{3}}{2}+2(x+1)^{4}+\frac{103(x+1)^{5}}{40}+\frac{129(x+1)^{6}}{40}\right)y'(-1) + O\left((x+1)^{6}\right)$$

Verified OK.

11.6.1 Maple step by step solution

Let's solve

$y''x+(2x+4)\,y'+(2+x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(2+x)y}{x}-\frac{2(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(2+x)y'}{x}+\frac{(2+x)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(2+x)}{x}, P_3(x)=\frac{2+x}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=4$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (2x+4)\, y' + (2+x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, x^{-1+r} + \left(a_1(1+r)\,(4+r) + 2a_0(1+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+4+r) + 2a_k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1(1+r)(4+r) + 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+4+r) + 2a_k k + 2a_k r + 2a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+5+r) + 2a_{k+1}(k+1) + 2ra_{k+1} + 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k+4a_{k+1}}{(k+2+r)(k+5+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-2a_{k+1}}{(k-1)(k+2)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-2a_{k+1}}{(k-1)(k+2)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+4a_{k+1}}{(k+2)(k+5)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+4a_{k+1}}{(k+2)(k+5)}, 4a_1 + 2a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(4+2*x)*diff(y(x),x)+(2+x)*y(x)=0,y(x),type='series',x=-1);
```

$$y(x) = \left(1 + \frac{(x+1)^2}{2} + \frac{2(x+1)^3}{3} + \frac{7(x+1)^4}{8} + \frac{17(x+1)^5}{15}\right) y(-1) + \left(x + 1 + (x+1)^2 + \frac{3(x+1)^3}{2} + 2(x+1)^4 + \frac{103(x+1)^5}{40}\right) D(y)(-1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 81

```
AsymptoticDSolveValue[(x)*y''[x]+(4+2*x)*y'[x]+(2+x)*y[x]==0,y[x],{x,-1,5}]
```

$$y(x) \to c_1\left(\frac{17}{15}(x+1)^5 + \frac{7}{8}(x+1)^4 + \frac{2}{3}(x+1)^3 + \frac{1}{2}(x+1)^2 + 1\right) + c_2\left(\frac{103}{40}(x+1)^5 + 2(x+1)^4 + \frac{3}{2}(x+1)^3 + (x+1)^2 + x + 1\right)$$

11.7 problem 17

Internal problem ID [1196]
Internal file name [`OUTPUT/1197_Sunday_June_05_2022_02_04_50_AM_23712595/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 2y'x - 3yx = 0}$$

With the expansion point for the power series method at $x = 2$.

The ODE is

$$x^2y'' + 2y'x - 3yx = 0$$

Or

$$x(xy'' - 3y + 2y') = 0$$

For $x \neq 0$ the above simplifies to

$$xy'' - 3y + 2y' = 0$$

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(2+t)^2 \left(\frac{d^2}{dt^2}y(t)\right) + 2(2+t)\left(\frac{d}{dt}y(t)\right) - 3(2+t)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2}\,f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (803) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (804) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2\frac{d}{dt}y(t) - 3y(t)}{2+t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(3t+12)\left(\frac{d}{dt}y(t)\right) - 9y(t)}{(2+t)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(-18t-60)\left(\frac{d}{dt}y(t)\right) + 9y(t)(t+6)}{(2+t)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(9t^2+144t+372)\left(\frac{d}{dt}y(t)\right) + (-72t-324)\,y(t)}{(2+t)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(-108t^2-1152t-2592)\left(\frac{d}{dt}y(t)\right) + 27y(t)(t^2+24t+84)}{(2+t)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) + \frac{3y(0)}{2} \\
F_1 &= -\frac{9y(0)}{4} + 3y'(0) \\
F_2 &= \frac{27y(0)}{4} - \frac{15y'(0)}{2} \\
F_3 &= -\frac{81y(0)}{4} + \frac{93y'(0)}{4} \\
F_4 &= \frac{567y(0)}{8} - 81y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \left(1 + \frac{3}{4}t^2 - \frac{3}{8}t^3 + \frac{9}{32}t^4 - \frac{27}{160}t^5 + \frac{63}{640}t^6\right) y(0) \\
& + \left(t - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{16}t^4 + \frac{31}{160}t^5 - \frac{9}{80}t^6\right) y'(0) + O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(t^2 + 4t + 4\right) \left(\frac{d^2}{dt^2}y(t)\right) + (2t + 4) \left(\frac{d}{dt}y(t)\right) + (-3t - 6)\, y(t) = 0
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(t^2 + 4t + 4\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + (2t + 4) \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (-3t - 6) \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}
$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2}\right) \\ + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 4n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} \left(-3t^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-6a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (1+n)\, t^n$$

$$\sum_{n=1}^{\infty} 4n a_n t^{n-1} = \sum_{n=0}^{\infty} 4(1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} \left(-3t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (1+n)\, t^n\right) \\ + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 4(1+n)\, a_{1+n} t^n\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} t^n\right) + \sum_{n=0}^{\infty} \left(-6a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 4a_1 - 6a_0 = 0$$

$$a_2 = \frac{3a_0}{4} - \frac{a_1}{2}$$

$n = 1$ gives

$$16a_2 + 24a_3 - 4a_1 - 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_0}{8} + \frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$\begin{aligned} &na_n(n-1) + 4(1+n)\, a_{1+n} n + 4(n+2)\, a_{n+2}(1+n) \\ &+ 2na_n + 4(1+n)\, a_{1+n} - 3a_{n-1} - 6a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 4n^2 a_{1+n} + na_n + 8na_{1+n} - 6a_n + 4a_{1+n} - 3a_{n-1}}{4\,(n+2)\,(1+n)} \\ &= -\frac{(n^2+n-6)\, a_n}{4\,(n+2)\,(1+n)} - \frac{(4n^2+8n+4)\, a_{1+n}}{4\,(n+2)\,(1+n)} + \frac{3a_{n-1}}{4\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$36a_3 + 48a_4 - 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{9a_0}{32} - \frac{5a_1}{16}$$

For $n = 3$ the recurrence equation gives

$$6a_3 + 64a_4 + 80a_5 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{27a_0}{160} + \frac{31a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$14a_4 + 100a_5 + 120a_6 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{63a_0}{640} - \frac{9a_1}{80}$$

For $n = 5$ the recurrence equation gives

$$24a_5 + 144a_6 + 168a_7 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{99a_0}{1792} + \frac{283a_1}{4480}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(\frac{3a_0}{4} - \frac{a_1}{2}\right) t^2 + \left(-\frac{3a_0}{8} + \frac{a_1}{2}\right) t^3 \\ + \left(\frac{9a_0}{32} - \frac{5a_1}{16}\right) t^4 + \left(-\frac{27a_0}{160} + \frac{31a_1}{160}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{3}{4}t^2 - \frac{3}{8}t^3 + \frac{9}{32}t^4 - \frac{27}{160}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{16}t^4 + \frac{31}{160}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{3}{4}t^2 - \frac{3}{8}t^3 + \frac{9}{32}t^4 - \frac{27}{160}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{16}t^4 + \frac{31}{160}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = -2 + x$ results in

$$y = \left(1 + \frac{3(-2+x)^2}{4} - \frac{3(-2+x)^3}{8} + \frac{9(-2+x)^4}{32} - \frac{27(-2+x)^5}{160} + \frac{63(-2+x)^6}{640}\right) y(2) + \left(-2 + x - \frac{(-2+x)^2}{2} + \frac{(-2+x)^3}{2} - \frac{5(-2+x)^4}{16} + \frac{31(-2+x)^5}{160} - \frac{9(-2+x)^6}{80}\right) y'(2) + O\left((-2+x)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{3(-2+x)^2}{4} - \frac{3(-2+x)^3}{8} + \frac{9(-2+x)^4}{32} - \frac{27(-2+x)^5}{160} + \frac{63(-2+x)^6}{640}\right) y(2) + \left(-2 + x - \frac{(-2+x)^2}{2} + \frac{(-2+x)^3}{2} - \frac{5(-2+x)^4}{16} + \frac{31(-2+x)^5}{160} - \frac{9(-2+x)^6}{80}\right) y'(2) + O\left((-2+x)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 + \frac{3(-2+x)^2}{4} - \frac{3(-2+x)^3}{8} + \frac{9(-2+x)^4}{32} - \frac{27(-2+x)^5}{160} + \frac{63(-2+x)^6}{640}\right) y(2) + \left(-2 + x - \frac{(-2+x)^2}{2} + \frac{(-2+x)^3}{2} - \frac{5(-2+x)^4}{16} + \frac{31(-2+x)^5}{160} - \frac{9(-2+x)^6}{80}\right) y'(2) + O\left((-2+x)^6\right)$$

Verified OK.

11.7.1 Maple step by step solution

Let's solve

$x^2y'' + 2y'x - 3yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{3y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{3}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x - 3y + 2y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) - 3a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+2+r) - 3a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{3a_k}{(k+1+r)(k+2+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{3a_k}{k(k+1)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{3a_k}{k(k+1)}\right]$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{3a_k}{(k+1)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{3a_k}{(k+1)(k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{1+k} = \frac{3a_k}{k(1+k)}, b_{1+k} = \frac{3b_k}{(1+k)(k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-3*x*y(x)=0,y(x),type='series',x=2);
```

$$y(x) = \left(1+\frac{3(-2+x)^2}{4}-\frac{3(-2+x)^3}{8}+\frac{9(-2+x)^4}{32}-\frac{27(-2+x)^5}{160}\right)y(2) + \left(-2+x-\frac{(-2+x)^2}{2}+\frac{(-2+x)^3}{2}-\frac{5(-2+x)^4}{16}+\frac{31(-2+x)^5}{160}\right)D(y)(2) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*y''[x]+2*x*y'[x]-3*x*y[x]==0,y[x],{x,2,5}]
```

$$y(x) \to c_1\left(-\frac{27}{160}(x-2)^5+\frac{9}{32}(x-2)^4-\frac{3}{8}(x-2)^3+\frac{3}{4}(x-2)^2+1\right) + c_2\left(\frac{31}{160}(x-2)^5-\frac{5}{16}(x-2)^4+\frac{1}{2}(x-2)^3-\frac{1}{2}(x-2)^2+x-2\right)$$

11.8 problem 18

Internal problem ID [1197]
Internal file name [OUTPUT/1198_Sunday_June_05_2022_02_04_52_AM_23130848/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(2-x)\,y''+2y=0}$$

With initial conditions

$$[y(0)=a_0, y'(0)=a_1]$$

With the expansion point for the power series method at $x=0$.

11.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{2}{2-x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y}{2-x} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{2-x}$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{806}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{807}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y}{-2+x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-4+2x)\, y' - 2y}{(-2+x)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-4x+8)\, y' + 4y(x-1)}{(-2+x)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(4x^2-4x-8)\, y' + (-16x+20)\, y}{(-2+x)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-24x^2+48x)\, y' + 8y(x^2+5x-8)}{(-2+x)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=a_0$ and $y'(0)=a_1$ gives

$$
\begin{aligned}
F_0 &= -a_0 \\
F_1 &= -\frac{a_0}{2} - a_1 \\
F_2 &= \frac{a_0}{2} - a_1 \\
F_3 &= \frac{5a_0}{4} - \frac{a_1}{2} \\
F_4 &= 2a_0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = xa_1 + a_0 - \frac{a_0x^2}{2} - \frac{x^3a_0}{12} - \frac{x^3a_1}{6} + \frac{x^4a_0}{48} - \frac{x^4a_1}{24} + \frac{x^5a_0}{96} - \frac{x^5a_1}{240} + \frac{a_0x^6}{360} + O(x^6)$$

$$y = xa_1 + a_0 - \frac{a_0x^2}{2} - \frac{x^3a_0}{12} - \frac{x^3a_1}{6} + \frac{x^4a_0}{48} - \frac{x^4a_1}{24} + \frac{x^5a_0}{96} - \frac{x^5a_1}{240} + \frac{a_0x^6}{360} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2-x)\,y'' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2-x)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-n\,x^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n\,x^{n-1} a_n(n-1)\right) = \sum_{n=1}^{\infty} \left(-(n+1)\,a_{n+1} n\,x^n\right)$$
$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=1}^{\infty} \left(-(n+1) a_{n+1} n x^{n}\right) + \left(\sum_{n=0}^{\infty} 2(n+2) a_{n+2}(n+1) x^{n}\right) + \left(\sum_{n=0}^{\infty} 2a_{n} x^{n}\right) = 0 \quad (3)$$

$n = 0$ gives

$$4a_2 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$-(n+1) a_{n+1} n + 2(n+2) a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{n+1} + n a_{n+1} - 2a_n}{2(n+2)(n+1)} \\ &= -\frac{a_n}{(n+2)(n+1)} + \frac{(n^2+n) a_{n+1}}{2(n+2)(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-2a_2 + 12a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{12} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$-6a_3 + 24a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{48} - \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$-12a_4 + 40a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{96} - \frac{a_1}{240}$$

For $n = 4$ the recurrence equation gives

$$-20a_5 + 60a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{360}$$

For $n = 5$ the recurrence equation gives

$$-30a_6 + 84a_7 + 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{1344} + \frac{a_1}{10080}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{a_0x^2}{2} + \left(-\frac{a_0}{12} - \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{48} - \frac{a_1}{24}\right)x^4 + \left(\frac{a_0}{96} - \frac{a_1}{240}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{96}x^5\right)a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{240}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{96}x^5\right)c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{240}x^5\right)c_2 + O(x^6)$$

$$y = a_0 - \frac{a_0 x^2}{2} - \frac{x^3 a_0}{12} + \frac{x^4 a_0}{48} + \frac{x^5 a_0}{96} + x a_1 - \frac{x^3 a_1}{6} - \frac{x^4 a_1}{24} - \frac{x^5 a_1}{240} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x a_1 + a_0 - \frac{a_0 x^2}{2} - \frac{x^3 a_0}{12} - \frac{x^3 a_1}{6} + \frac{x^4 a_0}{48} - \frac{x^4 a_1}{24} + \frac{x^5 a_0}{96} - \frac{x^5 a_1}{240} + \frac{a_0 x^6}{360} + O(x^6) \tag{1}$$

$$y = a_0 - \frac{a_0 x^2}{2} - \frac{x^3 a_0}{12} + \frac{x^4 a_0}{48} + \frac{x^5 a_0}{96} + x a_1 - \frac{x^3 a_1}{6} - \frac{x^4 a_1}{24} - \frac{x^5 a_1}{240} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x a_1 + a_0 - \frac{a_0 x^2}{2} - \frac{x^3 a_0}{12} - \frac{x^3 a_1}{6} + \frac{x^4 a_0}{48} - \frac{x^4 a_1}{24} + \frac{x^5 a_0}{96} - \frac{x^5 a_1}{240} + \frac{a_0 x^6}{360} + O(x^6)$$

Verified OK.

$$y = a_0 - \frac{a_0 x^2}{2} - \frac{x^3 a_0}{12} + \frac{x^4 a_0}{48} + \frac{x^5 a_0}{96} + x a_1 - \frac{x^3 a_1}{6} - \frac{x^4 a_1}{24} - \frac{x^5 a_1}{240} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
Order:=6;
dsolve([(2-x)*diff(y(x),x$2)+2*y(x)=0,y(0) = a__0, D(y)(0) = a__1],y(x),type='series',x=0);
```

$$y(x) = a_0 + a_1 x - \frac{1}{2} a_0 x^2 + \left(-\frac{a_1}{6} - \frac{a_0}{12}\right) x^3 + \left(\frac{a_0}{48} - \frac{a_1}{24}\right) x^4 + \left(-\frac{a_1}{240} + \frac{a_0}{96}\right) x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 79

```
AsymptoticDSolveValue[{(2-x)*y''[x]+2*y[x]==0,{y[0]==a0,y'[0]==a1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{1}{20}x^5\left(\frac{1}{6}\left(\frac{\text{a0}}{2}+\text{a1}\right)+\frac{\text{a0}}{8}-\frac{\text{a1}}{4}\right)+\frac{1}{12}x^4\left(\frac{\text{a0}}{4}-\frac{\text{a1}}{2}\right)+\frac{1}{6}x^3\left(-\frac{\text{a0}}{2}-\text{a1}\right)-\frac{\text{a0}x^2}{2}+\text{a0}+\text{a1}x$$

11.9 problem 19

Internal problem ID [1198]
Internal file name [`OUTPUT/1199_Sunday_June_05_2022_02_04_53_AM_91201647/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''+2(x-1)^2\,y'+3y=0}$$

With initial conditions

$$[y(1)=a_0, y'(1)=a_1]$$

With the expansion point for the power series method at $x=1$.

11.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2(x-1)^2}{x+1}\\q(x)&=\frac{3}{x+1}\\F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{2(x-1)^2 y'}{x+1} + \frac{3y}{x+1} = 0$$

The domain of $p(x) = \frac{2(x-1)^2}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{3}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(2+t) + 2\left(\frac{d}{dt}y(t)\right)t^2 + 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = a_0$$
$$y'(0) = a_1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (809) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (810) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$F_0 = -\frac{2\left(\frac{d}{dt}y(t)\right)t^2 + 3y(t)}{2+t}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(4t^4 - 2t^2 - 11t - 6)\left(\frac{d}{dt}y(t)\right) + (6t^2+3)\,y(t)}{(2+t)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(-8t^6 + 12t^4 + 60t^3 + 24t^2 + 6t - 4)\left(\frac{d}{dt}y(t)\right) - 12\left(t^4 - \frac{1}{2}t^2 - \frac{19}{4}t - 1\right)y(t)}{(2+t)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(16t^8 - 48t^6 - 228t^5 - 60t^4 + 90t^3 + 413t^2 + 210t + 48)\left(\frac{d}{dt}y(t)\right) + 24\left(t^6 - 2t^4 - \frac{23}{2}t^3 - \frac{13}{4}t^2 - \frac{9}{2}t +}{(2+t)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(-32t^{10} + 160t^8 + 736t^7 + 72t^6 - 1032t^5 - 3478t^4 - 1620t^3 - 646t^2 + 896t + 408)\left(\frac{d}{dt}y(t)\right) - 48\left(t^8}{(2+t)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=a_0$ and $y'(0)=a_1$ gives

$$\begin{aligned}
F_0 &= -\frac{3a_0}{2} \\
F_1 &= \frac{3a_0}{4} - \frac{3a_1}{2} \\
F_2 &= \frac{3a_0}{2} - \frac{a_1}{2} \\
F_3 &= \frac{45a_0}{8} + 3a_1 \\
F_4 &= -\frac{45a_0}{2} + \frac{51a_1}{4}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = ta_1 + a_0 - \frac{3a_0t^2}{4} + \frac{t^3a_0}{8} - \frac{t^3a_1}{4} + \frac{t^4a_0}{16} - \frac{t^4a_1}{48} + \frac{3t^5a_0}{64} + \frac{t^5a_1}{40} - \frac{t^6a_0}{32} + \frac{17t^6a_1}{960} + O(t^6)$$

$$y(t) = ta_1 + a_0 - \frac{3a_0t^2}{4} + \frac{t^3a_0}{8} - \frac{t^3a_1}{4} + \frac{t^4a_0}{16} - \frac{t^4a_1}{48} + \frac{3t^5a_0}{64} + \frac{t^5a_1}{40} - \frac{t^6a_0}{32} + \frac{17t^6a_1}{960} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2+t) + 2\left(\frac{d}{dt}y(t)\right)t^2 + 3y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(2+t) + 2\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)t^2 + 3\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=1}^{\infty} 2n\, t^{1+n} a_n = \sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

$n = 1$ gives

$$2a_2 + 12a_3 + 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{8} - \frac{a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + 2(n+2)\, a_{n+2}(1+n) + 2(n-1)\, a_{n-1} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_{1+n} + n a_{1+n} + 2n a_{n-1} + 3a_n - 2a_{n-1}}{2(n+2)(1+n)} \\
&= -\frac{3a_n}{2(n+2)(1+n)} - \frac{(n^2+n)\, a_{1+n}}{2(n+2)(1+n)} - \frac{(2n-2)\, a_{n-1}}{2(n+2)(1+n)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
6a_3 + 24a_4 + 2a_1 + 3a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = \frac{a_0}{16} - \frac{a_1}{48}
$$

For $n = 3$ the recurrence equation gives

$$
12a_4 + 40a_5 + 4a_2 + 3a_3 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{3a_0}{64} + \frac{a_1}{40}
$$

For $n = 4$ the recurrence equation gives

$$
20a_5 + 60a_6 + 6a_3 + 3a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_6 = -\frac{a_0}{32} + \frac{17a_1}{960}
$$

For $n = 5$ the recurrence equation gives

$$
30a_6 + 84a_7 + 8a_4 + 3a_5 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_7 = \frac{19a_0}{5376} - \frac{211a_1}{40320}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t - \frac{3a_0 t^2}{4} + \left(\frac{a_0}{8} - \frac{a_1}{4}\right) t^3 + \left(\frac{a_0}{16} - \frac{a_1}{48}\right) t^4 + \left(\frac{3a_0}{64} + \frac{a_1}{40}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - \frac{3}{4}t^2 + \frac{1}{8}t^3 + \frac{1}{16}t^4 + \frac{3}{64}t^5\right) a_0 + \left(t - \frac{1}{4}t^3 - \frac{1}{48}t^4 + \frac{1}{40}t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - \frac{3}{4}t^2 + \frac{1}{8}t^3 + \frac{1}{16}t^4 + \frac{3}{64}t^5\right) c_1 + \left(t - \frac{1}{4}t^3 - \frac{1}{48}t^4 + \frac{1}{40}t^5\right) c_2 + O(t^6)
$$

$$
y(t) = a_0 - \frac{3a_0 t^2}{4} + \frac{t^3 a_0}{8} + \frac{t^4 a_0}{16} + \frac{3t^5 a_0}{64} + t a_1 - \frac{t^3 a_1}{4} - \frac{t^4 a_1}{48} + \frac{t^5 a_1}{40} + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$
\begin{aligned}
y = (x-1)\, a_1 + a_0 - \frac{3a_0(x-1)^2}{4} + \frac{(x-1)^3 a_0}{8} - \frac{(x-1)^3 a_1}{4} + \frac{(x-1)^4 a_0}{16} - \frac{(x-1)^4 a_1}{48} \\
+ \frac{3(x-1)^5 a_0}{64} + \frac{(x-1)^5 a_1}{40} - \frac{(x-1)^6 a_0}{32} + \frac{17(x-1)^6 a_1}{960} + O\left((x-1)^6\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = (x-1)\, a_1 + a_0 - \frac{3a_0(x-1)^2}{4} + \frac{(x-1)^3 a_0}{8} - \frac{(x-1)^3 a_1}{4} \\
+ \frac{(x-1)^4 a_0}{16} - \frac{(x-1)^4 a_1}{48} + \frac{3(x-1)^5 a_0}{64} + \frac{(x-1)^5 a_1}{40} \\
- \frac{(x-1)^6 a_0}{32} + \frac{17(x-1)^6 a_1}{960} + O\left((x-1)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$y = (x-1)\,a_1 + a_0 - \frac{3a_0(x-1)^2}{4} + \frac{(x-1)^3\,a_0}{8} - \frac{(x-1)^3\,a_1}{4} + \frac{(x-1)^4\,a_0}{16} - \frac{(x-1)^4\,a_1}{48} + \frac{3(x-1)^5\,a_0}{64} + \frac{(x-1)^5\,a_1}{40} - \frac{(x-1)^6\,a_0}{32} + \frac{17(x-1)^6\,a_1}{960} + O\big((x-1)^6\big)$$

Verified OK.

11.9.2 Maple step by step solution

Let's solve

$\left[(x+1)\,y''+2(x-1)^2\,y'+3y=0, y(1)=a_0, y'\Big|_{\{x=1\}}=a_1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{x+1}-\frac{2(x-1)^2 y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x-1)^2 y'}{x+1}+\frac{3y}{x+1}=0$

□ Check to see if $x_0=-1$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(x-1)^2}{x+1}, P_3(x)=\frac{3}{x+1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=8$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x+1)\,y''+2(x-1)^2\,y'+3y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^{2}}{d u^{2}} y(u)\right)+\left(2 u^{2}-8 u+8\right)\left(\frac{d}{d u} y(u)\right)+3 y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)$ to series expansion for $m=0..2$

    $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r) u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$

  - Convert $u \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)$ to series expansion

    $u \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) u^{k+r-1}$

  - Shift index using $k->k+1$

    $u \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) u^{k+r}$

  Rewrite ODE with series expansions

  $a_{0} r(7+r) u^{-1+r}+\left(a_{1}(1+r)(8+r)-a_{0}(-3+8 r)\right) u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)(k+8+r)-a_{\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(7+r)=0$

- Values of r that satisfy the indicial equation

  $r \in\{-7,0\}$

- Each term must be 0

  $a_{1}(1+r)(8+r)-a_{0}(-3+8 r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+8+r)+\left(-8 a_{k}+2 a_{k-1}\right) k+\left(-8 a_{k}+2 a_{k-1}\right) r+3 a_{k}-2 a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+9+r)+\left(-8 a_{k+1}+2 a_{k}\right)(k+1)+\left(-8 a_{k+1}+2 a_{k}\right) r+3 a_{k+1}-2 a_{k}=0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k k - 8k a_{k+1} + 2a_k r - 8r a_{k+1} - 5a_{k+1}}{(k+2+r)(k+9+r)}$$

- Recursion relation for $r = -7$

$$a_{k+2} = -\frac{2a_k k - 8k a_{k+1} - 14a_k + 51a_{k+1}}{(k-5)(k+2)}$$

- Series not valid for $r = -7$, division by 0 in the recursion relation at $k = 5$

$$a_{k+2} = -\frac{2a_k k - 8k a_{k+1} - 14a_k + 51a_{k+1}}{(k-5)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2a_k k - 8k a_{k+1} - 5a_{k+1}}{(k+2)(k+9)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2a_k k - 8k a_{k+1} - 5a_{k+1}}{(k+2)(k+9)}, 8a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k , a_{k+2} = -\frac{2a_k k - 8k a_{k+1} - 5a_{k+1}}{(k+2)(k+9)}, 8a_1 + 3a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
Order:=6;
dsolve([(1+x)*diff(y(x),x$2)+2*(x-1)^2*diff(y(x),x)+3*y(x)=0,y(1) = a__0, D(y)(1) = a__1],y(
```

$$y(x) = a_0 + a_1(x-1) - \frac{3}{4}a_0(x-1)^2 + \left(\frac{a_0}{8} - \frac{a_1}{4}\right)(x-1)^3 + \left(\frac{a_0}{16} - \frac{a_1}{48}\right)(x-1)^4 + \left(\frac{3a_0}{64} + \frac{a_1}{40}\right)(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 95

```
AsymptoticDSolveValue[{(1+x)*y''[x]+2*(x-1)^2*y'[x]+3*y[x]==0,{y[1]==a0,y'[1]==a1}},y[x],{x,
```

$$y(x) \to \frac{1}{20}(x-1)^5\left(\frac{1}{4}\left(\frac{3\text{a1}}{2}-\frac{3\text{a0}}{4}\right)+\frac{9\text{a0}}{8}+\frac{\text{a1}}{8}\right)+\frac{1}{12}(x-1)^4\left(\frac{3\text{a0}}{4}-\frac{\text{a1}}{4}\right) + \frac{1}{6}(x-1)^3\left(\frac{3\text{a0}}{4}-\frac{3\text{a1}}{2}\right)-\frac{3}{4}\text{a0}(x-1)^2+\text{a0}+\text{a1}(x-1)$$

11.10 problem 21

Internal problem ID [1199]
Internal file name [`OUTPUT/1200_Sunday_June_05_2022_02_04_55_AM_17975047/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)\,y''+x(x+4)\,y'+(2-x)\,y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(2-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = -\frac{x+4}{x\,(x-1)}$$
$$q(x) = \frac{-2+x}{x^2\,(x-1)}$$

Table 449: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+4}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{-2+x}{x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x-1) + (x^2+4x)\, y' + (2-x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + (x^2+4x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (2-x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} 2 a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\ +\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} 2 a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)+4 x^{n+r} a_n(n+r)+2 a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 4x^r a_0 r + 2a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 4x^r r + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 + 3r + 2)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 + 3r + 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) + 4a_n(n+r) + 2a_n - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-1}(n-3)^2}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(-1+r)^2}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{r^2+5r+6}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2(-1+r)^2}{(r+3)^2(r+2)(r+4)}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{r^2+5r+6}$	2
a_2	$\frac{r^2(-1+r)^2}{(r+3)^2(r+2)(r+4)}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-1+r)^2 r^2 (r+1)^2}{(r+3)^2 (r+2) (r+4)^2 (r+5)}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{r^2+5r+6}$	2
a_2	$\frac{r^2(-1+r)^2}{(r+3)^2(r+2)(r+4)}$	$\frac{1}{3}$
a_3	$\frac{(-1+r)^2 r^2(r+1)^2}{(r+3)^2(r+2)(r+4)^2(r+5)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+2)(-1+r)^2 r^2 (r+1)^2}{(r+6)(r+5)^2 (r+4)^2 (r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{r^2+5r+6}$	2
a_2	$\frac{r^2(-1+r)^2}{(r+3)^2(r+2)(r+4)}$	$\frac{1}{3}$
a_3	$\frac{(-1+r)^2 r^2(r+1)^2}{(r+3)^2(r+2)(r+4)^2(r+5)}$	0
a_4	$\frac{(r+2)(-1+r)^2 r^2(r+1)^2}{(r+6)(r+5)^2(r+4)^2(r+3)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r+2)(-1+r)^2 r^2(r+1)^2}{(r+7)(r+6)^2(r+4)^2(r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{r^2+5r+6}$	2
a_2	$\frac{r^2(-1+r)^2}{(r+3)^2(r+2)(r+4)}$	$\frac{1}{3}$
a_3	$\frac{(-1+r)^2 r^2(r+1)^2}{(r+3)^2(r+2)(r+4)^2(r+5)}$	0
a_4	$\frac{(r+2)(-1+r)^2 r^2(r+1)^2}{(r+6)(r+5)^2(r+4)^2(r+3)^2}$	0
a_5	$\frac{(r+2)(-1+r)^2 r^2(r+1)^2}{(r+7)(r+6)^2(r+4)^2(r+5)^2}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + 2x + \frac{x^2}{3} + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{(-1+r)^2}{r^2+5r+6} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{(-1+r)^2}{r^2+5r+6} &= \lim_{r\to -2} \frac{(-1+r)^2}{r^2+5r+6} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $-y'' x^2(x-1) + (x^2+4x) y' + (2-x) y = 0$

gives

$$
\begin{aligned}
&-\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2(x-1)\\
&+\left(x^2+4x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(2-x)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(-y_1''(x)\,x^2(x-1)+\left(x^2+4x\right)y_1'(x)+(2-x)\,y_1(x)\right)\ln(x)\right.\\
&\left.-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x^2(x-1)+\frac{\left(x^2+4x\right)y_1(x)}{x}\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2(x-1)\\
&+\left(x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(2-x)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
-y_1''(x)\,x^2(x-1)+\left(x^2+4x\right)y_1'(x)+(2-x)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(-\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x^2(x-1)+\frac{\left(x^2+4x\right)y_1(x)}{x}\right)C\\
&-\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2(x-1)\\
&+\left(x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(2-x)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned} \tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (2x+3)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\ + \left(-x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)\left(-1+n+r_2\right)\right) \\ + \left(x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) - (-2+x)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = -1$ and $r_2 = -2$ then the above becomes

$$\left(-2x(x-1)\left(\sum_{n=0}^{\infty} x^{n-2} a_n (n-1)\right) + (2x+3)\left(\sum_{n=0}^{\infty} a_n x^{n-1}\right)\right) C \\ + \left(-x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(n-3)\right) \\ + \left(x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{n-3} b_n (n-2)\right) - (-2+x)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0 \tag{10}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2C\, x^n a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 2C a_n x^n\right) \\ + \left(\sum_{n=0}^{\infty} 3C\, x^{n-1} a_n\right) + \sum_{n=0}^{\infty}\left(-b_n x^{n-1}(n-2)(n-3)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-2)\right) \\ + \left(\sum_{n=0}^{\infty} 4x^{n-2} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 2b_n x^{n-2}\right) + \sum_{n=0}^{\infty}\left(-x^{n-1} b_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n}a_{n}(n-1)\right)=\sum_{n=2}^{\infty}\left(-2Ca_{n-2}(n-3)\,x^{n-2}\right)$$

$$\sum_{n=0}^{\infty}2C\,x^{n-1}a_{n}(n-1)=\sum_{n=1}^{\infty}2Ca_{n-1}(n-2)\,x^{n-2}$$

$$\sum_{n=0}^{\infty}2Ca_{n}x^{n}=\sum_{n=2}^{\infty}2Ca_{n-2}x^{n-2}$$

$$\sum_{n=0}^{\infty}3C\,x^{n-1}a_{n}=\sum_{n=1}^{\infty}3Ca_{n-1}x^{n-2}$$

$$\sum_{n=0}^{\infty}\left(-b_{n}x^{n-1}(n-2)\,(n-3)\right)=\sum_{n=1}^{\infty}\left(-b_{n-1}(-4+n)\,(n-3)\,x^{n-2}\right)$$

$$\sum_{n=0}^{\infty}x^{n-1}b_{n}(n-2)=\sum_{n=1}^{\infty}b_{n-1}(n-3)\,x^{n-2}$$

$$\sum_{n=0}^{\infty}\left(-x^{n-1}b_{n}\right)=\sum_{n=1}^{\infty}\left(-b_{n-1}x^{n-2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-2Ca_{n-2}(n-3)\,x^{n-2}\right)+\left(\sum_{n=1}^{\infty}2Ca_{n-1}(n-2)\,x^{n-2}\right)\\
&+\left(\sum_{n=2}^{\infty}2Ca_{n-2}x^{n-2}\right)+\left(\sum_{n=1}^{\infty}3Ca_{n-1}x^{n-2}\right)\\
&+\sum_{n=1}^{\infty}\left(-b_{n-1}(-4+n)\,(n-3)\,x^{n-2}\right)+\left(\sum_{n=0}^{\infty}x^{n-2}b_{n}\left(n^{2}-5n+6\right)\right)\\
&+\left(\sum_{n=1}^{\infty}b_{n-1}(n-3)\,x^{n-2}\right)+\left(\sum_{n=0}^{\infty}4x^{n-2}b_{n}(n-2)\right)\\
&+\left(\sum_{n=0}^{\infty}2b_{n}x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-b_{n-1}x^{n-2}\right)=0
\end{aligned}\tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 9 = 0$$

Which is solved for C. Solving for C gives

$$C = 9$$

For $n = 2$, Eq (2B) gives

$$(4a_0 + 3a_1) C - 4b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$90 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -45$$

For $n = 3$, Eq (2B) gives

$$(2a_1 + 5a_2) C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$96 + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -16$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = 0$$

For $n = 5$, Eq (2B) gives

$$(-2a_3 + 9a_4) C - b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 9$ and all b_n, then the second solution becomes

$$y_2(x) = 9\left(\frac{1 + 2x + \frac{x^2}{3} + O(x^6)}{x}\right)\ln(x) + \frac{1 - 45x^2 - 16x^3 + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 + 2x + \frac{x^2}{3} + O(x^6)\right)}{x} \\ &\quad + c_2\left(9\left(\frac{1 + 2x + \frac{x^2}{3} + O(x^6)}{x}\right)\ln(x) + \frac{1 - 45x^2 - 16x^3 + O(x^6)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1 + 2x + \frac{x^2}{3} + O(x^6)\right)}{x} \\ &\quad + c_2\left(\frac{9\left(1 + 2x + \frac{x^2}{3} + O(x^6)\right)\ln(x)}{x} + \frac{1 - 45x^2 - 16x^3 + O(x^6)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{c_1\left(1 + 2x + \frac{x^2}{3} + O(x^6)\right)}{x} \\ &\quad + c_2\left(\frac{9\left(1 + 2x + \frac{x^2}{3} + O(x^6)\right)\ln(x)}{x} + \frac{1 - 45x^2 - 16x^3 + O(x^6)}{x^2}\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1+2x+\frac{x^2}{3}+O(x^6)\right)}{x} + c_2\left(\frac{9\left(1+2x+\frac{x^2}{3}+O(x^6)\right)\ln(x)}{x} + \frac{1-45x^2-16x^3+O(x^6)}{x^2}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 55

```
Order:=6;
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{\ln(x)\left(9x+18x^2+3x^3+\mathrm{O}(x^6)\right)c_2 + c_1\left(1+2x+\frac{1}{3}x^2+\mathrm{O}(x^6)\right)x + \left(1-5x-55x^2-\frac{53}{3}x^3+\mathrm{O}(x^6)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 56

```
AsymptoticDSolveValue[x^2*(1-x)*y''[x]+x*(4+x)*y'[x]+(2-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{3(x^2+6x+3)\log(x)}{x} - \frac{21x^3+75x^2+15x-1}{x^2} \right) + c_2 \left(\frac{x}{3} + \frac{1}{x} + 2 \right)$$

11.11 problem 22

Internal problem ID [1200]
Internal file name [`OUTPUT/1201_Sunday_June_05_2022_02_04_58_AM_40370122/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x+1)\,y''+x(1+2x)\,y'-(6x+4)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{1+2x}{x\,(x+1)}\\ q(x)&=-\frac{2(3x+2)}{x^2\,(x+1)}\end{aligned}$$

Table 450: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+2x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{2(3x+2)}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + \left(2x^2+x\right) y' + (-6x-4)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(2x^2+x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-6x-4)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=1}^{\infty} \left(-6a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r r - 4x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 2a_{n-1}(n+r-1) + a_n(n+r) - 6a_{n-1} - 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-3)\,a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{(n-1)\,a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r+2}{-1+r}$$

Which for the root $r = 2$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+2}{-1+r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r-2}{r}$$

Which for the root $r = 2$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+2}{-1+r}$	0
a_2	$\frac{r-2}{r}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r+2}{1+r}$$

Which for the root $r = 2$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+2}{-1+r}$	0
a_2	$\frac{r-2}{r}$	0
a_3	$\frac{-r+2}{1+r}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r-2}{2+r}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+2}{-1+r}$	0
a_2	$\frac{r-2}{r}$	0
a_3	$\frac{-r+2}{1+r}$	0
a_4	$\frac{r-2}{2+r}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r+2}{3+r}$$

Which for the root $r = 2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r+2}{-1+r}$	0
a_2	$\frac{r-2}{r}$	0
a_3	$\frac{-r+2}{1+r}$	0
a_4	$\frac{r-2}{2+r}$	0
a_5	$\frac{-r+2}{3+r}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{r-2}{2+r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{r-2}{2+r} &= \lim_{r \to -2} \frac{r-2}{2+r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2) \right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \\ &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2) \right) \end{aligned}$$

Substituting these back into the given ode $x^2(x+1) y'' + (2x^2 + x) y' + (-6x - 4) y = 0$

gives

$$
\begin{aligned}
&x^2(x+1)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(2x^2+x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(-6x-4)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2(x+1)\,y_1''(x)+\left(2x^2+x\right)y_1'(x)+(-6x-4)\,y_1(x)\right)\ln(x)\right.\\
&\left.+x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(2x^2+x\right)y_1(x)}{x}\right)C\\
&+x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(2x^2+x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-6x-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2(x+1)\,y_1''(x)+\left(2x^2+x\right)y_1'(x)+(-6x-4)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(2x^2+x\right)y_1(x)}{x}\right)C\\
&+x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(2x^2+x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-6x-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x\right) C \\
&+x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \\
&+\left(2x^2+x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)+(-6x-4)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{9}
$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{1+n} a_n(n+2)\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) x\right) C \\
&+x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-2)(-3+n)\right) \\
&+\left(2x^2+x\right)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-2)\right)+(-6x-4)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n(n+2)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2)\right) \\
&+\left(\sum_{n=0}^{\infty} C\, x^{n+3} a_n\right)+\left(\sum_{n=0}^{\infty} x^{n-1} b_n\left(n^2-5n+6\right)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-2} b_n\left(n^2-5n+6\right)\right)+\left(\sum_{n=0}^{\infty} 2x^{n-1} b_n(n-2)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-2)\right)+\sum_{n=0}^{\infty}\left(-6x^{n-1} b_n\right)+\sum_{n=0}^{\infty}\left(-4b_n x^{n-2}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n(n+2) = \sum_{n=5}^{\infty} 2C a_{n-5}(-3+n)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n(n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n}(n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} C\, x^{n+3} a_n = \sum_{n=5}^{\infty} C a_{n-5} x^{n-2}$$

$$\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-5n+6\right) = \sum_{n=1}^{\infty} b_{n-1}\left((n-1)^2-5n+11\right) x^{n-2}$$

$$\sum_{n=0}^{\infty} 2x^{n-1} b_n(n-2) = \sum_{n=1}^{\infty} 2b_{n-1}(-3+n)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-6x^{n-1} b_n\right) = \sum_{n=1}^{\infty} \left(-6b_{n-1} x^{n-2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned} &\left(\sum_{n=5}^{\infty} 2C a_{n-5}(-3+n)\, x^{n-2}\right) + \left(\sum_{n=4}^{\infty} 2C a_{-4+n}(n-2)\, x^{n-2}\right) \\ &+ \left(\sum_{n=5}^{\infty} C a_{n-5} x^{n-2}\right) + \left(\sum_{n=1}^{\infty} b_{n-1}\left((n-1)^2-5n+11\right) x^{n-2}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=1}^{\infty} 2b_{n-1}(-3+n)\, x^{n-2}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n(n-2)\right) + \sum_{n=1}^{\infty} \left(-6b_{n-1} x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-4b_n x^{n-2}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-4b_0 - 3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4 - 3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{4}{3}$$

For $n = 2$, Eq (2B) gives

$$-6b_1 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = 2$$

For $n = 3$, Eq (2B) gives

$$-3b_3 - 6b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 - 12 = 0$$

Solving the above for b_3 gives

$$b_3 = -4$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 16 = 0$$

Which is solved for C. Solving for C gives

$$C = -4$$

For $n = 5$, Eq (2B) gives

$$(5a_0 + 6a_1) C + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20 + 5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 4$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -4$ and all b_n, then the second solution becomes

$$y_2(x) = (-4)\left(x^2\left(1 + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 - \frac{4x}{3} + 2x^2 - 4x^3 + 4x^5 + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 + O\left(x^6\right)\right) \\
&\quad + c_2\left((-4)\left(x^2\left(1 + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 - \frac{4x}{3} + 2x^2 - 4x^3 + 4x^5 + O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(1+O\left(x^6\right)\right) + c_2\left(-4x^2\left(1+O\left(x^6\right)\right)\ln(x) + \frac{1 - \frac{4x}{3} + 2x^2 - 4x^3 + 4x^5 + O(x^6)}{x^2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^2\left(1 + O\left(x^6\right)\right) \\
&\quad + c_2\left(-4x^2\left(1 + O\left(x^6\right)\right)\ln(x) + \frac{1 - \frac{4x}{3} + 2x^2 - 4x^3 + 4x^5 + O(x^6)}{x^2}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^2\left(1+O\left(x^6\right)\right) + c_2\left(-4x^2\left(1+O\left(x^6\right)\right)\ln(x) + \frac{1 - \frac{4x}{3} + 2x^2 - 4x^3 + 4x^5 + O(x^6)}{x^2}\right)$$

Verified OK.

11.11.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y'' + (2x^2 + x)\,y' + (-6x - 4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2(3x+2)y}{x^2(x+1)} - \frac{(1+2x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+2x)y'}{x(x+1)} - \frac{2(3x+2)y}{x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1+2x}{x(x+1)}, P_3(x) = -\frac{2(3x+2)}{x^2(x+1)}\right]$

- ○ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$

- ○ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- ○ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\,y'' + x(1+2x)\,y' + (-6x-4)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (2u^2 - 3u + 1)\left(\frac{d}{du}y(u)\right) + (-6u+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(2r^2+r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k^2+4kr+2r^2+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(2r^2+r-2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2-k-6)\, a_{k-1} + (-2k^2-k+2)\, a_k + a_{k+1}(k+1)^2 = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2-k-7\right) a_k + \left(-2(k+1)^2-k+1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1+2*x)*diff(y(x),x)-(4+6*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = c_1 x^2 (1 + \mathrm{O}(x^6)) + \frac{c_2(\ln(x)(576x^4 + \mathrm{O}(x^6)) + (-144 + 192x - 288x^2 + 576x^3 - 576x^4 - 576x^5 + \mathrm{O}(x^6)))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 48

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]+x*(1+2*x)*y'[x]-(4+6*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^2 + c_1 \left(\frac{3x^4 - 12x^3 + 6x^2 - 4x + 3}{3x^2} - 4x^2 \log(x)\right)$$

11.12 problem 23

Internal problem ID [1201]
Internal file name [`OUTPUT/1202_Sunday_June_05_2022_02_05_02_AM_53203217/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x+1)\,y''-x\left(-x^2-6x+1\right)y'+\left(x^2+6x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(x^3+6x^2-x\right)y'+\left(x^2+6x+1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+6x-1}{x\,(x+1)}$$
$$q(x)=\frac{x^2+6x+1}{x^2\,(x+1)}$$

Table 452: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+6x-1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{x^2+6x+1}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + \left(x^3 + 6x^2 - x\right) y' + \left(x^2 + 6x + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(x^3 + 6x^2 - x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 + 6x + 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6 x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 6 x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 6 x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 6 a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} 6 x^{1+n+r} a_n = \sum_{n=1}^{\infty} 6 a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)\,x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r^2 - 5r - 6}{r^2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) \\ &+ 6a_{n-1}(n+r-1) - a_n(n+r) + a_{n-2} + 6a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-1} + 2nr a_{n-1} + r^2 a_{n-1} + n a_{n-2} + 3n a_{n-1} + r a_{n-2} + 3r a_{n-1} - a_{n-2} + 2a_{n-1}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n^2 a_{n-1} + (-a_{n-2} - 5a_{n-1})\, n - 6a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-5r-6}{r^2}$	-12

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 + 11r^3 + 52r^2 + 102r + 72}{r^2 (1 + r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{119}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-5r-6}{r^2}$	-12
a_2	$\frac{r^4+11r^3+52r^2+102r+72}{r^2(1+r)^2}$	$\frac{119}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^6 - 19r^5 - 162r^4 - 759r^3 - 1979r^2 - 2648r - 1428}{r^2 (1 + r)^2 (r + 2)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{583}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-5r-6}{r^2}$	-12
a_2	$\frac{r^4+11r^3+52r^2+102r+72}{r^2(1+r)^2}$	$\frac{119}{2}$
a_3	$\frac{-r^6-19r^5-162r^4-759r^3-1979r^2-2648r-1428}{r^2(1+r)^2(r+2)^2}$	$-\frac{583}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 29r^7 + 383r^6 + 2966r^5 + 14534r^4 + 45437r^3 + 87166r^2 + 92772r + 41976}{r^2 (1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1981}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-5r-6}{r^2}$	-12
a_2	$\frac{r^4+11r^3+52r^2+102r+72}{r^2(1+r)^2}$	$\frac{119}{2}$
a_3	$\frac{-r^6-19r^5-162r^4-759r^3-1979r^2-2648r-1428}{r^2(1+r)^2(r+2)^2}$	$-\frac{583}{3}$
a_4	$\frac{r^8+29r^7+383r^6+2966r^5+14534r^4+45437r^3+87166r^2+92772r+41976}{r^2(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1981}{4}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^{10} - 41r^9 - 773r^8 - 8778r^7 - 66136r^6 - 343352r^5 - 1234958r^4 - 3013745r^3 - 4736076r^2 - 429}{r^2 (1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{80287}{75}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-5r-6}{r^2}$	-12
a_2	$\frac{r^4+11r^3+52r^2+102r+72}{r^2(1+r)^2}$	$\frac{119}{2}$
a_3	$\frac{-r^6-19r^5-162r^4-759r^3-1979r^2-2648r-1428}{r^2(1+r)^2(r+2)^2}$	$-\frac{583}{3}$
a_4	$\frac{r^8+29r^7+383r^6+2966r^5+14534r^4+45437r^3+87166r^2+92772r+41976}{r^2(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1981}{4}$
a_5	$\frac{-r^{10}-41r^9-773r^8-8778r^7-66136r^6-343352r^5-1234958r^4-3013745r^3-4736076r^2-4299660r-1711584}{r^2(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{80287}{75}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a$
b_0	1	1	N/A since
b_1	$\frac{-r^2-5r-6}{r^2}$	-12	$\frac{5r+12}{r^3}$
b_2	$\frac{r^4+11r^3+52r^2+102r+72}{r^2(1+r)^2}$	$\frac{119}{2}$	$\frac{-9r^4-93r^3-}{r^3(}$
b_3	$\frac{-r^6-19r^5-162r^4-759r^3-1979r^2-2648r-1428}{r^2(1+r)^2(r+2)^2}$	$-\frac{583}{3}$	$13r^7+259r^6+$
b_4	$\frac{r^8+29r^7+383r^6+2966r^5+14534r^4+45437r^3+87166r^2+92772r+41976}{r^2(1+r)^2(r+2)^2(r+3)^2}$	$\frac{1981}{4}$	$-17r^{10}-548r$
b_5	$\frac{-r^{10}-41r^9-773r^8-8778r^7-66136r^6-343352r^5-1234958r^4-3013745r^3-4736076r^2-4299660r-1711584}{r^2(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{80287}{75}$	$21r^{13}+996r^{12}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \ldots \\
&= x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)\ln(x) \\
&\quad + x\left(17x - \frac{471x^2}{4} + 445x^3 - \frac{118285x^4}{96} + \frac{702451x^5}{250} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(17x - \frac{471x^2}{4} + 445x^3 - \frac{118285x^4}{96} + \frac{702451x^5}{250} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(17x - \frac{471x^2}{4} + 445x^3 - \frac{118285x^4}{96} + \frac{702451x^5}{250} + O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(17x - \frac{471x^2}{4} + 445x^3 - \frac{118285x^4}{96} + \frac{702451x^5}{250} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right)$$
$$+ c_2 \left(x \left(1 - 12x + \frac{119x^2}{2} - \frac{583x^3}{3} + \frac{1981x^4}{4} - \frac{80287x^5}{75} + O(x^6)\right) \ln(x) + x \left(17x - \frac{471x^2}{4} + 445x^3 - \frac{118285x^4}{96} + \frac{702451x^5}{250} + O(x^6)\right)\right)$$

Verified OK.

11.12.1 Maple step by step solution

Let's solve

$x^2(x+1)\, y'' + (x^3 + 6x^2 - x)\, y' + (x^2 + 6x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+6x+1)y}{x^2(x+1)} - \frac{(x^2+6x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+6x-1)y'}{x(x+1)} + \frac{(x^2+6x+1)y}{x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2+6x-1}{x(x+1)}, P_3(x) = \frac{x^2+6x+1}{x^2(x+1)}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 6$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

$x^2(x+1)\,y''+x(x^2+6x-1)\,y'+(x^2+6x+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(u^3+3u^2-10u+6)\left(\frac{d}{du}y(u)\right)+(u^2+4u-4)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m}u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0r(5+r)\,u^{-1+r}+\left(a_1(1+r)\,(6+r)-2a_0(r^2+4r+2)\right)u^r+\left(a_2(2+r)\,(7+r)-2a_1(r^2+6r+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(5+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-5,0\}$

- The coefficients of each power of u must be 0

$[a_1(1+r)(6+r) - 2a_0(r^2+4r+2) = 0, a_2(2+r)(7+r) - 2a_1(r^2+6r+7) + a_0(r^2+2r+4)$

- Solve for the dependent coefficient(s)

$\left\{a_1 = \frac{2a_0(r^2+4r+2)}{r^2+7r+6}, a_2 = \frac{a_0(3r^4+31r^3+108r^2+120r+32)}{r^4+16r^3+83r^2+152r+84}\right\}$

- Each term in the series must be 0, giving the recursion relation

$(-2a_k + a_{k-1} + a_{k+1})k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})r - 8a_k + a_{k-2} + 7a_{k+1})k + (-2a_k + a_{k-1}$

- Shift index using $k->k+2$

$(-2a_{k+2} + a_{k+1} + a_{k+3})(k+2)^2 + ((-4a_{k+2} + 2a_{k+1} + 2a_{k+3})r - 8a_{k+2} + a_k + 7a_{k+3})(k+2) +$

- Recursion relation that defines series solution to ODE

$a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + 2kra_{k+1} - 4kra_{k+2} + r^2a_{k+1} - 2r^2a_{k+2} + ka_k + 4ka_{k+1} - 16ka_{k+2} + ra_k + 4ra_{k+1} - 16ra_{k+2} + a_k + 7a_{k+}}{k^2+2kr+r^2+11k+11r+24}$

- Recursion relation for $r = -5$

$a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + ka_k - 6ka_{k+1} + 4ka_{k+2} - 4a_k + 12a_{k+1} + 2a_{k+2}}{k^2+k-6}$

- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 2$

$a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + ka_k - 6ka_{k+1} + 4ka_{k+2} - 4a_k + 12a_{k+1} + 2a_{k+2}}{k^2+k-6}$

- Recursion relation for $r = 0$

$a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + ka_k + 4ka_{k+1} - 16ka_{k+2} + a_k + 7a_{k+1} - 28a_{k+2}}{k^2+11k+24}$

- Solution for $r = 0$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + ka_k + 4ka_{k+1} - 16ka_{k+2} + a_k + 7a_{k+1} - 28a_{k+2}}{k^2+11k+24}, a_1 = \frac{2a_0}{3}, a_2 = \frac{8a_0}{21}\right]$

- Revert the change of variables $u = x+1$

$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = -\frac{k^2a_{k+1} - 2k^2a_{k+2} + ka_k + 4ka_{k+1} - 16ka_{k+2} + a_k + 7a_{k+1} - 28a_{k+2}}{k^2+11k+24}, a_1 = \frac{2a_0}{3}, a_2 = \frac{8a_0}{21}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(1-6*x-x^2)*diff(y(x),x)+(1+6*x+x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \left((c_2 \ln(x) + c_1) \left(1 - 12x + \frac{119}{2}x^2 - \frac{583}{3}x^3 + \frac{1981}{4}x^4 - \frac{80287}{75}x^5 + \mathrm{O}(x^6) \right) + \left(17x - \frac{471}{4}x^2 + 445x^3 - \frac{118285}{96}x^4 + \frac{702451}{250}x^5 + \mathrm{O}(x^6) \right) c_2 \right) x$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 114

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-x*(1-6*x-x^2)*y'[x]+(1+6*x+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(-\frac{80287x^5}{75} + \frac{1981x^4}{4} - \frac{583x^3}{3} + \frac{119x^2}{2} - 12x + 1\right) + c_2 \left(x \left(\frac{702451x^5}{250} - \frac{118285x^4}{96} + 445x^3 - \frac{471x^2}{4} + 17x\right) + x \left(-\frac{80287x^5}{75} + \frac{1981x^4}{4} - \frac{583x^3}{3} + \frac{119x^2}{2} - 12x + 1\right) \log(x)\right)$$

11.13 problem 24

Internal problem ID [1202]
Internal file name [`OUTPUT/1203_Sunday_June_05_2022_02_05_04_AM_2392867/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2(3x+1)\,y''+x\left(x^2+12x+2\right)y'+2x(x+3)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$\left(3x^3+x^2\right)y''+\left(x^3+12x^2+2x\right)y'+\left(2x^2+6x\right)y=0$$

Or

$$x\left(y'x^2+3x^2y''+2yx+12y'x+xy''+6y+2y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(3x^2+x\right)y''+y'\left(x^2+12x+2\right)+2y(x+3)=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^3+x^2\right)y''+\left(x^3+12x^2+2x\right)y'+\left(2x^2+6x\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$
\begin{aligned}
p(x) &= \frac{x^2+12x+2}{x(3x+1)} \\
q(x) &= \frac{2x+6}{x(3x+1)}
\end{aligned}
$$

Table 454: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+12x+2}{x(3x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{3}$	"regular"

$q(x) = \frac{2x+6}{x(3x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{3}\right]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
x^2(3x+1)\, y'' + \left(x^3+12x^2+2x\right) y' + \left(2x^2+6x\right) y = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$x^2(3x+1)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}\right) + (x^3+12x^2+2x)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right) + (2x^2+6x)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}12x^{1+n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}2x^{n+r+2}a_n\right) + \left(\sum_{n=0}^{\infty}6x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty}3a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r) = \sum_{n=2}^{\infty}a_{n-2}(n+r-2)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}12x^{1+n+r}a_n(n+r) = \sum_{n=1}^{\infty}12a_{n-1}(n+r-1)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}2x^{n+r+2}a_n = \sum_{n=2}^{\infty}2a_{n-2}x^{n+r}$$

$$\sum_{n=0}^{\infty}6x^{1+n+r}a_n = \sum_{n=1}^{\infty}6a_{n-1}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\,x^{n+r}\right) \tag{2B}$$
$$+\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\right)+\left(\sum_{n=2}^{\infty} 2a_{n-2}x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 6a_{n-1}x^{n+r}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1)+2x^{n+r}a_n(n+r)=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+2x^r a_0 r=0$$

Or

$$(x^r r(-1+r)+2x^r r)\,a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=-1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(1+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x)=Cy_1(x)\ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -3$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &3a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) \\ &+ 12a_{n-1}(n+r-1) + 2a_n(n+r) + 2a_{n-2} + 6a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3na_{n-1} + 3ra_{n-1} + a_{n-2} + 3a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-3n-3)a_{n-1} - a_{n-2}}{1+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r + 26}{3 + r}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{26}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	$\frac{9r+26}{3+r}$	$\frac{26}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^2 - 183r - 303}{(3 + r)(4 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{101}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	$\frac{9r+26}{3+r}$	$\frac{26}{3}$
a_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{101}{4}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^3 + 945r^2 + 3592r + 4441}{(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{4441}{60}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	$\frac{9r+26}{3+r}$	$\frac{26}{3}$
a_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{101}{4}$
a_4	$\frac{81r^3+945r^2+3592r+4441}{(3+r)(4+r)(5+r)}$	$\frac{4441}{60}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-243r^4 - 4266r^3 - 27468r^2 - 76761r - 78423}{(3+r)(4+r)(5+r)(6+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{26141}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	$\frac{9r+26}{3+r}$	$\frac{26}{3}$
a_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{101}{4}$
a_4	$\frac{81r^3+945r^2+3592r+4441}{(3+r)(4+r)(5+r)}$	$\frac{4441}{60}$
a_5	$\frac{-243r^4-4266r^3-27468r^2-76761r-78423}{(3+r)(4+r)(5+r)(6+r)}$	$-\frac{26141}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - 3x + \frac{26x^2}{3} - \frac{101x^3}{4} + \frac{4441x^4}{60} - \frac{26141x^5}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -3 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -3 &= \lim_{r \to -1} -3 \\ &= -3 \end{aligned}$$

The limit is -3. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = -3$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &3b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) \\ &+ 12b_{n-1}(n+r-1) + 2b_n(n+r) + 2b_{n-2} + 6b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -1$ becomes

$$\begin{aligned} &3b_{n-1}(n-2)(n-3) + b_n(n-1)(n-2) + b_{n-2}(n-3) \\ &+ 12b_{n-1}(n-2) + 2b_n(n-1) + 2b_{n-2} + 6b_{n-1} = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{3nb_{n-1} + 3rb_{n-1} + b_{n-2} + 3b_{n-1}}{1 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{3nb_{n-1} + b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r + 26}{3 + r}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{17}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	$\frac{9r+26}{3+r}$	$\frac{17}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{3(9r^2 + 61r + 101)}{(3 + r)(4 + r)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{49}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	$\frac{9r+26}{3+r}$	$\frac{17}{2}$
b_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{49}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^3 + 945r^2 + 3592r + 4441}{(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{571}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	$\frac{9r+26}{3+r}$	$\frac{17}{2}$
b_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{49}{2}$
b_4	$\frac{81r^3+945r^2+3592r+4441}{(3+r)(4+r)(5+r)}$	$\frac{571}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{3(81r^4 + 1422r^3 + 9156r^2 + 25587r + 26141)}{(3 + r)(4 + r)(6 + r)(5 + r)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{8369}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	$\frac{9r+26}{3+r}$	$\frac{17}{2}$
b_3	$\frac{-27r^2-183r-303}{(3+r)(4+r)}$	$-\frac{49}{2}$
b_4	$\frac{81r^3+945r^2+3592r+4441}{(3+r)(4+r)(5+r)}$	$\frac{571}{8}$
b_5	$\frac{-243r^4-4266r^3-27468r^2-76761r-78423}{(3+r)(4+r)(5+r)(6+r)}$	$-\frac{8369}{40}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 3x + \frac{17x^2}{2} - \frac{49x^3}{2} + \frac{571x^4}{8} - \frac{8369x^5}{40} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - 3x + \frac{26x^2}{3} - \frac{101x^3}{4} + \frac{4441x^4}{60} - \frac{26141x^5}{120} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 - 3x + \frac{17x^2}{2} - \frac{49x^3}{2} + \frac{571x^4}{8} - \frac{8369x^5}{40} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - 3x + \frac{26x^2}{3} - \frac{101x^3}{4} + \frac{4441x^4}{60} - \frac{26141x^5}{120} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 - 3x + \frac{17x^2}{2} - \frac{49x^3}{2} + \frac{571x^4}{8} - \frac{8369x^5}{40} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - 3x + \frac{26x^2}{3} - \frac{101x^3}{4} + \frac{4441x^4}{60} - \frac{26141x^5}{120} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 - 3x + \frac{17x^2}{2} - \frac{49x^3}{2} + \frac{571x^4}{8} - \frac{8369x^5}{40} + O(x^6)\right)}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1 - 3x + \frac{26x^2}{3} - \frac{101x^3}{4} + \frac{4441x^4}{60} - \frac{26141x^5}{120} + O(x^6)\right) + \frac{c_2\left(1 - 3x + \frac{17x^2}{2} - \frac{49x^3}{2} + \frac{571x^4}{8} - \frac{8369x^5}{40} + O(x^6)\right)}{x}$$

Verified OK.

11.13.1 Maple step by step solution

Let's solve

$x^2(3x+1)\, y'' + (x^3 + 12x^2 + 2x)\, y' + (2x^2 + 6x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x+3)y}{x(3x+1)} - \frac{(x^2+12x+2)y'}{x(3x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+12x+2)y'}{x(3x+1)} + \frac{2(x+3)y}{x(3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2+12x+2}{x(3x+1)}, P_3(x) = \frac{2(x+3)}{x(3x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(3x+1)\, y'' + y'(x^2 + 12x + 2) + (2x + 6)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  ◦ Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) + 3a_0(2+r)\,(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+r+2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(2+r) + 3a_0(2+r)\,(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(\frac{a_{k+1}(k+r+2)}{3}+a_k(k+r+2)+\frac{a_{k-1}}{3}\right)(k+r+1)=0$$

- Shift index using $k->k+1$

$$3\left(\frac{a_{k+2}(k+3+r)}{3}+a_{k+1}(k+3+r)+\frac{a_k}{3}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{3ka_{k+1}+3ra_{k+1}+a_k+9a_{k+1}}{k+3+r}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{3ka_{k+1}+a_k+6a_{k+1}}{k+2}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-1}, a_{k+2}=-\frac{3ka_{k+1}+a_k+6a_{k+1}}{k+2}, 0=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{3ka_{k+1}+a_k+9a_{k+1}}{k+3}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k}, a_{k+2}=-\frac{3ka_{k+1}+a_k+9a_{k+1}}{k+3}, 2a_1+6a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k}\right), a_{k+2}=-\frac{3ka_{1+k}+a_k+6a_{1+k}}{k+2}, 0=0, b_{k+2}=-\frac{3kb_{1+k}+b_k+9b_{1+k}}{k+3}, 2b_1 \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(x^2*(1+3*x)*diff(y(x),x$2)+x*(2+12*x+x^2)*diff(y(x),x)+2*x*(3+x)*y(x)=0,y(x),type='se
```

$$y(x) = c_1\left(1-3x+\frac{26}{3}x^2-\frac{101}{4}x^3+\frac{4441}{60}x^4-\frac{26141}{120}x^5+\mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1-6x+\frac{35}{2}x^2-\frac{101}{2}x^3+\frac{1177}{8}x^4-\frac{17251}{40}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x^2*(1+3*x)*y''[x]+x*(2+12*x+x^2)*y'[x]+2*x*(3+x)*y[x]==0,y[x],{x,0,5}
```

$$y(x) \to c_1\left(\frac{571x^3}{8}-\frac{49x^2}{2}+\frac{17x}{2}+\frac{1}{x}-3\right)+c_2\left(\frac{4441x^4}{60}-\frac{101x^3}{4}+\frac{26x^2}{3}-3x+1\right)$$

11.14 problem 25

Internal problem ID [1203]
Internal file name [`OUTPUT/1204_Sunday_June_05_2022_02_05_07_AM_19121030/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2x^2+1)\,y''+x(2x^2+4)\,y'+2(-x^2+1)\,y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+x^2)\,y''+(2x^3+4x)\,y'+(-2x^2+2)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2+4}{x\,(2x^2+1)}$$
$$q(x)=-\frac{2(x^2-1)}{x^2\,(2x^2+1)}$$

Table 456: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+4}{x(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(2x^2+1)\,y'' + (2x^3+4x)\,y' + (-2x^2+2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(2x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3+4x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-2x^2+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-2x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)\,(n+r-1) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r} \\
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-2x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)\,(n+r-1)+4x^{n+r} a_n(n+r)+2a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 4x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 4x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ &+ 2a_{n-2}(n+r-2) + 4a_n(n+r) - 2a_{n-2} + 2a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}(n^2 + 2nr + r^2 - 4n - 4r + 3)}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{2a_{n-2}(n^2 - 6n + 8)}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2r^2 + 2}{r^2 + 7r + 12}$$

Which for the root $r = -1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r^2+2}{r^2+7r+12}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r^2+2}{r^2+7r+12}$	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4(-1+r)\,(1+r)^2}{(r+6)\,(r+5)\,(r+4)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r^2+2}{r^2+7r+12}$	0
a_3	0	0
a_4	$\frac{4(-1+r)(1+r)^2}{(r+6)(r+5)(r+4)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2r^2+2}{r^2+7r+12}$	0
a_3	0	0
a_4	$\frac{4(-1+r)(1+r)^2}{(r+6)(r+5)(r+4)}$	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -2} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-2}(n+r-2)(n-3+r)+b_n(n+r)(n+r-1) \\ &+2b_{n-2}(n+r-2)+4b_n(n+r)-2b_{n-2}+2b_n=0 \end{aligned} \tag{4}$$

Which for for the root $r = -2$ becomes

$$2b_{n-2}(n-4)(n-5)+b_n(n-2)(n-3)+2b_{n-2}(n-4)+4b_n(n-2)-2b_{n-2}+2b_n=0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2b_{n-2}(n^2+2nr+r^2-4n-4r+3)}{n^2+2nr+r^2+3n+3r+2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{2b_{n-2}(n^2-8n+15)}{n^2-n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{2(r^2-1)}{r^2+7r+12}$$

Which for the root $r = -2$ becomes

$$b_2 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r^2+2}{r^2+7r+12}$	-3

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r^2+2}{r^2+7r+12}$	-3
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^3 + 4r^2 - 4r - 4}{(r^2 + 11r + 30)(r + 4)}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r^2+2}{r^2+7r+12}$	-3
b_3	0	0
b_4	$\frac{4(-1+r)(1+r)^2}{(r+6)(r+5)(r+4)}$	$-\frac{1}{2}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-2r^2+2}{r^2+7r+12}$	-3
b_3	0	0
b_4	$\frac{4(-1+r)(1+r)^2}{(r+6)(r+5)(r+4)}$	$-\frac{1}{2}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - 3x^2 - \frac{x^4}{2} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1(1 + O(x^6))}{x} + \frac{c_2\left(1 - 3x^2 - \frac{x^4}{2} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1(1 + O(x^6))}{x} + \frac{c_2\left(1 - 3x^2 - \frac{x^4}{2} + O(x^6)\right)}{x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(1 + O(x^6))}{x} + \frac{c_2\left(1 - 3x^2 - \frac{x^4}{2} + O(x^6)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(1 + O(x^6))}{x} + \frac{c_2\left(1 - 3x^2 - \frac{x^4}{2} + O(x^6)\right)}{x^2}$$

Verified OK.

11.14.1 Maple step by step solution

Let's solve

$x^2(2x^2+1)\,y''+(2x^3+4x)\,y'+(-2x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2(x^2-1)y}{x^2(2x^2+1)}-\frac{2(x^2+2)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x^2+2)y'}{x(2x^2+1)}-\frac{2(x^2-1)y}{x^2(2x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(x^2+2)}{x(2x^2+1)}, P_3(x)=-\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=4$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=2$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x^2+1)\,y''+2(x^2+2)\,xy'+(-2x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2+r)\,(1+r)\, x^{r}+a_{1}(3+r)\,(2+r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(k+r+2)\,(k+r+1)+2a_{k-2}(k+r-1$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,-1\}$

- Each term must be 0

  $a_{1}(3+r)\,(2+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+2)\,(k+r+1)+2a_{k-2}(k+r-1)\,(k-3+r)=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+3+r)+2a_{k}(k+r+1)\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_{k}(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -1$; series terminates at $k = 2$

$$a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(1+k)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
Order:=6;
dsolve(x^2*(1+2*x^2)*diff(y(x),x$2)+x*(4+2*x^2)*diff(y(x),x)+2*(1-x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{c_1(1 + \mathrm{O}(x^6))\,x + c_2\left(1 - 3x^2 - \frac{1}{2}x^4 + \mathrm{O}(x^6)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 25

```
AsymptoticDSolveValue[x^2*(1+2*x^2)*y''[x]+x*(4+2*x^2)*y'[x]+2*(1-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^2}{2} + \frac{1}{x^2} - 3\right) + \frac{c_2}{x}$$

11.15 problem 26

Internal problem ID [1204]
Internal file name [`OUTPUT/1205_Sunday_June_05_2022_02_05_09_AM_47489370/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.1 Exercises. Page 318
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+2)\,y''+2x(x^2+5)\,y'+2(-x^2+3)\,y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+2x^2)\,y''+(2x^3+10x)\,y'+(-2x^2+6)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2+10}{(x^2+2)\,x}$$
$$q(x)=-\frac{2(x^2-3)}{x^2\,(x^2+2)}$$

Table 458: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+10}{(x^2+2)x}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = -\frac{2(x^2-3)}{x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i\sqrt{2}, i\sqrt{2}, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+2)\,y'' + (2x^3+10x)\,y' + (-2x^2+6)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}\right) + (2x^3+10x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-2x^2+6)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 10x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 10x^r a_0 r + 6a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 10x^r r + 6x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 + 8r + 6) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 8r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 + 8r + 6) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + 2a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) + 10a_n(n+r) - 2a_{n-2} + 6a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2+2nr+r^2-3n-3r)}{2(n^2+2nr+r^2+4n+4r+3)} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-2}(n^2-5n+4)}{2n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^2-r+2}{2r^2+16r+30}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-r+2}{2r^2+16r+30}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-r+2}{2r^2+16r+30}$	$\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 6r^3 + 7r^2 - 6r - 8}{4\left(r + 5\right)^2 \left(r + 3\right) \left(r + 7\right)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-r+2}{2r^2+16r+30}$	$\frac{1}{8}$
a_3	0	0
a_4	$\frac{r^4+6r^3+7r^2-6r-8}{4(r+5)^2(r+3)(r+7)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-r+2}{2r^2+16r+30}$	$\frac{1}{8}$
a_3	0	0
a_4	$\frac{r^4+6r^3+7r^2-6r-8}{4(r+5)^2(r+3)(r+7)}$	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{8} + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{-r^2 - r + 2}{2r^2 + 16r + 30} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-r^2 - r + 2}{2r^2 + 16r + 30} &= \lim_{r \to -3} \frac{-r^2 - r + 2}{2r^2 + 16r + 30} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2(x^2+2)\,y''+(2x^3+10x)\,y'+(-2x^2+6)\,y = 0$ gives

$$x^2(x^2+2)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (2x^3+10x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + (-2x^2+6)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2(x^2+2)\,y_1''(x) + (2x^3+10x)\,y_1'(x) + (-2x^2+6)\,y_1(x)\right)\ln(x) + x^2(x^2+2)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(2x^3+10x)\,y_1(x)}{x}\right)C + x^2(x^2+2)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (2x^3+10x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-2x^2+6)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2(x^2+2)\,y_1''(x)+(2x^3+10x)\,y_1'(x)+(-2x^2+6)\,y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^2(x^2+2)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3+10x)\,y_1(x)}{x}\right)C\\&+x^2(x^2+2)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+(2x^3+10x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-2x^2+6)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2x(x^2+2)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(x^2+8)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C\\&+(x^4+2x^2)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)\\&+2(x^3+5x)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)+2(-x^2+3)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{9}$$

Since $r_1=-1$ and $r_2=-3$ then the above becomes

$$\begin{aligned}&\left(2x(x^2+2)\left(\sum_{n=0}^{\infty}x^{n-2}a_n(n-1)\right)+(x^2+8)\left(\sum_{n=0}^{\infty}a_n x^{n-1}\right)\right)C\\&+(x^4+2x^2)\left(\sum_{n=0}^{\infty}x^{-5+n}b_n(n-3)(-4+n)\right)\\&+2(x^3+5x)\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-3)\right)+2(-x^2+3)\left(\sum_{n=0}^{\infty}b_n x^{n-3}\right)=0\end{aligned}\tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=0}^{\infty} C\, x^{1+n} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} 8C\, x^{n-1} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (-4+n)(n-3)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{n-3} b_n (-4+n)(n-3)\right) + \left(\sum_{n=0}^{\infty} 2x^{n-1} b_n (n-3)\right) \\
&+ \left(\sum_{n=0}^{\infty} 10x^{n-3} b_n (n-3)\right) + \sum_{n=0}^{\infty} \left(-2x^{n-1} b_n\right) + \left(\sum_{n=0}^{\infty} 6b_n x^{n-3}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-3$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-3} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (n-1) = \sum_{n=4}^{\infty} 2C a_{-4+n} (-5+n)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n (n-1) = \sum_{n=2}^{\infty} 4C a_{n-2} (n-3)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} C\, x^{1+n} a_n = \sum_{n=4}^{\infty} C a_{-4+n} x^{n-3}$$

$$\sum_{n=0}^{\infty} 8C\, x^{n-1} a_n = \sum_{n=2}^{\infty} 8C a_{n-2} x^{n-3}$$

$$\sum_{n=0}^{\infty} x^{n-1} b_n (-4+n)(n-3) = \sum_{n=2}^{\infty} b_{n-2} (-6+n)(-5+n)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} 2x^{n-1} b_n (n-3) = \sum_{n=2}^{\infty} 2b_{n-2} (-5+n)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} \left(-2x^{n-1} b_n\right) = \sum_{n=2}^{\infty} \left(-2b_{n-2} x^{n-3}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-3$.

$$
\begin{aligned}
&\left(\sum_{n=4}^{\infty} 2C a_{-4+n} (-5+n)\, x^{n-3}\right) + \left(\sum_{n=2}^{\infty} 4C a_{n-2} (n-3)\, x^{n-3}\right) \\
&+ \left(\sum_{n=4}^{\infty} C a_{-4+n} x^{n-3}\right) + \left(\sum_{n=2}^{\infty} 8C a_{n-2} x^{n-3}\right) \\
&+ \left(\sum_{n=2}^{\infty} b_{n-2}(-6+n)\,(-5+n)\, x^{n-3}\right) + \left(\sum_{n=0}^{\infty} 2x^{n-3} b_n (-4+n)\,(n-3)\right) \\
&+ \left(\sum_{n=2}^{\infty} 2b_{n-2}(-5+n)\, x^{n-3}\right) + \left(\sum_{n=0}^{\infty} 10x^{n-3} b_n (n-3)\right) \\
&+ \sum_{n=2}^{\infty} \left(-2b_{n-2} x^{n-3}\right) + \left(\sum_{n=0}^{\infty} 6b_n x^{n-3}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$4C + 4 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 3$, Eq (2B) gives

$$8Ca_1 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(-a_0 + 12a_2)\, C - 2b_2 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{2} + 16b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{32}$$

For $n = 5$, Eq (2B) gives

$$(a_1 + 16a_3)\, C - 2b_3 + 30b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1) \left(\frac{1 + \frac{x^2}{8} + O(x^6)}{x}\right) \ln(x) + \frac{1 + \frac{x^4}{32} + O(x^6)}{x^3}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1 \left(1 + \frac{x^2}{8} + O(x^6)\right)}{x} + c_2 \left((-1) \left(\frac{1 + \frac{x^2}{8} + O(x^6)}{x}\right) \ln(x) + \frac{1 + \frac{x^4}{32} + O(x^6)}{x^3}\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1+\frac{x^2}{8}+O(x^6)\right)}{x}+c_2\left(-\frac{\left(1+\frac{x^2}{8}+O(x^6)\right)\ln(x)}{x}+\frac{1+\frac{x^4}{32}+O(x^6)}{x^3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(1+\frac{x^2}{8}+O(x^6)\right)}{x}+c_2\left(-\frac{\left(1+\frac{x^2}{8}+O(x^6)\right)\ln(x)}{x}+\frac{1+\frac{x^4}{32}+O(x^6)}{x^3}\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\left(1+\frac{x^2}{8}+O(x^6)\right)}{x}+c_2\left(-\frac{\left(1+\frac{x^2}{8}+O(x^6)\right)\ln(x)}{x}+\frac{1+\frac{x^4}{32}+O(x^6)}{x^3}\right)
$$

Verified OK.

11.15.1 Maple step by step solution

Let's solve

$x^2(x^2+2)\,y''+(2x^3+10x)\,y'+(-2x^2+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2(x^2-3)y}{x^2(x^2+2)}-\frac{2(x^2+5)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x^2+5)y'}{x(x^2+2)}-\frac{2(x^2-3)y}{x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(x^2+5)}{(x^2+2)x}, P_3(x)=-\frac{2(x^2-3)}{x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 5$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+2)\, y'' + 2x(x^2+5)\, y' + (-2x^2+6)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$2a_0(3+r)(1+r)x^r + 2a_1(4+r)(2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2(3+r)(1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, -1\}$$

- Each term must be 0

  $$2a_1(4+r)(2+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$$

- Shift index using $k- >k+2$

  $$2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$$

- Recursion relation for $r = -3$; series terminates at $k = 4$

  $$a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$$

- Solution for $r = -3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r = -1$; series terminates at $k = 2$

  $$a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(-4+k)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(1+k)(k-2)}{2(4+k)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*(2+x^2)*diff(y(x),x$2)+2*x*(x^2+5)*diff(y(x),x)+2*(3-x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{c_1\left(1+\frac{1}{8}x^2+\mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2\left(\ln\left(x\right)\left(2x^2+\frac{1}{4}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-2-\frac{3}{2}x^2-\frac{1}{4}x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 51

```
AsymptoticDSolveValue[x^2*(2+x^2)*y''[x]+2*x*(x^2+5)*y'[x]+2*(3-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4+7x^2+4}{4x^3}-\frac{(x^2+8)\log(x)}{8x}\right)+c_2\left(\frac{x}{8}+\frac{1}{x}\right)$$

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12.1 problem 1

Internal problem ID [1205]
Internal file name [`OUTPUT/1206_Sunday_June_05_2022_02_05_12_AM_61059740/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+6y'x+6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{818}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{819}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\big|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{6(y'x+y)}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{36y'x^2 + 48yx - 12y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{240(-x^3+x)\, y' + 120(-3x^2+1)\, y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{360(5x^4 - 10x^2 + 1)\, y' + 2880(x^3 - x)\, y}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-15120x^5 + 50400x^3 - 15120x)\, y' - 25200y\left(x^4 - 2x^2 + \frac{1}{5}\right)}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -6y(0) \\
F_1 &= -12y'(0) \\
F_2 &= 120y(0) \\
F_3 &= 360y'(0) \\
F_4 &= -5040y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-7x^6 + 5x^4 - 3x^2 + 1\right) y(0) + \left(3x^5 - 2x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2+1\right) y'' + 6y'x + 6y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(x^2+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 6\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \left(\sum_{n=1}^{\infty} 6n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$6a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -2a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) + 6na_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n+3)\,a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$20a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 5a_0$$

For $n = 3$ the recurrence equation gives

$$30a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 3a_1$$

For $n = 4$ the recurrence equation gives

$$42a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -7a_0$$

For $n = 5$ the recurrence equation gives

$$56a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -4a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = 3a_1 x^5 + 5a_0 x^4 - 2a_1 x^3 - 3a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (5x^4 - 3x^2 + 1)\, a_0 + (3x^5 - 2x^3 + x)\, a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (5x^4 - 3x^2 + 1)\, c_1 + (3x^5 - 2x^3 + x)\, c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = (-7x^6 + 5x^4 - 3x^2 + 1)\, y(0) + (3x^5 - 2x^3 + x)\, y'(0) + O(x^6) \tag{1}$$
$$y = (5x^4 - 3x^2 + 1)\, c_1 + (3x^5 - 2x^3 + x)\, c_2 + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = (-7x^6 + 5x^4 - 3x^2 + 1)\, y(0) + (3x^5 - 2x^3 + x)\, y'(0) + O(x^6)$$

Verified OK.

$$y = (5x^4 - 3x^2 + 1)\, c_1 + (3x^5 - 2x^3 + x)\, c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+6*x*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (5x^4 - 3x^2 + 1)\, y(0) + (3x^5 - 2x^3 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+6*x*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2(3x^5 - 2x^3 + x) + c_1(5x^4 - 3x^2 + 1)$$

12.2 problem 2

Internal problem ID [1206]
Internal file name [`OUTPUT/1207_Sunday_June_05_2022_02_05_13_AM_51423085/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2y'x-2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (821) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (822) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(-y+y'x)}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{8(-y+y'x)\, x}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{8(-y+y'x)\,(5x^2-1)}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{240\left(x^2-\frac{3}{5}\right)(-y+y'x)\, x}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{48(-y+y'x)\,(35x^4-42x^2+3)}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 0 \\
F_2 &= -8y(0) \\
F_3 &= 0 \\
F_4 &= 144y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6\right) y(0) + y'(0)\, x + O\!\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y''+2y'x-2y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)x-2\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=0}^{\infty}(-2a_n x^n)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right)\\+\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=0}^{\infty}(-2a_n x^n)=0\end{gathered} \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 2na_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(n-1)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$10a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$18a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$28a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 - \frac{1}{3} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6\right) y(0) + y'(0)\, x + O(x^6) \tag{1}$$

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6\right) y(0) + y'(0)\, x + O(x^6)$$

Verified OK.

$$y = \left(1 + x^2 - \frac{1}{3}x^4\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 - \frac{1}{3}x^4\right) y(0) + D(y)(0)\, x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 23

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^4}{3} + x^2 + 1\right) + c_2 x$$

12.3 problem 3

Internal problem ID [1207]
Internal file name [`OUTPUT/1208_Sunday_June_05_2022_02_05_14_AM_24094619/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y'' - 8y'x + 20y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (824)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (825)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= \frac{8y'x - 20y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{36y'x^2 - 120yx - 12y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{96x^3y' - 360x^2y - 96y'x + 120y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{24(5x^4 - 10x^2 + 1)\, y' + 480(-x^3 + x)\, y}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -20y(0) \\
F_1 &= -12y'(0) \\
F_2 &= 120y(0) \\
F_3 &= 24y'(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = (5x^4 - 10x^2 + 1)\, y(0) + \left(x - 2x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y'' - 8y'x + 20y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) - 8\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 20\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-8n a_n x^n) + \left(\sum_{n=0}^{\infty} 20 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n\right)\\&+\sum_{n=1}^{\infty} (-8n a_n x^n) + \left(\sum_{n=0}^{\infty} 20 a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 20a_0 = 0$$

$$a_2 = -10a_0$$

$n = 1$ gives

$$6a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -2a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 8na_n + 20a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 9n + 20)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 5a_0$$

For $n = 3$ the recurrence equation gives

$$2a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 10a_0 x^2 - 2a_1 x^3 + 5a_0 x^4 + \frac{1}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (5x^4 - 10x^2 + 1)\, a_0 + \left(x - 2x^3 + \frac{1}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (5x^4 - 10x^2 + 1)\, c_1 + \left(x - 2x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (5x^4 - 10x^2 + 1)\, y(0) + \left(x - 2x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = (5x^4 - 10x^2 + 1)\, c_1 + \left(x - 2x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (5x^4 - 10x^2 + 1)\, y(0) + \left(x - 2x^3 + \frac{1}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (5x^4 - 10x^2 + 1)\, c_1 + \left(x - 2x^3 + \frac{1}{5}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)+20*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (5x^4 - 10x^2 + 1)\, y(0) + \left(x - 2x^3 + \frac{1}{5}x^5\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-8*x*y'[x]+20*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{5} - 2x^3 + x\right) + c_1(5x^4 - 10x^2 + 1)$$

12.4 problem 4

Internal problem ID [1208]
Internal file name [OUTPUT/1209_Sunday_June_05_2022_02_05_16_AM_56479052/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^{2}+1\right) y''-8y'x-12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{827}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{828}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{4(2y'x + 3y)}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{60y'x^2 + 120yx + 20y'}{(x^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{120(4x^3y' + 9x^2y + 4y'x + 3y)}{(x^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(4200x^4 + 8400x^2 + 840)\, y' + 10080xy(x^2 + 1)}{(x^2 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-40320x^5 - 134400x^3 - 40320x)\, y' - 100800\left(x^4 + 2x^2 + \frac{1}{5}\right) y}{(x^2 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 12y(0) \\
F_1 &= 20y'(0) \\
F_2 &= 360y(0) \\
F_3 &= 840y'(0) \\
F_4 &= 20160y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(28x^6 + 15x^4 + 6x^2 + 1\right) y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2+1)\, y'' - 8y'x - 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 8\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 12\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-8n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-12 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-8n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-12 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 12a_0 = 0$$

$$a_2 = 6a_0$$

$n = 1$ gives

$$6a_3 - 20a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{10a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 8na_n - 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 + 7n + 12)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-30a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 15a_0$$

For $n = 3$ the recurrence equation gives

$$-42a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 7a_1$$

For $n = 4$ the recurrence equation gives

$$-56a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 28a_0$$

For $n = 5$ the recurrence equation gives

$$-72a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 12a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 6a_0 x^2 + \frac{10}{3} a_1 x^3 + 15a_0 x^4 + 7a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (15x^4 + 6x^2 + 1) a_0 + \left(x + \frac{10}{3}x^3 + 7x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (15x^4 + 6x^2 + 1) c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (28x^6 + 15x^4 + 6x^2 + 1) y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = (15x^4 + 6x^2 + 1) c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (28x^6 + 15x^4 + 6x^2 + 1)\, y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = (15x^4 + 6x^2 + 1)\, c_1 + \left(x + \frac{10}{3}x^3 + 7x^5\right) c_2 + O(x^6)$$

Verified OK.

12.4.1 Maple step by step solution

Let's solve

$(-x^2 + 1)\, y'' - 8y'x - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 4$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

$$\left(x^2-1\right) y''+8y'x+12y=0$$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$\left(u^2-2u\right)\left(\frac{d^2}{du^2}y(u)\right)+(8u-8)\left(\frac{d}{du}y(u)\right)+12y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

◦ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

◦ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(3+r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r+4)+a_k(k+r+4)\,(k+r+3)\right) u^{k+r}\right)=$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(3+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-3,0\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left((-2k-2r-2)\, a_{k+1}+a_k(k+r+3)\right)(k+r+4)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+3)}{2(k+1+r)}$$

- Recursion relation for $r=-3$

$a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)-12*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (15x^4 + 6x^2 + 1)\,y(0) + \left(x + \frac{10}{3}x^3 + 7x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-8*x*y'[x]-12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2\left(7x^5 + \frac{10x^3}{3} + x\right) + c_1\left(15x^4 + 6x^2 + 1\right)$$

12.5 problem 5

Internal problem ID [1209]
Internal file name [`OUTPUT/1210_Sunday_June_05_2022_02_05_17_AM_79760735/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\, y''+7y'x+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (830) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (831) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{7y'x + 2y}{2x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{59y'x^2 + 22yx - 9y'}{(2x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-605x^3y' - 250x^2y + 275y'x + 40y}{(2x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(7365x^4 - 6660x^2 + 315)\, y' + (3210x^3 - 1530x)\, y}{(2x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-104055x^5 + 156150x^3 - 22095x)\, y' + (-46830x^4 + 44370x^2 - 2160)\, y}{(2x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -9y'(0) \\
F_2 &= 40y(0) \\
F_3 &= 315y'(0) \\
F_4 &= -2160y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{5}{3}x^4 - 3x^6\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2x^2+1)\, y'' + 7y'x + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 7\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 7n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ + \left(\sum_{n=1}^{\infty} 7n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 7na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{(2n+1)\, a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$20a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$35a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{21a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$54a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -3a_0$$

For $n = 5$ the recurrence equation gives

$$77a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{77a_1}{16}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{3}{2}a_1x^3 + \frac{5}{3}a_0x^4 + \frac{21}{8}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right)a_0 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right)c_1 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{5}{3}x^4 - 3x^6\right)y(0) + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right)c_1 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{5}{3}x^4 - 3x^6\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{5}{3}x^4\right) c_1 + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{5}{3}x^4\right) y(0) + \left(x - \frac{3}{2}x^3 + \frac{21}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{21x^5}{8} - \frac{3x^3}{2} + x\right) + c_1\left(\frac{5x^4}{3} - x^2 + 1\right)$$

12.6 problem 6

Internal problem ID [1210]
Internal file name [`OUTPUT/1211_Sunday_June_05_2022_02_05_18_AM_42177438/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}+1\right) y''+2y'x+\frac{y}{4}=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (833)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (834)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{8y'x + y}{4\left(x^2+1\right)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{23y'x^2 + 4yx - 9y'}{4\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-352x^3y' - 71x^2y + 416y'x + 25y}{16\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(1689x^4 - 4014x^2 + 441\right) y' + \left(372x^3 - 396x\right) y}{16\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-39048x^5 + 155376x^3 - 51336x\right) y' + \left(-9129x^4 + 19566x^2 - 2025\right) y}{64\left(x^2+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{4} \\
F_1 &= -\frac{9y'(0)}{4} \\
F_2 &= \frac{25y(0)}{16} \\
F_3 &= \frac{441y'(0)}{16} \\
F_4 &= -\frac{2025y(0)}{64}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4-\frac{45}{1024}x^6\right)y(0)+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2+1\right)y''+2y'x+\frac{y}{4}=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty}n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(x^2+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+2\left(\sum_{n=1}^{\infty}n a_n x^{n-1}\right)x+\frac{\left(\sum_{n=0}^{\infty}a_n x^n\right)}{4}=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty}2n a_n x^n\right)+\left(\sum_{n=0}^{\infty}\frac{a_n x^n}{4}\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\left(\sum_{n=0}^{\infty}\frac{a_n x^n}{4}\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 + \frac{a_0}{4} = 0$$

$$a_2 = -\frac{a_0}{8}$$

$n = 1$ gives

$$6a_3 + \frac{9a_1}{4} = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_1}{8}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(n+1) + 2n a_n + \frac{a_n}{4} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(4n^2+4n+1)}{4\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$\frac{25a_2}{4} + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{25a_0}{384}$$

For $n = 3$ the recurrence equation gives

$$\frac{49a_3}{4} + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{147a_1}{640}$$

For $n = 4$ the recurrence equation gives

$$\frac{81a_4}{4} + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{45a_0}{1024}$$

For $n = 5$ the recurrence equation gives

$$\frac{121a_5}{4} + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{847a_1}{5120}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{8}a_0x^2 - \frac{3}{8}a_1x^3 + \frac{25}{384}a_0x^4 + \frac{147}{640}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{8}x^2 + \frac{25}{384}x^4\right)a_0 + \left(x - \frac{3}{8}x^3 + \frac{147}{640}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4\right)c_1+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)c_2+O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4-\frac{45}{1024}x^6\right)y(0)+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)y'(0)+O\left(x^6\right) \quad (1)$$

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4\right)c_1+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)c_2+O\left(x^6\right) \quad (2)$$

Verification of solutions

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4-\frac{45}{1024}x^6\right)y(0)+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(1-\frac{1}{8}x^2+\frac{25}{384}x^4\right)c_1+\left(x-\frac{3}{8}x^3+\frac{147}{640}x^5\right)c_2+O\left(x^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)+1/4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{8}x^2 + \frac{25}{384}x^4\right) y(0) + \left(x - \frac{3}{8}x^3 + \frac{147}{640}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+2*x*y'[x]+1/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{147x^5}{640} - \frac{3x^3}{8} + x\right) + c_1\left(\frac{25x^4}{384} - \frac{x^2}{8} + 1\right)$$

12.7 problem 7

Internal problem ID [1211]
Internal file name [`OUTPUT/1212_Sunday_June_05_2022_02_05_19_AM_26816525/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{\left(-x^2+1\right) y''-5y'x-4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{836}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{837}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{5y'x+4y}{x^2-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{26y'x^2+28yx+9y'}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-154x^3y'-188x^2y-161y'x-64y}{(x^2-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(1044x^4+2196x^2+225)\,y'+(1368x^3+1404x)\,y}{(x^2-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-8028x^5-28296x^3-8721x)\,y'+(-11016x^4-22716x^2-2304)\,y}{(x^2-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 9y'(0) \\
F_2 &= 64y(0) \\
F_3 &= 225y'(0) \\
F_4 &= 2304y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1+2x^2+\frac{8}{3}x^4+\frac{16}{5}x^6\right) y(0) + \left(x+\frac{3}{2}x^3+\frac{15}{8}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 1)\, y'' - 5y'x - 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 5\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-5n a_n x^n) + \sum_{n=0}^{\infty} (-4a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-5n a_n x^n) + \sum_{n=0}^{\infty} (-4a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

$n = 1$ gives

$$6a_3 - 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{3a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$-na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 5na_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n+2)\,a_n}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-16a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{8a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$-25a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{15a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$-36a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{16a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$-49a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{35a_1}{16}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 + \frac{3}{2} a_1 x^3 + \frac{8}{3} a_0 x^4 + \frac{15}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{8}{3}x^4\right) a_0 + \left(x + \frac{3}{2}x^3 + \frac{15}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{8}{3}x^4\right) c_1 + \left(x + \frac{3}{2}x^3 + \frac{15}{8}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 2x^2 + \frac{8}{3}x^4 + \frac{16}{5}x^6\right) y(0) + \left(x + \frac{3}{2}x^3 + \frac{15}{8}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + 2x^2 + \frac{8}{3}x^4\right) c_1 + \left(x + \frac{3}{2}x^3 + \frac{15}{8}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+2x^2+\frac{8}{3}x^4+\frac{16}{5}x^6\right)y(0)+\left(x+\frac{3}{2}x^3+\frac{15}{8}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+2x^2+\frac{8}{3}x^4\right)c_1+\left(x+\frac{3}{2}x^3+\frac{15}{8}x^5\right)c_2+O(x^6)$$

Verified OK.

12.7.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-5y'x-4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5xy'}{x^2-1}-\frac{4y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5xy'}{x^2-1}+\frac{4y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{5x}{x^2-1}, P_3(x)=\frac{4}{x^2-1}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{5}{2}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$$(x^2-1)\,y''+5y'x+4y=0$$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(5u-5)\left(\frac{d}{du}y(u)\right)+4y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+5+2r)+a_k(k+r+2)^2\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+2)^2-2(k+1+r)\left(k+\frac{5}{2}+r\right)a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{a_k (k+2)^2}{(k+1)(2k+5)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k (k+2)^2}{(k+1)(2k+5)}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k (k+2)^2}{(k+1)(2k+5)}\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{3}{2}}\right), a_{1+k} = \frac{a_k (k+2)^2}{(1+k)(2k+5)}, b_{1+k} = \frac{b_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-5*x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + 2x^2 + \frac{8}{3}x^4\right) y(0) + \left(x + \frac{3}{2}x^3 + \frac{15}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-5*x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{15x^5}{8} + \frac{3x^3}{2} + x\right) + c_1\left(\frac{8x^4}{3} + 2x^2 + 1\right)$$

12.8 problem 8

Internal problem ID [1212]
Internal file name [`OUTPUT/1213_Sunday_June_05_2022_02_05_20_AM_93707444/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-10y'x+28y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{839}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{840}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{10y'x - 28y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{62y'x^2 - 224yx - 18y'}{\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{272x^3y' - 1064x^2y - 208y'x + 280y}{\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(840x^4 - 1008x^2 + 72\right) y' - 3360\left(x^2 - \frac{3}{5}\right) xy}{\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(1680x^5 - 2016x^3 + 144x\right) y' - 6720\left(x^2 - \frac{3}{5}\right) x^2 y}{\left(x^2+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -28y(0) \\
F_1 &= -18y'(0) \\
F_2 &= 280y(0) \\
F_3 &= 72y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) y(0) + \left(x - 3x^3 + \frac{3}{5}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' - 10y'x + 28y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 10\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 28\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-10 n a_n x^n) + \left(\sum_{n=0}^{\infty} 28 a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-10 n a_n x^n) + \left(\sum_{n=0}^{\infty} 28 a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 28a_0 = 0$$

$$a_2 = -14a_0$$

$n = 1$ gives

$$6a_3 + 18a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -3a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 10na_n + 28a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 11n + 28)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$10a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{35a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$4a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-2a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 14a_0 x^2 - 3a_1 x^3 + \frac{35}{3} a_0 x^4 + \frac{3}{5} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) a_0 + \left(x - 3x^3 + \frac{3}{5}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) c_1 + \left(x - 3x^3 + \frac{3}{5}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) y(0) + \left(x - 3x^3 + \frac{3}{5}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) c_1 + \left(x - 3x^3 + \frac{3}{5}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) y(0) + \left(x - 3x^3 + \frac{3}{5}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - 14x^2 + \frac{35}{3}x^4\right) c_1 + \left(x - 3x^3 + \frac{3}{5}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)-10*x*diff(y(x),x)+28*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{35}{3}x^4 - 14x^2\right) y(0) + \left(x - 3x^3 + \frac{3}{5}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-10*x*y'[x]+28*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{3x^5}{5} - 3x^3 + x\right) + c_1\left(\frac{35x^4}{3} - 14x^2 + 1\right)$$

12.9 problem 9(a)

Internal problem ID [1213]
Internal file name [`OUTPUT/1214_Sunday_June_05_2022_02_05_22_AM_14355239/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 9(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (842)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (843)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + 2yx - 3y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -x^3y' - 2x^2y + 7y'x + 8y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 12x^2 + 15\right) y' + 2\left(x^3 - 9x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 18x^3 - 57x\right) y' - 2y\left(x^4 - 15x^2 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -3y'(0) \\
F_2 &= 8y(0) \\
F_3 &= 15y'(0) \\
F_4 &= -48y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + n a_n + 2 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{3} a_0 x^4 + \frac{1}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Verified OK.

12.9.1 Maple step by step solution

Let's solve

$y'' = -y'x - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{3}x^4\right) y(0) + \left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8} - \frac{x^3}{2} + x\right) + c_1\left(\frac{x^4}{3} - x^2 + 1\right)$$

12.10 problem 10

Internal problem ID [1214]
Internal file name [OUTPUT/1215_Sunday_June_05_2022_02_05_23_AM_36955727/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y'x + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{845}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{846}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2y'x - 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 4y'x^2 + 6yx - 5y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -8x^3y' - 12x^2y + 24y'x + 21y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(16x^4 - 84x^2 + 45\right)y' + 24\left(x^3 - 4x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-32x^5 + 256x^3 - 354x\right)y' + \left(-48x^4 + 324x^2 - 231\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -3y(0) \\
F_1 &= -5y'(0) \\
F_2 &= 21y(0) \\
F_3 &= 45y'(0) \\
F_4 &= -231y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{77}{240}x^6\right)y(0) + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right)y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n+3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 5a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{5a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 7a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 9a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 11a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{77a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 13a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{13a_1}{112}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^2 - \frac{5}{6} a_1 x^3 + \frac{7}{8} a_0 x^4 + \frac{3}{8} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4\right) a_0 + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{77}{240}x^6\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 - \frac{77}{240}x^6\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) c_2 + O(x^6)$$

Verified OK.

12.10.1 Maple step by step solution

Let's solve

$y'' = -2y'x - 3y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y'x + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} + 2a_k k + 3a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(2k+3)}{k^2+3k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{2}x^2 + \frac{7}{8}x^4\right) y(0) + \left(x - \frac{5}{6}x^3 + \frac{3}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+2*x*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{3x^5}{8} - \frac{5x^3}{6} + x\right) + c_1\left(\frac{7x^4}{8} - \frac{3x^2}{2} + 1\right)$$

12.11 problem 11

Internal problem ID [1215]
Internal file name [`OUTPUT/1216_Sunday_June_05_2022_02_05_24_AM_30131521/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{(x^2+1)\,y''+y'x+y=0}$$

With initial conditions

$$[y(0)=2, y'(0)=-1]$$

With the expansion point for the power series method at $x=0$.

12.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$
\begin{aligned}
p(x) &= \frac{x}{x^2+1} \\
q(x) &= \frac{1}{x^2+1} \\
F &= 0
\end{aligned}
$$

Hence the ode is

$$
y'' + \frac{x y'}{x^2+1} + \frac{y}{x^2+1} = 0
$$

The domain of $p(x) = \frac{x}{x^2+1}$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x^2+1}$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$
y'' = f(x, y, y')
$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{848}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{849}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y' x + y}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{y' x^2 + 3yx - 2y'}{\left(x^2 + 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-10x^2 y + 15y' x + 5y}{\left(x^2 + 1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(-10x^4 - 95x^2 + 20\right) y' + \left(40x^3 - 65x\right) y}{\left(x^2 + 1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(90x^5 + 600x^3 - 435x\right) y' + \left(-190x^4 + 670x^2 - 85\right) y}{\left(x^2 + 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 2 \\
F_2 &= 10 \\
F_3 &= -20 \\
F_4 &= -170
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 - x + 2 + \frac{x^3}{3} + \frac{5x^4}{12} - \frac{x^5}{6} - \frac{17x^6}{72} + O\left(x^6\right)
$$

$$y = -x^2 - x + 2 + \frac{x^3}{3} + \frac{5x^4}{12} - \frac{x^5}{6} - \frac{17x^6}{72} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1) y'' + y'x + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\,a_{n+2}(n+1) + na_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2+1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$5a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$10a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{6}$$

For $n = 4$ the recurrence equation gives

$$17a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{17a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$26a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{13a_1}{126}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2}a_0 x^2 - \frac{1}{3}a_1 x^3 + \frac{5}{24}a_0 x^4 + \frac{1}{6}a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{6}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{6}x^5\right) c_2 + O(x^6)$$

$$y = 2 - x^2 + \frac{5x^4}{12} - x + \frac{x^3}{3} - \frac{x^5}{6} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = -x^2 - x + 2 + \frac{x^3}{3} + \frac{5x^4}{12} - \frac{x^5}{6} - \frac{17x^6}{72} + O(x^6) \tag{1}$$

$$y = 2 - x^2 + \frac{5x^4}{12} - x + \frac{x^3}{3} - \frac{x^5}{6} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -x^2 - x + 2 + \frac{x^3}{3} + \frac{5x^4}{12} - \frac{x^5}{6} - \frac{17x^6}{72} + O(x^6)$$

Verified OK.

$$y = 2 - x^2 + \frac{5x^4}{12} - x + \frac{x^3}{3} - \frac{x^5}{6} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+x^2)*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(0) = 2, D(y)(0) = -1],y(x),type='seri
```

$$y(x) = 2 - x - x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{1}{6}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(1+x^2)*y''[x]+x*y'[x]+y[x]==0,{y[0]==2,y'[0]==-1}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{6} + \frac{5x^4}{12} + \frac{x^3}{3} - x^2 - x + 2$$

12.12 problem 12

Internal problem ID [1216]
Internal file name [`OUTPUT/1217_Sunday_June_05_2022_02_05_26_AM_87113912/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+1)\, y'' - 9y'x - 6y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

With the expansion point for the power series method at $x = 0$.

12.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{9x}{2x^2+1} \\ q(x) &= -\frac{6}{2x^2+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{9xy'}{2x^2+1} - \frac{6y}{2x^2+1} = 0$$

The domain of $p(x) = -\frac{9x}{2x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{6}{2x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{851}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{852}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{9y'x + 6y}{2x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{75y'x^2 + 30yx + 15y'}{(2x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{435x^3y' + 270x^2y + 195y'x + 120y}{(2x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(1845x^4 + 1620x^2 + 315)\,y' + (450x^3 + 270x)\,y}{(2x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(2745x^5 + 3510x^3 + 1305x)\,y' + (6570x^4 + 7290x^2 + 2160)\,y}{(2x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= 6 \\
F_1 &= -15 \\
F_2 &= 120 \\
F_3 &= -315 \\
F_4 &= 2160
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 3x^2 - x + 1 - \frac{5x^3}{2} + 5x^4 - \frac{21x^5}{8} + 3x^6 + O(x^6)
$$

$$
y = 3x^2 - x + 1 - \frac{5x^3}{2} + 5x^4 - \frac{21x^5}{8} + 3x^6 + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(2x^2 + 1)\, y'' - 9y'x - 6y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(2x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 9\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-9n a_n x^n) + \sum_{n=0}^{\infty} (-6a_n x^n) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-9n a_n x^n) + \sum_{n=0}^{\infty} (-6a_n x^n) = 0 \tag{3}
$$

$n = 0$ gives

$$2a_2 - 6a_0 = 0$$

$$a_2 = 3a_0$$

$n = 1$ gives

$$6a_3 - 15a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+2)\,a_{n+2}(n+1) - 9na_n - 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(2n^2 - 11n - 6)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-20a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 5a_0$$

For $n = 3$ the recurrence equation gives

$$-21a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{21a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$-18a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 3a_0$$

For $n = 5$ the recurrence equation gives

$$-11a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_1}{16}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 3a_0x^2 + \frac{5}{2}a_1x^3 + 5a_0x^4 + \frac{21}{8}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (5x^4 + 3x^2 + 1)\, a_0 + \left(x + \frac{5}{2}x^3 + \frac{21}{8}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (5x^4 + 3x^2 + 1)\, c_1 + \left(x + \frac{5}{2}x^3 + \frac{21}{8}x^5\right) c_2 + O(x^6)$$

$$y = 5x^4 + 3x^2 + 1 - x - \frac{5x^3}{2} - \frac{21x^5}{8} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 3x^2 - x + 1 - \frac{5x^3}{2} + 5x^4 - \frac{21x^5}{8} + 3x^6 + O(x^6) \tag{1}$$

$$y = 5x^4 + 3x^2 + 1 - x - \frac{5x^3}{2} - \frac{21x^5}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 3x^2 - x + 1 - \frac{5x^3}{2} + 5x^4 - \frac{21x^5}{8} + 3x^6 + O(x^6)$$

Verified OK.

$$y = 5x^4 + 3x^2 + 1 - x - \frac{5x^3}{2} - \frac{21x^5}{8} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+2*x^2)*diff(y(x),x$2)-9*x*diff(y(x),x)-6*y(x)=0,y(0) = 1, D(y)(0) = -1],y(x),type
```

$$y(x) = 1 - x + 3x^2 - \frac{5}{2}x^3 + 5x^4 - \frac{21}{8}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(1+2*x^2)*y''[x]-9*x*y'[x]-6*y[x]==0,{y[0]==1,y'[0]==-1}},y[x],{x,0,5
```

$$y(x) \to -\frac{21x^5}{8} + 5x^4 - \frac{5x^3}{2} + 3x^2 - x + 1$$

12.13 problem 13

Internal problem ID [1217]
Internal file name [`OUTPUT/1218_Sunday_June_05_2022_02_05_28_AM_77980607/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(8x^2+1)\,y''+2y=0}$$

With initial conditions

$$[y(0)=2, y'(0)=-1]$$

With the expansion point for the power series method at $x=0$.

12.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{2}{8x^2+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y}{8x^2 + 1} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{8x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} &(1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' &(854)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(855)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f &(2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{2y}{8x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-16y'x^2 + 32yx - 2y'}{(8x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{512x^3y' - 736x^2y + 64y'x + 36y}{(8x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-18176x^4 - 1472x^2 + 100)\, y' + (22528x^3 - 3328x)\, y}{(8x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(761856x^5 - 6144x^3 - 12672x)\, y' + (-864768x^4 + 256896x^2 - 3528)\, y}{(8x^2+1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$\begin{aligned}
F_0 &= -4 \\
F_1 &= 2 \\
F_2 &= 72 \\
F_3 &= -100 \\
F_4 &= -7056
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = -2x^2 - x + 2 + \frac{x^3}{3} + 3x^4 - \frac{5x^5}{6} - \frac{49x^6}{5} + O(x^6)$$

$$y = -2x^2 - x + 2 + \frac{x^3}{3} + 3x^4 - \frac{5x^5}{6} - \frac{49x^6}{5} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(8x^2 + 1)\, y'' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(8x^2 + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 8x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 8x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$8na_n(n-1) + (n+2)\,a_{n+2}(n+1) + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(4n^2 - 4n + 1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$18a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$50a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_1}{6}$$

For $n = 4$ the recurrence equation gives

$$98a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{49a_0}{10}$$

For $n = 5$ the recurrence equation gives

$$162a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{45a_1}{14}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{3}{2} a_0 x^4 + \frac{5}{6} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{3}{2}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{5}{6}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{3}{2}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{5}{6}x^5\right) c_2 + O(x^6)$$

$$y = 3x^4 - 2x^2 + 2 - x + \frac{x^3}{3} - \frac{5x^5}{6} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -2x^2 - x + 2 + \frac{x^3}{3} + 3x^4 - \frac{5x^5}{6} - \frac{49x^6}{5} + O(x^6) \tag{1}$$

$$y = 3x^4 - 2x^2 + 2 - x + \frac{x^3}{3} - \frac{5x^5}{6} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -2x^2 - x + 2 + \frac{x^3}{3} + 3x^4 - \frac{5x^5}{6} - \frac{49x^6}{5} + O(x^6)$$

Verified OK.

$$y = 3x^4 - 2x^2 + 2 - x + \frac{x^3}{3} - \frac{5x^5}{6} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+8*x^2)*diff(y(x),x$2)+2*y(x)=0,y(0) = 2, D(y)(0) = -1],y(x),type='series',x=0);
```

$$y(x) = 2 - x - 2x^2 + \frac{1}{3}x^3 + 3x^4 - \frac{5}{6}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(1+8*x^2)*y''[x]+2*y[x]==0,{y[0]==2,y'[0]==-1}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{5x^5}{6} + 3x^4 + \frac{x^3}{3} - 2x^2 - x + 2$$

12.14 problem 16

Internal problem ID [1218]
Internal file name [`OUTPUT/1219_Sunday_June_05_2022_02_05_30_AM_44814531/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 3$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (857) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (858) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)} F_0 \\
&= \frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)} F_1 \\
&= y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)} F_2 \\
&= \frac{d}{dt}y(t) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)} F_3 \\
&= y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6\right) y(0) + \left(t + \frac{1}{6}t^3 + \frac{1}{120}t^5\right) y'(0) + O(t^6)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} a_n t^n \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} a_1 t^3 + \frac{1}{24} a_0 t^4 + \frac{1}{120} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2} t^2 + \frac{1}{24} t^4\right) a_0 + \left(t + \frac{1}{6} t^3 + \frac{1}{120} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2} t^2 + \frac{1}{24} t^4\right) c_1 + \left(t + \frac{1}{6} t^3 + \frac{1}{120} t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 3$ results in

$$\begin{aligned} y = &\left(1 + \frac{(x-3)^2}{2} + \frac{(x-3)^4}{24} + \frac{(x-3)^6}{720}\right) y(3) \\ &+ \left(x - 3 + \frac{(x-3)^3}{6} + \frac{(x-3)^5}{120}\right) y'(3) + O\left((x-3)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{(x-3)^2}{2} + \frac{(x-3)^4}{24} + \frac{(x-3)^6}{720}\right) y(3) + \left(x - 3 + \frac{(x-3)^3}{6} + \frac{(x-3)^5}{120}\right) y'(3) + O\left((x-3)^6\right) \tag{1}$$

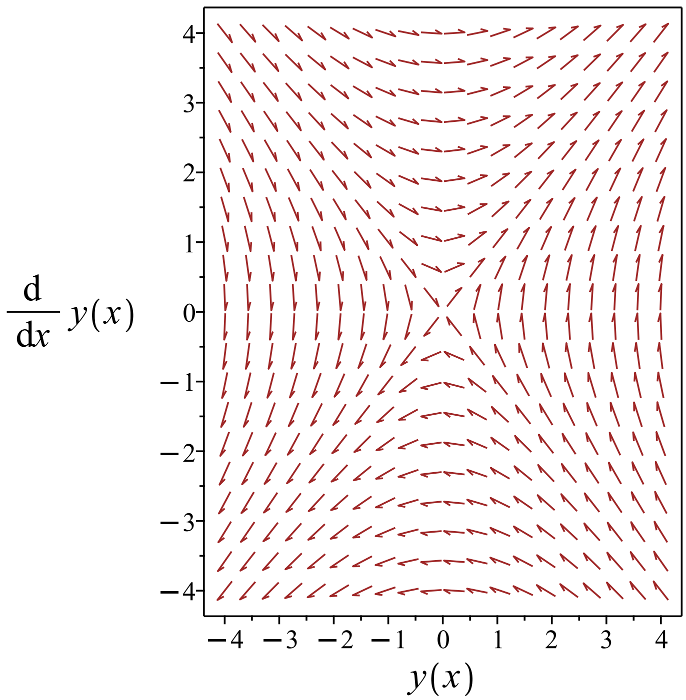

Figure 536: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{(x-3)^2}{2} + \frac{(x-3)^4}{24} + \frac{(x-3)^6}{720}\right) y(3) + \left(x - 3 + \frac{(x-3)^3}{6} + \frac{(x-3)^5}{120}\right) y'(3) + O\left((x-3)^6\right)$$

Verified OK.

12.14.1 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-y(x)=0,y(x),type='series',x=3);
```

$$y(x) = \left(1+\frac{(x-3)^{2}}{2}+\frac{(x-3)^{4}}{24}\right) y(3)+\left(x-3+\frac{(x-3)^{3}}{6}+\frac{(x-3)^{5}}{120}\right) D(y)(3)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 51

```
AsymptoticDSolveValue[y''[x]-y[x]==0,y[x],{x,3,5}]
```

$$y(x) \to c_1\left(\frac{1}{24}(x-3)^4+\frac{1}{2}(x-3)^2+1\right)+c_2\left(\frac{1}{120}(x-3)^5+\frac{1}{6}(x-3)^3+x-3\right)$$

12.15 problem 17

Internal problem ID [1219]
Internal file name [`OUTPUT/1220_Sunday_June_05_2022_02_05_31_AM_75916991/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - (x - 3)\, y' - y = 0}$$

With the expansion point for the power series method at $x = 3$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - t\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (860) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (861) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= t\left(\frac{d}{dt}y(t)\right) + y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\frac{d}{dt}y(t)\right)t^2 + y(t)\,t + 2\frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \left(t^3 + 5t\right)\left(\frac{d}{dt}y(t)\right) + y(t)\left(t^2 + 3\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(t^4 + 9t^2 + 8\right)\left(\frac{d}{dt}y(t)\right) + ty(t)\left(t^2 + 7\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(t^5 + 14t^3 + 33t\right)\left(\frac{d}{dt}y(t)\right) + y(t)\left(t^4 + 12t^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= 2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= 15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6\right)y(0) + \left(t + \frac{1}{3}t^3 + \frac{1}{15}t^5\right)y'(0) + O\left(t^6\right)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt} y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2} y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, t^n a_n) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) t^n\right) + \sum_{n=1}^{\infty} (-n\, t^n a_n) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{1}{2} a_0 t^2 + \frac{1}{3} a_1 t^3 + \frac{1}{8} a_0 t^4 + \frac{1}{15} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{8}t^4\right) a_0 + \left(t + \frac{1}{3}t^3 + \frac{1}{15}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{8}t^4\right) c_1 + \left(t + \frac{1}{3}t^3 + \frac{1}{15}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 3$ results in

$$\begin{aligned} y = &\left(1 + \frac{(x-3)^2}{2} + \frac{(x-3)^4}{8} + \frac{(x-3)^6}{48}\right) y(3) \\ &+ \left(x - 3 + \frac{(x-3)^3}{3} + \frac{(x-3)^5}{15}\right) y'(3) + O\left((x-3)^6\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(1+\frac{(x-3)^{2}}{2}+\frac{(x-3)^{4}}{8}+\frac{(x-3)^{6}}{48}\right) y(3) + \left(x-3+\frac{(x-3)^{3}}{3}+\frac{(x-3)^{5}}{15}\right) y'(3)+O\left((x-3)^{6}\right) \tag{1}$$

Verification of solutions

$$y = \left(1+\frac{(x-3)^{2}}{2}+\frac{(x-3)^{4}}{8}+\frac{(x-3)^{6}}{48}\right) y(3) + \left(x-3+\frac{(x-3)^{3}}{3}+\frac{(x-3)^{5}}{15}\right) y'(3)+O\left((x-3)^{6}\right)$$

Verified OK.

12.15.1 Maple step by step solution

Let's solve

$y''+(3-x)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  - Convert y'' to series expansion

    $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + 3a_{k+1}(k+1) - a_k(k+1)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1)\left(a_{k+2}(k+2) + 3a_{k+1} - a_k\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{-3a_{k+1}+a_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)-(x-3)*diff(y(x),x)-y(x)=0,y(x),type='series',x=3);
```

$$y(x) = \left(1+\frac{(x-3)^2}{2}+\frac{(x-3)^4}{8}\right) y(3)+\left(x-3+\frac{(x-3)^3}{3}+\frac{(x-3)^5}{15}\right) D(y)(3)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 51

```
AsymptoticDSolveValue[y''[x]-(x-3)*y'[x]-y[x]==0,y[x],{x,3,5}]
```

$$y(x) \rightarrow c_1\left(\frac{1}{8}(x-3)^4+\frac{1}{2}(x-3)^2+1\right)+c_2\left(\frac{1}{15}(x-3)^5+\frac{1}{3}(x-3)^3+x-3\right)$$

12.16 problem 18

Internal problem ID [1220]
Internal file name [OUTPUT/1221_Sunday_June_05_2022_02_05_33_AM_65793830/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(2x^2-4x+1)\,y''+10(x-1)\,y'+6y=0}$$

With the expansion point for the power series method at $x=1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(t+1)^2-4t-3\right)\left(\frac{d^2}{dt^2}y(t)\right)+10t\left(\frac{d}{dt}y(t)\right)+6y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (863) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (864) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2\left(5t\left(\frac{d}{dt}y(t)\right) + 3y(t)\right)}{2t^2 - 1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{108\left(\frac{d}{dt}y(t)\right)t^2 + 84y(t)\,t + 16\frac{d}{dt}y(t)}{\left(2t^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{-1344\left(\frac{d}{dt}y(t)\right)t^3 - 1152y(t)\,t^2 - 588t\left(\frac{d}{dt}y(t)\right) - 180y(t)}{\left(2t^2 - 1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{\left(19200t^4 + 16584t^2 + 768\right)\left(\frac{d}{dt}y(t)\right) + \left(17280t^3 + 7992t\right)y(t)}{\left(2t^2 - 1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{\left(-311040t^5 - 442944t^3 - 61128t\right)\left(\frac{d}{dt}y(t)\right) - 288000\left(t^4 + \frac{457}{500}t^2 + \frac{7}{160}\right)y(t)}{\left(2t^2 - 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 6y(0) \\
F_1 &= 16y'(0) \\
F_2 &= 180y(0) \\
F_3 &= 768y'(0) \\
F_4 &= 12600y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1+3t^2+\frac{15}{2}t^4+\frac{35}{2}t^6\right) y(0)+\left(t+\frac{8}{3}t^3+\frac{32}{5}t^5\right) y'(0)+O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right) (2t^2-1)+10t\left(\frac{d}{dt}y(t)\right)+6y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) (2t^2-1)+10t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+6\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right)+\sum_{n=2}^{\infty} \left(-n(n-1) a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty} 10n a_n t^n\right)+\left(\sum_{n=0}^{\infty} 6a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1) a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2) a_{n+2}(n+1) t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 10n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 6a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 6a_0 = 0$$

$$a_2 = 3a_0$$

$n = 1$ gives

$$-6a_3 + 16a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{8a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) - (n+2)\, a_{n+2}(n+1) + 10na_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2(n+3)\, a_n}{n+2} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$30a_2 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{15a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$48a_3 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{32a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$70a_4 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{35a_0}{2}$$

For $n = 5$ the recurrence equation gives

$$96a_5 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{512a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + 3a_0 t^2 + \frac{8}{3} a_1 t^3 + \frac{15}{2} a_0 t^4 + \frac{32}{5} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + 3t^2 + \frac{15}{2} t^4\right) a_0 + \left(t + \frac{8}{3} t^3 + \frac{32}{5} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + 3t^2 + \frac{15}{2}t^4\right) c_1 + \left(t + \frac{8}{3}t^3 + \frac{32}{5}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$\begin{aligned} y = &\left(1 + 3(x-1)^2 + \frac{15(x-1)^4}{2} + \frac{35(x-1)^6}{2}\right) y(1) \\ &+ \left(x - 1 + \frac{8(x-1)^3}{3} + \frac{32(x-1)^5}{5}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + 3(x-1)^2 + \frac{15(x-1)^4}{2} + \frac{35(x-1)^6}{2}\right) y(1) \\ &+ \left(x - 1 + \frac{8(x-1)^3}{3} + \frac{32(x-1)^5}{5}\right) y'(1) + O\left((x-1)^6\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 + 3(x-1)^2 + \frac{15(x-1)^4}{2} + \frac{35(x-1)^6}{2}\right) y(1) \\ &+ \left(x - 1 + \frac{8(x-1)^3}{3} + \frac{32(x-1)^5}{5}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Verified OK.

12.16.1 Maple step by step solution

Let's solve

$(2x^2 - 4x + 1) y'' + (10x - 10) y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y}{2x^2-4x+1} - \frac{10(x-1)y'}{2x^2-4x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{10(x-1)y'}{2x^2-4x+1}+\frac{6y}{2x^2-4x+1}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{10(x-1)}{2x^2-4x+1}, P_3(x)=\frac{6}{2x^2-4x+1}\right]$

  - $\left(x-1+\frac{\sqrt{2}}{2}\right)\cdot P_2(x)$ is analytic at $x=1-\frac{\sqrt{2}}{2}$

    $\left(\left(x-1+\frac{\sqrt{2}}{2}\right)\cdot P_2(x)\right)\Bigg|_{x=1-\frac{\sqrt{2}}{2}}=0$

  - $\left(x-1+\frac{\sqrt{2}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=1-\frac{\sqrt{2}}{2}$

    $\left(\left(x-1+\frac{\sqrt{2}}{2}\right)^2\cdot P_3(x)\right)\Bigg|_{x=1-\frac{\sqrt{2}}{2}}=0$

  - $x=1-\frac{\sqrt{2}}{2}$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=1-\frac{\sqrt{2}}{2}$

- Multiply by denominators

  $(2x^2-4x+1)\,y''+(10x-10)\,y'+6y=0$

- Change variables using $x=u+1-\frac{\sqrt{2}}{2}$ so that the regular singular point is at $u=0$

  $(2u^2-2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right)+(10u-5\sqrt{2})\left(\frac{d}{du}y(u)\right)+6y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-\sqrt{2}\, r(2r+3)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-\sqrt{2}\,(k+r+1)(2k+5+2r)\, a_{k+1} + 2a_k(k+r+3)(k+r+1\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\sqrt{2}\, r(2r+3) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(-a_{k+1}\left(k+r+\frac{5}{2}\right)\sqrt{2} + a_k(k+r+3)\right)(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)\sqrt{2}}{2k+5+2r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(k+3)\sqrt{2}}{2k+5}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)\sqrt{2}}{2k+5}\right]$$

- Revert the change of variables $u = x - 1 + \frac{\sqrt{2}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x - 1 + \frac{\sqrt{2}}{2}\right)^k, a_{k+1} = \frac{a_k(k+3)\sqrt{2}}{2k+5}\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{3}{2}\right)\sqrt{2}}{2k+2}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{3}{2}\right)\sqrt{2}}{2k+2}\right]$$

- Revert the change of variables $u = x - 1 + \frac{\sqrt{2}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x-1+\frac{\sqrt{2}}{2}\right)^{k-\frac{3}{2}}, a_{k+1}=\frac{a_k\left(k+\frac{3}{2}\right)\sqrt{2}}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x-1+\frac{\sqrt{2}}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x-1+\frac{\sqrt{2}}{2}\right)^{k-\frac{3}{2}}\right), a_{1+k}=\frac{a_k(k+3)\sqrt{2}}{2k+5}, b_{1+k}=\frac{b_k\left(k+\frac{3}{2}\right)\sqrt{2}}{2k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve((1-4*x+2*x^2)*diff(y(x),x$2)+10*(x-1)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=1);
```

$$y(x)=\left(1+3(x-1)^2+\frac{15(x-1)^4}{2}\right)y(1)+\left(x-1+\frac{8(x-1)^3}{3}+\frac{32(x-1)^5}{5}\right)D(y)(1)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[(1-4*x+2*x^2)*y''[x]+10*(x-1)*y'[x]+6*y[x]==0,y[x],{x,1,5}]
```

$$y(x)\to c_1\left(\frac{15}{2}(x-1)^4+3(x-1)^2+1\right)+c_2\left(\frac{32}{5}(x-1)^5+\frac{8}{3}(x-1)^3+x-1\right)$$

12.17 problem 19

Internal problem ID [1221]
Internal file name [`OUTPUT/1222_Sunday_June_05_2022_02_05_35_AM_4673127/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2 - 8x + 11)\, y'' - 16(-2 + x)\, y' + 36y = 0}$$

With the expansion point for the power series method at $x = 2$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(2+t)^2 - 5 - 8t\right)\left(\frac{d^2}{dt^2}y(t)\right) - 16t\left(\frac{d}{dt}y(t)\right) + 36y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (866) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (867) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{16t\left(\frac{d}{dt}y(t)\right) - 36y(t)}{2t^2+3} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{152\left(\frac{d}{dt}y(t)\right)t^2 - 432y(t)\,t - 60\frac{d}{dt}y(t)}{\left(2t^2+3\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{960\left(\frac{d}{dt}y(t)\right)t^3 - 2880y(t)\,t^2 - 864t\left(\frac{d}{dt}y(t)\right) + 864y(t)}{\left(2t^2+3\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{384\left(10\left(\frac{d}{dt}y(t)\right)t^3 - 30y(t)\,t^2 - 9t\left(\frac{d}{dt}y(t)\right) + 9y(t)\right)t}{\left(2t^2+3\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{3840\left(\frac{d}{dt}y(t)\right)t^3 - 11520y(t)\,t^2 - 3456t\left(\frac{d}{dt}y(t)\right) + 3456y(t)}{\left(2t^2+3\right)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -12y(0) \\
F_1 &= -\frac{20y'(0)}{3} \\
F_2 &= 32y(0) \\
F_3 &= 0 \\
F_4 &= \frac{128y(0)}{3}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - 6t^2 + \frac{4}{3}t^4 + \frac{8}{135}t^6\right) y(0) + \left(t - \frac{10}{9}t^3\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2 + 3) - 16t\left(\frac{d}{dt}y(t)\right) + 36y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right)(2t^2 + 3) - 16t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 36\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1) a_n t^{n-2}\right) + \sum_{n=1}^{\infty}(-16n a_n t^n) + \left(\sum_{n=0}^{\infty} 36a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} 3(n+2) a_{n+2}(n+1) t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n\,=1}^{\infty} (-16na_n t^n) + \left(\sum_{n=0}^{\infty} 36a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 + 36a_0 = 0$$

$$a_2 = -6a_0$$

$n = 1$ gives

$$18a_3 + 20a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{10a_1}{9}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + 3(n+2)\, a_{n+2}(n+1) - 16na_n + 36a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(n^2 - 9n + 18)}{3\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$8a_2 + 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{4a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-4a_4 + 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{8a_0}{135}$$

For $n = 5$ the recurrence equation gives

$$-4a_5 + 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - 6a_0 t^2 - \frac{10}{9} a_1 t^3 + \frac{4}{3} a_0 t^4 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - 6t^2 + \frac{4}{3}t^4\right) a_0 + \left(t - \frac{10}{9}t^3\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - 6t^2 + \frac{4}{3}t^4\right) c_1 + \left(t - \frac{10}{9}t^3\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = -2 + x$ results in

$$y = \left(1 - 6(-2+x)^2 + \frac{4(-2+x)^4}{3} + \frac{8(-2+x)^6}{135}\right) y(2) + \left(-2 + x - \frac{10(-2+x)^3}{9}\right) y'(2) + O\left((-2+x)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - 6(-2+x)^2 + \frac{4(-2+x)^4}{3} + \frac{8(-2+x)^6}{135}\right) y(2) + \left(-2 + x - \frac{10(-2+x)^3}{9}\right) y'(2) + O\left((-2+x)^6\right) \quad (1)$$

Verification of solutions

$$y = \left(1 - 6(-2+x)^2 + \frac{4(-2+x)^4}{3} + \frac{8(-2+x)^6}{135}\right) y(2) + \left(-2 + x - \frac{10(-2+x)^3}{9}\right) y'(2) + O\left((-2+x)^6\right)$$

Verified OK.

12.17.1 Maple step by step solution

Let's solve

$(2x^2 - 8x + 11) y'' + (-16x + 32) y' + 36y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{36y}{2x^2-8x+11} + \frac{16(-2+x)y'}{2x^2-8x+11}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{16(-2+x)y'}{2x^2-8x+11} + \frac{36y}{2x^2-8x+11} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{16(-2+x)}{2x^2-8x+11}, P_3(x) = \frac{36}{2x^2-8x+11}\right]$

  ◦ $\left(x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = 2 - \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

  ◦ $\left(x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = 2 - \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

  ◦ $x = 2 - \frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 2 - \frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2 - 8x + 11)\, y'' + (-16x + 32)\, y' + 36y = 0$

- Change variables using $x = u + 2 - \frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2 - 2\,\mathrm{I}u\sqrt{6}) \left(\frac{d^2}{du^2}y(u)\right) + (-16u + 8\,\mathrm{I}\sqrt{6}) \left(\frac{d}{du}y(u)\right) + 36y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\,r(r-5)\,a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\,r(r-5) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\,\mathrm{I}\sqrt{6}\,(k+1+r)(k-4+r)\,a_{k+1} + 2a_k(k+r-3)(k+r-6) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k\left(k^2+2kr+r^2-9k-9r+18\right)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$$

- Recursion relation for $r=0$; series terminates at $k=3$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k\left(k^2-9k+18\right)\sqrt{6}}{k^2-3k-4}$$

- Apply recursion relation for $k=0$

$$a_1 = \frac{3\,\mathrm{I}}{4}a_0\sqrt{6}$$

- Apply recursion relation for $k=1$

$$a_2 = \frac{5\,\mathrm{I}}{18}a_1\sqrt{6}$$

- Express in terms of a_0

$$a_2 = -\frac{5a_0}{4}$$

- Apply recursion relation for $k=2$

$$a_3 = \frac{\mathrm{I}}{9}a_2\sqrt{6}$$

- Express in terms of a_0

$$a_3 = -\frac{5\,\mathrm{I}}{36}a_0\sqrt{6}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 + \frac{3\,\mathrm{I}\sqrt{6}\,u}{4} - \frac{5u^2}{4} - \frac{5\,\mathrm{I}\sqrt{6}\,u^3}{36}\right)$$

- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y = -\frac{\mathrm{I}}{72} a_0 \sqrt{6}\, (10x^3 - 60x^2 + 111x - 62)\right]$

- Recursion relation for $r = 5$; series terminates at $k = 1$

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+k-2)\sqrt{6}}{k^2+7k+6}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{\mathrm{I}}{18} a_0 \sqrt{6}$

- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{\mathrm{I}\sqrt{6}\, u}{18}\right)$

- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y = a_0 \left(\frac{5}{6} + \frac{\mathrm{I}(-2+x)\sqrt{6}}{18}\right)\right]$

- Combine solutions and rename parameters

  $\left[y = -\frac{\mathrm{I} a_0 \sqrt{6}\, (10x^3-60x^2+111x-62)}{72} + b_0 \left(\frac{5}{6} + \frac{\mathrm{I}(-2+x)\sqrt{6}}{18}\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve((11-8*x+2*x^2)*diff(y(x),x$2)-16*(x-2)*diff(y(x),x)+36*y(x)=0,y(x),type='series',x=2)
```

$$y(x) = \left(1-6(-2+x)^2+\frac{4(-2+x)^4}{3}\right)y(2)+\left(-2+x-\frac{10(-2+x)^3}{9}\right)D(y)(2)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[(11-8*x+2*x^2)*y''[x]-16*(x-2)*y'[x]+36*y[x]==0,y[x],{x,2,5}]
```

$$y(x) \to c_1\left(\frac{4}{3}(x-2)^4-6(x-2)^2+1\right)+c_2\left(-\frac{10}{9}(x-2)^3+x-2\right)$$

12.18 problem 20

Internal problem ID [1222]
Internal file name [`OUTPUT/1223_Sunday_June_05_2022_02_05_36_AM_94614651/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(3x^2+6x+5)\,y''+9(x+1)\,y'+3y=0}$$

With the expansion point for the power series method at $x=-1$.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x+1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(3(-1+t)^2-1+6t\right)\left(\frac{d^2}{dt^2}y(t)\right)+9t\left(\frac{d}{dt}y(t)\right)+3y(t)=0$$

With its expansion point and initial conditions now at $t=0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} & (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' & (869) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f & (870) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f & (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f & (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3\left(3t\left(\frac{d}{dt}y(t)\right) + y(t)\right)}{3t^2 + 2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{99\left(\frac{d}{dt}y(t)\right)t^2 + 45y(t)\,t - 24\frac{d}{dt}y(t)}{\left(3t^2 + 2\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= -\frac{18\left(75\left(\frac{d}{dt}y(t)\right)t^3 + 39y(t)\,t^2 - 55t\left(\frac{d}{dt}y(t)\right) - 9y(t)\right)}{\left(3t^2 + 2\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{\left(22194t^4 - 32778t^2 + 2304\right)\left(\frac{d}{dt}y(t)\right) + \left(12474t^3 - 8694t\right)y(t)}{\left(3t^2 + 2\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{\left(-428652t^5 + 1061424t^3 - 224532t\right)\left(\frac{d}{dt}y(t)\right) - 253692y(t)\left(t^4 - \frac{122}{87}t^2 + \frac{25}{261}\right)}{\left(3t^2 + 2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2} \\
F_1 &= -6y'(0) \\
F_2 &= \frac{81y(0)}{4} \\
F_3 &= 144y'(0) \\
F_4 &= -\frac{6075y(0)}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - \frac{3}{4}t^2 + \frac{27}{32}t^4 - \frac{135}{128}t^6\right) y(0) + \left(t - t^3 + \frac{6}{5}t^5\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(3t^2 + 2) + 9t\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(3t^2 + 2) + 9t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 9n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 3t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 9n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

$n = 1$ gives

$$12a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -a_1$$

For $2 \leq n$, the recurrence equation is

$$3n a_n(n-1) + 2(n+2)\, a_{n+2}(n+1) + 9n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{3(n+1)\, a_n}{2\,(n+2)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$27a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{27a_0}{32}$$

For $n = 3$ the recurrence equation gives

$$48a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{6a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$75a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{135a_0}{128}$$

For $n = 5$ the recurrence equation gives

$$108a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{54a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{3}{4} a_0 t^2 - a_1 t^3 + \frac{27}{32} a_0 t^4 + \frac{6}{5} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{3}{4} t^2 + \frac{27}{32} t^4\right) a_0 + \left(t - t^3 + \frac{6}{5} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{3}{4}t^2 + \frac{27}{32}t^4\right) c_1 + \left(t - t^3 + \frac{6}{5}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$y = \left(1 - \frac{3(x+1)^2}{4} + \frac{27(x+1)^4}{32} - \frac{135(x+1)^6}{128}\right) y(-1) + \left(x + 1 - (x+1)^3 + \frac{6(x+1)^5}{5}\right) y'(-1) + O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{3(x+1)^2}{4} + \frac{27(x+1)^4}{32} - \frac{135(x+1)^6}{128}\right) y(-1) + \left(x + 1 - (x+1)^3 + \frac{6(x+1)^5}{5}\right) y'(-1) + O\left((x+1)^6\right) \tag{1}$$

Verification of solutions

$$y = \left(1 - \frac{3(x+1)^2}{4} + \frac{27(x+1)^4}{32} - \frac{135(x+1)^6}{128}\right) y(-1) + \left(x + 1 - (x+1)^3 + \frac{6(x+1)^5}{5}\right) y'(-1) + O\left((x+1)^6\right)$$

Verified OK.

12.18.1 Maple step by step solution

Let's solve

$(3x^2 + 6x + 5) y'' + (9x + 9) y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{3x^2+6x+5} - \frac{9(x+1)y'}{3x^2+6x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{9(x+1)y'}{3x^2+6x+5} + \frac{3y}{3x^2+6x+5} = 0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{9(x+1)}{3x^2+6x+5}, P_3(x) = \frac{3}{3x^2+6x+5}\right]$

  ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right) \cdot P_2(x)$ is analytic at $x = -1-\frac{\mathrm{I}\sqrt{6}}{3}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right) \cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{3}} = 0$

  ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = -1-\frac{\mathrm{I}\sqrt{6}}{3}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{3}} = 0$

  ○ $x = -1-\frac{\mathrm{I}\sqrt{6}}{3}$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1-\frac{\mathrm{I}\sqrt{6}}{3}$

- Multiply by denominators

  $(3x^2+6x+5)\, y'' + (9x+9)\, y' + 3y = 0$

- Change variables using $x = u-1-\frac{\mathrm{I}\sqrt{6}}{3}$ so that the regular singular point is at $u = 0$

  $(3u^2 - 2\,\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right) + (9u - 3\,\mathrm{I}\sqrt{6})\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum\limits_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum\limits_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-\mathrm{I}\sqrt{6}\,(2r+1)\, r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-\mathrm{I}\sqrt{6}\,(2k+3+2r)(k+r+1)\, a_{k+1} + 3a_k(k+r+1)^2\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\mathrm{I}\sqrt{6}\,(2r+1)\, r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(\mathrm{I}\left(k+r+\frac{3}{2}\right) a_{k+1}\sqrt{6} - \frac{3a_k(k+r+1)}{2}\right)(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k(k+r+1)\sqrt{6}}{2k+3+2r}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k(k+1)\sqrt{6}}{2k+3}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k(k+1)\sqrt{6}}{2k+3}\right]$$

- Revert the change of variables $u = x+1+\frac{\mathrm{I}\sqrt{6}}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k(k+1)\sqrt{6}}{2k+3}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k\left(k+\frac{1}{2}\right)\sqrt{6}}{2k+2}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{-\frac{\mathrm{I}}{2}a_k\left(k+\frac{1}{2}\right)\sqrt{6}}{2k+2}\right]$$

- Revert the change of variables $u = x+1+\frac{\mathrm{I}\sqrt{6}}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k-\frac{1}{2}}, a_{k+1}=\frac{-\frac{\mathrm{I}}{2} a_k \left(k+\frac{1}{2}\right)\sqrt{6}}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k-\frac{1}{2}}\right), a_{1+k}=\frac{-\frac{\mathrm{I}}{2} a_k (1+k)\sqrt{6}}{2k+3}, b_{1+k}=\frac{-\frac{\mathrm{I}}{2} b_k (k+\dots}{2k+\dots}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve((5+6*x+3*x^2)*diff(y(x),x$2)+9*(x+1)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=-1);
```

$$y(x)=\left(1-\frac{3(x+1)^2}{4}+\frac{27(x+1)^4}{32}\right)y(-1)+\left(x+1-(x+1)^3+\frac{6(x+1)^5}{5}\right)D(y)(-1)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(5+6*x+2*x^2)*y''[x]+9*(x+1)*y'[x]+3*y[x]==0,y[x],{x,-1,5}]
```

$$y(x)\to c_1\left(-\frac{93}{20}(x+1)^5+\frac{17}{8}(x+1)^4+(x+1)^3-\frac{3}{2}(x+1)^2+1\right)+c_2\left(\frac{9}{5}(x+1)^5+2(x+1)^4-2(x+1)^3+x+1\right)$$

12.19 problem 21

Internal problem ID [1223]
Internal file name [OUTPUT/1224_Sunday_June_05_2022_02_05_38_AM_55005998/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(x^2-4\right) y''-y'x-3y=0}$$

With initial conditions

$$[y(0)=-1, y'(0)=2]$$

With the expansion point for the power series method at $x=0$.

12.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x}{x^2-4} \\
q(x) &= -\frac{3}{x^2-4} \\
F &= 0
\end{aligned}$$

Hence the ode is

$$y'' - \frac{xy'}{x^2 - 4} - \frac{3y}{x^2 - 4} = 0$$

The domain of $p(x) = -\frac{x}{x^2-4}$ is

$$\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{3}{x^2-4}$ is

$$\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{872}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{873}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3y+y'x}{x^2-4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{3y'x^2-3yx-16y'}{(x^2-4)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{-6x^3y'+18x^2y+36y'x-36y}{(x^2-4)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{30x((x^3-6x)\,y'+(-3x^2+6)\,y)}{(x^2-4)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\frac{180\left(x^2+\frac{2}{3}\right)((x^3-6x)\,y'+(-3x^2+6)\,y)}{(x^2-4)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=-1$ and $y'(0)=2$ gives

$$
\begin{aligned}
F_0 &= \frac{3}{4} \\
F_1 &= -2 \\
F_2 &= -\frac{9}{16} \\
F_3 &= 0 \\
F_4 &= -\frac{45}{64}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 2x - 1 + \frac{3x^2}{8} - \frac{x^3}{3} - \frac{3x^4}{128} - \frac{x^6}{1024} + O(x^6)$$

$$y = 2x - 1 + \frac{3x^2}{8} - \frac{x^3}{3} - \frac{3x^4}{128} - \frac{x^6}{1024} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 - 4)\, y'' - y'x - 3y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 - 4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-4n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \sum_{n=0}^{\infty} (-3a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-4n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-4(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-8a_2 - 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{8}$$

$n = 1$ gives

$$-24a_3 - 4a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - 4(n+2)\, a_{n+2}(n+1) - n a_n - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{(n-3)\, a_n}{4n+8} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-3a_2 - 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{128}$$

For $n = 3$ the recurrence equation gives

$$-80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$5a_4 - 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{1024}$$

For $n = 5$ the recurrence equation gives

$$12a_5 - 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{8} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{3}{128} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{8}x^2 + \frac{3}{128}x^4\right) a_0 + \left(x - \frac{1}{6}x^3\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{8}x^2 + \frac{3}{128}x^4\right)c_1 + \left(x - \frac{1}{6}x^3\right)c_2 + O(x^6)$$

$$y = -1 + \frac{3x^2}{8} - \frac{3x^4}{128} + 2x - \frac{x^3}{3} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 2x - 1 + \frac{3x^2}{8} - \frac{x^3}{3} - \frac{3x^4}{128} - \frac{x^6}{1024} + O(x^6) \tag{1}$$

$$y = -1 + \frac{3x^2}{8} - \frac{3x^4}{128} + 2x - \frac{x^3}{3} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 2x - 1 + \frac{3x^2}{8} - \frac{x^3}{3} - \frac{3x^4}{128} - \frac{x^6}{1024} + O(x^6)$$

Verified OK.

$$y = -1 + \frac{3x^2}{8} - \frac{3x^4}{128} + 2x - \frac{x^3}{3} + O(x^6)$$

Verified OK.

12.19.2 Maple step by step solution

Let's solve

$$\left[(x^2 - 4)\,y'' - y'x - 3y = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{xy'}{x^2-4} + \frac{3y}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x^2-4} - \frac{3y}{x^2-4} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x}{x^2-4}, P_3(x) = -\frac{3}{x^2-4}\right]$

- $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2} = -\frac{1}{2}$

- $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

- $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $(x^2-4)\,y'' - y'x - 3y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d^2}{du^2}y(u)\right) + (-u+2)\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+r+1)\,(2k-1+2r)+a_k(k+r+1)\,(k+r-3)\right)u^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $((-4k-4r+2)\,a_{k+1}+a_k(k+r-3))\,(k+r+1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-3)}{2(2k-1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=3$

  $a_{k+1}=\frac{a_k(k-3)}{2(2k-1)}$
- Apply recursion relation for $k=0$

  $a_1=\frac{3a_0}{2}$
- Apply recursion relation for $k=1$

  $a_2=-a_1$
- Express in terms of a_0

  $a_2=-\frac{3a_0}{2}$
- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{6}$
- Express in terms of a_0

  $a_3=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1+\frac{3}{2}u-\frac{3}{2}u^2+\frac{1}{4}u^3\right)$
- Revert the change of variables $u=2+x$

  $\left[y=a_0\left(-\frac{3}{2}x+\frac{1}{4}x^3\right)\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{a_k\left(k-\frac{3}{2}\right)}{2(2k+2)}$

- Solution for $r=\frac{3}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{a_k\left(k-\frac{3}{2}\right)}{2(2k+2)}\right]$

- Revert the change of variables $u=2+x$

  $\left[y=\sum_{k=0}^{\infty} a_k(2+x)^{k+\frac{3}{2}}, a_{k+1}=\frac{a_k\left(k-\frac{3}{2}\right)}{2(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y=a_0\left(-\frac{3}{2}x+\frac{1}{4}x^3\right)+\left(\sum_{k=0}^{\infty} b_k(2+x)^{k+\frac{3}{2}}\right), b_{1+k}=\frac{b_k\left(k-\frac{3}{2}\right)}{2(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(x^2-4)*diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=0,y(0) = -1, D(y)(0) = 2],y(x),type='se
```

$$y(x)=-1+2x+\frac{3}{8}x^2-\frac{1}{3}x^3-\frac{3}{128}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 29

```
AsymptoticDSolveValue[{(x^2-4)*y''[x]-x*y'[x]-3*y[x]==0,{y[0]==-1,y'[0]==2}},y[x],{x,0,5}]
```

$$y(x)\rightarrow -\frac{3x^4}{128}-\frac{x^3}{3}+\frac{3x^2}{8}+2x-1$$

12.20 problem 22

Internal problem ID [1224]
Internal file name [`OUTPUT/1225_Sunday_June_05_2022_02_05_40_AM_69064895/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x-3)\,y' + 3y = 0}$$

With initial conditions

$$[y(3) = -2, y'(3) = 3]$$

With the expansion point for the power series method at $x = 3$.

12.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x - 3 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (x - 3)\, y' + 3y = 0$$

The domain of $p(x) = x - 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + t\left(\frac{d}{dt}y(t)\right) + 3y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = -2$$
$$y'(0) = 3$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (875) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (876) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -t\left(\frac{d}{dt}y(t)\right) - 3y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\frac{d}{dt}y(t)\right)t^2 + 3y(t)\,t - 4\frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= -\left(\frac{d}{dt}y(t)\right)t^3 - 3y(t)\,t^2 + 9t\left(\frac{d}{dt}y(t)\right) + 15y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(t^4 - 15t^2 + 24\right)\left(\frac{d}{dt}y(t)\right) + 3ty(t)\left(t^2 - 11\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(-t^5 + 22t^3 - 87t\right)\left(\frac{d}{dt}y(t)\right) - 3y(t)\left(t^4 - 18t^2 + 35\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = -2$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= 6 \\
F_1 &= -12 \\
F_2 &= -30 \\
F_3 &= 72 \\
F_4 &= 210
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = -2t^3 + 3t^2 + 3t - 2 - \frac{5t^4}{4} + \frac{3t^5}{5} + \frac{7t^6}{24} + O\left(t^6\right)
$$

$$y(t) = -2t^3 + 3t^2 + 3t - 2 - \frac{5t^4}{4} + \frac{3t^5}{5} + \frac{7t^6}{24} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, t^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} n\, t^n a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n+3)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{3}{2}a_0 t^2 - \frac{2}{3}a_1 t^3 + \frac{5}{8}a_0 t^4 + \frac{1}{5}a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{5}{8}t^4\right) a_0 + \left(t - \frac{2}{3}t^3 + \frac{1}{5}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{5}{8}t^4\right) c_1 + \left(t - \frac{2}{3}t^3 + \frac{1}{5}t^5\right) c_2 + O(t^6)$$

$$y(t) = -2 + 3t^2 - \frac{5t^4}{4} + 3t - 2t^3 + \frac{3t^5}{5} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 3$ results in

$$y = -2(x-3)^3 + 3(x-3)^2 + 3x - 11 - \frac{5(x-3)^4}{4} + \frac{3(x-3)^5}{5} + \frac{7(x-3)^6}{24} + O\left((x-3)^6\right)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-2(x-3)^3 + 3(x-3)^2 + 3x - 11 - \frac{5(x-3)^4}{4} \\
&+ \frac{3(x-3)^5}{5} + \frac{7(x-3)^6}{24} + O\left((x-3)^6\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = -2(x-3)^3 + 3(x-3)^2 + 3x - 11 - \frac{5(x-3)^4}{4} + \frac{3(x-3)^5}{5} + \frac{7(x-3)^6}{24} + O\left((x-3)^6\right)
$$

Verified OK.

12.20.2 Maple step by step solution

Let's solve

$\left[y'' + (x-3)\, y' + 3y = 0, y(3) = -2, y'\Big|_{\{x=3\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ∘ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ∘ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ∘ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\left(k+1\right)-3a_{k+1}(k+1)+a_k(k+3)\right)x^k=0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2a_{k+2}+\left(a_k-3a_{k+1}+3a_{k+2}\right)k+3a_k-3a_{k+1}+2a_{k+2}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k k-3a_{k+1}k+3a_k-3a_{k+1}}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+(x-3)*diff(y(x),x)+3*y(x)=0,y(3) = -2, D(y)(3) = 3],y(x),type='series
```

$$y(x) = -2 + 3(x-3) + 3(x-3)^2 - 2(x-3)^3 - \frac{5}{4}(x-3)^4 + \frac{3}{5}(x-3)^5 + \mathrm{O}\left((x-3)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[{y''[x]+(x-3)*y'[x]+3*y[x]==0,{y[3]==-2,y'[3]==3}},y[x],{x,3,5}]
```

$$y(x) \to \frac{3}{5}(x-3)^5 - \frac{5}{4}(x-3)^4 - 2(x-3)^3 + 3(x-3)^2 + 3(x-3) - 2$$

12.21 problem 23

Internal problem ID [1225]
Internal file name [`OUTPUT/1226_Sunday_June_05_2022_02_05_43_AM_10674045/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2-6x+5)\,y''+(x-1)\,y'+12y=0}$$

With initial conditions

$$[y(1)=-1, y'(1)=1]$$

With the expansion point for the power series method at $x=1$.

12.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x-1}{3x^2-6x+5}\\ q(x)&=\frac{12}{3x^2-6x+5}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(x-1)\,y'}{3x^2-6x+5} + \frac{12y}{3x^2-6x+5} = 0$$

The domain of $p(x) = \frac{x-1}{3x^2-6x+5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{12}{3x^2-6x+5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(3(t+1)^2 - 6t - 1\right)\left(\frac{d^2}{dt^2}y(t)\right) + t\left(\frac{d}{dt}y(t)\right) + 12y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= -1 \\ y'(0) &= 1 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (878) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (879) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right) + 12y(t)}{3t^2 + 2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{-32\left(\frac{d}{dt}y(t)\right)t^2 + 84y(t)\,t - 26\frac{d}{dt}y(t)}{\left(3t^2 + 2\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{476\left(\frac{d}{dt}y(t)\right)t^3 - 372y(t)\,t^2 + 378t\left(\frac{d}{dt}y(t)\right) + 480y(t)}{\left(3t^2 + 2\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(-5876t^4 - 2496t^2 + 1716\right)\left(\frac{d}{dt}y(t)\right) - 1248ty(t)\left(t^2 + \frac{47}{4}\right)}{\left(3t^2 + 2\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(72644t^5 - 46072t^3 - 82212t\right)\left(\frac{d}{dt}y(t)\right) + \left(89232t^4 + 330408t^2 - 49920\right)y(t)}{\left(3t^2 + 2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = -1$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 6 \\
F_1 &= -\frac{13}{2} \\
F_2 &= -60 \\
F_3 &= \frac{429}{4} \\
F_4 &= 1560
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 3t^2 + t - 1 - \frac{13t^3}{12} - \frac{5t^4}{2} + \frac{143t^5}{160} + \frac{13t^6}{6} + O(t^6)$$

$$y(t) = 3t^2 + t - 1 - \frac{13t^3}{12} - \frac{5t^4}{2} + \frac{143t^5}{160} + \frac{13t^6}{6} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(3t^2+2) + t\left(\frac{d}{dt}y(t)\right) + 12y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(3t^2+2) + t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 12\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(n+1)\,t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 3t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 12a_0 = 0$$

$$a_2 = -3a_0$$

$n = 1$ gives

$$12a_3 + 13a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{13a_1}{12}$$

For $2 \le n$, the recurrence equation is

$$3na_n(n-1) + 2(n+2)\, a_{n+2}(n+1) + na_n + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(3n^2 - 2n + 12)}{2\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$20a_2 + 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$33a_3 + 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{143a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$52a_4 + 60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$77a_5 + 84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1573a_1}{1920}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - 3a_0 t^2 - \frac{13}{12} a_1 t^3 + \frac{5}{2} a_0 t^4 + \frac{143}{160} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - 3t^2 + \frac{5}{2}t^4\right) a_0 + \left(t - \frac{13}{12}t^3 + \frac{143}{160}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - 3t^2 + \frac{5}{2}t^4\right)c_1 + \left(t - \frac{13}{12}t^3 + \frac{143}{160}t^5\right)c_2 + O(t^6)$$

$$y(t) = -1 + 3t^2 - \frac{5t^4}{2} + t - \frac{13t^3}{12} + \frac{143t^5}{160} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = 3(x-1)^2 + x - 2 - \frac{13(x-1)^3}{12} - \frac{5(x-1)^4}{2} + \frac{143(x-1)^5}{160} + \frac{13(x-1)^6}{6} + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = 3(x-1)^2 + x - 2 - \frac{13(x-1)^3}{12} - \frac{5(x-1)^4}{2} + \frac{143(x-1)^5}{160} + \frac{13(x-1)^6}{6} + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = 3(x-1)^2 + x - 2 - \frac{13(x-1)^3}{12} - \frac{5(x-1)^4}{2} + \frac{143(x-1)^5}{160} + \frac{13(x-1)^6}{6} + O\left((x-1)^6\right)$$

Verified OK.

12.21.2 Maple step by step solution

Let's solve

$$\left[(3x^2 - 6x + 5)y'' + (x-1)y' + 12y = 0, y(1) = -1, y'\Big|_{\{x=1\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y}{3x^2-6x+5} - \frac{(x-1)y'}{3x^2-6x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x-1)y'}{3x^2-6x+5} + \frac{12y}{3x^2-6x+5} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x-1}{3x^2-6x+5}, P_3(x) = \frac{12}{3x^2-6x+5}\right]$

- $\left(x - 1 + \frac{I\sqrt{6}}{3}\right) \cdot P_2(x)$ is analytic at $x = 1 - \frac{I\sqrt{6}}{3}$

  $\left(\left(x - 1 + \frac{I\sqrt{6}}{3}\right) \cdot P_2(x)\right)\bigg|_{x=1-\frac{I\sqrt{6}}{3}} = 0$

- $\left(x - 1 + \frac{I\sqrt{6}}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = 1 - \frac{I\sqrt{6}}{3}$

  $\left(\left(x - 1 + \frac{I\sqrt{6}}{3}\right)^2 \cdot P_3(x)\right)\bigg|_{x=1-\frac{I\sqrt{6}}{3}} = 0$

- $x = 1 - \frac{I\sqrt{6}}{3}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 1 - \frac{I\sqrt{6}}{3}$

- Multiply by denominators

  $(3x^2 - 6x + 5)\, y'' + (x - 1)\, y' + 12y = 0$

- Change variables using $x = u + 1 - \frac{I\sqrt{6}}{3}$ so that the regular singular point is at $u = 0$

  $(3u^2 - 2\,\mathrm{I}u\sqrt{6}) \left(\frac{d^2}{du^2}y(u)\right) + \left(u - \frac{I\sqrt{6}}{3}\right) \left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-\frac{\mathrm{I}\sqrt{6}\,r(6r-5)a_{0}u^{-1+r}}{3}+\left(\sum_{k=0}^{\infty}\left(-\frac{\mathrm{I}\sqrt{6}\,(k+1+r)(6k+1+6r)a_{k+1}}{3}+a_{k}(3k^{2}+6kr+3r^{2}-2k-2r+12)\right)u^{k+r}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-\frac{\mathrm{I}}{3}\sqrt{6}\,r(6r-5)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,\frac{5}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\,\mathrm{I}\left(k+r+\frac{1}{6}\right)(k+1+r)\,a_{k+1}\sqrt{6}+3\left(k^{2}+\left(2r-\frac{2}{3}\right)k+r^{2}-\frac{2r}{3}+4\right)a_{k}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}+6kr+3r^{2}-2k-2r+12\right)\sqrt{6}}{6k^{2}+12kr+6r^{2}+7k+7r+1}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}-2k+12\right)\sqrt{6}}{6k^{2}+7k+1}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}-2k+12\right)\sqrt{6}}{6k^{2}+7k+1}\right]$$

- Revert the change of variables $u=x-1+\frac{\mathrm{I}\sqrt{6}}{3}$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}\left(x-1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k},a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}-2k+12\right)\sqrt{6}}{6k^{2}+7k+1}\right]$$

- Recursion relation for $r=\frac{5}{6}$

  $$a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}+3k+\frac{149}{12}\right)\sqrt{6}}{6k^{2}+17k+11}$$

- Solution for $r=\frac{5}{6}$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+\frac{5}{6}},a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}+3k+\frac{149}{12}\right)\sqrt{6}}{6k^{2}+17k+11}\right]$$

- Revert the change of variables $u=x-1+\frac{\mathrm{I}\sqrt{6}}{3}$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}\left(x-1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k+\frac{5}{6}},a_{k+1}=\frac{-\frac{\mathrm{I}}{2}a_{k}\left(3k^{2}+3k+\frac{149}{12}\right)\sqrt{6}}{6k^{2}+17k+11}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x-1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x-1+\frac{\mathrm{I}\sqrt{6}}{3}\right)^{k+\frac{5}{6}}\right), a_{1+k}=\frac{-\frac{\mathrm{I}}{2}a_k(3k^2-2k+12)\sqrt{6}}{6k^2+7k+1}, b_{1+k}=\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(5-6*x+3*x^2)*diff(y(x),x$2)+(x-1)*diff(y(x),x)+12*y(x)=0,y(1) = -1, D(y)(1) = 1],y(
```

$$y(x)=-1+(x-1)+3(x-1)^2-\frac{13}{12}(x-1)^3-\frac{5}{2}(x-1)^4+\frac{143}{160}(x-1)^5+\mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{(5-6*x+3*x^2)*y''[x]+(x-1)*y'[x]+12*y[x]==0,{y[1]==-1,y'[1]==1}},y[x]
```

$$y(x) \to \frac{143}{160}(x-1)^5 - \frac{5}{2}(x-1)^4 - \frac{13}{12}(x-1)^3 + 3(x-1)^2 + x - 2$$

12.22 problem 24

Internal problem ID [1226]
Internal file name [`OUTPUT/1227_Sunday_June_05_2022_02_05_45_AM_387783/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(4x^2 - 24x + 37)\, y'' + y = 0}$$

With initial conditions

$$[y(3) = 4, y'(3) = -6]$$

With the expansion point for the power series method at $x = 3$.

12.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{4x^2 - 24x + 37} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y}{4x^2 - 24x + 37} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is inside this domain. The domain of $q(x) = \frac{1}{4x^2-24x+37}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(4(t+3)^2 - 24t - 35\right)\left(\frac{d^2}{dt^2}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 4$$
$$y'(0) = -6$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (881) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (882) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= -\frac{y(t)}{4t^2+1}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \frac{-4\left(\frac{d}{dt}y(t)\right)t^2+8y(t)\,t-\frac{d}{dt}y(t)}{\left(4t^2+1\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \frac{16(4t^3+t)\left(\frac{d}{dt}y(t)\right)+\left(-92t^2+9\right)y(t)}{\left(4t^2+1\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \frac{\left(-1136t^4-184t^2+25\right)\left(\frac{d}{dt}y(t)\right)+\left(1408t^3-416t\right)y(t)}{\left(4t^2+1\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \frac{\left(23808t^5-384t^3-1584t\right)\left(\frac{d}{dt}y(t)\right)+\left(-27024t^4+16056t^2-441\right)y(t)}{\left(4t^2+1\right)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=4$ and $y'(0)=-6$ gives

$$\begin{aligned}
F_0 &= -4\\
F_1 &= 6\\
F_2 &= 36\\
F_3 &= -150\\
F_4 &= -1764
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = t^3 - 2t^2 - 6t + 4 + \frac{3t^4}{2} - \frac{5t^5}{4} - \frac{49t^6}{20} + O(t^6)$$

$$y(t) = t^3 - 2t^2 - 6t + 4 + \frac{3t^4}{2} - \frac{5t^5}{4} - \frac{49t^6}{20} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(4t^2 + 1) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(4t^2 + 1) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 4t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$4na_n(n-1) + (n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(4n^2 - 4n + 1)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$9a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$25a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$49a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{49a_0}{80}$$

For $n = 5$ the recurrence equation gives

$$81a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{45a_1}{112}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{1}{2} a_0 t^2 - \frac{1}{6} a_1 t^3 + \frac{3}{8} a_0 t^4 + \frac{5}{24} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4\right) a_0 + \left(t - \frac{1}{6}t^3 + \frac{5}{24}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4\right) c_1 + \left(t - \frac{1}{6}t^3 + \frac{5}{24}t^5\right) c_2 + O(t^6)$$

$$y(t) = 4 - 2t^2 + \frac{3t^4}{2} - 6t + t^3 - \frac{5t^5}{4} + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x - 3$ results in

$$y = (x-3)^3 - 2(x-3)^2 - 6x + 22 + \frac{3(x-3)^4}{2} - \frac{5(x-3)^5}{4} - \frac{49(x-3)^6}{20} + O\left((x-3)^6\right)$$

Summary

The solution(s) found are the following

$$y = (x-3)^3 - 2(x-3)^2 - 6x + 22 + \frac{3(x-3)^4}{2} - \frac{5(x-3)^5}{4} - \frac{49(x-3)^6}{20} + O\left((x-3)^6\right) \tag{1}$$

Verification of solutions

$$y = (x-3)^3 - 2(x-3)^2 - 6x + 22 + \frac{3(x-3)^4}{2} - \frac{5(x-3)^5}{4} - \frac{49(x-3)^6}{20} + O\left((x-3)^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(4*x^2-24*x+37)*diff(y(x),x$2)+y(x)=0,y(3) = 4, D(y)(3) = -6],y(x),type='series',x=3
```

$$y(x) = 4 - 6(x-3) - 2(x-3)^2 + (x-3)^3 + \frac{3}{2}(x-3)^4 - \frac{5}{4}(x-3)^5 + \mathrm{O}\left((x-3)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{(4*x^2-24*x+37)*y''[x]+y[x]==0,{y[3]==4,y'[3]==-6}},y[x],{x,3,5}]
```

$$y(x) \to -\frac{5}{4}(x-3)^5 + \frac{3}{2}(x-3)^4 + (x-3)^3 - 2(x-3)^2 - 6(x-3) + 4$$

12.23 problem 25

Internal problem ID [1227]
Internal file name [`OUTPUT/1228_Sunday_June_05_2022_02_05_48_AM_82497750/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - 8x + 14)\, y'' - 8(x - 4)\, y' + 20y = 0}$$

With initial conditions

$$[y(4) = 3, y'(4) = -4]$$

With the expansion point for the power series method at $x = 4$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t + 4)^2 - 8t - 18\right) \left(\frac{d^2}{dt^2} y(t)\right) - 8t\left(\frac{d}{dt} y(t)\right) + 20y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$
\begin{aligned}
y(0) &= 3 \\
y'(0) &= -4
\end{aligned}
$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$
y'' = f(x, y, y')
$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (884) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (885) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8t\left(\frac{d}{dt}y(t)\right) - 20y(t)}{t^2 - 2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{36\left(\frac{d}{dt}y(t)\right)t^2 - 120y(t)\,t + 24\frac{d}{dt}y(t)}{\left(t^2 - 2\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{96\left(\frac{d}{dt}y(t)\right)t^3 - 360y(t)\,t^2 + 192t\left(\frac{d}{dt}y(t)\right) - 240y(t)}{\left(t^2 - 2\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{\left(120t^4 + 480t^2 + 96\right)\left(\frac{d}{dt}y(t)\right) - 480ty(t)\left(t^2 + 2\right)}{\left(t^2 - 2\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 3$ and $y'(0) = -4$ gives

$$
\begin{aligned}
F_0 &= 30 \\
F_1 &= -24 \\
F_2 &= 90 \\
F_3 &= -24 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = -4t^3 + 15t^2 - 4t + 3 + \frac{15t^4}{4} - \frac{t^5}{5} + O\left(t^6\right)
$$

$$y(t) = -4t^3 + 15t^2 - 4t + 3 + \frac{15t^4}{4} - \frac{t^5}{5} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 - 2) - 8t\left(\frac{d}{dt}y(t)\right) + 20y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 - 2) - 8t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 20\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty}\left(-2n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty}\left(-8n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 20 a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-2n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty}\left(-2(n+2)\, a_{n+2}(n+1)\, t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+2) a_{n+2}(n+1) t^n\right) + \sum_{n=1}^{\infty} \left(-8n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 20 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-4a_2 + 20a_0 = 0$$

$$a_2 = 5a_0$$

$n = 1$ gives

$$-12a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) - 2(n+2) a_{n+2}(n+1) - 8n a_n + 20a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - 9n + 20)}{2(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_2 - 24a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$2a_3 - 40a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$-60a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$-84a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + 5a_0 t^2 + a_1 t^3 + \frac{5}{4} a_0 t^4 + \frac{1}{20} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + 5t^2 + \frac{5}{4} t^4\right) a_0 + \left(t + t^3 + \frac{1}{20} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t=0$ the solution above becomes

$$y(t) = \left(1+5t^2+\frac{5}{4}t^4\right)c_1 + \left(t+t^3+\frac{1}{20}t^5\right)c_2 + O(t^6)$$

$$y(t) = -4t^3+15t^2-4t+3+\frac{15t^4}{4}-\frac{t^5}{5}+O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-4$ results in

$$y = -4(x-4)^3+15(x-4)^2-4x+19+\frac{15(x-4)^4}{4}-\frac{(x-4)^5}{5}+O\left((x-4)^6\right)$$

Summary

The solution(s) found are the following

$$y = -4(x-4)^3+15(x-4)^2-4x+19+\frac{15(x-4)^4}{4}-\frac{(x-4)^5}{5}+O\left((x-4)^6\right) \quad (1)$$

Verification of solutions

$$y = -4(x-4)^3+15(x-4)^2-4x+19+\frac{15(x-4)^4}{4}-\frac{(x-4)^5}{5}+O\left((x-4)^6\right)$$

Verified OK.

12.23.1 Maple step by step solution

Let's solve

$$\left[(x^2-8x+14)\,y''+(-8x+32)\,y'+20y=0, y(4)=3, y'\Big|_{\{x=4\}}=-4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{20y}{x^2-8x+14}+\frac{8(x-4)y'}{x^2-8x+14}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{8(x-4)y'}{x^2-8x+14}+\frac{20y}{x^2-8x+14}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{8(x-4)}{x^2-8x+14}, P_3(x) = \frac{20}{x^2-8x+14}\right]$
  - $(x-4+\sqrt{2}) \cdot P_2(x)$ is analytic at $x = 4-\sqrt{2}$

    $$\left((x-4+\sqrt{2}) \cdot P_2(x)\right)\bigg|_{x=4-\sqrt{2}} = 0$$
  - $(x-4+\sqrt{2})^2 \cdot P_3(x)$ is analytic at $x = 4-\sqrt{2}$

    $$\left((x-4+\sqrt{2})^2 \cdot P_3(x)\right)\bigg|_{x=4-\sqrt{2}} = 0$$
  - $x = 4-\sqrt{2}$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 4-\sqrt{2}$
- Multiply by denominators

  $$(x^2-8x+14)\, y'' + (-8x+32)\, y' + 20y = 0$$
- Change variables using $x = u+4-\sqrt{2}$ so that the regular singular point is at $u = 0$

  $$(u^2-2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right) + (-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right) + 20y(u) = 0$$
- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$
  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2\sqrt{2}\,r(r-5)\,a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)(k+r-4)\,a_{k+1} + a_k(k+r-4)(k+r-5)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\,r(r-5) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-4)\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-5)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{5a_0\sqrt{2}}{4}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{2}$

- Express in terms of a_0

  $a_2 = \frac{5a_0}{4}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2\sqrt{2}}{4}$

- Express in terms of a_0

  $a_3 = -\frac{5a_0\sqrt{2}}{16}$

- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3\sqrt{2}}{8}$

- Express in terms of a_0

  $a_4 = \frac{5a_0}{64}$

- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4\sqrt{2}}{20}$
- Express in terms of a_0

  $a_5 = -\frac{a_0\sqrt{2}}{256}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$
- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$
- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$
- Solution for $r = 5$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k\left(x - 4 + \sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k\left(x - 4 + \right.\right.\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(x^2-8*x+14)*diff(y(x),x$2)-8*(x-4)*diff(y(x),x)+20*y(x)=0,y(4) = 3, D(y)(4) = -4],y
```

$$y(x) = 3 - 4(x-4) + 15(x-4)^2 - 4(x-4)^3 + \frac{15}{4}(x-4)^4 - \frac{1}{5}(x-4)^5 + \mathrm{O}\left((x-4)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 44

```
AsymptoticDSolveValue[{(x^2-8*x+14)*y''[x]+8*(x-4)*y'[x]+20*y[x]==0,{y[4]==3,y'[4]==-4}},y[x
```

$$y(x) \to -\frac{35}{3}(x-4)^5 + \frac{95}{4}(x-4)^4 - \frac{28}{3}(x-4)^3 + 15(x-4)^2 - 4(x-4) + 3$$

12.24 problem 26

Internal problem ID [1228]
Internal file name [`OUTPUT/1229_Sunday_June_05_2022_02_05_51_AM_43820621/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+4x+5)\, y''-20(x+1)\, y'+60y=0}$$

With initial conditions

$$[y(-1)=3, y'(-1)=-3]$$

With the expansion point for the power series method at $x=-1$.

12.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{-20x-20}{2x^2+4x+5}\\ q(x)&=\frac{60}{2x^2+4x+5}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(-20x - 20)\, y'}{2x^2 + 4x + 5} + \frac{60y}{2x^2 + 4x + 5} = 0$$

The domain of $p(x) = \frac{-20x-20}{2x^2+4x+5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{60}{2x^2+4x+5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(-1+t)^2 + 1 + 4t\right)\left(\frac{d^2}{dt^2}y(t)\right) - 20t\left(\frac{d}{dt}y(t)\right) + 60y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 3 \\ y'(0) &= -3 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (887) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (888) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{20t\left(\frac{d}{dt}y(t)\right) - 60y(t)}{2t^2+3} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{240\left(\frac{d}{dt}y(t)\right)t^2 - 960y(t)\,t - 120\frac{d}{dt}y(t)}{\left(2t^2+3\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{1920\left(\frac{d}{dt}y(t)\right)t^3 - 8640y(t)\,t^2 - 2880t\left(\frac{d}{dt}y(t)\right) + 4320y(t)}{\left(2t^2+3\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{\left(9600t^4 - 28800t^2 + 4320\right)\left(\frac{d}{dt}y(t)\right) - 46080\left(t^2 - \frac{3}{2}\right)ty(t)}{\left(2t^2+3\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{\left(23040t^5 - 115200t^3 + 51840t\right)\left(\frac{d}{dt}y(t)\right) - 115200\left(t^4 - 3t^2 + \frac{9}{20}\right)y(t)}{\left(2t^2+3\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 3$ and $y'(0) = -3$ gives

$$
\begin{aligned}
F_0 &= -60 \\
F_1 &= 40 \\
F_2 &= 480 \\
F_3 &= -160 \\
F_4 &= -640
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -30t^2 - 3t + 3 + \frac{20t^3}{3} + 20t^4 - \frac{4t^5}{3} - \frac{8t^6}{9} + O(t^6)$$

$$y(t) = -30t^2 - 3t + 3 + \frac{20t^3}{3} + 20t^4 - \frac{4t^5}{3} - \frac{8t^6}{9} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2+3) - 20t\left(\frac{d}{dt}y(t)\right) + 60y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(2t^2+3) - 20t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 60\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} (-20n a_n t^n) + \left(\sum_{n=0}^{\infty} 60 a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, t^n\right) + \sum_{n=1}^{\infty} (-20 n a_n t^n) + \left(\sum_{n=0}^{\infty} 60 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 + 60a_0 = 0$$

$$a_2 = -10a_0$$

$n = 1$ gives

$$18a_3 + 40a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{20a_1}{9}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + 3(n+2)\, a_{n+2}(n+1) - 20na_n + 60a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(n^2 - 11n + 30)}{3\,(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$24a_2 + 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{20a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$12a_3 + 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{4a_1}{9}$$

For $n = 4$ the recurrence equation gives

$$4a_4 + 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{8a_0}{27}$$

For $n = 5$ the recurrence equation gives

$$126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - 10a_0 t^2 - \frac{20}{9} a_1 t^3 + \frac{20}{3} a_0 t^4 + \frac{4}{9} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - 10t^2 + \frac{20}{3} t^4\right) a_0 + \left(t - \frac{20}{9} t^3 + \frac{4}{9} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t=0$ the solution above becomes

$$y(t) = \left(1-10t^2+\frac{20}{3}t^4\right)c_1+\left(t-\frac{20}{9}t^3+\frac{4}{9}t^5\right)c_2+O\left(t^6\right)$$

$$y(t) = 20t^4-30t^2+3-3t+\frac{20t^3}{3}-\frac{4t^5}{3}+O\left(t^6\right)$$

Replacing t in the above with the original independent variable $x$susing $t=x+1$ results in

$$y = -30(x+1)^2-3x+\frac{20(x+1)^3}{3}+20(x+1)^4-\frac{4(x+1)^5}{3}-\frac{8(x+1)^6}{9}+O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$y = -30(x+1)^2-3x+\frac{20(x+1)^3}{3}+20(x+1)^4-\frac{4(x+1)^5}{3}-\frac{8(x+1)^6}{9}+O\left((x+1)^6\right) \tag{1}$$

Verification of solutions

$$y = -30(x+1)^2-3x+\frac{20(x+1)^3}{3}+20(x+1)^4-\frac{4(x+1)^5}{3}-\frac{8(x+1)^6}{9}+O\left((x+1)^6\right)$$

Verified OK.

12.24.2 Maple step by step solution

Let's solve

$$\left[\left(2x^2+4x+5\right)y''+(-20x-20)\,y'+60y=0, y(-1)=3, y'\Big|_{\{x=-1\}}=-3\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y''=-\frac{60y}{2x^2+4x+5}+\frac{20(x+1)y'}{2x^2+4x+5}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{20(x+1)y'}{2x^2+4x+5} + \frac{60y}{2x^2+4x+5} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{20(x+1)}{2x^2+4x+5}, P_3(x) = \frac{60}{2x^2+4x+5}\right]$
  - $\left(x+1+\frac{I\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = -1 - \frac{I\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{I\sqrt{6}}{2}\right) \cdot P_2(x)\right)\Bigg|_{x=-1-\frac{I\sqrt{6}}{2}} = 0$
  - $\left(x+1+\frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -1 - \frac{I\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{I\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-1-\frac{I\sqrt{6}}{2}} = 0$
  - $x = -1 - \frac{I\sqrt{6}}{2}$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1 - \frac{I\sqrt{6}}{2}$
- Multiply by denominators

  $(2x^2+4x+5)\, y'' + (-20x-20)\, y' + 60y = 0$
- Change variables using $x = u - 1 - \frac{I\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2 - 2\,\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right) + \left(-20u + 10\,\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right) + 60y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\,(r-6)\, r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+r-5)(k+1+r)\, a_{k+1} + 2a_k(k+r-5)(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\,(r-6)\, r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(\mathrm{I}(k+1+r)\, a_{k+1}\sqrt{6} - a_k(k+r-6)\right)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{5} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}, a_{k+1}=\frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{5} a_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^{k+6}\right), a_{1+k}=\frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{1+k}, b_{1+k}=\frac{-\frac{\mathrm{I}}{6}b_k k\sqrt{6}}{k+7}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2*x^2+4*x+5)*diff(y(x),x$2)-20*(x+1)*diff(y(x),x)+60*y(x)=0,y(-1) = 3, D(y)(-1) = -
```

$$y(x)=3-3(x+1)-30(x+1)^2+\frac{20}{3}(x+1)^3+20(x+1)^4-\frac{4}{3}(x+1)^5+\mathrm{O}\left((x+1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[{(2*x^2+4*x+5)*y''[x]-20*(x+1)*y'[x]+60*y[x]==0,{y[-1]==3,y'[-1]==-3}}
```

$$y(x)\to -\frac{4}{3}(x+1)^5+20(x+1)^4+\frac{20}{3}(x+1)^3-30(x+1)^2-3(x+1)+3$$

12.25 problem 27

Internal problem ID [1229]
Internal file name [`OUTPUT/1230_Sunday_June_05_2022_02_05_53_AM_57799692/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+1)\, y''+4y'x+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{890}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{891}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(y + 2y'x)}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18y'x^2 + 12yx - 6y'}{(x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-96x^3y' - 72x^2y + 96y'x + 24y}{(x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{120(5x^4 - 10x^2 + 1)\, y' + 480(x^3 - x)\, y}{(x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x^5 + 14400x^3 - 4320x)\, y' - 3600y\left(x^4 - 2x^2 + \frac{1}{5}\right)}{(x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -6y'(0) \\
F_2 &= 24y(0) \\
F_3 &= 120y'(0) \\
F_4 &= -720y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-x^6 + x^4 - x^2 + 1\right) y(0) + \left(x^5 - x^3 + x\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y''+4y'x+2y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^n$$

Then

$$\begin{aligned}y' &= \sum_{n=1}^{\infty}n a_n x^{n-1}\\ y'' &= \sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+4\left(\sum_{n=1}^{\infty}n a_n x^{n-1}\right)x+2\left(\sum_{n=0}^{\infty}a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty}4n a_n x^n\right)+\left(\sum_{n=0}^{\infty}2a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty}x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right)\\&+\left(\sum_{n=1}^{\infty}4n a_n x^n\right)+\left(\sum_{n=0}^{\infty}2a_n x^n\right)=0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -a_1$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(n+1) + 4na_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -a_n \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = a_0$$

For $n = 3$ the recurrence equation gives

$$20a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = a_1$$

For $n = 4$ the recurrence equation gives

$$30a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -a_0$$

For $n = 5$ the recurrence equation gives

$$42a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x^5 + a_0 x^4 - a_1 x^3 - a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^4 - x^2 + 1) a_0 + (x^5 - x^3 + x) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (-x^6 + x^4 - x^2 + 1) y(0) + (x^5 - x^3 + x) y'(0) + O(x^6) \tag{1}$$

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-x^6 + x^4 - x^2 + 1) y(0) + (x^5 - x^3 + x) y'(0) + O(x^6)$$

Verified OK.

$$y = (x^4 - x^2 + 1) c_1 + (x^5 - x^3 + x) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (x^4 - x^2 + 1) y(0) + (x^5 - x^3 + x) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 30

```
AsymptoticDSolveValue[(1+x^2)*y''[x]+4*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2(x^5 - x^3 + x) + c_1(x^4 - x^2 + 1)$$

12.26 problem 31

Internal problem ID [1230]
Internal file name [`OUTPUT/1231_Sunday_June_05_2022_02_05_55_AM_27204929/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x + 2\alpha y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{893}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{894}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2y'x - 2\alpha y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(4x^2 - 2\alpha + 2\right) y' - 4y\alpha x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(8x^3 - 8\alpha x + 12x\right) y' - 8\left(x^2 - \frac{\alpha}{2} + 1\right) \alpha y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 4\left(3 + \alpha^2 + 2\left(-3x^2 - 2\right) \alpha + 4x^4 + 12x^2\right) y' - 16\left(x^2 - \alpha + \frac{5}{2}\right) x\alpha y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 32\left(\frac{3\alpha^2}{4} + \left(-2x^2 - \frac{15}{4}\right) \alpha + x^4 + 5x^2 + \frac{15}{4}\right) xy' - 32\alpha y\left(2 + \frac{\alpha^2}{4} + \frac{3(-x^2 - 1)\alpha}{2} + x^4 + \frac{9x^2}{2}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0)\,\alpha \\
F_1 &= -2y'(0)\,\alpha + 2y'(0) \\
F_2 &= 4y(0)\,\alpha^2 - 8y(0)\,\alpha \\
F_3 &= 4y'(0)\,\alpha^2 - 16y'(0)\,\alpha + 12y'(0) \\
F_4 &= -8y(0)\,\alpha^3 + 48y(0)\,\alpha^2 - 64y(0)\,\alpha
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \alpha x^2 + \frac{1}{6}x^4\alpha^2 - \frac{1}{3}x^4\alpha - \frac{1}{90}x^6\alpha^3 + \frac{1}{15}x^6\alpha^2 - \frac{4}{45}x^6\alpha\right) y(0) \\
&+ \left(x - \frac{1}{3}x^3\alpha + \frac{1}{3}x^3 + \frac{1}{30}x^5\alpha^2 - \frac{2}{15}x^5\alpha + \frac{1}{10}x^5\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\alpha\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2\alpha a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2\alpha a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2\alpha a_0 + 2a_2 = 0$$

$$a_2 = -\alpha a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - 2na_n + 2\alpha a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_n(\alpha - n)}{(n+2)(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2\alpha a_1 - 2a_1 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{3}\alpha a_1 + \frac{1}{3}a_1$$

For $n = 2$ the recurrence equation gives

$$2\alpha a_2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{6}\alpha^2 a_0 - \frac{1}{3}\alpha a_0$$

For $n = 3$ the recurrence equation gives

$$2\alpha a_3 - 6a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{30}\alpha^2 a_1 - \frac{2}{15}\alpha a_1 + \frac{1}{10}a_1$$

For $n = 4$ the recurrence equation gives

$$2\alpha a_4 - 8a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{90}\alpha^3 a_0 + \frac{1}{15}\alpha^2 a_0 - \frac{4}{45}\alpha a_0$$

For $n = 5$ the recurrence equation gives

$$2\alpha a_5 - 10a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{630}\alpha^3 a_1 + \frac{1}{70}\alpha^2 a_1 - \frac{23}{630}\alpha a_1 + \frac{1}{42}a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x - \alpha a_0 x^2 + \left(-\frac{1}{3}\alpha a_1 + \frac{1}{3}a_1\right) x^3 \\ + \left(\frac{1}{6}\alpha^2 a_0 - \frac{1}{3}\alpha a_0\right) x^4 + \left(\frac{1}{30}\alpha^2 a_1 - \frac{2}{15}\alpha a_1 + \frac{1}{10}a_1\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \alpha x^2 + \left(\frac{1}{6}\alpha^2 - \frac{1}{3}\alpha\right) x^4\right) a_0 \\ + \left(x + \left(-\frac{\alpha}{3} + \frac{1}{3}\right) x^3 + \left(\frac{1}{30}\alpha^2 - \frac{2}{15}\alpha + \frac{1}{10}\right) x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \alpha x^2 + \left(\frac{1}{6}\alpha^2 - \frac{1}{3}\alpha\right) x^4\right) c_1 \\ + \left(x + \left(-\frac{\alpha}{3} + \frac{1}{3}\right) x^3 + \left(\frac{1}{30}\alpha^2 - \frac{2}{15}\alpha + \frac{1}{10}\right) x^5\right) c_2 + O(x^6) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1-\alpha \,x^{2}+\frac{1}{6}x^{4} \alpha^{2}-\frac{1}{3}x^{4} \alpha -\frac{1}{90}x^{6} \alpha^{3}+\frac{1}{15}x^{6} \alpha^{2}-\frac{4}{45}x^{6} \alpha \right) y(0) \\
&+\left(x -\frac{1}{3}x^{3} \alpha +\frac{1}{3}x^{3}+\frac{1}{30}x^{5} \alpha^{2}-\frac{2}{15}x^{5} \alpha +\frac{1}{10}x^{5}\right) y^{\prime}(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = &\left(1-\alpha \,x^{2}+\left(\frac{1}{6}\alpha^{2}-\frac{1}{3}\alpha \right) x^{4}\right) c_{1} \\
&+\left(x +\left(-\frac{\alpha}{3}+\frac{1}{3}\right) x^{3}+\left(\frac{1}{30}\alpha^{2}-\frac{2}{15}\alpha +\frac{1}{10}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1-\alpha \,x^{2}+\frac{1}{6}x^{4} \alpha^{2}-\frac{1}{3}x^{4} \alpha -\frac{1}{90}x^{6} \alpha^{3}+\frac{1}{15}x^{6} \alpha^{2}-\frac{4}{45}x^{6} \alpha \right) y(0) \\
&+\left(x -\frac{1}{3}x^{3} \alpha +\frac{1}{3}x^{3}+\frac{1}{30}x^{5} \alpha^{2}-\frac{2}{15}x^{5} \alpha +\frac{1}{10}x^{5}\right) y^{\prime}(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y = &\left(1-\alpha \,x^{2}+\left(\frac{1}{6}\alpha^{2}-\frac{1}{3}\alpha \right) x^{4}\right) c_{1} \\
&+\left(x +\left(-\frac{\alpha}{3}+\frac{1}{3}\right) x^{3}+\left(\frac{1}{30}\alpha^{2}-\frac{2}{15}\alpha +\frac{1}{10}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

12.26.1 Maple step by step solution

Let's solve

$y'' = 2y'x - 2\alpha y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x + 2\alpha y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$$

  Rewrite DE with series expansions

  $$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(\alpha - k)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - 2a_k(k - \alpha) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k(\alpha-k)}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 63

```
Order:=6;
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+2*alpha*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1-\alpha\,x^2+\frac{\alpha(\alpha-2)\,x^4}{6}\right)y(0) + \left(x-\frac{(\alpha-1)\,x^3}{3}+\frac{(\alpha^2-4\alpha+3)\,x^5}{30}\right)D(y)\,(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[y''[x]-2*x*y'[x]+2*\[Alpha]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{\alpha^2x^5}{30}-\frac{2\alpha x^5}{15}+\frac{x^5}{10}-\frac{\alpha x^3}{3}+\frac{x^3}{3}+x\right)+c_1\left(\frac{\alpha^2x^4}{6}-\frac{\alpha x^4}{3}-\alpha x^2+1\right)$$

12.27 problem 33

Internal problem ID [1231]
Internal file name [`OUTPUT/1232_Sunday_June_05_2022_02_05_56_AM_31970484/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (896) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (897) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x + y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 y + 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(y'x + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= yx^3 + 6y'x + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{6}\right) a_0 + \left(x + \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

12.27.1 Maple step by step solution

Let's solve

$y'' = yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_{k-1} = 0$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1+\frac{x^3}{6}\right)y(0)+\left(x+\frac{1}{12}x^4\right)D(y)(0)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^4}{12}+x\right)+c_1\left(\frac{x^3}{6}+1\right)$$

12.28 problem 34

Internal problem ID [1232]
Internal file name [`OUTPUT/1233_Sunday_June_05_2022_02_05_57_AM_40351558/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(-2x^{3}+1\right)y''-10y'x^{2}-8yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (899) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (900) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2x(5y'x + 4y)}{2x^3 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{104y'x^4 + 112yx^3 + 28y'x + 8y}{(2x^3 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{-1232y'x^6 - 1504yx^5 - 1072x^3y' - 656x^2y - 36y'}{(2x^3 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(16704x^8 + 31168x^5 + 4880x^2)\,y' + 21888xy\left(x^6 + \frac{395}{342}x^3 + \frac{25}{342}\right)}{(2x^3 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(-256896x^{10} - 852992x^7 - 324320x^4 - 11360x)\,y' + (-352512x^9 - 807040x^6 - 175360x^3 - 1600)\,y}{(2x^3 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 8y(0) \\
F_2 &= 36y'(0) \\
F_3 &= 0 \\
F_4 &= 1600y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{4}{3}x^3 + \frac{20}{9}x^6\right) y(0) + \left(x + \frac{3}{2}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-2x^3+1\right) y'' - 10y'x^2 - 8yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-2x^3+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 10\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 8\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-2n\, x^{1+n} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty}\left(-10n\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty}\left(-8x^{1+n} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-2n\, x^{1+n} a_n (n-1)\right) = \sum_{n=3}^{\infty}\left(-2(n-1)\, a_{n-1}(n-2)\, x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty}\left(-10n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty}\left(-10(n-1)\, a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty}\left(-8x^{1+n} a_n\right) = \sum_{n=1}^{\infty}\left(-8a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=3}^{\infty} \left(-2(n-1)\, a_{n-1}(n-2)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \sum_{n=2}^{\infty} \left(-10(n-1)\, a_{n-1} x^{n}\right) + \sum_{n=1}^{\infty} \left(-8 a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 - 8a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_0}{3}$$

$n = 2$ gives

$$12a_4 - 18a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{3a_1}{2}$$

For $3 \leq n$, the recurrence equation is

$$-2(n-1)\, a_{n-1}(n-2) + (n+2)\, a_{n+2}(1+n) - 10(n-1)\, a_{n-1} - 8a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2(1+n)\, a_{n-1}}{n+2} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$-32a_2 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$-50a_3 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{20a_0}{9}$$

For $n = 5$ the recurrence equation gives

$$-72a_4 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{18a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{4}{3}a_0x^3 + \frac{3}{2}a_1x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{4x^3}{3}\right)a_0 + \left(x + \frac{3}{2}x^4\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{4x^3}{3}\right)c_1 + \left(x + \frac{3}{2}x^4\right)c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{4}{3}x^3 + \frac{20}{9}x^6\right)y(0) + \left(x + \frac{3}{2}x^4\right)y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{4x^3}{3}\right)c_1 + \left(x + \frac{3}{2}x^4\right)c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{4}{3}x^3+\frac{20}{9}x^6\right)y(0)+\left(x+\frac{3}{2}x^4\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{4x^3}{3}\right)c_1+\left(x+\frac{3}{2}x^4\right)c_2+O(x^6)$$

Verified OK.

12.28.1 Maple step by step solution

Let's solve

$(-2x^3+1)\,y''-10y'x^2-8yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{10x^2y'}{2x^3-1}-\frac{8xy}{2x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{10x^2y'}{2x^3-1}+\frac{8xy}{2x^3-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{10x^2}{2x^3-1}, P_3(x)=\frac{8x}{2x^3-1}\right]$

  ◦ $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)$ is analytic at $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}}=0$

  ◦ $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2\cdot P_3(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}}=0$

  ◦ $x=-\frac{2^{\frac{2}{3}}}{4}-\frac{I\sqrt{3}\,2^{\frac{2}{3}}}{4}$ is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = -\frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\, 2^{\frac{2}{3}}}{4}$

- Multiply by denominators

$y''(2x^3-1)+10y'x^2+8yx=0$

- Change variables using $x = u - \frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\, 2^{\frac{2}{3}}}{4}$ so that the regular singular point is at $u = 0$

$\left(2u^3 - \frac{3u^2 2^{\frac{2}{3}}}{2} - \frac{3\,\mathrm{I}u^2\sqrt{3}\, 2^{\frac{2}{3}}}{2} - \frac{3u 2^{\frac{1}{3}}}{2} + \frac{3\,\mathrm{I}u\sqrt{3}\, 2^{\frac{1}{3}}}{2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(10u^2 - 5u2^{\frac{2}{3}} - 5\,\mathrm{I}u\sqrt{3}\, 2^{\frac{2}{3}} - \frac{5\, 2^{\frac{1}{3}}}{2} + \frac{5\,\mathrm{I}}{}\right.$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- > k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$\frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(3r+2)r a_0 u^{-1+r}}{2} + \left(\frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(5+3r)(r+1)a_1}{2} - \frac{2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(r+1)(4+3r)a_0}{2}\right)u^r + \left(\sum_{k=1}^{\infty}\left(\frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(}{}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(3r+2)r}{2} = 0$$

- Values of r that satisfy the indicial equation

$r \in \left\{0, -\frac{2}{3}\right\}$

- Each term must be 0

$$\frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(5+3r)(r+1)a_1}{2} - \frac{2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(r+1)(4+3r)a_0}{2} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{3(k+1+r)\left(-\left(k+r+\frac{5}{3}\right)a_{k+1}\left(\mathrm{I}\sqrt{3}-1\right)2^{\frac{1}{3}}+\left(k+\frac{4}{3}+r\right)a_k\left(1+\mathrm{I}\sqrt{3}\right)2^{\frac{2}{3}}-\frac{4a_{k-1}(k+1+r)}{3}\right)}{2} = 0$$

- Shift index using $k->k+1$

$$-\frac{3(k+r+2)\left(-\left(k+\frac{8}{3}+r\right)a_{k+2}\left(\mathrm{I}\sqrt{3}-1\right)2^{\frac{1}{3}}+\left(k+\frac{7}{3}+r\right)a_{k+1}\left(1+\mathrm{I}\sqrt{3}\right)2^{\frac{2}{3}}-\frac{4a_k(k+r+2)}{3}\right)}{2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}r\sqrt{3}+7\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+3a_{k+1}2^{\frac{2}{3}}r+7a_{k+1}2^{\frac{2}{3}}-4ka_k-4ra_k-8a_k\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}+3\,\mathrm{I}r\sqrt{3}+8\,\mathrm{I}\sqrt{3}-3k-3r-8\right)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+7\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+7a_{k+1}2^{\frac{2}{3}}-4ka_k-8a_k\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}-8+8\,\mathrm{I}\sqrt{3}-3k\right)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+7\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+7a_{k+1}2^{\frac{2}{3}}-4ka_k-8a_k\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}-8+8\,\mathrm{I}\sqrt{3}-3k\right)}, \frac{5\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)a_1}{2} - 2\right.$$

- Revert the change of variables $u = x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^k, a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+7\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+7a_{k+1}2^{\frac{2}{3}}-4ka_k-8a_k\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}-8+8\,\mathrm{I}\sqrt{3}-3k\right)}, \frac{5\,2^{\frac{1}{3}}\right.$$

- Recursion relation for $r = -\frac{2}{3}$

$$a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+5\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+5a_{k+1}2^{\frac{2}{3}}-4ka_k-\frac{16a_k}{3}\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}+6\,\mathrm{I}\sqrt{3}-3k-6\right)}$$

- Solution for $r = -\frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{2}{3}}, a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+5\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+5a_{k+1}2^{\frac{2}{3}}-4ka_k-\frac{16a_k}{3}\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}+6\,\mathrm{I}\sqrt{3}-3k-6\right)}, \frac{2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)a_1}{2} - \right.$$

- Revert the change of variables $u = x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k-\frac{2}{3}}, a_{k+2} = \frac{\left(3\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}k\sqrt{3}+5\,\mathrm{I}a_{k+1}2^{\frac{2}{3}}\sqrt{3}+3a_{k+1}2^{\frac{2}{3}}k+5a_{k+1}2^{\frac{2}{3}}-4ka_k-\frac{16a_k}{3}\right)2^{\frac{2}{3}}}{2\left(3\,\mathrm{I}k\sqrt{3}+6\,\mathrm{I}\sqrt{3}-3k-6\right)},\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k-\frac{2}{3}}\right), a_{k+2} = \frac{\left(3\,\mathrm{I}a_{1+k}2^{\frac{2}{3}}k\sqrt{3}+7\,\mathrm{I}a_{1+k}\right.}{2\left(\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((1-2*x^3)*diff(y(x),x$2)-10*x^2*diff(y(x),x)-8*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{4x^3}{3}\right) y(0) + \left(x + \frac{3}{2}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[(1-2*x^3)*y''[x]-10*x^2*y'[x]-8*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{3x^4}{2} + x\right) + c_1 \left(\frac{4x^3}{3} + 1\right)$$

12.29 problem 35

Internal problem ID [1233]
Internal file name [`OUTPUT/1234_Sunday_June_05_2022_02_05_58_AM_42837116/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3+1)\,y''+7y'x^2+9yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{902}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{903}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(7y'x+9y)}{x^3+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{47y'x^4 + 81yx^3 - 23y'x - 9y}{(x^3+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-342y'x^6 - 666yx^5 + 536x^3y' + 504x^2y - 32y'}{(x^3+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(2754x^8 - 9182x^5 + 2624x^2)\, y' + 5742\left(x^6 - \frac{59}{29}x^3 + \frac{72}{319}\right) xy}{(x^3+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-24552x^{10} + 144640x^7 - 100904x^4 + 6544x)\, y' - 53496\left(x^9 - \frac{3004}{743}x^6 + \frac{1175}{743}x^3 - \frac{18}{743}\right) y}{(x^3+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -9y(0) \\
F_2 &= -32y'(0) \\
F_3 &= 0 \\
F_4 &= 1296y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{3}{2}x^3 + \frac{9}{5}x^6\right) y(0) + \left(x - \frac{4}{3}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^3+1\right) y''+7y' x^2+9yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(x^3+1\right)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)+7\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2+9\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 7n\, x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} 9x^{1+n} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1) = \sum_{n=3}^{\infty} (n-1) a_{n-1} (n-2)\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 7n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 7(n-1) a_{n-1} x^n$$

$$\sum_{n=0}^{\infty} 9x^{1+n} a_n = \sum_{n=1}^{\infty} 9a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=3}^{\infty} (n-1)\, a_{n-1}(n-2)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) \\
+\left(\sum_{n=2}^{\infty} 7(n-1)\, a_{n-1} x^{n}\right)+\left(\sum_{n=1}^{\infty} 9 a_{n-1} x^{n}\right) = 0 \tag{3}
$$

$n = 1$ gives

$$
6a_3 + 9a_0 = 0
$$

Which after substituting earlier equations, simplifies to

$$
a_3 = -\frac{3a_0}{2}
$$

$n = 2$ gives

$$
12a_4 + 16a_1 = 0
$$

Which after substituting earlier equations, simplifies to

$$
a_4 = -\frac{4a_1}{3}
$$

For $3 \leq n$, the recurrence equation is

$$
(n-1)\, a_{n-1}(n-2) + (n+2)\, a_{n+2}(1+n) + 7(n-1)\, a_{n-1} + 9a_{n-1} = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
a_{n+2} = -\frac{(n+2)\, a_{n-1}}{1+n} \tag{5}
$$

For $n = 3$ the recurrence equation gives

$$
25a_2 + 20a_5 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = 0
$$

For $n = 4$ the recurrence equation gives

$$36a_3 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{9a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$49a_4 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{14a_1}{9}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2} a_0 x^3 - \frac{4}{3} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3x^3}{2}\right) a_0 + \left(x - \frac{4}{3}x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3x^3}{2}\right) c_1 + \left(x - \frac{4}{3}x^4\right) c_2 + O(x^6)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 - \frac{3}{2}x^3 + \frac{9}{5}x^6\right) y(0) + \left(x - \frac{4}{3}x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{3x^3}{2}\right) c_1 + \left(x - \frac{4}{3}x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{3}{2}x^{3}+\frac{9}{5}x^{6}\right) y(0)+\left(x-\frac{4}{3}x^{4}\right) y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{3x^{3}}{2}\right) c_{1}+\left(x-\frac{4}{3}x^{4}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

12.29.1 Maple step by step solution

Let's solve

$\left(x^{3}+1\right) y''+7y'x^{2}+9yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{7x^{2}y'}{x^{3}+1}-\frac{9xy}{x^{3}+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{7x^{2}y'}{x^{3}+1}+\frac{9xy}{x^{3}+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_{2}(x)=\frac{7x^{2}}{x^{3}+1}, P_{3}(x)=\frac{9x}{x^{3}+1}\right]$

  ◦ $(x+1)\cdot P_{2}(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_{2}(x)\right)\bigg|_{x=-1}=\frac{7}{3}$

  ◦ $(x+1)^{2}\cdot P_{3}(x)$ is analytic at $x=-1$

    $\left((x+1)^{2}\cdot P_{3}(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_{0}=-1$

- Multiply by denominators

$\left(x^{3}+1\right) y''+7 y' x^{2}+9 y x=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$\left(u^{3}-3 u^{2}+3 u\right)\left(\frac{d^{2}}{d u^{2}} y(u)\right)+\left(7 u^{2}-14 u+7\right)\left(\frac{d}{d u} y(u)\right)+(9 u-9) y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^{m} \cdot y(u)$ to series expansion for $m=0..1$

  $u^{m} \cdot y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r+m}$

  - Shift index using $k->k-m$

  $u^{m} \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)$ to series expansion for $m=0..2$

  $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r) u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $u^{m} \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$

  - Convert $u^{m} \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)$ to series expansion for $m=1..3$

  $u^{m} \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

  $u^{m} \cdot\left(\frac{d^{2}}{d u^{2}} y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$

Rewrite ODE with series expansions

$a_{0} r(4+3 r) u^{-1+r}+\left(a_{1}(1+r)(7+3 r)-a_{0}\left(3 r^{2}+11 r+9\right)\right) u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)(3 k+7\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$r(4+3 r)=0$

- Values of r that satisfy the indicial equation

$r \in\left\{0,-\frac{4}{3}\right\}$

- Each term must be 0

  $a_1(1+r)(7+3r) - a_0(3r^2+11r+9) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(3k+7+3r) - a_k(3k^2+6kr+3r^2+11k+11r+9) + a_{k-1}(k+2+r)^2 = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(3k+10+3r) - a_{k+1}\left(3(k+1)^2+6(k+1)r+3r^2+11k+20+11r\right) + a_k(k+$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+2kra_k-6kra_{k+1}+r^2a_k-3r^2a_{k+1}+6ka_k-17ka_{k+1}+6ra_k-17ra_{k+1}+9a_k-23a_{k+1}}{(k+2+r)(3k+10+3r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1-9a_0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1-9a_0 = 0\right]$
- Recursion relation for $r = -\frac{4}{3}$

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$
- Solution for $r = -\frac{4}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3} = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1+\frac{a_0}{3} = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k^2a_k-3k^2a_{1+k}+6ka_k-17ka_{1+k}+9a_k-23a_{1+k}}{(k+2)(3k+10)}, 7a_1\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((1+x^3)*diff(y(x),x$2)+7*x^2*diff(y(x),x)+9*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3x^3}{2}\right) y(0) + \left(x - \frac{4}{3}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[(1+x^3)*y''[x]+7*x^2*y'[x]+9*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{4x^4}{3}\right) + c_1\left(1 - \frac{3x^3}{2}\right)$$

12.30 problem 36

Internal problem ID [1234]
Internal file name [`OUTPUT/1235_Sunday_June_05_2022_02_06_00_AM_18016655/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(-2x^3+1\right)y''+6y'x^2+24yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (905) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (906) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= \frac{6x(y'x+4y)}{2x^3-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{36y'x+24y}{2x^3-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{120x^3y'+720x^2y-60y'}{(2x^3-1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{1440\left(x^3-\frac{1}{2}\right)x^2y' + \left(-2880x^4-2880x\right)y}{(2x^3-1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{4320(-2x^7-x^4+x)\,y' + 63360\left(x^6+\frac{7}{11}x^3+\frac{1}{22}\right)y}{(2x^3-1)^4}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= -24y(0) \\
F_2 &= -60y'(0) \\
F_3 &= 0 \\
F_4 &= 2880y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(4x^6-4x^3+1\right)y(0) + \left(x-\frac{5}{2}x^4\right)y'(0) + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-2x^3+1\right) y'' + 6y'x^2 + 24yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-2x^3+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 6\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + 24\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-2n\, x^{1+n} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 24x^{1+n} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-2n\, x^{1+n} a_n (n-1)\right) = \sum_{n=3}^{\infty}\left(-2(n-1)\, a_{n-1}(n-2)\, x^n\right)$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=1}^{\infty} 6n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 6(n-1)\, a_{n-1} x^n$$
$$\sum_{n=0}^{\infty} 24x^{1+n} a_n = \sum_{n=1}^{\infty} 24 a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=3}^{\infty} \left(-2(n-1)\, a_{n-1}(n-2)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \left(\sum_{n=2}^{\infty} 6(n-1)\, a_{n-1} x^{n}\right) + \left(\sum_{n=1}^{\infty} 24 a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + 24a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -4a_0$$

$n = 2$ gives

$$12a_4 + 30a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{5a_1}{2}$$

For $3 \le n$, the recurrence equation is

$$-2(n-1)\, a_{n-1}(n-2) + (n+2)\, a_{n+2}(1+n) + 6(n-1)\, a_{n-1} + 24a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{2(n-7)\, a_{n-1}}{n+2} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$32a_2 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_3 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 4a_0$$

For $n = 5$ the recurrence equation gives

$$24a_4 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{10a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 4a_0 x^3 - \frac{5}{2} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-4x^3 + 1\right) a_0 + \left(x - \frac{5}{2}x^4\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-4x^3 + 1\right) c_1 + \left(x - \frac{5}{2}x^4\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(4x^6 - 4x^3 + 1\right) y(0) + \left(x - \frac{5}{2}x^4\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(-4x^3 + 1\right) c_1 + \left(x - \frac{5}{2}x^4\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(4x^6 - 4x^3 + 1\right) y(0) + \left(x - \frac{5}{2}x^4\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(-4x^3 + 1\right) c_1 + \left(x - \frac{5}{2}x^4\right) c_2 + O\left(x^6\right)$$

Verified OK.

12.30.1 Maple step by step solution

Let's solve

$\left(-2x^3 + 1\right) y'' + 6y'x^2 + 24yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6x^2y'}{2x^3-1} + \frac{24xy}{2x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6x^2y'}{2x^3-1} - \frac{24xy}{2x^3-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{6x^2}{2x^3-1}, P_3(x) = -\frac{24x}{2x^3-1}\right]$

  ◦ $\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}} = 0$

  ◦ $\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

  $\left(\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}} = 0$

  ◦ $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = -\frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

- Multiply by denominators

  $y''(2x^3 - 1) - 6y'x^2 - 24yx = 0$

- Change variables using $x = u - \frac{2^{\frac{2}{3}}}{4} - \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ so that the regular singular point is at $u = 0$

  $\left(2u^3 - \frac{3u^2 2^{\frac{2}{3}}}{2} - \frac{3\,\mathrm{I}u^2\sqrt{3}\,2^{\frac{2}{3}}}{2} - \frac{3u2^{\frac{1}{3}}}{2} + \frac{3\,\mathrm{I}u\sqrt{3}\,2^{\frac{1}{3}}}{2}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-6u^2 + 3u2^{\frac{2}{3}} + 3\,\mathrm{I}u\sqrt{3}\,2^{\frac{2}{3}} + \frac{3\,2^{\frac{1}{3}}}{2} - 3\right.$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(r-2)ra_0u^{r-1}}{2} + \left(\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(r-1)(1+r)a_1}{2} - \frac{3\,2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(1+r)(-4+r)a_0}{2}\right)u^r + \left(\sum_{k=1}^{\infty}\left(\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right.}{}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(r-2)r}{2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(r-1)(1+r)a_1}{2} - \frac{3\,2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(1+r)(-4+r)a_0}{2} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-\frac{3(k+1+r)\left(-(k+r-1)a_{k+1}\left(\mathrm{I}\sqrt{3}-1\right)2^{\frac{1}{3}}+\left(1+\mathrm{I}\sqrt{3}\right)a_k(k-4+r)2^{\frac{2}{3}}-\frac{4a_{k-1}(k-7+r)}{3}\right)}{2} = 0$$

- Shift index using $k->k+1$

$$-\frac{3(k+r+2)\left(-(k+r)a_{k+2}\left(\mathrm{I}\sqrt{3}-1\right)2^{\frac{1}{3}}+\left(1+\mathrm{I}\sqrt{3}\right)a_{k+1}(k-3+r)2^{\frac{2}{3}}-\frac{4a_k(k+r-6)}{3}\right)}{2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}+3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,r a_{k+1}-9\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}+3\,2^{\frac{2}{3}}\,r a_{k+1}-9a_{k+1}2^{\frac{2}{3}}-4ka_k-4ra_k+24a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k+\mathrm{I}\sqrt{3}\,r-k-r\right)}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}-9\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}-9a_{k+1}2^{\frac{2}{3}}-4ka_k+24a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k-k\right)}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

$$a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}-9\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}-9a_{k+1}2^{\frac{2}{3}}-4ka_k+24a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k-k\right)}$$

- Recursion relation for $r=2$

$$a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}-3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}-3a_{k+1}2^{\frac{2}{3}}-4ka_k+16a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k+2\,\mathrm{I}\sqrt{3}-k-2\right)}$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}-3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}-3a_{k+1}2^{\frac{2}{3}}-4ka_k+16a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k+2\,\mathrm{I}\sqrt{3}-k-2\right)}, \frac{9\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)a_1}{2}\right.$$

- Revert the change of variables $u = x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k+2}, a_{k+2} = \frac{\left(3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,k a_{k+1}-3\,\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}\,a_{k+1}+3\,2^{\frac{2}{3}}\,k a_{k+1}-3a_{k+1}2^{\frac{2}{3}}-4ka_k+16a_k\right)2^{\frac{2}{3}}}{6\left(\mathrm{I}\sqrt{3}\,k+2\,\mathrm{I}\sqrt{3}-k-2\right)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((1-2*x^3)*diff(y(x),x$2)+6*x^2*diff(y(x),x)+24*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(-4x^3+1\right) y(0) + \left(x - \frac{5}{2}x^4\right) D(y)\left(0\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 26

```
AsymptoticDSolveValue[(1-2*x^3)*y''[x]+6*x^2*y'[x]+24*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{5x^4}{2}\right) + c_1\left(1-4x^3\right)$$

12.31 problem 37

Internal problem ID [1235]
Internal file name [`OUTPUT/1236_Sunday_June_05_2022_02_06_01_AM_10713628/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(-x^{3}+1\right) y''+15y'x^{2}-36yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{908}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{909}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3x(5y'x - 12y)}{x^3 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{174y'x^4 - 468yx^3 + 6y'x + 36y}{(x^3 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{6(299x^6 - 22x^3 - 7)\, y' + 972(-5x^5 + x^2)\, y}{(x^3 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(16668x^8 - 6120x^5 - 828x^2)\, y' - 45144xy\left(x^6 - \frac{103}{209}x^3 + \frac{2}{209}\right)}{(x^3 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(138204x^{10} - 114912x^7 + 3780x^4 + 2088x)\, y' + (-374328x^9 + 358344x^6 - 54432x^3 + 432)\, y}{(x^3 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 36y(0) \\
F_2 &= 42y'(0) \\
F_3 &= 0 \\
F_4 &= -432y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 6x^3 - \frac{3}{5}x^6\right) y(0) + \left(x + \frac{7}{4}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^3+1\right) y'' + 15y'x^2 - 36yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^3+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 15\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 36\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-n\, x^{1+n} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 15n\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty}\left(-36 x^{1+n} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-n\, x^{1+n} a_n (n-1)\right) = \sum_{n=3}^{\infty}\left(-(n-1)\, a_{n-1}(n-2)\, x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} 15n\, x^{1+n} a_n = \sum_{n=2}^{\infty} 15(n-1)\, a_{n-1} x^n$$

$$\sum_{n=0}^{\infty}\left(-36 x^{1+n} a_n\right) = \sum_{n=1}^{\infty}\left(-36 a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=3}^{\infty} \left(-(n-1)\, a_{n-1}(n-2)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \left(\sum_{n=2}^{\infty} 15(n-1)\, a_{n-1} x^{n}\right) + \sum_{n=1}^{\infty} \left(-36 a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 - 36a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 6a_0$$

$n = 2$ gives

$$12a_4 - 21a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{7a_1}{4}$$

For $3 \le n$, the recurrence equation is

$$-(n-1)\, a_{n-1}(n-2) + (n+2)\, a_{n+2}(1+n) + 15(n-1)\, a_{n-1} - 36a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}(n^2 - 18n + 53)}{(n+2)\,(1+n)} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$-8a_2 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$3a_3 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{3a_0}{5}$$

For $n = 5$ the recurrence equation gives

$$12a_4 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{2}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 6a_0 x^3 + \frac{7}{4} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = (6x^3 + 1) a_0 + \left(x + \frac{7}{4} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (6x^3 + 1) c_1 + \left(x + \frac{7}{4} x^4\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 + 6x^3 - \frac{3}{5} x^6\right) y(0) + \left(x + \frac{7}{4} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = (6x^3 + 1) c_1 + \left(x + \frac{7}{4} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+6x^3-\frac{3}{5}x^6\right) y(0)+\left(x+\frac{7}{4}x^4\right) y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(6x^3+1\right) c_1+\left(x+\frac{7}{4}x^4\right) c_2+O\left(x^6\right)$$

Verified OK.

12.31.1 Maple step by step solution

Let's solve

$\left(-x^3+1\right) y''+15y'x^2-36yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{15x^2y'}{x^3-1}-\frac{36xy}{x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{15x^2y'}{x^3-1}+\frac{36xy}{x^3-1}=0$

- ☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{15x^2}{x^3-1}, P_3(x)=\frac{36x}{x^3-1}\right]$

  - $(x-1)\cdot P_2(x)$ is analytic at $x=1$

    $\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-5$

  - $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

    $\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

  - $x=1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=1$

- Multiply by denominators

$y''(x^3-1)-15y'x^2+36yx=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$(u^3+3u^2+3u)\left(\frac{d^2}{du^2}y(u)\right)+(-15u^2-30u-15)\left(\frac{d}{du}y(u)\right)+(36u+36)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0r(-6+r)\,u^{-1+r}+(3a_1(1+r)\,(-5+r)+3a_0(r^2-11r+12))\,u^r+\left(\sum_{k=1}^{\infty}\left(3a_{k+1}(k+1+r)\,(k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-6+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,6\}$

- Each term must be 0

  $3a_1(1+r)(-5+r)+3a_0(r^2-11r+12)=0$

- Each term in the series must be 0, giving the recursion relation

  $(3a_k+a_{k-1}+3a_{k+1})k^2+(2(3a_k+a_{k-1}+3a_{k+1})r-33a_k-18a_{k-1}-12a_{k+1})k+(3a_k+a_{k-1}$

- Shift index using $k->k+1$

  $(3a_{k+1}+a_k+3a_{k+2})(k+1)^2+(2(3a_{k+1}+a_k+3a_{k+2})r-33a_{k+1}-18a_k-12a_{k+2})(k+1)+($

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+2kra_k+6kra_{k+1}+r^2a_k+3r^2a_{k+1}-16ka_k-27ka_{k+1}-16ra_k-27ra_{k+1}+36a_k+6a_{k+1}}{3(k^2+2kr+r^2-2k-2r-8)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}-16ka_k-27ka_{k+1}+36a_k+6a_{k+1}}{3(k^2-2k-8)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=4$

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}-16ka_k-27ka_{k+1}+36a_k+6a_{k+1}}{3(k^2-2k-8)}$

- Recursion relation for $r=6$

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}-4ka_k+9ka_{k+1}-24a_k-48a_{k+1}}{3(k^2+10k+16)}$

- Solution for $r=6$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+6}, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}-4ka_k+9ka_{k+1}-24a_k-48a_{k+1}}{3(k^2+10k+16)}, 21a_1-54a_0=0\right]$$

- Revert the change of variables $u=x-1$

  $$\left[y=\sum_{k=0}^{\infty}a_k(x-1)^{k+6}, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}-4ka_k+9ka_{k+1}-24a_k-48a_{k+1}}{3(k^2+10k+16)}, 21a_1-54a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve((1-x^3)*diff(y(x),x$2)+15*x^2*diff(y(x),x)-36*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (6x^3 + 1)\, y(0) + \left(x + \frac{7}{4}x^4\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[(1-2*x^3)*y''[x]-10*x^2*y'[x]-8*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{3x^4}{2} + x\right) + c_1\left(\frac{4x^3}{3} + 1\right)$$

12.32 problem 39

Internal problem ID [1236]
Internal file name [OUTPUT/1237_Sunday_June_05_2022_02_06_02_AM_35795779/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^5+1)\, y''+14y'x^4+10yx^3=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\, f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\, f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\, \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{911}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{912}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2x^3(7y'x + 5y)}{2x^5 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(204x^8 - 66x^3)\, y' + 180x^2\left(x^5 - \frac{1}{6}\right) y}{(2x^5 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-3312x^{12} + 3600x^7 - 228x^2)\, y' - 3120xy\left(x^{10} - \frac{10}{13}x^5 + \frac{1}{52}\right)}{(2x^5 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(60000x^{16} - 146064x^{11} + 36600x^6 - 516x)\, y' + 58080\left(x^{15} - \frac{43}{22}x^{10} + \frac{153}{484}x^5 - \frac{1}{968}\right) y}{(2x^5 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-1203840x^{20} + 5465088x^{15} - 3220704x^{10} + 264672x^5 - 576)\, y' - 1180800\left(x^{15} - \frac{799}{205}x^{10} + \frac{285}{164}x^5 -\right.}{(2x^5 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= -60y(0) \\
F_4 &= -576y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^5}{2}\right) y(0) + \left(x - \frac{4}{5}x^6\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(2x^5+1\right)y''+14y'x^4+10yx^3=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_nx^n$$

Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}$$
$$y''=\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}$$

Substituting the above back into the ode gives

$$\left(2x^5+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}\right)+14\left(\sum_{n=1}^{\infty}na_nx^{n-1}\right)x^4+10\left(\sum_{n=0}^{\infty}a_nx^n\right)x^3=0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty}2n\,x^{n+3}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}\right)+\left(\sum_{n=1}^{\infty}14n\,x^{n+3}a_n\right)+\left(\sum_{n=0}^{\infty}10x^{n+3}a_n\right)=0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty}2n\,x^{n+3}a_n(n-1)=\sum_{n=5}^{\infty}2(n-3)\,a_{n-3}(n-4)\,x^n$$

$$\sum_{n=2}^{\infty}n(n-1)\,a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n$$

$$\sum_{n=1}^{\infty}14n\,x^{n+3}a_n=\sum_{n=4}^{\infty}14(n-3)\,a_{n-3}x^n$$

$$\sum_{n=0}^{\infty}10x^{n+3}a_n=\sum_{n=3}^{\infty}10a_{n-3}x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=5}^{\infty} 2(n-3)\, a_{n-3}(n-4)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=4}^{\infty} 14(n-3)\, a_{n-3} x^{n}\right)+\left(\sum_{n=3}^{\infty} 10 a_{n-3} x^{n}\right) = 0 \tag{3}$$

$n=3$ gives

$$20 a_5 + 10 a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{a_0}{2}$$

$n=4$ gives

$$30 a_6 + 24 a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{4 a_1}{5}$$

For $5 \leq n$, the recurrence equation is

$$2(n-3)\, a_{n-3}(n-4)+(n+2)\, a_{n+2}(n+1)+14(n-3)\, a_{n-3}+10 a_{n-3} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2(n-2)\, a_{n-3}}{n+1} \tag{5}$$

For $n=5$ the recurrence equation gives

$$42 a_2 + 42 a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x - \frac{1}{2} a_0 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{x^5}{2}\right) a_0 + a_1 x + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{x^5}{2}\right) c_1 + c_2 x + O(x^6)
$$

Summary

The solution(s) found are the following

$$
y = \left(1 - \frac{x^5}{2}\right) y(0) + \left(x - \frac{4}{5} x^6\right) y'(0) + O(x^6) \tag{1}
$$

$$
y = \left(1 - \frac{x^5}{2}\right) c_1 + c_2 x + O(x^6) \tag{2}
$$

Verification of solutions

$$
y = \left(1 - \frac{x^5}{2}\right) y(0) + \left(x - \frac{4}{5} x^6\right) y'(0) + O(x^6)
$$

Verified OK.

$$
y = \left(1 - \frac{x^5}{2}\right) c_1 + c_2 x + O(x^6)
$$

Verified OK.

12.32.1 Maple step by step solution

Let's solve

$(2x^5+1)y''+14y'x^4+10yx^3=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{14x^4y'}{2x^5+1}-\frac{10x^3y}{2x^5+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{14x^4y'}{2x^5+1}+\frac{10x^3y}{2x^5+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{14x^4}{2x^5+1}, P_3(x)=\frac{10x^3}{2x^5+1}\right]$

  ◦ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

  $\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

  ◦ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

  $\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

  ◦ $x=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$ is

  Check to see if x_0 is a regular singular point

  $x_0=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$

- Multiply by denominators

  $(2x^5+1)y''+14y'x^4+10yx^3=0$

- Change variables using $x = u + \left(\left(\left(-\frac{\sqrt{5}}{8} - \frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right) - \frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}} + \mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}\right.\right.$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..3$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..4$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..5$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3\cos\left(\frac{\pi}{5}\right) - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - \mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 + 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2 \mathrm{si}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3\cos\left(\frac{\pi}{5}\right) - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - \mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 + 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2 \mathrm{si}\right.$

- Values of r that satisfy the indicial equation

  $r = r$

- The coefficients of each power of u must be 0

$$\left[-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right)+\cos\left(\frac{\pi}{5}\right)+1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^{3}\cos\left(\frac{\pi}{5}\right)-4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^{3}-\mathrm{I}\sin\left(\frac{\pi}{5}\right)^{3}+3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^{2}\mathrm{s}\right.$$

- Solve for the dependent coefficient(s)
- Each term in the series must be 0, giving the recursion relation

$$\frac{5\,\mathrm{I}\left(-(k+r+1)\left(\sqrt{5}+1\right)a_{k+1}\left(k+r+\frac{7}{5}\right)2^{\frac{1}{5}}-\left(\sqrt{5}+1\right)\left(k^{2}+\left(2r+\frac{3}{5}\right)k+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k-2}2^{\frac{4}{5}}+4\left(k^{2}+\left(2r+\frac{9}{5}\right)k+r^{2}+\frac{9r}{5}+\frac{1}{2}\right)a_{k}2^{\frac{2}{5}}+4\left(-\frac{7}{1}\right.\right.}{4}$$

- Shift index using $k->k+3$

$$\frac{5\,\mathrm{I}\left(-(k+4+r)\left(\sqrt{5}+1\right)a_{k+4}\left(k+\frac{22}{5}+r\right)2^{\frac{1}{5}}-\left(\sqrt{5}+1\right)\left((k+3)^{2}+\left(2r+\frac{3}{5}\right)(k+3)+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k+1}2^{\frac{4}{5}}+4\left((k+3)^{2}+\left(2r+\frac{9}{5}\right)(k+3)+r^{2}-\right.\right.}{4}$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{2\left(20\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}kr\sqrt{5}+80a_{k+1}2^{\frac{4}{5}}kr-160a_{k+2}2^{\frac{3}{5}}kr+160a_{k+3}2^{\frac{2}{5}}kr+149\,\mathrm{I}\sqrt{5-\sqrt{5}}\,2^{\frac{9}{10}}a_{k+3}+86\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}-\right.}{}$$

- Recursion relation for $r=r$

$$a_{k+4}=-\frac{2\left(20\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}kr\sqrt{5}+80a_{k+1}2^{\frac{4}{5}}kr-160a_{k+2}2^{\frac{3}{5}}kr+160a_{k+3}2^{\frac{2}{5}}kr+149\,\mathrm{I}\sqrt{5-\sqrt{5}}\,2^{\frac{9}{10}}a_{k+3}+86\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}-\right.}{}$$

- Solution for $r=r$
- Revert the change of variables $u=x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
Order:=6;
dsolve((1+2*x^5)*diff(y(x),x$2)+14*x^4*diff(y(x),x)+10*x^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^5}{2}\right) y(0) + D(y)(0)\, x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[(1+2*x^5)*y''[x]+14*x^4*y'[x]+10*x^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(1 - \frac{x^5}{2}\right) + c_2x$$

12.33 problem 40

Internal problem ID [1237]
Internal file name [`OUTPUT/1238_Sunday_June_05_2022_02_06_04_AM_55445381/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (914)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (915)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -x(2y + y'x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - 4y'x - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12x^3 y' - x^2 y\left(x^4 - 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O\left(x^6\right)$$

Verified OK.

12.33.1 Maple step by step solution

Let's solve

$y'' = -x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + x^2 y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- > k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(x - \frac{x^5}{20}\right) + c_1 \left(1 - \frac{x^4}{12}\right)$$

12.34 problem 41

Internal problem ID [1238]
Internal file name [`OUTPUT/1239_Sunday_June_05_2022_02_06_05_AM_57468465/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x^6 + 7yx^5 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{917}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{918}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y' x^6 - 7 y x^5 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^4 \left(\left(x^8 - 13x \right) y' + 7y \left(x^7 - 5 \right) \right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\left(\left(x^{15} - 32x^8 + 100x \right) y' + 7y \left(x^{14} - 24x^7 + 20 \right) \right) x^3 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(x^{22} - 57x^{15} + 620x^8 - 540x \right) y' + 7y \left(x^{21} - 49x^{14} + 340x^7 - 60 \right) \right) x^2 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -x \left(\left(x^{29} - 88x^{22} + 1932x^{15} - 9120x^8 + 2040x \right) y' + 7y \left(x^{28} - 80x^{21} + 1404x^{14} - 3600x^7 + 120 \right) \right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + y'(0)\, x + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^6 - 7\left(\sum_{n=0}^{\infty} a_n x^n\right) x^5 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{5+n} a_n\right) + \left(\sum_{n=0}^{\infty} 7 x^{5+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\
\sum_{n=1}^{\infty} n\, x^{5+n} a_n &= \sum_{n=6}^{\infty} (n-5)\, a_{n-5} x^n \\
\sum_{n=0}^{\infty} 7 x^{5+n} a_n &= \sum_{n=5}^{\infty} 7 a_{n-5} x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=6}^{\infty} (n-5)\, a_{n-5} x^n\right) + \left(\sum_{n=5}^{\infty} 7 a_{n-5} x^n\right) = 0 \tag{3}
$$

$n = 5$ gives

$$
42 a_7 + 7 a_0 = 0
$$

Which after substituting earlier equations, simplifies to

$$a_7 = -\frac{a_0}{6}$$

For $6 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + (n-5)\, a_{n-5} + 7a_{n-5} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-5}}{n+1} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = a_1x + a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_2x + c_1 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = y(0) + y'(0)\, x + O(x^6) \tag{1}$$
$$y = c_2x + c_1 + O(x^6) \tag{2}$$

Verification of solutions

$$y = y(0) + y'(0)\, x + O(x^6)$$

Verified OK.

$$y = c_2x + c_1 + O(x^6)$$

Verified OK.

12.34.1 Maple step by step solution

Let's solve

$y'' = -y' x^6 - 7y x^5$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' x^6 + 7y x^5 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^5 \cdot y$ to series expansion

  $x^5 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+5}$

  ◦ Shift index using $k- > k-5$

  $x^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} x^k$

  ◦ Convert $x^6 \cdot y'$ to series expansion

  $x^6 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$

  ◦ Shift index using $k- > k-5$

  $x^6 \cdot y' = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-5}(k+2)\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $$\left[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0\right]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)\left(k a_{k+2} + a_{k-5} + a_{k+2}\right) = 0$$

- Shift index using $k- > k+5$

  $$(k+7)\left((k+5)\, a_{k+7} + a_k + a_{k+7}\right) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
Order:=6;
dsolve(diff(y(x),x$2)+x^6*diff(y(x),x)+7*x^5*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\, x$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[y''[x]+x^6*y'[x]+7*x^5*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x + c_1$$

12.35 problem 42

Internal problem ID [1239]
Internal file name [`OUTPUT/1240_Sunday_June_05_2022_02_06_06_AM_66728715/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^8+1)\, y''-16y' x^7+72y x^6=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2} y''(x_0)+\frac{(x-x_0)^3}{3!} y'''(x_0)+\cdots \\
&= y_0+x y_0'+\frac{x^2}{2} f|_{x_0,y_0,y_0'}+\frac{x^3}{3!} f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+x y_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (920)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (921)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x, y, y') \qquad (4) \\ F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\ &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{8x^6(2y'x - 9y)}{x^8 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(168x^{14} + 40x^6)\, y' - 1008\left(x^8 + \frac{3}{7}\right) x^5 y}{(x^8 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{48x^4(28y'x^9 - 189yx^8 - 4y'x - 45y)}{(x^8 + 1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{240x^3(35y'x^9 - 252yx^8 - 13y'x - 36y)}{(x^8 + 1)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{960x^2(42y'x^9 - 315yx^8 - 22y'x - 27y)}{(x^8 + 1)^2}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = y(0) + y'(0)\, x + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^8 + 1)\, y'' - 16y'x^7 + 72yx^6 = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^8 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 16\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^7 + 72\left(\sum_{n=0}^{\infty} a_n x^n\right) x^6 = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n+6} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-16n\, x^{n+6} a_n\right) + \left(\sum_{n=0}^{\infty} 72x^{n+6} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n+6} a_n (n-1) = \sum_{n=8}^{\infty} (n-6)\, a_{n-6}(n-7)\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-16n\, x^{n+6} a_n\right) = \sum_{n=7}^{\infty} \left(-16(n-6)\, a_{n-6} x^n\right)$$

$$\sum_{n=0}^{\infty} 72x^{n+6} a_n = \sum_{n=6}^{\infty} 72a_{n-6} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=8}^{\infty} (n-6)\, a_{n-6}(n-7)\, x^{n}\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\ +\sum_{n=7}^{\infty} \left(-16(n-6)\, a_{n-6} x^{n}\right)+\left(\sum_{n=6}^{\infty} 72 a_{n-6} x^{n}\right) = 0 \tag{3}$$

For $8 \leq n$, the recurrence equation is

$$(n-6)\, a_{n-6}(n-7)+(n+2)\, a_{n+2}(n+1)-16(n-6)\, a_{n-6}+72 a_{n-6} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-6}\left(n^{2}-29n+210\right)}{(n+2)\,(n+1)} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_{n} x^{n} \\ &= a_{3} x^{3}+a_{2} x^{2}+a_{1} x+a_{0}+\dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_{1} x+a_{0}+\dots$$

Collecting terms, the solution becomes

$$y = a_{1} x+a_{0}+O\left(x^{6}\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_{2} x+c_{1}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$y = y(0)+y'(0)\, x+O\left(x^{6}\right) \tag{1}$$
$$y = c_{2} x+c_{1}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = y(0) + y'(0)\,x + O(x^6)$$

Verified OK.

$$y = c_2x + c_1 + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
Order:=6;
dsolve((1+x^8)*diff(y(x),x$2)-16*x^7*diff(y(x),x)+72*x^6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)\,(0)\,x$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[(1+x^8)*y''[x]-16*x^7*y'[x]+72*x^6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2x + c_1$$

12.36 problem 43

Internal problem ID [1240]
Internal file name [`OUTPUT/1241_Sunday_June_05_2022_02_06_08_AM_56327843/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(-x^{6}+1\right) y''-12y' x^{5}-30y x^{4}=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (923) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (924) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{6x^4(2y'x+5y)}{x^6-1}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{6x^3(21y'x^7+70yx^6+15y'x+20y)}{(x^6-1)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -\frac{1344\left(\left(x^{13}+\frac{5}{2}x^7+\frac{5}{14}x\right)y' + \frac{15\left(x^{12}+\frac{3}{2}x^6+\frac{1}{14}\right)y}{4}\right)x^2}{(x^6-1)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{15120x\left(\left(x^{19}+\frac{35}{6}x^{13}+\frac{10}{3}x^7+\frac{5}{42}x\right)y' + 4\left(x^{18}+\frac{49}{12}x^{12}+\frac{4}{3}x^6+\frac{1}{84}\right)y\right)}{(x^6-1)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{-181440\left(x^{24}+\frac{34}{3}x^{18}+\frac{31}{2}x^{12}+3x^6+\frac{1}{42}\right)xy' - 756000y\left(x^{24}+\frac{646}{75}x^{18}+\frac{403}{50}x^{12}+\frac{21}{25}x^6+\frac{1}{1050}\right)}{(x^6-1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 720y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(x^6+1\right)y(0) + y'(0)\,x + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^6+1\right) y'' - 12y' x^5 - 30y x^4 = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^6+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 12\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^5 - 30\left(\sum_{n=0}^{\infty} a_n x^n\right) x^4 = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-n\, x^{n+4} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-12n\, x^{n+4} a_n\right) + \sum_{n=0}^{\infty} \left(-30 x^{n+4} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n\, x^{n+4} a_n (n-1)\right) = \sum_{n=6}^{\infty} \left(-(n-4)\, a_{n-4}(n-5)\, x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty}\left(-12n\,x^{n+4}a_n\right) = \sum_{n=5}^{\infty}\left(-12(n-4)\,a_{n-4}x^n\right)$$

$$\sum_{n=0}^{\infty}\left(-30x^{n+4}a_n\right) = \sum_{n=4}^{\infty}\left(-30a_{n-4}x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=6}^{\infty}\left(-(n-4)\,a_{n-4}(n-5)\,x^n\right) + \left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right) + \sum_{n=5}^{\infty}\left(-12(n-4)\,a_{n-4}x^n\right) + \sum_{n=4}^{\infty}\left(-30a_{n-4}x^n\right) = 0 \tag{3}$$

$n = 4$ gives

$$30a_6 - 30a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = a_0$$

$n = 5$ gives

$$42a_7 - 42a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = a_1$$

For $6 \leq n$, the recurrence equation is

$$-(n-4)\,a_{n-4}(n-5) + (n+2)\,a_{n+2}(n+1) - 12(n-4)\,a_{n-4} - 30a_{n-4} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = a_{n-4} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = a_1 x + a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_2 x + c_1 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (x^6 + 1)\, y(0) + y'(0)\, x + O(x^6) \tag{1}$$

$$y = c_2 x + c_1 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (x^6 + 1)\, y(0) + y'(0)\, x + O(x^6)$$

Verified OK.

$$y = c_2 x + c_1 + O(x^6)$$

Verified OK.

12.36.1 Maple step by step solution

Let's solve

$(-x^6 + 1)\, y'' - 12y'x^5 - 30yx^4 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12x^5 y'}{x^6-1} - \frac{30x^4 y}{x^6-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{12x^5 y'}{x^6-1} + \frac{30x^4 y}{x^6-1} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{12x^5}{x^6-1}, P_3(x)=\frac{30x^4}{x^6-1}\right]$

- $\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$

  $\left(\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}}=0$

- $\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$

  $\left(\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}}=0$

- $x=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$

- Multiply by denominators

  $y''(x^6-1)+12y'x^5+30yx^4=0$

- Change variables using $x=u-\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$ so that the regular singular point is at $u=0$

  $\left(u^6-3u^5\sqrt{-2\,\mathrm{I}\sqrt{3}-2}-\frac{15\,\mathrm{I}u^4\sqrt{3}}{2}-\frac{15u^4}{2}+\frac{15\,\mathrm{I}u^2\sqrt{3}}{2}+5u^3\sqrt{-2\,\mathrm{I}\sqrt{3}-2}-\frac{15u^2}{2}-\frac{3\,\mathrm{I}\sqrt{-2\,\mathrm{I}\sqrt{3}-2}\,u\sqrt{3}}{2}\right.$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..4$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..5$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..6$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-3\left(1+\mathrm{I}\sqrt{3}\right) r(1+r)\, a_0 u^{-1+r} + \left(-3\left(1+\mathrm{I}\sqrt{3}\right)(1+r)\,(2+r)\, a_1 + \frac{15\left(\mathrm{I}\sqrt{3}-1\right)(1+r)(2+r)a_0}{2}\right) u^r + \left(\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3\left(1+\mathrm{I}\sqrt{3}\right) r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- The coefficients of each power of u must be 0

$$\left[-3\left(1+\mathrm{I}\sqrt{3}\right)(1+r)\,(2+r)\, a_1 + \frac{15\left(\mathrm{I}\sqrt{3}-1\right)(1+r)(2+r)a_0}{2} = 0, -3\left(1+\mathrm{I}\sqrt{3}\right)(2+r)\,(3+r)\, a_2 + \frac{15(}{}\right.$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{5a_0\left(\mathrm{I}\sqrt{3}-1\right)}{2\left(1+\mathrm{I}\sqrt{3}\right)}, a_2 = \frac{5a_0\left(-14-14\,\mathrm{I}\sqrt{3}\right)}{12\left(2\,\mathrm{I}\sqrt{3}-2\right)}, a_3 = -\frac{35a_0}{24}, a_4 = \frac{\left(952-952\,\mathrm{I}\sqrt{3}\right)a_0}{144\left(-8-8\,\mathrm{I}\sqrt{3}\right)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{15\left(\mathrm{I}\left(a_k+\frac{2a_{k-3}}{5}-a_{k-2}-\frac{2a_{k+1}}{5}\right)\sqrt{3}-a_k+\frac{2a_{k-4}}{15}-\frac{2a_{k-3}}{5}-a_{k-2}+\frac{8a_{k-1}}{3}-\frac{2a_{k+1}}{5}\right)(k+2+r)(k+1+r)}{2} = 0$$

- Shift index using $k->k+4$

$$\frac{15\left(\mathrm{I}\left(a_{k+4}+\frac{2a_{k+1}}{5}-a_{k+2}-\frac{2a_{k+5}}{5}\right)\sqrt{3}-a_{k+4}+\frac{2a_k}{15}-\frac{2a_{k+1}}{5}-a_{k+2}+\frac{8a_{k+3}}{3}-\frac{2a_{k+5}}{5}\right)(k+r+6)(k+r+5)}{2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+5} = \frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Recursion relation for $r = -1$

$$a_{k+5} = \frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Solution for $r=-1$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+5}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}, a_1=\frac{5a_0\left(\right.}{2\left(1\right.}\right.$$

- Revert the change of variables $u=x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^{k-1}, a_{k+5}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k}}{6\left(1+\mathrm{I}\sqrt{3}\right)}\right.$$

- Recursion relation for $r=0$

$$a_{k+5}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k}, a_{k+5}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}, a_1=\frac{5a_0\left(\mathrm{I}\sqrt{\right.}}{2\left(1+\mathrm{I}\right.}\right.$$

- Revert the change of variables $u=x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^{k}, a_{k+5}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{k+1}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}\sqrt{3}\,a_{k+4}+2a_k-6a_{k+1}-15a_{k+2}+40a_{k+3}-15a_{k+4}}{6\left(1+\mathrm{I}\sqrt{3}\right)}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+\frac{\sqrt{-2\,\mathrm{I}\sqrt{3}-2}}{2}\right)^{k}\right), a_{5+k}=\frac{6\,\mathrm{I}\sqrt{3}\,a_{1+k}-15\,\mathrm{I}\sqrt{3}\,a_{k+2}+15\,\mathrm{I}}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve((1-x^6)*diff(y(x),x$2)-12*x^5*diff(y(x),x)-30*x^4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[(1-x^6)*y''[x]-12*x^5*y'[x]-30*x^4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x + c_1$$

12.37 problem 44

Internal problem ID [1241]
Internal file name [`OUTPUT/1242_Sunday_June_05_2022_02_06_09_AM_92488736/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I. Exercises 7.2. Page 329
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x^5 + 6yx^4 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (926)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (927)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y' x^5 - 6y x^4 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(\left(x^7 - 11x\right) y' + 6y\left(x^6 - 4\right)\right) x^3 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\left(\left(x^{13} - 27x^7 + 68x\right) y' + 6y\left(x^{12} - 20x^6 + 12\right)\right) x^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\left(x^{19} - 48x^{13} + 431x^7 - 276x\right) y' + 6y\left(x^{18} - 41x^{12} + 228x^6 - 24\right)\right) x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^{25} + 74x^{19} - 1349x^{13} + 5092x^7 - 696x\right) y' - 6y\left(x^{24} - 67x^{18} + 964x^{12} - 1872x^6 + 24\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= -144y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^6}{5}\right) y(0) + y'(0)\, x + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^5 - 6\left(\sum_{n=0}^{\infty} a_n x^n\right) x^4 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{4+n} a_n\right) + \left(\sum_{n=0}^{\infty} 6x^{4+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

$$\sum_{n=1}^{\infty} n\, x^{4+n} a_n = \sum_{n=5}^{\infty} (n-4) a_{n-4} x^n$$

$$\sum_{n=0}^{\infty} 6x^{4+n} a_n = \sum_{n=4}^{\infty} 6a_{n-4} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=5}^{\infty} (n-4) a_{n-4} x^n\right) + \left(\sum_{n=4}^{\infty} 6a_{n-4} x^n\right) = 0 \tag{3}$$

$n = 4$ gives

$$30a_6 + 6a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{a_0}{5}$$

For $5 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + (n-4)\, a_{n-4} + 6a_{n-4} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-4}}{n+1} \tag{5}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = a_1 x + a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = c_2 x + c_1 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{x^6}{5}\right) y(0)+y'(0)\, x+O\left(x^6\right) \tag{1}$$

$$y = c_2 x+c_1+O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{x^6}{5}\right) y(0)+y'(0)\, x+O\left(x^6\right)$$

Verified OK.

$$y = c_2 x+c_1+O\left(x^6\right)$$

Verified OK.

12.37.1 Maple step by step solution

Let's solve

$y'' = -y' x^5-6 y x^4$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y' x^5+6 y x^4 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^4 \cdot y$ to series expansion

    $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  ◦ Shift index using $k- >k-4$

    $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

  ◦ Convert $x^5 \cdot y'$ to series expansion

    $x^5 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+4}$

- Shift index using $k->k-4$

  $$x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4}(k-4)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_{k-4}(k+2))\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)\,(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$$

- Shift index using $k->k+4$

  $$(k+6)\,((k+4)\, a_{k+6} + a_k + a_{k+6}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve(diff(y(x),x$2)+x^5*diff(y(x),x)+6*x^4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 10

```
AsymptoticDSolveValue[y''[x]+x^5*y'[x]+6*x^4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2x + c_1$$

13 Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338

13.1 problem 1

Internal problem ID [1242]
Internal file name [`OUTPUT/1243_Sunday_June_05_2022_02_06_10_AM_30732153/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x+1)\,y''+y'x+2y=0}$$

With initial conditions

$$[y(0)=2,y'(0)=-3]$$

With the expansion point for the power series method at $x=0$.

13.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x}{3x+1}\\q(x)&=\frac{2}{3x+1}\\F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{3x+1} + \frac{2y}{3x+1} = 0$$

The domain of $p(x) = \frac{x}{3x+1}$ is

$$\left\{x < -\frac{1}{3} \vee -\frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{3x+1}$ is

$$\left\{x < -\frac{1}{3} \vee -\frac{1}{3} < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\ &= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (929)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (930)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y + y'x}{3x+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 - 6x - 3)\, y' + 2y(x+3)}{(3x+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^3 + 12x^2 + 43x + 18)\, y' - 2y(x^2 - 3x + 14)}{(3x+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^4 - 18x^3 - 66x^2 - 330x - 147)\, y' + 2y(x^3 - 9x^2 - 63x + 111)}{(3x+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^5 + 24x^4 + 72x^3 + 276x^2 + 3525x + 1656)\, y' - 2y(x^4 - 15x^3 - 123x^2 - 879x + 1248)}{(3x+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -3$ gives

$$
\begin{aligned}
F_0 &= -4 \\
F_1 &= 21 \\
F_2 &= -110 \\
F_3 &= 885 \\
F_4 &= -9960
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -2x^2 - 3x + 2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - \frac{83x^6}{6} + O(x^6)
$$

$$y = -2x^2 - 3x + 2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - \frac{83x^6}{6} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(3x+1)\, y'' + y'x + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(3x+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, x^n$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, x^{n}\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) \\
+\left(\sum_{n=1}^{\infty} n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 2 a_{n} x^{n}\right)=0 \tag{3}
$$

$n=0$ gives

$$
2a_{2}+2a_{0}=0
$$

$$
a_{2}=-a_{0}
$$

For $1 \leq n$, the recurrence equation is

$$
3(n+1)\, a_{n+1} n+(n+2)\, a_{n+2}(n+1)+n a_{n}+2 a_{n}=0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{3n^{2} a_{n+1}+n a_{n}+3n a_{n+1}+2 a_{n}}{(n+2)\,(n+1)} \\
&= -\frac{a_{n}}{n+1}-\frac{\left(3n^{2}+3n\right) a_{n+1}}{(n+2)\,(n+1)}
\end{aligned} \tag{5}
$$

For $n=1$ the recurrence equation gives

$$
6a_{2}+6a_{3}+3a_{1}=0
$$

Which after substituting the earlier terms found becomes

$$
a_{3}=a_{0}-\frac{a_{1}}{2}
$$

For $n=2$ the recurrence equation gives

$$
18a_{3}+12a_{4}+4a_{2}=0
$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{7a_0}{6} + \frac{3a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$36a_4 + 20a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{37a_0}{20} - \frac{49a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$60a_5 + 30a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{52a_0}{15} + \frac{23a_1}{10}$$

For $n = 5$ the recurrence equation gives

$$90a_6 + 42a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5981a_0}{840} - \frac{7937a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 + \left(a_0 - \frac{a_1}{2}\right)x^3 + \left(-\frac{7a_0}{6} + \frac{3a_1}{4}\right)x^4 + \left(\frac{37a_0}{20} - \frac{49a_1}{40}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + x^3 - \frac{7}{6}x^4 + \frac{37}{20}x^5\right) a_0 + \left(x - \frac{1}{2}x^3 + \frac{3}{4}x^4 - \frac{49}{40}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + x^3 - \frac{7}{6}x^4 + \frac{37}{20}x^5\right) c_1 + \left(x - \frac{1}{2}x^3 + \frac{3}{4}x^4 - \frac{49}{40}x^5\right) c_2 + O(x^6)$$

$$y = 2 - 2x^2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - 3x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -2x^2 - 3x + 2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - \frac{83x^6}{6} + O(x^6) \quad (1)$$

$$y = 2 - 2x^2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - 3x + O(x^6) \quad (2)$$

Verification of solutions

$$y = -2x^2 - 3x + 2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - \frac{83x^6}{6} + O(x^6)$$

Verified OK.

$$y = 2 - 2x^2 + \frac{7x^3}{2} - \frac{55x^4}{12} + \frac{59x^5}{8} - 3x + O(x^6)$$

Verified OK.

13.1.2 Maple step by step solution

Let's solve

$$\left[(3x+1)\, y'' + y'x + 2y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y}{3x+1} - \frac{xy'}{3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{xy'}{3x+1} + \frac{2y}{3x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

◦ Define functions

$\left[P_2(x) = \frac{x}{3x+1}, P_3(x) = \frac{2}{3x+1}\right]$

◦ $\left(x+\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{3}$

$\left(\left(x+\frac{1}{3}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{1}{3}} = -\frac{1}{9}$

◦ $\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{3}$

$\left(\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-\frac{1}{3}} = 0$

◦ $x = -\frac{1}{3}$is a regular singular point

Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

$x_0 = -\frac{1}{3}$

- Multiply by denominators

$(3x+1)\,y'' + y'x + 2y = 0$

- Change variables using $x = u - \frac{1}{3}$ so that the regular singular point is at $u = 0$

$3u\left(\frac{d^2}{du^2}y(u)\right) + \left(u - \frac{1}{3}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

◦ Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

◦ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r(-10+9r)u^{-1+r}}{3} + \left(\sum_{k=0}^{\infty} \left(\frac{a_{k+1}(k+1+r)(9k-1+9r)}{3} + a_k(k+r+2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r(-10+9r)}{3} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{10}{9}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k - \frac{1}{9} + r\right)(k+1+r)\, a_{k+1} + a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r+2)}{(9k-1+9r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^k, a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{10}{9}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}$$

- Solution for $r = \frac{10}{9}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_{k} \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1}=-\frac{3 a_{k} \left(k+\frac{28}{9}\right)}{\left(9 k+9\right) \left(k+\frac{19}{9}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_{k} \left(x+\frac{1}{3}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_{k} \left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{1+k}=-\frac{3 a_{k} \left(k+2\right)}{\left(9 k-1\right) \left(1+k\right)}, b_{1+k}=-\frac{3 b_{k} \left(k+\frac{28}{9}\right)}{\left(9 k+9\right) \left(k+\frac{19}{9}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+3*x)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(0) = 2, D(y)(0) = -3],y(x),type='se
```

$$y(x) = 2 - 3x - 2x^2 + \frac{7}{2}x^3 - \frac{55}{12}x^4 + \frac{59}{8}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(1+3*x)*y''[x]+x*y'[x]+2*y[x]==0,{y[0]==2,y'[0]==-3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{59x^5}{8} - \frac{55x^4}{12} + \frac{7x^3}{2} - 2x^2 - 3x + 2$$

13.2 problem 2

Internal problem ID [1243]
Internal file name [`OUTPUT/1244_Sunday_June_05_2022_02_06_12_AM_96010497/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(2x^2 + x + 1)\, y'' + (2 + 8x)\, y' + 4y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

With the expansion point for the power series method at $x = 0$.

13.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2+8x}{2x^2+x+1} \\ q(x) &= \frac{4}{2x^2+x+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(2+8x)\, y'}{2x^2+x+1} + \frac{4y}{2x^2+x+1} = 0$$

The domain of $p(x) = \frac{2+8x}{2x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{4}{2x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (932)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (933)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n |_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(4y'x + y' + 2y)}{2x^2 + x + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(72x^2 + 36x - 6)\, y' + (48x + 12)\, y}{(2x^2 + x + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-768x^3 - 576x^2 + 192x + 72)\, y' - 576\left(x^2 + \frac{1}{2}x - \frac{1}{12}\right) y}{(2x^2 + x + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(9600x^4 + 9600x^3 - 4800x^2 - 3600x - 120)\, y' + 7680\left(x + \frac{1}{4}\right)\left(x^2 + \frac{1}{2}x - \frac{3}{8}\right) y}{(2x^2 + x + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-138240x^5 - 172800x^4 + 115200x^3 + 129600x^2 + 8640x - 3600)\, y' - 115200y\left(x^4 + x^3 - \frac{1}{2}x^2 - \frac{3}{8}\right.}{(2x^2 + x + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -1$ and $y'(0) = 2$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -24 \\
F_2 &= 96 \\
F_3 &= 480 \\
F_4 &= -8640
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -12x^6 + 4x^5 + 4x^4 - 4x^3 + 2x - 1 + O(x^6)
$$

$$y = -12x^6 + 4x^5 + 4x^4 - 4x^3 + 2x - 1 + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2x^2 + x + 1)\, y'' + (2 + 8x)\, y' + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2x^2 + x + 1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (2 + 8x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 8n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right)+\left(\sum_{n=1}^{\infty} (n+1) a_{n+1} n\, x^n\right)+\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1)\, x^n\right) \\ +\left(\sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n\right)+\left(\sum_{n=1}^{\infty} 8n a_n x^n\right)+\left(\sum_{n=0}^{\infty} 4a_n x^n\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 4a_0 = 0$$

$$a_2 = -2a_0 - a_1$$

$n = 1$ gives

$$6a_2 + 6a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 2a_0 - a_1$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + (n+1) a_{n+1}n + (n+2) a_{n+2}(n+1) + 2(n+1) a_{n+1} + 8na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -2a_n - a_{n+1} \\ &= -2a_n - a_{n+1} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$24a_2 + 12a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 2a_0 + 3a_1$$

For $n = 3$ the recurrence equation gives

$$40a_3 + 20a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -6a_0 - a_1$$

For $n = 4$ the recurrence equation gives

$$60a_4 + 30a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 2a_0 - 5a_1$$

For $n = 5$ the recurrence equation gives

$$84a_5 + 42a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 10a_0 + 7a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + (-2a_0 - a_1) x^2 + (2a_0 - a_1) x^3 + (2a_0 + 3a_1) x^4 + (-6a_0 - a_1) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-6x^5 + 2x^4 + 2x^3 - 2x^2 + 1\right) a_0 + \left(-x^5 + 3x^4 - x^3 - x^2 + x\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(-6x^5 + 2x^4 + 2x^3 - 2x^2 + 1\right) c_1 + \left(-x^5 + 3x^4 - x^3 - x^2 + x\right) c_2 + O\left(x^6\right)$$

$$y = 4x^5 + 4x^4 - 4x^3 - 1 + 2x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -12x^6 + 4x^5 + 4x^4 - 4x^3 + 2x - 1 + O(x^6) \tag{1}$$
$$y = 4x^5 + 4x^4 - 4x^3 - 1 + 2x + O(x^6) \tag{2}$$

Verification of solutions

$$y = -12x^6 + 4x^5 + 4x^4 - 4x^3 + 2x - 1 + O(x^6)$$

Verified OK.

$$y = 4x^5 + 4x^4 - 4x^3 - 1 + 2x + O(x^6)$$

Verified OK.

13.2.2 Maple step by step solution

Let's solve

$\left[(2x^2 + x + 1)\, y'' + (2 + 8x)\, y' + 4y = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{2x^2+x+1} - \frac{2(4x+1)y'}{2x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(4x+1)y'}{2x^2+x+1} + \frac{4y}{2x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2(4x+1)}{2x^2+x+1}, P_3(x) = \frac{4}{2x^2+x+1}\right]$

- ◦ $\left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{I\sqrt{7}}{4} - \frac{1}{4}$

  $\left(\left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}} = 0$

- ◦ $\left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{I\sqrt{7}}{4} - \frac{1}{4}$

$$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^{2}\cdot P_{3}(x)\right)\Bigg|_{x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}}=0$$

- $x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{I\sqrt{7}}{4}-\frac{1}{4}$

- Multiply by denominators

  $(2x^2+x+1)\,y''+(2+8x)\,y'+4y=0$

- Change variables using $x=u-\frac{I\sqrt{7}}{4}-\frac{1}{4}$ so that the regular singular point is at $u=0$

  $(2u^2-Iu\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right)+(8u-2\,I\sqrt{7})\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-I\sqrt{7}\,r(r+1)\,a_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-I\sqrt{7}\,(k+r+1)\,(k+r+2)\,a_{k+1}+2a_k(k+r+2)\,(k+r+1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-I\sqrt{7}\,r(r+1)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $-\left(\mathrm{I}\sqrt{7}\, a_{k+1} - 2a_k\right)(k+r+2)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}\right]$
- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k-1}, a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}\right]$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}\right]$
- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k}, a_{k+1} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k}\right), a_{1+k} = -\frac{2\,\mathrm{I}}{7} a_k \sqrt{7}, b_{1+k} = -\frac{2\,\mathrm{I}}{7} b_k \sqrt{7}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(1+x+2*x^2)*diff(y(x),x$2)+(2+8*x)*diff(y(x),x)+4*y(x)=0,y(0) = -1, D(y)(0) = 2],y(x
```

$$y(x) = -1 + 2x - 4x^3 + 4x^4 + 4x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 23

```
AsymptoticDSolveValue[{(1+x+2*x^2)*y''[x]+(2+8*x)*y'[x]+4*y[x]==0,{y[0]==-1,y'[0]==2}},y[x],
```

$$y(x) \to 4x^5 + 4x^4 - 4x^3 + 2x - 1$$

13.3 problem 3

Internal problem ID [1244]
Internal file name [OUTPUT/1245_Sunday_June_05_2022_02_06_14_AM_91059259/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(-2x^2+1)\,y''+(2-6x)\,y'-2y=0}$$

With initial conditions

$$[y(0)=1, y'(0)=0]$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (935) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (936) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(3y'x - y' + y)}{2x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(44x^2 - 32x + 12)\, y' + (20x - 4)\, y}{(2x^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-400x^3 + 464x^2 - 340x + 60)\, y' + (-208x^2 + 96x - 44)\, y}{(2x^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(4384x^4 - 7104x^3 + 7688x^2 - 2784x + 504)\, y' + 2464y\left(x^3 - \frac{59}{77}x^2 + \frac{29}{44}x - \frac{27}{308}\right)}{(2x^2 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-56448x^5 + 118656x^4 - 169344x^3 + 93824x^2 - 33656x + 4008)\, y' - 33408y\left(x^4 - \frac{32}{29}x^3 + \frac{79}{58}x^2 - \cdots\right.}{(2x^2 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= -4 \\
F_2 &= 44 \\
F_3 &= -216 \\
F_4 &= 2632
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + \frac{329x^6}{90} + O(x^6)
$$

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + \frac{329x^6}{90} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-2x^2 + 1) y'' + (2 - 6x) y' - 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-2x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + (2 - 6x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-2x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1} \right) + \sum_{n=1}^{\infty} (-6n a_n x^n) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty}\left(-2x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^{n}\right) + \left(\sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n}\right)+\sum_{n=1}^{\infty}\left(-6 n a_{n} x^{n}\right)+\sum_{n=0}^{\infty}\left(-2 a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+2a_1-2a_0=0$$

$$a_2=a_0-a_1$$

$n=1$ gives

$$6a_3+4a_2-8a_1=0$$

Which after substituting earlier equations, simplifies to

$$a_3=-\frac{2a_0}{3}+2a_1$$

For $2\leq n$, the recurrence equation is

$$-2na_n(n-1)+(n+2)\,a_{n+2}(n+1)+2(n+1)\,a_{n+1}-6na_n-2a_n=0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2na_n+2a_n-2a_{n+1}}{n+2} \\ &= \frac{2(n+1)\,a_n}{n+2}-\frac{2a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n=2$ the recurrence equation gives

$$-18a_2+12a_4+6a_3=0$$

Which after substituting the earlier terms found becomes

$$a_4=\frac{11a_0}{6}-\frac{5a_1}{2}$$

For $n = 3$ the recurrence equation gives

$$-32a_3 + 20a_5 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{9a_0}{5} + \frac{21a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$-50a_4 + 30a_6 + 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{329a_0}{90} - \frac{167a_1}{30}$$

For $n = 5$ the recurrence equation gives

$$-72a_5 + 42a_7 + 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1301a_0}{315} + \frac{923a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + (a_0 - a_1)x^2 + \left(-\frac{2a_0}{3} + 2a_1\right)x^3 \\ + \left(\frac{11a_0}{6} - \frac{5a_1}{2}\right)x^4 + \left(-\frac{9a_0}{5} + \frac{21a_1}{5}\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(x^2 + 1 - \frac{2}{3}x^3 + \frac{11}{6}x^4 - \frac{9}{5}x^5\right) a_0 + \left(x - x^2 + 2x^3 - \frac{5}{2}x^4 + \frac{21}{5}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(x^2 + 1 - \frac{2}{3}x^3 + \frac{11}{6}x^4 - \frac{9}{5}x^5\right) c_1 + \left(x - x^2 + 2x^3 - \frac{5}{2}x^4 + \frac{21}{5}x^5\right) c_2 + O(x^6)$$

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + \frac{329x^6}{90} + O(x^6) \quad (1)$$

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + O(x^6) \quad (2)$$

<u>Verification of solutions</u>

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + \frac{329x^6}{90} + O(x^6)$$

Verified OK.

$$y = x^2 + 1 - \frac{2x^3}{3} + \frac{11x^4}{6} - \frac{9x^5}{5} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(1-2*x^2)*diff(y(x),x$2)+(2-6*x)*diff(y(x),x)-2*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),t
```

$$y(x) = 1 + x^2 - \frac{2}{3}x^3 + \frac{11}{6}x^4 - \frac{9}{5}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 29

```
AsymptoticDSolveValue[{(1-2*x^2)*y''[x]+(2-6*x)*y'[x]-2*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,
```

$$y(x) \to -\frac{9x^5}{5} + \frac{11x^4}{6} - \frac{2x^3}{3} + x^2 + 1$$

13.4 problem 4

Internal problem ID [1245]
Internal file name [`OUTPUT/1246_Sunday_June_05_2022_02_06_21_AM_14952766/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2 + x + 1)\, y'' + (2 + 15x)\, y' + 12y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

13.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2 + 15x}{3x^2 + x + 1} \\ q(x) &= \frac{12}{3x^2 + x + 1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(2+15x)\,y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$$

The domain of $p(x) = \frac{2+15x}{3x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{12}{3x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{938}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{939}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{15y'x + 2y' + 12y}{3x^2 + x + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(234x^2 + 60x - 21)\, y' + (252x + 36)\, y}{(3x^2 + x + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-4158x^3 - 1548x^2 + 1143x + 180)\, y' - 5076y\left(x^2 + \frac{13}{47}x - \frac{4}{47}\right)}{(3x^2 + x + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(84564x^4 + 40824x^3 - 47304x^2 - 14580x + 675)\, y' + 110808\left(x^3 + \frac{23}{57}x^2 - \frac{89}{342}x - \frac{5}{114}\right) y}{(3x^2 + x + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-1950804x^5 - 1148904x^4 + 1845828x^3 + 837864x^2 - 81729x - 23490)\, y' - 2676888\left(x^4 + \frac{241}{459}x^3 - \right.}{(3x^2 + x + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= -21 \\
F_2 &= 180 \\
F_3 &= 675 \\
F_4 &= -23490
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 + x - \frac{7x^3}{2} + \frac{15x^4}{2} + \frac{45x^5}{8} - \frac{261x^6}{8} + O(x^6)
$$

$$
y = -x^{2}+x -\frac{7 x^{3}}{2}+\frac{15 x^{4}}{2}+\frac{45 x^{5}}{8}-\frac{261 x^{6}}{8}+O\left(x^{6}\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(3x^{2}+x +1\right) y''+\left(2+15 x \right) y'+12 y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n =0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n =1}^{\infty} n a_{n} x^{n -1} \\
y'' &= \sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(3x^{2}+x +1\right) \left(\sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}\right)+\left(2+15 x \right) \left(\sum_{n =1}^{\infty} n a_{n} x^{n -1}\right)+12\left(\sum_{n =0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n =2}^{\infty} 3 x^{n} a_{n} n \left(n -1\right)\right)+\left(\sum_{n =2}^{\infty} n \,x^{n -1} a_{n} \left(n -1\right)\right)+\left(\sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2}\right) \\
&+\left(\sum_{n =1}^{\infty} 2 n a_{n} x^{n -1}\right)+\left(\sum_{n =1}^{\infty} 15 n a_{n} x^{n}\right)+\left(\sum_{n =0}^{\infty} 12 a_{n} x^{n}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n =2}^{\infty} n \,x^{n -1} a_{n} \left(n -1\right) &= \sum_{n =1}^{\infty} \left(n +1\right) a_{n +1} n \,x^{n} \\
\sum_{n =2}^{\infty} n \left(n -1\right) a_{n} x^{n -2} &= \sum_{n =0}^{\infty} \left(n +2\right) a_{n +2} \left(n +1\right) x^{n} \\
\sum_{n =1}^{\infty} 2 n a_{n} x^{n -1} &= \sum_{n =0}^{\infty} 2 \left(n +1\right) a_{n +1} x^{n}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right)+\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right)+\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) \\ +\left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^n\right)+\left(\sum_{n=1}^{\infty} 15 n a_n x^n\right)+\left(\sum_{n=0}^{\infty} 12 a_n x^n\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 12a_0 = 0$$

$$a_2 = -6a_0 - a_1$$

$n = 1$ gives

$$6a_2 + 6a_3 + 27a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 6a_0 - \frac{7a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$3na_n(n-1)+(n+1)\, a_{n+1}n+(n+2)\, a_{n+2}(n+1)+2(n+1)\, a_{n+1}+15na_n+12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3na_n + na_{n+1} + 6a_n + a_{n+1}}{n+1} \\ &= -\frac{(3n+6)\, a_n}{n+1} - a_{n+1} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$48a_2 + 12a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 18a_0 + \frac{15a_1}{2}$$

For $n = 3$ the recurrence equation gives

$$75a_3 + 20a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{81a_0}{2} + \frac{45a_1}{8}$$

For $n = 4$ the recurrence equation gives

$$108a_4 + 30a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{243a_0}{10} - \frac{261a_1}{8}$$

For $n = 5$ the recurrence equation gives

$$147a_5 + 42a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3321a_0}{20} + \frac{207a_1}{16}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + (-6a_0 - a_1)\, x^2 + \left(6a_0 - \frac{7a_1}{2}\right) x^3 \\ + \left(18a_0 + \frac{15a_1}{2}\right) x^4 + \left(-\frac{81a_0}{2} + \frac{45a_1}{8}\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1-6x^2+6x^3+18x^4-\frac{81}{2}x^5\right)a_0+\left(-x^2+x-\frac{7}{2}x^3+\frac{15}{2}x^4+\frac{45}{8}x^5\right)a_1+O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1-6x^2+6x^3+18x^4-\frac{81}{2}x^5\right)c_1+\left(-x^2+x-\frac{7}{2}x^3+\frac{15}{2}x^4+\frac{45}{8}x^5\right)c_2+O\left(x^6\right)$$

$$y = -x^2+x-\frac{7x^3}{2}+\frac{15x^4}{2}+\frac{45x^5}{8}+O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = -x^2+x-\frac{7x^3}{2}+\frac{15x^4}{2}+\frac{45x^5}{8}-\frac{261x^6}{8}+O\left(x^6\right) \quad (1)$$

$$y = -x^2+x-\frac{7x^3}{2}+\frac{15x^4}{2}+\frac{45x^5}{8}+O\left(x^6\right) \quad (2)$$

Verification of solutions

$$y = -x^2+x-\frac{7x^3}{2}+\frac{15x^4}{2}+\frac{45x^5}{8}-\frac{261x^6}{8}+O\left(x^6\right)$$

Verified OK.

$$y = -x^2+x-\frac{7x^3}{2}+\frac{15x^4}{2}+\frac{45x^5}{8}+O\left(x^6\right)$$

Verified OK.

13.4.2 Maple step by step solution

Let's solve

$$\left[(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0, y(0)=0, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{12y}{3x^2+x+1} - \frac{(2+15x)y'}{3x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2+15x)y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

- ○ $\left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right) \cdot P_2(x)$ is analytic at $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{I\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ○ $\left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(\frac{I\sqrt{11}}{6} + x + \frac{1}{6}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{I\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ○ $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

- Multiply by denominators

  $(3x^2 + x + 1)\, y'' + (2 + 15x)\, y' + 12y = 0$

- Change variables using $x = u - \frac{I\sqrt{11}}{6} - \frac{1}{6}$ so that the regular singular point is at $u = 0$

  $(3u^2 - Iu\sqrt{11})\left(\frac{d^2}{du^2}y(u)\right) + \left(-\frac{1}{2} + 15u - \frac{5\,I\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\, u^{k+r-1+m}$

- ○ Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right)r a_0 u^{-1+r}}{22} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)(k+1+r)a_{k+1}}{22} + 3a_k(k+r+2)^2\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$\frac{\mathrm{I}}{22}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right)r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3a_k(k+r+2)^2 - (k+1+r)\,a_{k+1}\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}$

  $$\left[y = \sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Recursion relation for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

  $$a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+\frac{111\,\mathrm{I}\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}\right.$$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2\right.}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}\right), a_{1+k}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
Order:=6;
dsolve([(1+x+3*x^2)*diff(y(x),x$2)+(2+15*x)*diff(y(x),x)+12*y(x)=0,y(0) = 0, D(y)(0) = 1],y(
```

$$y(x) = x - x^2 - \frac{7}{2}x^3 + \frac{15}{2}x^4 + \frac{45}{8}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 31

```
AsymptoticDSolveValue[{(1+x+3*x^2)*y''[x]+(2+15*x)*y'[x]+12*y[x]==0,{y[0]==0,y'[0]==1}},y[x]
```

$$y(x) \to \frac{45x^5}{8} + \frac{15x^4}{2} - \frac{7x^3}{2} - x^2 + x$$

13.5 problem 5

Internal problem ID [1246]
Internal file name [`OUTPUT/1247_Sunday_June_05_2022_02_06_25_AM_51456129/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2+x)\,y'' + (x+1)\,y' + 3y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = 3]$$

With the expansion point for the power series method at $x = 0$.

13.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{x+1}{2+x} \\ q(x) &= \frac{3}{2+x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(x+1)\,y'}{2+x} + \frac{3y}{2+x} = 0$$

The domain of $p(x) = \frac{x+1}{2+x}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{3}{2+x}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{941}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{942}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x + y' + 3y}{2 + x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x - 3)\, y' + 3y}{2 + x} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^2 + 5x + 14)\, y' - 3(-2 + x)\, y}{(2 + x)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 - 7x^2 - 28x - 20)\, y' + 3y(x^2 - 4x - 20)}{(2 + x)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^4 + 9x^3 + 42x^2 - 8x - 96)\, y' - 3y(x^3 - 6x^2 - 40x - 72)}{(2 + x)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 4$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= -\frac{15}{2} \\
F_1 &= \frac{3}{2} \\
F_2 &= \frac{33}{2} \\
F_3 &= -\frac{75}{2} \\
F_4 &= 36
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 3x + 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + \frac{x^6}{20} + O(x^6)$$

$$y = 3x + 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + \frac{x^6}{20} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2 + x) y'' + (x + 1) y' + 3y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2 + x) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (x + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + a_1 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4} - \frac{a_1}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 2(n+2)\, a_{n+2}(n+1) + n a_n + (n+1)\, a_{n+1} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_n + 2n a_{n+1} + 3a_n + a_{n+1}}{2\,(n+2)\,(n+1)} \\ &= -\frac{(n+3)\, a_n}{2\,(n+2)\,(n+1)} - \frac{(n^2+2n+1)\, a_{n+1}}{2\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$4a_2 + 12a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{4} - \frac{a_1}{4}$$

For $n = 2$ the recurrence equation gives

$$9a_3 + 24a_4 + 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{16} + \frac{7a_1}{48}$$

For $n = 3$ the recurrence equation gives

$$16a_4 + 40a_5 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{16} - \frac{a_1}{48}$$

For $n = 4$ the recurrence equation gives

$$25a_5 + 60a_6 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{3a_0}{160} - \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$36a_6 + 84a_7 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_0}{480} + \frac{a_1}{180}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \left(-\frac{3a_0}{4} - \frac{a_1}{4}\right)x^2 + \left(\frac{a_0}{4} - \frac{a_1}{4}\right)x^3 + \left(\frac{a_0}{16} + \frac{7a_1}{48}\right)x^4 + \left(-\frac{a_0}{16} - \frac{a_1}{48}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^4 - \frac{1}{16}x^5\right)a_0 + \left(x - \frac{1}{4}x^2 - \frac{1}{4}x^3 + \frac{7}{48}x^4 - \frac{1}{48}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^4 - \frac{1}{16}x^5\right)c_1 + \left(x - \frac{1}{4}x^2 - \frac{1}{4}x^3 + \frac{7}{48}x^4 - \frac{1}{48}x^5\right)c_2 + O(x^6)$$

$$y = 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + 3x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 3x + 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + \frac{x^6}{20} + O(x^6) \tag{1}$$
$$y = 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + 3x + O(x^6) \tag{2}$$

Verification of solutions

$$y = 3x + 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + \frac{x^6}{20} + O(x^6)$$

Verified OK.

$$y = 4 - \frac{15x^2}{4} + \frac{x^3}{4} + \frac{11x^4}{16} - \frac{5x^5}{16} + 3x + O(x^6)$$

Verified OK.

13.5.2 Maple step by step solution

Let's solve

$$\left[(2+x)\, y''+(x+1)\, y'+3y=0, y(0)=4, y'\bigg|_{\{x=0\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{2+x}-\frac{(x+1)y'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+1)y'}{2+x}+\frac{3y}{2+x}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+1}{2+x}, P_3(x)=\frac{3}{2+x}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=-1$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(2+x)\, y''+(x+1)\, y'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2+x)*diff(y(x),x$2)+(1+x)*diff(y(x),x)+3*y(x)=0,y(0) = 4, D(y)(0) = 3],y(x),type='s
```

$$y(x) = 4 + 3x - \frac{15}{4}x^2 + \frac{1}{4}x^3 + \frac{11}{16}x^4 - \frac{5}{16}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(2+x)*y''[x]+(1+x)*y'[x]+3*y[x]==0,{y[0]==4,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{5x^5}{16} + \frac{11x^4}{16} + \frac{x^3}{4} - \frac{15x^2}{4} + 3x + 4$$

13.6 problem 6

Internal problem ID [1247]
Internal file name [OUTPUT/1248_Sunday_June_05_2022_02_06_27_AM_57198332/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(x^{2}+3x+3\right) y''+\left(6+4x\right) y'+2y=0}$$

With initial conditions

$$[y(0)=7, y'(0)=3]$$

With the expansion point for the power series method at $x=0$.

13.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= \frac{6+4x}{x^2+3x+3}\\ q(x) &= \frac{2}{x^2+3x+3}\\ F &= 0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(6+4x)\,y'}{x^2+3x+3} + \frac{2y}{x^2+3x+3} = 0$$

The domain of $p(x) = \frac{6+4x}{x^2+3x+3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{x^2+3x+3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots\\ &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}f'|_{x_0,y_0,y'_0} + \cdots\\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (944)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (945)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(2y'x + 3y' + y)}{x^2 + 3x + 3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{18(x^2 + 3x + 2)\, y' + 6(2x + 3)\, y}{(x^2 + 3x + 3)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-96x^3 - 432x^2 - 576x - 216)\, y' - 72(2 + x)\, y(x + 1)}{(x^2 + 3x + 3)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(600x^4 + 3600x^3 + 7200x^2 + 5400x + 1080)\, y' + 480\left(x^2 + 3x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) y}{(x^2 + 3x + 3)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x^5 - 32400x^4 - 86400x^3 - 97200x^2 - 38880x)\, y' - 3600\left(x^4 + 6x^3 + 12x^2 + 9x + \frac{9}{5}\right) y}{(x^2 + 3x + 3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 7$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= -\frac{32}{3} \\
F_1 &= 26 \\
F_2 &= -\frac{184}{3} \\
F_3 &= \frac{400}{3} \\
F_4 &= -\frac{560}{3}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 7 + 3x - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} - \frac{7x^6}{27} + O(x^6)$$

$$y = 7 + 3x - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} - \frac{7x^6}{27} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 3x + 3)\, y'' + (6 + 4x)\, y' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 3x + 3)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (6 + 4x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 6n a_n x^{n-1} = \sum_{n=0}^{\infty} 6(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, x^n\right)\\&+\left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 6(n+1)\, a_{n+1} x^n\right)\\&+\left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$6a_2 + 6a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{3} - a_1$$

$n = 1$ gives

$$18a_2 + 18a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{3} + \frac{2a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 3(n+1) a_{n+1} n + 3(n+2) a_{n+2} (n+1) + 6(n+1) a_{n+1} + 4 n a_n + 2 a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{a_n}{3} - a_{n+1} \\ &= -\frac{a_n}{3} - a_{n+1} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12 a_2 + 36 a_3 + 36 a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{2 a_0}{9} - \frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20 a_3 + 60 a_4 + 60 a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{9} + \frac{a_1}{9}$$

For $n = 4$ the recurrence equation gives

$$30 a_4 + 90 a_5 + 90 a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{27}$$

For $n = 5$ the recurrence equation gives

$$42 a_5 + 126 a_6 + 126 a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{27}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{3} - a_1\right) x^2 + \left(\frac{a_0}{3} + \frac{2a_1}{3}\right) x^3 + \left(-\frac{2a_0}{9} - \frac{a_1}{3}\right) x^4 + \left(\frac{a_0}{9} + \frac{a_1}{9}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{3}x^2 + \frac{1}{3}x^3 - \frac{2}{9}x^4 + \frac{1}{9}x^5\right) a_0 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{9}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{3}x^2 + \frac{1}{3}x^3 - \frac{2}{9}x^4 + \frac{1}{9}x^5\right) c_1 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{9}x^5\right) c_2 + O(x^6)$$

$$y = 7 - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} + 3x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 7 + 3x - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} - \frac{7x^6}{27} + O(x^6) \quad (1)$$

$$y = 7 - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} + 3x + O(x^6) \quad (2)$$

Verification of solutions

$$y = 7 + 3x - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} - \frac{7x^6}{27} + O(x^6)$$

Verified OK.

$$y = 7 - \frac{16x^2}{3} + \frac{13x^3}{3} - \frac{23x^4}{9} + \frac{10x^5}{9} + 3x + O(x^6)$$

Verified OK.

13.6.2 Maple step by step solution

Let's solve

$$\left[\left(x^2+3x+3\right)y''+\left(6+4x\right)y'+2y=0, y(0)=7, y'\Big|_{\{x=0\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x^2+3x+3}-\frac{2(2x+3)y'}{x^2+3x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(2x+3)y'}{x^2+3x+3}+\frac{2y}{x^2+3x+3}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(2x+3)}{x^2+3x+3}, P_3(x)=\frac{2}{x^2+3x+3}\right]$

  ◦ $\left(\frac{I\sqrt{3}}{2}+x+\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{I\sqrt{3}}{2}-\frac{3}{2}$

  $\left(\left(\frac{I\sqrt{3}}{2}+x+\frac{3}{2}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{I\sqrt{3}}{2}-\frac{3}{2}}=0$

  ◦ $\left(\frac{I\sqrt{3}}{2}+x+\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{I\sqrt{3}}{2}-\frac{3}{2}$

  $\left(\left(\frac{I\sqrt{3}}{2}+x+\frac{3}{2}\right)^2\cdot P_3(x)\right)\Bigg|_{x=-\frac{I\sqrt{3}}{2}-\frac{3}{2}}=0$

  ◦ $x=-\frac{I\sqrt{3}}{2}-\frac{3}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{I\sqrt{3}}{2}-\frac{3}{2}$

- Multiply by denominators

  $\left(x^2+3x+3\right)y''+\left(6+4x\right)y'+2y=0$

- Change variables using $x=u-\frac{I\sqrt{3}}{2}-\frac{3}{2}$ so that the regular singular point is at $u=0$

  $\left(u^2-Iu\sqrt{3}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(4u-2I\sqrt{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-\mathrm{I}\sqrt{3}\, r(r+1)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-\mathrm{I}\sqrt{3}\,(k+r+1)\,(k+r+2)\, a_{k+1} + a_k(k+r+2)\,(k+r+1)\right) u$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-\mathrm{I}\sqrt{3}\, r(r+1) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $\left(-\mathrm{I}\sqrt{3}\, a_{k+1} + a_k\right)(k+r+2)\,(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{\mathrm{I}}{3} a_k \sqrt{3}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{\mathrm{I}}{3} a_k \sqrt{3}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = -\frac{\mathrm{I}}{3} a_k \sqrt{3}\right]$
- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{3}}{2} + x + \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x + \frac{3}{2}\right)^{k-1}, a_{k+1} = -\frac{I}{3} a_k \sqrt{3}\right]$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{I}{3} a_k \sqrt{3}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{I}{3} a_k \sqrt{3}\right]$$

- Revert the change of variables $u = \frac{I\sqrt{3}}{2} + x + \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x + \frac{3}{2}\right)^{k}, a_{k+1} = -\frac{I}{3} a_k \sqrt{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x + \frac{3}{2}\right)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{3}}{2} + x + \frac{3}{2}\right)^{k}\right), a_{1+k} = -\frac{I}{3} a_k \sqrt{3}, b_{1+k} = -\frac{I}{3} b_k \sqrt{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(3+3*x+x^2)*diff(y(x),x$2)+(6+4*x)*diff(y(x),x)+2*y(x)=0,y(0) = 7, D(y)(0) = 3],y(x)
```

$$y(x) = 7 + 3x - \frac{16}{3}x^2 + \frac{13}{3}x^3 - \frac{23}{9}x^4 + \frac{10}{9}x^5 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(3+3*x+x^2)*y''[x]+(6+4*x)*y'[x]+2*y[x]==0,{y[0]==7,y'[0]==3}},y[x],{
```

$$y(x) \to \frac{10x^5}{9} - \frac{23x^4}{9} + \frac{13x^3}{3} - \frac{16x^2}{3} + 3x + 7$$

13.7 problem 7

Internal problem ID [1248]
Internal file name [`OUTPUT/1249_Sunday_June_05_2022_02_06_29_AM_3004114/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+4)\,y''+(2+x)\,y'+2y=0}$$

With initial conditions

$$[y(0)=2,y'(0)=5]$$

With the expansion point for the power series method at $x=0$.

13.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2+x}{x+4}\\ q(x)&=\frac{2}{x+4}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(2+x)\,y'}{x+4} + \frac{2y}{x+4} = 0$$

The domain of $p(x) = \frac{2+x}{x+4}$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2}{x+4}$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{947}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{948}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x+2y'+2y}{x+4} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{(x^2+2x-6)\,y'+2y(x+3)}{(x+4)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(-x^2+2x+14)\,y'-2y(x-1)}{(x+4)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(x^2-6x-10)\,y'+2y(x-5)}{(x+4)^2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(-x^3+6x^2+34x-24)\,y'-2y(x+3)\,(x-8)}{(x+4)^3}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=2$ and $y'(0)=5$ gives

$$
\begin{aligned}
F_0 &= -\frac{7}{2} \\
F_1 &= -\frac{9}{8} \\
F_2 &= \frac{37}{8} \\
F_3 &= -\frac{35}{8} \\
F_4 &= -\frac{3}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 5x + 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} - \frac{x^6}{1920} + O(x^6)$$

$$y = 5x + 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} - \frac{x^6}{1920} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x+4)\, y'' + (2+x)\, y' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x+4)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (2+x)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, x^{n}$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2} (n+1)\, x^{n}\right) + \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n}\right) + \left(\sum_{n=0}^{\infty} 2 a_n x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 2a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4} - \frac{a_1}{4}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 4(n+2)\, a_{n+2} (n+1) + 2(n+1)\, a_{n+1} + n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_{n+1} + a_n + a_{n+1}}{4\,(n+1)} \\ &= -\frac{a_n}{4\,(n+1)} - \frac{a_{n+1}}{4} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 + 24a_3 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{16} - \frac{a_1}{16}$$

For $n = 2$ the recurrence equation gives

$$12a_3 + 48a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{192} + \frac{7a_1}{192}$$

For $n = 3$ the recurrence equation gives

$$20a_4 + 80a_5 + 5a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{192} - \frac{a_1}{192}$$

For $n = 4$ the recurrence equation gives

$$30a_5 + 120a_6 + 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{960} - \frac{a_1}{1920}$$

For $n = 5$ the recurrence equation gives

$$42a_6 + 168a_7 + 7a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_0}{23040} + \frac{a_1}{2880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{4} - \frac{a_1}{4}\right) x^2 + \left(\frac{a_0}{16} - \frac{a_1}{16}\right) x^3 + \left(\frac{a_0}{192} + \frac{7a_1}{192}\right) x^4 + \left(-\frac{a_0}{192} - \frac{a_1}{192}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$\begin{aligned} y = &\left(1 - \frac{1}{4}x^2 + \frac{1}{16}x^3 + \frac{1}{192}x^4 - \frac{1}{192}x^5\right) a_0 \\ &+ \left(x - \frac{1}{4}x^2 - \frac{1}{16}x^3 + \frac{7}{192}x^4 - \frac{1}{192}x^5\right) a_1 + O(x^6) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{4}x^2 + \frac{1}{16}x^3 + \frac{1}{192}x^4 - \frac{1}{192}x^5\right) c_1 + \left(x - \frac{1}{4}x^2 - \frac{1}{16}x^3 + \frac{7}{192}x^4 - \frac{1}{192}x^5\right) c_2 + O(x^6)$$

$$y = 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} + 5x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 5x + 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} - \frac{x^6}{1920} + O(x^6) \tag{1}$$

$$y = 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} + 5x + O(x^6) \tag{2}$$

Verification of solutions

$$y = 5x + 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} - \frac{x^6}{1920} + O(x^6)$$

Verified OK.

$$y = 2 - \frac{7x^2}{4} - \frac{3x^3}{16} + \frac{37x^4}{192} - \frac{7x^5}{192} + 5x + O(x^6)$$

Verified OK.

13.7.2 Maple step by step solution

Let's solve

$$\left[(x+4)\,y''+(2+x)\,y'+2y=0, y(0)=2, y'\Big|_{\{x=0\}}=5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x+4}-\frac{(2+x)y'}{x+4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2+x)y'}{x+4}+\frac{2y}{x+4}=0$

☐ Check to see if $x_0=-4$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2+x}{x+4}, P_3(x)=\frac{2}{x+4}\right]$

  ◦ $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

  $\left((x+4)\cdot P_2(x)\right)\Big|_{x=-4}=-2$

  ◦ $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

  $\left((x+4)^2\cdot P_3(x)\right)\Big|_{x=-4}=0$

  ◦ $x=-4$is a regular singular point

  Check to see if $x_0=-4$ is a regular singular point

  $x_0=-4$

- Multiply by denominators

  $(x+4)\,y''+(2+x)\,y'+2y=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-2+u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$
- Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$
- Solution for $r = 3$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1}=-\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u=x+4$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+4)^{k+3}, a_{k+1}=-\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
Order:=6;
dsolve([(4+x)*diff(y(x),x$2)+(2+x)*diff(y(x),x)+2*y(x)=0,y(0) = 2, D(y)(0) = 5],y(x),type='s
```

$$y(x)=2+5x-\frac{7}{4}x^2-\frac{3}{16}x^3+\frac{37}{192}x^4-\frac{7}{192}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(4+x)*y''[x]+(2+x)*y'[x]+2*y[x]==0,{y[0]==4,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x)\to -\frac{7x^5}{192}+\frac{25x^4}{192}+\frac{x^3}{16}-\frac{7x^2}{4}+3x+4$$

13.8 problem 8

Internal problem ID [1249]
Internal file name [`OUTPUT/1250_Sunday_June_05_2022_02_06_31_AM_53835349/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(2x^2 - 3x + 2)\, y'' - (4 - 6x)\, y' + 2y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = -1]$$

With the expansion point for the power series method at $x = 1$.

13.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{6x - 4}{2x^2 - 3x + 2} \\ q(x) &= \frac{2}{2x^2 - 3x + 2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(6x-4)\, y'}{2x^2-3x+2} + \frac{2y}{2x^2-3x+2} = 0$$

The domain of $p(x) = \frac{6x-4}{2x^2-3x+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{2x^2-3x+2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(t+1)^2 - 3t - 1\right)\left(\frac{d^2}{dt^2}y(t)\right) + (6t+2)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = -1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (950) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (951) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2\left(3t\left(\frac{d}{dt}y(t)\right) + \frac{d}{dt}y(t) + y(t)\right)}{2t^2 + t + 1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(44t^2 + 30t - 2\right)\left(\frac{d}{dt}y(t)\right) + \left(20t + 6\right)y(t)}{\left(2t^2 + t + 1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-400t^3 - 416t^2 + 52t + 44\right)\left(\frac{d}{dt}y(t)\right) - 208\left(t^2 + \frac{8}{13}t - \frac{3}{52}\right)y(t)}{\left(2t^2 + t + 1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(4384t^4 + 6160t^3 - 1096t^2 - 1948t - 156\right)\left(\frac{d}{dt}y(t)\right) + 2464\left(t^3 + \frac{145}{154}t^2 - \frac{51}{308}t - \frac{9}{88}\right)y(t)}{\left(2t^2 + t + 1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-56448t^5 - 100224t^4 + 22752t^3 + 63232t^2 + 10320t - 1264\right)\left(\frac{d}{dt}y(t)\right) - 33408y(t)\left(t^4 + \frac{37}{29}t^3 - \frac{37}{116}t\right.}{\left(2t^2 + t + 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 8 \\
F_2 &= -32 \\
F_3 &= -96 \\
F_4 &= 2176
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 - t + \frac{4t^3}{3} - \frac{4t^4}{3} - \frac{4t^5}{5} + \frac{136t^6}{45} + O(t^6)$$

$$y(t) = 1 - t + \frac{4t^3}{3} - \frac{4t^4}{3} - \frac{4t^5}{5} + \frac{136t^6}{45} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2 + t + 1) + (6t + 2)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\ \frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right)(2t^2 + t + 1) + (6t + 2)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) \\ &+ \left(\sum_{n=1}^{\infty} 6n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

$$\sum_{n=1}^{\infty} 2n a_n t^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 6n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 2a_0 = 0$$

$$a_2 = -a_0 - a_1$$

$n = 1$ gives

$$6a_2 + 6a_3 + 8a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_0 - \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$2n a_n(n-1) + (n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + 6n a_n + 2(n+1)\, a_{n+1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{2na_n + na_{n+1} + 2a_n + 2a_{n+1}}{n+2} \\
&= -\frac{(2n+2)\, a_n}{n+2} - a_{n+1}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$18a_2 + 12a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2} + \frac{11a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$32a_3 + 20a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{21a_0}{10} - \frac{13a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$50a_4 + 30a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19a_0}{15} - \frac{79a_1}{45}$$

For $n = 5$ the recurrence equation gives

$$72a_5 + 42a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{7a_0}{3} + \frac{251a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + (-a_0 - a_1) t^2 + \left(a_0 - \frac{a_1}{3}\right) t^3 + \left(\frac{a_0}{2} + \frac{11 a_1}{6}\right) t^4 + \left(-\frac{21 a_0}{10} - \frac{13 a_1}{10}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - t^2 + t^3 + \frac{1}{2} t^4 - \frac{21}{10} t^5\right) a_0 + \left(t - t^2 - \frac{1}{3} t^3 + \frac{11}{6} t^4 - \frac{13}{10} t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - t^2 + t^3 + \frac{1}{2} t^4 - \frac{21}{10} t^5\right) c_1 + \left(t - t^2 - \frac{1}{3} t^3 + \frac{11}{6} t^4 - \frac{13}{10} t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 + \frac{4t^3}{3} - \frac{4t^4}{3} - \frac{4t^5}{5} - t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = 2 - x + \frac{4(x-1)^3}{3} - \frac{4(x-1)^4}{3} - \frac{4(x-1)^5}{5} + \frac{136(x-1)^6}{45} + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = 2 - x + \frac{4(x-1)^3}{3} - \frac{4(x-1)^4}{3} - \frac{4(x-1)^5}{5} + \frac{136(x-1)^6}{45} + O\left((x-1)^6\right) \quad (1)$$

Verification of solutions

$$y = 2 - x + \frac{4(x-1)^3}{3} - \frac{4(x-1)^4}{3} - \frac{4(x-1)^5}{5} + \frac{136(x-1)^6}{45} + O\left((x-1)^6\right)$$

Verified OK.

13.8.2 Maple step by step solution

Let's solve

$$\left[\left(2x^2-3x+2\right)y''+\left(6x-4\right)y'+2y=0, y(1)=1, y'\Big|_{\{x=1\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{2x^2-3x+2}-\frac{2(3x-2)y'}{2x^2-3x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(3x-2)y'}{2x^2-3x+2}+\frac{2y}{2x^2-3x+2}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{2(3x-2)}{2x^2-3x+2}, P_3(x)=\frac{2}{2x^2-3x+2}\right]$

- ○ $\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{3}{4}\right)\cdot P_2(x)$ is analytic at $x=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}$

  $\left(\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{3}{4}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}}=0$

- ○ $\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{3}{4}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}$

  $\left(\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{3}{4}\right)^2\cdot P_3(x)\right)\Bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}}=0$

- ○ $x=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}$

- Multiply by denominators

  $\left(2x^2-3x+2\right)y''+\left(6x-4\right)y'+2y=0$

- Change variables using $x=u-\frac{\mathrm{I}\sqrt{7}}{4}+\frac{3}{4}$ so that the regular singular point is at $u=0$

  $\left(2u^2-\mathrm{I}u\sqrt{7}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(6u-\frac{3\,\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$
  - Shift index using $k->k+2-m$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-\frac{\mathrm{I}\sqrt{7}\, r\left(\mathrm{I}\sqrt{7}+7+14r\right) a_0 u^{-1+r}}{14} + \left(\sum_{k=0}^{\infty}\left(-\frac{\mathrm{I}\sqrt{7}\,(k+r+1)\left(\mathrm{I}\sqrt{7}+14k+21+14r\right) a_{k+1}}{14} + 2a_k(k+r+1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-\frac{\mathrm{I}}{14}\sqrt{7}\, r\left(\mathrm{I}\sqrt{7}+7+14r\right) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{14}\right\}$$
- Each term in the series must be 0, giving the recursion relation

  $$-(k+r+1)\left(\mathrm{I}\left(k+r+\frac{3}{2}\right) a_{k+1}\sqrt{7} - \frac{a_{k+1}}{2} + (-2k-2r-2)\, a_k\right) = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{4a_k(k+r+1)}{-1+2\,\mathrm{I}\sqrt{7}\,k+2\,\mathrm{I}\sqrt{7}\,r+3\,\mathrm{I}\sqrt{7}}$$
- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{4a_k(k+1)}{-1+2\,\mathrm{I}\sqrt{7}\,k+3\,\mathrm{I}\sqrt{7}}$$
- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{4a_k(k+1)}{-1+2\,\mathrm{I}\sqrt{7}\,k+3\,\mathrm{I}\sqrt{7}}\right]$$
- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{7}}{4} + x - \frac{3}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4}+x-\frac{3}{4}\right)^k, a_{k+1}=\frac{4a_k(k+1)}{-1+2I\sqrt{7}\,k+3I\sqrt{7}}\right]$$

- Recursion relation for $r=-\frac{1}{2}-\frac{I\sqrt{7}}{14}$

$$a_{k+1}=\frac{4a_k\left(k+\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)}{-1+2I\sqrt{7}\,k+2I\sqrt{7}\left(-\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)+3I\sqrt{7}}$$

- Solution for $r=-\frac{1}{2}-\frac{I\sqrt{7}}{14}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}-\frac{I\sqrt{7}}{14}}, a_{k+1}=\frac{4a_k\left(k+\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)}{-1+2I\sqrt{7}\,k+2I\sqrt{7}\left(-\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)+3I\sqrt{7}}\right]$$

- Revert the change of variables $u=\frac{I\sqrt{7}}{4}+x-\frac{3}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4}+x-\frac{3}{4}\right)^{k-\frac{1}{2}-\frac{I\sqrt{7}}{14}}, a_{k+1}=\frac{4a_k\left(k+\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)}{-1+2I\sqrt{7}\,k+2I\sqrt{7}\left(-\frac{1}{2}-\frac{I\sqrt{7}}{14}\right)+3I\sqrt{7}}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4}+x-\frac{3}{4}\right)^k\right)+\left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{7}}{4}+x-\frac{3}{4}\right)^{k-\frac{1}{2}-\frac{I\sqrt{7}}{14}}\right), a_{1+k}=\frac{4a_k(1+k)}{-1+2I\sqrt{7}\,k+3I\sqrt{7}}, b_{1+k}=$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(2-3*x+2*x^2)*diff(y(x),x$2)-(4-6*x)*diff(y(x),x)+2*y(x)=0,y(1) = 1, D(y)(1) = -1],y
```

$$y(x) = 1 - (x-1) + \frac{4}{3}(x-1)^3 - \frac{4}{3}(x-1)^4 - \frac{4}{5}(x-1)^5 + O\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[{(2-3*x+2*x^2)*y''[x]-(4-6*x)*y'[x]+2*y[x]==0,{y[1]==1,y'[1]==-1}},y[x
```

$$y(x) \to -\frac{4}{5}(x-1)^5 - \frac{4}{3}(x-1)^4 + \frac{4}{3}(x-1)^3 - x + 2$$

13.9 problem 9

Internal problem ID [1250]
Internal file name [OUTPUT/1251_Sunday_June_05_2022_02_06_37_AM_37683215/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+3x)\, y''+10(x+1)\, y'+8y=0}$$

With initial conditions

$$[y(-1)=1, y'(-1)=-1]$$

With the expansion point for the power series method at $x=-1$.

13.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{10x+10}{2x^2+3x} \\ q(x) &= \frac{8}{2x^2+3x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(10x + 10)\, y'}{2x^2 + 3x} + \frac{8y}{2x^2 + 3x} = 0$$

The domain of $p(x) = \frac{10x+10}{2x^2+3x}$ is

$$\left\{-\infty \le x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \le \infty\right\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{8}{2x^2+3x}$ is

$$\left\{-\infty \le x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \le \infty\right\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(-1 + t)^2 - 3 + 3t\right) \left(\frac{d^2}{dt^2} y(t)\right) + 10t \left(\frac{d}{dt} y(t)\right) + 8y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = -1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (953) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (954) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y'' \nonumber\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2\left(5t\left(\frac{d}{dt}y(t)\right) + 4y(t)\right)}{2t^2 - t - 1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(104t^2 + 8t + 18\right)\left(\frac{d}{dt}y(t)\right) + \left(112t - 8\right)y(t)}{\left(2t^2 - t - 1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-1232t^3 - 256t^2 - 628t + 36\right)\left(\frac{d}{dt}y(t)\right) - 1504\left(t^2 - \frac{7}{94}t + \frac{17}{94}\right)y(t)}{\left(2t^2 - t - 1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(16704t^4 + 6336t^3 + 16848t^2 - 1376t + 1008\right)\left(\frac{d}{dt}y(t)\right) + 21888\left(t^3 - \frac{1}{38}t^2 + \frac{10}{19}t - \frac{1}{18}\right)y(t)}{\left(2t^2 - t - 1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-256896t^5 - 149760t^4 - 429408t^3 + 34336t^2 - 74336t + 6624\right)\left(\frac{d}{dt}y(t)\right) - 352512\left(t^4 + \frac{19}{306}t^3 + \frac{35}{34}t^2 \right.}{\left(2t^2 - t - 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= 8 \\
F_1 &= -26 \\
F_2 &= 308 \\
F_3 &= -2224 \\
F_4 &= 31072
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 4t^2 - t + 1 - \frac{13t^3}{3} + \frac{77t^4}{6} - \frac{278t^5}{15} + \frac{1942t^6}{45} + O(t^6)$$

$$y(t) = 4t^2 - t + 1 - \frac{13t^3}{3} + \frac{77t^4}{6} - \frac{278t^5}{15} + \frac{1942t^6}{45} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2 - t - 1) + 10t\left(\frac{d}{dt}y(t)\right) + 8y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(2t^2 - t - 1) + 10t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-n\, t^{n-1} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 10 n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 8 a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n\, t^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-(n+1)\, a_{n+1} n\, t^n\right)$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (n+1)\, t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-(n+1)\, a_{n+1} n\, t^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 10 n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 8 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 8a_0 = 0$$

$$a_2 = 4a_0$$

$n = 1$ gives

$$-2a_2 - 6a_3 + 18a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_0}{3} + 3a_1$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) - (n+1)\, a_{n+1} n - (n+2)\, a_{n+2}(n+1) + 10na_n + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{2n^2 a_n - n^2 a_{n+1} + 8n a_n - n a_{n+1} + 8a_n}{(n+2)(n+1)} \\
&= \frac{(2n^2 + 8n + 8) a_n}{(n+2)(n+1)} + \frac{(-n^2 - n) a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$32a_2 - 6a_3 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{34a_0}{3} - \frac{3a_1}{2}$$

For $n = 3$ the recurrence equation gives

$$50a_3 - 12a_4 - 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{152a_0}{15} + \frac{42a_1}{5}$$

For $n = 4$ the recurrence equation gives

$$72a_4 - 20a_5 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1528a_0}{45} - \frac{46a_1}{5}$$

For $n = 5$ the recurrence equation gives

$$98a_5 - 30a_6 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{15088a_0}{315} + \frac{916a_1}{35}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + 4a_0 t^2 + \left(-\frac{4a_0}{3} + 3a_1\right) t^3 + \left(\frac{34a_0}{3} - \frac{3a_1}{2}\right) t^4 + \left(-\frac{152a_0}{15} + \frac{42a_1}{5}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + 4t^2 - \frac{4}{3}t^3 + \frac{34}{3}t^4 - \frac{152}{15}t^5\right) a_0 + \left(t + 3t^3 - \frac{3}{2}t^4 + \frac{42}{5}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + 4t^2 - \frac{4}{3}t^3 + \frac{34}{3}t^4 - \frac{152}{15}t^5\right) c_1 + \left(t + 3t^3 - \frac{3}{2}t^4 + \frac{42}{5}t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 + 4t^2 - \frac{13t^3}{3} + \frac{77t^4}{6} - \frac{278t^5}{15} - t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$y = 4(x+1)^2 - x - \frac{13(x+1)^3}{3} + \frac{77(x+1)^4}{6} - \frac{278(x+1)^5}{15} + \frac{1942(x+1)^6}{45} + O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = 4(x+1)^2 - x - \frac{13(x+1)^3}{3} + \frac{77(x+1)^4}{6} \\ - \frac{278(x+1)^5}{15} + \frac{1942(x+1)^6}{45} + O\left((x+1)^6\right) \end{aligned} \quad (1)$$

Verification of solutions

$$y = 4(x+1)^2 - x - \frac{13(x+1)^3}{3} + \frac{77(x+1)^4}{6} - \frac{278(x+1)^5}{15} + \frac{1942(x+1)^6}{45} + O\left((x+1)^6\right)$$

Verified OK.

13.9.2 Maple step by step solution

Let's solve

$$\left[\left(2x^2+3x\right)y''+\left(10x+10\right)y'+8y=0, y(-1)=1, y'\Big|_{\{x=-1\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{x(2x+3)}-\frac{10(x+1)y'}{x(2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{10(x+1)y'}{x(2x+3)}+\frac{8y}{x(2x+3)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{10(x+1)}{x(2x+3)}, P_3(x)=\frac{8}{x(2x+3)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{10}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(2x+3)+(10x+10)y'+8y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(7+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(7+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{7}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$$

- Recursion relation for $r = -\frac{7}{3}$

$$a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = -\frac{7}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k \left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{1+k} = -\frac{2a_k (k+2)^2}{(1+k)(3k+10)}, b_{1+k} = -\frac{2b_k \left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(3*x+2*x^2)*diff(y(x),x$2)+10*(1+x)*diff(y(x),x)+8*y(x)=0,y(-1) = 1, D(y)(-1) = -1],
```

$$y(x) = 1-(x+1)+4(x+1)^2-\frac{13}{3}(x+1)^3+\frac{77}{6}(x+1)^4-\frac{278}{15}(x+1)^5+\mathrm{O}\left((x+1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[{(3*x+2*x^2)*y''[x]+10*(1+x)*y'[x]+8*y[x]==0,{y[-1]==1,y'[-1]==-1}},y[
```

$$y(x)\to -\frac{278}{15}(x+1)^5+\frac{77}{6}(x+1)^4-\frac{13}{3}(x+1)^3+4(x+1)^2-x$$

13.10 problem 10

Internal problem ID [1251]
Internal file name [`OUTPUT/1252_Sunday_June_05_2022_02_06_40_AM_33631006/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2 - x + 1)\, y'' - (-4x + 1)\, y' + 2y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = -1]$$

With the expansion point for the power series method at $x = 1$.

13.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{4x - 1}{x^2 - x + 1}$$
$$q(x) = \frac{2}{x^2 - x + 1}$$
$$F = 0$$

Hence the ode is

$$y'' + \frac{(4x-1)\, y'}{x^2 - x + 1} + \frac{2y}{x^2 - x + 1} = 0$$

The domain of $p(x) = \frac{4x-1}{x^2-x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{x^2-x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t+1)^2 - t\right)\left(\frac{d^2}{dt^2} y(t)\right) + (4t+3)\left(\frac{d}{dt} y(t)\right) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 2 \\ y'(0) &= -1 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (956) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (957) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{4t\left(\frac{d}{dt}y(t)\right) + 3\frac{d}{dt}y(t) + 2y(t)}{t^2+t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(18t^2+28t+6\right)\left(\frac{d}{dt}y(t)\right) + \left(12t+8\right)y(t)}{\left(t^2+t+1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-96t^3-230t^2-104t+6\right)\left(\frac{d}{dt}y(t)\right) - 72y(t)\left(t^2+\frac{25}{18}t+\frac{2}{9}\right)}{\left(t^2+t+1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(600t^4+1956t^3+1380t^2-116t-156\right)\left(\frac{d}{dt}y(t)\right) + 480\left(t^3+\frac{43}{20}t^2+\frac{3}{4}t-\frac{2}{15}\right)y(t)}{\left(t^2+t+1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-4320t^5-17892t^4-17352t^3+1572t^2+5624t+912\right)\left(\frac{d}{dt}y(t)\right) - 3600\left(t^4+\frac{147}{50}t^3+\frac{41}{25}t^2-\frac{12}{25}t-\frac{5}{22}\right.}{\left(t^2+t+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=2$ and $y'(0)=-1$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= 10 \\
F_2 &= -38 \\
F_3 &= 28 \\
F_4 &= 944
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -t + 2 - \frac{t^2}{2} + \frac{5t^3}{3} - \frac{19t^4}{12} + \frac{7t^5}{30} + \frac{59t^6}{45} + O(t^6)$$

$$y(t) = -t + 2 - \frac{t^2}{2} + \frac{5t^3}{3} - \frac{19t^4}{12} + \frac{7t^5}{30} + \frac{59t^6}{45} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 + t + 1) + (4t + 3)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(t^2 + t + 1) + (4t + 3)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 3n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n$$

$$\sum_{n=1}^{\infty} 3n a_n t^{n-1} = \sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 4n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} 2 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_1 + 2a_0 = 0$$

$$a_2 = -a_0 - \frac{3a_1}{2}$$

$n = 1$ gives

$$8a_2 + 6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{4a_0}{3} + a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + (n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + 4n a_n + 3(n+1)\, a_{n+1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{na_n + na_{n+1} + 2a_n + 3a_{n+1}}{n+2}$$

(5)

$$= -a_n - \frac{(n+3)\, a_{n+1}}{n+2}$$

For $n = 2$ the recurrence equation gives

$$12a_2 + 15a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{2a_0}{3} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_3 + 24a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{8a_0}{15} - \frac{13a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_4 + 35a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{58a_0}{45} + \frac{19a_1}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_5 + 48a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{296a_0}{315} - \frac{31a_1}{210}$$

And so on. Therefore the solution is

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$
$$= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-a_0 - \frac{3a_1}{2}\right) t^2 + \left(\frac{4a_0}{3} + a_1\right) t^3 + \left(-\frac{2a_0}{3} + \frac{a_1}{4}\right) t^4 + \left(-\frac{8a_0}{15} - \frac{13a_1}{10}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 - \frac{8}{15}t^5\right) a_0 + \left(t - \frac{3}{2}t^2 + t^3 + \frac{1}{4}t^4 - \frac{13}{10}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 - \frac{8}{15}t^5\right) c_1 + \left(t - \frac{3}{2}t^2 + t^3 + \frac{1}{4}t^4 - \frac{13}{10}t^5\right) c_2 + O(t^6)$$

$$y(t) = 2 - \frac{t^2}{2} + \frac{5t^3}{3} - \frac{19t^4}{12} + \frac{7t^5}{30} - t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = -x + 3 - \frac{(x-1)^2}{2} + \frac{5(x-1)^3}{3} - \frac{19(x-1)^4}{12} + \frac{7(x-1)^5}{30} + \frac{59(x-1)^6}{45} + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = -x + 3 - \frac{(x-1)^2}{2} + \frac{5(x-1)^3}{3} - \frac{19(x-1)^4}{12} + \frac{7(x-1)^5}{30} + \frac{59(x-1)^6}{45} + O\left((x-1)^6\right) \quad (1)$$

Verification of solutions

$$y = -x + 3 - \frac{(x-1)^2}{2} + \frac{5(x-1)^3}{3} - \frac{19(x-1)^4}{12} + \frac{7(x-1)^5}{30} + \frac{59(x-1)^6}{45} + O\left((x-1)^6\right)$$

Verified OK.

13.10.2 Maple step by step solution

Let's solve

$$\left[\left(x^2-x+1\right)y''+\left(4x-1\right)y'+2y=0, y(1)=2, y'\Big|_{\{x=1\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2y}{x^2-x+1}-\frac{(4x-1)y'}{x^2-x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(4x-1)y'}{x^2-x+1}+\frac{2y}{x^2-x+1}=0$

☐ Check to see if x_0 is a regular singular point

◦ Define functions

$\left[P_2(x)=\frac{4x-1}{x^2-x+1}, P_3(x)=\frac{2}{x^2-x+1}\right]$

◦ $\left(\frac{\mathrm{I}\sqrt{3}}{2}+x-\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

$\left(\left(\frac{\mathrm{I}\sqrt{3}}{2}+x-\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

◦ $\left(\frac{\mathrm{I}\sqrt{3}}{2}+x-\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

$\left(\left(\frac{\mathrm{I}\sqrt{3}}{2}+x-\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}=0$

◦ $x=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

- Multiply by denominators

$\left(x^2-x+1\right)y''+\left(4x-1\right)y'+2y=0$

- Change variables using $x=u+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$ so that the regular singular point is at $u=0$

$\left(u^2-\mathrm{I}u\sqrt{3}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(4u+1-2\,\mathrm{I}\sqrt{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$
  - Shift index using $k->k+2-m$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-\frac{\mathrm{I}\sqrt{3}\, r\left(\mathrm{I}\sqrt{3}+3+3r\right) a_0 u^{-1+r}}{3} + \left(\sum_{k=0}^{\infty}\left(-\frac{\mathrm{I}\sqrt{3}\,(k+r+1)\left(\mathrm{I}\sqrt{3}+3k+6+3r\right) a_{k+1}}{3} + a_k(k+r+2)\,(k+r+1)\right) u^{k+r}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-\frac{\mathrm{I}}{3}\sqrt{3}\, r\left(\mathrm{I}\sqrt{3}+3+3r\right) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -1-\frac{\mathrm{I}\sqrt{3}}{3}\right\}$$
- Each term in the series must be 0, giving the recursion relation

  $$\left(-\mathrm{I}(k+r+2)\, a_{k+1}\sqrt{3} + a_{k+1} - (-k-2-r)\, a_k\right)(k+r+1) = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r+2)}{-1+\mathrm{I}\sqrt{3}\, k+\mathrm{I}r\sqrt{3}+2\,\mathrm{I}\sqrt{3}}$$
- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k+2)}{-1+\mathrm{I}\sqrt{3}\, k+2\,\mathrm{I}\sqrt{3}}$$
- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)}{-1+\mathrm{I}\sqrt{3}\, k+2\,\mathrm{I}\sqrt{3}}\right]$$
- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{3}}{2} + x - \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x - \frac{1}{2}\right)^k, a_{k+1} = \frac{a_k(k+2)}{-1 + I\sqrt{3}\,k + 2I\sqrt{3}}\right]$$

- Recursion relation for $r = -1 - \frac{I\sqrt{3}}{3}$

$$a_{k+1} = \frac{a_k\left(k+1-\frac{I\sqrt{3}}{3}\right)}{-1 + I\sqrt{3}\,k + I\left(-1 - \frac{I\sqrt{3}}{3}\right)\sqrt{3} + 2I\sqrt{3}}$$

- Solution for $r = -1 - \frac{I\sqrt{3}}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1-\frac{I\sqrt{3}}{3}}, a_{k+1} = \frac{a_k\left(k+1-\frac{I\sqrt{3}}{3}\right)}{-1 + I\sqrt{3}\,k + I\left(-1 - \frac{I\sqrt{3}}{3}\right)\sqrt{3} + 2I\sqrt{3}}\right]$$

- Revert the change of variables $u = \frac{I\sqrt{3}}{2} + x - \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x - \frac{1}{2}\right)^{k-1-\frac{I\sqrt{3}}{3}}, a_{k+1} = \frac{a_k\left(k+1-\frac{I\sqrt{3}}{3}\right)}{-1 + I\sqrt{3}\,k + I\left(-1 - \frac{I\sqrt{3}}{3}\right)\sqrt{3} + 2I\sqrt{3}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{3}}{2} + x - \frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{3}}{2} + x - \frac{1}{2}\right)^{k-1-\frac{I\sqrt{3}}{3}}\right), a_{1+k} = \frac{a_k(k+2)}{-1 + I\sqrt{3}\,k + 2I\sqrt{3}}, b_{1+k} = \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form is not straightforward to achieve - returning hypergeometric solution
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1-x+x^2)*diff(y(x),x$2)-(1-4*x)*diff(y(x),x)+2*y(x)=0,y(1) = 2, D(y)(1) = -1],y(x),
```

$$y(x) = 2 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{5}{3}(x-1)^3 - \frac{19}{12}(x-1)^4 + \frac{7}{30}(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 44

```
AsymptoticDSolveValue[{(1-x+x^2)*y''[x]-(1-4*x)*y'[x]+2*y[x]==0,{y[1]==2,y'[1]==-1}},y[x],{x
```

$$y(x) \to \frac{7}{30}(x-1)^5 - \frac{19}{12}(x-1)^4 + \frac{5}{3}(x-1)^3 - \frac{1}{2}(x-1)^2 - x + 3$$

13.11 problem 11

Internal problem ID [1252]
Internal file name [`OUTPUT/1253_Sunday_June_05_2022_02_06_46_AM_60515970/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(2+x)\,y''+(2+x)\,y'+y=0}$$

With initial conditions

$$[y(-1)=-2, y'(-1)=3]$$

With the expansion point for the power series method at $x=-1$.

13.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=1\\ q(x)&=\frac{1}{2+x}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + y' + \frac{y}{2+x} = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{1}{2+x}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + (t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= -2 \\ y'(0) &= 3 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (959)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (960)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\;\;\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{t\left(\frac{d}{dt}y(t)\right) + \frac{d}{dt}y(t) + y(t)}{t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(t^2+t)\left(\frac{d}{dt}y(t)\right) + (2+t)\,y(t)}{(t+1)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(-t^3 - t^2 + 3t + 3)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^2+2t+3)}{(t+1)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(t+1)\,(t^3 - 7t - 12)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^3+2t^2-t+4)}{(t+1)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{-(t+1)\,(t^4 - 12t^2 - 32t - 45)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^4+2t^3-6t^2-26t+5)}{(t+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = -2$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= -4 \\
F_2 &= 15 \\
F_3 &= -44 \\
F_4 &= 145
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -2 + 3t - \frac{t^2}{2} - \frac{2t^3}{3} + \frac{5t^4}{8} - \frac{11t^5}{30} + \frac{29t^6}{144} + O(t^6)$$

$$y(t) = -2 + 3t - \frac{t^2}{2} - \frac{2t^3}{3} + \frac{5t^4}{8} - \frac{11t^5}{30} + \frac{29t^6}{144} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + (t+1)\left(\frac{d}{dt}y(t)\right) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(t+1) + (t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \qquad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \qquad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, t^{n-1} a_n(n-1) &= \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n \\
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n \\
\sum_{n=1}^{\infty} n a_n t^{n-1} &= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}
$$

$n = 0$ gives

$$
2a_2 + a_1 + a_0 = 0
$$

$$
a_2 = -\frac{a_0}{2} - \frac{a_1}{2}
$$

For $1 \leq n$, the recurrence equation is

$$
(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + n a_n + (n+1)\, a_{n+1} + a_n = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n a_{n+1} + a_n + a_{n+1}}{n+2} \\
&= -\frac{a_n}{n+2} - \frac{(n+1)\, a_{n+1}}{n+2}
\end{aligned} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$4a_2 + 6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$9a_3 + 12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{8} + \frac{a_1}{8}$$

For $n = 3$ the recurrence equation gives

$$16a_4 + 20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{30} - \frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$25a_5 + 30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{144} + \frac{a_1}{16}$$

For $n = 5$ the recurrence equation gives

$$36a_6 + 42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{840} - \frac{11a_1}{280}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) t^2 + \frac{a_0 t^3}{3} + \left(-\frac{a_0}{8} + \frac{a_1}{8}\right) t^4 + \left(\frac{a_0}{30} - \frac{a_1}{10}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{8}t^4 + \frac{1}{30}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{10}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{8}t^4 + \frac{1}{30}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{10}t^5\right) c_2 + O(t^6)$$

$$y(t) = -2 - \frac{t^2}{2} - \frac{2t^3}{3} + \frac{5t^4}{8} - \frac{11t^5}{30} + 3t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$y = 1 + 3x - \frac{(x+1)^2}{2} - \frac{2(x+1)^3}{3} + \frac{5(x+1)^4}{8} - \frac{11(x+1)^5}{30} + \frac{29(x+1)^6}{144} + O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = 1 + 3x - \frac{(x+1)^2}{2} - \frac{2(x+1)^3}{3} + \frac{5(x+1)^4}{8} \\ - \frac{11(x+1)^5}{30} + \frac{29(x+1)^6}{144} + O\left((x+1)^6\right)\end{aligned} \quad (1)$$

Verification of solutions

$$y = 1 + 3x - \frac{(x+1)^2}{2} - \frac{2(x+1)^3}{3} + \frac{5(x+1)^4}{8} - \frac{11(x+1)^5}{30} + \frac{29(x+1)^6}{144} + O\left((x+1)^6\right)$$

Verified OK.

13.11.2 Maple step by step solution

Let's solve

$$\left[(2+x)\,y''+(2+x)\,y'+y=0, y(-1)=-2, y'\Big|_{\{x=-1\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-y'-\frac{y}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{y}{2+x}=0$

☐ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=1, P_3(x)=\frac{1}{2+x}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=0$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(2+x)\,y''+(2+x)\,y'+y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+u\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a_k(k+1+r)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+1+r)\,(a_{k+1}(k+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+r}$$

- Recursion relation for $r=0$

  $$a_{k+1} = -\frac{a_k}{k}$$

- Solution for $r=0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{a_k}{k}\right]$$

- Revert the change of variables $u = 2+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(2+x)^k, a_{k+1} = -\frac{a_k}{k}\right]$$

- Recursion relation for $r=1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r=1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{1+k}\right), a_{1+k} = -\frac{a_k}{k}, b_{1+k} = -\frac{b_k}{1+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2+x)*diff(y(x),x$2)+(2+x)*diff(y(x),x)+y(x)=0,y(-1) = -2, D(y)(-1) = 3],y(x),type='
```

$$y(x) = -2 + 3(x+1) - \frac{1}{2}(x+1)^2 - \frac{2}{3}(x+1)^3 + \frac{5}{8}(x+1)^4 - \frac{11}{30}(x+1)^5 + \mathrm{O}\left((x+1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(2+x)*y''[x]+(2+x)*y'[x]+y[x]==0,{y[-1]==-2,y'[-1]==3}},y[x],{x,-1,5}
```

$$y(x) \rightarrow -\frac{11}{30}(x+1)^5 + \frac{5}{8}(x+1)^4 - \frac{2}{3}(x+1)^3 - \frac{1}{2}(x+1)^2 + 3(x+1) - 2$$

13.12 problem 12

Internal problem ID [1253]
Internal file name [OUTPUT/1254_Sunday_June_05_2022_02_06_49_AM_34265106/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - (6 - 7x) y' + 8y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = -2]$$

With the expansion point for the power series method at $x = 1$.

13.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{-6 + 7x}{x^2}$$
$$q(x) = \frac{8}{x^2}$$
$$F = 0$$

Hence the ode is

$$y'' + \frac{(-6+7x)\,y'}{x^2} + \frac{8y}{x^2} = 0$$

The domain of $p(x) = \frac{-6+7x}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{8}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+1)^2 \left(\frac{d^2}{dt^2}y(t)\right) + (7t+1)\left(\frac{d}{dt}y(t)\right) + 8y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = -2$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (962) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (963) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &\quad (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &\quad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &\quad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{7t\left(\frac{d}{dt}y(t)\right)+\frac{d}{dt}y(t)+8y(t)}{(t+1)^2}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \frac{48\left(\frac{d}{dt}y(t)\right)t^2+72y(t)\,t-12\frac{d}{dt}y(t)+24y(t)}{(t+1)^4}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \frac{(-360t^3+120t^2+348t+84)\left(\frac{d}{dt}y(t)\right)+(-600t^2-240t+72)\,y(t)}{(t+1)^6}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \frac{(3000t^4-2400t^3-6624t^2-2688t-168)\left(\frac{d}{dt}y(t)\right)+5280\left(t^3+\frac{3}{11}t^2-\frac{36}{55}t-\frac{14}{55}\right)y(t)}{(t+1)^8}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \frac{(-27720t^5+37800t^4+110016t^3+56736t^2+1944t-2520)\left(\frac{d}{dt}y(t)\right)+(-50400t^4+87264t^2+5587}{(t+1)^{10}}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=1$ and $y'(0)=-2$ gives

$$
\begin{aligned}
F_0 &= -6\\
F_1 &= 48\\
F_2 &= -96\\
F_3 &= -1008\\
F_4 &= 13680
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -4t^4 + 8t^3 - 3t^2 - 2t + 1 - \frac{42t^5}{5} + 19t^6 + O(t^6)$$

$$y(t) = -4t^4 + 8t^3 - 3t^2 - 2t + 1 - \frac{42t^5}{5} + 19t^6 + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(t^2 + 2t + 1) \left(\frac{d^2}{dt^2}y(t)\right) + (7t + 1) \left(\frac{d}{dt}y(t)\right) + 8y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(t^2 + 2t + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + (7t + 1) \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 7n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 8a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 7n a_n t^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} 8a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + 8a_0 = 0$$

$$a_2 = -4a_0 - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_2 + 6a_3 + 15a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 4a_0 - 2a_1$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + 2(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + 7n a_n + (n+1)\, a_{n+1} + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_n + 2n^2 a_{n+1} + 6n a_n + 3n a_{n+1} + 8a_n + a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{(n^2+6n+8)\, a_n}{(n+2)(n+1)} - \frac{(2n^2+3n+1)\, a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
24a_2 + 15a_3 + 12a_4 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_4 = 3a_0 + \frac{7a_1}{2}
$$

For $n = 3$ the recurrence equation gives

$$
35a_3 + 28a_4 + 20a_5 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_5 = -\frac{56a_0}{5} - \frac{7a_1}{5}
$$

For $n = 4$ the recurrence equation gives

$$
48a_4 + 45a_5 + 30a_6 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_6 = 12a_0 - \frac{7a_1}{2}
$$

For $n = 5$ the recurrence equation gives

$$
63a_5 + 66a_6 + 42a_7 = 0
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{72a_0}{35} + \frac{38a_1}{5}
$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-4a_0 - \frac{a_1}{2}\right) t^2 + (4a_0 - 2a_1)\, t^3 + \left(3a_0 + \frac{7a_1}{2}\right) t^4 + \left(-\frac{56a_0}{5} - \frac{7a_1}{5}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - 4t^2 + 4t^3 + 3t^4 - \frac{56}{5}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 - 2t^3 + \frac{7}{2}t^4 - \frac{7}{5}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - 4t^2 + 4t^3 + 3t^4 - \frac{56}{5}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 - 2t^3 + \frac{7}{2}t^4 - \frac{7}{5}t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 - 3t^2 + 8t^3 - 4t^4 - \frac{42t^5}{5} - 2t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = -4(x-1)^4 + 8(x-1)^3 - 3(x-1)^2 - 2x + 3 - \frac{42(x-1)^5}{5} + 19(x-1)^6 + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = -4(x-1)^4 + 8(x-1)^3 - 3(x-1)^2 - 2x + 3 - \frac{42(x-1)^5}{5} + 19(x-1)^6 + O\left((x-1)^6\right) \quad (1)$$

Verification of solutions

$$y = -4(x-1)^4 + 8(x-1)^3 - 3(x-1)^2 - 2x + 3 - \frac{42(x-1)^5}{5} + 19(x-1)^6 + O\left((x-1)^6\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([x^2*diff(y(x),x$2)-(6-7*x)*diff(y(x),x)+8*y(x)=0,y(1) = 1, D(y)(1) = -2],y(x),type='
```

$$y(x) = 1 - 2(x-1) - 3(x-1)^2 + 8(x-1)^3 - 4(x-1)^4 - \frac{42}{5}(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{x^2*y''[x]-(6-7*x)*y'[x]+8*y[x]==0,{y[1]==1,y'[1]==-2}},y[x],{x,1,5}]
```

$$y(x) \to -\frac{42}{5}(x-1)^5 - 4(x-1)^4 + 8(x-1)^3 - 3(x-1)^2 - 2(x-1) + 1$$

13.13 problem 13

Internal problem ID [1254]
Internal file name [`OUTPUT/1255_Sunday_June_05_2022_02_06_51_AM_91343712/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2 + x + 1)\, y'' + (1 + 7x)\, y' + 2y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0]$$

With the expansion point for the power series method at $x = 1$.

13.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1 + 7x}{2x^2 + x + 1} \\ q(x) &= \frac{2}{2x^2 + x + 1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(1+7x)\, y'}{2x^2+x+1} + \frac{2y}{2x^2+x+1} = 0$$

The domain of $p(x) = \frac{1+7x}{2x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{2x^2+x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(t+1)^2 + t + 2\right)\left(\frac{d^2}{dt^2}y(t)\right) + (8+7t)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \qquad (965)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (966)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f \qquad (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{7t\left(\frac{d}{dt}y(t)\right)+8\frac{d}{dt}y(t)+2y(t)}{2t^2+5t+4}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0\\
&= \frac{\left(59t^2+134t+68\right)\left(\frac{d}{dt}y(t)\right)+\left(22t+26\right)y(t)}{\left(2t^2+5t+4\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1\\
&= \frac{\left(-605t^3-2052t^2-2072t-584\right)\left(\frac{d}{dt}y(t)\right)+\left(-250t^2-586t-308\right)y(t)}{\left(2t^2+5t+4\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2\\
&= \frac{\left(7365t^4+33198t^3+50094t^2+28092t+3912\right)\left(\frac{d}{dt}y(t)\right)+3210\left(t^3+\frac{1869}{535}t^2+\frac{390}{107}t+\frac{574}{535}\right)y(t)}{\left(2t^2+5t+4\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3\\
&= \frac{\left(-104055t^5-584808t^4-1173210t^3-983004t^2-271320t+16608\right)\left(\frac{d}{dt}y(t)\right)-46830\left(t^4+\frac{5167}{1115}t^3+\right.}{\left(2t^2+5t+4\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=1$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= -\frac{1}{2}\\
F_1 &= \frac{13}{8}\\
F_2 &= -\frac{77}{16}\\
F_3 &= \frac{861}{64}\\
F_4 &= -\frac{1869}{64}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 - \frac{t^2}{4} + \frac{13t^3}{48} - \frac{77t^4}{384} + \frac{287t^5}{2560} - \frac{623t^6}{15360} + O(t^6)$$

$$y(t) = 1 - \frac{t^2}{4} + \frac{13t^3}{48} - \frac{77t^4}{384} + \frac{287t^5}{2560} - \frac{623t^6}{15360} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2 + 5t + 4) + (8 + 7t)\left(\frac{d}{dt}y(t)\right) + 2y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right)(2t^2 + 5t + 4) + (8 + 7t)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 5n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 8n a_n t^{n-1}\right) + \left(\sum_{n=1}^{\infty} 7n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 5n\,t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} 5(n+1)\,a_{n+1} n\,t^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\,a_n t^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\,a_{n+2}(n+1)\,t^n$$

$$\sum_{n=1}^{\infty} 8n a_n t^{n-1} = \sum_{n=0}^{\infty} 8(n+1)\,a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 5(n+1)\,a_{n+1} n\,t^n\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\,a_{n+2}(n+1)\,t^n\right) + \left(\sum_{n=0}^{\infty} 8(n+1)\,a_{n+1} t^n\right) + \left(\sum_{n=1}^{\infty} 7n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 8a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4} - a_1$$

$n = 1$ gives

$$26a_2 + 24a_3 + 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{13a_0}{48} + \frac{17a_1}{24}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1)+5(n+1)\, a_{n+1}n+4(n+2)\, a_{n+2}(n+1)+8(n+1)\, a_{n+1}+7na_n+2a_n=0 \quad (4)$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{2n^2a_n+5n^2a_{n+1}+5na_n+13na_{n+1}+2a_n+8a_{n+1}}{4\,(n+2)\,(n+1)} \\
&= -\frac{(2n^2+5n+2)\, a_n}{4\,(n+2)\,(n+1)}-\frac{(5n^2+13n+8)\, a_{n+1}}{4\,(n+2)\,(n+1)}
\end{aligned} \quad (5)
$$

For $n = 2$ the recurrence equation gives

$$20a_2+54a_3+48a_4=0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{77a_0}{384}-\frac{73a_1}{192}$$

For $n = 3$ the recurrence equation gives

$$35a_3+92a_4+80a_5=0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{287a_0}{2560}+\frac{163a_1}{1280}$$

For $n = 4$ the recurrence equation gives

$$54a_4+140a_5+120a_6=0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{623a_0}{15360}+\frac{173a_1}{7680}$$

For $n = 5$ the recurrence equation gives

$$77a_5+198a_6+168a_7=0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{3072} - \frac{913a_1}{10752}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t + \left(-\frac{a_0}{4} - a_1\right) t^2 + \left(\frac{13a_0}{48} + \frac{17a_1}{24}\right) t^3 \\ + \left(-\frac{77a_0}{384} - \frac{73a_1}{192}\right) t^4 + \left(\frac{287a_0}{2560} + \frac{163a_1}{1280}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y(t) = \left(1 - \frac{1}{4}t^2 + \frac{13}{48}t^3 - \frac{77}{384}t^4 + \frac{287}{2560}t^5\right) a_0 \\ + \left(t - t^2 + \frac{17}{24}t^3 - \frac{73}{192}t^4 + \frac{163}{1280}t^5\right) a_1 + O(t^6) \end{aligned} \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{4}t^2 + \frac{13}{48}t^3 - \frac{77}{384}t^4 + \frac{287}{2560}t^5\right) c_1 + \left(t - t^2 + \frac{17}{24}t^3 - \frac{73}{192}t^4 + \frac{163}{1280}t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 - \frac{t^2}{4} + \frac{13t^3}{48} - \frac{77t^4}{384} + \frac{287t^5}{2560} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$y = 1 - \frac{(x-1)^2}{4} + \frac{13(x-1)^3}{48} - \frac{77(x-1)^4}{384} + \frac{287(x-1)^5}{2560} - \frac{623(x-1)^6}{15360} + O\left((x-1)^6\right)$$

Summary
The solution(s) found are the following

$$y = 1-\frac{(x-1)^{2}}{4}+\frac{13(x-1)^{3}}{48}-\frac{77(x-1)^{4}}{384}+\frac{287(x-1)^{5}}{2560}-\frac{623(x-1)^{6}}{15360}+O\left((x-1)^{6}\right) \tag{1}$$

Verification of solutions

$$y = 1-\frac{(x-1)^{2}}{4}+\frac{13(x-1)^{3}}{48}-\frac{77(x-1)^{4}}{384}+\frac{287(x-1)^{5}}{2560}-\frac{623(x-1)^{6}}{15360}+O\left((x-1)^{6}\right)$$

Verified OK.

13.13.2 Maple step by step solution

Let's solve

$$\left[\left(2x^{2}+x+1\right)y''+(1+7x)\,y'+2y=0, y(1)=1, y'\Big|_{\{x=1\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{2x^{2}+x+1}-\frac{(1+7x)y'}{2x^{2}+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+7x)y'}{2x^{2}+x+1}+\frac{2y}{2x^{2}+x+1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+7x}{2x^{2}+x+1}, P_3(x)=\frac{2}{2x^{2}+x+1}\right]$

  - $\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)\cdot P_2(x)$ is analytic at $x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}$

    $$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}}=0$$

  - $\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^{2}\cdot P_3(x)$ is analytic at $x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}$

    $$\left(\left(\frac{I\sqrt{7}}{4}+x+\frac{1}{4}\right)^{2}\cdot P_3(x)\right)\Bigg|_{x=-\frac{I\sqrt{7}}{4}-\frac{1}{4}}=0$$

- $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

- Multiply by denominators

  $(2x^2 + x + 1) y'' + (1 + 7x) y' + 2y = 0$

- Change variables using $x = u - \frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$ so that the regular singular point is at $u = 0$

  $\left(2u^2 - \mathrm{I}u\sqrt{7}\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(-\frac{3}{4} + 7u - \frac{7\,\mathrm{I}\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)(k + r - 1)\, u^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)(k + 1 - m + r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{\mathrm{I}\sqrt{7}\, r\left(3\,\mathrm{I}\sqrt{7} - 21 - 28r\right) a_0 u^{-1+r}}{28} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{7}\,(k+1+r)\left(3\,\mathrm{I}\sqrt{7} - 28k - 49 - 28r\right) a_{k+1}}{28} + a_k(k + r + 2)(2k + 2r + 1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{\mathrm{I}}{28}\sqrt{7}\, r\left(3\,\mathrm{I}\sqrt{7} - 21 - 28r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3\,\mathrm{I}\sqrt{7}}{28} - \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$-\mathrm{I}\left(k+r+\frac{7}{4}\right)(k+1+r)\,a_{k+1}\sqrt{7}+\frac{(-3k-3r-3)a_{k+1}}{4}+2(k+r+2)\left(k+r+\frac{1}{2}\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{4a_k\left(2k^2+4kr+2r^2+5k+5r+2\right)}{3+4\mathrm{I}\sqrt{7}\,k^2+8\mathrm{I}\sqrt{7}\,kr+4\mathrm{I}\sqrt{7}\,r^2+11\mathrm{I}\sqrt{7}\,k+11\mathrm{I}\sqrt{7}\,r+7\mathrm{I}\sqrt{7}+3k+3r}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4\mathrm{I}\sqrt{7}\,k^2+11\mathrm{I}\sqrt{7}\,k+7\mathrm{I}\sqrt{7}+3k}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4\mathrm{I}\sqrt{7}\,k^2+11\mathrm{I}\sqrt{7}\,k+7\mathrm{I}\sqrt{7}+3k}\right]$$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)^k, a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4\mathrm{I}\sqrt{7}\,k^2+11\mathrm{I}\sqrt{7}\,k+7\mathrm{I}\sqrt{7}+3k}\right]$$

- Recursion relation for $r=\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

$$a_{k+1}=\frac{4a_k\left(2k^2+4k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\mathrm{I}\sqrt{7}\,k^2+8\mathrm{I}\sqrt{7}\,k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\mathrm{I}\sqrt{7}\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\mathrm{I}\sqrt{7}\,k+11\mathrm{I}\sqrt{7}\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\mathrm{I}\sqrt{7}}{28}+3k}$$

- Solution for $r=\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1}=\frac{4a_k\left(2k^2+4k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\mathrm{I}\sqrt{7}\,k^2+8\mathrm{I}\sqrt{7}\,k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\mathrm{I}\sqrt{7}\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\mathrm{I}\sqrt{7}\,k+11\mathrm{I}\sqrt{7}\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\right.$$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)^{k+\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1}=\frac{4a_k\left(2k^2+4k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\mathrm{I}\sqrt{7}}{28}\right.}{\frac{3}{4}+4\mathrm{I}\sqrt{7}\,k^2+8\mathrm{I}\sqrt{7}\,k\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\mathrm{I}\sqrt{7}\left(\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\mathrm{I}\sqrt{7}\,k+11\mathrm{I}\sqrt{7}}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)^{k+\frac{3\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}\right), a_{1+k}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4\mathrm{I}\sqrt{7}\,k^2+11\mathrm{I}\sqrt{7}\,k+7\mathrm{I}\sqrt{7}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
Order:=6;
dsolve([(1+x+2*x^2)*diff(y(x),x$2)+(1+7*x)*diff(y(x),x)+2*y(x)=0,y(1) = 1, D(y)(1) = 0],y(x)
```

$$y(x) = 1 - \frac{1}{4}(x-1)^2 + \frac{13}{48}(x-1)^3 - \frac{77}{384}(x-1)^4 + \frac{287}{2560}(x-1)^5 + O\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 41

```
AsymptoticDSolveValue[{(1+x+2*x^2)*y''[x]+(1+7*x)*y'[x]+2*y[x]==0,{y[1]==1,y'[1]==0}},y[x],{
```

$$y(x) \to \frac{287(x-1)^5}{2560} - \frac{77}{384}(x-1)^4 + \frac{13}{48}(x-1)^3 - \frac{1}{4}(x-1)^2 + 1$$

13.14 problem 14

Internal problem ID [1255]
Internal file name [`OUTPUT/1256_Sunday_June_05_2022_02_06_56_AM_72253966/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+3)\,y''+(1+2x)\,y'-(2-x)\,y=0}$$

With initial conditions

$$[y(-1)=1, y'(-1)=0]$$

With the expansion point for the power series method at $x=-1$.

13.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{1+2x}{x+3}\\ q(x)&=\frac{-2+x}{x+3}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(1+2x)\, y'}{x+3} + \frac{(-2+x)\, y}{x+3} = 0$$

The domain of $p(x) = \frac{1+2x}{x+3}$ is

$$\{x < -3 \vee -3 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{-2+x}{x+3}$ is

$$\{x < -3 \vee -3 < x\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2} y(t)\right)(2+t) + (2t-1)\left(\frac{d}{dt} y(t)\right) + (t-3)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (968) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (969) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)\,t + 2t\left(\frac{d}{dt}y(t)\right) - 3y(t) - \frac{d}{dt}y(t)}{2+t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{(3t^2 - 3t + 2)\left(\frac{d}{dt}y(t)\right) + 2\left(t^2 - \frac{7}{2}t - 1\right)y(t)}{\left(2+t\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(-4t^3 + 6t^2 - 8t - 12)\left(\frac{d}{dt}y(t)\right) - 3y(t)\left(t^3 - 4t^2 - \frac{4}{3}t + \frac{4}{3}\right)}{\left(2+t\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{5(t^4 - 2t^3 + 4t^2 + 12t)\left(\frac{d}{dt}y(t)\right) + 4\left(t^4 - \frac{9}{2}t^3 - t^2 + 7t - 4\right)y(t)}{\left(2+t\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(-6t^5 + 15t^4 - 40t^3 - 180t^2 + 88)\left(\frac{d}{dt}y(t)\right) - 5y(t)\left(t^5 - 5t^4 + 20t^2 - 16t - 24\right)}{\left(2+t\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= \frac{3}{2} \\
F_1 &= -\frac{1}{2} \\
F_2 &= -\frac{1}{2} \\
F_3 &= -1 \\
F_4 &= \frac{15}{4}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 + \frac{3t^2}{4} - \frac{t^3}{12} - \frac{t^4}{48} - \frac{t^5}{120} + \frac{t^6}{192} + O(t^6)$$

$$y(t) = 1 + \frac{3t^2}{4} - \frac{t^3}{12} - \frac{t^4}{48} - \frac{t^5}{120} + \frac{t^6}{192} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2+t) + (2t-1)\left(\frac{d}{dt}y(t)\right) + (t-3)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(2+t) + (2t-1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t-3)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) \tag{2}$$
$$+ \sum_{n=1}^{\infty} (-n a_n t^{n-1}) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-3 a_n t^n) = 0$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} (1+n)\,a_{1+n} n\,t^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(1+n)\,t^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(1+n)\,a_{1+n} t^n\right)$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\,a_{1+n} n\,t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(1+n)\,t^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \sum_{n=0}^{\infty} \left(-(1+n)\,a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \sum_{n=0}^{\infty} \left(-3 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 - a_1 - 3a_0 = 0$$

$$a_2 = \frac{3a_0}{4} + \frac{a_1}{4}$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\,a_{1+n} n + 2(n+2)\,a_{n+2}(1+n) + 2n a_n - (1+n)\,a_{1+n} + a_{n-1} - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{1+n} + 2n a_n - 3a_n - a_{1+n} + a_{n-1}}{2\,(n+2)\,(1+n)} \\ &= -\frac{(2n-3)\,a_n}{2\,(n+2)\,(1+n)} - \frac{(n^2-1)\,a_{1+n}}{2\,(n+2)\,(1+n)} - \frac{a_{n-1}}{2\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$12a_3 - a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{12} - \frac{a_0}{12}$$

For $n = 2$ the recurrence equation gives

$$3a_3 + 24a_4 + a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{16} - \frac{a_0}{48}$$

For $n = 3$ the recurrence equation gives

$$8a_4 + 40a_5 + 3a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$15a_5 + 60a_6 + 5a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{192} + \frac{11a_1}{2880}$$

For $n = 5$ the recurrence equation gives

$$24a_6 + 84a_7 + 7a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{20160} - \frac{a_1}{2880}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(\frac{3a_0}{4} + \frac{a_1}{4}\right) t^2 + \left(\frac{a_1}{12} - \frac{a_0}{12}\right) t^3 + \left(-\frac{a_1}{16} - \frac{a_0}{48}\right) t^4 - \frac{a_0 t^5}{120} + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{3}{4}t^2 - \frac{1}{12}t^3 - \frac{1}{48}t^4 - \frac{1}{120}t^5\right) a_0 + \left(t + \frac{1}{4}t^2 + \frac{1}{12}t^3 - \frac{1}{16}t^4\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{3}{4}t^2 - \frac{1}{12}t^3 - \frac{1}{48}t^4 - \frac{1}{120}t^5\right) c_1 + \left(t + \frac{1}{4}t^2 + \frac{1}{12}t^3 - \frac{1}{16}t^4\right) c_2 + O(t^6)$$

$$y(t) = 1 + \frac{3t^2}{4} - \frac{t^3}{12} - \frac{t^4}{48} - \frac{t^5}{120} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$y = 1 + \frac{3(x+1)^2}{4} - \frac{(x+1)^3}{12} - \frac{(x+1)^4}{48} - \frac{(x+1)^5}{120} + \frac{(x+1)^6}{192} + O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{3(x+1)^2}{4} - \frac{(x+1)^3}{12} - \frac{(x+1)^4}{48} - \frac{(x+1)^5}{120} + \frac{(x+1)^6}{192} + O\left((x+1)^6\right) \quad (1)$$

Verification of solutions

$$y = 1 + \frac{3(x+1)^2}{4} - \frac{(x+1)^3}{12} - \frac{(x+1)^4}{48} - \frac{(x+1)^5}{120} + \frac{(x+1)^6}{192} + O\left((x+1)^6\right)$$

Verified OK.

13.14.2 Maple step by step solution

Let's solve

$$\left[(x+3)\,y''+(1+2x)\,y'+(-2+x)\,y=0, y(-1)=1, y'\Big|_{\{x=-1\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-2+x)y}{x+3}-\frac{(1+2x)y'}{x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+2x)y'}{x+3}+\frac{(-2+x)y}{x+3}=0$

□ Check to see if $x_0=-3$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+2x}{x+3}, P_3(x)=\frac{-2+x}{x+3}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=-5$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

  Check to see if $x_0=-3$ is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $(x+3)\,y''+(1+2x)\,y'+(-2+x)\,y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(-5+2u)\left(\frac{d}{du}y(u)\right)+(-5+u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- > k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k + 1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-6+r)\, u^{-1+r} + \left(a_1(1+r)\,(-5+r) + a_0(-5+2r)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k-5+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 6\}$

- Each term must be 0

  $a_1(1+r)\,(-5+r) + a_0(-5+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $a_{k+2}(k+2+r)\,(k-4+r) + 2a_{k+1}(k+1) + 2r a_{k+1} - 5a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$$

- Recursion relation for $r = 6$

$$a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(3+x)*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)-(2-x)*y(x)=0,y(-1) = 1, D(y)(-1) = 0],y(x)
```

$$y(x) = 1 + \frac{3}{4}(x+1)^2 - \frac{1}{12}(x+1)^3 - \frac{1}{48}(x+1)^4 - \frac{1}{120}(x+1)^5 + O\left((x+1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 41

```
AsymptoticDSolveValue[{(3+x)*y''[x]+(1+2*x)*y'[x]-(2-x)*y[x]==0,{y[-1]==1,y'[-1]==0}},y[x],{
```

$$y(x) \to -\frac{1}{120}(x+1)^5 - \frac{1}{48}(x+1)^4 - \frac{1}{12}(x+1)^3 + \frac{3}{4}(x+1)^2 + 1$$

13.15 problem 15

Internal problem ID [1256]
Internal file name [`OUTPUT/1257_Sunday_June_05_2022_02_06_59_AM_54892672/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y'x + (2x^2 + 4)\, y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

13.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3x \\ q(x) &= 2x^2 + 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y'x + \left(2x^2 + 4\right) y = 0$$

The domain of $p(x) = 3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x^2 + 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{971}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{972}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2x^2y - 3y'x - 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 6yx^3 + 7y'x^2 + 8yx - 7y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-15x^3 + 43x\right) y' + \left(-14x^4 + 4x^2 + 36\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(31x^4 - 170x^2 + 79\right) y' + 30\left(x^4 - \frac{41}{15}x^2 - \frac{82}{15}\right) xy \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-63x^5 + 552x^3 - 741x\right) y' + \left(-62x^6 + 366x^4 + 276x^2 - 480\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= -4 \\
F_1 &= 0 \\
F_2 &= 36 \\
F_3 &= 0 \\
F_4 &= -480
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -2x^2 + 1 + \frac{3x^4}{2} - \frac{2x^6}{3} + O\left(x^6\right)
$$

$$
y = -2x^2 + 1 + \frac{3x^4}{2} - \frac{2x^6}{3} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - 3\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 4a_0 = 0$$

$$a_2 = -2a_0$$

$n = 1$ gives

$$6a_3 + 7a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{7a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1) + 3na_n + 2a_{n-2} + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3na_n + 4a_n + 2a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{(3n+4)\,a_n}{(n+2)\,(n+1)} - \frac{2a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 10a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 13a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{79a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 16a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{2a_0}{3}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 19a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{407a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - 2a_0x^2 - \frac{7}{6}a_1x^3 + \frac{3}{2}a_0x^4 + \frac{79}{120}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-2x^2 + 1 + \frac{3}{2}x^4\right)a_0 + \left(x - \frac{7}{6}x^3 + \frac{79}{120}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-2x^2 + 1 + \frac{3}{2}x^4\right)c_1 + \left(x - \frac{7}{6}x^3 + \frac{79}{120}x^5\right)c_2 + O(x^6)$$

$$y = -2x^2 + 1 + \frac{3x^4}{2} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -2x^2 + 1 + \frac{3x^4}{2} - \frac{2x^6}{3} + O(x^6) \tag{1}$$

$$y = -2x^2 + 1 + \frac{3x^4}{2} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -2x^2 + 1 + \frac{3x^4}{2} - \frac{2x^6}{3} + O(x^6)$$

Verified OK.

$$y = -2x^2 + 1 + \frac{3x^4}{2} + O(x^6)$$

Verified OK.

13.15.2 Maple step by step solution

Let's solve

$\left[y'' = -2x^2y - 3y'x - 4y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-2x^2 - 4)\, y - 3y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3y'x + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 4a_0 + (6a_3 + 7a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(3k+4) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3k a_{k+2} + 2a_k + 10a_{k+2}}{k^2 + 7k + 12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+3*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='s
```

$$y(x) = 1 - 2x^2 + \frac{3}{2}x^4 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y''[x]+3*x*y'[x]+(4+2*x^2)*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{3x^4}{2} - 2x^2 + 1$$

13.16 problem 19

Internal problem ID [1257]
Internal file name [`OUTPUT/1258_Sunday_June_05_2022_02_07_01_AM_10248380/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(4x+2)\,y''-4y'-(6+4x)\,y=0}$$

With initial conditions

$$[y(0)=2,y'(0)=-7]$$

With the expansion point for the power series method at $x=0$.

13.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=-\frac{4}{4x+2}\\ q(x)&=\frac{-4x-6}{4x+2}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' - \frac{4y'}{4x+2} + \frac{(-4x-6)\, y}{4x+2} = 0$$

The domain of $p(x) = -\frac{4}{4x+2}$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{-4x-6}{4x+2}$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (974)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (975)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2yx + 2y' + 3y}{1 + 2x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(2x + 3)\, y' + 2y}{1 + 2x} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{4y' + (2x + 5)\, y}{1 + 2x} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(2x + 5)\, y' + 4y}{1 + 2x} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{6y' + (2x + 7)\, y}{1 + 2x}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -7$ gives

$$
\begin{aligned}
F_0 &= -8 \\
F_1 &= -17 \\
F_2 &= -18 \\
F_3 &= -27 \\
F_4 &= -28
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -4x^2 - 7x + 2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - \frac{7x^6}{180} + O(x^6)
$$

$$y = -4x^2 - 7x + 2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - \frac{7x^6}{180} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(4x + 2) y'' - 4y' + (-4x - 6) y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(4x + 2) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + (-4x - 6) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-4n a_n x^{n-1}) + \sum_{n=0}^{\infty} (-4x^{1+n} a_n) + \sum_{n=0}^{\infty} (-6 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(1+n) a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2) a_{n+2} (1+n)\, x^n$$

$$\sum_{n=1}^{\infty}\left(-4n a_n x^{n-1}\right) = \sum_{n=0}^{\infty}\left(-4(1+n)\, a_{1+n} x^n\right)$$

$$\sum_{n=0}^{\infty}\left(-4x^{1+n} a_n\right) = \sum_{n=1}^{\infty}\left(-4a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, x^n\right) \\ + \sum_{n=0}^{\infty}\left(-4(1+n)\, a_{1+n} x^n\right) + \sum_{n=1}^{\infty}\left(-4a_{n-1} x^n\right) + \sum_{n=0}^{\infty}\left(-6a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 - 4a_1 - 6a_0 = 0$$

$$a_2 = \frac{3a_0}{2} + a_1$$

For $1 \leq n$, the recurrence equation is

$$4(1+n)\, a_{1+n} n + 2(n+2)\, a_{n+2}(1+n) - 4(1+n)\, a_{1+n} - 4a_{n-1} - 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_{1+n} - 3a_n - 2a_{1+n} - 2a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{3a_n}{(n+2)\,(1+n)} - \frac{(2n^2-2)\, a_{1+n}}{(n+2)\,(1+n)} + \frac{2a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$12a_3 - 4a_0 - 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{3} + \frac{a_1}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_3 + 24a_4 - 4a_1 - 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{24} + \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$32a_4 + 40a_5 - 4a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{30} + \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$60a_5 + 60a_6 - 4a_3 - 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{720} + \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$96a_6 + 84a_7 - 4a_4 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_0}{840} + \frac{a_1}{720}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{3a_0}{2} + a_1\right) x^2 + \left(\frac{a_0}{3} + \frac{a_1}{2}\right) x^3 + \left(\frac{5a_0}{24} + \frac{a_1}{6}\right) x^4 + \left(\frac{a_0}{30} + \frac{a_1}{24}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{30}x^5\right) a_0 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{30}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

$$y = 2 - 4x^2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - 7x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -4x^2 - 7x + 2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - \frac{7x^6}{180} + O(x^6) \quad (1)$$

$$y = 2 - 4x^2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - 7x + O(x^6) \quad (2)$$

Verification of solutions

$$y = -4x^2 - 7x + 2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - \frac{7x^6}{180} + O(x^6)$$

Verified OK.

$$y = 2 - 4x^2 - \frac{17x^3}{6} - \frac{3x^4}{4} - \frac{9x^5}{40} - 7x + O(x^6)$$

Verified OK.

13.16.2 Maple step by step solution

Let's solve

$$\left[(4x+2)\, y'' - 4y' + (-4x-6)\, y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -7\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{(2x+3)y}{1+2x} + \frac{2y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2y'}{1+2x} - \frac{(2x+3)y}{1+2x} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

◦ Define functions

$\left[P_2(x) = -\frac{2}{1+2x}, P_3(x) = -\frac{2x+3}{1+2x}\right]$

◦ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

$\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}} = -1$

◦ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

$\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}} = 0$

◦ $x = -\frac{1}{2}$is a regular singular point

Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

$x_0 = -\frac{1}{2}$

- Multiply by denominators

$(1+2x)\, y'' - 2y' + (-3-2x)\, y = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

$2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2-2u)\, y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0 r(-2+r)\, u^{-1+r} + \left(2a_1(1+r)\,(-1+r) - 2a_0\right) u^r + \left(\sum_{k=1}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term must be 0

  $$2a_1(1+r)\,(-1+r) - 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k - 2a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$2a_{k+2}(k+2+r)\,(k+r) - 2a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2+4*x)*diff(y(x),x$2)-4*diff(y(x),x)-(6+4*x)*y(x)=0,y(0) = 2, D(y)(0) = -7],y(x),ty
```

$$y(x) = 2 - 7x - 4x^2 - \frac{17}{6}x^3 - \frac{3}{4}x^4 - \frac{9}{40}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(2+4*x)*y''[x]-4*y'[x]-(6+4*x)*y[x]==0,{y[0]==2,y'[0]==-7}},y[x],{x,0
```

$$y(x) \to -\frac{9x^5}{40} - \frac{3x^4}{4} - \frac{17x^3}{6} - 4x^2 - 7x + 2$$

13.17 problem 20

Internal problem ID [1258]
Internal file name [`OUTPUT/1259_Sunday_June_05_2022_02_07_03_AM_53593430/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+2x)\,y''-(1-2x)\,y'-(-2x+3)\,y=0}$$

With initial conditions

$$[y(1)=1, y'(1)=-2]$$

With the expansion point for the power series method at $x=1$.

13.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2x-1}{1+2x}\\ q(x)&=\frac{-3+2x}{1+2x}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(2x-1)\,y'}{1+2x} + \frac{(-3+2x)\,y}{1+2x} = 0$$

The domain of $p(x) = \frac{2x-1}{1+2x}$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{-3+2x}{1+2x}$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(2t+3)\left(\frac{d^2}{dt^2}y(t)\right) + (1+2t)\left(\frac{d}{dt}y(t)\right) + (2t-1)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = -2$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (977)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (978)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2y(t)\,t + 2t\left(\frac{d}{dt}y(t)\right) - y(t) + \frac{d}{dt}y(t)}{2t+3} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{y(t)\left(-3+2t\right)}{2t+3} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{4\left(\frac{d}{dt}y(t)\right)t^2 + 12y(t) - 9\frac{d}{dt}y(t)}{\left(2t+3\right)^2} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(-8t^3 - 4t^2 + 66t + 81\right)\left(\frac{d}{dt}y(t)\right) - 8\left(t^3 - \frac{1}{2}t^2 - \frac{9}{4}t + \frac{57}{8}\right)y(t)}{\left(2t+3\right)^3} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-144t^2 - 576t - 540\right)\left(\frac{d}{dt}y(t)\right) + 16\left(t^4 - \frac{27}{2}t^2 - 9t + \frac{477}{16}\right)y(t)}{\left(2t+3\right)^4}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = -2$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -1 \\
F_2 &= \frac{10}{3} \\
F_3 &= -\frac{73}{9} \\
F_4 &= \frac{173}{9}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 + \frac{t^2}{2} - 2t - \frac{t^3}{6} + \frac{5t^4}{36} - \frac{73t^5}{1080} + \frac{173t^6}{6480} + O(t^6)$$

$$y(t) = 1 + \frac{t^2}{2} - 2t - \frac{t^3}{6} + \frac{5t^4}{36} - \frac{73t^5}{1080} + \frac{173t^6}{6480} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2t+3)\left(\frac{d^2}{dt^2}y(t)\right) + (1+2t)\left(\frac{d}{dt}y(t)\right) + (2t-1)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(2t+3)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + (1+2t)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (2t-1)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2t^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} 2t^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(1+n)\, t^n\right) \\ &+ \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} t^n\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$6a_2 + a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{6} - \frac{a_1}{6}$$

For $1 \leq n$, the recurrence equation is

$$2(1+n)\, a_{1+n} n + 3(n+2)\, a_{n+2}(1+n) + (1+n)\, a_{1+n} + 2n a_n + 2a_{n-1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_{1+n} + 2n a_n + 3n a_{1+n} - a_n + a_{1+n} + 2a_{n-1}}{3\,(n+2)\,(1+n)} \\ &= -\frac{(2n-1)\, a_n}{3\,(n+2)\,(1+n)} - \frac{(2n^2+3n+1)\, a_{1+n}}{3\,(n+2)\,(1+n)} - \frac{2a_{n-1}}{3\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 + 18a_3 + a_1 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$15a_3 + 36a_4 + 3a_2 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{18} - \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$28a_4 + 60a_5 + 5a_3 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{19a_0}{1080} + \frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$45a_5 + 90a_6 + 7a_4 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{53a_0}{6480} - \frac{a_1}{108}$$

For $n = 5$ the recurrence equation gives

$$66a_6 + 126a_7 + 9a_5 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_0}{4860} + \frac{169a_1}{45360}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t + \left(\frac{a_0}{6} - \frac{a_1}{6}\right) t^2 - \frac{a_0 t^3}{6} + \left(\frac{a_0}{18} - \frac{a_1}{24}\right) t^4 + \left(-\frac{19 a_0}{1080} + \frac{a_1}{40}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 + \frac{1}{6} t^2 - \frac{1}{6} t^3 + \frac{1}{18} t^4 - \frac{19}{1080} t^5\right) a_0 + \left(t - \frac{1}{6} t^2 - \frac{1}{24} t^4 + \frac{1}{40} t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 + \frac{1}{6} t^2 - \frac{1}{6} t^3 + \frac{1}{18} t^4 - \frac{19}{1080} t^5\right) c_1 + \left(t - \frac{1}{6} t^2 - \frac{1}{24} t^4 + \frac{1}{40} t^5\right) c_2 + O(t^6)
$$

$$
y(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{5 t^4}{36} - \frac{73 t^5}{1080} - 2t + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$
y = 3 + \frac{(x-1)^2}{2} - 2x - \frac{(x-1)^3}{6} + \frac{5(x-1)^4}{36} - \frac{73(x-1)^5}{1080} + \frac{173(x-1)^6}{6480} + O\left((x-1)^6\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = 3 &+ \frac{(x-1)^2}{2} - 2x - \frac{(x-1)^3}{6} + \frac{5(x-1)^4}{36} \\
&- \frac{73(x-1)^5}{1080} + \frac{173(x-1)^6}{6480} + O\left((x-1)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$
y = 3 + \frac{(x-1)^2}{2} - 2x - \frac{(x-1)^3}{6} + \frac{5(x-1)^4}{36} - \frac{73(x-1)^5}{1080} + \frac{173(x-1)^6}{6480} + O\left((x-1)^6\right)
$$

Verified OK.

13.17.2 Maple step by step solution

Let's solve

$$\left[(1+2x)\,y''+(2x-1)\,y'+(-3+2x)\,y=0, y(1)=1, y'\Big|_{\{x=1\}}=-2\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y''=-\frac{(-3+2x)y}{1+2x}-\frac{(2x-1)y'}{1+2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''+\frac{(2x-1)y'}{1+2x}+\frac{(-3+2x)y}{1+2x}=0$$

☐ Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x)=\frac{2x-1}{1+2x}, P_3(x)=\frac{-3+2x}{1+2x}\right]$$

  ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

$$\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}}=-1$$

  ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

$$\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}}=0$$

  ◦ $x=-\frac{1}{2}$is a regular singular point

Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

$$x_0=-\frac{1}{2}$$

- Multiply by denominators

$$(1+2x)\,y''+(2x-1)\,y'+(-3+2x)\,y=0$$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$$2u\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)+(-4+2u)\,y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k + 1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\, u^{-1+r} + (2a_1(1+r)\,(-1+r) + 2a_0(-2+r))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$2a_1(1+r)\,(-1+r) + 2a_0(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k+1}(k+1+r)\,(k+r-1) + a_k(2k+2r-4) + 2a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$2a_{k+2}(k+2+r)\,(k+r) + a_{k+1}(2k-2+2r) + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k a_{k+1} + r a_{k+1} + a_k - a_{k+1}}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k a_{k+1} + a_k - a_{k+1}}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k a_{k+1} + a_k - a_{k+1}}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+4)(k+2)}, 6a_1 = 0\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k+2}, a_{k+2} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+4)(k+2)}, 6a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+2*x)*diff(y(x),x$2)-(1-2*x)*diff(y(x),x)-(3-2*x)*y(x)=0,y(1) = 1, D(y)(1) = -2],y
```

$$y(x) = 1 - 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{5}{36}(x-1)^4 - \frac{73}{1080}(x-1)^5 + O\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(1+2*x)*y''[x]-(1-2*x)*y'[x]-(3-2*x)*y[x]==0,{y[1]==1,y'[1]==-2}},y[x
```

$$y(x) \to -\frac{73(x-1)^5}{1080} + \frac{5}{36}(x-1)^4 - \frac{1}{6}(x-1)^3 + \frac{1}{2}(x-1)^2 - 2(x-1) + 1$$

13.18 problem 21

Internal problem ID [1259]
Internal file name [`OUTPUT/1260_Sunday_June_05_2022_02_07_06_AM_69075082/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x+5)\,y''-y'+(x+5)\,y=0}$$

With initial conditions

$$[y(-2)=2, y'(-2)=-1]$$

With the expansion point for the power series method at $x=-2$.

13.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=-\frac{1}{2x+5}\\ q(x)&=\frac{x+5}{2x+5}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{2x+5} + \frac{(x+5)\,y}{2x+5} = 0$$

The domain of $p(x) = -\frac{1}{2x+5}$ is

$$\left\{x < -\frac{5}{2} \lor -\frac{5}{2} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = \frac{x+5}{2x+5}$ is

$$\left\{x < -\frac{5}{2} \lor -\frac{5}{2} < x\right\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = 2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(1+2t)\left(\frac{d^2}{dt^2}y(t)\right) - \frac{d}{dt}y(t) + (t+3)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = -1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (980) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (981) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)\,t + 3y(t) - \frac{d}{dt}y(t)}{1+2t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(-2t^2-7t-4\right)\left(\frac{d}{dt}y(t)\right) - y(t)\left(t-2\right)}{\left(1+2t\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-4t^2+6t+7\right)\left(\frac{d}{dt}y(t)\right) + 2y(t)\left(t^3+\frac{13}{2}t^2+\frac{27}{2}t+\frac{3}{2}\right)}{\left(1+2t\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(4t^4+28t^3+71t^2+7t-26\right)\left(\frac{d}{dt}y(t)\right) + 4\left(t^3-\frac{7}{2}t^2-\frac{107}{4}t-3\right)y(t)}{\left(1+2t\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(12t^4-36t^3-357t^2-24t+177\right)\left(\frac{d}{dt}y(t)\right) - 4y(t)\left(t^5+10t^4+\frac{163}{4}t^3+38t^2-\frac{619}{4}t-\frac{67}{4}\right)}{\left(1+2t\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -7 \\
F_1 &= 8 \\
F_2 &= -1 \\
F_3 &= 2 \\
F_4 &= -43
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -t + 2 - \frac{7t^2}{2} + \frac{4t^3}{3} - \frac{t^4}{24} + \frac{t^5}{60} - \frac{43t^6}{720} + O(t^6)$$

$$y(t) = -t + 2 - \frac{7t^2}{2} + \frac{4t^3}{3} - \frac{t^4}{24} + \frac{t^5}{60} - \frac{43t^6}{720} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(1 + 2t) \left(\frac{d^2}{dt^2} y(t)\right) - \frac{d}{dt} y(t) + (t + 3)\, y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt} y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2} y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(1 + 2t) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t + 3) \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=1}^{\infty} \left(-n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} t^n\right)$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n\right) + \sum_{n=0}^{\infty} \left(-(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 3 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_1 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2} + \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$2(1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) - (1+n)\, a_{1+n} + a_{n-1} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_{1+n} + n a_{1+n} + 3a_n - a_{1+n} + a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{3a_n}{(n+2)\,(1+n)} - \frac{(2n^2+n-1)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 6a_3 + a_0 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{3} - \frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$9a_3 + 12a_4 + a_1 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8} + \frac{7a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_4 + 20a_5 + a_2 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{10} - \frac{13a_1}{60}$$

For $n = 4$ the recurrence equation gives

$$35a_5 + 30a_6 + a_3 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{67a_0}{720} + \frac{59a_1}{240}$$

For $n = 5$ the recurrence equation gives

$$54a_6 + 42a_7 + a_4 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{97a_0}{840} - \frac{155a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y(t) = a_0 + a_1 t &+ \left(-\frac{3a_0}{2} + \frac{a_1}{2}\right) t^2 + \left(\frac{a_0}{3} - \frac{2a_1}{3}\right) t^3 \\ &+ \left(\frac{a_0}{8} + \frac{7a_1}{24}\right) t^4 + \left(-\frac{a_0}{10} - \frac{13a_1}{60}\right) t^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{8}t^4 - \frac{1}{10}t^5\right) a_0 + \left(t + \frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{7}{24}t^4 - \frac{13}{60}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{8}t^4 - \frac{1}{10}t^5\right) c_1 + \left(t + \frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{7}{24}t^4 - \frac{13}{60}t^5\right) c_2 + O(t^6)$$

$$y(t) = 2 - \frac{7t^2}{2} + \frac{4t^3}{3} - \frac{t^4}{24} + \frac{t^5}{60} - t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = 2 + x$ results in

$$y = -x - \frac{7(2+x)^2}{2} + \frac{4(2+x)^3}{3} - \frac{(2+x)^4}{24} + \frac{(2+x)^5}{60} - \frac{43(2+x)^6}{720} + O\left((2+x)^6\right)$$

Summary

The solution(s) found are the following

$$y = -x - \frac{7(2+x)^2}{2} + \frac{4(2+x)^3}{3} - \frac{(2+x)^4}{24} + \frac{(2+x)^5}{60} - \frac{43(2+x)^6}{720} + O\left((2+x)^6\right) \quad (1)$$

Verification of solutions

$$y=-x-\frac{7(2+x)^{2}}{2}+\frac{4(2+x)^{3}}{3}-\frac{(2+x)^{4}}{24}+\frac{(2+x)^{5}}{60}-\frac{43(2+x)^{6}}{720}+O\left((2+x)^{6}\right)$$

Verified OK.

13.18.2 Maple step by step solution

Let's solve

$\left[(2x+5)\, y''-y'+(x+5)\, y=0, y(-2)=2, y'\Big|_{\{x=-2\}}=-1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x+5)y}{2x+5}+\frac{y'}{2x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{2x+5}+\frac{(x+5)y}{2x+5}=0$

- □ Check to see if $x_0=-\frac{5}{2}$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{1}{2x+5}, P_3(x)=\frac{x+5}{2x+5}\right]$

  - $\left(x+\frac{5}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{5}{2}$

    $\left(\left(x+\frac{5}{2}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{5}{2}}=-\frac{1}{2}$

  - $\left(x+\frac{5}{2}\right)^{2}\cdot P_3(x)$ is analytic at $x=-\frac{5}{2}$

    $\left(\left(x+\frac{5}{2}\right)^{2}\cdot P_3(x)\right)\Bigg|_{x=-\frac{5}{2}}=0$

  - $x=-\frac{5}{2}$is a regular singular point

  Check to see if $x_0=-\frac{5}{2}$ is a regular singular point

  $x_0=-\frac{5}{2}$

- Multiply by denominators

  $(2x+5)\, y''-y'+(x+5)\, y=0$

- Change variables using $x=u-\frac{5}{2}$ so that the regular singular point is at $u=0$

$$2u\left(\frac{d^2}{du^2}y(u)\right) - \frac{d}{du}y(u) + \left(u + \frac{5}{2}\right)y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ○ Shift index using $k- > k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ○ Convert $\frac{d}{du}y(u)$ to series expansion

$$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$$

- ○ Shift index using $k- > k + 1$

$$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- ○ Shift index using $k- > k + 1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, u^{-1+r} + \left(a_1(1+r)\,(-1+2r) + \frac{5a_0}{2}\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + \frac{5a}{2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(-1+2r) + \frac{5a_0}{2} = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k-\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+\frac{5a_k}{2}+a_{k-1}=0$$

- Shift index using $k->k+1$

  $$2\left(k+\frac{1}{2}+r\right)(k+2+r)\,a_{k+2}+\frac{5a_{k+1}}{2}+a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+1+2r)(k+2+r)}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+1)(k+2)}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+1)(k+2)}, -a_1+\frac{5a_0}{2}=0\right]$$

- Revert the change of variables $u=x+\frac{5}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{5}{2}\right)^k, a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+1)(k+2)}, -a_1+\frac{5a_0}{2}=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

  $$a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{7}{2}\right)}$$

- Solution for $r=\frac{3}{2}$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{3}{2}}, a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{7}{2}\right)}, 5a_1+\frac{5a_0}{2}=0\right]$$

- Revert the change of variables $u=x+\frac{5}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{5}{2}\right)^{k+\frac{3}{2}}, a_{k+2}=-\frac{5a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{7}{2}\right)}, 5a_1+\frac{5a_0}{2}=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{5}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{5}{2}\right)^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{5a_{1+k}+2a_k}{2(2k+1)(k+2)}, -a_1+\frac{5a_0}{2}=0, b_{k+2}=-\frac{}{2(}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(5+2*x)*diff(y(x),x$2)-diff(y(x),x)+(5+x)*y(x)=0,y(-2) = 2, D(y)(-2) = -1],y(x),type
```

$$y(x) = 2 - (2 + x) - \frac{7}{2}(2 + x)^2 + \frac{4}{3}(2 + x)^3 - \frac{1}{24}(2 + x)^4 + \frac{1}{60}(2 + x)^5 + O\left((2 + x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 43

```
AsymptoticDSolveValue[{(5+2*x)*y''[x]-y'[x]+(5+x)*y[x]==0,{y[-2]==2,y'[-2]==-1}},y[x],{x,-2,
```

$$y(x) \to \frac{1}{60}(x + 2)^5 - \frac{1}{24}(x + 2)^4 + \frac{4}{3}(x + 2)^3 - \frac{7}{2}(x + 2)^2 - x$$

13.19 problem 22

Internal problem ID [1260]
Internal file name [OUTPUT/1261_Sunday_June_05_2022_02_07_09_AM_58608132/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+4)\,y''-(2x+4)\,y'+(x+6)\,y=0}$$

With initial conditions

$$[y(-3)=2, y'(-3)=-2]$$

With the expansion point for the power series method at $x=-3$.

13.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x)&=\frac{-2x-4}{x+4}\\ q(x)&=\frac{x+6}{x+4}\\ F&=0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-2x-4)\,y'}{x+4} + \frac{(x+6)\,y}{x+4} = 0$$

The domain of $p(x) = \frac{-2x-4}{x+4}$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = -3$ is inside this domain. The domain of $q(x) = \frac{x+6}{x+4}$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = -3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + (-2t+2)\left(\frac{d}{dt}y(t)\right) + (t+3)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = -2$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\left.\frac{d^nf}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}y'' && (983)\\
&= \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}f && (984)\\
\frac{d^2f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right)+\frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y'+\frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2f}{dx^2}\right)+\left(\frac{\partial}{\partial y}\frac{d^2f}{dx^2}\right)y'+\frac{\partial}{\partial y'}\left(\frac{d^2f}{dx^2}\right)f && (3)\\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)\,t - 2t\left(\frac{d}{dt}y(t)\right) + 3y(t) + 2\frac{d}{dt}y(t)}{t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(3t^2 - 12t + 5\right)\left(\frac{d}{dt}y(t)\right) - 2y(t)\left(t^2 + 2t - 4\right)}{\left(t+1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{4\left(t^3 - 9t^2 + 14t - 6\right)\left(\frac{d}{dt}y(t)\right) - 3\left(t^3 - t^2 - \frac{31}{3}t + \frac{35}{3}\right)y(t)}{\left(t+1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(5t^4 - 80t^3 + 266t^2 - 348t + 141\right)\left(\frac{d}{dt}y(t)\right) - 4y(t)\left(t^4 - 6t^3 - 10t^2 + 50t - 52\right)}{\left(t+1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(6t^5 - 150t^4 + 856t^3 - 2160t^2 + 2562t - 986\right)\left(\frac{d}{dt}y(t)\right) - 5y(t)\left(t^5 - 13t^4 + \frac{66}{5}t^3 + \frac{458}{5}t^2 - \frac{1583}{5}t + 29\right.}{\left(t+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = -2$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 6 \\
F_2 &= -22 \\
F_3 &= 134 \\
F_4 &= -938
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = t^3 - t^2 - 2t + 2 - \frac{11t^4}{12} + \frac{67t^5}{60} - \frac{469t^6}{360} + O(t^6)$$

$$y(t) = t^3 - t^2 - 2t + 2 - \frac{11t^4}{12} + \frac{67t^5}{60} - \frac{469t^6}{360} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t+1) + (-2t+2)\left(\frac{d}{dt}y(t)\right) + (t+3)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(t+1) + (-2t+2)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t+3)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=2}^{\infty} n\,t^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + \sum_{n=1}^{\infty}(-2n a_n t^n) \\ + \left(\sum_{n=1}^{\infty} 2n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=1}^{\infty} 2n a_n t^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n\right) + \sum_{n=1}^{\infty} (-2n a_n t^n) + \left(\sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2} - a_1$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) - 2n a_n + 2(1+n)\, a_{1+n} + a_{n-1} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{1+n} - 2n a_n + 3n a_{1+n} + 3a_n + 2a_{1+n} + a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{(-2n+3)\, a_n}{(n+2)\,(1+n)} - \frac{(n^2+3n+2)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 + 6a_3 + a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{4a_0}{3} + \frac{5a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_3 + 12a_4 - a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{35a_0}{24} - a_1$$

For $n = 3$ the recurrence equation gives

$$20a_4 + 20a_5 - 3a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{26a_0}{15} + \frac{47a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_5 + 30a_6 - 5a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{97a_0}{48} - \frac{493a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$42a_6 + 42a_7 - 7a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{211a_0}{90} + \frac{8009a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{3a_0}{2} - a_1\right) t^2 + \left(\frac{4a_0}{3} + \frac{5a_1}{6}\right) t^3 + \left(-\frac{35a_0}{24} - a_1\right) t^4 + \left(\frac{26a_0}{15} + \frac{47a_1}{40}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{4}{3}t^3 - \frac{35}{24}t^4 + \frac{26}{15}t^5\right) a_0 + \left(t - t^2 + \frac{5}{6}t^3 - t^4 + \frac{47}{40}t^5\right) a_1 + O\left(t^6\right) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{3}{2}t^2 + \frac{4}{3}t^3 - \frac{35}{24}t^4 + \frac{26}{15}t^5\right) c_1 + \left(t - t^2 + \frac{5}{6}t^3 - t^4 + \frac{47}{40}t^5\right) c_2 + O\left(t^6\right)$$

$$y(t) = 2 - t^2 + t^3 - \frac{11t^4}{12} + \frac{67t^5}{60} - 2t + O\left(t^6\right)$$

Replacing t in the above with the original independent variable $x$susing $t = x + 3$ results in

$$y = (x+3)^3 - (x+3)^2 - 2x - 4 - \frac{11(x+3)^4}{12} + \frac{67(x+3)^5}{60} - \frac{469(x+3)^6}{360} + O\left((x+3)^6\right)$$

Summary
The solution(s) found are the following

$$y = (x+3)^3 - (x+3)^2 - 2x - 4 - \frac{11(x+3)^4}{12} + \frac{67(x+3)^5}{60} - \frac{469(x+3)^6}{360} + O\left((x+3)^6\right) \quad (1)$$

Verification of solutions

$$y = (x+3)^3 - (x+3)^2 - 2x - 4 - \frac{11(x+3)^4}{12} + \frac{67(x+3)^5}{60} - \frac{469(x+3)^6}{360} + O((x+3)^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(4+x)*diff(y(x),x$2)-(4+2*x)*diff(y(x),x)+(6+x)*y(x)=0,y(-3) = 2, D(y)(-3) = -2],y(x
```

$$y(x) = 2 - 2(x+3) - (x+3)^2 + (x+3)^3 - \frac{11}{12}(x+3)^4 + \frac{67}{60}(x+3)^5 + O\left((x+3)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{(4+x)*y''[x]-(4+2*x)*y'[x]+(6+x)*y[x]==0,{y[-3]==2,y'[-3]==-2}},y[x],
```

$$y(x) \to \frac{67}{60}(x+3)^5 - \frac{11}{12}(x+3)^4 + (x+3)^3 - (x+3)^2 - 2(x+3) + 2$$

13.20 problem 23

Internal problem ID [1261]
Internal file name [`OUTPUT/1262_Sunday_June_05_2022_02_07_12_AM_20866627/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x+2)\,y''-y'x+2yx=0}$$

With initial conditions

$$[y(0)=-1, y'(0)=2]$$

With the expansion point for the power series method at $x=0$.

13.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=-\frac{x}{3x+2}\\ q(x)&=\frac{2x}{3x+2}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' - \frac{xy'}{3x+2} + \frac{2xy}{3x+2} = 0$$

The domain of $p(x) = -\frac{x}{3x+2}$ is

$$\left\{x < -\frac{2}{3} \lor -\frac{2}{3} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2x}{3x+2}$ is

$$\left\{x < -\frac{2}{3} \lor -\frac{2}{3} < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (986)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (987)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(-y'+2y)}{3x+2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(-5x^2-4x+2\right) y' + \left(-2x^2-4\right) y}{\left(3x+2\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-11x^3-8x^2-18x-28\right) y' + 10(2+x)\left(x^2-\frac{6}{5}x+\frac{6}{5}\right) y}{\left(3x+2\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(19x^4+36x^3-80x^2+96x+264\right) y' + 22\left(x^4+\frac{8}{11}x^3+\frac{36}{11}x^2+\frac{80}{11}x-\frac{120}{11}\right) y}{\left(3x+2\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(85x^5+128x^4+212x^3+1416x^2-1320x-3456\right) y' - 38y\left(x^5+\frac{36}{19}x^4-\frac{144}{19}x^3+\frac{264}{19}x^2+\frac{840}{19}x-\frac{1600}{19}\right.}{\left(3x+2\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=-1$ and $y'(0)=2$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 2 \\
F_2 &= -10 \\
F_3 &= 48 \\
F_4 &= -316
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 2x - 1 + \frac{x^3}{3} - \frac{5x^4}{12} + \frac{2x^5}{5} - \frac{79x^6}{180} + O\left(x^6\right)
$$

$$
y = 2x - 1 + \frac{x^3}{3} - \frac{5x^4}{12} + \frac{2x^5}{5} - \frac{79x^6}{180} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(3x+2)\,y'' - y'x + 2yx = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(3x+2)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2}\right) \\
&+ \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 3(1+n)\,a_{1+n} n\, x^n
$$

$$
\sum_{n=2}^{\infty} 2n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\,a_{n+2}(1+n)\,x^n
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 3(1+n)\, a_{1+n} n\, x^{n}\right)+\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \sum_{n\,=1}^{\infty} \left(-n a_{n} x^{n}\right)+\left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n}\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$3(1+n)\, a_{1+n} n+2(n+2)\, a_{n+2}(1+n)-n a_{n}+2 a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3 n^{2} a_{1+n}-n a_{n}+3 n a_{1+n}+2 a_{n-1}}{2\,(n+2)\,(1+n)} \\ &= \frac{n a_{n}}{2\,(n+2)\,(1+n)}-\frac{\left(3 n^{2}+3 n\right) a_{1+n}}{2\,(n+2)\,(1+n)}-\frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6 a_{2}+12 a_{3}-a_{1}+2 a_{0} = 0$$

Which after substituting the earlier terms found becomes

$$a_{3} = \frac{a_{1}}{12}-\frac{a_{0}}{6}$$

For $n = 2$ the recurrence equation gives

$$18 a_{3}+24 a_{4}-2 a_{2}+2 a_{1} = 0$$

Which after substituting the earlier terms found becomes

$$a_{4} = -\frac{7 a_{1}}{48}+\frac{a_{0}}{8}$$

For $n = 3$ the recurrence equation gives

$$36 a_{4}+40 a_{5}-3 a_{3}+2 a_{2} = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11a_1}{80} - \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$60a_5 + 60a_6 - 4a_4 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{3a_1}{20} + \frac{5a_0}{36}$$

For $n = 5$ the recurrence equation gives

$$90a_6 + 84a_7 - 5a_5 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{695a_1}{4032} - \frac{107a_0}{672}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \left(\frac{a_1}{12} - \frac{a_0}{6}\right)x^3 + \left(-\frac{7a_1}{48} + \frac{a_0}{8}\right)x^4 + \left(\frac{11a_1}{80} - \frac{a_0}{8}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{8}x^5\right)a_0 + \left(x + \frac{1}{12}x^3 - \frac{7}{48}x^4 + \frac{11}{80}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{8}x^5\right)c_1 + \left(x + \frac{1}{12}x^3 - \frac{7}{48}x^4 + \frac{11}{80}x^5\right)c_2 + O(x^6)$$

$$y=-1+\frac{x^{3}}{3}-\frac{5x^{4}}{12}+\frac{2x^{5}}{5}+2x+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$y=2x-1+\frac{x^{3}}{3}-\frac{5x^{4}}{12}+\frac{2x^{5}}{5}-\frac{79x^{6}}{180}+O\left(x^{6}\right) \tag{1}$$

$$y=-1+\frac{x^{3}}{3}-\frac{5x^{4}}{12}+\frac{2x^{5}}{5}+2x+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y=2x-1+\frac{x^{3}}{3}-\frac{5x^{4}}{12}+\frac{2x^{5}}{5}-\frac{79x^{6}}{180}+O\left(x^{6}\right)$$

Verified OK.

$$y=-1+\frac{x^{3}}{3}-\frac{5x^{4}}{12}+\frac{2x^{5}}{5}+2x+O\left(x^{6}\right)$$

Verified OK.

13.20.2 Maple step by step solution

Let's solve

$\left[(3x+2)\,y''-y'x+2yx=0, y(0)=-1, y'\Big|_{\{x=0\}}=2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy}{3x+2}+\frac{xy'}{3x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{xy'}{3x+2}+\frac{2xy}{3x+2}=0$

□ Check to see if $x_0=-\frac{2}{3}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{3x+2}, P_3(x)=\frac{2x}{3x+2}\right]$

  ◦ $\left(x+\frac{2}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{2}{3}$

$$\left(\left(x+\frac{2}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{2}{3}} = \frac{2}{9}$$

- ○ $\left(x+\frac{2}{3}\right)^2 \cdot P_3(x)$ is analytic at $x=-\frac{2}{3}$

  $$\left(\left(x+\frac{2}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{2}{3}} = 0$$

- ○ $x=-\frac{2}{3}$is a regular singular point

  Check to see if $x_0=-\frac{2}{3}$ is a regular singular point

  $x_0=-\frac{2}{3}$

- • Multiply by denominators

  $(3x+2)\,y''-y'x+2yx=0$

- • Change variables using $x=u-\frac{2}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d^2}{du^2}y(u)\right)+\left(-u+\frac{2}{3}\right)\left(\frac{d}{du}y(u)\right)+\left(2u-\frac{4}{3}\right)y(u)=0$

- • Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $$u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - ○ Shift index using $k->k-m$

    $$u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - ○ Shift index using $k->k+1-m$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{a_0 r(-7+9r)u^{-1+r}}{3} + \left(\frac{a_1(1+r)(2+9r)}{3} - \frac{a_0(4+3r)}{3}\right)u^r + \left(\sum_{k=1}^{\infty}\left(\frac{a_{k+1}(k+1+r)(9k+2+9r)}{3} - \frac{a_k(3k+3r+4)}{3} + 2a_{k-1}\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{r(-7+9r)}{3} = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{7}{9}\right\}$

- Each term must be 0

  $\frac{a_1(1+r)(2+9r)}{3} - \frac{a_0(4+3r)}{3} = 0$

- Each term in the series must be 0, giving the recursion relation

  $3(k+1+r)\left(k+\frac{2}{9}+r\right)a_{k+1} - a_k k - a_k r - \frac{4a_k}{3} + 2a_{k-1} = 0$

- Shift index using $k- >k+1$

  $3(k+2+r)\left(k+\frac{11}{9}+r\right)a_{k+2} - a_{k+1}(k+1) - ra_{k+1} - \frac{4a_{k+1}}{3} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{3ka_{k+1}+3ra_{k+1}-6a_k+7a_{k+1}}{(k+2+r)(9k+11+9r)}$

- Recursion relation for $r=0$

  $a_{k+2} = \frac{3ka_{k+1}-6a_k+7a_{k+1}}{(k+2)(9k+11)}$

- Solution for $r=0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{3ka_{k+1}-6a_k+7a_{k+1}}{(k+2)(9k+11)}, \frac{2a_1}{3} - \frac{4a_0}{3} = 0\right]$

- Revert the change of variables $u = x+\frac{2}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{2}{3}\right)^k, a_{k+2} = \frac{3ka_{k+1}-6a_k+7a_{k+1}}{(k+2)(9k+11)}, \frac{2a_1}{3} - \frac{4a_0}{3} = 0\right]$

- Recursion relation for $r=\frac{7}{9}$

  $a_{k+2} = \frac{3ka_{k+1}-6a_k+\frac{28}{3}a_{k+1}}{\left(k+\frac{25}{9}\right)(9k+18)}$

- Solution for $r=\frac{7}{9}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{7}{9}}, a_{k+2}=\frac{3k a_{k+1}-6a_k+\frac{28}{3}a_{k+1}}{\left(k+\frac{25}{9}\right)(9k+18)}, \frac{16a_1}{3}-\frac{19a_0}{9}=0\right]$$

- Revert the change of variables $u=x+\frac{2}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{2}{3}\right)^{k+\frac{7}{9}}, a_{k+2}=\frac{3k a_{k+1}-6a_k+\frac{28}{3}a_{k+1}}{\left(k+\frac{25}{9}\right)(9k+18)}, \frac{16a_1}{3}-\frac{19a_0}{9}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x+\frac{2}{3}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+\frac{2}{3}\right)^{k+\frac{7}{9}}\right), a_{k+2}=\frac{3k a_{1+k}-6a_k+7a_{1+k}}{(k+2)(9k+11)}, \frac{2a_1}{3}-\frac{4a_0}{3}=0, b_{k+2}=\frac{3k\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(2+3*x)*diff(y(x),x$2)-x*diff(y(x),x)+2*x*y(x)=0,y(0) = -1, D(y)(0) = 2],y(x),type='
```

$$y(x)=-1+2x+\frac{1}{3}x^3-\frac{5}{12}x^4+\frac{2}{5}x^5+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 29

```
AsymptoticDSolveValue[{(2+3*x)*y''[x]-x*y'[x]+2*x*y[x]==0,{y[0]==-1,y'[0]==2}},y[x],{x,0,5}]
```

$$y(x) \to \frac{2x^5}{5} - \frac{5x^4}{12} + \frac{x^3}{3} + 2x - 1$$

13.21 problem 24

Internal problem ID [1262]
Internal file name [`OUTPUT/1263_Sunday_June_05_2022_02_07_14_AM_97472436/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x+3)\,y''+3y'-yx=0}$$

With initial conditions

$$[y(-1)=2, y'(-1)=-3]$$

With the expansion point for the power series method at $x=-1$.

13.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x)&=\frac{3}{2x+3}\\ q(x)&=-\frac{x}{2x+3}\\ F&=0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y'}{2x+3} - \frac{xy}{2x+3} = 0$$

The domain of $p(x) = \frac{3}{2x+3}$ is

$$\left\{x < -\frac{3}{2} \lor -\frac{3}{2} < x\right\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = -\frac{x}{2x+3}$ is

$$\left\{x < -\frac{3}{2} \lor -\frac{3}{2} < x\right\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(1+2t)\left(\frac{d^2}{dt^2}y(t)\right) + 3\frac{d}{dt}y(t) - y(t)\,(-1+t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = -3$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (989)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (990)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y(t)\,t - y(t) - 3\frac{d}{dt}y(t)}{1+2t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(2t^2-t+14\right)\left(\frac{d}{dt}y(t)\right) - 3y(t)\left(t-2\right)}{\left(1+2t\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-12t^2+18t-93\right)\left(\frac{d}{dt}y(t)\right) + 2\left(t^3-\frac{3}{2}t^2+\frac{21}{2}t-\frac{41}{2}\right)y(t)}{\left(1+2t\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(4t^4-4t^3+99t^2-211t+814\right)\left(\frac{d}{dt}y(t)\right) - 12\left(t^3-\frac{7}{2}t^2+\frac{67}{4}t-30\right)y(t)}{\left(1+2t\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-36t^4+108t^3-1065t^2+2616t-8805\right)\left(\frac{d}{dt}y(t)\right) + 4\left(t^5-2t^4+\frac{127}{4}t^3-\frac{257}{2}t^2+\frac{2315}{4}t-\frac{3895}{4}\right)y(t)}{\left(1+2t\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=2$ and $y'(0)=-3$ gives

$$
\begin{aligned}
F_0 &= 7 \\
F_1 &= -30 \\
F_2 &= 197 \\
F_3 &= -1722 \\
F_4 &= 18625
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 2 - 3t + \frac{7t^2}{2} - 5t^3 + \frac{197t^4}{24} - \frac{287t^5}{20} + \frac{3725t^6}{144} + O(t^6)$$

$$y(t) = 2 - 3t + \frac{7t^2}{2} - 5t^3 + \frac{197t^4}{24} - \frac{287t^5}{20} + \frac{3725t^6}{144} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(1 + 2t) \left(\frac{d^2}{dt^2} y(t)\right) + 3 \frac{d}{dt} y(t) + (1 - t) y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt} y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$\frac{d^2}{dt^2} y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(1 + 2t) \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + 3 \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (1 - t) \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) + \sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=1}^{\infty} 3n a_n t^{n-1} = \sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - \frac{3a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$2(1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) + 3(1+n)\, a_{1+n} + a_n - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_{1+n} + 5n a_{1+n} + a_n + 3a_{1+n} - a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{(n+2)\,(1+n)} - \frac{(2n^2+5n+3)\, a_{1+n}}{(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$10a_2 + 6a_3 + a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = a_0 + \frac{7a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$21a_3 + 12a_4 + a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{41a_0}{24} - \frac{31a_1}{8}$$

For $n = 3$ the recurrence equation gives

$$36a_4 + 20a_5 + a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 3a_0 + \frac{407a_1}{60}$$

For $n = 4$ the recurrence equation gives

$$55a_5 + 30a_6 + a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{779a_0}{144} - \frac{587a_1}{48}$$

For $n = 5$ the recurrence equation gives

$$78a_6 + 42a_7 + a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1669a_0}{168} + \frac{56593a_1}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{3a_1}{2}\right) t^2 + \left(a_0 + \frac{7a_1}{3}\right) t^3 \\ + \left(-\frac{41a_0}{24} - \frac{31a_1}{8}\right) t^4 + \left(3a_0 + \frac{407a_1}{60}\right) t^5 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + t^3 - \frac{41}{24}t^4 + 3t^5\right) a_0 + \left(t - \frac{3}{2}t^2 + \frac{7}{3}t^3 - \frac{31}{8}t^4 + \frac{407}{60}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + t^3 - \frac{41}{24}t^4 + 3t^5\right) c_1 + \left(t - \frac{3}{2}t^2 + \frac{7}{3}t^3 - \frac{31}{8}t^4 + \frac{407}{60}t^5\right) c_2 + O(t^6)$$

$$y(t) = 2 + \frac{7t^2}{2} - 5t^3 + \frac{197t^4}{24} - \frac{287t^5}{20} - 3t + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x+1$ results in

$$\begin{aligned}y = -1 - 3x + \frac{7(x+1)^2}{2} - 5(x+1)^3 + \frac{197(x+1)^4}{24} \\ - \frac{287(x+1)^5}{20} + \frac{3725(x+1)^6}{144} + O\left((x+1)^6\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -1 - 3x + \frac{7(x+1)^2}{2} - 5(x+1)^3 + \frac{197(x+1)^4}{24} - \frac{287(x+1)^5}{20} + \frac{3725(x+1)^6}{144} + O\left((x+1)^6\right) \tag{1}$$

Verification of solutions

$$y = -1 - 3x + \frac{7(x+1)^2}{2} - 5(x+1)^3 + \frac{197(x+1)^4}{24} - \frac{287(x+1)^5}{20} + \frac{3725(x+1)^6}{144} + O\left((x+1)^6\right)$$

Verified OK.

13.21.2 Maple step by step solution

Let's solve

$\left[(2x+3)\, y'' + 3y' - yx = 0, y(-1) = 2, y'\Big|_{\{x=-1\}} = -3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{xy}{2x+3} - \frac{3y'}{2x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x+3} - \frac{xy}{2x+3} = 0$

□ Check to see if $x_0 = -\frac{3}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{2x+3}, P_3(x) = -\frac{x}{2x+3}\right]$

  ◦ $\left(x+\frac{3}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{3}{2}$

    $\left(\left(x+\frac{3}{2}\right) \cdot P_2(x)\right)\Big|_{x=-\frac{3}{2}} = \frac{3}{2}$

  ◦ $\left(x+\frac{3}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{3}{2}$

    $\left(\left(x+\frac{3}{2}\right)^2 \cdot P_3(x)\right)\Big|_{x=-\frac{3}{2}} = 0$

- $x=-\frac{3}{2}$is a regular singular point

  Check to see if $x_0=-\frac{3}{2}$ is a regular singular point

  $x_0=-\frac{3}{2}$

- Multiply by denominators

  $(2x+3)\,y''+3y'-yx=0$

- Change variables using $x=u-\frac{3}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d^2}{du^2}y(u)\right)+3\frac{d}{du}y(u)+\left(-u+\frac{3}{2}\right)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(1+2r)\,u^{-1+r}+\left(a_1(1+r)\,(3+2r)+\frac{3a_0}{2}\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(2k+3+2r)+\frac{3a_k}{2}-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term must be 0

  $a_1(1+r)(3+2r) + \frac{3a_0}{2} = 0$
- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+1} + \frac{3a_k}{2} - a_{k-1} = 0$
- Shift index using $k->k+1$

  $2(k+2+r)\left(k+\frac{5}{2}+r\right)a_{k+2} + \frac{3a_{k+1}}{2} - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{-3a_{k+1}+2a_k}{2(k+2+r)(2k+5+2r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{-3a_{k+1}+2a_k}{2(k+2)(2k+5)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{-3a_{k+1}+2a_k}{2(k+2)(2k+5)}, 3a_1 + \frac{3a_0}{2} = 0\right]$
- Revert the change of variables $u = x+\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{3}{2}\right)^k, a_{k+2} = \frac{-3a_{k+1}+2a_k}{2(k+2)(2k+5)}, 3a_1 + \frac{3a_0}{2} = 0\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{-3a_{k+1}+2a_k}{2\left(k+\frac{3}{2}\right)(2k+4)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+2} = \frac{-3a_{k+1}+2a_k}{2\left(k+\frac{3}{2}\right)(2k+4)}, a_1 + \frac{3a_0}{2} = 0\right]$
- Revert the change of variables $u = x+\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{3}{2}\right)^{k-\frac{1}{2}}, a_{k+2} = \frac{-3a_{k+1}+2a_k}{2\left(k+\frac{3}{2}\right)(2k+4)}, a_1 + \frac{3a_0}{2} = 0\right]$
- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x+\frac{3}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+\frac{3}{2}\right)^{k-\frac{1}{2}}\right), a_{k+2}=\frac{-3a_{1+k}+2a_k}{2(k+2)(2k+5)}, 3a_1+\frac{3a_0}{2}=0, b_{k+2}=\frac{-3b_{1+k}}{2(k+\frac{3}{2})}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(3+2*x)*diff(y(x),x$2)+3*diff(y(x),x)-x*y(x)=0,y(-1) = 2, D(y)(-1) = -3],y(x),type='
```

$$y(x)=2-3(x+1)+\frac{7}{2}(x+1)^2-5(x+1)^3+\frac{197}{24}(x+1)^4-\frac{287}{20}(x+1)^5+\mathrm{O}\left((x+1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 44

```
AsymptoticDSolveValue[{(3+2*x)*y''[x]+3*y'[x]-x*y[x]==0,{y[-1]==2,y'[-1]==-3}},y[x],{x,-1,5}
```

$$y(x)\to -\frac{287}{20}(x+1)^5+\frac{197}{24}(x+1)^4-5(x+1)^3+\frac{7}{2}(x+1)^2-3(x+1)+2$$

13.22 problem 25

Internal problem ID [1263]
Internal file name [`OUTPUT/1264_Sunday_June_05_2022_02_07_17_AM_38759375/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x+3)\,y''-3y'-(2+x)\,y=0}$$

With initial conditions

$$[y(-2)=-2, y'(-2)=3]$$

With the expansion point for the power series method at $x=-2$.

13.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=-\frac{3}{2x+3}\\ q(x)&=\frac{-x-2}{2x+3}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' - \frac{3y'}{2x+3} + \frac{(-x-2)\,y}{2x+3} = 0$$

The domain of $p(x) = -\frac{3}{2x+3}$ is

$$\left\{x < -\frac{3}{2} \lor -\frac{3}{2} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = \frac{-x-2}{2x+3}$ is

$$\left\{x < -\frac{3}{2} \lor -\frac{3}{2} < x\right\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = 2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(2t-1)\left(\frac{d^2}{dt^2}y(t)\right) - 3\frac{d}{dt}y(t) - y(t)\,t = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = -2$$
$$y'(0) = 3$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (992) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (993) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n\big|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3\frac{d}{dt}y(t)+y(t)\,t}{2t-1}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0\\
&= \frac{(2t^2-t+3)\left(\frac{d}{dt}y(t)\right)+y(t)\left(-1+3t\right)}{\left(2t-1\right)^2}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1\\
&= \frac{(12t^2-10t-1)\left(\frac{d}{dt}y(t)\right)+2\left(t^3-\frac{1}{2}t^2-\frac{3}{2}t+\frac{1}{2}\right)y(t)}{\left(2t-1\right)^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2\\
&= \frac{(4t^4-4t^3+7t^2-9t+12)\left(\frac{d}{dt}y(t)\right)+12y(t)\left(t^3-\frac{7}{6}t^2+\frac{13}{12}t-\frac{1}{4}\right)}{\left(2t-1\right)^4}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3\\
&= \frac{(36t^4-60t^3+45t^2-6t-48)\left(\frac{d}{dt}y(t)\right)+4\left(t^5-t^4-\frac{17}{4}t^3+\frac{11}{4}t^2-\frac{19}{2}t+\frac{11}{4}\right)y(t)}{\left(2t-1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=-2$ and $y'(0)=3$ gives

$$
\begin{aligned}
F_0 &= -9\\
F_1 &= 11\\
F_2 &= 5\\
F_3 &= 42\\
F_4 &= 166
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -2+3t-\frac{9t^2}{2}+\frac{11t^3}{6}+\frac{5t^4}{24}+\frac{7t^5}{20}+\frac{83t^6}{360}+O(t^6)$$

$$y(t) = -2+3t-\frac{9t^2}{2}+\frac{11t^3}{6}+\frac{5t^4}{24}+\frac{7t^5}{20}+\frac{83t^6}{360}+O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2t-1)\left(\frac{d^2}{dt^2}y(t)\right)-3\frac{d}{dt}y(t)-y(t)\,t = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(2t-1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)-3\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n t^n\right)t = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1)\, a_n t^{n-2}\right) + \sum_{n=1}^{\infty}\left(-3n a_n t^{n-1}\right)+\sum_{n=0}^{\infty}\left(-t^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right)$$

$$\sum_{n=1}^{\infty} \left(-3n a_n t^{n-1}\right) = \sum_{n=0}^{\infty} \left(-3(1+n)\, a_{1+n} t^n\right)$$

$$\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, t^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right) + \sum_{n=0}^{\infty} \left(-3(1+n)\, a_{1+n} t^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 - 3a_1 = 0$$

$$a_2 = -\frac{3a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$2(1+n)\, a_{1+n} n - (n+2)\, a_{n+2}(1+n) - 3(1+n)\, a_{1+n} - a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2n^2 a_{1+n} - n a_{1+n} - 3a_{1+n} - a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{(2n^2 - n - 3)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-2a_2 - 6a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} - \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - 12a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{24} - \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$12a_4 - 20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{10} - \frac{a_0}{40}$$

For $n = 4$ the recurrence equation gives

$$25a_5 - 30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{15} - \frac{11a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_6 - 42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{331a_1}{5040} - \frac{a_0}{70}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{3a_1 t^2}{2} + \left(\frac{a_1}{2} - \frac{a_0}{6}\right) t^3 + \left(\frac{a_1}{24} - \frac{a_0}{24}\right) t^4 + \left(\frac{a_1}{10} - \frac{a_0}{40}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{6}t^3 - \frac{1}{24}t^4 - \frac{1}{40}t^5\right) a_0 + \left(t - \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 + \frac{1}{10}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{6}t^3 - \frac{1}{24}t^4 - \frac{1}{40}t^5\right) c_1 + \left(t - \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 + \frac{1}{10}t^5\right) c_2 + O(t^6)$$

$$y(t) = -2 + \frac{11t^3}{6} + \frac{5t^4}{24} + \frac{7t^5}{20} + 3t - \frac{9t^2}{2} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = 2 + x$ results in

$$y = 4 + 3x - \frac{9(2+x)^2}{2} + \frac{11(2+x)^3}{6} + \frac{5(2+x)^4}{24} + \frac{7(2+x)^5}{20} + \frac{83(2+x)^6}{360} + O\left((2+x)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = 4 + 3x - \frac{9(2+x)^2}{2} + \frac{11(2+x)^3}{6} + \frac{5(2+x)^4}{24} \\ + \frac{7(2+x)^5}{20} + \frac{83(2+x)^6}{360} + O\left((2+x)^6\right) \end{aligned} \quad (1)$$

Verification of solutions

$$y = 4 + 3x - \frac{9(2+x)^2}{2} + \frac{11(2+x)^3}{6} + \frac{5(2+x)^4}{24} + \frac{7(2+x)^5}{20} + \frac{83(2+x)^6}{360} + O\left((2+x)^6\right)$$

Verified OK.

13.22.2 Maple step by step solution

Let's solve

$$\left[(2x+3)\, y'' - 3y' + (-x-2)\, y = 0, y(-2) = -2, y'\Big|_{\{x=-2\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{(2+x)y}{2x+3} + \frac{3y'}{2x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{3y'}{2x+3} - \frac{(2+x)y}{2x+3} = 0$

☐ Check to see if $x_0 = -\frac{3}{2}$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{3}{2x+3}, P_3(x) = -\frac{2+x}{2x+3}\right]$

- ◦ $\left(x+\frac{3}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{3}{2}$

  $\left(\left(x+\frac{3}{2}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{3}{2}} = -\frac{3}{2}$

- ◦ $\left(x+\frac{3}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{3}{2}$

  $\left(\left(x+\frac{3}{2}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-\frac{3}{2}} = 0$

- ◦ $x = -\frac{3}{2}$is a regular singular point

Check to see if $x_0 = -\frac{3}{2}$ is a regular singular point

$x_0 = -\frac{3}{2}$

- Multiply by denominators

$(2x+3)\, y'' - 3y' + (-x-2)\, y = 0$

- Change variables using $x = u - \frac{3}{2}$ so that the regular singular point is at $u = 0$

$2u\left(\frac{d^2}{du^2}y(u)\right) - 3\frac{d}{du}y(u) + \left(-u - \frac{1}{2}\right) y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..1$

  $u^{m}\cdot y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r+m}$

- Shift index using $k->k-m$

  $u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-5+2r)\, u^{-1+r}+\left(a_{1}(1+r)\,(-3+2r)-\frac{a_{0}}{2}\right) u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(2k-3+2r)-\frac{a_{k}}{2}\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{5}{2}\right\}$

- Each term must be 0

  $a_{1}(1+r)\,(-3+2r)-\frac{a_{0}}{2}=0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{3}{2}+r\right)(k+1+r)\, a_{k+1}-\frac{a_{k}}{2}-a_{k-1}=0$

- Shift index using $k->k+1$

  $2\left(k-\frac{1}{2}+r\right)(k+2+r)\, a_{k+2}-\frac{a_{k+1}}{2}-a_{k}=0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k-1+2r)(k+2+r)}$

- Recursion relation for $r = 0$

$a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k-1)(k+2)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k-1)(k+2)}, -3a_1 - \frac{a_0}{2} = 0\right]$$

- Revert the change of variables $u = x + \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{3}{2}\right)^k, a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k-1)(k+2)}, -3a_1 - \frac{a_0}{2} = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

$a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{9}{2}\right)}$

- Solution for $r = \frac{5}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{5}{2}}, a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{9}{2}\right)}, 7a_1 - \frac{a_0}{2} = 0\right]$$

- Revert the change of variables $u = x + \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{3}{2}\right)^{k+\frac{5}{2}}, a_{k+2} = \frac{a_{k+1}+2a_k}{2(2k+4)\left(k+\frac{9}{2}\right)}, 7a_1 - \frac{a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{3}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{3}{2}\right)^{k+\frac{5}{2}}\right), a_{k+2} = \frac{a_{1+k}+2a_k}{2(2k-1)(k+2)}, -3a_1 - \frac{a_0}{2} = 0, b_{k+2} = \frac{b_{1+k}}{2(2k+4}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(3+2*x)*diff(y(x),x$2)-3*diff(y(x),x)-(2+x)*y(x)=0,y(-2) = -2, D(y)(-2) = 3],y(x),ty
```

$$y(x) = -2 + 3(2 + x) - \frac{9}{2}(2 + x)^2 + \frac{11}{6}(2 + x)^3 + \frac{5}{24}(2 + x)^4 + \frac{7}{20}(2 + x)^5 + O\left((2 + x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(3+2*x)*y''[x]-3*y'[x]-(2+x)*y[x]==0,{y[-2]==-2,y'[-2]==3}},y[x],{x,-
```

$$y(x) \to \frac{7}{20}(x+2)^5 + \frac{5}{24}(x+2)^4 + \frac{11}{6}(x+2)^3 - \frac{9}{2}(x+2)^2 + 3(x+2) - 2$$

13.23 problem 26

Internal problem ID [1264]
Internal file name [`OUTPUT/1265_Sunday_June_05_2022_02_07_20_AM_17878250/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(10-2x)\,y''+(x+1)\,y=0}$$

With initial conditions

$$[y(2)=2, y'(2)=-4]$$

With the expansion point for the power series method at $x=2$.

13.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=0\\q(x)&=\frac{x+1}{10-2x}\\F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(x+1)\, y}{10 - 2x} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{x+1}{10-2x}$ is

$$\{x < 5 \lor 5 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = -2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(6 - 2t) \left(\frac{d^2}{dt^2} y(t)\right) + (t+3)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 2$$
$$y'(0) = -4$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (995)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (996)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{(t+3)\, y(t)}{2t-6} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\,(t)}F_0 \\
&= \frac{\left(\frac{d}{dt}y(t)\right)t^2 - 9\frac{d}{dt}y(t) - 6y(t)}{2\,(t-3)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\,(t)}F_1 \\
&= \frac{(-24t+72)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^3+3t^2-9t-3)}{4\,(t-3)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\,(t)}F_2 \\
&= \frac{(t^4-18t^2+72t-135)\left(\frac{d}{dt}y(t)\right) + (-24t^2+144)\,y(t)}{4\,(t-3)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\,(t)}F_3 \\
&= \frac{72(-t^3+3t^2+t-3)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^5+3t^4-18t^3+114t^2+369t-1557)}{8\,(t-3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = -4$ gives

$$
\begin{aligned}
F_0 &= -1 \\
F_1 &= \frac{4}{3} \\
F_2 &= \frac{49}{18} \\
F_3 &= \frac{23}{9} \\
F_4 &= \frac{125}{108}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 2 - 4t - \frac{t^2}{2} + \frac{2t^3}{9} + \frac{49t^4}{432} + \frac{23t^5}{1080} + \frac{25t^6}{15552} + O(t^6)$$

$$y(t) = 2 - 4t - \frac{t^2}{2} + \frac{2t^3}{9} + \frac{49t^4}{432} + \frac{23t^5}{1080} + \frac{25t^6}{15552} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(6 - 2t)\left(\frac{d^2}{dt^2}y(t)\right) + (t+3)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(6 - 2t)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + (t+3)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-2n\,t^{n-1}a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} 6n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} t^{1+n}a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty}\left(-2n\,t^{n-1}a_{n}(n-1)\right) = \sum_{n=1}^{\infty}\left(-2(1+n)\,a_{1+n} n\,t^{n}\right)$$

$$\sum_{n=2}^{\infty} 6n(n-1)\,a_{n} t^{n-2} = \sum_{n=0}^{\infty} 6(n+2)\,a_{n+2}(1+n)\,t^{n}$$

$$\sum_{n=0}^{\infty} t^{1+n} a_{n} = \sum_{n=1}^{\infty} a_{n-1} t^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\sum_{n=1}^{\infty}\left(-2(1+n)\,a_{1+n} n\,t^{n}\right) + \left(\sum_{n=0}^{\infty} 6(n+2)\,a_{n+2}(1+n)\,t^{n}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^{n}\right) + \left(\sum_{n=0}^{\infty} 3a_{n} t^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$12a_{2} + 3a_{0} = 0$$

$$a_{2} = -\frac{a_{0}}{4}$$

For $1 \leq n$, the recurrence equation is

$$-2(1+n)\,a_{1+n} n + 6(n+2)\,a_{n+2}(1+n) + a_{n-1} + 3a_{n} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2n^{2}a_{1+n} + 2na_{1+n} - 3a_{n} - a_{n-1}}{6\,(n+2)\,(1+n)} \\ &= -\frac{a_{n}}{2\,(n+2)\,(1+n)} + \frac{(2n^{2}+2n)\,a_{1+n}}{6\,(n+2)\,(1+n)} - \frac{a_{n-1}}{6\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-4a_2 + 36a_3 + a_0 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{18} - \frac{a_1}{12}$$

For $n = 2$ the recurrence equation gives

$$-12a_3 + 72a_4 + a_1 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{864} - \frac{a_1}{36}$$

For $n = 3$ the recurrence equation gives

$$-24a_4 + 120a_5 + a_2 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{270} - \frac{a_1}{288}$$

For $n = 4$ the recurrence equation gives

$$-40a_5 + 180a_6 + a_3 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{173a_0}{155520} + \frac{a_1}{6480}$$

For $n = 5$ the recurrence equation gives

$$-60a_6 + 252a_7 + a_4 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{353a_0}{1632960} + \frac{41a_1}{217728}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t - \frac{a_0 t^2}{4} + \left(-\frac{a_0}{18} - \frac{a_1}{12}\right) t^3 + \left(\frac{a_0}{864} - \frac{a_1}{36}\right) t^4 + \left(\frac{a_0}{270} - \frac{a_1}{288}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - \frac{1}{4}t^2 - \frac{1}{18}t^3 + \frac{1}{864}t^4 + \frac{1}{270}t^5\right) a_0 + \left(t - \frac{1}{12}t^3 - \frac{1}{36}t^4 - \frac{1}{288}t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - \frac{1}{4}t^2 - \frac{1}{18}t^3 + \frac{1}{864}t^4 + \frac{1}{270}t^5\right) c_1 + \left(t - \frac{1}{12}t^3 - \frac{1}{36}t^4 - \frac{1}{288}t^5\right) c_2 + O(t^6)
$$

$$
y(t) = 2 - \frac{t^2}{2} + \frac{2t^3}{9} + \frac{49t^4}{432} + \frac{23t^5}{1080} - 4t + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = -2 + x$ results in

$$
\begin{aligned}
y = 10 - 4x &- \frac{(-2+x)^2}{2} + \frac{2(-2+x)^3}{9} + \frac{49(-2+x)^4}{432} \\
&+ \frac{23(-2+x)^5}{1080} + \frac{25(-2+x)^6}{15552} + O\left((-2+x)^6\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = 10 - 4x &- \frac{(-2+x)^2}{2} + \frac{2(-2+x)^3}{9} + \frac{49(-2+x)^4}{432} \\
&+ \frac{23(-2+x)^5}{1080} + \frac{25(-2+x)^6}{15552} + O\left((-2+x)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$y = 10 - 4x - \frac{(-2+x)^2}{2} + \frac{2(-2+x)^3}{9} + \frac{49(-2+x)^4}{432} + \frac{23(-2+x)^5}{1080} + \frac{25(-2+x)^6}{15552} + O\left((-2+x)^6\right)$$

Verified OK.

13.23.2 Maple step by step solution

Let's solve

$\left[(10-2x)\, y'' + (x+1)\, y = 0, y(2) = 2, y'\Big|_{\{x=2\}} = -4\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+1)y}{2(x-5)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+1)y}{2(x-5)} = 0$

□ Check to see if $x_0 = 5$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{x+1}{2(x-5)}\right]$

  - $(x-5) \cdot P_2(x)$ is analytic at $x = 5$

    $\left((x-5) \cdot P_2(x)\right)\Big|_{x=5} = 0$

  - $(x-5)^2 \cdot P_3(x)$ is analytic at $x = 5$

    $\left((x-5)^2 \cdot P_3(x)\right)\Big|_{x=5} = 0$

  - $x = 5$is a regular singular point

  Check to see if $x_0 = 5$ is a regular singular point

  $x_0 = 5$

- Multiply by denominators

  $y''(2x-10) + (-x-1)\, y = 0$

- Change variables using $x = u + 5$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + (-u-6)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+r)\,u^{-1+r} + (2a_1(1+r)\,r - 6a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r) - 6a_k - a_{k-1})\,u^{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$
- Each term must be 0

  $2a_1(1+r)\,r - 6a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)\,(k+r) - 6a_k - a_{k-1} = 0$
- Shift index using $k->k+1$

  $2a_{k+2}(k+2+r)\,(k+1+r) - 6a_{k+1} - a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+2+r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+2)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+2)(k+1)}, -6a_0 = 0\right]$$

- Revert the change of variables $u = x - 5$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-5)^k, a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+2)(k+1)}, -6a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+3)(k+2)}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+3)(k+2)}, 4a_1 - 6a_0 = 0\right]$$

- Revert the change of variables $u = x - 5$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-5)^{k+1}, a_{k+2} = \frac{6a_{k+1}+a_k}{2(k+3)(k+2)}, 4a_1 - 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-5)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-5)^{1+k}\right), a_{k+2} = \frac{6a_{1+k}+a_k}{2(k+2)(1+k)}, -6a_0 = 0, b_{k+2} = \frac{6b_{1+k}+b_k}{2(k+3)(k+2)}, 4\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(10-2*x)*diff(y(x),x$2)+(1+x)*y(x)=0,y(2) = 2, D(y)(2) = -4],y(x),type='series',x=2)
```

$$
\begin{aligned}
y(x) = &\, 2 - 4(-2+x) - \frac{1}{2}(-2+x)^2 + \frac{2}{9}(-2+x)^3 \\
&+ \frac{49}{432}(-2+x)^4 + \frac{23}{1080}(-2+x)^5 + \mathrm{O}\left((-2+x)^6\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(10-2*x)*y''[x]+(1+x)*y[x]==0,{y[2]==2,y'[2]==-4}},y[x],{x,2,5}]
```

$$
y(x) \to \frac{23(x-2)^5}{1080} + \frac{49}{432}(x-2)^4 + \frac{2}{9}(x-2)^3 - \frac{1}{2}(x-2)^2 - 4(x-2) + 2
$$

13.24 problem 27

Internal problem ID [1265]
Internal file name [`OUTPUT/1266_Sunday_June_05_2022_02_07_23_AM_82174801/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(7+x)\,y''+(2x+8)\,y'+(x+5)\,y=0}$$

With initial conditions

$$[y(-4)=1, y'(-4)=2]$$

With the expansion point for the power series method at $x=-4$.

13.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2x+8}{7+x}\\ q(x)&=\frac{x+5}{7+x}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(2x+8)\,y'}{7+x} + \frac{(x+5)\,y}{7+x} = 0$$

The domain of $p(x) = \frac{2x+8}{7+x}$ is

$$\{x < -7 \vee -7 < x\}$$

And the point $x_0 = -4$ is inside this domain. The domain of $q(x) = \frac{x+5}{7+x}$ is

$$\{x < -7 \vee -7 < x\}$$

And the point $x_0 = -4$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 4$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+3)\left(\frac{d^2}{dt^2}y(t)\right) + 2t\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (998) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (999) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$F_0 = -\frac{2t\left(\frac{d}{dt}y(t)\right) + y(t)\,t + y(t)}{t+3}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(3t^2-4t-9\right)\left(\frac{d}{dt}y(t)\right) + 2y(t)\left(t^2+t-1\right)}{\left(t+3\right)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{\left(-4t^3+16t^2+44t\right)\left(\frac{d}{dt}y(t)\right) - 3\left(t^2-\frac{4}{3}t-\frac{19}{3}\right)\left(t+1\right)y(t)}{\left(t+3\right)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{\left(5t^4-40t^3-114t^2+96t+189\right)\left(\frac{d}{dt}y(t)\right) + 4y(t)\left(t^4-3t^3-22t^2-21t+3\right)}{\left(t+3\right)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{\left(-6t^5+80t^4+204t^3-672t^2-1590t-432\right)\left(\frac{d}{dt}y(t)\right) - 5\left(t^5-7t^4-\frac{214}{5}t^3-\frac{86}{5}t^2+\frac{561}{5}t+\frac{489}{5}\right)y(t)}{\left(t+3\right)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=1$ and $y'(0)=2$ gives

$$\begin{aligned}
F_0 &= -\frac{1}{3} \\
F_1 &= -\frac{20}{9} \\
F_2 &= \frac{19}{27} \\
F_3 &= \frac{130}{27} \\
F_4 &= -\frac{451}{81}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 + 2t - \frac{t^2}{6} - \frac{10t^3}{27} + \frac{19t^4}{648} + \frac{13t^5}{324} - \frac{451t^6}{58320} + O(t^6)$$

$$y(t) = 1 + 2t - \frac{t^2}{6} - \frac{10t^3}{27} + \frac{19t^4}{648} + \frac{13t^5}{324} - \frac{451t^6}{58320} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(t+3)\left(\frac{d^2}{dt^2}y(t)\right) + 2t\left(\frac{d}{dt}y(t)\right) + (t+1)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$(t+3)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + 2t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t+1)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\,t^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n(n-1)\,a_n t^{n-2}\right) \\
&+ \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0
\end{aligned} \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=0}^{\infty} t^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{6}$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + 3(n+2)\, a_{n+2}(1+n) + 2n a_n + a_{n-1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{1+n} + 2n a_n + n a_{1+n} + a_n + a_{n-1}}{3\,(n+2)\,(1+n)} \\ &= -\frac{(2n+1)\, a_n}{3\,(n+2)\,(1+n)} - \frac{(n^2+n)\, a_{1+n}}{3\,(n+2)\,(1+n)} - \frac{a_{n-1}}{3\,(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 18a_3 + 3a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{27} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 36a_4 + 5a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{19a_0}{648}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 60a_5 + 7a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{810} + \frac{7a_1}{360}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 90a_6 + 9a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{163a_0}{58320} - \frac{a_1}{405}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 126a_7 + 11a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{199a_0}{612360} - \frac{151a_1}{136080}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t - \frac{a_0 t^2}{6} + \left(-\frac{a_0}{27} - \frac{a_1}{6}\right) t^3 + \frac{19 a_0 t^4}{648} + \left(\frac{a_0}{810} + \frac{7 a_1}{360}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - \frac{1}{6} t^2 - \frac{1}{27} t^3 + \frac{19}{648} t^4 + \frac{1}{810} t^5\right) a_0 + \left(t - \frac{1}{6} t^3 + \frac{7}{360} t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - \frac{1}{6} t^2 - \frac{1}{27} t^3 + \frac{19}{648} t^4 + \frac{1}{810} t^5\right) c_1 + \left(t - \frac{1}{6} t^3 + \frac{7}{360} t^5\right) c_2 + O(t^6)
$$

$$
y(t) = 1 - \frac{t^2}{6} - \frac{10 t^3}{27} + \frac{19 t^4}{648} + \frac{13 t^5}{324} + 2t + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = x + 4$ results in

$$
y = 9 + 2x - \frac{(x+4)^2}{6} - \frac{10(x+4)^3}{27} + \frac{19(x+4)^4}{648} + \frac{13(x+4)^5}{324} - \frac{451(x+4)^6}{58320} + O\left((x+4)^6\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = 9 + 2x - \frac{(x+4)^2}{6} - \frac{10(x+4)^3}{27} + \frac{19(x+4)^4}{648} \\
+ \frac{13(x+4)^5}{324} - \frac{451(x+4)^6}{58320} + O\left((x+4)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$
y = 9 + 2x - \frac{(x+4)^2}{6} - \frac{10(x+4)^3}{27} + \frac{19(x+4)^4}{648} + \frac{13(x+4)^5}{324} - \frac{451(x+4)^6}{58320} + O\left((x+4)^6\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(7+x)*diff(y(x),x$2)+(8+2*x)*diff(y(x),x)+(5+x)*y(x)=0,y(-4) = 1, D(y)(-4) = 2],y(x)
```

$$y(x) = 1 + 2(x+4) - \frac{1}{6}(x+4)^2 - \frac{10}{27}(x+4)^3 + \frac{19}{648}(x+4)^4 + \frac{13}{324}(x+4)^5 + O\left((x+4)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(7+x)*y''[x]+(8+2*x)*y'[x]+(5+x)*y[x]==0,{y[-4]==1,y'[-4]==2}},y[x],{
```

$$y(x) \to \frac{13}{324}(x+4)^5 + \frac{19}{648}(x+4)^4 - \frac{10}{27}(x+4)^3 - \frac{1}{6}(x+4)^2 + 2(x+4) + 1$$

13.25 problem 28

Internal problem ID [1266]
Internal file name [`OUTPUT/1267_Sunday_June_05_2022_02_07_26_AM_11076969/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(6+4x)\,y''+(1+2x)\,y=0}$$

With initial conditions

$$[y(-1)=-1, y'(-1)=2]$$

With the expansion point for the power series method at $x=-1$.

13.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1+2x}{6+4x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(1+2x)\,y}{6+4x} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{1+2x}{6+4x}$ is

$$\left\{x < -\frac{3}{2} \lor -\frac{3}{2} < x\right\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x + 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(4t+2)\left(\frac{d^2}{dt^2}y(t)\right) + (2t-1)\,y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = -1$$
$$y'(0) = 2$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1001) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1002) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{(2t-1)\,y(t)}{2\,(1+2t)} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\,(t)}F_0 \\
&= \frac{-2\left(\frac{d}{dt}y(t)\right)t^2 - 2y(t) + \frac{\frac{d}{dt}y(t)}{2}}{(1+2t)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\,(t)}F_1 \\
&= \frac{4(-2t-1)\left(\frac{d}{dt}y(t)\right) + 2\left(t+\frac{3}{2}\right)\left(t^2-2t+\frac{11}{4}\right)y(t)}{(1+2t)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\,(t)}F_2 \\
&= \frac{(16t^4-8t^2+192t+97)\left(\frac{d}{dt}y(t)\right) + (64t^2-208)\,y(t)}{4\,(1+2t)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\,(t)}F_3 \\
&= \frac{(384t^3+192t^2-3168t-1584)\left(\frac{d}{dt}y(t)\right) - 32\left(t^5-\frac{1}{2}t^4-\frac{1}{2}t^3+\frac{113}{4}t^2-\frac{127}{16}t-\frac{3425}{32}\right)y(t)}{8\,(1+2t)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=-1$ and $y'(0)=2$ gives

$$
\begin{aligned}
F_0 &= -\frac{1}{2} \\
F_1 &= 3 \\
F_2 &= -\frac{65}{4} \\
F_3 &= \frac{201}{2} \\
F_4 &= -\frac{6593}{8}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 2t - 1 - \frac{t^2}{4} + \frac{t^3}{2} - \frac{65t^4}{96} + \frac{67t^5}{80} - \frac{6593t^6}{5760} + O(t^6)$$

$$y(t) = 2t - 1 - \frac{t^2}{4} + \frac{t^3}{2} - \frac{65t^4}{96} + \frac{67t^5}{80} - \frac{6593t^6}{5760} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(4t+2)\left(\frac{d^2}{dt^2}y(t)\right) + (2t-1)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(4t+2)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + (2t-1)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4n\,t^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2t^{1+n} a_n\right) + \sum_{n=0}^{\infty}(-a_n t^n) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, t^n$$

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, t^n$$

$$\sum_{n=0}^{\infty} 2t^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} t^n\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{4}$$

For $1 \leq n$, the recurrence equation is

$$4(1+n)\, a_{1+n} n + 2(n+2)\, a_{n+2}(1+n) + 2a_{n-1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{4n^2 a_{1+n} + 4n a_{1+n} - a_n + 2a_{n-1}}{2\,(n+2)\,(1+n)} \\ &= \frac{a_n}{2\,(n+2)\,(1+n)} - \frac{(4n^2+4n)\, a_{1+n}}{2\,(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$8a_2 + 12a_3 + 2a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{3} + \frac{a_1}{12}$$

For $n = 2$ the recurrence equation gives

$$24a_3 + 24a_4 + 2a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{11a_0}{32} - \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$48a_4 + 40a_5 + 2a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13a_0}{30} + \frac{97a_1}{480}$$

For $n = 4$ the recurrence equation gives

$$80a_5 + 60a_6 + 2a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{685a_0}{1152} - \frac{11a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$120a_6 + 84a_7 + 2a_4 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{2899a_0}{3360} + \frac{16097a_1}{40320}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{a_0 t^2}{4} + \left(-\frac{a_0}{3} + \frac{a_1}{12}\right) t^3 + \left(\frac{11 a_0}{32} - \frac{a_1}{6}\right) t^4 + \left(-\frac{13 a_0}{30} + \frac{97 a_1}{480}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{4} t^2 - \frac{1}{3} t^3 + \frac{11}{32} t^4 - \frac{13}{30} t^5\right) a_0 + \left(t + \frac{1}{12} t^3 - \frac{1}{6} t^4 + \frac{97}{480} t^5\right) a_1 + O\left(t^6\right) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{4} t^2 - \frac{1}{3} t^3 + \frac{11}{32} t^4 - \frac{13}{30} t^5\right) c_1 + \left(t + \frac{1}{12} t^3 - \frac{1}{6} t^4 + \frac{97}{480} t^5\right) c_2 + O\left(t^6\right)$$

$$y(t) = -1 - \frac{t^2}{4} + \frac{t^3}{2} - \frac{65 t^4}{96} + \frac{67 t^5}{80} + 2t + O\left(t^6\right)$$

Replacing t in the above with the original independent variable $x$susing $t = x+1$ results in

$$y = 2x+1 - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{2} - \frac{65(x+1)^4}{96} + \frac{67(x+1)^5}{80} - \frac{6593(x+1)^6}{5760} + O\left((x+1)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = 2x + 1 - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{2} - \frac{65(x+1)^4}{96} \\ + \frac{67(x+1)^5}{80} - \frac{6593(x+1)^6}{5760} + O\left((x+1)^6\right)\end{aligned} \quad (1)$$

Verification of solutions

$$y = 2x+1 - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{2} - \frac{65(x+1)^4}{96} + \frac{67(x+1)^5}{80} - \frac{6593(x+1)^6}{5760} + O\left((x+1)^6\right)$$

Verified OK.

13.25.2 Maple step by step solution

Let's solve

$\left[(6+4x)\,y''+(1+2x)\,y=0, y(-1)=-1, y'\Big|_{\{x=-1\}}=2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(1+2x)y}{2(2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+2x)y}{2(2x+3)}=0$

□ Check to see if $x_0=-\frac{3}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=0, P_3(x)=\frac{1+2x}{2(2x+3)}\right]$

  ◦ $\left(x+\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{3}{2}$

    $\left(\left(x+\frac{3}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{3}{2}}=0$

  ◦ $\left(x+\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{3}{2}$

    $\left(\left(x+\frac{3}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{3}{2}}=0$

  ◦ $x=-\frac{3}{2}$is a regular singular point

  Check to see if $x_0=-\frac{3}{2}$ is a regular singular point

  $x_0=-\frac{3}{2}$

- Multiply by denominators

  $(6+4x)\,y''+(1+2x)\,y=0$

- Change variables using $x=u-\frac{3}{2}$ so that the regular singular point is at $u=0$

  $4u\left(\frac{d^2}{du^2}y(u)\right)+(-2+2u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$
  - Shift index using $k->k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0 r(-1+r)\,u^{-1+r} + (4a_1(1+r)\,r - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+r) - 2a_k + 2a_{k-1})\,u^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$
- Each term must be 0

  $4a_1(1+r)\,r - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4a_{k+1}(k+1+r)(k+r) - 2a_k + 2a_{k-1} = 0$
- Shift index using $k->k+1$

  $4a_{k+2}(k+2+r)(k+1+r) - 2a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{-a_{k+1}+a_k}{2(k+2+r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{-a_{k+1}+a_k}{2(k+2)(k+1)}$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{-a_{k+1}+a_k}{2(k+2)(k+1)}, -2a_0=0\right]$$

- Revert the change of variables $u=x+\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{3}{2}\right)^k, a_{k+2}=-\frac{-a_{k+1}+a_k}{2(k+2)(k+1)}, -2a_0=0\right]$$

- Recursion relation for $r=1$

$$a_{k+2}=-\frac{-a_{k+1}+a_k}{2(k+3)(k+2)}$$

- Solution for $r=1$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2}=-\frac{-a_{k+1}+a_k}{2(k+3)(k+2)}, 8a_1-2a_0=0\right]$$

- Revert the change of variables $u=x+\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{3}{2}\right)^{k+1}, a_{k+2}=-\frac{-a_{k+1}+a_k}{2(k+3)(k+2)}, 8a_1-2a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x+\frac{3}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+\frac{3}{2}\right)^{1+k}\right), a_{k+2}=-\frac{-a_{1+k}+a_k}{2(k+2)(1+k)}, -2a_0=0, b_{k+2}=-\frac{-b_{1+k}+b}{2(k+3)(k+}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(6+4*x)*diff(y(x),x$2)+(1+2*x)*y(x)=0,y(-1) = -1, D(y)(-1) = 2],y(x),type='series',x
```

$$y(x) = -1 + 2(x+1) - \frac{1}{4}(x+1)^2 + \frac{1}{2}(x+1)^3 - \frac{65}{96}(x+1)^4 + \frac{67}{80}(x+1)^5 + O\left((x+1)^6\right)$$

✓ Solution by Mathematica
Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(6+4*x)*y''[x]+(1+2*x)*y[x]==0,{y[-1]==-1,y'[-1]==2}},y[x],{x,-1,5}]
```

$$y(x) \to \frac{67}{80}(x+1)^5 - \frac{65}{96}(x+1)^4 + \frac{1}{2}(x+1)^3 - \frac{1}{4}(x+1)^2 + 2(x+1) - 1$$

13.26 problem 29

Internal problem ID [1267]
Internal file name [`OUTPUT/1268_Sunday_June_05_2022_02_07_29_AM_65047568/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(\beta\,x^2+\alpha x+1\right)y''+\left(\delta x+\gamma\right)y'+\epsilon y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x)&=y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&=y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots\\
&=y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1004}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1005}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$F_0 = -\frac{y'\delta x + y'\gamma + \epsilon y}{\beta\, x^2 + \alpha x + 1}$$

$$\begin{aligned} F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ &= \frac{\left(\left(-\beta\epsilon + \delta(\beta+\delta)\right) x^2 + \left(-\alpha\epsilon + 2\gamma(\beta+\delta)\right) x + \alpha\gamma + \gamma^2 - \delta - \epsilon\right) y' + 2\left(\left(\beta + \frac{\delta}{2}\right) x + \frac{\alpha}{2} + \frac{\gamma}{2}\right) y\epsilon}{\left(\beta\, x^2 + \alpha x + 1\right)^2} \end{aligned}$$

$$\begin{aligned} F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\ &= \frac{\left(-2\left(\beta + \frac{\delta}{2}\right)\left(-2\beta\epsilon + \delta(\beta+\delta)\right) x^3 + \left(\left(\left(6\alpha + 2\gamma\right)\beta + 2\alpha\delta\right)\epsilon - 6\left(\beta + \frac{\delta}{2}\right)\gamma(\beta+\delta)\right) x^2 + \left(\left(2\alpha^2 + 2\alpha\gamma\right.\right.}{} \end{aligned}$$

$$\begin{aligned} F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\ &= \frac{\left(\left(\beta^2\epsilon^2 + \left(-18\beta^3 - 14\delta\,\beta^2 - 3\beta\,\delta^2\right)\epsilon + 6\beta^3\delta + 11\beta^2\delta^2 + 6\beta\,\delta^3 + \delta^4\right) x^4 + \left(2\beta\alpha\,\epsilon^2 + \left(\left(-36\alpha - 18\gamma\right)\right.\right.\right.}{} \end{aligned}$$

$$\begin{aligned} F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\ &= \text{Expression too large to display} \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$F_0 = -y(0)\,\epsilon - y'(0)\,\gamma$$
$$F_1 = y(0)\,\alpha\epsilon + y(0)\,\epsilon\gamma + y'(0)\,\alpha\gamma + y'(0)\,\gamma^2 - y'(0)\,\delta - y'(0)\,\epsilon$$
$$F_2 = -2y(0)\,\alpha^2\epsilon - 3y(0)\,\alpha\epsilon\gamma - y(0)\,\epsilon\,\gamma^2 - 2y'(0)\,\alpha^2\gamma - 3y'(0)\,\alpha\,\gamma^2 - y'(0)\,\gamma^3 + 2y(0)\,\beta\epsilon + 2y(0)\,\epsilon\delta + \epsilon^2 y$$
$$F_3 = 11y(0)\,\alpha^2\epsilon\gamma + 6y(0)\,\alpha\epsilon\,\gamma^2 - 9y(0)\,\alpha\epsilon\delta - 8y(0)\,\beta\epsilon\gamma - 5y(0)\,\epsilon\delta\gamma - 14y'(0)\,\alpha\delta\gamma - 9y'(0)\,\alpha\epsilon\gamma - 12y(0)$$
$$F_4 = 80y(0)\,\alpha\beta\epsilon\gamma + 41y(0)\,\alpha\epsilon\delta\gamma + 44y(0)\,\alpha^2\epsilon\delta + 15\epsilon^2 y(0)\,\alpha\gamma + 20y(0)\,\beta\epsilon\,\gamma^2 + 9y(0)\,\epsilon\delta\,\gamma^2 - 32y(0)\,\beta\epsilon\delta -$$

Substituting all the above in (7) and simplifying gives the solution as

Expression too large to display

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\beta\,x^2+\alpha x+1\right) y''+\left(\delta x+\gamma\right) y'+\epsilon y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(\beta\,x^2+\alpha x+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\delta x+\gamma\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\epsilon\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n \beta n(n-1)\right)+\left(\sum_{n=2}^{\infty} n\alpha\, x^{n-1} a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\ +\left(\sum_{n=1}^{\infty} n a_n x^n \delta\right)+\left(\sum_{n=1}^{\infty} n\gamma\, x^{n-1} a_n\right)+\left(\sum_{n=0}^{\infty} \epsilon a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\alpha\, x^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} \alpha(n+1)\, a_{n+1} n\, x^n$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=1}^{\infty} n\gamma\, x^{n-1} a_n = \sum_{n=0}^{\infty} \gamma(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^{n} a_{n} \beta n(n-1)\right)+\left(\sum_{n=1}^{\infty} \alpha(n+1) a_{n+1} n\, x^{n}\right)\\&+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^{n}\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n} \delta\right)\\&+\left(\sum_{n=0}^{\infty} \gamma(n+1) a_{n+1} x^{n}\right)+\left(\sum_{n=0}^{\infty} \epsilon a_{n} x^{n}\right)=0\end{aligned} \tag{3}$$

$n=0$ gives

$$\epsilon a_{0}+\gamma a_{1}+2 a_{2}=0$$

$$a_{2}=-\frac{\epsilon a_{0}}{2}-\frac{\gamma a_{1}}{2}$$

$n=1$ gives

$$2 \alpha a_{2}+\delta a_{1}+\epsilon a_{1}+2 \gamma a_{2}+6 a_{3}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=\frac{1}{6} \alpha \epsilon a_{0}+\frac{1}{6} \alpha \gamma a_{1}+\frac{1}{6} \gamma \epsilon a_{0}+\frac{1}{6} \gamma^{2} a_{1}-\frac{1}{6} \delta a_{1}-\frac{1}{6} \epsilon a_{1}$$

For $2 \leq n$, the recurrence equation is

$$\beta n a_{n}(n-1)+\alpha(n+1) a_{n+1} n+(n+2) a_{n+2}(n+1)+\delta n a_{n}+\gamma(n+1) a_{n+1}+\epsilon a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}a_{n+2}&=-\frac{\alpha\, n^{2} a_{n+1}+\beta\, n^{2} a_{n}+\alpha n a_{n+1}-\beta n a_{n}+\delta n a_{n}+\gamma n a_{n+1}+\epsilon a_{n}+\gamma a_{n+1}}{(n+2)(n+1)}\\&=-\frac{\left(\beta\, n^{2}-\beta n+\delta n+\epsilon\right) a_{n}}{(n+2)(n+1)}-\frac{\left(\alpha\, n^{2}+\alpha n+\gamma n+\gamma\right) a_{n+1}}{(n+2)(n+1)}\end{aligned} \tag{5}$$

For $n=2$ the recurrence equation gives

$$6 \alpha a_{3}+2 \beta a_{2}+2 \delta a_{2}+\epsilon a_{2}+3 \gamma a_{3}+12 a_{4}=0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{12}\alpha^2\epsilon a_0 - \frac{1}{12}\alpha^2\gamma a_1 - \frac{1}{8}\alpha\gamma\epsilon a_0 - \frac{1}{8}\alpha\,\gamma^2 a_1 + \frac{1}{12}\alpha\delta a_1 + \frac{1}{12}\alpha\epsilon a_1 + \frac{1}{12}\beta\epsilon a_0 + \frac{1}{12}\beta\gamma a_1 + \frac{1}{12}\delta\epsilon a_0 + \frac{1}{8}\delta\gamma a_1 + \frac{1}{24}\epsilon^2 a_0 + \frac{1}{12}\epsilon\gamma a_1 - \frac{1}{24}\gamma^2\epsilon a_0 - \frac{1}{24}\gamma^3 a_1$$

For $n = 3$ the recurrence equation gives

$$12\alpha a_4 + 6\beta a_3 + 3\delta a_3 + \epsilon a_3 + 4\gamma a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11}{120}\alpha^2\gamma\epsilon a_0 - \frac{1}{10}\alpha\beta\epsilon a_0 - \frac{1}{10}\alpha\beta\gamma a_1 - \frac{3}{40}\alpha\delta\epsilon a_0 - \frac{7}{60}\alpha\delta\gamma a_1 - \frac{3}{40}\alpha\epsilon\gamma a_1 + \frac{1}{20}\alpha\,\gamma^2\epsilon a_0 - \frac{1}{15}\beta\gamma\epsilon a_0 - \frac{1}{24}\delta\gamma\epsilon a_0 + \frac{1}{120}\gamma^3\epsilon a_0 + \frac{1}{20}\alpha^3\epsilon a_0 + \frac{1}{20}\alpha^3\gamma a_1 + \frac{11}{120}\alpha^2\gamma^2 a_1 - \frac{1}{20}\alpha^2\delta a_1 - \frac{1}{20}\alpha^2\epsilon a_1 - \frac{1}{30}\alpha\,\epsilon^2 a_0 + \frac{1}{20}\alpha\,\gamma^3 a_1 - \frac{1}{15}\beta\,\gamma^2 a_1 + \frac{1}{20}\beta\delta a_1 + \frac{1}{20}\beta\epsilon a_1 - \frac{1}{20}\delta\,\gamma^2 a_1 + \frac{1}{30}\delta\epsilon a_1 - \frac{1}{60}\gamma\,\epsilon^2 a_0 - \frac{1}{40}\epsilon\,\gamma^2 a_1 + \frac{1}{40}\delta^2 a_1 + \frac{1}{120}\epsilon^2 a_1 + \frac{1}{120}\gamma^4 a_1$$

For $n = 4$ the recurrence equation gives

$$20\alpha a_5 + 12\beta a_4 + 4\delta a_4 + \epsilon a_4 + 5\gamma a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{9}\alpha\beta\gamma\epsilon a_0 + \frac{41}{720}\alpha\delta\gamma\epsilon a_0 + \frac{1}{30}\alpha\epsilon\,\gamma^2 a_1 - \frac{2}{45}\beta\delta\epsilon a_0 - \frac{5}{72}\beta\delta\gamma a_1 - \frac{2}{45}\beta\epsilon\gamma a_1 + \frac{1}{36}\beta\,\gamma^2\epsilon a_0 - \frac{1}{48}\delta\epsilon\gamma a_1 + \frac{1}{80}\delta\,\gamma^2\epsilon a_0 - \frac{5}{72}\alpha^3\gamma\epsilon a_0 + \frac{1}{10}\alpha^2\beta\epsilon a_0 + \frac{1}{10}\alpha^2\beta\gamma a_1 + \frac{11}{180}\alpha^2\delta\epsilon a_0 + \frac{7}{72}\alpha^2\delta\gamma a_1 + \frac{11}{180}\alpha^2\epsilon\gamma a_1 - \frac{7}{144}\alpha^2\gamma^2\epsilon a_0 - \frac{1}{72}\alpha\,\gamma^3\epsilon a_0 + \frac{1}{9}\alpha\beta\,\gamma^2 a_1 - \frac{1}{15}\alpha\beta\delta a_1 - \frac{1}{15}\alpha\beta\epsilon a_1 + \frac{5}{72}\alpha\delta\,\gamma^2 a_1 - \frac{13}{360}\alpha\delta\epsilon a_1 + \frac{1}{48}\alpha\gamma\,\epsilon^2 a_0 - \frac{1}{720}\gamma^5 a_1 - \frac{1}{720}\epsilon^3 a_0 - \frac{1}{240}\epsilon^2\gamma a_1 + \frac{1}{240}\gamma^2\epsilon^2 a_0 + \frac{1}{180}\epsilon\,\gamma^3 a_1 - \frac{1}{720}\gamma^4\epsilon a_0 - \frac{1}{36}\alpha\,\delta^2 a_1 - \frac{1}{30}\alpha^4\epsilon a_0 - \frac{1}{30}\alpha^4\gamma a_1 - \frac{5}{72}\alpha^3\gamma^2 a_1 + \frac{1}{30}\alpha^3\delta a_1 + \frac{1}{30}\alpha^3\epsilon a_1 + \frac{1}{40}\alpha^2\epsilon^2 a_0 - \frac{7}{144}\alpha^2\gamma^3 a_1 - \frac{1}{120}\alpha\,\epsilon^2 a_1 - \frac{1}{72}\alpha\,\gamma^4 a_1 - \frac{1}{30}\beta^2\epsilon a_0 - \frac{1}{30}\beta^2\gamma a_1 - \frac{7}{360}\beta\,\epsilon^2 a_0 + \frac{1}{36}\beta\,\gamma^3 a_1 - \frac{1}{90}\delta^2\epsilon a_0 - \frac{1}{48}\delta^2\gamma a_1 - \frac{1}{120}\delta\,\epsilon^2 a_0 + \frac{1}{72}\delta\,\gamma^3 a_1$$

For $n = 5$ the recurrence equation gives

$$30\alpha a_6 + 20\beta a_5 + 5\delta a_5 + \epsilon a_5 + 6\gamma a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

Expression too large to display

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x &+ \left(-\frac{\epsilon a_0}{2} - \frac{\gamma a_1}{2}\right) x^2 + \left(\frac{1}{6}\alpha\epsilon a_0 + \frac{1}{6}\alpha\gamma a_1 + \frac{1}{6}\gamma\epsilon a_0 + \frac{1}{6}\gamma^2 a_1 - \frac{1}{6}\delta a_1 - \frac{1}{6}\epsilon a_1\right) x^3 \\ &+ \left(-\frac{1}{12}\alpha^2\epsilon a_0 - \frac{1}{12}\alpha^2\gamma a_1 - \frac{1}{8}\alpha\gamma\epsilon a_0 - \frac{1}{8}\alpha\gamma^2 a_1 + \frac{1}{12}\alpha\delta a_1 + \frac{1}{12}\alpha\epsilon a_1 + \frac{1}{12}\beta\epsilon a_0 \right. \\ &\quad \left. + \frac{1}{12}\beta\gamma a_1 + \frac{1}{12}\delta\epsilon a_0 + \frac{1}{8}\delta\gamma a_1 + \frac{1}{24}\epsilon^2 a_0 + \frac{1}{12}\epsilon\gamma a_1 - \frac{1}{24}\gamma^2\epsilon a_0 - \frac{1}{24}\gamma^3 a_1\right) x^4 \\ &+ \left(\frac{11}{120}\alpha^2\gamma\epsilon a_0 - \frac{1}{10}\alpha\beta\epsilon a_0 - \frac{1}{10}\alpha\beta\gamma a_1 - \frac{3}{40}\alpha\delta\epsilon a_0 - \frac{7}{60}\alpha\delta\gamma a_1 - \frac{3}{40}\alpha\epsilon\gamma a_1 + \frac{1}{20}\alpha\gamma^2\epsilon a_0 \right. \\ &\quad - \frac{1}{15}\beta\gamma\epsilon a_0 - \frac{1}{24}\delta\gamma\epsilon a_0 + \frac{1}{120}\gamma^3\epsilon a_0 + \frac{1}{20}\alpha^3\epsilon a_0 + \frac{1}{20}\alpha^3\gamma a_1 + \frac{11}{120}\alpha^2\gamma^2 a_1 - \frac{1}{20}\alpha^2\delta a_1 \\ &\quad - \frac{1}{20}\alpha^2\epsilon a_1 - \frac{1}{30}\alpha\epsilon^2 a_0 + \frac{1}{20}\alpha\gamma^3 a_1 - \frac{1}{15}\beta\gamma^2 a_1 + \frac{1}{20}\beta\delta a_1 + \frac{1}{20}\beta\epsilon a_1 - \frac{1}{20}\delta\gamma^2 a_1 \\ &\quad \left. + \frac{1}{30}\delta\epsilon a_1 - \frac{1}{60}\gamma\epsilon^2 a_0 - \frac{1}{40}\epsilon\gamma^2 a_1 + \frac{1}{40}\delta^2 a_1 + \frac{1}{120}\epsilon^2 a_1 + \frac{1}{120}\gamma^4 a_1\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y=\left(1-\frac{\epsilon\,x^{2}}{2}+\left(\frac{1}{6}\alpha\epsilon+\frac{1}{6}\epsilon\gamma\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\epsilon-\frac{1}{8}\alpha\gamma\epsilon+\frac{1}{12}\beta\epsilon+\frac{1}{12}\delta\epsilon+\frac{1}{24}\epsilon^{2}-\frac{1}{24}\gamma^{2}\epsilon\right)x^{4}+\left(\frac{11}{120}\alpha^{2}\gamma\epsilon-\frac{1}{10}\beta\alpha\epsilon-\frac{3}{40}\alpha\delta\epsilon+\frac{1}{20}\alpha\,\gamma^{2}\epsilon-\frac{1}{15}\beta\epsilon\gamma-\frac{1}{24}\delta\epsilon\gamma+\frac{1}{120}\gamma^{3}\epsilon+\frac{1}{20}\alpha^{3}\epsilon-\frac{1}{30}\alpha\,\epsilon^{2}-\frac{1}{60}\gamma\,\epsilon^{2}\right)x^{5}\right)a_{0}+\left(x-\frac{x^{2}\gamma}{2}+\left(\frac{1}{6}\alpha\gamma+\frac{1}{6}\gamma^{2}-\frac{1}{6}\delta-\frac{1}{6}\epsilon\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\gamma-\frac{1}{8}\alpha\,\gamma^{2}+\frac{1}{12}\alpha\delta+\frac{1}{12}\alpha\epsilon+\frac{1}{12}\beta\gamma+\frac{1}{8}\delta\gamma+\frac{1}{12}\epsilon\gamma-\frac{1}{24}\gamma^{3}\right)x^{4}+\left(-\frac{1}{10}\alpha\beta\gamma-\frac{7}{60}\alpha\delta\gamma-\frac{3}{40}\alpha\gamma\epsilon+\frac{1}{20}\alpha^{3}\gamma+\frac{11}{120}\alpha^{2}\gamma^{2}-\frac{1}{20}\alpha^{2}\delta-\frac{1}{20}\alpha^{2}\epsilon+\frac{1}{20}\alpha\,\gamma^{3}-\frac{1}{15}\beta\,\gamma^{2}+\frac{1}{20}\beta\delta+\frac{1}{20}\beta\epsilon-\frac{1}{20}\delta\,\gamma^{2}+\frac{1}{30}\delta\epsilon-\frac{1}{40}\gamma^{2}\epsilon+\frac{1}{40}\delta^{2}+\frac{1}{120}\epsilon^{2}+\frac{1}{120}\gamma^{4}\right)x^{5}\right)a_{1}+O\!\left(x^{6}\right) \tag{3}$$

At $x=0$ the solution above becomes

$$y=\left(1-\frac{\epsilon\,x^{2}}{2}+\left(\frac{1}{6}\alpha\epsilon+\frac{1}{6}\epsilon\gamma\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\epsilon-\frac{1}{8}\alpha\gamma\epsilon+\frac{1}{12}\beta\epsilon+\frac{1}{12}\delta\epsilon+\frac{1}{24}\epsilon^{2}-\frac{1}{24}\gamma^{2}\epsilon\right)x^{4}+\left(\frac{11}{120}\alpha^{2}\gamma\epsilon-\frac{1}{10}\beta\alpha\epsilon-\frac{3}{40}\alpha\delta\epsilon+\frac{1}{20}\alpha\,\gamma^{2}\epsilon-\frac{1}{15}\beta\epsilon\gamma-\frac{1}{24}\delta\epsilon\gamma+\frac{1}{120}\gamma^{3}\epsilon+\frac{1}{20}\alpha^{3}\epsilon-\frac{1}{30}\alpha\,\epsilon^{2}-\frac{1}{60}\gamma\,\epsilon^{2}\right)x^{5}\right)c_{1}+\left(x-\frac{x^{2}\gamma}{2}+\left(\frac{1}{6}\alpha\gamma+\frac{1}{6}\gamma^{2}-\frac{1}{6}\delta-\frac{1}{6}\epsilon\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\gamma-\frac{1}{8}\alpha\,\gamma^{2}+\frac{1}{12}\alpha\delta+\frac{1}{12}\alpha\epsilon+\frac{1}{12}\beta\gamma+\frac{1}{8}\delta\gamma+\frac{1}{12}\epsilon\gamma-\frac{1}{24}\gamma^{3}\right)x^{4}+\left(-\frac{1}{10}\alpha\beta\gamma-\frac{7}{60}\alpha\delta\gamma-\frac{3}{40}\alpha\gamma\epsilon+\frac{1}{20}\alpha^{3}\gamma+\frac{11}{120}\alpha^{2}\gamma^{2}-\frac{1}{20}\alpha^{2}\delta-\frac{1}{20}\alpha^{2}\epsilon+\frac{1}{20}\alpha\,\gamma^{3}-\frac{1}{15}\beta\,\gamma^{2}+\frac{1}{20}\beta\delta+\frac{1}{20}\beta\epsilon-\frac{1}{20}\delta\,\gamma^{2}+\frac{1}{30}\delta\epsilon-\frac{1}{40}\gamma^{2}\epsilon+\frac{1}{40}\delta^{2}+\frac{1}{120}\epsilon^{2}+\frac{1}{120}\gamma^{4}\right)x^{5}\right)c_{2}+O\!\left(x^{6}\right)$$

Summary
The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

$$y = \left(1-\frac{\epsilon x^{2}}{2}+\left(\frac{1}{6}\alpha\epsilon+\frac{1}{6}\epsilon\gamma\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\epsilon-\frac{1}{8}\alpha\gamma\epsilon+\frac{1}{12}\beta\epsilon+\frac{1}{12}\delta\epsilon+\frac{1}{24}\epsilon^{2}-\frac{1}{24}\gamma^{2}\epsilon\right)x^{4}+\left(\frac{11}{120}\alpha^{2}\gamma\epsilon-\frac{1}{10}\beta\alpha\epsilon-\frac{3}{40}\alpha\delta\epsilon+\frac{1}{20}\alpha\gamma^{2}\epsilon-\frac{1}{15}\beta\epsilon\gamma-\frac{1}{24}\delta\epsilon\gamma+\frac{1}{120}\gamma^{3}\epsilon+\frac{1}{20}\alpha^{3}\epsilon-\frac{1}{30}\alpha\epsilon^{2}-\frac{1}{60}\gamma\epsilon^{2}\right)x^{5}\right)c_{1}+\left(x-\frac{x^{2}\gamma}{2}+\left(\frac{1}{6}\alpha\gamma+\frac{1}{6}\gamma^{2}-\frac{1}{6}\delta-\frac{1}{6}\epsilon\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\gamma-\frac{1}{8}\alpha\gamma^{2}+\frac{1}{12}\alpha\delta+\frac{1}{12}\alpha\epsilon+\frac{1}{12}\beta\gamma+\frac{1}{8}\delta\gamma+\frac{1}{12}\epsilon\gamma-\frac{1}{24}\gamma^{3}\right)x^{4}+\left(-\frac{1}{10}\alpha\beta\gamma-\frac{7}{60}\alpha\delta\gamma-\frac{3}{40}\alpha\gamma\epsilon+\frac{1}{20}\alpha^{3}\gamma+\frac{11}{120}\alpha^{2}\gamma^{2}-\frac{1}{20}\alpha^{2}\delta-\frac{1}{20}\alpha^{2}\epsilon+\frac{1}{20}\alpha\gamma^{3}-\frac{1}{15}\beta\gamma^{2}+\frac{1}{20}\beta\delta+\frac{1}{20}\beta\epsilon-\frac{1}{20}\delta\gamma^{2}+\frac{1}{30}\delta\epsilon-\frac{1}{40}\gamma^{2}\epsilon+\frac{1}{40}\delta^{2}+\frac{1}{120}\epsilon^{2}+\frac{1}{120}\gamma^{4}\right)x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$\text{Expression too large to display}$$

Verified OK.

$$y = \left(1-\frac{\epsilon x^{2}}{2}+\left(\frac{1}{6}\alpha\epsilon+\frac{1}{6}\epsilon\gamma\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\epsilon-\frac{1}{8}\alpha\gamma\epsilon+\frac{1}{12}\beta\epsilon+\frac{1}{12}\delta\epsilon+\frac{1}{24}\epsilon^{2}-\frac{1}{24}\gamma^{2}\epsilon\right)x^{4}+\left(\frac{11}{120}\alpha^{2}\gamma\epsilon-\frac{1}{10}\beta\alpha\epsilon-\frac{3}{40}\alpha\delta\epsilon+\frac{1}{20}\alpha\gamma^{2}\epsilon-\frac{1}{15}\beta\epsilon\gamma-\frac{1}{24}\delta\epsilon\gamma+\frac{1}{120}\gamma^{3}\epsilon+\frac{1}{20}\alpha^{3}\epsilon-\frac{1}{30}\alpha\epsilon^{2}-\frac{1}{60}\gamma\epsilon^{2}\right)x^{5}\right)c_{1}+\left(x-\frac{x^{2}\gamma}{2}+\left(\frac{1}{6}\alpha\gamma+\frac{1}{6}\gamma^{2}-\frac{1}{6}\delta-\frac{1}{6}\epsilon\right)x^{3}+\left(-\frac{1}{12}\alpha^{2}\gamma-\frac{1}{8}\alpha\gamma^{2}+\frac{1}{12}\alpha\delta+\frac{1}{12}\alpha\epsilon+\frac{1}{12}\beta\gamma+\frac{1}{8}\delta\gamma+\frac{1}{12}\epsilon\gamma-\frac{1}{24}\gamma^{3}\right)x^{4}+\left(-\frac{1}{10}\alpha\beta\gamma-\frac{7}{60}\alpha\delta\gamma-\frac{3}{40}\alpha\gamma\epsilon+\frac{1}{20}\alpha^{3}\gamma+\frac{11}{120}\alpha^{2}\gamma^{2}-\frac{1}{20}\alpha^{2}\delta-\frac{1}{20}\alpha^{2}\epsilon+\frac{1}{20}\alpha\gamma^{3}-\frac{1}{15}\beta\gamma^{2}+\frac{1}{20}\beta\delta+\frac{1}{20}\beta\epsilon-\frac{1}{20}\delta\gamma^{2}+\frac{1}{30}\delta\epsilon-\frac{1}{40}\gamma^{2}\epsilon+\frac{1}{40}\delta^{2}+\frac{1}{120}\epsilon^{2}+\frac{1}{120}\gamma^{4}\right)x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

13.26.1 Maple step by step solution

Let's solve

$(\beta x^2+\alpha x+1) y''+(\delta x+\gamma) y'+\epsilon y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\epsilon y}{\beta x^2+\alpha x+1}-\frac{(\delta x+\gamma)y'}{\beta x^2+\alpha x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(\delta x+\gamma)y'}{\beta x^2+\alpha x+1}+\frac{\epsilon y}{\beta x^2+\alpha x+1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{\delta x+\gamma}{\beta x^2+\alpha x+1}, P_3(x)=\frac{\epsilon}{\beta x^2+\alpha x+1}\right]$

  - $\left(x-\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)\cdot P_2(x)$ is analytic at $x=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$

    $\left(\left(x-\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}}=0$

  - $\left(x-\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$

    $\left(\left(x-\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}}=0$

  - $x=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$

- Multiply by denominators

  $(\beta x^2+\alpha x+1) y''+(\delta x+\gamma) y'+\epsilon y=0$

- Change variables using $x=u+\frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$ so that the regular singular point is at $u=0$

  $(\beta u^2+u\sqrt{\alpha^2-4\beta})\left(\frac{d^2}{du^2}y(u)\right)+\left(\delta u-\frac{\delta\alpha}{2\beta}+\frac{\delta\sqrt{\alpha^2-4\beta}}{2\beta}+\gamma\right)\left(\frac{d}{du}y(u)\right)+\epsilon y(u)=0$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r\left(2\sqrt{\alpha^2-4\beta}\,\beta r-2\sqrt{\alpha^2-4\beta}\,\beta+\delta\sqrt{\alpha^2-4\beta}-\alpha\delta+2\beta\gamma\right)u^{-1+r}}{2\beta} + \left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)\left(2\sqrt{\alpha^2-4\beta}\,\beta(k+1)+2\sqrt{\alpha^2-4\beta}\,\beta r-2\sqrt{}\right.}{2\beta}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r\left(2\sqrt{\alpha^2-4\beta}\,\beta r-2\sqrt{\alpha^2-4\beta}\,\beta+\delta\sqrt{\alpha^2-4\beta}-\alpha\delta+2\beta\gamma\right)}{2\beta} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2\sqrt{\alpha^2-4\beta}\,\beta-\delta\sqrt{\alpha^2-4\beta}+\alpha\delta-2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{2\left((k+r)\beta+\frac{\delta}{2}\right)(k+1+r)a_{k+1}\sqrt{\alpha^2-4\beta}+2a_k(k+r)(k+r-1)\beta^2+(2\gamma(k+1+r)a_{k+1}+2a_k(\delta k+\delta r+\epsilon))\beta-a_{k+1}\alpha\delta(k+1+r)}{2\beta} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2\beta a_k\left(\beta k^2+2\beta kr+\beta r^2-\beta k-\beta r+\delta k+\delta r+\epsilon\right)}{2\sqrt{\alpha^2-4\beta}\,\beta k^2+4\sqrt{\alpha^2-4\beta}\,\beta kr+2\sqrt{\alpha^2-4\beta}\,\beta r^2+2\sqrt{\alpha^2-4\beta}\,\beta k+2\sqrt{\alpha^2-4\beta}\,\beta r+\sqrt{\alpha^2-4\beta}\,\delta k+\sqrt{\alpha^2-4\beta}\,\delta r-\alpha\delta k-\alpha\delta r-}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2\beta a_k\left(\beta k^2-\beta k+\delta k+\epsilon\right)}{2\sqrt{\alpha^2-4\beta}\,\beta k^2+2\sqrt{\alpha^2-4\beta}\,\beta k+\sqrt{\alpha^2-4\beta}\,\delta k-\alpha\delta k+2\beta\gamma k+\delta\sqrt{\alpha^2-4\beta}-\alpha\delta+2\beta\gamma}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{2\beta a_k (\beta k^2 - \beta k + \delta k + \epsilon)}{2\sqrt{\alpha^2-4\beta}\,\beta k^2 + 2\sqrt{\alpha^2-4\beta}\,\beta k + \sqrt{\alpha^2-4\beta}\,\delta k - \alpha\delta k + 2\beta\gamma k + \delta\sqrt{\alpha^2-4\beta} - \alpha\delta + 2\beta\gamma}\right]$$

- Revert the change of variables $u = x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^k, a_{k+1} = -\frac{2\beta a_k (\beta k^2 - \beta k + \delta k + \epsilon)}{2\sqrt{\alpha^2-4\beta}\,\beta k^2 + 2\sqrt{\alpha^2-4\beta}\,\beta k + \sqrt{\alpha^2-4\beta}\,\delta k - \alpha\delta k + 2\beta\gamma k + \delta\sqrt{\alpha^2-4\beta} - \alpha\delta +$$

- Recursion relation for $r = \frac{2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}$

$$a_{k+1} = -\frac{2\beta a_k \left(\beta k^2 + \frac{k\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma\right)}{\sqrt{\alpha^2-4\beta}} + \frac{\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\right.}}{4\beta\left(\alpha^2-4\beta\right)}\right.}{2\sqrt{\alpha^2-4\beta}\,\beta k^2 + 2k\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma\right) + \frac{\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma\right)^2}{2\sqrt{\alpha^2-4\beta}\,\beta} + 2\sqrt{\alpha^2-4\beta}\,\beta k + 2\sqrt{\alpha^2-}}$$

- Solution for $r = \frac{2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k + \frac{2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}}, a_{k+1} = -\frac{2\beta a_k \left(\beta k^2 + \right.}{2\sqrt{\alpha^2-4\beta}\,\beta k^2 + 2k\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma\right) + }\right.$$

- Revert the change of variables $u = x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^{k + \frac{2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}}, a_{k+1} = -\frac{}{2\sqrt{\alpha^2-4\beta}\,\beta k^2 + 2k\left(2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-\right.}}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{-\alpha+\sqrt{\alpha^2-4\beta}}{2\beta}\right)^{k + \frac{2\sqrt{\alpha^2-4\beta}\,\beta - \delta\sqrt{\alpha^2-4\beta} + \alpha\delta - 2\beta\gamma}{2\sqrt{\alpha^2-4\beta}\,\beta}}\right), a_{1+k}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ <u>Solution by Maple</u>

Time used: 0.031 (sec). Leaf size: 366

```
Order:=6;
dsolve((1+alpha*x+beta*x^2)*diff(y(x),x$2)+(gamma+delta*x)*diff(y(x),x)+epsilon*y(x)=0,y(x),
```

$$y(x) = \left(1-\frac{\epsilon\,x^{2}}{2}+\frac{\epsilon(\alpha+\gamma)\,x^{3}}{6}+\frac{\epsilon\left(-\alpha^{2}-\frac{3}{2}\alpha\gamma-\frac{1}{2}\gamma^{2}+\beta+\delta+\frac{1}{2}\epsilon\right)x^{4}}{12} - \frac{\epsilon\left(\frac{\left(\alpha+\frac{\gamma}{2}\right)\epsilon}{3}-\frac{\gamma^{3}}{12}-\frac{\alpha\,\gamma^{2}}{2}+\frac{\left(-\frac{11\alpha^{2}}{4}+2\beta+\frac{5\delta}{4}\right)\gamma}{3}+\alpha\left(-\frac{\alpha^{2}}{2}+\beta+\frac{3\delta}{4}\right)\right)x^{5}}{10}\right)y(0)$$

$$+\left(x-\frac{\gamma\,x^{2}}{2}+\frac{\left(\alpha\gamma+\gamma^{2}-\delta-\epsilon\right)x^{3}}{6} + \frac{\left((2\alpha+2\gamma)\,\epsilon-\gamma^{3}-3\alpha\,\gamma^{2}+\left(-2\alpha^{2}+2\beta+3\delta\right)\gamma+2\delta\alpha\right)x^{4}}{24}\right.$$

$$+\frac{\left(\epsilon^{2}+\left(-6\alpha^{2}-9\alpha\gamma-3\gamma^{2}+6\beta+4\delta\right)\epsilon+\gamma^{4}+6\alpha\,\gamma^{3}+\left(11\alpha^{2}-8\beta-6\delta\right)\gamma^{2}-12\alpha\left(-\frac{\alpha^{2}}{2}+\beta+\frac{7\delta}{6}\right)\gamma\right.}{120}$$

$$+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 561

```
AsymptoticDSolveValue[(1+\[Alpha]*x+\[Beta]*x^2)*y''[x]+(\[Gamma]+\[Delta]*x)*y'[x]+\[Epsilo
```

$$
\begin{aligned}
y(x) \to\; & c_1\left(-\frac{1}{20}x^5\epsilon(2\alpha\beta-\alpha^3)-\frac{1}{20}x^5\epsilon(\alpha\delta-\gamma(\alpha^2-\beta))+\frac{1}{60}\gamma x^5\epsilon(\alpha^2-\beta)\right.\\
&+\frac{1}{120}\alpha\gamma^2x^5\epsilon+\frac{1}{40}\alpha x^5\epsilon(\alpha\gamma-\delta)+\frac{1}{24}\gamma x^5\epsilon(\alpha\gamma-\delta)-\frac{1}{30}\alpha x^5\epsilon^2+\frac{1}{120}\gamma^3x^5\epsilon\\
&-\frac{1}{60}\gamma x^5\epsilon^2-\frac{1}{12}x^4\epsilon(\alpha^2-\beta)-\frac{1}{12}x^4\epsilon(\alpha\gamma-\delta)-\frac{1}{24}\alpha\gamma x^4\epsilon-\frac{1}{24}\gamma^2x^4\epsilon+\frac{x^4\epsilon^2}{24}\\
&\left.+\frac{1}{6}\alpha x^3\epsilon+\frac{1}{6}\gamma x^3\epsilon-\frac{x^2\epsilon}{2}+1\right)\\
&+c_2\left(\frac{1}{60}\gamma x^5(\gamma(\alpha^2-\beta)-\alpha\delta)-\frac{1}{20}\gamma x^5(\alpha\delta-\gamma(\alpha^2-\beta))-\frac{1}{20}x^5\epsilon(\alpha^2-\beta)\right.\\
&-\frac{1}{20}x^5(\alpha^3(-\gamma)+\alpha^2\delta+2\alpha\beta\gamma-\beta\delta)+\frac{1}{24}\gamma^2x^5(\alpha\gamma-\delta)-\frac{1}{120}\gamma^2x^5(\delta-\alpha\gamma)\\
&-\frac{1}{40}x^5\epsilon(\alpha\gamma-\delta)+\frac{1}{120}x^5\epsilon(\delta-\alpha\gamma)-\frac{1}{40}x^5(\alpha\gamma-\delta)(\delta-\alpha\gamma)-\frac{1}{24}\alpha\gamma x^5\epsilon+\frac{\gamma^4x^5}{120}\\
&-\frac{1}{40}\gamma^2x^5\epsilon+\frac{x^5\epsilon^2}{120}-\frac{1}{12}x^4(\gamma(\alpha^2-\beta)-\alpha\delta)-\frac{1}{12}\gamma x^4(\alpha\gamma-\delta)+\frac{1}{24}\gamma x^4(\delta-\alpha\gamma)\\
&\left.+\frac{1}{12}\alpha x^4\epsilon-\frac{\gamma^3x^4}{24}+\frac{1}{12}\gamma x^4\epsilon-\frac{1}{6}x^3(\delta-\alpha\gamma)+\frac{\gamma^2x^3}{6}-\frac{x^3\epsilon}{6}-\frac{\gamma x^2}{2}+x\right)
\end{aligned}
$$

13.27 problem 31(a)

Internal problem ID [1268]
Internal file name [`OUTPUT/1269_Sunday_June_05_2022_02_07_30_AM_94733438/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 31(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(2x^2+3x+1)\,y''+(6+8x)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1007)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1008)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(4y'x+3y'+2y)}{2x^2+3x+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{(72x^2+108x+42)\,y'+(48x+36)\,y}{(2x^2+3x+1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(-768x^3-1728x^2-1344x-360)\,y'-576y\left(x^2+\frac{3}{2}x+\frac{7}{12}\right)}{(2x^2+3x+1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{(9600x^4+28800x^3+33600x^2+18000x+3720)\,y'+7680\left(x^2+\frac{3}{2}x+\frac{5}{8}\right)y\left(\frac{3}{4}+x\right)}{(2x^2+3x+1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{(-138240x^5-518400x^4-806400x^3-648000x^2-267840x-45360)\,y'-115200\left(x^4+3x^3+\frac{7}{2}x^2 -\right.}{(2x^2+3x+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0)-6y'(0)\\
F_1 &= 36y(0)+42y'(0)\\
F_2 &= -336y(0)-360y'(0)\\
F_3 &= 3600y(0)+3720y'(0)\\
F_4 &= -44640y(0)-45360y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y &= \left(-62x^6+30x^5-14x^4+6x^3-2x^2+1\right)y(0)\\
&\quad+\left(-63x^6+31x^5-15x^4+7x^3-3x^2+x\right)y'(0)+O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(2x^2 + 3x + 1) y'' + (6 + 8x) y' + 4y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(2x^2 + 3x + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (6 + 8x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 6n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 8n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 3n\, x^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} 3(n+1) a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

$$\sum_{n=1}^{\infty} 6n a_n x^{n-1} = \sum_{n=0}^{\infty} 6(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 3(n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 6(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=1}^{\infty} 8n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 6a_1 + 4a_0 = 0$$

$$a_2 = -2a_0 - 3a_1$$

$n = 1$ gives

$$18a_2 + 6a_3 + 12a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = 6a_0 + 7a_1$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) + 3(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + 6(n+1)\, a_{n+1} + 8na_n + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -2a_n - 3a_{n+1} \\ &= -2a_n - 3a_{n+1} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$24a_2 + 36a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -14a_0 - 15a_1$$

For $n = 3$ the recurrence equation gives

$$40a_3 + 60a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 30a_0 + 31a_1$$

For $n = 4$ the recurrence equation gives

$$60a_4 + 90a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -62a_0 - 63a_1$$

For $n = 5$ the recurrence equation gives

$$84a_5 + 126a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 126a_0 + 127a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + (-2a_0 - 3a_1) x^2 + (6a_0 + 7a_1) x^3 + (-14a_0 - 15a_1) x^4 + (30a_0 + 31a_1) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(30x^5 - 14x^4 + 6x^3 - 2x^2 + 1\right) a_0 + \left(31x^5 - 15x^4 + 7x^3 - 3x^2 + x\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(30x^5 - 14x^4 + 6x^3 - 2x^2 + 1\right) c_1 + \left(31x^5 - 15x^4 + 7x^3 - 3x^2 + x\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-62x^6+30x^5-14x^4+6x^3-2x^2+1\right) y(0) + \left(-63x^6+31x^5-15x^4+7x^3-3x^2+x\right) y'(0)+O\left(x^6\right) \tag{1}$$

$$y = \left(30x^5-14x^4+6x^3-2x^2+1\right) c_1+\left(31x^5-15x^4+7x^3-3x^2+x\right) c_2+O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(-62x^6+30x^5-14x^4+6x^3-2x^2+1\right) y(0) + \left(-63x^6+31x^5-15x^4+7x^3-3x^2+x\right) y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(30x^5-14x^4+6x^3-2x^2+1\right) c_1+\left(31x^5-15x^4+7x^3-3x^2+x\right) c_2+O\left(x^6\right)$$

Verified OK.

13.27.1 Maple step by step solution

Let's solve

$(2x^2+3x+1) y''+(6+8x) y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{2x^2+3x+1}-\frac{2(4x+3)y'}{2x^2+3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(4x+3)y'}{2x^2+3x+1}+\frac{4y}{2x^2+3x+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(4x+3)}{2x^2+3x+1}, P_3(x)=\frac{4}{2x^2+3x+1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=2$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(2x^2+3x+1)\,y''+(6+8x)\,y'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(-2+8u)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0r(1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}(-a_{k+1}(k+r+1)\,(k+r+2)+2a_k(k+r+2)\,(k+r+1))\,u^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)(k+r+1)(-a_{k+1}+2a_k)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=2a_k$$

- Recursion relation for $r=-1$

$$a_{k+1}=2a_k$$

- Solution for $r=-1$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1}=2a_k\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k-1}, a_{k+1}=2a_k\right]$$

- Recursion relation for $r=0$

$$a_{k+1}=2a_k$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k}, a_{k+1}=2a_k\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k}, a_{k+1}=2a_k\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k}\right), a_{1+k}=2a_k, b_{1+k}=2b_k\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((1+3*x+2*x^2)*diff(y(x),x$2)+(6+8*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (30x^5 - 14x^4 + 6x^3 - 2x^2 + 1)\, y(0) + (31x^5 - 15x^4 + 7x^3 - 3x^2 + x)\, D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[(1+3*x+2*x^2)*y''[x]+(6+8*x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(30x^5 - 14x^4 + 6x^3 - 2x^2 + 1) + c_2(31x^5 - 15x^4 + 7x^3 - 3x^2 + x)$$

13.28 problem 31(b)

Internal problem ID [1269]
Internal file name [`OUTPUT/1270_Sunday_June_05_2022_02_07_32_AM_92575816/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 31(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(6x^2-5x+1)\,y''-(10-24x)\,y'+12y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1010) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1011) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(12y'x - 5y' + 6y)}{6x^2 - 5x + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(648x^2 - 540x + 114)\, y' + (432x - 180)\, y}{(6x^2 - 5x + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-20736x^3 + 25920x^2 - 10944x + 1560)\, y' - 15552\left(x^2 - \frac{5}{6}x + \frac{19}{108}\right) y}{(6x^2 - 5x + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(777600x^4 - 1296000x^3 + 820800x^2 - 234000x + 25320)\, y' + 622080\left(x^2 - \frac{5}{6}x + \frac{13}{72}\right) y \left(x - \frac{5}{12}\right)}{(6x^2 - 5x + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-33592320x^5 + 69984000x^4 - 59097600x^3 + 25272000x^2 - 5469120x + 478800)\, y' - 27993600y(x}{(6x^2 - 5x + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -12y(0) + 10y'(0) \\
F_1 &= -180y(0) + 114y'(0) \\
F_2 &= -2736y(0) + 1560y'(0) \\
F_3 &= -46800y(0) + 25320y'(0) \\
F_4 &= -911520y(0) + 478800y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(-1266x^6 - 390x^5 - 114x^4 - 30x^3 - 6x^2 + 1\right) y(0) \\
&+ \left(665x^6 + 211x^5 + 65x^4 + 19x^3 + 5x^2 + x\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(6x^2 - 5x + 1\right) y'' + (24x - 10)\, y' + 12y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(6x^2 - 5x + 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (24x - 10) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 12\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=2}^{\infty} 6x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-5n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\ + \left(\sum_{n=1}^{\infty} 24n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-10n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-5n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-5(n+1)\, a_{n+1} n\, x^n\right)$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-10n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-10(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 6x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-5(n+1) a_{n+1} n x^n\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 24 n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-10(n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 10a_1 + 12a_0 = 0$$

$$a_2 = -6a_0 + 5a_1$$

$n = 1$ gives

$$-30a_2 + 6a_3 + 36a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -30a_0 + 19a_1$$

For $2 \leq n$, the recurrence equation is

$$6na_n(n-1) - 5(n+1) a_{n+1} n + (n+2) a_{n+2}(n+1) + 24na_n - 10(n+1) a_{n+1} + 12a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -6a_n + 5a_{n+1} \\ &= -6a_n + 5a_{n+1} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$72a_2 - 60a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -114a_0 + 65a_1$$

For $n = 3$ the recurrence equation gives

$$120a_3 - 100a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -390a_0 + 211a_1$$

For $n = 4$ the recurrence equation gives

$$180a_4 - 150a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -1266a_0 + 665a_1$$

For $n = 5$ the recurrence equation gives

$$252a_5 - 210a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -3990a_0 + 2059a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + (-6a_0 + 5a_1) x^2 + (-30a_0 + 19a_1) x^3 \\ + (-114a_0 + 65a_1) x^4 + (-390a_0 + 211a_1) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1\right) a_0 + \left(211x^5 + 65x^4 + 19x^3 + 5x^2 + x\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) c_1 + (211x^5 + 65x^4 + 19x^3 + 5x^2 + x) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (-1266x^6 - 390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) y(0) + (665x^6 + 211x^5 + 65x^4 + 19x^3 + 5x^2 + x) y'(0) + O(x^6) \tag{1}$$

$$y = (-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) c_1 + (211x^5 + 65x^4 + 19x^3 + 5x^2 + x) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-1266x^6 - 390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) y(0) + (665x^6 + 211x^5 + 65x^4 + 19x^3 + 5x^2 + x) y'(0) + O(x^6)$$

Verified OK.

$$y = (-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) c_1 + (211x^5 + 65x^4 + 19x^3 + 5x^2 + x) c_2 + O(x^6)$$

Verified OK.

13.28.1 Maple step by step solution

Let's solve

$(6x^2 - 5x + 1) y'' + (24x - 10) y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y}{6x^2-5x+1} - \frac{2(12x-5)y'}{6x^2-5x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(12x-5)y'}{6x^2-5x+1} + \frac{12y}{6x^2-5x+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(12x-5)}{6x^2-5x+1}, P_3(x) = \frac{12}{6x^2-5x+1}\right]$

  ◦ $\left(x - \frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{2}$

$$\left(\left(x-\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}}=2$$

- $\left(x-\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}$

  $$\left(\left(x-\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}}=0$$

- $x=\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=\frac{1}{2}$

- Multiply by denominators

  $$\left(6x^2-5x+1\right)y''+\left(24x-10\right)y'+12y=0$$

- Change variables using $x=u+\frac{1}{2}$ so that the regular singular point is at $u=0$

  $$\left(6u^2+u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(24u+2\right)\left(\frac{d}{du}y(u)\right)+12y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+r+1)\,(k+r+2)+6a_k(k+r+2)\,(k+r+1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(k+r+1)(a_{k+1}+6a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -6a_k$

- Recursion relation for $r = -1$

  $a_{k+1} = -6a_k$

- Solution for $r = -1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = -6a_k\right]$$

- Revert the change of variables $u = x - \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2}\right)^{k-1}, a_{k+1} = -6a_k\right]$$

- Recursion relation for $r = 0$

  $a_{k+1} = -6a_k$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+1} = -6a_k\right]$$

- Revert the change of variables $u = x - \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2}\right)^{k}, a_{k+1} = -6a_k\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2}\right)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{1}{2}\right)^{k}\right), a_{1+k} = -6a_k, b_{1+k} = -6b_k\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve((1-5*x+6*x^2)*diff(y(x),x$2)-(10-24*x)*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = (-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1)\,y(0) + (211x^5 + 65x^4 + 19x^3 + 5x^2 + x)\,D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[(1-5*x+6*x^2)*y''[x]-(10-24*x)*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(-390x^5 - 114x^4 - 30x^3 - 6x^2 + 1) + c_2(211x^5 + 65x^4 + 19x^3 + 5x^2 + x)$$

13.29 problem 31(c)

Internal problem ID [1270]
Internal file name [`OUTPUT/1271_Sunday_June_05_2022_02_07_33_AM_99022394/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 31(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(4x^2-4x+1\right) y''-(8-16x)\, y'+8y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1013) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1014) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{8(2y'x-y'+y)}{(2x-1)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{(144x-72)\,y'+96y}{(2x-1)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{(-1536x+768)\,y'-1152y}{(2x-1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{(19200x-9600)\,y'+15360y}{(2x-1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{(-276480x+138240)\,y'-230400y}{(2x-1)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -8y(0)+8y'(0) \\
F_1 &= -96y(0)+72y'(0) \\
F_2 &= -1152y(0)+768y'(0) \\
F_3 &= -15360y(0)+9600y'(0) \\
F_4 &= -230400y(0)+138240y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(-320x^6-128x^5-48x^4-16x^3-4x^2+1\right)y(0) \\
&+\left(192x^6+80x^5+32x^4+12x^3+4x^2+x\right)y'(0)+O\left(x^6\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(4x^2 - 4x + 1)\, y'' + (16x - 8)\, y' + 8y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(4x^2 - 4x + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (16x - 8) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-4n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \tag{2}$$
$$+ \left(\sum_{n=1}^{\infty} 16 n a_n x^n\right) + \sum_{n=1}^{\infty} \left(-8 n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-4n\, x^{n-1} a_n (n-1)\right) = \sum_{n=1}^{\infty} \left(-4(n+1)\, a_{n+1} n\, x^n\right)$$
$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$
$$\sum_{n=1}^{\infty} \left(-8 n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-8(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 4x^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-4(n+1) a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 16 n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-8(n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} 8 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 8a_1 + 8a_0 = 0$$

$$a_2 = -4a_0 + 4a_1$$

$n = 1$ gives

$$-24a_2 + 6a_3 + 24a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -16a_0 + 12a_1$$

For $2 \leq n$, the recurrence equation is

$$4na_n(n-1) - 4(n+1) a_{n+1} n + (n+2) a_{n+2}(n+1) + 16na_n - 8(n+1) a_{n+1} + 8a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -4a_n + 4a_{n+1} \\ &= -4a_n + 4a_{n+1} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$48a_2 - 48a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -48a_0 + 32a_1$$

For $n = 3$ the recurrence equation gives

$$80a_3 - 80a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -128a_0 + 80a_1$$

For $n = 4$ the recurrence equation gives

$$120a_4 - 120a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -320a_0 + 192a_1$$

For $n = 5$ the recurrence equation gives

$$168a_5 - 168a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -768a_0 + 448a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + (-4a_0 + 4a_1) x^2 + (-16a_0 + 12a_1) x^3 \\ + (-48a_0 + 32a_1) x^4 + (-128a_0 + 80a_1) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(-128x^5 - 48x^4 - 16x^3 - 4x^2 + 1\right) a_0 + \left(80x^5 + 32x^4 + 12x^3 + 4x^2 + x\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x=0$ the solution above becomes

$$y = \left(-128x^5-48x^4-16x^3-4x^2+1\right) c_1+\left(80x^5+32x^4+12x^3+4x^2+x\right) c_2+O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(-320x^6-128x^5-48x^4-16x^3-4x^2+1\right) y(0) + \left(192x^6+80x^5+32x^4+12x^3+4x^2+x\right) y'(0)+O\left(x^6\right) \tag{1}$$

$$y = \left(-128x^5-48x^4-16x^3-4x^2+1\right) c_1+\left(80x^5+32x^4+12x^3+4x^2+x\right) c_2+O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(-320x^6-128x^5-48x^4-16x^3-4x^2+1\right) y(0) + \left(192x^6+80x^5+32x^4+12x^3+4x^2+x\right) y'(0)+O\left(x^6\right)$$

Verified OK.

$$y = \left(-128x^5-48x^4-16x^3-4x^2+1\right) c_1+\left(80x^5+32x^4+12x^3+4x^2+x\right) c_2+O\left(x^6\right)$$

Verified OK.

13.29.1 Maple step by step solution

Let's solve

$(4x^2-4x+1) y''+(16x-8) y'+8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{4x^2-4x+1}-\frac{8y'}{2x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{8y'}{2x-1}+\frac{8y}{4x^2-4x+1}=0$

□ Check to see if $x_0=\frac{1}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{8}{2x-1}, P_3(x)=\frac{8}{4x^2-4x+1}\right]$

  ◦ $\left(x-\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{2}$

$\left(\left(x-\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}} = 4$

- $\left(x-\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}$

  $\left(\left(x-\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}} = 2$

- $x=\frac{1}{2}$is a regular singular point

  Check to see if $x_0=\frac{1}{2}$ is a regular singular point

  $x_0=\frac{1}{2}$

- Multiply by denominators

  $$(2x-1)\,y''(4x^2-4x+1)+(32x^2-32x+8)\,y'+(16x-8)\,y=0$$

- Change variables using $x=u+\frac{1}{2}$ so that the regular singular point is at $u=0$

  $$8u^3\left(\frac{d^2}{du^2}y(u)\right)+32u^2\left(\frac{d}{du}y(u)\right)+16uy(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite DE with series expansions

- Convert $u\cdot y(u)$ to series expansion

  $$u\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+1}$$

- Shift index using $k->k-1$

  $$u\cdot y(u)=\sum_{k=1}^{\infty}a_{k-1}u^{k+r}$$

- Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $$u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$$

- Shift index using $k->k-1$

  $$u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$$

- Convert $u^3\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u^3\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,u^{k+r+1}$$

- Shift index using $k->k-1$

$$u^3 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)(k-2+r)\, u^{k+r}$$

Rewrite DE with series expansions

$$\sum_{k=1}^{\infty} 8a_{k-1}(k+r+1)(k+r)\, u^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$8a_{k-1}(k+1)\, k = 0$$

- Shift index using $k- >k+1$

  $$8a_k(k+2)(k+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_k = 0$$

- Recursion relation for $r = 0$

  $$a_k = 0$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2}\right)^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((1-4*x+4*x^2)*diff(y(x),x$2)-(8-16*x)*diff(y(x),x)+8*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-128x^5 - 48x^4 - 16x^3 - 4x^2 + 1) y(0) + (80x^5 + 32x^4 + 12x^3 + 4x^2 + x) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 54

```
AsymptoticDSolveValue[(1-4*x+4*x^2)*y''[x]-(8-16*x)*y'[x]+8*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(-128x^5 - 48x^4 - 16x^3 - 4x^2 + 1) + c_2(80x^5 + 32x^4 + 12x^3 + 4x^2 + x)$$

13.30 problem 31(d)

Internal problem ID [1271]
Internal file name [OUTPUT/1272_Sunday_June_05_2022_02_07_34_AM_29640191/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 31(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+4x+4)\,y''+(8+4x)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1016) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1017) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(2y'x+4y'+y)}{(2+x)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(18x+36)\, y' + 12y}{(2+x)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-96x-192)\, y' - 72y}{(2+x)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{600(2+x)\, y' + 480y}{(2+x)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-4320x-8640)\, y' - 3600y}{(2+x)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{y(0)}{2} - 2y'(0) \\
F_1 &= \frac{3y(0)}{2} + \frac{9y'(0)}{2} \\
F_2 &= -\frac{9y(0)}{2} - 12y'(0) \\
F_3 &= 15y(0) + \frac{75y'(0)}{2} \\
F_4 &= -\frac{225y(0)}{4} - 135y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned} y = &\left(1-\frac{1}{4}x^{2}+\frac{1}{4}x^{3}-\frac{3}{16}x^{4}+\frac{1}{8}x^{5}-\frac{5}{64}x^{6}\right) y(0) \\ &+\left(x-x^{2}+\frac{3}{4}x^{3}-\frac{1}{2}x^{4}+\frac{5}{16}x^{5}-\frac{3}{16}x^{6}\right) y'(0)+O\left(x^{6}\right) \end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^{2}+4x+4\right) y''+\left(8+4x\right) y'+2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_{n} x^{n}$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\left(x^{2}+4x+4\right) \left(\sum_{n=2}^{\infty} n \left(n-1\right) a_{n} x^{n-2}\right)+\left(8+4x\right) \left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} x^{n} a_{n} n \left(n-1\right)\right)+\left(\sum_{n=2}^{\infty} 4n\, x^{n-1} a_{n} \left(n-1\right)\right)+\left(\sum_{n=2}^{\infty} 4n \left(n-1\right) a_{n} x^{n-2}\right) \\ &+\left(\sum_{n=1}^{\infty} 8n a_{n} x^{n-1}\right)+\left(\sum_{n=1}^{\infty} 4n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 2a_{n} x^{n}\right) = 0 \end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 4n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 8n a_n x^{n-1} = \sum_{n=0}^{\infty} 8(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 4(n+1)\, a_{n+1} n\, x^n\right) + \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 8(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$8a_2 + 8a_1 + 2a_0 = 0$$

$$a_2 = -\frac{a_0}{4} - a_1$$

$n = 1$ gives

$$24a_2 + 24a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{4} + \frac{3a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 4(n+1)\, a_{n+1} n + 4(n+2)\, a_{n+2}(n+1) + 8(n+1)\, a_{n+1} + 4n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{a_n}{4} - a_{n+1} \\
&= -\frac{a_n}{4} - a_{n+1}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$12a_2 + 48a_3 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_0}{16} - \frac{a_1}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_3 + 80a_4 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{8} + \frac{5a_1}{16}$$

For $n = 4$ the recurrence equation gives

$$30a_4 + 120a_5 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{5a_0}{64} - \frac{3a_1}{16}$$

For $n = 5$ the recurrence equation gives

$$42a_5 + 168a_6 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3a_0}{64} + \frac{7a_1}{64}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \left(-\frac{a_0}{4} - a_1\right) x^2 + \left(\frac{a_0}{4} + \frac{3a_1}{4}\right) x^3 + \left(-\frac{3a_0}{16} - \frac{a_1}{2}\right) x^4 + \left(\frac{a_0}{8} + \frac{5a_1}{16}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5\right) a_0 + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5\right) a_1 + O(x^6) \quad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5\right) c_1 + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5 - \frac{5}{64}x^6\right) y(0) \\
& + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5 - \frac{3}{16}x^6\right) y'(0) + O(x^6)
\end{aligned} \quad (1)
$$

$$
y = \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5\right) c_1 + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5\right) c_2 + O(x^6) \quad (2)
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5 - \frac{5}{64}x^6\right) y(0) \\
& + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5 - \frac{3}{16}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{16}x^4 + \frac{1}{8}x^5\right) c_1 + \left(x - x^2 + \frac{3}{4}x^3 - \frac{1}{2}x^4 + \frac{5}{16}x^5\right) c_2 + O(x^6)
$$

Verified OK.

13.30.1 Maple step by step solution

Let's solve

$(x^2+4x+4)\, y''+(8+4x)\, y'+2y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2y}{x^2+4x+4}-\frac{4y'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{4y'}{2+x}+\frac{2y}{x^2+4x+4}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4}{2+x}, P_3(x)=\frac{2}{x^2+4x+4}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=4$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=2$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

$(2+x)\, y''(x^2+4x+4)+(4x^2+16x+16)\, y'+(2x+4)\, y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u^3\left(\frac{d^2}{du^2}y(u)\right)+4u^2\left(\frac{d}{du}y(u)\right)+2uy(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

  ◦ Convert $u\cdot y(u)$ to series expansion

$$u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$$

- Shift index using $k- >k - 1$

$$u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$$

- Convert $u^2 \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion

$$u^2 \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r+1}$$

- Shift index using $k- >k - 1$

$$u^2 \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, u^{k+r}$$

- Convert $u^3 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

$$u^3 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, u^{k+r+1}$$

- Shift index using $k- >k - 1$

$$u^3 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\,(k-2+r)\, u^{k+r}$$

Rewrite DE with series expansions

$$\sum_{k=1}^{\infty} a_{k-1} (k+r+1)\,(k+r)\, u^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-1}(k+1)\, k = 0$$

- Shift index using $k- >k + 1$

$$a_k (k+2)\,(k+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_k = 0$$

- Recursion relation for $r = 0$

$$a_k = 0$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_k=0\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (2+x)^k, a_k=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
Order:=6;
dsolve((4+4*x+x^2)*diff(y(x),x$2)+(8+4*x)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{1}{4}x^2+\frac{1}{4}x^3-\frac{3}{16}x^4+\frac{1}{8}x^5\right)y(0)+\left(x-x^2+\frac{3}{4}x^3-\frac{1}{2}x^4+\frac{5}{16}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 68

```
AsymptoticDSolveValue[(4+4*x+x^2)*y''[x]+(8+4*x)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^5}{8}-\frac{3x^4}{16}+\frac{x^3}{4}-\frac{x^2}{4}+1\right)+c_2\left(\frac{5x^5}{16}-\frac{x^4}{2}+\frac{3x^3}{4}-x^2+x\right)$$

13.31 problem 31(e)

Internal problem ID [1272]
Internal file name [`OUTPUT/1273_Sunday_June_05_2022_02_07_36_AM_89784834/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 31(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(3x^2+8x+4\right)y''+(16+12x)\,y'+6y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1019) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1020) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{2(6y'x+8y'+3y)}{3x^2+8x+4}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0\\
&= \frac{(162x^2+432x+312)\,y'+(108x+144)\,y}{(3x^2+8x+4)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1\\
&= \frac{(-2592x^3-10368x^2-14976x-7680)\,y'-1944\left(x^2+\frac{8}{3}x+\frac{52}{27}\right)y}{(3x^2+8x+4)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2\\
&= \frac{(48600x^4+259200x^3+561600x^2+576000x+232320)\,y'+38880\left(\frac{4}{3}+x\right)\left(x^2+\frac{8}{3}x+\frac{20}{9}\right)y}{(3x^2+8x+4)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3\\
&= \frac{-1049760\left(x^2+\frac{8}{3}x+\frac{52}{27}\right)\left(\frac{4}{3}+x\right)\left(x^2+\frac{8}{3}x+\frac{28}{9}\right)y'-874800\left(x^4+\frac{16}{3}x^3+\frac{104}{9}x^2+\frac{320}{27}x+\frac{1936}{405}\right)y}{(3x^2+8x+4)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -\frac{3y(0)}{2}-4y'(0)\\
F_1 &= 9y(0)+\frac{39y'(0)}{2}\\
F_2 &= -\frac{117y(0)}{2}-120y'(0)\\
F_3 &= 450y(0)+\frac{1815y'(0)}{2}\\
F_4 &= -\frac{16335y(0)}{4}-8190y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5 - \frac{363}{64}x^6\right) y(0) + \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5 - \frac{91}{8}x^6\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(3x^2 + 8x + 4) y'' + (16 + 12x) y' + 6y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(3x^2 + 8x + 4) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (16 + 12x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 6\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 8n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 16 n a_n x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 12 n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 6 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 8n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 8(n+1)\, a_{n+1} n\, x^n$$

$$\sum_{n=2}^{\infty} 4n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} 16n a_n x^{n-1} = \sum_{n=0}^{\infty} 16(n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 8(n+1)\, a_{n+1} n\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} 4(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 16(n+1)\, a_{n+1} x^n\right) \\ &+ \left(\sum_{n=1}^{\infty} 12n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 6a_n x^n\right) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$8a_2 + 16a_1 + 6a_0 = 0$$

$$a_2 = -\frac{3a_0}{4} - 2a_1$$

$n = 1$ gives

$$48a_2 + 24a_3 + 18a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{3a_0}{2} + \frac{13a_1}{4}$$

For $2 \le n$, the recurrence equation is

$$3na_n(n-1) + 8(n+1)\, a_{n+1} n + 4(n+2)\, a_{n+2}(n+1) + 16(n+1)\, a_{n+1} + 12na_n + 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{3a_n}{4} - 2a_{n+1} \\
&= -\frac{3a_n}{4} - 2a_{n+1}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$36a_2 + 96a_3 + 48a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{39a_0}{16} - 5a_1$$

For $n = 3$ the recurrence equation gives

$$60a_3 + 160a_4 + 80a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{15a_0}{4} + \frac{121a_1}{16}$$

For $n = 4$ the recurrence equation gives

$$90a_4 + 240a_5 + 120a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{363a_0}{64} - \frac{91a_1}{8}$$

For $n = 5$ the recurrence equation gives

$$126a_5 + 336a_6 + 168a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{273a_0}{32} + \frac{1093a_1}{64}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x + \left(-\frac{3a_0}{4} - 2a_1\right) x^2 + \left(\frac{3a_0}{2} + \frac{13a_1}{4}\right) x^3 \\
+ \left(-\frac{39a_0}{16} - 5a_1\right) x^4 + \left(\frac{15a_0}{4} + \frac{121a_1}{16}\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5\right) a_0 + \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5\right) a_1 + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5\right) c_1 + \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5 - \frac{363}{64}x^6\right) y(0) \\
+ \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5 - \frac{91}{8}x^6\right) y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
y = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5\right) c_1 + \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5\right) c_2 + O(x^6) \tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{3}{4}x^{2}+\frac{3}{2}x^{3}-\frac{39}{16}x^{4}+\frac{15}{4}x^{5}-\frac{363}{64}x^{6}\right)y(0) + \left(x-2x^{2}+\frac{13}{4}x^{3}-5x^{4}+\frac{121}{16}x^{5}-\frac{91}{8}x^{6}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{3}{4}x^{2}+\frac{3}{2}x^{3}-\frac{39}{16}x^{4}+\frac{15}{4}x^{5}\right)c_{1}+\left(x-2x^{2}+\frac{13}{4}x^{3}-5x^{4}+\frac{121}{16}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

13.31.1 Maple step by step solution

Let's solve

$(3x^{2}+8x+4)\,y''+(16+12x)\,y'+6y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{6y}{3x^{2}+8x+4}-\frac{4(3x+4)y'}{3x^{2}+8x+4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{4(3x+4)y'}{3x^{2}+8x+4}+\frac{6y}{3x^{2}+8x+4}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_{2}(x)=\frac{4(3x+4)}{3x^{2}+8x+4}, P_{3}(x)=\frac{6}{3x^{2}+8x+4}\right]$

  ○ $(2+x)\cdot P_{2}(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_{2}(x)\right)\Big|_{x=-2}=2$

  ○ $(2+x)^{2}\cdot P_{3}(x)$ is analytic at $x=-2$

    $\left((2+x)^{2}\cdot P_{3}(x)\right)\Big|_{x=-2}=0$

  ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $\left(3x^2+8x+4\right)y''+\left(16+12x\right)y'+6y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $\left(3u^2-4u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-8+12u\right)\left(\frac{d}{du}y(u)\right)+6y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-4a_0r(1+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-4a_{k+1}(k+r+1)\,(k+r+2)+3a_k(k+r+2)\,(k+r+1)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-4r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\,(k+r+1)\,(-4a_{k+1}+3a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{3a_k}{4}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{3a_k}{4}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = \frac{3a_k}{4}\right]$
- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-1}, a_{k+1} = \frac{3a_k}{4}\right]$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{3a_k}{4}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+1} = \frac{3a_k}{4}\right]$
- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k}, a_{k+1} = \frac{3a_k}{4}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k}\right), a_{1+k} = \frac{3a_k}{4}, b_{1+k} = \frac{3b_k}{4}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((4+8*x+3*x^2)*diff(y(x),x$2)+(16+12*x)*diff(y(x),x)+6*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{3}{4}x^2 + \frac{3}{2}x^3 - \frac{39}{16}x^4 + \frac{15}{4}x^5\right) y(0) + \left(x - 2x^2 + \frac{13}{4}x^3 - 5x^4 + \frac{121}{16}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 66

```
AsymptoticDSolveValue[(4+8*x+3*x^2)*y''[x]+(16+12*x)*y'[x]+6*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{15x^5}{4} - \frac{39x^4}{16} + \frac{3x^3}{2} - \frac{3x^2}{4} + 1\right) + c_2\left(\frac{121x^5}{16} - 5x^4 + \frac{13x^3}{4} - 2x^2 + x\right)$$

13.32 problem 32

Internal problem ID [1273]
Internal file name [`OUTPUT/1274_Sunday_June_05_2022_02_07_37_AM_11537175/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y'x + (2x^2 + 3)\, y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

13.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= 2x^2 + 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y'x + (2x^2 + 3)\, y = 0$$

The domain of $p(x) = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x^2 + 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1022}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1023}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2x^2y - 2y'x - 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 4yx^3 + 2y'x^2 + 2yx - 5y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= 16y'x - 4\left(x^4 - 4x^2 - \frac{17}{4}\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(-4x^4 - 16x^2 + 33\right) y' + \left(-48x^3 - 16x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(8x^5 - 32x^3 - 114x\right) y' + 8y\left(x^6 + \frac{11}{2}x^4 - \frac{81}{4}x^2 - \frac{115}{8}\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = -2$ gives

$$
\begin{aligned}
F_0 &= -3 \\
F_1 &= 10 \\
F_2 &= 17 \\
F_3 &= -66 \\
F_4 &= -115
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - 2x - \frac{3x^2}{2} + \frac{5x^3}{3} + \frac{17x^4}{24} - \frac{11x^5}{20} - \frac{23x^6}{144} + O\left(x^6\right)
$$

$$
y = 1 - 2x - \frac{3x^2}{2} + \frac{5x^3}{3} + \frac{17x^4}{24} - \frac{11x^5}{20} - \frac{23x^6}{144} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{2}$$

$n = 1$ gives

$$6a_3 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2na_n + 2a_{n-2} + 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2na_n + 3a_n + 2a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{(2n+3)\, a_n}{(n+2)\,(n+1)} - \frac{2a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 7a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{17a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 9a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{11a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 11a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{23a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 13a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{229a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{3}{2}a_0 x^2 - \frac{5}{6}a_1 x^3 + \frac{17}{24}a_0 x^4 + \frac{11}{40}a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{17}{24}x^4\right) a_0 + \left(x - \frac{5}{6}x^3 + \frac{11}{40}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{3}{2}x^2 + \frac{17}{24}x^4\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{11}{40}x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{3x^2}{2} + \frac{17x^4}{24} - 2x + \frac{5x^3}{3} - \frac{11x^5}{20} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 1 - 2x - \frac{3x^2}{2} + \frac{5x^3}{3} + \frac{17x^4}{24} - \frac{11x^5}{20} - \frac{23x^6}{144} + O(x^6) \tag{1}$$

$$y = 1 - \frac{3x^2}{2} + \frac{17x^4}{24} - 2x + \frac{5x^3}{3} - \frac{11x^5}{20} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - 2x - \frac{3x^2}{2} + \frac{5x^3}{3} + \frac{17x^4}{24} - \frac{11x^5}{20} - \frac{23x^6}{144} + O(x^6)$$

Verified OK.

$$y = 1 - \frac{3x^2}{2} + \frac{17x^4}{24} - 2x + \frac{5x^3}{3} - \frac{11x^5}{20} + O(x^6)$$

Verified OK.

13.32.2 Maple step by step solution

Let's solve

$$\left[y'' = -2x^2y - 2y'x - 3y, y(0) = 1, y'\Big|_{\{x=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-2x^2 - 3)\,y - 2y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y'x + (2x^2 + 3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 3a_0 + (6a_3 + 5a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(2k+3) + 2a_{k-2}) \, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 3a_0 = 0, 6a_3 + 5a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{3a_0}{2}, a_3 = -\frac{5a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} + 2a_k k + 3a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k+2) + 3a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2} + 2a_k + 7a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{3a_0}{2}, a_3 = -\frac{5a_1}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+2*x*diff(y(x),x)+(3+2*x^2)*y(x)=0,y(0) = 1, D(y)(0) = -2],y(x),type='
```

$$y(x) = 1 - 2x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{17}{24}x^4 - \frac{11}{20}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{y''[x]+2*x*y'[x]+(3+2*x^2)*y[x]==0,{y[0]==1,y'[0]==-2}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{11x^5}{20} + \frac{17x^4}{24} + \frac{5x^3}{3} - \frac{3x^2}{2} - 2x + 1$$

13.33 problem 33

Internal problem ID [1274]
Internal file name [`OUTPUT/1275_Sunday_June_05_2022_02_07_39_AM_40863679/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y'x + (2x^2 + 5)\, y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

13.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3x \\ q(x) &= 2x^2 + 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 3y'x + (2x^2 + 5)\, y = 0$$

The domain of $p(x) = -3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x^2 + 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1025)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1026)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2x^2y + 3y'x - 5y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= -6yx^3 + 7y'x^2 - 19yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(15x^3 - 11x\right)y' + \left(-14x^4 - 49x^2 - 9\right)y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(31x^4 - 37x^2 - 20\right)y' + \left(-30x^5 - 109x^3 - 43x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(63x^5 - 96x^3 - 177x\right)y' + \left(-62x^6 - 231x^4 - 102x^2 + 57\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=-2$ gives

$$
\begin{aligned}
F_0 &= -5 \\
F_1 &= 4 \\
F_2 &= -9 \\
F_3 &= 40 \\
F_4 &= 57
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - 2x - \frac{5x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{8} + \frac{x^5}{3} + \frac{19x^6}{240} + O\left(x^6\right)
$$

$$
y = 1 - 2x - \frac{5x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{8} + \frac{x^5}{3} + \frac{19x^6}{240} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) + 3\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 5\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-3n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 5a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-3n\, x^n a_n) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 5a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 5a_0 = 0$$

$$a_2 = -\frac{5a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 3na_n + 2a_{n-2} + 5a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{3na_n - 5a_n - 2a_{n-2}}{(n+2)\,(n+1)} \\ &= \frac{(3n-5)\, a_n}{(n+2)\,(n+1)} - \frac{2a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{6}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 7a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 10a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{42}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{5}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 - \frac{3}{8} a_0 x^4 - \frac{1}{6} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{5}{2}x^2 - \frac{3}{8}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{5}{2}x^2 - \frac{3}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{6}x^5\right) c_2 + O(x^6)$$

$$y = 1 - \frac{5x^2}{2} - \frac{3x^4}{8} - 2x + \frac{2x^3}{3} + \frac{x^5}{3} + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = 1 - 2x - \frac{5x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{8} + \frac{x^5}{3} + \frac{19x^6}{240} + O\left(x^6\right) \tag{1}$$

$$y = 1 - \frac{5x^2}{2} - \frac{3x^4}{8} - 2x + \frac{2x^3}{3} + \frac{x^5}{3} + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = 1 - 2x - \frac{5x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{8} + \frac{x^5}{3} + \frac{19x^6}{240} + O\left(x^6\right)$$

Verified OK.

$$y = 1 - \frac{5x^2}{2} - \frac{3x^4}{8} - 2x + \frac{2x^3}{3} + \frac{x^5}{3} + O\left(x^6\right)$$

Verified OK.

13.33.2 Maple step by step solution

Let's solve

$$\left[y'' = -2x^2y + 3y'x - 5y, y(0) = 1, y'\Big|_{\{x=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = (-2x^2 - 5)\, y + 3y'x$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 3y'x + (2x^2 + 5)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + 5a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k(3k-5) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 + 5a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - 3a_k k + 5a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2}(k+2) + 5a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3k a_{k+2} - 2a_k + a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)-3*x*diff(y(x),x)+(5+2*x^2)*y(x)=0,y(0) = 1, D(y)(0) = -2],y(x),type='
```

$$y(x) = 1 - 2x - \frac{5}{2}x^2 + \frac{2}{3}x^3 - \frac{3}{8}x^4 + \frac{1}{3}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{y''[x]-3*x*y'[x]+(5+2*x^2)*y[x]==0,{y[0]==1,y'[0]==-2}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{3} - \frac{3x^4}{8} + \frac{2x^3}{3} - \frac{5x^2}{2} - 2x + 1$$

13.34 problem 34

Internal problem ID [1275]
Internal file name [`OUTPUT/1276_Sunday_June_05_2022_02_07_41_AM_99699570/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 5y'x - (-x^2 + 3)\, y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

13.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 5x \\ q(x) &= x^2 - 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 5y'x + (x^2 - 3)\, y = 0$$

The domain of $p(x) = 5x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2 - 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1028}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1029}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2y - 5y'x + 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 5yx^3 + 24y'x^2 - 17yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(-115x^3 + 41x\right)y' + \left(-24x^4 + 89x^2 - 23\right)y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(551x^4 - 461x^2 + 18\right)y' + \left(115x^5 - 482x^3 + 301x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-2640x^5 + 4027x^3 - 711x\right)y' + \left(-551x^6 + 2689x^4 - 2847x^2 + 355\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=6$ and $y'(0)=-2$ gives

$$
\begin{aligned}
F_0 &= 18 \\
F_1 &= 4 \\
F_2 &= -138 \\
F_3 &= -36 \\
F_4 &= 2130
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 9x^2 - 2x + 6 + \frac{2x^3}{3} - \frac{23x^4}{4} - \frac{3x^5}{10} + \frac{71x^6}{24} + O(x^6)
$$

$$
y = 9x^2 - 2x + 6 + \frac{2x^3}{3} - \frac{23x^4}{4} - \frac{3x^5}{10} + \frac{71x^6}{24} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) - 5\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 5n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) + \sum_{n=0}^{\infty} (-3 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 5n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \sum_{n=0}^{\infty} (-3 a_n x^n) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - 3a_0 = 0$$

$$a_2 = \frac{3a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 5na_n + a_{n-2} - 3a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{5na_n - 3a_n + a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{(5n-3)\, a_n}{(n+2)\,(n+1)} - \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 7a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{23a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 12a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 17a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{71a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 22a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{89a_1}{1260}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{3}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 - \frac{23}{24} a_0 x^4 + \frac{3}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{3}{2}x^2 - \frac{23}{24}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{3}{20}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{3}{2}x^2 - \frac{23}{24}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{3}{20}x^5\right) c_2 + O(x^6)$$

$$y = 6 + 9x^2 - \frac{23x^4}{4} - 2x + \frac{2x^3}{3} - \frac{3x^5}{10} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 9x^2 - 2x + 6 + \frac{2x^3}{3} - \frac{23x^4}{4} - \frac{3x^5}{10} + \frac{71x^6}{24} + O(x^6) \tag{1}$$

$$y = 6 + 9x^2 - \frac{23x^4}{4} - 2x + \frac{2x^3}{3} - \frac{3x^5}{10} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 9x^2 - 2x + 6 + \frac{2x^3}{3} - \frac{23x^4}{4} - \frac{3x^5}{10} + \frac{71x^6}{24} + O(x^6)$$

Verified OK.

$$y = 6 + 9x^2 - \frac{23x^4}{4} - 2x + \frac{2x^3}{3} - \frac{3x^5}{10} + O(x^6)$$

Verified OK.

13.34.2 Maple step by step solution

Let's solve

$\left[y'' = -x^2 y - 5y'x + 3y, y(0) = 6, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -5y'x + (-x^2 + 3)\, y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 5y'x + (x^2 - 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - 3a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(5k-3) + a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 3a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{3a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 5a_k k - 3a_k + a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 5a_{k+2}(k+2) - 3a_{k+2} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5k a_{k+2} + a_k + 7a_{k+2}}{k^2 + 7k + 12}, a_2 = \frac{3a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+5*x*diff(y(x),x)-(3-x^2)*y(x)=0,y(0) = 6, D(y)(0) = -2],y(x),type='se
```

$$y(x) = 6 - 2x + 9x^2 + \frac{2}{3}x^3 - \frac{23}{4}x^4 - \frac{3}{10}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{y''[x]+5*x*y'[x]-(3-x^2)*y[x]==0,{y[0]==6,y'[0]==-2}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{3x^5}{10} - \frac{23x^4}{4} + \frac{2x^3}{3} + 9x^2 - 2x + 6$$

13.35 problem 35

Internal problem ID [1276]
Internal file name [`OUTPUT/1277_Sunday_June_05_2022_02_07_50_AM_61947886/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'x - (3x^2 + 2)\, y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -5]$$

With the expansion point for the power series method at $x = 0$.

13.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= -3x^2 - 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y'x + \left(-3x^2 - 2\right) y = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -3x^2 - 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1031) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1032) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 3x^2y + 2y'x + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 6yx^3 + 7y'x^2 + 10yx + 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(20x^3 + 32x\right) y' + y\left(21x^4 + 44x^2 + 18\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(61x^4 + 168x^2 + 50\right) y' + 60\left(x^4 + \frac{11}{3}x^2 + \frac{38}{15}\right) xy \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(182x^5 + 800x^3 + 588x\right) y' + y\left(183x^6 + 926x^4 + 1146x^2 + 252\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -5$ gives

$$
\begin{aligned}
F_0 &= 4 \\
F_1 &= -20 \\
F_2 &= 36 \\
F_3 &= -250 \\
F_4 &= 504
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 2x^2 - 5x + 2 - \frac{10x^3}{3} + \frac{3x^4}{2} - \frac{25x^5}{12} + \frac{7x^6}{10} + O\left(x^6\right)
$$

$$
y = 2x^2 - 5x + 2 - \frac{10x^3}{3} + \frac{3x^4}{2} - \frac{25x^5}{12} + \frac{7x^6}{10} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 3x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) + 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=0}^{\infty} \left(-3x^{n+2} a_n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} \left(-3x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n\, x^n a_n) + \sum_{n=2}^{\infty} \left(-3a_{n-2} x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$6a_3 - 4a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{2a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2na_n - 3a_{n-2} - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2na_n + 2a_n + 3a_{n-2}}{(n+2)\,(n+1)} \\ &= \frac{(2n+2)\, a_n}{(n+2)\,(n+1)} + \frac{3a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_2 - 3a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{3a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_3 - 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{5a_1}{12}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{20}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_5 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \frac{2}{3} a_1 x^3 + \frac{3}{4} a_0 x^4 + \frac{5}{12} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 + \frac{3}{4} x^4\right) a_0 + \left(x + \frac{2}{3} x^3 + \frac{5}{12} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 + \frac{3}{4} x^4\right) c_1 + \left(x + \frac{2}{3} x^3 + \frac{5}{12} x^5\right) c_2 + O(x^6)$$

$$y = 2 + 2x^2 + \frac{3x^4}{2} - 5x - \frac{10x^3}{3} - \frac{25x^5}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 2x^2 - 5x + 2 - \frac{10x^3}{3} + \frac{3x^4}{2} - \frac{25x^5}{12} + \frac{7x^6}{10} + O(x^6) \tag{1}$$

$$y = 2 + 2x^2 + \frac{3x^4}{2} - 5x - \frac{10x^3}{3} - \frac{25x^5}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 2x^2 - 5x + 2 - \frac{10x^3}{3} + \frac{3x^4}{2} - \frac{25x^5}{12} + \frac{7x^6}{10} + O(x^6)$$

Verified OK.

$$y = 2 + 2x^2 + \frac{3x^4}{2} - 5x - \frac{10x^3}{3} - \frac{25x^5}{12} + O(x^6)$$

Verified OK.

13.35.2 Maple step by step solution

Let's solve

$$\left[y'' = 3x^2y + 2y'x + 2y, y(0) = 2, y'\Big|_{\{x=0\}} = -5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 2y'x + (3x^2 + 2)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y'x + (-3x^2 - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 4a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(k+1) - 3a_{k-2}) \, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 4a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = a_0, a_3 = \frac{2a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) \, a_{k+2} - 2a_k k - 2a_k - 3a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 2a_{k+2}(k+2) - 2a_{k+2} - 3a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} + 3a_k + 6a_{k+2}}{k^2 + 7k + 12}, a_2 = a_0, a_3 = \frac{2a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)-2*x*diff(y(x),x)-(2+3*x^2)*y(x)=0,y(0) = 2, D(y)(0) = -5],y(x),type='
```

$$y(x) = 2 - 5x + 2x^2 - \frac{10}{3}x^3 + \frac{3}{2}x^4 - \frac{25}{12}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{y''[x]-2*x*y'[x]-(2+3*x^2)*y[x]==0,{y[0]==2,y'[0]==-5}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{25x^5}{12} + \frac{3x^4}{2} - \frac{10x^3}{3} + 2x^2 - 5x + 2$$

13.36 problem 36

Internal problem ID [1277]
Internal file name [`OUTPUT/1278_Sunday_June_05_2022_02_07_53_AM_30782941/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y'x + \left(4x^2 + 2\right) y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 6]$$

With the expansion point for the power series method at $x = 0$.

13.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3x \\ q(x) &= 4x^2 + 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y'x + (4x^2 + 2) y = 0$$

The domain of $p(x) = 3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4x^2 + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1034) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1035) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -4x^2 y - 3y' x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 5\left(x^2 - 1\right) y' + 2\left(6x^3 - x\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-3x^3 + 23x\right) y' + \left(-20x^4 + 46x^2 + 8\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(-11x^4 - 32x^2 + 31\right) y' + 12x\left(x^4 - \frac{83}{6}x^2 + \frac{23}{6}\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(45x^5 - 114x^3 - 111x\right) y' + \left(44x^6 + 210x^4 - 558x^2 - 16\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 3$ and $y'(0) = 6$ gives

$$
\begin{aligned}
F_0 &= -6 \\
F_1 &= -30 \\
F_2 &= 24 \\
F_3 &= 186 \\
F_4 &= -48
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^4 - 5x^3 - 3x^2 + 6x + 3 + \frac{31x^5}{20} - \frac{x^6}{15} + O\left(x^6\right)
$$

$$
y = x^4 - 5x^3 - 3x^2 + 6x + 3 + \frac{31x^5}{20} - \frac{x^6}{15} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -4x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - 3\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=0}^{\infty} 4x^{n+2} a_n &= \sum_{n=2}^{\infty} 4a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 3n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 5a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 3na_n + 4a_{n-2} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3na_n + 2a_n + 4a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{(3n+2)\, a_n}{(n+2)\,(n+1)} - \frac{4a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 8a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 11a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{31a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 14a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 17a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{127a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{5}{6}a_1x^3 + \frac{1}{3}a_0x^4 + \frac{31}{120}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) a_0 + \left(x - \frac{5}{6}x^3 + \frac{31}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^4\right) c_1 + \left(x - \frac{5}{6}x^3 + \frac{31}{120}x^5\right) c_2 + O(x^6)$$

$$y = x^4 - 3x^2 + 3 + 6x - 5x^3 + \frac{31x^5}{20} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^4 - 5x^3 - 3x^2 + 6x + 3 + \frac{31x^5}{20} - \frac{x^6}{15} + O(x^6) \tag{1}$$

$$y = x^4 - 3x^2 + 3 + 6x - 5x^3 + \frac{31x^5}{20} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^4 - 5x^3 - 3x^2 + 6x + 3 + \frac{31x^5}{20} - \frac{x^6}{15} + O(x^6)$$

Verified OK.

$$y = x^4 - 3x^2 + 3 + 6x - 5x^3 + \frac{31x^5}{20} + O(x^6)$$

Verified OK.

13.36.2 Maple step by step solution

Let's solve

$\left[y'' = -4x^2y - 3y'x - 2y, y(0) = 3, y'\Big|_{\{x=0\}} = 6\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-4x^2 - 2)\, y - 3y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 3y'x + (4x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + 2a_0 + (6a_3 + 5a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(3k+2) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 + 2a_0 = 0, 6a_3 + 5a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -a_0, a_3 = -\frac{5a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2 + 3k + 2\right) a_{k+2} + 3a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 3a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3k a_{k+2} + 4a_k + 8a_{k+2}}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -\frac{5a_1}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+3*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(0) = 3, D(y)(0) = 6],y(x),type='s
```

$$y(x) = 3 + 6x - 3x^2 - 5x^3 + x^4 + \frac{31}{20}x^5 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[{y''[x]+3*x*y'[x]+(2+4*x^2)*y[x]==0,{y[0]==3,y'[0]==6}},y[x],{x,0,5}]
```

$$y(x) \to \frac{31x^5}{20} + x^4 - 5x^3 - 3x^2 + 6x + 3$$

13.37 problem 37

Internal problem ID [1278]
Internal file name [`OUTPUT/1279_Sunday_June_05_2022_02_07_55_AM_13752740/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' + 5y'x + (2x^2 + 4)\, y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -2]$$

With the expansion point for the power series method at $x = 0$.

13.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{5x}{2} \\ q(x) &= x^2 + 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{5y'x}{2} + \left(x^2 + 2\right) y = 0$$

The domain of $p(x) = \frac{5x}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2 + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1037}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1038}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y - \frac{5y' x}{2} - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{3(7x^2-6)\, y'}{4} + \frac{(5x^3+6x)\, y}{2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-85x^3+198x)\, y'}{8} + \frac{3(-7x^4+2x^2+16)\, y}{4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(341x^4-1476x^2+588)\, y'}{16} + \frac{(85x^5-196x^3-372x)\, y}{8} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{21(-65x^5+444x^3-492x)\, y'}{32} + \frac{(-341x^6+1644x^4+1188x^2-1920)\, y}{16}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=3$ and $y'(0)=-2$ gives

$$
\begin{aligned}
F_0 &= -6 \\
F_1 &= 9 \\
F_2 &= 36 \\
F_3 &= -\frac{147}{2} \\
F_4 &= -360
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -3x^2 - 2x + 3 + \frac{3x^3}{2} + \frac{3x^4}{2} - \frac{49x^5}{80} - \frac{x^6}{2} + O\left(x^6\right)
$$

$$y = -3x^2 - 2x + 3 + \frac{3x^3}{2} + \frac{3x^4}{2} - \frac{49x^5}{80} - \frac{x^6}{2} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{5\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x}{2} - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 5n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 5n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$4a_2 + 4a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$12a_3 + 9a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{3a_1}{4}$$

For $2 \leq n$, the recurrence equation is

$$2(n+2)\, a_{n+2}(n+1) + 5na_n + 2a_{n-2} + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{5na_n + 4a_n + 2a_{n-2}}{2\,(n+2)\,(n+1)} \\ &= -\frac{(5n+4)\, a_n}{2\,(n+2)\,(n+1)} - \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$24a_4 + 14a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$40a_5 + 19a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{49a_1}{160}$$

For $n = 4$ the recurrence equation gives

$$60a_6 + 24a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$84a_7 + 29a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1181a_1}{13440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - a_0x^2 - \frac{3}{4}a_1x^3 + \frac{1}{2}a_0x^4 + \frac{49}{160}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right)a_0 + \left(x - \frac{3}{4}x^3 + \frac{49}{160}x^5\right)a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right)c_1 + \left(x - \frac{3}{4}x^3 + \frac{49}{160}x^5\right)c_2 + O(x^6)$$

$$y = 3 - 3x^2 + \frac{3x^4}{2} - 2x + \frac{3x^3}{2} - \frac{49x^5}{80} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = -3x^2 - 2x + 3 + \frac{3x^3}{2} + \frac{3x^4}{2} - \frac{49x^5}{80} - \frac{x^6}{2} + O(x^6) \tag{1}$$

$$y = 3 - 3x^2 + \frac{3x^4}{2} - 2x + \frac{3x^3}{2} - \frac{49x^5}{80} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -3x^2 - 2x + 3 + \frac{3x^3}{2} + \frac{3x^4}{2} - \frac{49x^5}{80} - \frac{x^6}{2} + O(x^6)$$

Verified OK.

$$y = 3 - 3x^2 + \frac{3x^4}{2} - 2x + \frac{3x^3}{2} - \frac{49x^5}{80} + O(x^6)$$

Verified OK.

13.37.2 Maple step by step solution

Let's solve

$\left[y'' = -x^2 y - \frac{5y'x}{2} - 2y, y(0) = 3, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x^2 - 2)\, y - \frac{5y'x}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'x}{2} + (x^2 + 2)\, y = 0$

- Multiply by denominators

  $2y'' + 5y'x + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum_{k=2}^{\infty} (2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5k a_{k+2} + 2a_k + 14a_{k+2}}{2(k^2 + 7k + 12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([2*diff(y(x),x$2)+5*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(0) = 3, D(y)(0) = -2],y(x),type
```

$$y(x) = 3 - 2x - 3x^2 + \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{49}{80}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{2*y''[x]+5*x*y'[x]+(4+2*x^2)*y[x]==0,{y[0]==3,y'[0]==-2}},y[x],{x,0,5
```

$$y(x) \to -\frac{49x^5}{80} + \frac{3x^4}{2} + \frac{3x^3}{2} - 3x^2 - 2x + 3$$

13.38 problem 38

Internal problem ID [1279]
Internal file name [`OUTPUT/1280_Sunday_June_05_2022_02_07_57_AM_73715578/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3y'' + 2y'x + \left(-x^2 + 4\right) y = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = 3]$$

With the expansion point for the power series method at $x = 0$.

13.38.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2x}{3} \\ q(x) &= -\frac{x^2}{3} + \frac{4}{3} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y''+\frac{2y'x}{3}+\left(-\frac{x^{2}}{3}+\frac{4}{3}\right)y = 0$$

The domain of $p(x)=\frac{2x}{3}$ is

$$\{-\infty <x<\infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x)=-\frac{x^{2}}{3}+\frac{4}{3}$ is

$$\{-\infty <x<\infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2}\,f|_{x_0,y_0,y'_0} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1040}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1041}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{x^2 y}{3} - \frac{2y' x}{3} - \frac{4y}{3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -\frac{2y x^3}{9} + \frac{7y' x^2}{9} + \frac{14yx}{9} - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{20(-x^3 + 6x)\, y'}{27} + \frac{y(7x^4 - 64x^2 + 114)}{27} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(61x^4 - 612x^2 + 702)\, y'}{81} - \frac{20\left(x^4 - \frac{71}{5}x^2 + \frac{216}{5}\right) xy}{81} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{2(-91x^5 + 1404x^3 - 3834x)\, y'}{243} + \frac{y(61x^6 - 1156x^4 + 5706x^2 - 5400)}{243}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -2$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= \frac{8}{3} \\
F_1 &= -6 \\
F_2 &= -\frac{76}{9} \\
F_3 &= 26 \\
F_4 &= \frac{400}{9}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 3x - 2 + \frac{4x^2}{3} - x^3 - \frac{19x^4}{54} + \frac{13x^5}{60} + \frac{5x^6}{81} + O\left(x^6\right)
$$

$$
y = 3x - 2 + \frac{4x^2}{3} - x^3 - \frac{19x^4}{54} + \frac{13x^5}{60} + \frac{5x^6}{81} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right)}{3} - \frac{2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x}{3} - \frac{4\left(\sum_{n=0}^{\infty} a_n x^n\right)}{3} \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n
$$

$$
\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n\, x^n a_n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}
$$

$n = 0$ gives

$$6a_2 + 4a_0 = 0$$

$$a_2 = -\frac{2a_0}{3}$$

$n = 1$ gives

$$18a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$3(n+2)\, a_{n+2}(n+1) + 2na_n - a_{n-2} + 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2na_n + 4a_n - a_{n-2}}{3\,(n+2)\,(n+1)} \\ &= -\frac{(2n+4)\, a_n}{3\,(n+2)\,(n+1)} + \frac{a_{n-2}}{3\,(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$36a_4 + 8a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{19a_0}{108}$$

For $n = 3$ the recurrence equation gives

$$60a_5 + 10a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{13a_1}{180}$$

For $n = 4$ the recurrence equation gives

$$90a_6 + 12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{5a_0}{162}$$

For $n = 5$ the recurrence equation gives

$$126a_7 + 14a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{121a_1}{11340}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{2}{3} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{19}{108} a_0 x^4 + \frac{13}{180} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{2}{3}x^2 + \frac{19}{108}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{13}{180}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{2}{3}x^2 + \frac{19}{108}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{13}{180}x^5\right) c_2 + O(x^6)$$

$$y = -2+\frac{4x^{2}}{3}-\frac{19x^{4}}{54}+3x-x^{3}+\frac{13x^{5}}{60}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$y = 3x-2+\frac{4x^{2}}{3}-x^{3}-\frac{19x^{4}}{54}+\frac{13x^{5}}{60}+\frac{5x^{6}}{81}+O\left(x^{6}\right) \tag{1}$$

$$y = -2+\frac{4x^{2}}{3}-\frac{19x^{4}}{54}+3x-x^{3}+\frac{13x^{5}}{60}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = 3x-2+\frac{4x^{2}}{3}-x^{3}-\frac{19x^{4}}{54}+\frac{13x^{5}}{60}+\frac{5x^{6}}{81}+O\left(x^{6}\right)$$

Verified OK.

$$y = -2+\frac{4x^{2}}{3}-\frac{19x^{4}}{54}+3x-x^{3}+\frac{13x^{5}}{60}+O\left(x^{6}\right)$$

Verified OK.

13.38.2 Maple step by step solution

Let's solve

$\left[y'' = \frac{x^{2}y}{3}-\frac{2y'x}{3}-\frac{4y}{3}, y(0) = -2, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \left(\frac{x^{2}}{3}-\frac{4}{3}\right)y-\frac{2y'x}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'x}{3}+\left(-\frac{x^{2}}{3}+\frac{4}{3}\right)y = 0$

- Multiply by denominators

  $3y''+2y'x+\left(-x^{2}+4\right)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k} x^{k}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k + 2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$6a_2 + 4a_0 + (18a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(3a_{k+2}(k+2)\,(k+1) + 2a_k(k+2) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[6a_2 + 4a_0 = 0, 18a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -\frac{2a_0}{3}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(3k^2 + 9k + 6)\, a_{k+2} + 2a_k k + 4a_k - a_{k-2} = 0$$

- Shift index using $k- >k + 2$

  $$\left(3(k+2)^2 + 9k + 24\right) a_{k+4} + 2a_{k+2}(k+2) + 4a_{k+2} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2} - a_k + 8a_{k+2}}{3(k^2+7k+12)}, a_2 = -\frac{2a_0}{3}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([3*diff(y(x),x$2)+2*x*diff(y(x),x)+(4-x^2)*y(x)=0,y(0) = -2, D(y)(0) = 3],y(x),type='
```

$$y(x) = -2 + 3x + \frac{4}{3}x^2 - x^3 - \frac{19}{54}x^4 + \frac{13}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{3*y''[x]+2*x*y'[x]+(4-x^2)*y[x]==0,{y[0]==-2,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{13x^5}{60} - \frac{19x^4}{54} - x^3 + \frac{4x^2}{3} + 3x - 2$$

13.39 problem 39 (a)

Internal problem ID [1280]
Internal file name [`OUTPUT/1281_Sunday_June_05_2022_02_07_59_AM_93449407/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 39 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y'x + \left(4x^2 + 2\right) y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

13.39.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4x \\ q(x) &= 4x^2 + 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y'x + \left(4x^2 + 2\right) y = 0$$

The domain of $p(x) = 4x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4x^2 + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1043}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1044}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n |_{x_0, y_0, y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -4x^2y - 4y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 16yx^3 + 12y'x^2 - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(-32x^3 + 48x\right)y' - 48\left(x^4 - x^2 - \frac{1}{4}\right)y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= 128yx^5 + 80y'x^4 - 320yx^3 - 240y'x^2 + 60y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-192x^5 + 960x^3 - 720x\right)y' - 320\left(x^6 - \frac{9}{2}x^4 + \frac{9}{4}x^2 + \frac{3}{8}\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 0 \\
F_2 &= 12 \\
F_3 &= 0 \\
F_4 &= -120
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + O(x^6)
$$

$$
y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -4x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} 4x^{n+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=1}^{\infty} 4n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -a_1$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 4na_n + 4a_{n-2} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(2na_n + a_n + 2a_{n-2})}{(n+2)\,(n+1)} \\ &= -\frac{2(2n+1)\, a_n}{(n+2)\,(n+1)} - \frac{4a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 10a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 14a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 18a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 22a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{1}{2} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2} x^4\right) a_0 + \left(x - x^3 + \frac{1}{2} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2} x^4\right) c_1 + \left(x - x^3 + \frac{1}{2} x^5\right) c_2 + O(x^6)$$

$$y = 1 - x^2 + \frac{x^4}{2} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + O(x^6) \tag{1}$$

$$y = 1 - x^2 + \frac{x^4}{2} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + O(x^6)$$

Verified OK.

$$y = 1 - x^2 + \frac{x^4}{2} + O(x^6)$$

Verified OK.

13.39.2 Maple step by step solution

Let's solve

$$\left[y'' = -4x^2y - 4y'x - 2y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-4x^2 - 2)\,y - 4y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y'x + (4x^2 + 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='s
```

$$y(x) = 1 - x^2 + \frac{1}{2}x^4 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{x^4}{2} - x^2 + 1$$

13.40 problem 39 (b)

Internal problem ID [1281]
Internal file name [`OUTPUT/1282_Sunday_June_05_2022_02_08_01_AM_44876535/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 39 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y'x + (4x^2 + 2)\, y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

13.40.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 4x$$
$$q(x) = 4x^2 + 2$$
$$F = 0$$

Hence the ode is

$$y'' + 4y'x + (4x^2 + 2)\, y = 0$$

The domain of $p(x) = 4x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4x^2 + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1046}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1047}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -4x^2y - 4y'x - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 16yx^3 + 12y'x^2 - 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \left(-32x^3 + 48x\right)y' - 48\left(x^4 - x^2 - \frac{1}{4}\right)y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= 128yx^5 + 80y'x^4 - 320yx^3 - 240y'x^2 + 60y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \left(-192x^5 + 960x^3 - 720x\right)y' - 320\left(x^6 - \frac{9}{2}x^4 + \frac{9}{4}x^2 + \frac{3}{8}\right)y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -6 \\
F_2 &= 0 \\
F_3 &= 60 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - x^3 + \frac{x^5}{2} + O\left(x^6\right)
$$

$$
y = x - x^3 + \frac{x^5}{2} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -4x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - 4\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} 4x^{n+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 4n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

$n = 1$ gives

$$6a_3 + 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -a_1$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 4na_n + 4a_{n-2} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2(2na_n + a_n + 2a_{n-2})}{(n+2)\,(n+1)} \\ &= -\frac{2(2n+1)\, a_n}{(n+2)\,(n+1)} - \frac{4a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 10a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 14a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 18a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 22a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 - a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{1}{2} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) a_0 + \left(x - x^3 + \frac{1}{2}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{2}x^4\right) c_1 + \left(x - x^3 + \frac{1}{2}x^5\right) c_2 + O\left(x^6\right)$$

$$y = x - x^3 + \frac{x^5}{2} + O\left(x^6\right)$$

Summary
The solution(s) found are the following

$$y = x - x^3 + \frac{x^5}{2} + O(x^6) \tag{1}$$

$$y = x - x^3 + \frac{x^5}{2} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - x^3 + \frac{x^5}{2} + O(x^6)$$

Verified OK.

$$y = x - x^3 + \frac{x^5}{2} + O(x^6)$$

Verified OK.

13.40.2 Maple step by step solution

Let's solve

$\left[y'' = -4x^2y - 4y'x - 2y, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-4x^2 - 2)\, y - 4y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y'x + (4x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k->k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2k a_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='s
```

$$y(x) = x - x^3 + \frac{1}{2}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{x^5}{2} - x^3 + x$$

13.41 problem 40

Internal problem ID [1282]
Internal file name [`OUTPUT/1283_Sunday_June_05_2022_02_08_03_AM_71948828/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''+y'x^2+(1+2x)\,y=0}$$

With initial conditions

$$[y(0)=-2, y'(0)=3]$$

With the expansion point for the power series method at $x=0$.

13.41.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x^2}{x+1}\\ q(x)&=\frac{1+2x}{x+1}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{x^2 y'}{x+1} + \frac{(1+2x)\,y}{x+1} = 0$$

The domain of $p(x) = \frac{x^2}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1+2x}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1049}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1050}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x^2 + 2yx + y}{x+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^4 - 3x^2 - 5x - 1)\, y' + y(2x^3 + x^2 - 1)}{(x+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^6 + 7x^4 + 12x^3 + 2x^2 - 2x - 4)\, y' - 2\left(x^5 + \frac{1}{2}x^4 - 4x^3 - \frac{19}{2}x^2 - \frac{9}{2}x - \frac{3}{2}\right) y}{(x+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^8 - 12x^6 - 21x^5 + 12x^4 + 57x^3 + 66x^2 + 20x + 13)\, y' + 2\left(x^7 + \frac{1}{2}x^6 - 9x^5 - 21x^4 - 10x^3 + \frac{7}{2}x^2 + \right.}{(x+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^{10} + 18x^8 + 32x^7 - 53x^6 - 210x^5 - 233x^4 - 42x^3 + 63x^2 + 106x - 28)\, y' - 2\left(x^9 + \frac{1}{2}x^8 - 15x^7 \right.}{(x+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -2$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= 2 \\
F_1 &= -1 \\
F_2 &= -18 \\
F_3 &= 31 \\
F_4 &= -86
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x^2 + 3x - 2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} - \frac{43x^6}{360} + O(x^6)
$$

$$
y = x^2 + 3x - 2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} - \frac{43x^6}{360} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(x+1)\, y'' + y' x^2 + (1+2x)\, y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$
$$
y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(x+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 + (1+2x)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\
&+ \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0
\end{aligned} \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^n
$$

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n
$$

$$\sum_{n=1}^{\infty} n\, x^{1+n} a_{n} = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^{n}$$

$$\sum_{n=0}^{\infty} 2 x^{1+n} a_{n} = \sum_{n=1}^{\infty} 2 a_{n-1} x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^{n}\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2 a_{2} + a_{0} = 0$$

$$a_{2} = -\frac{a_{0}}{2}$$

$n = 1$ gives

$$2 a_{2} + 6 a_{3} + a_{1} + 2 a_{0} = 0$$

Which after substituting earlier equations, simplifies to

$$a_{3} = -\frac{a_{0}}{6} - \frac{a_{1}}{6}$$

For $2 \le n$, the recurrence equation is

$$(1+n)\, a_{1+n} n + (n+2)\, a_{n+2} (1+n) + (n-1)\, a_{n-1} + a_{n} + 2 a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^{2} a_{1+n} + n a_{1+n} + n a_{n-1} + a_{n} + a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_{n}}{(n+2)\,(1+n)} - \frac{(n^{2}+n)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{a_{n-1}}{n+2} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 12a_4 + 3a_1 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8} - \frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 20a_5 + 4a_2 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{30} + \frac{13a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 30a_6 + 5a_3 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720} - \frac{7a_1}{180}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 42a_7 + 6a_4 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{560} + \frac{247a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{8} - \frac{a_1}{6}\right) x^4 + \left(\frac{a_0}{30} + \frac{13 a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{13}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{13}{120}x^5\right) c_2 + O(x^6)$$

$$y = -2 + x^2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} + 3x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x^2 + 3x - 2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} - \frac{43x^6}{360} + O(x^6) \quad (1)$$

$$y = -2 + x^2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} + 3x + O(x^6) \quad (2)$$

Verification of solutions

$$y = x^2 + 3x - 2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} - \frac{43x^6}{360} + O(x^6)$$

Verified OK.

$$y = -2 + x^2 - \frac{x^3}{6} - \frac{3x^4}{4} + \frac{31x^5}{120} + 3x + O(x^6)$$

Verified OK.

13.41.2 Maple step by step solution

Let's solve

$\left[(x+1)\,y''+y' x^2+(1+2x)\,y=0, y(0)=-2, y'\Big|_{\{x=0\}}=3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{x^2 y'}{x+1}-\frac{(1+2x)y}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{x^2 y'}{x+1}+\frac{(1+2x)y}{x+1}=0$

☐ Check to see if $x_0=-1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x^2}{x+1}, P_3(x)=\frac{1+2x}{x+1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x+1)\,y''+y' x^2+(1+2x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+\left(u^2-2u+1\right)\left(\frac{d}{du}y(u)\right)+(-1+2u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k + 1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}(k -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+1} + (-2a_k + a_{k-1} + 2a_{k+1})\, k - a_k + a_{k-1} + a_{k+1} = 0$$

- Shift index using $k- >k + 1$

$$(k+1)^2 a_{k+2} + (-2a_{k+1} + a_k + 2a_{k+2})\,(k+1) - a_{k+1} + a_k + a_{k+2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k k - 2k a_{k+1} + 2a_k - 3a_{k+1}}{k^2 + 4k + 4}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k k - 2k a_{k+1} + 2a_k - 3a_{k+1}}{k^2 + 4k + 4}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{a_k k - 2k a_{k+1} + 2a_k - 3a_{k+1}}{k^2 + 4k + 4}, a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{a_k k - 2k a_{k+1} + 2a_k - 3a_{k+1}}{k^2 + 4k + 4}, a_1 - a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+x)*diff(y(x),x$2)+x^2*diff(y(x),x)+(1+2*x)*y(x)=0,y(0) = -2, D(y)(0) = 3],y(x),ty
```

$$y(x) = -2 + 3x + x^2 - \frac{1}{6}x^3 - \frac{3}{4}x^4 + \frac{31}{120}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(1+x)*y''[x]+x^2*y'[x]+(1+2*x)*y[x]==0,{y[0]==-2,y'[0]==3}},y[x],{x,0
```

$$y(x) \to \frac{31x^5}{120} - \frac{3x^4}{4} - \frac{x^3}{6} + x^2 + 3x - 2$$

13.42 problem 41

Internal problem ID [1283]
Internal file name [`OUTPUT/1284_Sunday_June_05_2022_02_08_06_AM_2747193/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (x^2 + 2x + 1)\, y' + 2y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

With the expansion point for the power series method at $x = 0$.

13.42.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= (x+1)^2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (x+1)^2 y' + 2y = 0$$

The domain of $p(x) = (x+1)^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1052}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1053}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x^2 - 2y'x - y' - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^4 + 4x^3 + 6x^2 + 2x - 3\right) y' + 2y(x+1)^2 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^6 - 6x^5 - 15x^4 - 14x^3 + 7x^2 + 20x + 7\right) y' - 2y\left(x^4 + 4x^3 + 6x^2 - 5\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^8 + 8x^7 + 28x^6 + 44x^5 + 4x^4 - 88x^3 - 108x^2 - 20x + 23\right) y' + 2y\left(x^6 + 6x^5 + 15x^4 + 10x^3 - 19x^2
\\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^{10} - 10x^9 - 45x^8 - 100x^7 - 62x^6 + 216x^5 + 530x^4 + 360x^3 - 177x^2 - 306x - 57\right) y' - 2y\left(x^8 +
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = 3$ gives

$$
\begin{aligned}
F_0 &= -7 \\
F_1 &= -5 \\
F_2 &= 41 \\
F_3 &= 41 \\
F_4 &= -391
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 3x + 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} - \frac{391x^6}{720} + O\left(x^6\right)
$$

$$
y = 3x + 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} - \frac{391x^6}{720} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x^2 - 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$
$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^{n}\right) \\ +\left(\sum_{n=1}^{\infty} 2n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^{n}\right)+\left(\sum_{n=0}^{\infty} 2 a_{n} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$2a_{2}+a_{1}+2a_{0}=0$$

$$a_{2}=-a_{0}-\frac{a_{1}}{2}$$

$n=1$ gives

$$6a_{3}+4a_{1}+2a_{2}=0$$

Which after substituting earlier equations, simplifies to

$$a_{3}=\frac{a_{0}}{3}-\frac{a_{1}}{2}$$

For $2\leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n)+(n-1)\, a_{n-1}+2n a_{n}+(1+n)\, a_{1+n}+2a_{n}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2}&=-\frac{2n a_{n}+n a_{1+n}+n a_{n-1}+2a_{n}+a_{1+n}-a_{n-1}}{(n+2)\,(1+n)} \\ &=-\frac{(2n+2)\, a_{n}}{(n+2)\,(1+n)}-\frac{a_{1+n}}{n+2}-\frac{(n-1)\, a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n=2$ the recurrence equation gives

$$12a_{4}+a_{1}+6a_{2}+3a_{3}=0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_1}{24} + \frac{5a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_2 + 8a_3 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_0}{60} + \frac{23a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 3a_3 + 10a_4 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{11a_0}{72} - \frac{19a_1}{240}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_4 + 12a_5 + 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{359a_1}{5040} + \frac{13a_0}{840}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-a_0 - \frac{a_1}{2}\right) x^2 + \left(\frac{a_0}{3} - \frac{a_1}{2}\right) x^3 + \left(\frac{7a_1}{24} + \frac{5a_0}{12}\right) x^4 + \left(-\frac{7a_0}{60} + \frac{23a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{7}{60}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{7}{24}x^4 + \frac{23}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{7}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{7}{24}x^4 + \frac{23}{120}x^5\right) c_2 + O(x^6)$$

$$y = 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} + 3x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 3x + 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} - \frac{391x^6}{720} + O(x^6) \tag{1}$$

$$y = 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} + 3x + O(x^6) \tag{2}$$

Verification of solutions

$$y = 3x + 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} - \frac{391x^6}{720} + O(x^6)$$

Verified OK.

$$y = 2 - \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{41x^4}{24} + \frac{41x^5}{120} + 3x + O(x^6)$$

Verified OK.

13.42.2 Maple step by step solution

Let's solve

$$\left[y'' = -y'x^2 - 2y'x - y' - 2y, y(0) = 2, y'\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''+(x^2+2x+1)\,y'+2y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m=0..2$

$$x^m \cdot y'=\sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

  - Convert y'' to series expansion

$$y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  - Shift index using $k->k+2$

$$y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2+a_1+2a_0+\left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1)+a_{k+1}(k+1)+2a_k(k+1)+a_{k-1}(k-1)\right) x^k\right)=0$$

- Each term must be 0

$$2a_2+a_1+2a_0=0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2a_{k+2}+(2a_k+a_{k-1}+a_{k+1}+3a_{k+2})\,k+2a_k-a_{k-1}+a_{k+1}+2a_{k+2}=0$$

- Shift index using $k->k+1$

$$(k+1)^2\, a_{k+3}+(2a_{k+1}+a_k+a_{k+2}+3a_{k+3})\,(k+1)+2a_{k+1}-a_k+a_{k+2}+2a_{k+3}=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{a_k k+2a_{k+1}k+k a_{k+2}+4a_{k+1}+2a_{k+2}}{k^2+5k+6}, 2a_2+a_1+2a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+(1+2*x+x^2)*diff(y(x),x)+2*y(x)=0,y(0) = 2, D(y)(0) = 3],y(x),type='s
```

$$y(x) = 2 + 3x - \frac{7}{2}x^2 - \frac{5}{6}x^3 + \frac{41}{24}x^4 + \frac{41}{120}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{y''[x]+(1+2*x+x^2)*y'[x]+2*y[x]==0,{y[0]==2,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{41x^5}{120} + \frac{41x^4}{24} - \frac{5x^3}{6} - \frac{7x^2}{2} + 3x + 2$$

13.43 problem 42

Internal problem ID [1284]
Internal file name [`OUTPUT/1285_Sunday_June_05_2022_02_08_09_AM_82858837/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''+(x^2+2)\, y'+yx=0}$$

With initial conditions

$$[y(0)=-3, y'(0)=5]$$

With the expansion point for the power series method at $x=0$.

13.43.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x^2+2}{x^2+1}\\ q(x)&=\frac{x}{x^2+1}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(x^2+2)\,y'}{x^2+1} + \frac{xy}{x^2+1} = 0$$

The domain of $p(x) = \frac{x^2+2}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1055}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1056}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y' x^2 + yx + 2y'}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^4 - x^3 + 4x^2 + x + 4)\, y' + y(x^3 + x^2 + 2x - 1)}{(x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^6 + 2x^5 - 4x^4 - 18x^2 - 8x - 8)\, y' - y(x^5 + 6x^3 + 4x^2 - 2x - 2)}{(x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^8 - 3x^7 + 4x^6 - 9x^5 + 32x^4 + 60x^3 + 82x^2 + 30x + 10)\, y' + y(x^7 - x^6 + 4x^5 + 13x^4 + 34x^3 - 20x}{(x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-x^{10} + 4x^9 - 4x^8 + 20x^7 - 22x^6 - 108x^5 - 498x^4 - 492x^3 - 222x^2 + 12x + 12)\, y' - y(x^9 - 2x^8 +}{(x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -3$ and $y'(0) = 5$ gives

$$
\begin{aligned}
F_0 &= -10 \\
F_1 &= 23 \\
F_2 &= -46 \\
F_3 &= 44 \\
F_4 &= 96
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -5x^2 + 5x - 3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + \frac{2x^6}{15} + O(x^6)
$$

$$
y = -5x^2+5x-3+\frac{23x^3}{6}-\frac{23x^4}{12}+\frac{11x^5}{30}+\frac{2x^6}{15}+O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2+1\right)y''+\left(x^2+2\right)y'+yx = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(x^2+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(x^2+2\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)x = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) \\
&+\left(\sum_{n=1}^{\infty} n\,x^{1+n} a_n\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^n \\
\sum_{n=1}^{\infty} n\,x^{1+n} a_n &= \sum_{n=2}^{\infty} (n-1)\,a_{n-1} x^n
\end{aligned}
$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n\right) + \left(\sum_{n=2}^{\infty} (n-1) a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} 2(1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 = 0$$

$$a_2 = -a_1$$

$n = 1$ gives

$$6a_3 + 4a_2 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{2a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2) a_{n+2}(1+n) + (n-1) a_{n-1} + 2(1+n) a_{1+n} + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n - n a_n + 2n a_{1+n} + n a_{n-1} + 2a_{1+n}}{(n+2)(1+n)} \\ &= -\frac{(n^2-n) a_n}{(n+2)(1+n)} - \frac{(2n+2) a_{1+n}}{(n+2)(1+n)} - \frac{n a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$2a_2 + 12a_4 + 2a_1 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{3} + \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$6a_3 + 20a_5 + 3a_2 + 8a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{60} + \frac{a_1}{12}$$

For $n = 4$ the recurrence equation gives

$$12a_4 + 30a_6 + 4a_3 + 10a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{60} - \frac{a_0}{60}$$

For $n = 5$ the recurrence equation gives

$$20a_5 + 42a_7 + 5a_4 + 12a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{11a_0}{840} - \frac{a_1}{210}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_1 x^2 + \left(-\frac{a_0}{6} + \frac{2a_1}{3}\right) x^3 + \left(-\frac{a_1}{3} + \frac{a_0}{12}\right) x^4 + \left(\frac{a_0}{60} + \frac{a_1}{12}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5\right) a_0 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{12}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5\right) c_1 + \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{12}x^5\right) c_2 + O(x^6)$$

$$y = -3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + 5x - 5x^2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = -5x^2 + 5x - 3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + \frac{2x^6}{15} + O(x^6) \quad (1)$$

$$y = -3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + 5x - 5x^2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = -5x^2 + 5x - 3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + \frac{2x^6}{15} + O(x^6)$$

Verified OK.

$$y = -3 + \frac{23x^3}{6} - \frac{23x^4}{12} + \frac{11x^5}{30} + 5x - 5x^2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+x^2)*diff(y(x),x$2)+(2+x^2)*diff(y(x),x)+x*y(x)=0,y(0) = -3, D(y)(0) = 5],y(x),ty
```

$$y(x) = -3 + 5x - 5x^2 + \frac{23}{6}x^3 - \frac{23}{12}x^4 + \frac{11}{30}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{(1+x^2)*y''[x]+(2+x^2)*y'[x]+x*y[x]==0,{y[0]==-3,y'[0]==5}},y[x],{x,0
```

$$y(x) \to \frac{11x^5}{30} - \frac{23x^4}{12} + \frac{23x^3}{6} - 5x^2 + 5x - 3$$

13.44 problem 43

Internal problem ID [1285]
Internal file name [`OUTPUT/1286_Sunday_June_05_2022_02_08_11_AM_57431340/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''+\left(2x^{2}-3x+1\right)y'-(x-4)\,y=0}$$

With initial conditions

$$[y(1)=-2, y'(1)=3]$$

With the expansion point for the power series method at $x=1$.

13.44.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{2x^{2}-3x+1}{x+1}\\ q(x)&=\frac{-x+4}{x+1}\\ F&=0\end{aligned}$$

Hence the ode is

$$y'' + \frac{(2x^2 - 3x + 1)\, y'}{x+1} + \frac{(-x+4)\, y}{x+1} = 0$$

The domain of $p(x) = \frac{2x^2-3x+1}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{-x+4}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2} y(t)\right) (2+t) + \left(2(t+1)^2 - 3t - 2\right) \left(\frac{d}{dt} y(t)\right) + (-t+3)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = -2$$
$$y'(0) = 3$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1058) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1059) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{2\left(\frac{d}{dt}y(t)\right)t^2 + t\left(\frac{d}{dt}y(t)\right) - y(t)\,t + 3y(t)}{2+t}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{(4t^4 + 4t^3 - 9t - 8)\left(\frac{d}{dt}y(t)\right) - 2\left(t^3 - \frac{5}{2}t^2 - \frac{3}{2}t - \frac{5}{2}\right)y(t)}{(2+t)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{(-8t^6 - 12t^5 + 2t^4 + 55t^3 + 62t^2 + 28t + 8)\left(\frac{d}{dt}y(t)\right) + 4\left(t^5 - 2t^4 - \frac{7}{2}t^3 - \frac{21}{4}t^2 + 9t + 5\right)y(t)}{(2+t)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{(16t^8 + 32t^7 - 12t^6 - 232t^5 - 327t^4 - 151t^3 + 218t^2 + 276t + 72)\left(\frac{d}{dt}y(t)\right) - 8\left(t^7 - \frac{3}{2}t^6 - \frac{23}{4}t^5 - \frac{81}{8}t\right.}{(2+t)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{(-32t^{10} - 80t^9 + 48t^8 + 824t^7 + 1380t^6 + 426t^5 - 2610t^4 - 3790t^3 - 2436t^2 - 504t + 240)\left(\frac{d}{dt}y(t)\right)}{(2+t)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = -2$ and $y'(0) = 3$ gives

$$\begin{aligned}
F_0 &= 3 \\
F_1 &= -\frac{17}{2} \\
F_2 &= -2 \\
F_3 &= 15 \\
F_4 &= 62
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -2 + 3t + \frac{3t^2}{2} - \frac{17t^3}{12} - \frac{t^4}{12} + \frac{t^5}{8} + \frac{31t^6}{360} + O(t^6)$$

$$y(t) = -2 + 3t + \frac{3t^2}{2} - \frac{17t^3}{12} - \frac{t^4}{12} + \frac{t^5}{8} + \frac{31t^6}{360} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2+t) + (2t^2+t)\left(\frac{d}{dt}y(t)\right) + (-t+3)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(2+t) + (2t^2+t)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (-t+3)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,t^{n-1}a_n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n\,t^{1+n}a_n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \sum_{n=0}^{\infty}\left(-t^{1+n}a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) &= \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n \\
\sum_{n=2}^{\infty} 2n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (1+n)\, t^n \\
\sum_{n=1}^{\infty} 2n\, t^{1+n} a_n &= \sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} t^n \\
\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) + \left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2} (1+n)\, t^n\right) \\
&+ \left(\sum_{n=2}^{\infty} 2(n-1)\, a_{n-1} t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 3 a_n t^n\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3a_0}{4}$$

$n = 1$ gives

$$2a_2 + 12a_3 + 4a_1 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{5a_0}{24} - \frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} n + 2(n+2) a_{n+2}(1+n) + 2(n-1) a_{n-1} + n a_n - a_{n-1} + 3a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{1+n} + n a_n + n a_{1+n} + 2n a_{n-1} + 3a_n - 3a_{n-1}}{2(n+2)(1+n)} \\ &= -\frac{(n+3) a_n}{2(n+2)(1+n)} - \frac{(n^2+n) a_{1+n}}{2(n+2)(1+n)} - \frac{(2n-3) a_{n-1}}{2(n+2)(1+n)} \end{aligned} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 24a_4 + a_1 + 5a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{5a_0}{48} + \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 40a_5 + 3a_2 + 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{160} + \frac{3a_1}{80}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 60a_6 + 5a_3 + 7a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{79a_0}{2880} + \frac{a_1}{96}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 84a_7 + 7a_4 + 8a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{23a_0}{13440} - \frac{31a_1}{2880}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{3a_0 t^2}{4} + \left(\frac{5a_0}{24} - \frac{a_1}{3}\right) t^3 + \left(\frac{5a_0}{48} + \frac{a_1}{24}\right) t^4 + \left(-\frac{a_0}{160} + \frac{3a_1}{80}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{3}{4}t^2 + \frac{5}{24}t^3 + \frac{5}{48}t^4 - \frac{1}{160}t^5\right) a_0 + \left(t - \frac{1}{3}t^3 + \frac{1}{24}t^4 + \frac{3}{80}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{3}{4}t^2 + \frac{5}{24}t^3 + \frac{5}{48}t^4 - \frac{1}{160}t^5\right) c_1 + \left(t - \frac{1}{3}t^3 + \frac{1}{24}t^4 + \frac{3}{80}t^5\right) c_2 + O(t^6)$$

$$y(t) = -2 + \frac{3t^2}{2} - \frac{17t^3}{12} - \frac{t^4}{12} + \frac{t^5}{8} + 3t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = -5 + 3x + \frac{3(x-1)^2}{2} - \frac{17(x-1)^3}{12} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{8} + \frac{31(x-1)^6}{360} + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = -5 + 3x + \frac{3(x-1)^2}{2} - \frac{17(x-1)^3}{12} - \frac{(x-1)^4}{12} \\ + \frac{(x-1)^5}{8} + \frac{31(x-1)^6}{360} + O\left((x-1)^6\right) \end{aligned} \quad (1)$$

Verification of solutions

$$y = -5+3x+\frac{3(x-1)^2}{2}-\frac{17(x-1)^3}{12}-\frac{(x-1)^4}{12}+\frac{(x-1)^5}{8}+\frac{31(x-1)^6}{360}+O\left((x-1)^6\right)$$

Verified OK.

13.44.2 Maple step by step solution

Let's solve

$\left[(x+1)\,y''+(2x^2-3x+1)\,y'+(-x+4)\,y=0, y(1)=-2, y'\Big|_{\{x=1\}}=3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x-4)y}{x+1}-\frac{(2x^2-3x+1)y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x^2-3x+1)y'}{x+1}-\frac{(x-4)y}{x+1}=0$

□ Check to see if $x_0=-1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2x^2-3x+1}{x+1}, P_3(x)=-\frac{x-4}{x+1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=6$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x+1)\,y''+(2x^2-3x+1)\,y'+(-x+4)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$u\left(\frac{d^2}{du^2}y(u)\right)+(2u^2-7u+6)\left(\frac{d}{du}y(u)\right)+(-u+5)\,y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- ○ Shift index using $k->k+1$

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(5+r)\,u^{-1+r}+(a_1(1+r)\,(6+r)-a_0(-5+7r))\,u^r+\left(\sum_{k=1}^{\infty}(a_{k+1}(k+1+r)\,(k+6+r)-a_{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(5+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-5,0\}$$

- Each term must be 0

$$a_1(1+r)\,(6+r)-a_0(-5+7r)=0$$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+6+r)+(-7a_k+2a_{k-1})k+(-7a_k+2a_{k-1})r+5a_k-3a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+7+r)+(-7a_{k+1}+2a_k)(k+1)+(-7a_{k+1}+2a_k)r+5a_{k+1}-3a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_k k-7ka_{k+1}+2a_k r-7ra_{k+1}-a_k-2a_{k+1}}{(k+2+r)(k+7+r)}$

- Recursion relation for $r=-5$

  $a_{k+2}=-\frac{2a_k k-7ka_{k+1}-11a_k+33a_{k+1}}{(k-3)(k+2)}$

- Series not valid for $r=-5$, division by 0 in the recursion relation at $k=3$

  $a_{k+2}=-\frac{2a_k k-7ka_{k+1}-11a_k+33a_{k+1}}{(k-3)(k+2)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{2a_k k-7ka_{k+1}-a_k-2a_{k+1}}{(k+2)(k+7)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+2}=-\frac{2a_k k-7ka_{k+1}-a_k-2a_{k+1}}{(k+2)(k+7)}, 6a_1+5a_0=0\right]$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^k, a_{k+2}=-\frac{2a_k k-7ka_{k+1}-a_k-2a_{k+1}}{(k+2)(k+7)}, 6a_1+5a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+x)*diff(y(x),x$2)+(1-3*x+2*x^2)*diff(y(x),x)-(x-4)*y(x)=0,y(1) = -2, D(y)(1) = 3]
```

$$y(x) = -2 + 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{17}{12}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{8}(x-1)^5 + \mathrm{O}\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(1+x)*y''[x]+(1-3*x+x^2)*y'[x]-(x-4)*y[x]==0,{y[1]==-2,y'[1]==3}},y[x
```

$$y(x) \to -\frac{13}{240}(x-1)^5 - \frac{1}{96}(x-1)^4 - \frac{2}{3}(x-1)^3 + \frac{9}{4}(x-1)^2 + 3(x-1) - 2$$

13.45 problem 44

Internal problem ID [1286]
Internal file name [`OUTPUT/1287_Sunday_June_05_2022_02_08_15_AM_98761729/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (3x^2 + 12x + 13)\, y' + (2x + 5)\, y = 0}$$

With initial conditions

$$[y(-2) = 2, y'(-2) = -3]$$

With the expansion point for the power series method at $x = -2$.

13.45.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3x^2 + 12x + 13 \\ q(x) &= 2x + 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (3x^2 + 12x + 13)\, y' + (2x + 5)\, y = 0$$

The domain of $p(x) = 3x^2 + 12x + 13$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = 2x + 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = 2 + x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + \left(3(t-2)^2 + 12t - 11\right)\left(\frac{d}{dt}y(t)\right) + (1 + 2t)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 2 \\ y'(0) &= -3 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1061)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1062)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -3\left(\frac{d}{dt}y(t)\right)t^2 - 2y(t)\,t - \frac{d}{dt}y(t) - y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(9t^4 + 6t^2 - 8t\right)\left(\frac{d}{dt}y(t)\right) + 6\left(t^3 + \frac{1}{2}t^2 + \frac{1}{3}t - \frac{1}{6}\right)y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \left(-27t^6 - 27t^4 + 66t^3 - 3t^2 + 22t - 9\right)\left(\frac{d}{dt}y(t)\right) - 18\left(t^5 + \frac{1}{2}t^4 + \frac{2}{3}t^3 - \frac{14}{9}t^2 - \frac{7}{9}t - \frac{1}{9}\right)y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(81t^8 + 108t^6 - 378t^5 + 27t^4 - 252t^3 + 256t^2 - 14t + 33\right)\left(\frac{d}{dt}y(t)\right) + 54\left(t^7 + \frac{1}{2}t^6 + t^5 - \frac{65}{18}t^4 - \frac{1}{9}\right. \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \left(-243t^{10} - 405t^8 + 1836t^7 - 162t^6 + 1836t^5 - 2880t^4 + 306t^3 - 1188t^2 + 578t - 24\right)\left(\frac{d}{dt}y(t)\right) - 1
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = -3$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -2 \\
F_2 &= 31 \\
F_3 &= -53 \\
F_4 &= 110
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = 2 - 3t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{31t^4}{24} - \frac{53t^5}{120} + \frac{11t^6}{72} + O\left(t^6\right)
$$

$$
y(t)=2-3t+\frac{t^{2}}{2}-\frac{t^{3}}{3}+\frac{31t^{4}}{24}-\frac{53t^{5}}{120}+\frac{11t^{6}}{72}+O(t^{6})
$$

Since the expansion point $t=0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t)=\sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -3\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) t^2 - 2\left(\sum_{n=0}^{\infty} a_n t^n\right) t - \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} 3n\, t^{1+n} a_n\right) \\
&+ \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) + \left(\sum_{n=0}^{\infty} 2t^{1+n} a_n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n \\
\sum_{n=1}^{\infty} 3n\, t^{1+n} a_n &= \sum_{n=2}^{\infty} 3(n-1)\, a_{n-1} t^n \\
\sum_{n=1}^{\infty} n a_n t^{n-1} &= \sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n \\
\sum_{n=0}^{\infty} 2t^{1+n} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} t^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^{n}\right) + \left(\sum_{n=2}^{\infty} 3(n-1)\, a_{n-1} t^{n}\right) + \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^{n}\right) + \left(\sum_{n=0}^{\infty} a_{n} t^{n}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} t^{n}\right) = 0 \tag{3}
$$

$n = 0$ gives

$$
2a_2 + a_1 + a_0 = 0
$$

$$
a_2 = -\frac{a_0}{2} - \frac{a_1}{2}
$$

$n = 1$ gives

$$
6a_3 + 2a_2 + a_1 + 2a_0 = 0
$$

Which after substituting earlier equations, simplifies to

$$
a_3 = -\frac{a_0}{6}
$$

For $2 \leq n$, the recurrence equation is

$$
(n+2)\, a_{n+2}(1+n) + 3(n-1)\, a_{n-1} + (1+n)\, a_{1+n} + a_n + 2a_{n-1} = 0 \tag{4}
$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n a_{1+n} + 3n a_{n-1} + a_n + a_{1+n} - a_{n-1}}{(n+2)\,(1+n)} \\
&= -\frac{a_n}{(n+2)\,(1+n)} - \frac{a_{1+n}}{n+2} - \frac{(3n-1)\, a_{n-1}}{(n+2)\,(1+n)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$
12a_4 + 5a_1 + 3a_3 + a_2 = 0
$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_1}{8} + \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 8a_2 + 4a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{23a_0}{120} + \frac{11a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 11a_3 + 5a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19a_0}{720} - \frac{a_1}{30}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 14a_4 + 6a_6 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{69a_1}{560} - \frac{13a_0}{360}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) t^2 - \frac{a_0 t^3}{6} + \left(-\frac{3a_1}{8} + \frac{a_0}{12}\right) t^4 + \left(\frac{23a_0}{120} + \frac{11a_1}{40}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1-\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{12}t^4+\frac{23}{120}t^5\right)a_0+\left(t-\frac{1}{2}t^2-\frac{3}{8}t^4+\frac{11}{40}t^5\right)a_1+O(t^6) \quad (3)$$

At $t=0$ the solution above becomes

$$y(t) = \left(1-\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{12}t^4+\frac{23}{120}t^5\right)c_1+\left(t-\frac{1}{2}t^2-\frac{3}{8}t^4+\frac{11}{40}t^5\right)c_2+O(t^6)$$

$$y(t) = 2+\frac{t^2}{2}-\frac{t^3}{3}+\frac{31t^4}{24}-\frac{53t^5}{120}-3t+O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t=2+x$ results in

$$y = -4-3x+\frac{(2+x)^2}{2}-\frac{(2+x)^3}{3}+\frac{31(2+x)^4}{24}-\frac{53(2+x)^5}{120}+\frac{11(2+x)^6}{72}+O\left((2+x)^6\right)$$

Summary

The solution(s) found are the following

$$y = -4-3x+\frac{(2+x)^2}{2}-\frac{(2+x)^3}{3}+\frac{31(2+x)^4}{24}-\frac{53(2+x)^5}{120}+\frac{11(2+x)^6}{72}+O\left((2+x)^6\right) \quad (1)$$

Verification of solutions

$$y = -4-3x+\frac{(2+x)^2}{2}-\frac{(2+x)^3}{3}+\frac{31(2+x)^4}{24}-\frac{53(2+x)^5}{120}+\frac{11(2+x)^6}{72}+O\left((2+x)^6\right)$$

Verified OK.

13.45.2 Maple step by step solution

Let's solve

$$\left[y''+(3x^2+12x+13)\,y'+(2x+5)\,y=0, y(-2)=2, y'\Big|_{\{x=-2\}}=-3\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k + 1 - m)\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k - 1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k + 2)\,(k + 1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 13a_1 + 5a_0 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k + 2)\,(k + 1) + 13a_{k+1}(k + 1) + a_k(12k + 5) + a_{k-1}(3k - 1))\, x^k\right.$

- Each term must be 0

  $2a_2 + 13a_1 + 5a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (12a_k + 3a_{k-1} + 13a_{k+1} + 3a_{k+2})\, k + 5a_k - a_{k-1} + 13a_{k+1} + 2a_{k+2} = 0$

- Shift index using $k- >k + 1$

  $(k + 1)^2\, a_{k+3} + (12a_{k+1} + 3a_k + 13a_{k+2} + 3a_{k+3})\,(k + 1) + 5a_{k+1} - a_k + 13a_{k+2} + 2a_{k+3} = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{3a_k k+12a_{k+1}k+13ka_{k+2}+2a_k+17a_{k+1}+26a_{k+2}}{k^2+5k+6}, 2a_2+13a_1+5a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)+(13+12*x+3*x^2)*diff(y(x),x)+(5+2*x)*y(x)=0,y(-2) = 2, D(y)(-2) = -3]
```

$$y(x)=2-3(2+x)+\frac{1}{2}(2+x)^2-\frac{1}{3}(2+x)^3+\frac{31}{24}(2+x)^4-\frac{53}{120}(2+x)^5+\mathrm{O}\left((2+x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{y''[x]+(13+12*x+3*x^2)*y'[x]+(5+2*x)*y[x]==0,{y[-2]==2,y'[-2]==-3}},y
```

$$y(x) \to -\frac{53}{120}(x+2)^5 + \frac{31}{24}(x+2)^4 - \frac{1}{3}(x+2)^3 + \frac{1}{2}(x+2)^2 - 3(x+2) + 2$$

13.46 problem 45

Internal problem ID [1287]
Internal file name [OUTPUT/1288_Sunday_June_05_2022_02_08_18_AM_70568683/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2+2x+1)\,y''+(-x^2+2)\,y'+(x+1)\,y=0}$$

With initial conditions

$$[y(0)=1, y'(0)=-2]$$

With the expansion point for the power series method at $x=0$.

13.46.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{-x^2+2}{3x^2+2x+1} \\ q(x) &= \frac{x+1}{3x^2+2x+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-x^2+2)\, y'}{3x^2+2x+1} + \frac{(x+1)\, y}{3x^2+2x+1} = 0$$

The domain of $p(x) = \frac{-x^2+2}{3x^2+2x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{x+1}{3x^2+2x+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1064}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1065}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y'x^2 - yx - 2y' - y}{3x^2 + 2x + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^4 - 3x^3 - 7x^2 + 11x + 7)\, y' - y(x^3 - 2x^2 - 8x - 3)}{(3x^2 + 2x + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(x^6 - 6x^5 + 8x^4 + 84x^3 - 60x^2 - 128x - 28)\, y' - y(x^5 - 5x^4 + 4x^3 + 79x^2 + 66x + 11)}{(3x^2 + 2x + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^8 - 9x^7 + 43x^6 + 27x^5 - 1086x^4 + 96x^3 + 2140x^2 + 1064x + 85)\, y' - y(x^7 - 8x^6 + 36x^5 + 51x^4 -}{(3x^2 + 2x + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(x^{10} - 12x^9 + 98x^8 - 352x^7 - 1523x^6 + 16080x^5 + 8816x^4 - 37212x^3 - 32483x^2 - 5756x + 242)\, y'}{(3x^2 + 2x +}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = -2$ gives

$$
\begin{aligned}
F_0 &= 3 \\
F_1 &= -11 \\
F_2 &= 45 \\
F_3 &= -142 \\
F_4 &= -333
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 - 2x + \frac{3x^2}{2} - \frac{11x^3}{6} + \frac{15x^4}{8} - \frac{71x^5}{60} - \frac{37x^6}{80} + O(x^6)
$$

$$
y = 1-2x+\frac{3x^2}{2}-\frac{11x^3}{6}+\frac{15x^4}{8}-\frac{71x^5}{60}-\frac{37x^6}{80}+O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(3x^2+2x+1)\,y''+(-x^2+2)\,y'+(x+1)\,y = 0
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$
$$
y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}
$$

Substituting the above back into the ode gives

$$
(3x^2+2x+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+(-x^2+2)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+(x+1)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) \\
&+\sum_{n=1}^{\infty}\left(-n\, x^{1+n} a_n\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-n\, x^{1+n} a_n\right) = \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right)$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{gathered}\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, x^n\right) \\ + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} x^n\right) \\ + \left(\sum_{n=0}^{\infty} 2(1+n)\, a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0\end{gathered} \tag{3}$$

$n = 0$ gives

$$2a_2 + 2a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - a_1$$

$n = 1$ gives

$$8a_2 + 6a_3 + a_0 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{2} + \frac{7a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$\begin{aligned} 3na_n(n-1) + 2(1+n)\, a_{1+n}n + (n+2)\, a_{n+2}(1+n) \\ -(n-1)\, a_{n-1} + 2(1+n)\, a_{1+n} + a_{n-1} + a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{3n^2a_n + 2n^2a_{1+n} - 3na_n + 4na_{1+n} - na_{n-1} + a_n + 2a_{1+n} + 2a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{(3n^2-3n+1)\, a_n}{(n+2)\,(1+n)} - \frac{(2n^2+4n+2)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{(-n+2)\, a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 18a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{11a_0}{24} - \frac{7a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$19a_3 + 32a_4 + 20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_0}{30} + \frac{17a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$37a_4 + 50a_5 + 30a_6 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{151a_0}{720} + \frac{121a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$61a_5 + 72a_6 + 42a_7 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{737a_0}{1008} - \frac{8509a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1x + \left(-\frac{a_0}{2} - a_1\right)x^2 + \left(\frac{a_0}{2} + \frac{7a_1}{6}\right)x^3 \\ + \left(-\frac{11a_0}{24} - \frac{7a_1}{6}\right)x^4 + \left(\frac{7a_0}{30} + \frac{17a_1}{24}\right)x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4 + \frac{7}{30}x^5\right)a_0 + \left(x - x^2 + \frac{7}{6}x^3 - \frac{7}{6}x^4 + \frac{17}{24}x^5\right)a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{11}{24}x^4 + \frac{7}{30}x^5\right)c_1 + \left(x - x^2 + \frac{7}{6}x^3 - \frac{7}{6}x^4 + \frac{17}{24}x^5\right)c_2 + O(x^6)$$

$$y = 1 + \frac{3x^2}{2} - \frac{11x^3}{6} + \frac{15x^4}{8} - \frac{71x^5}{60} - 2x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1-2x+\frac{3x^{2}}{2}-\frac{11x^{3}}{6}+\frac{15x^{4}}{8}-\frac{71x^{5}}{60}-\frac{37x^{6}}{80}+O\left(x^{6}\right) \tag{1}$$

$$y = 1+\frac{3x^{2}}{2}-\frac{11x^{3}}{6}+\frac{15x^{4}}{8}-\frac{71x^{5}}{60}-2x+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$y = 1-2x+\frac{3x^{2}}{2}-\frac{11x^{3}}{6}+\frac{15x^{4}}{8}-\frac{71x^{5}}{60}-\frac{37x^{6}}{80}+O\left(x^{6}\right)$$

Verified OK.

$$y = 1+\frac{3x^{2}}{2}-\frac{11x^{3}}{6}+\frac{15x^{4}}{8}-\frac{71x^{5}}{60}-2x+O\left(x^{6}\right)$$

Verified OK.

13.46.2 Maple step by step solution

Let's solve

$\left[\left(3x^{2}+2x+1\right)y''+\left(-x^{2}+2\right)y'+\left(x+1\right)y=0, y(0)=1, y'\Big|_{\{x=0\}}=-2\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y''=\frac{(x^{2}-2)y'}{3x^{2}+2x+1}-\frac{(x+1)y}{3x^{2}+2x+1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^{2}-2)y'}{3x^{2}+2x+1}+\frac{(x+1)y}{3x^{2}+2x+1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^{2}-2}{3x^{2}+2x+1}, P_3(x)=\frac{x+1}{3x^{2}+2x+1}\right]$
- ◦ $\left(\frac{\mathrm{I}\sqrt{2}}{3}+x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}-\frac{\mathrm{I}\sqrt{2}}{3}$

  $\left(\left(\frac{\mathrm{I}\sqrt{2}}{3}+x+\frac{1}{3}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{3}-\frac{\mathrm{I}\sqrt{2}}{3}}=0$

- $\left(\frac{I\sqrt{2}}{3}+x+\frac{1}{3}\right)^{2}\cdot P_{3}(x)$ is analytic at $x=-\frac{1}{3}-\frac{I\sqrt{2}}{3}$

  $\left(\left(\frac{I\sqrt{2}}{3}+x+\frac{1}{3}\right)^{2}\cdot P_{3}(x)\right)\Big|_{x=-\frac{1}{3}-\frac{I\sqrt{2}}{3}}=0$

- $x=-\frac{1}{3}-\frac{I\sqrt{2}}{3}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{3}-\frac{I\sqrt{2}}{3}$

- Multiply by denominators

  $(3x^2+2x+1)\,y''+(-x^2+2)\,y'+(x+1)\,y=0$

- Change variables using $x=u-\frac{1}{3}-\frac{I\sqrt{2}}{3}$ so that the regular singular point is at $u=0$

  $(3u^2-2\,\mathrm{I}u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right)+\left(-u^2+\frac{2u}{3}+\frac{2\,\mathrm{I}u\sqrt{2}}{3}+\frac{19}{9}-\frac{2\,\mathrm{I}\sqrt{2}}{9}\right)\left(\frac{d}{du}y(u)\right)+\left(u+\frac{2}{3}-\frac{I\sqrt{2}}{3}\right)y(u)=$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-\frac{\mathrm{I}\sqrt{2}\left(19\,\mathrm{I}\sqrt{2}-32+36r\right)r a_0 u^{-1+r}}{18}+\left(-\frac{\mathrm{I}\sqrt{2}\left(19\,\mathrm{I}\sqrt{2}+4+36r\right)(1+r)a_1}{18}-\frac{\left(\mathrm{I}\sqrt{2}+4-9r\right)\left(\mathrm{I}\sqrt{2}-1+3r\right)a_0}{9}\right)u^r+\left(\sum_{k=1}^{\infty}\left(-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\frac{\mathrm{I}}{18}\sqrt{2}\left(19\,\mathrm{I}\sqrt{2}-32+36r\right)r=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right\}$$

- Each term must be 0

$$-\frac{\mathrm{I}\sqrt{2}\left(19\,\mathrm{I}\sqrt{2}+4+36r\right)(1+r)a_1}{18}-\frac{\left(\mathrm{I}\sqrt{2}+4-9r\right)\left(\mathrm{I}\sqrt{2}-1+3r\right)a_0}{9}=0$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{2\,\mathrm{I}\left(-3\left(k+r+\frac{1}{9}\right)(k+1+r)a_{k+1}+\left(k+r-\frac{1}{2}\right)a_k\right)\sqrt{2}}{3}+\frac{19(k+1+r)a_{k+1}}{9}+\frac{\left(2+9k^2+(18r-7)k+9r^2-7r\right)a_k}{3}-a_{k-1}(k-2+r)$$

- Shift index using $k->k+1$

$$\frac{2\,\mathrm{I}\left(-3\left(k+\frac{10}{9}+r\right)(k+2+r)a_{k+2}+\left(k+\frac{1}{2}+r\right)a_{k+1}\right)\sqrt{2}}{3}+\frac{19(k+2+r)a_{k+2}}{9}+\frac{\left(2+9(k+1)^2+(18r-7)(k+1)+9r^2-7r\right)a_{k+1}}{3}-a_k(k$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{3\left(2\,\mathrm{I}\sqrt{2}\,a_{k+1}k+2\,\mathrm{I}\sqrt{2}\,a_{k+1}r+\mathrm{I}\sqrt{2}\,a_{k+1}+9k^2a_{k+1}+18kra_{k+1}+9r^2a_{k+1}-3ka_k+11ka_{k+1}-3ra_k+11ra_{k+1}+3a_k+4a_{k+1}\right)}{-38+56\,\mathrm{I}\sqrt{2}\,k+56\,\mathrm{I}\sqrt{2}\,r+40\,\mathrm{I}\sqrt{2}+18\,\mathrm{I}\sqrt{2}\,r^2+36\,\mathrm{I}\sqrt{2}\,kr+18\,\mathrm{I}\sqrt{2}\,k^2-19k-19r}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{3\left(2\,\mathrm{I}\sqrt{2}\,a_{k+1}k+\mathrm{I}\sqrt{2}\,a_{k+1}+9k^2a_{k+1}-3ka_k+11ka_{k+1}+3a_k+4a_{k+1}\right)}{-38+56\,\mathrm{I}\sqrt{2}\,k+40\,\mathrm{I}\sqrt{2}+18\,\mathrm{I}\sqrt{2}\,k^2-19k}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+2}=\frac{3\left(2\,\mathrm{I}\sqrt{2}\,a_{k+1}k+\mathrm{I}\sqrt{2}\,a_{k+1}+9k^2a_{k+1}-3ka_k+11ka_{k+1}+3a_k+4a_{k+1}\right)}{-38+56\,\mathrm{I}\sqrt{2}\,k+40\,\mathrm{I}\sqrt{2}+18\,\mathrm{I}\sqrt{2}\,k^2-19k}, -\frac{\mathrm{I}\sqrt{2}\left(19\,\mathrm{I}\sqrt{2}+4\right)a_1}{18}-\right.$$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{2}}{3}+x+\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{2}}{3}+x+\frac{1}{3}\right)^k, a_{k+2}=\frac{3\left(2\,\mathrm{I}\sqrt{2}\,a_{k+1}k+\mathrm{I}\sqrt{2}\,a_{k+1}+9k^2a_{k+1}-3ka_k+11ka_{k+1}+3a_k+4a_{k+1}\right)}{-38+56\,\mathrm{I}\sqrt{2}\,k+40\,\mathrm{I}\sqrt{2}+18\,\mathrm{I}\sqrt{2}\,k^2-19k}, -\frac{\mathrm{I}\sqrt{2}\left(19\,\mathrm{I}\right.}{1}\right.$$

- Recursion relation for $r=\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}$

$$a_{k+2}=\frac{3\left(2\,\mathrm{I}\sqrt{2}\,a_{k+1}k+2\,\mathrm{I}\sqrt{2}\,a_{k+1}\left(\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right)+\mathrm{I}\sqrt{2}\,a_{k+1}+9k^2a_{k+1}+18k\left(\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right)a_{k+1}+9\left(\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right)^2a_{k+1}-3ka_k+11ka_k\right.}{-\frac{494}{9}+56\,\mathrm{I}\sqrt{2}\,k+56\,\mathrm{I}\sqrt{2}\left(\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right)+\frac{1801\,\mathrm{I}\sqrt{2}}{36}+18\,\mathrm{I}\sqrt{2}\left(\frac{8}{9}-\frac{19\,\mathrm{I}\sqrt{2}}{36}\right)^2+36\,\mathrm{I}\sqrt{2}\,k\left(\frac{8}{9}-\frac{1}{}\right.}$$

- Solution for $r = \frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{8}{9}-\frac{19 \mathrm{I}\sqrt{2}}{36}}, a_{k+2} = \frac{3\left(2 \mathrm{I}\sqrt{2}\, a_{k+1} k + 2 \mathrm{I}\sqrt{2}\, a_{k+1}\left(\frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}\right) + \mathrm{I}\sqrt{2}\, a_{k+1} + 9k^2 a_{k+1} + 18k\left(\frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}\right) a_{k+1}\right.}{-\frac{494}{9} + 56 \mathrm{I}\sqrt{2}\, k + 56 \mathrm{I}\sqrt{2}\left(\frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}\right) + \frac{1801 \mathrm{I}\sqrt{2}}{36} + 1}\right.$$

- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{2}}{3} + x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{\mathrm{I}\sqrt{2}}{3} + x + \frac{1}{3}\right)^{k+\frac{8}{9}-\frac{19 \mathrm{I}\sqrt{2}}{36}}, a_{k+2} = \frac{3\left(2 \mathrm{I}\sqrt{2}\, a_{k+1} k + 2 \mathrm{I}\sqrt{2}\, a_{k+1}\left(\frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}\right) + \mathrm{I}\sqrt{2}\, a_{k+1} + 9k^2 a_{k+1} + 18k\left(\frac{8}{9}\right.\right.}{-\frac{494}{9} + 56 \mathrm{I}\sqrt{2}\, k + 56 \mathrm{I}\sqrt{2}\left(\frac{8}{9} - \frac{19 \mathrm{I}\sqrt{2}}{36}\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{\mathrm{I}\sqrt{2}}{3} + x + \frac{1}{3}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{\mathrm{I}\sqrt{2}}{3} + x + \frac{1}{3}\right)^{k+\frac{8}{9}-\frac{19 \mathrm{I}\sqrt{2}}{36}}\right), a_{k+2} = \frac{3\left(2 \mathrm{I}\sqrt{2}\, a_{1+k} k + \mathrm{I}\sqrt{2}\, a_{1+k} +\right.}{-38 + 56 \mathrm{I}\sqrt{2}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+2*x+3*x^2)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(1+x)*y(x)=0,y(0) = 1, D(y)(0) = -
```

$$y(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{11}{6}x^3 + \frac{15}{8}x^4 - \frac{71}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(1+2*x+3*x^2)*y''[x]+(2-x^2)*y'[x]+(1+x)*y[x]==0,{y[0]==1,y'[0]==-2}}
```

$$y(x) \to -\frac{71x^5}{60} + \frac{15x^4}{8} - \frac{11x^3}{6} + \frac{3x^2}{2} - 2x + 1$$

13.47 problem 46

Internal problem ID [1288]
Internal file name [`OUTPUT/1289_Sunday_June_05_2022_02_08_21_AM_59726315/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+4x+3)\,y''-(-x^2+4x+5)\,y'-(2+x)\,y=0}$$

With initial conditions

$$[y(-2)=2, y'(-2)=-1]$$

With the expansion point for the power series method at $x=-2$.

13.47.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{x^2-4x-5}{x^2+4x+3}\\ q(x)&=\frac{-x-2}{x^2+4x+3}\\ F&=0\end{aligned}$$

Hence the ode is

$$y''+\frac{(x^2-4x-5)\,y'}{x^2+4x+3}+\frac{(-x-2)\,y}{x^2+4x+3}=0$$

The domain of $p(x)=\frac{x^2-4x-5}{x^2+4x+3}$ is

$$\{-\infty\le x<-3,-3<x<-1,-1<x\le\infty\}$$

And the point $x_0=-2$ is inside this domain. The domain of $q(x)=\frac{-x-2}{x^2+4x+3}$ is

$$\{-\infty\le x<-3,-3<x<-1,-1<x\le\infty\}$$

And the point $x_0=-2$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=2+x$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left((t-2)^2+4t-5\right)\left(\frac{d^2}{dt^2}y(t)\right)+\left((t-2)^2-4t+3\right)\left(\frac{d}{dt}y(t)\right)-y(t)\,t=0$$

With its expansion point and initial conditions now at $t=0$. With initial conditions now becoming

$$y(0)=2$$
$$y'(0)=-1$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1067) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1068) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\left(\frac{d}{dt}y(t)\right)t^2 - 8t\left(\frac{d}{dt}y(t)\right) - y(t)\,t + 7\frac{d}{dt}y(t)}{t^2 - 1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{(t^4 - 15t^3 + 70t^2 - 97t + 41)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^3 - 7t^2 + 7t + 1)}{(t^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(-t^6 + 22t^5 - 175t^4 + 612t^3 - 979t^2 + 710t - 189)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^5 - 14t^4 + 56t^3 - 73t^2 + 31t +}{(t^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(t+1)\,(-1+t)\left((-1+t)\,(t^7 - 28t^6 + 294t^5 - 1461t^4 + 3634t^3 - 4601t^2 + 2815t - 606)\left(\frac{d}{dt}y(t)\right) -\right.}{(t^2 - 1)^5} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= -\frac{(t+1)\,(-1+t)\left((-1+t)\,(t^9 - 35t^8 + 476t^7 - 3260t^6 + 12243t^5 - 26241t^4 + 32822t^3 - 23242t^2 +\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -7 \\
F_1 &= -43 \\
F_2 &= -203 \\
F_3 &= -668 \\
F_4 &= -960
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -t + 2 - \frac{7t^2}{2} - \frac{43t^3}{6} - \frac{203t^4}{24} - \frac{167t^5}{30} - \frac{4t^6}{3} + O(t^6)$$

$$y(t) = -t + 2 - \frac{7t^2}{2} - \frac{43t^3}{6} - \frac{203t^4}{24} - \frac{167t^5}{30} - \frac{4t^6}{3} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(t^2 - 1) + (t^2 - 8t + 7)\left(\frac{d}{dt}y(t)\right) - y(t)\,t = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(t^2 - 1) + (t^2 - 8t + 7)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) t = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty}\left(-n(n-1)\,a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\,t^{1+n} a_n\right) \\ + \sum_{n=1}^{\infty}\left(-8n a_n t^n\right) + \left(\sum_{n=1}^{\infty} 7n a_n t^{n-1}\right) + \sum_{n=0}^{\infty}\left(-t^{1+n} a_n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right)$$

$$\sum_{n=1}^{\infty} n\, t^{1+n} a_n = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} t^n$$

$$\sum_{n=1}^{\infty} 7n a_n t^{n-1} = \sum_{n=0}^{\infty} 7(1+n)\, a_{1+n} t^n$$

$$\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} t^n\right) + \sum_{n=1}^{\infty} \left(-8n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 7(1+n)\, a_{1+n} t^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 7a_1 = 0$$

$$a_2 = \frac{7a_1}{2}$$

$n = 1$ gives

$$-6a_3 - 8a_1 + 14a_2 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} + \frac{41a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_{n}(n-1)-(n+2) a_{n+2}(1+n)+(n-1) a_{n-1}-8 n a_{n}+7(1+n) a_{1+n}-a_{n-1}=0 \quad (4)$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{n^{2} a_{n}-9 n a_{n}+7 n a_{1+n}+n a_{n-1}+7 a_{1+n}-2 a_{n-1}}{(n+2)(1+n)} \\
&= \frac{(n^{2}-9 n) a_{n}}{(n+2)(1+n)}+\frac{(7 n+7) a_{1+n}}{(n+2)(1+n)}+\frac{(n-2) a_{n-1}}{(n+2)(1+n)}
\end{aligned} \quad (5)
$$

For $n = 2$ the recurrence equation gives

$$-14 a_{2}-12 a_{4}+21 a_{3}=0$$

Which after substituting the earlier terms found becomes

$$a_{4}=\frac{63 a_{1}}{8}-\frac{7 a_{0}}{24}$$

For $n = 3$ the recurrence equation gives

$$-18 a_{3}-20 a_{5}+a_{2}+28 a_{4}=0$$

Which after substituting the earlier terms found becomes

$$a_{5}=-\frac{31 a_{0}}{120}+\frac{101 a_{1}}{20}$$

For $n = 4$ the recurrence equation gives

$$-20 a_{4}-30 a_{6}+2 a_{3}+35 a_{5}=0$$

Which after substituting the earlier terms found becomes

$$a_{6}=\frac{79 a_{1}}{72}-\frac{17 a_{0}}{144}$$

For $n = 5$ the recurrence equation gives

$$-20 a_{5}-42 a_{7}+3 a_{4}+42 a_{6}=0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_0}{63} - \frac{751a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{7a_1 t^2}{2} + \left(-\frac{a_0}{6} + \frac{41a_1}{6}\right) t^3 + \left(\frac{63a_1}{8} - \frac{7a_0}{24}\right) t^4 + \left(-\frac{31a_0}{120} + \frac{101a_1}{20}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{6}t^3 - \frac{7}{24}t^4 - \frac{31}{120}t^5\right) a_0 + \left(t + \frac{7}{2}t^2 + \frac{41}{6}t^3 + \frac{63}{8}t^4 + \frac{101}{20}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{6}t^3 - \frac{7}{24}t^4 - \frac{31}{120}t^5\right) c_1 + \left(t + \frac{7}{2}t^2 + \frac{41}{6}t^3 + \frac{63}{8}t^4 + \frac{101}{20}t^5\right) c_2 + O(t^6)$$

$$y(t) = 2 - \frac{43t^3}{6} - \frac{203t^4}{24} - \frac{167t^5}{30} - t - \frac{7t^2}{2} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = 2 + x$ results in

$$y = -x - \frac{7(2+x)^2}{2} - \frac{43(2+x)^3}{6} - \frac{203(2+x)^4}{24} - \frac{167(2+x)^5}{30} - \frac{4(2+x)^6}{3} + O\left((2+x)^6\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = -x - \frac{7(2+x)^2}{2} - \frac{43(2+x)^3}{6} - \frac{203(2+x)^4}{24} \\ - \frac{167(2+x)^5}{30} - \frac{4(2+x)^6}{3} + O\left((2+x)^6\right) \end{aligned} \quad (1)$$

Verification of solutions

$$y = -x - \frac{7(2+x)^2}{2} - \frac{43(2+x)^3}{6} - \frac{203(2+x)^4}{24} - \frac{167(2+x)^5}{30} - \frac{4(2+x)^6}{3} + O\left((2+x)^6\right)$$

Verified OK.

13.47.2 Maple step by step solution

Let's solve

$$\left[(x^2+4x+3)\, y'' + (x^2-4x-5)\, y' + (-x-2)\, y = 0, y(-2) = 2, y'\Big|_{\{x=-2\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+x)y}{x^2+4x+3} - \frac{y'(x-5)}{x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'(x-5)}{x+3} - \frac{(2+x)y}{x^2+4x+3} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-5}{x+3}, P_3(x) = -\frac{2+x}{x^2+4x+3}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = -8$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3} = 0$

  ◦ $x=-3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -3$

- Multiply by denominators

  $(x+3)\, y''(x^2+4x+3) + (x-5)\,(x^2+4x+3)\, y' - (2+x)\,(x+3)\, y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

$$\left(u^3 - 2u^2\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(u^3 - 10u^2 + 16u\right)\left(\frac{d}{du}y(u)\right) + \left(-u^2 + u\right)y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 1..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, u^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-9 + r)\, u^r + \left(-2a_1(1 + r)\,(-8 + r) + a_0\left(r^2 - 11r + 1\right)\right) u^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k + r)\,(k + r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-9 + r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 9\}$$

- Each term must be 0

$$-2a_1(1 + r)\,(-8 + r) + a_0\left(r^2 - 11r + 1\right) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{a_0(r^2-11r+1)}{2(r^2-7r-8)}$$

- Each term in the series must be 0, giving the recursion relation

$$(-2a_k + a_{k-1})\, k^2 + ((-4a_k + 2a_{k-1})\, r + 18a_k + a_{k-2} - 13a_{k-1})\, k + (-2a_k + a_{k-1})\, r^2 + (18a_k +$$

- Shift index using $k- >k+2$

$$(-2a_{k+2} + a_{k+1})\,(k+2)^2 + ((-4a_{k+2} + 2a_{k+1})\, r + 18a_{k+2} + a_k - 13a_{k+1})\,(k+2) + (-2a_{k+2} + a$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}+ka_k-9ka_{k+1}+ra_k-9ra_{k+1}-a_k-9a_{k+1}}{2(k^2+2kr+r^2-5k-5r-14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2a_{k+1}+ka_k-9ka_{k+1}-a_k-9a_{k+1}}{2(k^2-5k-14)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 7$

$$a_{k+2} = \frac{k^2a_{k+1}+ka_k-9ka_{k+1}-a_k-9a_{k+1}}{2(k^2-5k-14)}$$

- Recursion relation for $r = 9$

$$a_{k+2} = \frac{k^2a_{k+1}+ka_k+9ka_{k+1}+8a_k-9a_{k+1}}{2(k^2+13k+22)}$$

- Solution for $r = 9$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+9}, a_{k+2} = \frac{k^2a_{k+1}+ka_k+9ka_{k+1}+8a_k-9a_{k+1}}{2(k^2+13k+22)}, a_1 = -\frac{17a_0}{20}\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k+9}, a_{k+2} = \frac{k^2a_{k+1}+ka_k+9ka_{k+1}+8a_k-9a_{k+1}}{2(k^2+13k+22)}, a_1 = -\frac{17a_0}{20}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(3+4*x+x^2)*diff(y(x),x$2)-(5+4*x-x^2)*diff(y(x),x)-(2+x)*y(x)=0,y(-2) = 2, D(y)(-2)
```

$$y(x)=2-(2+x)-\frac{7}{2}(2+x)^2-\frac{43}{6}(2+x)^3-\frac{203}{24}(2+x)^4-\frac{167}{30}(2+x)^5+\mathrm{O}\left((2+x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 43

```
AsymptoticDSolveValue[{(3+4*x+x^2)*y''[x]-(5+4*x-x^2)*y'[x]-(2+x)*y[x]==0,{y[-2]==2,y'[-2]==
```

$$y(x) \to -\frac{167}{30}(x+2)^5 - \frac{203}{24}(x+2)^4 - \frac{43}{6}(x+2)^3 - \frac{7}{2}(x+2)^2 - x$$

13.48 problem 47

Internal problem ID [1289]
Internal file name [`OUTPUT/1290_Sunday_June_05_2022_02_08_24_AM_12516137/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2x+1)\, y''+(1-x)\, y=0}$$

With initial conditions

$$[y(0)=2, y'(0)=-1]$$

With the expansion point for the power series method at $x=0$.

13.48.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1-x}{(x+1)^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(1-x)\,y}{(x+1)^2} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1-x}{(x+1)^2}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1070}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1071}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n |_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{(x-1)\,y}{(x+1)^2} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{y'x^2 - yx + 3y - y'}{(x+1)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(x-3)\left((-2x-2)\,y' + y(x+3)\right)}{(x+1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 + 5x^2 - 25x - 29)\,y' - 4\left(x^2 - \frac{5}{2}x - \frac{15}{2}\right) y}{(x+1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-6x^3 - 6x^2 + 150x + 150)\,y' + y(x^3 + 15x^2 - 81x - 111)}{(x+1)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 2$ and $y'(0) = -1$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 7 \\
F_2 &= -24 \\
F_3 &= 89 \\
F_4 &= -372
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 - x + 2 + \frac{7x^3}{6} - x^4 + \frac{89x^5}{120} - \frac{31x^6}{60} + O\left(x^6\right)
$$

$$y = -x^2 - x + 2 + \frac{7x^3}{6} - x^4 + \frac{89x^5}{120} - \frac{31x^6}{60} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 2x + 1) y'' + (1 - x) y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 2x + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + (1 - x) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(1+n) a_{1+n} n\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=1}^{\infty} 2(1+n) a_{1+n} n x^{n}\right)$$
$$+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2}(1+n) x^{n}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n}\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2+a_0=0$$

$$a_2=-\frac{a_0}{2}$$

$n=1$ gives

$$4a_2+6a_3+a_1-a_0=0$$

Which after substituting earlier equations, simplifies to

$$a_3=\frac{a_0}{2}-\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_{n}(n-1)+2(1+n) a_{1+n} n+(n+2) a_{n+2}(1+n)+a_{n}-a_{n-1}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &=-\frac{n^{2} a_{n}+2 n^{2} a_{1+n}-n a_{n}+2 n a_{1+n}+a_{n}-a_{n-1}}{(n+2)(1+n)} \\ &=-\frac{\left(n^{2}-n+1\right) a_{n}}{(n+2)(1+n)}-\frac{\left(2 n^{2}+2 n\right) a_{1+n}}{(n+2)(1+n)}+\frac{a_{n-1}}{(n+2)(1+n)} \end{aligned} \tag{5}$$

For $n=2$ the recurrence equation gives

$$3a_2+12a_3+12a_4-a_1=0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_0}{8} + \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 24a_4 + 20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{4} - \frac{29a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$13a_4 + 40a_5 + 30a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{37a_0}{240} + \frac{5a_1}{24}$$

For $n = 5$ the recurrence equation gives

$$21a_5 + 60a_6 + 42a_7 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{29a_0}{336} - \frac{41a_1}{240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(\frac{a_0}{2} - \frac{a_1}{6}\right) x^3 + \left(-\frac{3a_0}{8} + \frac{a_1}{4}\right) x^4 + \left(\frac{a_0}{4} - \frac{29a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{3}{8}x^4+\frac{1}{4}x^5\right)a_0+\left(x-\frac{1}{6}x^3+\frac{1}{4}x^4-\frac{29}{120}x^5\right)a_1+O\left(x^6\right) \tag{3}$$

At $x=0$ the solution above becomes

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{3}{8}x^4+\frac{1}{4}x^5\right)c_1+\left(x-\frac{1}{6}x^3+\frac{1}{4}x^4-\frac{29}{120}x^5\right)c_2+O\left(x^6\right)$$

$$y = 2-x^2+\frac{7x^3}{6}-x^4+\frac{89x^5}{120}-x+O\left(x^6\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -x^2-x+2+\frac{7x^3}{6}-x^4+\frac{89x^5}{120}-\frac{31x^6}{60}+O\left(x^6\right) \tag{1}$$
$$y = 2-x^2+\frac{7x^3}{6}-x^4+\frac{89x^5}{120}-x+O\left(x^6\right) \tag{2}$$

<u>Verification of solutions</u>

$$y = -x^2-x+2+\frac{7x^3}{6}-x^4+\frac{89x^5}{120}-\frac{31x^6}{60}+O\left(x^6\right)$$

Verified OK.

$$y = 2-x^2+\frac{7x^3}{6}-x^4+\frac{89x^5}{120}-x+O\left(x^6\right)$$

Verified OK.

13.48.2 Maple step by step solution

Let's solve

$$\left[\left(x^2+2x+1\right)y''+(1-x)\,y=0, y(0)=2, y'\Big|_{\{x=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-1)y}{x^2+2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y}{x^2+2x+1} = 0$

- ☐ Check to see if $x_0 = -1$ is a regular singular point
  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{x-1}{x^2+2x+1}\right]$
  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 0$
  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 2$
  - $x = -1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2 + 2x + 1)\, y'' + (1 - x)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $u^2 \left(\frac{d^2}{du^2} y(u)\right) + (2 - u)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- > k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

$$u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0(r^2-r+2)\, u^r + \left(\sum_{k=1}^{\infty} \left(a_k(k^2+2kr+r^2-k-r+2) - a_{k-1}\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 - r + 2 = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}, \frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + (2r-1)\,k + r^2 - r + 2)\, a_k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2 + (2r-1)(k+1) + r^2 - r + 2\right) a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k^2+2kr+r^2+k+r+2}$$

- Recursion relation for $r = -\frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{k^2+2k\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)+\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)^2+k-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{5}{2}}$$

- Solution for $r = -\frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k^2+2k\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)+\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)^2+k-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{5}{2}}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k^2+2k\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)+\left(-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)^2+k-\frac{\mathrm{I}\sqrt{7}}{2}+\frac{5}{2}}\right]$$

- Recursion relation for $r = \frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{k^2+2k\left(\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)+\left(\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)^2+k+\frac{\mathrm{I}\sqrt{7}}{2}+\frac{5}{2}}$$

- Solution for $r = \frac{\mathrm{I}\sqrt{7}}{2} + \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k^2+2k\left(\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)+\left(\frac{\mathrm{I}\sqrt{7}}{2}+\frac{1}{2}\right)^2+k+\frac{\mathrm{I}\sqrt{7}}{2}+\frac{5}{2}}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{I\sqrt{7}}{2}+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k^2+2k\left(\frac{I\sqrt{7}}{2}+\frac{1}{2}\right)+\left(\frac{I\sqrt{7}}{2}+\frac{1}{2}\right)^2+k+\frac{I\sqrt{7}}{2}+\frac{5}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{I\sqrt{7}}{2}+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{I\sqrt{7}}{2}+\frac{1}{2}}\right), a_{1+k} = \frac{a_k}{k^2+2k\left(-\frac{I\sqrt{7}}{2}+\frac{1}{2}\right)+\left(-\frac{I\sqrt{7}}{2}+\frac{1}{2}\right)^2+k-\frac{I\sqrt{}}{2}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([(1+2*x+x^2)*diff(y(x),x$2)+(1-x)*y(x)=0,y(0) = 2, D(y)(0) = -1],y(x),type='series',x
```

$$y(x) = 2 - x - x^2 + \frac{7}{6}x^3 - x^4 + \frac{89}{120}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(1+2*x+x^2)*y''[x]+(1-x)*y[x]==0,{y[0]==2,y'[0]==-1}},y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{89x^5}{120} - x^4 + \frac{7x^3}{6} - x^2 - x + 2$$

13.49 problem 48

Internal problem ID [1290]
Internal file name [`OUTPUT/1291_Sunday_June_05_2022_02_08_26_AM_77405407/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(-2x^{2}+x\right) y''+\left(-x^{2}+3x+1\right) y'+\left(2+x\right) y=0}$$

With initial conditions

$$[y(1)=1, y'(1)=0]$$

With the expansion point for the power series method at $x=1$.

13.49.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{-x^{2}+3x+1}{-2x^{2}+x} \\ q(x) &= \frac{2+x}{-2x^{2}+x} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(-x^2+3x+1)\, y'}{-2x^2+x} + \frac{(2+x)\, y}{-2x^2+x} = 0$$

The domain of $p(x) = \frac{-x^2+3x+1}{-2x^2+x}$ is

$$\left\{-\infty \le x < 0, 0 < x < \frac{1}{2}, \frac{1}{2} < x \le \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2+x}{-2x^2+x}$ is

$$\left\{-\infty \le x < 0, 0 < x < \frac{1}{2}, \frac{1}{2} < x \le \infty\right\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(-2(t+1)^2 + t + 1\right)\left(\frac{d^2}{dt^2}y(t)\right) + \left(-(t+1)^2 + 3t + 4\right)\left(\frac{d}{dt}y(t)\right) + (t+3)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1073) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1074) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = -\frac{\left(\frac{d}{dt}y(t)\right)t^2 - t\left(\frac{d}{dt}y(t)\right) - y(t)\,t - 3\frac{d}{dt}y(t) - 3y(t)}{2t^2 + 3t + 1}$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\,(t)}F_0 \\
&= \frac{(t^4 - t^2 + 2t + 4)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^3 + 4t^2 + 6t - 1)}{(2t^2 + 3t + 1)^2}
\end{aligned}$$

$$\begin{aligned}
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\,(t)}F_1 \\
&= \frac{(-t^6 - t^5 - t^4 - 20t^3 - 37t^2 - 33t - 9)\left(\frac{d}{dt}y(t)\right) + y(t)\,t(t^4 + 5t^3 + 12t^2 + 32t + 12)}{(2t^2 + 3t + 1)^3}
\end{aligned}$$

$$\begin{aligned}
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\,(t)}F_2 \\
&= \frac{(t^8 + 2t^7 + 3t^6 + 48t^5 + 231t^4 + 360t^3 + 314t^2 + 136t + 21)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^7 + 6t^6 + 18t^5 + 75t^4 +}{(2t^2 + 3t + 1)^4}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\,(t)}F_3 \\
&= -\frac{2(t+1)\left((t^{10} + 3t^9 + 5t^8 + 80t^7 + 640t^6 + 2580t^5 + 4343t^4 + 4203t^3 + 2353t^2 + 664t + 68)\left(\frac{d}{dt}y(t\right.\right.}{(2t^2 + 3t}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$\begin{aligned}
F_0 &= 3 \\
F_1 &= 1 \\
F_2 &= 0 \\
F_3 &= -15 \\
F_4 &= 127
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 + \frac{3t^2}{2} + \frac{t^3}{6} - \frac{t^5}{8} + \frac{127t^6}{720} + O(t^6)$$

$$y(t) = 1 + \frac{3t^2}{2} + \frac{t^3}{6} - \frac{t^5}{8} + \frac{127t^6}{720} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-2t^2 - 3t - 1)\left(\frac{d^2}{dt^2}y(t)\right) + (-t^2 + t + 3)\left(\frac{d}{dt}y(t)\right) + (t + 3)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(-2t^2 - 3t - 1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + (-t^2 + t + 3)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t + 3)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-2t^n a_n n(n-1)\right) + \sum_{n=2}^{\infty}\left(-3n\,t^{n-1}a_n(n-1)\right) \\
&+ \sum_{n=2}^{\infty}\left(-n(n-1)\,a_n t^{n-2}\right) + \sum_{n=1}^{\infty}\left(-n\,t^{1+n}a_n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) \\
&+ \left(\sum_{n=1}^{\infty} 3n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n}a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n t^n\right) = 0
\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} \left(-3n\, t^{n-1} a_n (n-1)\right) &= \sum_{n=1}^{\infty} \left(-3(1+n)\, a_{1+n} n\, t^n\right) \\
\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n t^{n-2}\right) &= \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right) \\
\sum_{n=1}^{\infty} \left(-n\, t^{1+n} a_n\right) &= \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} t^n\right) \\
\sum_{n=1}^{\infty} 3n a_n t^{n-1} &= \sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} t^n \\
\sum_{n=0}^{\infty} t^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} t^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \left(-2t^n a_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-3(1+n)\, a_{1+n} n\, t^n\right) \\
&+ \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, t^n\right) + \sum_{n=2}^{\infty} \left(-(n-1)\, a_{n-1} t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) \\
&+ \left(\sum_{n=0}^{\infty} 3(1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 3 a_n t^n\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$
-2a_2 + 3a_1 + 3a_0 = 0
$$

$$
a_2 = \frac{3a_0}{2} + \frac{3a_1}{2}
$$

$n = 1$ gives

$$
-6a_3 + 4a_1 + a_0 = 0
$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} + \frac{2a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$\begin{aligned} -2na_n(n-1) - 3(1+n)\, a_{1+n}n - (n+2)\, a_{n+2}(1+n) \\ - (n-1)\, a_{n-1} + na_n + 3(1+n)\, a_{1+n} + a_{n-1} + 3a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2a_n + 3n^2a_{1+n} - 3na_n + na_{n-1} - 3a_n - 3a_{1+n} - 2a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{(2n^2-3n-3)\, a_n}{(n+2)\,(1+n)} - \frac{(3n^2-3)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{(n-2)\, a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_2 - 9a_3 - 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3a_1}{8}$$

For $n = 3$ the recurrence equation gives

$$-6a_3 - 24a_4 - 20a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{8} + \frac{7a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$-17a_4 - 45a_5 - 30a_6 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{17a_1}{180} + \frac{127a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$-32a_5 - 72a_6 - 42a_7 - 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{29a_0}{140} + \frac{31a_1}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(\frac{3a_0}{2} + \frac{3a_1}{2}\right) t^2 + \left(\frac{a_0}{6} + \frac{2a_1}{3}\right) t^3 - \frac{3a_1 t^4}{8} + \left(-\frac{a_0}{8} + \frac{7a_1}{40}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{3}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{8}t^5\right) a_0 + \left(t + \frac{3}{2}t^2 + \frac{2}{3}t^3 - \frac{3}{8}t^4 + \frac{7}{40}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{3}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{8}t^5\right) c_1 + \left(t + \frac{3}{2}t^2 + \frac{2}{3}t^3 - \frac{3}{8}t^4 + \frac{7}{40}t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 + \frac{3t^2}{2} + \frac{t^3}{6} - \frac{t^5}{8} + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$y = 1 + \frac{3(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^5}{8} + \frac{127(x-1)^6}{720} + O\left((x-1)^6\right)$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{3(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^5}{8} + \frac{127(x-1)^6}{720} + O\left((x-1)^6\right) \tag{1}$$

Verification of solutions

$$y = 1 + \frac{3(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^5}{8} + \frac{127(x-1)^6}{720} + O\left((x-1)^6\right)$$

Verified OK.

13.49.2 Maple step by step solution

Let's solve

$\left[(-2x^2+x)\, y'' + (-x^2+3x+1)\, y' + (2+x)\, y = 0, y(1) = 1, y'\Big|_{\{x=1\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+x)y}{x(2x-1)} - \frac{(x^2-3x-1)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-3x-1)y'}{x(2x-1)} - \frac{(2+x)y}{x(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2-3x-1}{x(2x-1)}, P_3(x) = -\frac{2+x}{x(2x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x(2x-1)+(x^2-3x-1)\,y'+(-x-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=0..2$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=1..2$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r^2 x^{-1+r}+\left(-a_1(1+r)^2+a_0(2r^2-5r-2)\right)x^r+\left(\sum_{k=1}^{\infty}\left(-a_{k+1}(k+1+r)^2+a_k(2k^2+4kr+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term must be 0

  $-a_1(1+r)^2+a_0(2r^2-5r-2)=0$

- Each term in the series must be 0, giving the recursion relation

  $$(2a_k - a_{k+1})k^2 + (-5a_k + a_{k-1} - 2a_{k+1})k - 2a_k - 2a_{k-1} - a_{k+1} = 0$$

- Shift index using $k- > k+1$

  $$(2a_{k+1} - a_{k+2})(k+1)^2 + (-5a_{k+1} + a_k - 2a_{k+2})(k+1) - 2a_{k+1} - 2a_k - a_{k+2} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2k^2a_{k+1} + ka_k - ka_{k+1} - a_k - 5a_{k+1}}{k^2+4k+4}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2k^2a_{k+1} + ka_k - ka_{k+1} - a_k - 5a_{k+1}}{k^2+4k+4}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k^2a_{k+1} + ka_k - ka_{k+1} - a_k - 5a_{k+1}}{k^2+4k+4}, -a_1 - 2a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([(x-2*x^2)*diff(y(x),x$2)+(1+3*x-x^2)*diff(y(x),x)+(2+x)*y(x)=0,y(1) = 1, D(y)(1) = 0
```

$$y(x) = 1 + \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{8}(x-1)^5 + O\left((x-1)^6\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{(x-2*x^2)*y''[x]+(1+3*x-x^2)*y'[x]+(2+x)*y[x]==0,{y[1]==1,y'[1]==0}},
```

$$y(x) \rightarrow -\frac{1}{8}(x-1)^5 + \frac{1}{6}(x-1)^3 + \frac{3}{2}(x-1)^2 + 1$$

13.50 problem 49

Internal problem ID [1291]
Internal file name [OUTPUT/1292_Sunday_June_05_2022_02_08_30_AM_68085191/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II. Exercises 7.3. Page 338
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2 - 11x + 16)\, y'' + (x^2 - 6x + 10)\, y' - (2 - x)\, y = 0}$$

With initial conditions

$$[y(3) = 1, y'(3) = -2]$$

With the expansion point for the power series method at $x = 3$.

13.50.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{x^2 - 6x + 10}{2x^2 - 11x + 16} \\ q(x) &= \frac{-2 + x}{2x^2 - 11x + 16} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y''+\frac{(x^2-6x+10)\,y'}{2x^2-11x+16}+\frac{(-2+x)\,y}{2x^2-11x+16}=0$$

The domain of $p(x)=\frac{x^2-6x+10}{2x^2-11x+16}$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0=3$ is inside this domain. The domain of $q(x)=\frac{-2+x}{2x^2-11x+16}$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0=3$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t=x-3$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(2(t+3)^2-11t-17\right)\left(\frac{d^2}{dt^2}y(t)\right)+\left((t+3)^2-6t-8\right)\left(\frac{d}{dt}y(t)\right)+(t+1)\,y(t)=0$$

With its expansion point and initial conditions now at $t=0$. With initial conditions now becoming

$$y(0)=1$$
$$y'(0)=-2$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the

case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1076) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1077) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\left(\frac{d}{dt}y(t)\right)t^2 + y(t)\,t + \frac{d}{dt}y(t) + y(t)}{2t^2 + t + 1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{(t^4 - 2t^3 - 2t^2 + 1)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^3 + 3t^2 + 5t + 1)}{\left(2t^2 + t + 1\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(-t^6 + 4t^5 + 14t^4 + 26t^3 + 5t^2 - 6t - 2)\left(\frac{d}{dt}y(t)\right) - y(t)\,(t^5 + t^4 + 7t^3 + 25t^2 + 8t - 2)}{\left(2t^2 + t + 1\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(t^8 - 6t^7 - 27t^6 - 100t^5 - 199t^4 - 52t^3 + 101t^2 + 46t + 4)\left(\frac{d}{dt}y(t)\right) + y(t)\,(t^7 - t^6 - 16t^5 - 4t^4 + 16}{\left(2t^2 + t + 1\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(-t^{10} + 8t^9 + 41t^8 + 172t^7 + 705t^6 + 1792t^5 + 449t^4 - 1800t^3 - 1096t^2 - 108t + 14)\left(\frac{d}{dt}y(t)\right) - y(t}{\left(2t^2 + t + 1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = -2$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -1 \\
F_2 &= 6 \\
F_3 &= -20 \\
F_4 &= -34
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = 1 + \frac{t^2}{2} - 2t - \frac{t^3}{6} + \frac{t^4}{4} - \frac{t^5}{6} - \frac{17t^6}{360} + O(t^6)$$

$$y(t) = 1 + \frac{t^2}{2} - 2t - \frac{t^3}{6} + \frac{t^4}{4} - \frac{t^5}{6} - \frac{17t^6}{360} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(2t^2 + t + 1) + (t^2 + 1)\left(\frac{d}{dt}y(t)\right) + (t + 1)\,y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right)(2t^2 + t + 1) + (t^2 + 1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + (t + 1)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) \\ + \left(\sum_{n=1}^{\infty} n\, t^{1+n} a_n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} t^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0\end{aligned} \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) &= \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n \\
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, t^n \\
\sum_{n=1}^{\infty} n\, t^{1+n} a_n &= \sum_{n=2}^{\infty} (n-1)\, a_{n-1} t^n \\
\sum_{n=1}^{\infty} n a_n t^{n-1} &= \sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n \\
\sum_{n=0}^{\infty} t^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} t^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^n\right) \\
&+ \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, t^n\right) + \left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} t^n\right) \\
&+ \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} t^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$
2a_2 + a_1 + a_0 = 0
$$

$$
a_2 = -\frac{a_0}{2} - \frac{a_1}{2}
$$

$n = 1$ gives

$$
4a_2 + 6a_3 + a_0 + a_1 = 0
$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} + \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$\begin{aligned} &2na_n(n-1) + (1+n)\, a_{1+n} n + (n+2)\, a_{n+2}(1+n) \\ &+ (n-1)\, a_{n-1} + (1+n)\, a_{1+n} + a_{n-1} + a_n = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{2n^2 a_n + n^2 a_{1+n} - 2na_n + 2na_{1+n} + na_{n-1} + a_n + a_{1+n}}{(n+2)\,(1+n)} \\ &= -\frac{(2n^2 - 2n + 1)\, a_n}{(n+2)\,(1+n)} - \frac{(n^2 + 2n + 1)\, a_{1+n}}{(n+2)\,(1+n)} - \frac{na_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$5a_2 + 9a_3 + 12a_4 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{12} - \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$13a_3 + 16a_4 + 20a_5 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{10} + \frac{a_1}{30}$$

For $n = 4$ the recurrence equation gives

$$25a_4 + 25a_5 + 30a_6 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{120} + \frac{7a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$41a_5 + 36a_6 + 42a_7 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{239a_0}{2520} - \frac{11a_1}{280}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3t^3 + a_2t^2 + a_1t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) t^2 + \left(\frac{a_0}{6} + \frac{a_1}{6}\right) t^3 + \left(\frac{a_0}{12} - \frac{a_1}{12}\right) t^4 + \left(-\frac{a_0}{10} + \frac{a_1}{30}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{10}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^4 + \frac{1}{30}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{10}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^4 + \frac{1}{30}t^5\right) c_2 + O(t^6)$$

$$y(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{4} - \frac{t^5}{6} - 2t + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 3$ results in

$$y = 7 + \frac{(x-3)^2}{2} - 2x - \frac{(x-3)^3}{6} + \frac{(x-3)^4}{4} - \frac{(x-3)^5}{6} - \frac{17(x-3)^6}{360} + O\left((x-3)^6\right)$$

Summary

The solution(s) found are the following

$$y = 7+\frac{(x-3)^{2}}{2}-2x-\frac{(x-3)^{3}}{6}+\frac{(x-3)^{4}}{4}-\frac{(x-3)^{5}}{6}-\frac{17(x-3)^{6}}{360}+O\left((x-3)^{6}\right) \tag{1}$$

Verification of solutions

$$y = 7+\frac{(x-3)^{2}}{2}-2x-\frac{(x-3)^{3}}{6}+\frac{(x-3)^{4}}{4}-\frac{(x-3)^{5}}{6}-\frac{17(x-3)^{6}}{360}+O\left((x-3)^{6}\right)$$

Verified OK.

13.50.2 Maple step by step solution

Let's solve

$$\left[\left(2x^{2}-11x+16\right) y''+\left(x^{2}-6x+10\right) y'+\left(-2+x\right) y=0, y(3)=1, y'\Big|_{\{x=3\}}=-2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y''=-\frac{(-2+x)y}{2x^{2}-11x+16}-\frac{(x^{2}-6x+10)y'}{2x^{2}-11x+16}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^{2}-6x+10)y'}{2x^{2}-11x+16}+\frac{(-2+x)y}{2x^{2}-11x+16}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{x^{2}-6x+10}{2x^{2}-11x+16}, P_3(x)=\frac{-2+x}{2x^{2}-11x+16}\right]$
- ◦ $\left(\frac{I\sqrt{7}}{4}+x-\frac{11}{4}\right)\cdot P_2(x)$ is analytic at $x=-\frac{I\sqrt{7}}{4}+\frac{11}{4}$

  $\left(\left(\frac{I\sqrt{7}}{4}+x-\frac{11}{4}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{I\sqrt{7}}{4}+\frac{11}{4}}=0$
- ◦ $\left(\frac{I\sqrt{7}}{4}+x-\frac{11}{4}\right)^{2}\cdot P_3(x)$ is analytic at $x=-\frac{I\sqrt{7}}{4}+\frac{11}{4}$

  $\left(\left(\frac{I\sqrt{7}}{4}+x-\frac{11}{4}\right)^{2}\cdot P_3(x)\right)\Big|_{x=-\frac{I\sqrt{7}}{4}+\frac{11}{4}}=0$
- ◦ $x=-\frac{I\sqrt{7}}{4}+\frac{11}{4}$ is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = -\frac{I\sqrt{7}}{4} + \frac{11}{4}$

- Multiply by denominators

$(2x^2 - 11x + 16)\, y'' + (x^2 - 6x + 10)\, y' + (-2 + x)\, y = 0$

- Change variables using $x = u - \frac{I\sqrt{7}}{4} + \frac{11}{4}$ so that the regular singular point is at $u = 0$

$(2u^2 - Iu\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right) + \left(u^2 - \frac{Iu\sqrt{7}}{2} - \frac{u}{2} + \frac{5}{8} + \frac{I\sqrt{7}}{8}\right)\left(\frac{d}{du}y(u)\right) + \left(\frac{3}{4} + u - \frac{I\sqrt{7}}{4}\right) y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-\frac{I\sqrt{7}\, r\left(5\, I\sqrt{7} - 63 + 56r\right) a_0 u^{-1+r}}{56} + \left(-\frac{I\sqrt{7}\,(1+r)\left(5\, I\sqrt{7} - 7 + 56r\right) a_1}{56} - \frac{\left(-3 + 2\, Ir\sqrt{7} + I\sqrt{7} - 8r^2 + 10r\right) a_0}{4}\right) u^r + \left(\sum_{k=1}^{\infty} \left(\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-\frac{I}{56}\sqrt{7}\, r\left(5\, I\sqrt{7} - 63 + 56r\right) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right\}$

- Each term must be 0

$-\frac{\mathrm{I}\sqrt{7}\,(1+r)\left(5\,\mathrm{I}\sqrt{7}-7+56r\right)a_1}{56}-\frac{\left(-3+2\,\mathrm{I}r\sqrt{7}+\mathrm{I}\sqrt{7}-8r^2+10r\right)a_0}{4}=0$

- Each term in the series must be 0, giving the recursion relation

$\frac{\mathrm{I}(-2(k+r-\frac{1}{8})(k+r+1)a_{k+1}-(k+r+\frac{1}{2})a_k)\sqrt{7}}{2}+\frac{5(k+r+1)a_{k+1}}{8}+2\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{2}\right)a_k+a_{k-1}(k+r)=$

- Shift index using $k->k+1$

$\frac{\mathrm{I}(-2(k+\frac{7}{8}+r)(k+2+r)a_{k+2}-(k+\frac{3}{2}+r)a_{k+1})\sqrt{7}}{2}+\frac{5(k+2+r)a_{k+2}}{8}+2\left(k+\frac{1}{4}+r\right)\left(k+r+\frac{1}{2}\right)a_{k+1}+a_k(k+r$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+2\,\mathrm{I}\sqrt{7}\,a_{k+1}r+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-16kra_{k+1}-8r^2a_{k+1}-4ka_k-6ka_{k+1}-4ra_k-6ra_{k+1}-4a_k-a_{k+1}\right)}{-10+14\,\mathrm{I}\sqrt{7}+8\,\mathrm{I}\sqrt{7}\,k^2+16\,\mathrm{I}\sqrt{7}\,kr+23\,\mathrm{I}\sqrt{7}\,k+23\,\mathrm{I}\sqrt{7}\,r+8\,\mathrm{I}\sqrt{7}\,r^2-5k-5r}$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-4ka_k-6ka_{k+1}-4a_k-a_{k+1}\right)}{-10+14\,\mathrm{I}\sqrt{7}+8\,\mathrm{I}\sqrt{7}\,k^2+23\,\mathrm{I}\sqrt{7}\,k-5k}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-4ka_k-6ka_{k+1}-4a_k-a_{k+1}\right)}{-10+14\,\mathrm{I}\sqrt{7}+8\,\mathrm{I}\sqrt{7}\,k^2+23\,\mathrm{I}\sqrt{7}\,k-5k}, -\frac{\mathrm{I}\sqrt{7}\left(5\,\mathrm{I}\sqrt{7}-7\right)a_1}{56}-\right.$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{11}{4}$

$\left[y=\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{11}{4}\right)^k, a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-4ka_k-6ka_{k+1}-4a_k-a_{k+1}\right)}{-10+14\,\mathrm{I}\sqrt{7}+8\,\mathrm{I}\sqrt{7}\,k^2+23\,\mathrm{I}\sqrt{7}\,k-5k}, -\frac{\mathrm{I}\sqrt{7}\,(5}{}\right.$

- Recursion relation for $r=\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}$

$a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+2\,\mathrm{I}\sqrt{7}\,a_{k+1}\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-16k\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)a_{k+1}-8\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)^2a_{k+1}-4ka_k-6ka_k\right.}{-\frac{125}{8}+\frac{809\,\mathrm{I}\sqrt{7}}{56}+8\,\mathrm{I}\sqrt{7}\,k^2+16\,\mathrm{I}\sqrt{7}\,k\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+23\,\mathrm{I}\sqrt{7}\,k+23\,\mathrm{I}\sqrt{7}\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+8\,\mathrm{I}\sqrt{}}$

- Solution for $r=\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}}, a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+2\,\mathrm{I}\sqrt{7}\,a_{k+1}\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-16k\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)a_{k+}\right.}{-\frac{125}{8}+\frac{809\,\mathrm{I}\sqrt{7}}{56}+8\,\mathrm{I}\sqrt{7}\,k^2+16\,\mathrm{I}\sqrt{7}\,k\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+}\right.$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{11}{4}$

$\left[y=\sum_{k=0}^{\infty}a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x-\frac{11}{4}\right)^{k+\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}}, a_{k+2}=-\frac{2\left(2\,\mathrm{I}\sqrt{7}\,a_{k+1}k+2\,\mathrm{I}\sqrt{7}\,a_{k+1}\left(\frac{9}{8}-\frac{5\,\mathrm{I}\sqrt{7}}{56}\right)+3\,\mathrm{I}\sqrt{7}\,a_{k+1}-8k^2a_{k+1}-16k\right.}{-\frac{125}{8}+\frac{809\,\mathrm{I}\sqrt{7}}{56}+8\,\mathrm{I}\sqrt{7}\,k^2+16\,\mathrm{I}\sqrt{}}\right.$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x - \frac{11}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{7}}{4} + x - \frac{11}{4}\right)^{k+\frac{9}{8}-\frac{5I\sqrt{7}}{56}}\right), a_{k+2} = -\frac{2\left(2I\sqrt{7}\, a_{1+k}k + 3I\sqrt{7}\, a\right.}{-10+14}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
Order:=6;
dsolve([(16-11*x+2*x^2)*diff(y(x),x$2)+(10-6*x+x^2)*diff(y(x),x)-(2-x)*y(x)=0,y(3) = 1, D(y)
```

$$y(x) = 1 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{6}(x-3)^3 + \frac{1}{4}(x-3)^4 - \frac{1}{6}(x-3)^5 + O\left((x-3)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 46

```
AsymptoticDSolveValue[{(16-11*x+2*x^2)*y''[x]+(10-6*x+x^2)*y'[x]-(2-x)*y[x]==0,{y[3]==1,y'[3
```

$$y(x) \to -\frac{1}{6}(x-3)^5 + \frac{1}{4}(x-3)^4 - \frac{1}{6}(x-3)^3 + \frac{1}{2}(x-3)^2 - 2(x-3) + 1$$

14 Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358

14.1 problem Example 7.5.1 page 353

Internal problem ID [1292]
Internal file name [OUTPUT/1293_Sunday_June_05_2022_02_08_34_AM_4314040/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: Example 7.5.1 page 353.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{11x^2+11x+9}{2\,(x^2+x+1)\,x}$$
$$q(x)=\frac{7x^2+10x+6}{2x^2\,(x^2+x+1)}$$

Table 530: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11x^2+11x+9}{2(x^2+x+1)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

$q(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2\left(x^2+x+1\right)y'' + \left(11x^3+11x^2+9x\right)y' + \left(7x^2+10x+6\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(x^2+x+1\right)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(11x^3+11x^2+9x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(7x^2+10x+6\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 11x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 10x^{1+n+r} a_n\right)+\left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\,(n+r-1)=\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\,(n+r-1)=\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r)=\sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 11x^{1+n+r} a_n (n+r)=\sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 7x^{n+r+2} a_n=\sum_{n=2}^{\infty} 7a_{n-2}x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 10x^{1+n+r} a_n=\sum_{n=1}^{\infty} 10a_{n-1}x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)\,(n+r-1) + 9x^{n+r}a_n(n+r) + 6a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 9x^r a_0 r + 6a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 9x^r r + 6x^r)\,a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2+7r+6\right)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2+7r+6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{3}{2}$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2+7r+6\right)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r-2}{r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-2}(n+r-2)(n-3+r)+2a_{n-1}(n+r-1)(n+r-2)+2a_n(n+r)(n+r-1) \\ &+11a_{n-2}(n+r-2)+11a_{n-1}(n+r-1)+9a_n(n+r)+7a_{n-2}+10a_{n-1}+6a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-2}+na_{n-1}+ra_{n-2}+ra_{n-1}-a_{n-2}+a_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_n = \frac{(-2a_{n-2}-2a_{n-1})\,n+5a_{n-2}+a_{n-1}}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4 + r}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_2 = \frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^2 + 3r + 1}{(r + 3)(5 + r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_3 = -\frac{5}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^3 - 8r^2 - 19r - 13}{(6 + r)(r + 3)(4 + r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_4 = \frac{7}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$
a_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{7}{135}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2r^3 + 18r^2 + 52r + 49}{(r+3)(4+r)(5+r)(7+r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_5 = \frac{76}{1155}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$
a_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{7}{135}$
a_5	$\frac{2r^3+18r^2+52r+49}{(r+3)(4+r)(5+r)(7+r)}$	$\frac{76}{1155}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \frac{1}{x^{\frac{3}{2}}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= \frac{1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)}{x^{\frac{3}{2}}}
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-r-2}{r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-2}(n+r-2)(n-3+r)+2b_{n-1}(n+r-1)(n+r-2)+2b_n(n+r)(n+r-1) \\ &+11b_{n-2}(n+r-2)+11b_{n-1}(n+r-1)+9b_n(n+r)+7b_{n-2}+10b_{n-1}+6b_n=0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{nb_{n-2} + nb_{n-1} + rb_{n-2} + rb_{n-1} - b_{n-2} + b_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = -2$ becomes

$$b_n = \frac{(-b_{n-2} - b_{n-1})\, n + 3b_{n-2} + b_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4+r}$$

Which for the root $r = -2$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r^2 + 3r + 1}{(r+3)(5+r)}$$

Which for the root $r = -2$ becomes

$$b_3 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-r^3 - 8r^2 - 19r - 13}{(6+r)(r+3)(4+r)}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$
b_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2r^3 + 18r^2 + 52r + 49}{(r+3)(4+r)(5+r)(7+r)}$$

Which for the root $r = -2$ becomes

$$b_5 = \frac{1}{30}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$
b_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{1}{8}$
b_5	$\frac{2r^3+18r^2+52r+49}{(r+3)(4+r)(5+r)(7+r)}$	$\frac{1}{30}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x^{\frac{3}{2}}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)\right)}{x^{\frac{3}{2}}} + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)\right)}{x^{\frac{3}{2}}} + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)\right)}{x^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \left(1-\frac{x}{3}+\frac{2 x^2}{5}-\frac{5 x^3}{21}+\frac{7 x^4}{135}+\frac{76 x^5}{1155}+O\left(x^6\right)\right)}{x^{\frac{3}{2}}} + \frac{c_2 \left(1+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{30}+O\left(x^6\right)\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \left(1-\frac{x}{3}+\frac{2 x^2}{5}-\frac{5 x^3}{21}+\frac{7 x^4}{135}+\frac{76 x^5}{1155}+O\left(x^6\right)\right)}{x^{\frac{3}{2}}} + \frac{c_2 \left(1+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{30}+O\left(x^6\right)\right)}{x^2}
$$

Verified OK.

14.1.1 Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\, y''+(11x^3+11x^2+9x)\, y'+(7x^2+10x+6)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)}-\frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}+\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{11x^2+11x+9}{2(x^2+x+1)x}, P_3(x)=\frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{9}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=3$

  - $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot y''$ to series expansion for $m=2..4$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

  $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(3+2r)\,x^r+\left(a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(2k+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(3+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{-2,-\frac{3}{2}\right\}$

- Each term must be 0

$$a_1(3+r)(5+2r)+a_0(5+2r)(2+r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{(2+r)a_0}{3+r}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r+\frac{3}{2}\right)\left((a_k+a_{k-2}+a_{k-1})k+(a_k+a_{k-2}+a_{k-1})r+2a_k-a_{k-2}+a_{k-1}\right)=0$$

- Shift index using $k->k+2$

$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})(k+2)+(a_{k+2}+a_k+a_{k+1})r+2a_{k+2}-a_k+a_{k+1}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$$

- Recursion relation for $r=-2$

$$a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-2},a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2},a_1=0\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}},a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}},a_1=-\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-\frac{3}{2}}\right),a_{k+2}=-\frac{ka_k+ka_{1+k}-a_k+a_{1+k}}{k+2},a_1=0,b_{k+2}=-\frac{kb_k+kb_{1+k}-\frac{1}{2}b_k+}{k+\frac{5}{2}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*(1+x+x^2)*diff(y(x),x$2)+x*(9+11*x+11*x^2)*diff(y(x),x)+(6+10*x+7*x^2)*y(x)=0,y
```

$$y(x) = \frac{c_1\left(1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{8}x^4+\frac{1}{30}x^5+\mathrm{O}\left(x^6\right)\right)}{x^2} + \frac{c_2\left(1-\frac{1}{3}x+\frac{2}{5}x^2-\frac{5}{21}x^3+\frac{7}{135}x^4+\frac{76}{1155}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 83

```
AsymptoticDSolveValue[2*x^2*(1+x+x^2)*y''[x]+x*(9+11*x+11*x^2)*y'[x]+(6+10*x+7*x^2)*y[x]==0,
```

$$y(x) \to \frac{c_2\left(\frac{x^5}{30}+\frac{x^4}{8}-\frac{x^3}{3}+\frac{x^2}{2}+1\right)}{x^2}+\frac{c_1\left(\frac{76x^5}{1155}+\frac{7x^4}{135}-\frac{5x^3}{21}+\frac{2x^2}{5}-\frac{x}{3}+1\right)}{x^{3/2}}$$

14.2 problem Example 7.5.2 page 354

Internal problem ID [1293]
Internal file name [`OUTPUT/1294_Sunday_June_05_2022_02_08_39_AM_34933700/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: Example 7.5.2 page 354.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(x+3)\,y''+5x(x+1)\,y'-(-4x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+3x^2\right)y''+\left(5x^2+5x\right)y'+(4x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5x+5}{x\,(x+3)}$$
$$q(x)=\frac{4x-1}{x^2\,(x+3)}$$

Table 532: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5x+5}{x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4x-1}{x^2(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+3)\,y'' + \left(5x^2+5x\right)y' + (4x-1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+3)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(5x^2+5x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (4x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right)\\
&+\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\\
\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\\
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)\\
&+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right)\\
&+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right)+\left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
3x^{n+r} a_n (n+r)(n+r-1)+5x^{n+r} a_n (n+r)-a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 5x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)\,(n+r-2) + 3a_n(n+r)\,(n+r-1) \\ + 5a_{n-1}(n+r-1) + 5a_n(n+r) + 4a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(1+n+r)\,a_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(4+3n)\,a_{n-1}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2-r}{2+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{7}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+5r+6}{9r^2+21r+10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{35}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 9r^2 - 26r - 24}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{455}{2187}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1820}{19683}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1820}{19683}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{6916}{177147}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1820}{19683}$
a_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{6916}{177147}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ &+ 5b_{n-1}(n+r-1) + 5b_n(n+r) + 4b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(1+n+r)\,b_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{n b_{n-1}}{3n-4} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2 - r}{2 + 3r}$$

Which for the root $r = -1$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 + 5r + 6}{9r^2 + 21r + 10}$$

Which for the root $r = -1$ becomes

$$b_2 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^3 - 9r^2 - 26r - 24}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{3}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{3}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$-\frac{3}{10}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{3}{22}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$-\frac{3}{10}$
b_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{3}{22}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + x - x^2 + \frac{3x^3}{5} - \frac{3x^4}{10} + \frac{3x^5}{22} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O(x^6)\right) \\
&+ \frac{c_2\left(1 + x - x^2 + \frac{3x^3}{5} - \frac{3x^4}{10} + \frac{3x^5}{22} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O(x^6)\right) \\
&+ \frac{c_2\left(1 + x - x^2 + \frac{3x^3}{5} - \frac{3x^4}{10} + \frac{3x^5}{22} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O(x^6)\right) + \frac{c_2 \left(1 + x - x^2 + \frac{3x^3}{5} - \frac{3x^4}{10} + \frac{3x^5}{22} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O(x^6)\right) + \frac{c_2 \left(1 + x - x^2 + \frac{3x^3}{5} - \frac{3x^4}{10} + \frac{3x^5}{22} + O(x^6)\right)}{x}$$

Verified OK.

14.2.1 Maple step by step solution

Let's solve

$x^2(x+3)\,y'' + (5x^2+5x)\,y' + (4x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-1)y}{x^2(x+3)} - \frac{5(x+1)y'}{x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5(x+1)y'}{x(x+3)} + \frac{(4x-1)y}{x^2(x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5(x+1)}{x(x+3)}, P_3(x) = \frac{4x-1}{x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = \frac{10}{3}$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3} = 0$

- $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $x^2(x+3)\,y''+5x(x+1)\,y'+(4x-1)\,y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^3-6u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-25u+30)\left(\frac{d}{du}y(u)\right)+(4u-13)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0r(7+3r)\,u^{-1+r}+(3a_1(1+r)\,(10+3r)-a_0(13+6r)\,(1+r))\,u^r+\left(\sum_{k=1}^{\infty}(3a_{k+1}(k+r+1)\,(3\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(7+3r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, -\frac{7}{3}\}$

- Each term must be 0

  $3a_1(1+r)(10+3r) - a_0(13+6r)(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $-6\left(\left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right)k + \left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right)r + \frac{13a_k}{6} - \frac{a_{k-1}}{6} - 5a_{k+1}\right)(k+r+1) = 0$

- Shift index using $k->k+1$

  $-6\left(\left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right)(k+1) + \left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right)r + \frac{13a_{k+1}}{6} - \frac{a_k}{6} - 5a_{k+2}\right)(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k - 6ka_{k+1} + ra_k - 6ra_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13+3r)}$

- Recursion relation for $r=0$

  $a_{k+2} = -\frac{ka_k - 6ka_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}$

- Solution for $r=0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{ka_k - 6ka_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}, 30a_1 - 13a_0 = 0\right]$

- Revert the change of variables $u = x+3$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{ka_k - 6ka_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}, 30a_1 - 13a_0 = 0\right]$

- Recursion relation for $r = -\frac{7}{3}$

  $a_{k+2} = -\frac{ka_k - 6ka_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}$

- Solution for $r = -\frac{7}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{3}}, a_{k+2} = -\frac{ka_k - 6ka_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -12a_1 - \frac{4a_0}{3} = 0\right]$

- Revert the change of variables $u = x+3$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{7}{3}}, a_{k+2} = -\frac{ka_k - 6ka_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -12a_1 - \frac{4a_0}{3} = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{7}{3}}\right), a_{k+2} = -\frac{k a_k - 6k a_{1+k} + 2a_k - 19a_{1+k}}{3(3k+13)}, 30a_1 - 13a_0 = 0, b\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - returning with
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*(3+x)*diff(y(x),x$2)+5*x*(1+x)*diff(y(x),x)-(1-4*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}}\left(1 - \frac{7}{9}x + \frac{35}{81}x^2 - \frac{455}{2187}x^3 + \frac{1820}{19683}x^4 - \frac{6916}{177147}x^5 + O(x^6)\right) + c_1\left(1 + x - x^2 + \frac{3}{5}x^3 - \frac{3}{10}x^4 + \frac{3}{22}x^5 + O\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 82

```
AsymptoticDSolveValue[x^2*(3+x)*y''[x]+5*x*(1+x)*y'[x]-(1-4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{6916x^5}{177147} + \frac{1820x^4}{19683} - \frac{455x^3}{2187} + \frac{35x^2}{81} - \frac{7x}{9} + 1\right) + \frac{c_2\left(\frac{3x^5}{22} - \frac{3x^4}{10} + \frac{3x^3}{5} - x^2 + x + 1\right)}{x}$$

14.3 problem Example 7.5.3 page 356

Internal problem ID [1294]
Internal file name [`OUTPUT/1295_Sunday_June_05_2022_02_08_42_AM_15958053/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: Example 7.5.3 page 356.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x^2+2)\,y''-x(4x^2+3)\,y'+(2+2x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^4+2x^2)\,y''+(-4x^3-3x)\,y'+(2+2x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{4x^2+3}{x\,(x^2-2)} \\ q(x) &= -\frac{2(x+1)}{x^2\,(x^2-2)} \end{aligned}$$

Table 534: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2+3}{x(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = -\frac{2(x+1)}{x^2(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \sqrt{2}, -\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x^2-2) + (-4x^3-3x)\,y' + (2+2x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(x^2-2) + (-4x^3-3x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (2+2x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)(n+r-1)\right) &= \sum_{n=2}^{\infty} \left(-a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n (n+r)\right) &= \sum_{n=2}^{\infty} \left(-4a_{n-2} (n+r-2)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \left(-a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\
&\quad + \sum_{n=2}^{\infty} \left(-4a_{n-2} (n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - 3x^r a_0 r + 2a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) - 3x^r r + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2}{2r^2 - r - 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-2}(n+r-2)\,(n-3+r)+2a_n(n+r)\,(n+r-1) \\ -4a_{n-2}(n+r-2)-3a_n(n+r)+2a_n+2a_{n-1}=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-2} + 2nra_{n-2} + r^2 a_{n-2} - na_{n-2} - ra_{n-2} - 2a_{n-2} - 2a_{n-1}}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{n^2 a_{n-2} + 3na_{n-2} - 2a_{n-1}}{n\,(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r^4 + 5r^3 - 4r^2 - 3r + 4}{4r^4 + 4r^3 - 5r^2 - 3r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{27}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$\frac{27}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 28r^2 - 10r - 8}{(2r+5)\,r\,(4r^3 + 4r^2 - 5r - 3)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{34}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$\frac{27}{35}$
a_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$-\frac{34}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^7 + 48r^6 + 197r^5 + 293r^4 - 3r^3 - 233r^2 + 90r + 216}{(2r + 5) r (4r^3 + 4r^2 - 5r - 3) (2r^2 + 11r + 14)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{584}{1155}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$\frac{27}{35}$
a_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$-\frac{34}{105}$
a_4	$\frac{4r^7+48r^6+197r^5+293r^4-3r^3-233r^2+90r+216}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)}$	$\frac{584}{1155}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{6(4r^7 + 64r^6 + 403r^5 + 1276r^4 + 2134r^3 + 1837r^2 + 882r + 408)}{(2r + 5) r (4r^3 + 4r^2 - 5r - 3) (2r^2 + 11r + 14) (2r^2 + 15r + 27)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{768}{3575}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2-r-1}$	$-\frac{2}{5}$
a_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$\frac{27}{35}$
a_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$-\frac{34}{105}$
a_4	$\frac{4r^7+48r^6+197r^5+293r^4-3r^3-233r^2+90r+216}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)}$	$\frac{584}{1155}$
a_5	$-\frac{6(4r^7+64r^6+403r^5+1276r^4+2134r^3+1837r^2+882r+408)}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)(2r^2+15r+27)}$	$-\frac{768}{3575}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 - \frac{2x}{5} + \frac{27x^2}{35} - \frac{34x^3}{105} + \frac{584x^4}{1155} - \frac{768x^5}{3575} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = -\frac{2}{2r^2 - r - 1}
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&-b_{n-2}(n+r-2)\,(n-3+r) + 2b_n(n+r)\,(n+r-1) \\
&- 4b_{n-2}(n+r-2) - 3b_n(n+r) + 2b_n + 2b_{n-1} = 0
\end{aligned}
\tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{n^2 b_{n-2} + 2nrb_{n-2} + r^2 b_{n-2} - nb_{n-2} - rb_{n-2} - 2b_{n-2} - 2b_{n-1}}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2}
\tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = \frac{4n^2 b_{n-2} - 9b_{n-2} - 8b_{n-1}}{8n^2 - 12n}
\tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2r^4 + 5r^3 - 4r^2 - 3r + 4}{4r^4 + 4r^3 - 5r^2 - 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{9}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$-\frac{9}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-8r^3 - 28r^2 - 10r - 8}{(2r + 5)\, r\, (4r^3 + 4r^2 - 5r - 3)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{7}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$-\frac{9}{8}$
b_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$\frac{7}{4}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^7 + 48r^6 + 197r^5 + 293r^4 - 3r^3 - 233r^2 + 90r + 216}{(2r + 5)\, r\, (4r^3 + 4r^2 - 5r - 3)\, (2r^2 + 11r + 14)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{607}{640}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$-\frac{9}{8}$
b_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$\frac{7}{4}$
b_4	$\frac{4r^7+48r^6+197r^5+293r^4-3r^3-233r^2+90r+216}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)}$	$-\frac{607}{640}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{6(4r^7 + 64r^6 + 403r^5 + 1276r^4 + 2134r^3 + 1837r^2 + 882r + 408)}{(2r+5)\, r\,(4r^3 + 4r^2 - 5r - 3)\,(2r^2 + 11r + 14)\,(2r^2 + 15r + 27)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{13347}{11200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2-r-1}$	2
b_2	$\frac{2r^4+5r^3-4r^2-3r+4}{4r^4+4r^3-5r^2-3r}$	$-\frac{9}{8}$
b_3	$\frac{-8r^3-28r^2-10r-8}{(2r+5)r(4r^3+4r^2-5r-3)}$	$\frac{7}{4}$
b_4	$\frac{4r^7+48r^6+197r^5+293r^4-3r^3-233r^2+90r+216}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)}$	$-\frac{607}{640}$
b_5	$-\frac{6(4r^7+64r^6+403r^5+1276r^4+2134r^3+1837r^2+882r+408)}{(2r+5)r(4r^3+4r^2-5r-3)(2r^2+11r+14)(2r^2+15r+27)}$	$\frac{13347}{11200}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + 2x - \frac{9x^2}{8} + \frac{7x^3}{4} - \frac{607x^4}{640} + \frac{13347x^5}{11200} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{2x}{5} + \frac{27x^2}{35} - \frac{34x^3}{105} + \frac{584x^4}{1155} - \frac{768x^5}{3575} + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\left(1 + 2x - \frac{9x^2}{8} + \frac{7x^3}{4} - \frac{607x^4}{640} + \frac{13347x^5}{11200} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{2x}{5} + \frac{27x^2}{35} - \frac{34x^3}{105} + \frac{584x^4}{1155} - \frac{768x^5}{3575} + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\left(1 + 2x - \frac{9x^2}{8} + \frac{7x^3}{4} - \frac{607x^4}{640} + \frac{13347x^5}{11200} + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x^2 &\left(1 - \frac{2x}{5} + \frac{27x^2}{35} - \frac{34x^3}{105} + \frac{584x^4}{1155} - \frac{768x^5}{3575} + O(x^6)\right) \\
&+ c_2 \sqrt{x}\left(1 + 2x - \frac{9x^2}{8} + \frac{7x^3}{4} - \frac{607x^4}{640} + \frac{13347x^5}{11200} + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 x^2 &\left(1 - \frac{2x}{5} + \frac{27x^2}{35} - \frac{34x^3}{105} + \frac{584x^4}{1155} - \frac{768x^5}{3575} + O(x^6)\right) \\
&+ c_2 \sqrt{x}\left(1 + 2x - \frac{9x^2}{8} + \frac{7x^3}{4} - \frac{607x^4}{640} + \frac{13347x^5}{11200} + O(x^6)\right)
\end{aligned}$$

Verified OK.

14.3.1 Maple step by step solution

Let's solve

$-y''x^2(x^2-2)+(-4x^3-3x)\,y'+(2+2x)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{2(x+1)y}{x^2(x^2-2)}-\frac{(4x^2+3)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(4x^2+3)y'}{x(x^2-2)}-\frac{2(x+1)y}{x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{4x^2+3}{x(x^2-2)}, P_3(x)=-\frac{2(x+1)}{x^2(x^2-2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{3}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$y''x^2(x^2-2)+x(4x^2+3)\,y'+(-2x-2)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(-1+2r)\,(-2+r)\, x^{r}+\left(-a_{1}(1+2r)\,(-1+r)-2a_{0}\right) x^{1+r}+\left(\sum_{k=2}^{\infty} \left(-a_{k}(2k+2r-1)\,(k+r\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{2, \frac{1}{2}\right\}$
- Each term must be 0

  $-a_{1}(1+2r)\,(-1+r)-2a_{0}=0$
- Solve for the dependent coefficient(s)

  $a_{1}=-\frac{2a_{0}}{2r^{2}-r-1}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k-2}(k+r-2)\,(k+1+r)-2(k+r-2)\left(k+r-\frac{1}{2}\right) a_{k}-2a_{k-1}=0$
- Shift index using $k->k+2$

  $a_{k}(k+r)\,(k+r+3)-2(k+r)\left(k+\frac{3}{2}+r\right) a_{k+2}-2a_{k+1}=0$
- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_k + 2kra_k + r^2 a_k + 3ka_k + 3ra_k - 2a_{k+1}}{(k+r)(2k+3+2r)}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k^2 a_k + 7ka_k + 10a_k - 2a_{k+1}}{(k+2)(2k+7)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_k + 7ka_k + 10a_k - 2a_{k+1}}{(k+2)(2k+7)}, a_1 = -\frac{2a_0}{5}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{k^2 a_k + 4ka_k + \frac{7}{4}a_k - 2a_{k+1}}{(k+\frac{1}{2})(2k+4)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{k^2 a_k + 4ka_k + \frac{7}{4}a_k - 2a_{k+1}}{(k+\frac{1}{2})(2k+4)}, a_1 = 2a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{k^2 a_k + 7ka_k + 10a_k - 2a_{1+k}}{(k+2)(2k+7)}, a_1 = -\frac{2a_0}{5}, b_{k+2} = \frac{k^2 b_k + 4kb_k + \frac{7}{4}b}{(k+\frac{1}{2})(2k}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-x*(3+4*x^2)*diff(y(x),x)+(2+2*x)*y(x)=0,y(x),type='series'
```

$$y(x) = c_1\sqrt{x}\left(1+2x-\frac{9}{8}x^2+\frac{7}{4}x^3-\frac{607}{640}x^4+\frac{13347}{11200}x^5+\mathrm{O}\left(x^6\right)\right) + c_2x^2\left(1-\frac{2}{5}x+\frac{27}{35}x^2-\frac{34}{105}x^3+\frac{584}{1155}x^4-\frac{768}{3575}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 86

```
AsymptoticDSolveValue[x^2*(2-x^2)*y''[x]-x*(3+4*x^2)*y'[x]+(2+2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{768x^5}{3575}+\frac{584x^4}{1155}-\frac{34x^3}{105}+\frac{27x^2}{35}-\frac{2x}{5}+1\right)x^2 + c_2\left(\frac{13347x^5}{11200}-\frac{607x^4}{640}+\frac{7x^3}{4}-\frac{9x^2}{8}+2x+1\right)\sqrt{x}$$

14.4 problem 1

Internal problem ID [1295]
Internal file name [`OUTPUT/1296_Sunday_June_05_2022_02_08_46_AM_84605145/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+x+1)\,y''+x(5x^2+3x+3)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+2x^3+2x^2)\,y''+(5x^3+3x^2+3x)\,y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{5x^2+3x+3}{2\,(x^2+x+1)\,x}\\ q(x)&=\frac{1}{2x^2\,(x^2+x+1)}\end{aligned}$$

Table 536: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5x^2+3x+3}{2(x^2+x+1)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

$q(x) = \frac{1}{2x^2(x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2\left(x^2 + x + 1\right) y'' + \left(5x^3 + 3x^2 + 3x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(x^2 + x + 1\right) \left(\sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2}\right) + \left(5x^3 + 3x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 5a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1) + 3x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 3x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 + r + 1) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{i\sqrt{7}}{4} - \frac{1}{4}$$
$$r_2 = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 + r + 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{i\sqrt{7}}{4}-\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{i\sqrt{7}}{4}-\frac{1}{4}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2r^2 - r}{2r^2 + 5r + 4}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-2}(n+r-2)(n-3+r) + 2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ &+ 5a_{n-2}(n+r-2) + 3a_{n-1}(n+r-1) + 3a_n(n+r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2n^2 a_{n-2} + 2n^2 a_{n-1} + 4nr a_{n-2} + 4nr a_{n-1} + 2r^2 a_{n-2} + 2r^2 a_{n-1} - 5n a_{n-2} - 3n a_{n-1} - 5r a_{n-2} - 3r}{2n^2 + 4nr + 2r^2 + n + r + 1} \tag{4}$$

Which for the root $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ becomes

$$a_n = \frac{i(2(-a_{n-2} - a_{n-1})\,n + 3a_{n-2} + 2a_{n-1})\sqrt{7} + 4(-a_{n-2} - a_{n-1})\,n^2 + 4(3a_{n-2} + 2a_{n-1})\,n - 5a_{n-2} - 2}{2n\left(i\sqrt{7} + 2n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-r}{2r^2+5r+4}$	$\frac{1}{i\sqrt{7}+2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{(2r+3)^2 r}{4r^4 + 28r^3 + 75r^2 + 91r + 44}$$

Which for the root $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ becomes

$$a_2 = -\frac{5\left(i\sqrt{7} - 1\right)\left(i\sqrt{7} + \frac{9}{5}\right)}{16\left(i\sqrt{7} + 2\right)\left(i\sqrt{7} + 4\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-r}{2r^2+5r+4}$	$\frac{1}{i\sqrt{7}+2}$
a_2	$-\frac{(2r+3)^2 r}{4r^4+28r^3+75r^2+91r+44}$	$-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^6+76r^5+282r^4+505r^3+433r^2+145r}{8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968}$$

Which for the root $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ becomes

$$a_3 = \frac{49i\sqrt{7}+89}{432-444i\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-r}{2r^2+5r+4}$	$\frac{1}{i\sqrt{7}+2}$
a_2	$-\frac{(2r+3)^2 r}{4r^4+28r^3+75r^2+91r+44}$	$-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}$
a_3	$\frac{8r^6+76r^5+282r^4+505r^3+433r^2+145r}{8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968}$	$\frac{49i\sqrt{7}+89}{432-444i\sqrt{7}}$

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{(16r^7+240r^6+1480r^5+4800r^4+8645r^3+8269r^2+3466r+273)\,r}{(2r^2+17r+37)\,(8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968)}$$

Which for the root $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ becomes

$$a_4 = \frac{1553i+395\sqrt{7}}{26256i+12480\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-r}{2r^2+5r+4}$	$\frac{1}{i\sqrt{7}+2}$
a_2	$-\frac{(2r+3)^2 r}{4r^4+28r^3+75r^2+91r+44}$	$-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}$
a_3	$\frac{8r^6+76r^5+282r^4+505r^3+433r^2+145r}{8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968}$	$\frac{49i\sqrt{7}+89}{432-444i\sqrt{7}}$
a_4	$-\frac{(16r^7+240r^6+1480r^5+4800r^4+8645r^3+8269r^2+3466r+273)r}{(2r^2+17r+37)(8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968)}$	$\frac{1553i+395\sqrt{7}}{26256i+12480\sqrt{7}}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{64\left(r^7+18r^6+136r^5+559r^4+\frac{21571}{16}r^3+\frac{7619}{4}r^2+\frac{23343}{16}r+\frac{15003}{32}\right)\left(r+\frac{9}{2}\right)r}{(2r^2+17r+37)(8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968)(2r^2+21r+56)}$$

Which for the root $r = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ becomes

$$a_5 = \frac{42423i\sqrt{7}+45275}{492720i\sqrt{7}-1749600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-r}{2r^2+5r+4}$	$\frac{1}{i\sqrt{7}+2}$
a_2	$-\frac{(2r+3)^2 r}{4r^4+28r^3+75r^2+91r+44}$	$-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}$
a_3	$\frac{8r^6+76r^5+282r^4+505r^3+433r^2+145r}{8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968}$	$\frac{49i\sqrt{7}+89}{432-444i\sqrt{7}}$
a_4	$-\frac{(16r^7+240r^6+1480r^5+4800r^4+8645r^3+8269r^2+3466r+273)r}{(2r^2+17r+37)(8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968)}$	$\frac{1553i+395\sqrt{7}}{26256i+12480\sqrt{7}}$
a_5	$-\frac{64\left(r^7+18r^6+136r^5+559r^4+\frac{21571}{16}r^3+\frac{7619}{4}r^2+\frac{23343}{16}r+\frac{15003}{32}\right)\left(r+\frac{9}{2}\right)r}{(2r^2+17r+37)(8r^6+108r^5+602r^4+1773r^3+2921r^2+2574r+968)(2r^2+21r+56)}$	$\frac{42423i\sqrt{7}+45275}{492720i\sqrt{7}-1749600}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$

$$= x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{i\sqrt{7}+2}-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}+\frac{\left(49i\sqrt{7}+89\right)x^{3}}{432-444i\sqrt{7}} + \frac{\left(1553i+395\sqrt{7}\right)x^{4}}{26256i+12480\sqrt{7}}+\frac{\left(42423i\sqrt{7}+45275\right)x^{5}}{492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right)$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{-i\sqrt{7}+2}-\frac{5\left(-i\sqrt{7}-1\right)\left(-i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(-i\sqrt{7}+2\right)\left(-i\sqrt{7}+4\right)}+\frac{\left(-49i\sqrt{7}+89\right)x^{3}}{432+444i\sqrt{7}} + \frac{\left(-1553i+395\sqrt{7}\right)x^{4}}{-26256i+12480\sqrt{7}}+\frac{\left(-42423i\sqrt{7}+45275\right)x^{5}}{-492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{i\sqrt{7}+2}-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}+\frac{\left(49i\sqrt{7}+89\right)x^{3}}{432-444i\sqrt{7}} + \frac{\left(1553i+395\sqrt{7}\right)x^{4}}{26256i+12480\sqrt{7}}+\frac{\left(42423i\sqrt{7}+45275\right)x^{5}}{492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right) \\ &+ c_2 x^{-\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{-i\sqrt{7}+2}-\frac{5\left(-i\sqrt{7}-1\right)\left(-i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(-i\sqrt{7}+2\right)\left(-i\sqrt{7}+4\right)}+\frac{\left(-49i\sqrt{7}+89\right)x^{3}}{432+444i\sqrt{7}} + \frac{\left(-1553i+395\sqrt{7}\right)x^{4}}{-26256i+12480\sqrt{7}}+\frac{\left(-42423i\sqrt{7}+45275\right)x^{5}}{-492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{i\sqrt{7}+2}-\frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)}+\frac{\left(49i\sqrt{7}+89\right)x^{3}}{432-444i\sqrt{7}} + \frac{\left(1553i+395\sqrt{7}\right)x^{4}}{26256i+12480\sqrt{7}}+\frac{\left(42423i\sqrt{7}+45275\right)x^{5}}{492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right) \\ &+ c_2 x^{-\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1+\frac{x}{-i\sqrt{7}+2}-\frac{5\left(-i\sqrt{7}-1\right)\left(-i\sqrt{7}+\frac{9}{5}\right)x^{2}}{16\left(-i\sqrt{7}+2\right)\left(-i\sqrt{7}+4\right)}+\frac{\left(-49i\sqrt{7}+89\right)x^{3}}{432+444i\sqrt{7}} + \frac{\left(-1553i+395\sqrt{7}\right)x^{4}}{-26256i+12480\sqrt{7}}+\frac{\left(-42423i\sqrt{7}+45275\right)x^{5}}{-492720i\sqrt{7}-1749600}+O\left(x^{6}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1 + \frac{x}{i\sqrt{7}+2} - \frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)x^2}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)} + \frac{\left(49i\sqrt{7}+89\right)x^3}{432-444i\sqrt{7}} + \frac{\left(1553i+395\sqrt{7}\right)x^4}{26256i+12480\sqrt{7}} + \frac{\left(42423i\sqrt{7}+45275\right)x^5}{492720i\sqrt{7}-1749600} + O\left(x^6\right)\right) + c_2 x^{-\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1 + \frac{x}{-i\sqrt{7}+2} - \frac{5\left(-i\sqrt{7}-1\right)\left(-i\sqrt{7}+\frac{9}{5}\right)x^2}{16\left(-i\sqrt{7}+2\right)\left(-i\sqrt{7}+4\right)} + \frac{\left(-49i\sqrt{7}+89\right)x^3}{432+444i\sqrt{7}} + \frac{\left(-1553i+395\sqrt{7}\right)x^4}{-26256i+12480\sqrt{7}} + \frac{\left(-42423i\sqrt{7}+45275\right)x^5}{-492720i\sqrt{7}-1749600} + O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1 + \frac{x}{i\sqrt{7}+2} - \frac{5\left(i\sqrt{7}-1\right)\left(i\sqrt{7}+\frac{9}{5}\right)x^2}{16\left(i\sqrt{7}+2\right)\left(i\sqrt{7}+4\right)} + \frac{\left(49i\sqrt{7}+89\right)x^3}{432-444i\sqrt{7}} + \frac{\left(1553i+395\sqrt{7}\right)x^4}{26256i+12480\sqrt{7}} + \frac{\left(42423i\sqrt{7}+45275\right)x^5}{492720i\sqrt{7}-1749600} + O\left(x^6\right)\right) + c_2 x^{-\frac{i\sqrt{7}}{4}-\frac{1}{4}} \left(1 + \frac{x}{-i\sqrt{7}+2} - \frac{5\left(-i\sqrt{7}-1\right)\left(-i\sqrt{7}+\frac{9}{5}\right)x^2}{16\left(-i\sqrt{7}+2\right)\left(-i\sqrt{7}+4\right)} + \frac{\left(-49i\sqrt{7}+89\right)x^3}{432+444i\sqrt{7}} + \frac{\left(-1553i+395\sqrt{7}\right)x^4}{-26256i+12480\sqrt{7}} + \frac{\left(-42423i\sqrt{7}+45275\right)x^5}{-492720i\sqrt{7}-1749600} + O\left(x^6\right)\right)$$

Verified OK.

14.4.1 Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\,y'' + (5x^3+3x^2+3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x^2(x^2+x+1)} - \frac{(5x^2+3x+3)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(5x^2+3x+3)y'}{2x(x^2+x+1)} + \frac{y}{2x^2(x^2+x+1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5x^2+3x+3}{2(x^2+x+1)x}, P_3(x) = \frac{1}{2x^2(x^2+x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $2x^2(x^2+x+1)\, y'' + x(5x^2+3x+3)\, y' + y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(2r^2+r+1)\,x^r+\left(\left(2r^2+5r+4\right)a_1+a_0r(1+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k\left(2k^2+4kr+2r^2+k+r\right.\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r^2+r+1=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4},\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right\}$$

- Each term must be 0

$$\left(2r^2+5r+4\right)a_1+a_0r(1+2r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{a_0r(1+2r)}{2r^2+5r+4}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(2a_k+2a_{k-2}+2a_{k-1}\right)k^2+\left(\left(4a_k+4a_{k-2}+4a_{k-1}\right)r+a_k-5a_{k-2}-3a_{k-1}\right)k+\left(2a_k+2a_{k-2}+\right.$$

- Shift index using $k->k+2$

$$\left(2a_{k+2}+2a_k+2a_{k+1}\right)(k+2)^2+\left(\left(4a_{k+2}+4a_k+4a_{k+1}\right)r+a_{k+2}-5a_k-3a_{k+1}\right)(k+2)+\left(2a_k\right.$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k+2k^2a_{k+1}+4kra_k+4kra_{k+1}+2r^2a_k+2r^2a_{k+1}+3ka_k+5ka_{k+1}+3ra_k+5ra_{k+1}+3a_{k+1}}{2k^2+4kr+2r^2+9k+9r+11}$$

- Recursion relation for $r=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}$

$$a_{k+2}=-\frac{2k^2a_k+2k^2a_{k+1}+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_k+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_{k+1}+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_k+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_{k+1}+3ka_k+5ka_{k+1}+3\left(-\right.}{2k^2+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2+9k-\frac{9\,\mathrm{I}\sqrt{7}}{4}+\frac{35}{4}}$$

- Solution for $r=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}},a_{k+2}=-\frac{2k^2a_k+2k^2a_{k+1}+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_k+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_{k+1}+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_k+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right.}{2k^2+4k\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)+2\left(-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2}\right.$$

- Recursion relation for $r=\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}$

$$a_{k+2}=-\frac{2k^2a_k+2k^2a_{k+1}+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_k+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_{k+1}+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_k+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_{k+1}+3ka_k+5ka_{k+1}+3\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)}{2k^2+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2+9k+\frac{9\,\mathrm{I}\sqrt{7}}{4}+\frac{35}{4}}$$

- Solution for $r=\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}},a_{k+2}=-\frac{2k^2a_k+2k^2a_{k+1}+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_k+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)a_{k+1}+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_k+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2a_{k+1}}{2k^2+4k\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)+2\left(\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}\right)^2+9k+}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{I\sqrt{7}}{4}-\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{I\sqrt{7}}{4}-\frac{1}{4}}\right), a_{k+2} = -\frac{2k^2 a_k + 2k^2 a_{1+k} + 4k\left(-\frac{I\sqrt{7}}{4}-\frac{1}{4}\right)a_k + 4k\left(-\frac{I\sqrt{7}}{4}-\frac{1}{4}\right)a_{1+}}{2}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 1243

```
Order:=6;
dsolve(2*x^2*(1+x+x^2)*diff(y(x),x$2)+x*(3+3*x+5*x^2)*diff(y(x),x)+y(x)=0,y(x),type='series'
```

$$y(x) = \frac{c_2 x^{\frac{i\sqrt{7}}{4}}\left(1 + \frac{1}{2+i\sqrt{7}}x + \frac{1}{4}\frac{11-i\sqrt{7}}{\left(2+i\sqrt{7}\right)\left(i\sqrt{7}+4\right)}x^2 - \frac{1}{12}\frac{49i\sqrt{7}+89}{\left(2+i\sqrt{7}\right)\left(i\sqrt{7}+4\right)\left(i\sqrt{7}+6\right)}x^3 + \frac{1}{48}\frac{395i\sqrt{7}-1553}{\left(2+i\sqrt{7}\right)\left(i\sqrt{7}+4\right)\left(i\sqrt{7}+6\right)\left(i\sqrt{7}+8\right)}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 4838

```
AsymptoticDSolveValue[2*x^2*(1+x+x^2)*y''[x]+x*(3+3*x+5*x^2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

Too large to display

14.5 problem 2

Internal problem ID [1296]
Internal file name [`OUTPUT/1297_Sunday_June_05_2022_02_08_51_AM_48638098/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
  _with_symmetry_[0,F(x)]`]]
```

$$\boxed{3x^2y'' + 2x(-2x^2 + x + 1)\, y' + (-8x^2 + 2x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + (-4x^3 + 2x^2 + 2x)\, y' + (-8x^2 + 2x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{2(2x^2 - x - 1)}{3x} \\ q(x) &= -\frac{2(4x - 1)}{3x} \end{aligned}$$

Table 538: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(2x^2-x-1)}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{2(4x-1)}{3x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + \left(-4x^3 + 2x^2 + 2x\right)y' + \left(-8x^2 + 2x\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-4x^3+2x^2+2x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-8x^2+2x\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(-4x^{n+r+2} a_n (n+r)\right) &= \sum_{n=2}^{\infty} \left(-4a_{n-2}(n+r-2)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r}\right) \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-4a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$3x^{n+r}a_n(n+r)(n+r-1) + 2x^{n+r}a_n(n+r) = 0$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 2x^r a_0 r = 0$$

Or

$$(3x^r r(-1+r) + 2x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-1+3r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-1+3r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2}{3r+2}$$

For $2 \leq n$ the recursive equation is

$$3a_n(n+r)(n+r-1) - 4a_{n-2}(n+r-2) + 2a_{n-1}(n+r-1) + 2a_n(n+r) - 8a_{n-2} + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4a_{n-2} - 2a_{n-1}}{3n - 1 + 3r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{4a_{n-2} - 2a_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3r+2}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{12 + 12r}{9r^2 + 21r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{8}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3r+2}$	$-\frac{2}{3}$
a_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{8}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-64 - 48r}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{40}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3r+2}$	$-\frac{2}{3}$
a_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{8}{9}$
a_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{40}{81}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{144r^2 + 624r + 512}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{92}{243}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3r+2}$	$-\frac{2}{3}$
a_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{8}{9}$
a_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{40}{81}$
a_4	$\frac{144r^2+624r+512}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{92}{243}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-864r^2 - 4128r - 3840}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{664}{3645}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3r+2}$	$-\frac{2}{3}$
a_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{8}{9}$
a_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{40}{81}$
a_4	$\frac{144r^2+624r+512}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{92}{243}$
a_5	$\frac{-864r^2-4128r-3840}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{664}{3645}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}} \left(1 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{40x^3}{81} + \frac{92x^4}{243} - \frac{664x^5}{3645} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{2}{3r+2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &3b_n(n+r)(n+r-1) - 4b_{n-2}(n+r-2) \\ &+ 2b_{n-1}(n+r-1) + 2b_n(n+r) - 8b_{n-2} + 2b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{4b_{n-2} - 2b_{n-1}}{3n - 1 + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{4b_{n-2} - 2b_{n-1}}{3n - 1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3r+2}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{12 + 12r}{9r^2 + 21r + 10}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{6}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3r+2}$	-1
b_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{6}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-64 - 48r}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{4}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3r+2}$	-1
b_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{6}{5}$
b_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{4}{5}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{144r^2 + 624r + 512}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{32}{55}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3r+2}$	-1
b_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{6}{5}$
b_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{4}{5}$
b_4	$\frac{144r^2+624r+512}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{32}{55}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-864r^2 - 4128r - 3840}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{24}{77}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3r+2}$	-1
b_2	$\frac{12+12r}{9r^2+21r+10}$	$\frac{6}{5}$
b_3	$\frac{-64-48r}{27r^3+135r^2+198r+80}$	$-\frac{4}{5}$
b_4	$\frac{144r^2+624r+512}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{32}{55}$
b_5	$\frac{-864r^2-4128r-3840}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{24}{77}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x + \frac{6x^2}{5} - \frac{4x^3}{5} + \frac{32x^4}{55} - \frac{24x^5}{77} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{40x^3}{81} + \frac{92x^4}{243} - \frac{664x^5}{3645} + O(x^6)\right) \\
&\quad + c_2 \left(1 - x + \frac{6x^2}{5} - \frac{4x^3}{5} + \frac{32x^4}{55} - \frac{24x^5}{77} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(1 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{40x^3}{81} + \frac{92x^4}{243} - \frac{664x^5}{3645} + O(x^6)\right) \\
&\quad + c_2 \left(1 - x + \frac{6x^2}{5} - \frac{4x^3}{5} + \frac{32x^4}{55} - \frac{24x^5}{77} + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{40x^3}{81} + \frac{92x^4}{243} - \frac{664x^5}{3645} + O(x^6)\right) \\
&+ c_2 \left(1 - x + \frac{6x^2}{5} - \frac{4x^3}{5} + \frac{32x^4}{55} - \frac{24x^5}{77} + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{40x^3}{81} + \frac{92x^4}{243} - \frac{664x^5}{3645} + O(x^6)\right) \\
&+ c_2 \left(1 - x + \frac{6x^2}{5} - \frac{4x^3}{5} + \frac{32x^4}{55} - \frac{24x^5}{77} + O(x^6)\right)
\end{aligned}$$

Verified OK.

14.5.1 Maple step by step solution

Let's solve

$3x^2y'' + (-4x^3 + 2x^2 + 2x)\, y' + (-8x^2 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(4x-1)y}{3x} + \frac{2(2x^2-x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(2x^2-x-1)y'}{3x} - \frac{2(4x-1)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(2x^2-x-1)}{3x}, P_3(x) = -\frac{2(4x-1)}{3x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{2}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3y''x + (-4x^2 + 2x + 2)\, y' + (-8x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x^m\cdot y'$ to series expansion for $m=0..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+3r)\,x^{-1+r}+(a_1(1+r)\,(2+3r)+2a_0(1+r))\,x^r+\left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(3k+2+3r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{3}\right\}$
- Each term must be 0

  $a_1(1+r)\,(2+3r)+2a_0(1+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(3ka_{k+1}+3ra_{k+1}+2a_k-4a_{k-1}+2a_{k+1})=0$
- Shift index using $k->k+1$

  $(k+r+2)\,(3(k+1)\,a_{k+2}+3ra_{k+2}+2a_{k+1}-4a_k+2a_{k+2})=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{2(-a_{k+1}+2a_k)}{3k+5}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1} + 2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = \frac{2(-a_{k+1} + 2a_k)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1} + 2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{1+k} + 2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{1+k} + 2b_k)}{3k+6}, 4b_1 +$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 44

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+2*x*(1+x-2*x^2)*diff(y(x),x)+(2*x-8*x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{2}{3}x + \frac{8}{9}x^2 - \frac{40}{81}x^3 + \frac{92}{243}x^4 - \frac{664}{3645}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 - x + \frac{6}{5}x^2 - \frac{4}{5}x^3 + \frac{32}{55}x^4 - \frac{24}{77}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 83

```
AsymptoticDSolveValue[3*x^2*y''[x]+2*x*(1+x-2*x^2)*y'[x]+(2*x-8*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{664x^5}{3645}+\frac{92x^4}{243}-\frac{40x^3}{81}+\frac{8x^2}{9}-\frac{2x}{3}+1\right)+c_2\left(-\frac{24x^5}{77}+\frac{32x^4}{55}-\frac{4x^3}{5}+\frac{6x^2}{5}-x+1\right)$$

14.6 problem 3

Internal problem ID [1297]
Internal file name [`OUTPUT/1298_Sunday_June_05_2022_02_08_54_AM_14302186/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2\left(x^2+3x+3\right)y''+x\left(7x^2+8x+5\right)y'-\left(-9x^2-2x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4+3x^3+3x^2\right)y''+\left(7x^3+8x^2+5x\right)y'+\left(9x^2+2x-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2+8x+5}{x\left(x^2+3x+3\right)}$$
$$q(x)=\frac{9x^2+2x-1}{x^2\left(x^2+3x+3\right)}$$

Table 540: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+8x+5}{x(x^2+3x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{3}}{2} - \frac{3}{2}$	"regular"
$x = \frac{i\sqrt{3}}{2} - \frac{3}{2}$	"regular"

$q(x) = \frac{9x^2+2x-1}{x^2(x^2+3x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{3}}{2} - \frac{3}{2}$	"regular"
$x = \frac{i\sqrt{3}}{2} - \frac{3}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{3}}{2} - \frac{3}{2}, \frac{i\sqrt{3}}{2} - \frac{3}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2\left(x^2+3x+3\right)y'' + \left(7x^3+8x^2+5x\right)y' + \left(9x^2+2x-1\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(x^2+3x+3\right)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(7x^3+8x^2+5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(9x^2+2x-1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 7 x^{n+r+2} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 5 x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 9 x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2 x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)=\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 3 x^{1+n+r} a_n (n+r)(n+r-1)=\sum_{n=1}^{\infty} 3 a_{n-1}(n+r-1)(n+r-2) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 7 x^{n+r+2} a_n (n+r)=\sum_{n=2}^{\infty} 7 a_{n-2}(n+r-2) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n (n+r)=\sum_{n=1}^{\infty} 8 a_{n-1}(n+r-1) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 9 x^{n+r+2} a_n=\sum_{n=2}^{\infty} 9 a_{n-2} x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 2 x^{1+n+r} a_n=\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 9a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r}a_n(n+r)(n+r-1) + 5x^{n+r}a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + 5x^r r - x^r\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-1-r}{r+2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r)+3a_{n-1}(n+r-1)(n+r-2)+3a_n(n+r)(n+r-1) \\ +7a_{n-2}(n+r-2)+8a_{n-1}(n+r-1)+5a_n(n+r)+9a_{n-2}+2a_{n-1}-a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + 3n^2 a_{n-1} + 2nra_{n-2} + 6nra_{n-1} + r^2 a_{n-2} + 3r^2 a_{n-1} + 2na_{n-2} - na_{n-1} + 2ra_{n-2} - ra_{n-1}}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{(-9a_{n-2} - 27a_{n-1})n^2 + (-24a_{n-2} - 9a_{n-1})n - 16a_{n-2}}{27n^2 + 36n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r+2}$	$-\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r^2 + 2r - 4}{3r^2 + 14r + 15}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{7}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r+2}$	$-\frac{4}{7}$
a_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	$-\frac{7}{45}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-3r^4 - 2r^3 + 69r^2 + 192r + 144}{9r^4 + 93r^3 + 346r^2 + 552r + 320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{970}{2457}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r+2}$	$-\frac{4}{7}$
a_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	$-\frac{7}{45}$
a_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{970}{2457}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3r^6 - 28r^5 - 606r^4 - 3196r^3 - 7445r^2 - 7976r - 3152}{(3r^2 + 26r + 55)(9r^3 + 57r^2 + 118r + 80)(r + 3)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = -\frac{5707}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r+2}$	$-\frac{4}{7}$
a_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	$-\frac{7}{45}$
a_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{970}{2457}$
a_4	$\frac{3r^6-28r^5-606r^4-3196r^3-7445r^2-7976r-3152}{(3r^2+26r+55)(9r^3+57r^2+118r+80)(r+3)}$	$-\frac{5707}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{180r^6 + 3066r^5 + 20450r^4 + 66790r^3 + 107050r^2 + 68544r + 2720}{(3r^2 + 32r + 84)(r + 3)(9r^2 + 39r + 40)(3r + 11)(r + 4)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{13568}{300105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r+2}$	$-\frac{4}{7}$
a_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	$-\frac{7}{45}$
a_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{970}{2457}$
a_4	$\frac{3r^6-28r^5-606r^4-3196r^3-7445r^2-7976r-3152}{(3r^2+26r+55)(9r^3+57r^2+118r+80)(r+3)}$	$-\frac{5707}{22680}$
a_5	$\frac{180r^6+3066r^5+20450r^4+66790r^3+107050r^2+68544r+2720}{(3r^2+32r+84)(r+3)(9r^2+39r+40)(3r+11)(r+4)}$	$\frac{13568}{300105}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{4x}{7} - \frac{7x^2}{45} + \frac{970x^3}{2457} - \frac{5707x^4}{22680} + \frac{13568x^5}{300105} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-1-r}{r+2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}b_{n-2}(n+r-2)\,(n-3+r)+3b_{n-1}(n+r-1)\,(n+r-2)+3b_n(n+r)\,(n+r-1)\\ + 7b_{n-2}(n+r-2) + 8b_{n-1}(n+r-1) + 5b_n(n+r) + 9b_{n-2} + 2b_{n-1} - b_n = 0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-2} + 3n^2 b_{n-1} + 2nrb_{n-2} + 6nrb_{n-1} + r^2 b_{n-2} + 3r^2 b_{n-1} + 2nb_{n-2} - nb_{n-1} + 2rb_{n-2} - rb_{n-1} +}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{(-b_{n-2} - 3b_{n-1})\,n^2 + 7nb_{n-1} - 4b_{n-1}}{3n^2 - 4n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r+2}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2r^2 + 2r - 4}{3r^2 + 14r + 15}$$

Which for the root $r = -1$ becomes

$$b_2 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r+2}$	0
b_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	-1

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-3r^4 - 2r^3 + 69r^2 + 192r + 144}{9r^4 + 93r^3 + 346r^2 + 552r + 320}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r+2}$	0
b_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	-1
b_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{3r^6 - 28r^5 - 606r^4 - 3196r^3 - 7445r^2 - 7976r - 3152}{(3r^2 + 26r + 55)(9r^3 + 57r^2 + 118r + 80)(r + 3)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r+2}$	0
b_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	-1
b_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{2}{3}$
b_4	$\frac{3r^6-28r^5-606r^4-3196r^3-7445r^2-7976r-3152}{(3r^2+26r+55)(9r^3+57r^2+118r+80)(r+3)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{180r^6 + 3066r^5 + 20450r^4 + 66790r^3 + 107050r^2 + 68544r + 2720}{(3r^2 + 32r + 84)(r + 3)(9r^2 + 39r + 40)(3r + 11)(r + 4)}$$

Which for the root $r=-1$ becomes

$$b_5 = -\frac{10}{33}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{r+2}$	0
b_2	$\frac{2r^2+2r-4}{3r^2+14r+15}$	-1
b_3	$\frac{-3r^4-2r^3+69r^2+192r+144}{9r^4+93r^3+346r^2+552r+320}$	$\frac{2}{3}$
b_4	$\frac{3r^6-28r^5-606r^4-3196r^3-7445r^2-7976r-3152}{(3r^2+26r+55)(9r^3+57r^2+118r+80)(r+3)}$	0
b_5	$\frac{180r^6+3066r^5+20450r^4+66790r^3+107050r^2+68544r+2720}{(3r^2+32r+84)(r+3)(9r^2+39r+40)(3r+11)(r+4)}$	$-\frac{10}{33}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\ &= \frac{1-x^2+\frac{2x^3}{3}-\frac{10x^5}{33}+O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x^{\frac{1}{3}}\left(1-\frac{4x}{7}-\frac{7x^2}{45}+\frac{970x^3}{2457}-\frac{5707x^4}{22680}+\frac{13568x^5}{300105}+O(x^6)\right) \\ &+ \frac{c_2\left(1-x^2+\frac{2x^3}{3}-\frac{10x^5}{33}+O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^{\frac{1}{3}}\left(1-\frac{4x}{7}-\frac{7x^2}{45}+\frac{970x^3}{2457}-\frac{5707x^4}{22680}+\frac{13568x^5}{300105}+O(x^6)\right) \\ &+ \frac{c_2\left(1-x^2+\frac{2x^3}{3}-\frac{10x^5}{33}+O(x^6)\right)}{x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{4x}{7} - \frac{7x^2}{45} + \frac{970x^3}{2457} - \frac{5707x^4}{22680} + \frac{13568x^5}{300105} + O(x^6)\right) + \frac{c_2 \left(1 - x^2 + \frac{2x^3}{3} - \frac{10x^5}{33} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{4x}{7} - \frac{7x^2}{45} + \frac{970x^3}{2457} - \frac{5707x^4}{22680} + \frac{13568x^5}{300105} + O(x^6)\right) + \frac{c_2 \left(1 - x^2 + \frac{2x^3}{3} - \frac{10x^5}{33} + O(x^6)\right)}{x}$$

Verified OK.

14.6.1 Maple step by step solution

Let's solve

$x^2(x^2+3x+3)\,y'' + (7x^3+8x^2+5x)\,y' + (9x^2+2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+2x-1)y}{x^2(x^2+3x+3)} - \frac{(7x^2+8x+5)y'}{x(x^2+3x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(7x^2+8x+5)y'}{x(x^2+3x+3)} + \frac{(9x^2+2x-1)y}{x^2(x^2+3x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{7x^2+8x+5}{x(x^2+3x+3)}, P_3(x) = \frac{9x^2+2x-1}{x^2(x^2+3x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = -\frac{1}{3}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+3x+3)\,y'' + x(7x^2+8x+5)\,y' + (9x^2+2x-1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+3r)\,x^r + \left(a_1(2+r)\,(2+3r) + a_0(1+r)\,(2+3r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(3k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(2+r)(2+3r) + a_0(1+r)(2+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(1+r)a_0}{2+r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-2}(k+r+1)^2 + 3\left(k+r-\frac{1}{3}\right)(a_k(k+r+1) + a_{k-1}(k+r)) = 0$$

- Shift index using $k- >k+2$

$$a_k(k+r+3)^2 + 3\left(k+\frac{5}{3}+r\right)(a_{k+2}(k+r+3) + a_{k+1}(k+2+r)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k+3k^2a_{k+1}+2kra_k+6kra_{k+1}+r^2a_k+3r^2a_{k+1}+6ka_k+11ka_{k+1}+6ra_k+11ra_{k+1}+9a_k+10a_{k+1}}{(3k+5+3r)(k+r+3)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{k^2a_k+3k^2a_{k+1}+4ka_k+5ka_{k+1}+4a_k+2a_{k+1}}{(3k+2)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{k^2a_k+3k^2a_{k+1}+4ka_k+5ka_{k+1}+4a_k+2a_{k+1}}{(3k+2)(k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{k^2a_k+3k^2a_{k+1}+\frac{20}{3}ka_k+13ka_{k+1}+\frac{100}{9}a_k+14a_{k+1}}{(3k+6)\left(k+\frac{10}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{k^2a_k+3k^2a_{k+1}+\frac{20}{3}ka_k+13ka_{k+1}+\frac{100}{9}a_k+14a_{k+1}}{(3k+6)\left(k+\frac{10}{3}\right)}, a_1 = -\frac{4a_0}{7}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{k^2a_k+3k^2a_{1+k}+4ka_k+5ka_{1+k}+4a_k+2a_{1+k}}{(3k+2)(k+2)}, a_1 = 0, b_{k+2} =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 43

```
Order:=6;
dsolve(x^2*(3+3*x+x^2)*diff(y(x),x$2)+x*(5+8*x+7*x^2)*diff(y(x),x)-(1-2*x-9*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}}\left(1-\frac{4}{7}x-\frac{7}{45}x^2+\frac{970}{2457}x^3-\frac{5707}{22680}x^4+\frac{13568}{300105}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-x^2+\frac{2}{3}x^3-\frac{10}{33}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x^2*(3+3*x+x^2)*y''[x]+x*(5+8*x+7*x^2)*y'[x]-(1-2*x-9*x^2)*y[x]==0,y[x
```

$$y(x) \to \frac{c_2\left(-\frac{10x^5}{33}+\frac{2x^3}{3}-x^2+1\right)}{x}+c_1\sqrt[3]{x}\left(\frac{13568x^5}{300105}-\frac{5707x^4}{22680}+\frac{970x^3}{2457}-\frac{7x^2}{45}-\frac{4x}{7}+1\right)$$

14.7 problem 4

Internal problem ID [1298]
Internal file name [`OUTPUT/1299_Sunday_June_05_2022_02_08_58_AM_10712497/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + x\left(4x^2+2x+7\right)y' - \left(-7x^2-4x+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(4x^3+2x^2+7x\right)y' + \left(7x^2+4x-1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4x^2+2x+7}{4x}$$
$$q(x) = \frac{7x^2+4x-1}{4x^2}$$

Table 542: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2+2x+7}{4x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{7x^2+4x-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (4x^3 + 2x^2 + 7x)\, y' + (7x^2 + 4x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2} \right) + (4x^3 + 2x^2 + 7x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (7x^2 + 4x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n = \sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) + 7x^{n+r}a_n(n+r) - a_nx^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 7x^r a_0 r - a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 7x^r r - x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 3r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2}{4r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4a_n(n+r)(n+r-1) + 4a_{n-2}(n+r-2) \\ &+ 2a_{n-1}(n+r-1) + 7a_n(n+r) + 7a_{n-2} + 4a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4na_{n-2} + 2na_{n-1} + 4ra_{n-2} + 2ra_{n-1} - a_{n-2} + 2a_{n-1}}{4n^2 + 8nr + 4r^2 + 3n + 3r - 1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = \frac{(-8a_{n-2} - 4a_{n-1})\, n - 5a_{n-1}}{8n^2 + 10n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{4r+3}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-16r^2 - 36r - 9}{16r^3 + 88r^2 + 141r + 63}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -\frac{19}{104}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{4r+3}$	$-\frac{1}{2}$
a_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{19}{104}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64r^3 + 440r^2 + 892r + 534}{64r^5 + 784r^4 + 3644r^3 + 7931r^2 + 7905r + 2772}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = \frac{1571}{10608}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{4r+3}$	$-\frac{1}{2}$
a_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{19}{104}$
a_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{1571}{10608}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^5 + 3136r^4 + 13968r^3 + 26804r^2 + 19001r + 600}{256r^7 + 5376r^6 + 46816r^5 + 218064r^4 + 582505r^3 + 882588r^2 + 689895r + 207900}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{3225}{198016}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{4r+3}$	$-\frac{1}{2}$
a_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{19}{104}$
a_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{1571}{10608}$
a_4	$\frac{256r^5+3136r^4+13968r^3+26804r^2+19001r+600}{256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900}$	$\frac{3225}{198016}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1536r^6 - 30208r^5 - 236640r^4 - 938240r^3 - 1962774r^2 - 2015622r - 768150}{(4r^2 + 43r + 114)\left(256r^7 + 5376r^6 + 46816r^5 + 218064r^4 + 582505r^3 + 882588r^2 + 689895r + 2079\right.}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = -\frac{752183}{29702400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{4r+3}$	$-\frac{1}{2}$
a_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{19}{104}$
a_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{1571}{10608}$
a_4	$\frac{256r^5+3136r^4+13968r^3+26804r^2+19001r+600}{256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900}$	$\frac{3225}{198016}$
a_5	$\frac{-1536r^6-30208r^5-236640r^4-938240r^3-1962774r^2-2015622r-768150}{(4r^2+43r+114)(256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900)}$	$-\frac{752183}{29702400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 - \frac{x}{2} - \frac{19x^2}{104} + \frac{1571x^3}{10608} + \frac{3225x^4}{198016} - \frac{752183x^5}{29702400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{2}{4r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4b_n(n+r)(n+r-1) + 4b_{n-2}(n+r-2) \\ &+ 2b_{n-1}(n+r-1) + 7b_n(n+r) + 7b_{n-2} + 4b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4nb_{n-2} + 2nb_{n-1} + 4rb_{n-2} + 2rb_{n-1} - b_{n-2} + 2b_{n-1}}{4n^2 + 8nr + 4r^2 + 3n + 3r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{(-4b_{n-2} - 2b_{n-1})\,n + 5b_{n-2}}{4n^2 - 5n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{4r+3}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-16r^2 - 36r - 9}{16r^3 + 88r^2 + 141r + 63}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{11}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{4r+3}$	2
b_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{11}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{64r^3 + 440r^2 + 892r + 534}{64r^5 + 784r^4 + 3644r^3 + 7931r^2 + 7905r + 2772}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{7}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{4r+3}$	2
b_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{11}{6}$
b_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$-\frac{1}{7}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256r^5 + 3136r^4 + 13968r^3 + 26804r^2 + 19001r + 600}{256r^7 + 5376r^6 + 46816r^5 + 218064r^4 + 582505r^3 + 882588r^2 + 689895r + 207900}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{895}{1848}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{4r+3}$	2
b_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{11}{6}$
b_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$-\frac{1}{7}$
b_4	$\frac{256r^5+3136r^4+13968r^3+26804r^2+19001r+600}{256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900}$	$\frac{895}{1848}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-1536r^6 - 30208r^5 - 236640r^4 - 938240r^3 - 1962774r^2 - 2015622r - 768150}{(4r^2 + 43r + 114)(256r^7 + 5376r^6 + 46816r^5 + 218064r^4 + 582505r^3 + 882588r^2 + 689895r + 2079}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{499}{13860}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{4r+3}$	2
b_2	$\frac{-16r^2-36r-9}{16r^3+88r^2+141r+63}$	$-\frac{11}{6}$
b_3	$\frac{64r^3+440r^2+892r+534}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$-\frac{1}{7}$
b_4	$\frac{256r^5+3136r^4+13968r^3+26804r^2+19001r+600}{256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900}$	$\frac{895}{1848}$
b_5	$\frac{-1536r^6-30208r^5-236640r^4-938240r^3-1962774r^2-2015622r-768150}{(4r^2+43r+114)(256r^7+5376r^6+46816r^5+218064r^4+582505r^3+882588r^2+689895r+207900)}$	$-\frac{499}{13860}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{4}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\ &= \frac{1+2x-\frac{11x^2}{6}-\frac{x^3}{7}+\frac{895x^4}{1848}-\frac{499x^5}{13860}+O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x)+c_2y_2(x) \\ &= c_1x^{\frac{1}{4}}\left(1-\frac{x}{2}-\frac{19x^2}{104}+\frac{1571x^3}{10608}+\frac{3225x^4}{198016}-\frac{752183x^5}{29702400}+O\left(x^6\right)\right) \\ &+ \frac{c_2\left(1+2x-\frac{11x^2}{6}-\frac{x^3}{7}+\frac{895x^4}{1848}-\frac{499x^5}{13860}+O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^{\frac{1}{4}}\left(1-\frac{x}{2}-\frac{19x^2}{104}+\frac{1571x^3}{10608}+\frac{3225x^4}{198016}-\frac{752183x^5}{29702400}+O\left(x^6\right)\right) \\ &+ \frac{c_2\left(1+2x-\frac{11x^2}{6}-\frac{x^3}{7}+\frac{895x^4}{1848}-\frac{499x^5}{13860}+O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1x^{\frac{1}{4}}\left(1-\frac{x}{2}-\frac{19x^2}{104}+\frac{1571x^3}{10608}+\frac{3225x^4}{198016}-\frac{752183x^5}{29702400}+O\left(x^6\right)\right) \\ &+ \frac{c_2\left(1+2x-\frac{11x^2}{6}-\frac{x^3}{7}+\frac{895x^4}{1848}-\frac{499x^5}{13860}+O(x^6)\right)}{x} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1x^{\frac{1}{4}}\left(1-\frac{x}{2}-\frac{19x^2}{104}+\frac{1571x^3}{10608}+\frac{3225x^4}{198016}-\frac{752183x^5}{29702400}+O\left(x^6\right)\right) \\ &+ \frac{c_2\left(1+2x-\frac{11x^2}{6}-\frac{x^3}{7}+\frac{895x^4}{1848}-\frac{499x^5}{13860}+O(x^6)\right)}{x} \end{aligned}$$

Verified OK.

14.7.1 Maple step by step solution

Let's solve

$4x^2y'' + (4x^3 + 2x^2 + 7x)\, y' + (7x^2 + 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+4x-1)y}{4x^2} - \frac{(4x^2+2x+7)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2+2x+7)y'}{4x} + \frac{(7x^2+4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+2x+7}{4x}, P_3(x) = \frac{7x^2+4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{7}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + x(4x^2 + 2x + 7)\, y' + (7x^2 + 4x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-1+4r)\, x^{r}+(a_{1}(2+r)\,(3+4r)+2a_{0}(2+r))\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(k+r+1)\,(4k+4r-\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+4r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{4}\right\}$

- Each term must be 0

  $a_{1}(2+r)\,(3+4r)+2a_{0}(2+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=-\frac{2a_{0}}{3+4r}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{4}\right)(k+r+1)\, a_{k}+(4a_{k-2}+2a_{k-1})\, k+(4a_{k-2}+2a_{k-1})\, r-a_{k-2}+2a_{k-1}=0$

- Shift index using $k->k+2$

  $4\left(k+\frac{7}{4}+r\right)(k+3+r)\, a_{k+2}+(4a_{k}+2a_{k+1})\,(k+2)+(4a_{k}+2a_{k+1})\, r-a_{k}+2a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4ka_{k}+2ka_{k+1}+4ra_{k}+2ra_{k+1}+7a_{k}+6a_{k+1}}{(4k+4r+7)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{4ka_{k}+2ka_{k+1}+3a_{k}+4a_{k+1}}{(4k+3)(k+2)}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{4k a_k + 2k a_{k+1} + 3a_k + 4a_{k+1}}{(4k+3)(k+2)}, a_1 = 2a_0\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -\frac{4k a_k + 2k a_{k+1} + 8a_k + \frac{13}{2} a_{k+1}}{(4k+8)\left(k+\frac{13}{4}\right)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k a_k + 2k a_{k+1} + 8a_k + \frac{13}{2} a_{k+1}}{(4k+8)\left(k+\frac{13}{4}\right)}, a_1 = -\frac{a_0}{2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{4k a_k + 2k a_{1+k} + 3a_k + 4a_{1+k}}{(4k+3)(k+2)}, a_1 = 2a_0, b_{k+2} = -\frac{4k b_k + 2k b_{1}}{(4k+}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+x*(7+2*x+4*x^2)*diff(y(x),x)-(1-4*x-7*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{c_2 x^{\frac{5}{4}}\left(1-\frac{1}{2}x-\frac{19}{104}x^2+\frac{1571}{10608}x^3+\frac{3225}{198016}x^4-\frac{752183}{29702400}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+2x-\frac{11}{6}x^2-\frac{1}{7}x^3+\frac{895}{1848}x^4-\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*y''[x]+x*(7+2*x+4*x^2)*y'[x]-(1-4*x-7*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[4]{x}\left(-\frac{752183x^5}{29702400}+\frac{3225x^4}{198016}+\frac{1571x^3}{10608}-\frac{19x^2}{104}-\frac{x}{2}+1\right)+\frac{c_2\left(-\frac{499x^5}{13860}+\frac{895x^4}{1848}-\frac{x^3}{7}-\frac{11x^2}{6}+2x+1\right)}{x}$$

14.8 problem 5

Internal problem ID [1299]
Internal file name [OUTPUT/1300_Sunday_June_05_2022_02_09_01_AM_25601244/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{12x^2(x+1)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x^2+35x+11}{12x\,(x+1)}$$
$$q(x)=\frac{5x^2+10x-1}{12x^2\,(x+1)}$$

Table 544: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2+35x+11}{12x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{5x^2+10x-1}{12x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$12x^2(x+1)\, y'' + \left(3x^3 + 35x^2 + 11x\right) y' + \left(5x^2 + 10x - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &12x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ \left(3x^3 + 35x^2 + 11x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) \\ &+ \left(5x^2 + 10x - 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 12x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 35x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 11x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 10x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 35x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 35a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 10x^{1+n+r} a_n = \sum_{n=1}^{\infty} 10a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 35a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 11x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1}x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$12x^{n+r}a_n(n+r)(n+r-1) + 11x^{n+r}a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$12x^r a_0 r(-1+r) + 11x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(12x^r r(-1+r) + 11x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(12r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$12r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(12r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{12}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -1$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &12a_{n-1}(n+r-1)(n+r-2) + 12a_n(n+r)(n+r-1) + 3a_{n-2}(n+r-2) \\ &+ 35a_{n-1}(n+r-1) + 11a_n(n+r) + 5a_{n-2} + 10a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4na_{n-1} + 4ra_{n-1} + a_{n-2} + a_{n-1}}{4n + 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{(-12n - 7)\, a_{n-1} - 3a_{n-2}}{12n + 7} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r+8}{9+4r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{28}{31}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{4r+8}{9+4r}$	$\frac{28}{31}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-16r^2-80r-95}{16r^2+88r+117}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{1111}{1333}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{4r+8}{9+4r}$	$\frac{28}{31}$
a_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{1111}{1333}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64r^3+576r^2+1656r+1511}{64r^3+624r^2+1964r+1989}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{57493}{73315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{4r+8}{9+4r}$	$\frac{28}{31}$
a_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{1111}{1333}$
a_4	$\frac{64r^3+576r^2+1656r+1511}{64r^3+624r^2+1964r+1989}$	$\frac{57493}{73315}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{4(64r^4 + 896r^3 + 4532r^2 + 9770r + 7529)}{(16r^2 + 88r + 117)(17 + 4r)(21 + 4r)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{3668716}{4912105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{4r+8}{9+4r}$	$\frac{28}{31}$
a_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{1111}{1333}$
a_4	$\frac{64r^3+576r^2+1656r+1511}{64r^3+624r^2+1964r+1989}$	$\frac{57493}{73315}$
a_5	$-\frac{4(64r^4+896r^3+4532r^2+9770r+7529)}{(16r^2+88r+117)(17+4r)(21+4r)}$	$-\frac{3668716}{4912105}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - x + \frac{28x^2}{31} - \frac{1111x^3}{1333} + \frac{57493x^4}{73315} - \frac{3668716x^5}{4912105} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -1$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}12b_{n-1}(n+r-1)(n+r-2) + 12b_n(n+r)(n+r-1) + 3b_{n-2}(n+r-2) \\ + 35b_{n-1}(n+r-1) + 11b_n(n+r) + 5b_{n-2} + 10b_{n-1} - b_n = 0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4nb_{n-1} + 4rb_{n-1} + b_{n-2} + b_{n-1}}{4n + 4r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_n = \frac{-4nb_{n-1} - b_{n-2}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r + 8}{9 + 4r}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_2 = \frac{7}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	$\frac{4r+8}{9+4r}$	$\frac{7}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-16r^2 - 80r - 95}{16r^2 + 88r + 117}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_3 = -\frac{19}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	$\frac{4r+8}{9+4r}$	$\frac{7}{8}$
b_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{19}{24}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{64r^3 + 576r^2 + 1656r + 1511}{64r^3 + 624r^2 + 1964r + 1989}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_4 = \frac{283}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	$\frac{4r+8}{9+4r}$	$\frac{7}{8}$
b_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{19}{24}$
b_4	$\frac{64r^3+576r^2+1656r+1511}{64r^3+624r^2+1964r+1989}$	$\frac{283}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{4(64r^4 + 896r^3 + 4532r^2 + 9770r + 7529)}{(16r^2 + 88r + 117)(17 + 4r)(21 + 4r)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_5 = -\frac{1339}{1920}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	$\frac{4r+8}{9+4r}$	$\frac{7}{8}$
b_3	$\frac{-16r^2-80r-95}{16r^2+88r+117}$	$-\frac{19}{24}$
b_4	$\frac{64r^3+576r^2+1656r+1511}{64r^3+624r^2+1964r+1989}$	$\frac{283}{384}$
b_5	$-\frac{4(64r^4+896r^3+4532r^2+9770r+7529)}{(16r^2+88r+117)(17+4r)(21+4r)}$	$-\frac{1339}{1920}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - x + \frac{7x^2}{8} - \frac{19x^3}{24} + \frac{283x^4}{384} - \frac{1339x^5}{1920} + O(x^6)}{x^{\frac{1}{4}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}}\left(1 - x + \frac{28x^2}{31} - \frac{1111x^3}{1333} + \frac{57493x^4}{73315} - \frac{3668716x^5}{4912105} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - x + \frac{7x^2}{8} - \frac{19x^3}{24} + \frac{283x^4}{384} - \frac{1339x^5}{1920} + O(x^6)\right)}{x^{\frac{1}{4}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}}\left(1 - x + \frac{28x^2}{31} - \frac{1111x^3}{1333} + \frac{57493x^4}{73315} - \frac{3668716x^5}{4912105} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - x + \frac{7x^2}{8} - \frac{19x^3}{24} + \frac{283x^4}{384} - \frac{1339x^5}{1920} + O(x^6)\right)}{x^{\frac{1}{4}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1 - x + \frac{28x^2}{31} - \frac{1111x^3}{1333} + \frac{57493x^4}{73315} - \frac{3668716x^5}{4912105} + O(x^6)\right) + \frac{c_2 \left(1 - x + \frac{7x^2}{8} - \frac{19x^3}{24} + \frac{283x^4}{384} - \frac{1339x^5}{1920} + O(x^6)\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - x + \frac{28x^2}{31} - \frac{1111x^3}{1333} + \frac{57493x^4}{73315} - \frac{3668716x^5}{4912105} + O(x^6)\right) + \frac{c_2 \left(1 - x + \frac{7x^2}{8} - \frac{19x^3}{24} + \frac{283x^4}{384} - \frac{1339x^5}{1920} + O(x^6)\right)}{x^{\frac{1}{4}}}$$

Verified OK.

14.8.1 Maple step by step solution

Let's solve

$12x^2(x+1)\, y'' + (3x^3 + 35x^2 + 11x)\, y' + (5x^2 + 10x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+10x-1)y}{12x^2(x+1)} - \frac{(3x^2+35x+11)y'}{12x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+35x+11)y'}{12x(x+1)} + \frac{(5x^2+10x-1)y}{12x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3x^2+35x+11}{12x(x+1)}, P_3(x) = \frac{5x^2+10x-1}{12x^2(x+1)}\right]$

- ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{4}$

- ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $12x^2(x+1)\,y'' + x(3x^2+35x+11)\,y' + (5x^2+10x-1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(12u^3-24u^2+12u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right) + (5u^2-6)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$3a_0r(3+4r)\,u^{-1+r}+\left(3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r)\right)u^r+\left(3a_2(2+r)\,(11+4r)-2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(3+4r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{4}\right\}$

- The coefficients of each power of u must be 0

  $[3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r)=0,3a_2(2+r)\,(11+4r)-2a_1(7+4r)\,(4+3r)+2a$

- Solve for the dependent coefficient(s)

  $\left\{a_1=\frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)},a_2=\frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $12(-2a_k+a_{k-1}+a_{k+1})\,k^2+\left(24(-2a_k+a_{k-1}+a_{k+1})\,r-26a_k+3a_{k-2}-10a_{k-1}+33a_{k+1}\right)k+$

- Shift index using $k->k+2$

  $12(-2a_{k+2}+a_{k+1}+a_{k+3})\,(k+2)^2+\left(24(-2a_{k+2}+a_{k+1}+a_{k+3})\,r-26a_{k+2}+3a_k-10a_{k+1}+3\right.$

- Recursion relation that defines series solution to ODE

  $a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+24kra_{k+1}-48kra_{k+2}+12r^2a_{k+1}-24r^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+3ra_k+38ra_{k+1}-122r}{3(4k^2+8kr+4r^2+27k+27r+45)}$

- Recursion relation for $r=0$

  $a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)},a_1=\frac{2a_0}{7},a_2=\right.$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^k,a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)},a_1=\frac{2a_0}{7},\right.$

- Recursion relation for $r=-\frac{3}{4}$

  $a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}$

- Solution for $r=-\frac{3}{4}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{3}{4}},a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)},a_1=0,a_2\right.$

- Revert the change of variables $u=x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2 a_{k+1} - 24k^2 a_{k+2} + 3k a_k + 20k a_{k+1} - 86k a_{k+2} + \frac{11}{4} a_k + \frac{17}{4} a_{k+1} - 76 a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0,\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{3}{4}}\right), a_{k+3} = -\frac{12k^2 a_{1+k} - 24k^2 a_{k+2} + 3k a_k + 38k a_{1+k} - 122k a_{k+2} + 5a}{3(4k^2+27k+45)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 47

```
Order:=6;
dsolve(12*x^2*(1+x)*diff(y(x),x$2)+x*(11+35*x+3*x^2)*diff(y(x),x)-(1-10*x-5*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_1\left(1-x+\frac{7}{8}x^2-\frac{19}{24}x^3+\frac{283}{384}x^4-\frac{1339}{1920}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{4}}} + c_2x^{\frac{1}{3}}\left(1-x+\frac{28}{31}x^2-\frac{1111}{1333}x^3+\frac{57493}{73315}x^4-\frac{3668716}{4912105}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 86

```
AsymptoticDSolveValue[12*x^2*(1+x)*y''[x]+x*(11+35*x+3*x^2)*y'[x]-(1-10*x-5*x^2)*y[x]==0,y[x
```

$$y(x) \to c_1\sqrt[3]{x}\left(-\frac{3668716x^5}{4912105}+\frac{57493x^4}{73315}-\frac{1111x^3}{1333}+\frac{28x^2}{31}-x+1\right) + \frac{c_2\left(-\frac{1339x^5}{1920}+\frac{283x^4}{384}-\frac{19x^3}{24}+\frac{7x^2}{8}-x+1\right)}{\sqrt[4]{x}}$$

14.9 problem 6

Internal problem ID [1300]
Internal file name [OUTPUT/1301_Sunday_June_05_2022_02_09_05_AM_93571922/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2(10x^2+x+5)\,y''+x(48x^2+3x+4)\,y'+(36x^2+x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$(10x^4+x^3+5x^2)\,y''+(48x^3+3x^2+4x)\,y'+(36x^2+x)\,y=0$$

Or

$$x(10y''x^3+48y'x^2+x^2y''+36yx+3y'x+5y''x+y+4y')=0$$

For $x\neq 0$ the above simplifies to

$$(10x^3+x^2+5x)\,y''+(48x^2+3x+4)\,y'+(36x+1)\,y=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(10x^4+x^3+5x^2)\,y''+(48x^3+3x^2+4x)\,y'+(36x^2+x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{48x^2 + 3x + 4}{x\,(10x^2 + x + 5)} \\ q(x) &= \frac{36x + 1}{x\,(10x^2 + x + 5)} \end{aligned}$$

Table 546: Table $p(x), q(x)$ singularites.

$p(x) = \frac{48x^2+3x+4}{x(10x^2+x+5)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{199}}{20} - \frac{1}{20}$	"regular"
$x = \frac{i\sqrt{199}}{20} - \frac{1}{20}$	"regular"

$q(x) = \frac{36x+1}{x(10x^2+x+5)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{199}}{20} - \frac{1}{20}$	"regular"
$x = \frac{i\sqrt{199}}{20} - \frac{1}{20}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{199}}{20} - \frac{1}{20}, \frac{i\sqrt{199}}{20} - \frac{1}{20}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2\left(10x^2 + x + 5\right) y'' + \left(48x^3 + 3x^2 + 4x\right) y' + \left(36x^2 + x\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2} \end{aligned}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^{2} \left(10 x^{2}+x +5\right) \left(\sum_{n =0}^{\infty} \left(n +r \right) \left(n +r -1\right) a_{n} x^{n +r -2}\right)\\
&+\left(48 x^{3}+3 x^{2}+4 x \right) \left(\sum_{n =0}^{\infty} \left(n +r \right) a_{n} x^{n +r -1}\right)+\left(36 x^{2}+x \right) \left(\sum_{n =0}^{\infty} a_{n} x^{n +r}\right) = 0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n =0}^{\infty} 10 x^{n +r +2} a_{n} \left(n +r \right) \left(n +r -1\right)\right)+\left(\sum_{n =0}^{\infty} x^{1+n +r} a_{n} \left(n +r \right) \left(n +r -1\right)\right)\\
&+\left(\sum_{n =0}^{\infty} 5 x^{n +r} a_{n} \left(n +r \right) \left(n +r -1\right)\right)+\left(\sum_{n =0}^{\infty} 48 x^{n +r +2} a_{n} \left(n +r \right)\right)\\
&+\left(\sum_{n =0}^{\infty} 3 x^{1+n +r} a_{n} \left(n +r \right)\right)+\left(\sum_{n =0}^{\infty} 4 x^{n +r} a_{n} \left(n +r \right)\right)\\
&+\left(\sum_{n =0}^{\infty} 36 x^{n +r +2} a_{n}\right)+\left(\sum_{n =0}^{\infty} x^{1+n +r} a_{n}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n +r$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n +r}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n =0}^{\infty} 10 x^{n +r +2} a_{n} \left(n +r \right) \left(n +r -1\right) &= \sum_{n =2}^{\infty} 10 a_{n -2} \left(n +r -2\right) \left(n -3+r \right) x^{n +r} \\
\sum_{n =0}^{\infty} x^{1+n +r} a_{n} \left(n +r \right) \left(n +r -1\right) &= \sum_{n =1}^{\infty} a_{n -1} \left(n +r -1\right) \left(n +r -2\right) x^{n +r} \\
\sum_{n =0}^{\infty} 48 x^{n +r +2} a_{n} \left(n +r \right) &= \sum_{n =2}^{\infty} 48 a_{n -2} \left(n +r -2\right) x^{n +r} \\
\sum_{n =0}^{\infty} 3 x^{1+n +r} a_{n} \left(n +r \right) &= \sum_{n =1}^{\infty} 3 a_{n -1} \left(n +r -1\right) x^{n +r} \\
\sum_{n =0}^{\infty} 36 x^{n +r +2} a_{n} &= \sum_{n =2}^{\infty} 36 a_{n -2} x^{n +r}
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 10 a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 5 x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 48 a_{n-2}(n+r-2) x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 3 a_{n-1}(n+r-1) x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4 x^{n+r} a_n(n+r)\right)+\left(\sum_{n=2}^{\infty} 36 a_{n-2} x^{n+r}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
5 x^{n+r} a_n(n+r)(n+r-1)+4 x^{n+r} a_n(n+r)=0
$$

When $n=0$ the above becomes

$$
5 x^r a_0 r(-1+r)+4 x^r a_0 r=0
$$

Or

$$
\left(5 x^r r(-1+r)+4 x^r r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
x^r r(-1+5 r)=0
$$

Since the above is true for all x then the indicial equation becomes

$$
5 r^2-r=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{1}{5} \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-1+5r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{5}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n=1$ in Eq. (2B) gives

$$a_1 = \frac{-1-r}{5r+4}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 10a_{n-2}(n+r-2)(n-3+r)+a_{n-1}(n+r-1)(n+r-2)+5a_n(n+r)(n+r-1) \\ +48a_{n-2}(n+r-2)+3a_{n-1}(n+r-1)+4a_n(n+r)+36a_{n-2}+a_{n-1}=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{10na_{n-2}+na_{n-1}+10ra_{n-2}+ra_{n-1}-2a_{n-2}}{5n-1+5r} \tag{4}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_n = \frac{(-50a_{n-2}-5a_{n-1})\,n-a_{n-1}}{25n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{5r+4}$	$-\frac{6}{25}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-49r^2 - 127r - 70}{25r^2 + 65r + 36}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_2 = -\frac{1217}{625}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{5r+4}$	$-\frac{6}{25}$
a_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{1217}{625}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{99r^3 + 554r^2 + 933r + 462}{125r^3 + 675r^2 + 1090r + 504}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_3 = \frac{41972}{46875}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{5r+4}$	$-\frac{6}{25}$
a_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{1217}{625}$
a_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{41972}{46875}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2351r^4 + 21570r^3 + 68329r^2 + 86470r + 35392}{625r^4 + 5750r^3 + 18275r^2 + 23230r + 9576}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_4 = \frac{1447799}{390625}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{5r+4}$	$-\frac{6}{25}$
a_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{1217}{625}$
a_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{41972}{46875}$
a_4	$\frac{2351r^4+21570r^3+68329r^2+86470r+35392}{625r^4+5750r^3+18275r^2+23230r+9576}$	$\frac{1447799}{390625}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-7301r^5 - 103595r^4 - 551337r^3 - 1357653r^2 - 1517298r - 598304}{3125r^5 + 43750r^4 + 229375r^3 + 554750r^2 + 605400r + 229824}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_5 = -\frac{375253322}{146484375}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{5r+4}$	$-\frac{6}{25}$
a_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{1217}{625}$
a_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{41972}{46875}$
a_4	$\frac{2351r^4+21570r^3+68329r^2+86470r+35392}{625r^4+5750r^3+18275r^2+23230r+9576}$	$\frac{1447799}{390625}$
a_5	$\frac{-7301r^5-103595r^4-551337r^3-1357653r^2-1517298r-598304}{3125r^5+43750r^4+229375r^3+554750r^2+605400r+229824}$	$-\frac{375253322}{146484375}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{1}{5}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \right)$$
$$= x^{\frac{1}{5}} \left(1 - \frac{6x}{25} - \frac{1217x^2}{625} + \frac{41972x^3}{46875} + \frac{1447799x^4}{390625} - \frac{375253322x^5}{146484375} + O(x^6)\right)$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-1 - r}{5r + 4}$$

For $2 \leq n$ the recursive equation is

$$10b_{n-2}(n+r-2)(n-3+r) + b_{n-1}(n+r-1)(n+r-2) + 5b_n(n+r)(n+r-1) + 48b_{n-2}(n+r-2) + 3b_{n-1}(n+r-1) + 4b_n(n+r) + 36b_{n-2} + b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{10nb_{n-2} + nb_{n-1} + 10rb_{n-2} + rb_{n-1} - 2b_{n-2}}{5n - 1 + 5r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(-10b_{n-2} - b_{n-1})\,n + 2b_{n-2}}{5n - 1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{5r+4}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-49r^2 - 127r - 70}{25r^2 + 65r + 36}$$

Which for the root $r = 0$ becomes

$$b_2 = -\frac{35}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{5r+4}$	$-\frac{1}{4}$
b_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{35}{18}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{99r^3 + 554r^2 + 933r + 462}{125r^3 + 675r^2 + 1090r + 504}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{11}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{5r+4}$	$-\frac{1}{4}$
b_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{35}{18}$
b_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{11}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2351r^4 + 21570r^3 + 68329r^2 + 86470r + 35392}{625r^4 + 5750r^3 + 18275r^2 + 23230r + 9576}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{632}{171}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{5r+4}$	$-\frac{1}{4}$
b_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{35}{18}$
b_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{11}{12}$
b_4	$\frac{2351r^4+21570r^3+68329r^2+86470r+35392}{625r^4+5750r^3+18275r^2+23230r+9576}$	$\frac{632}{171}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-7301r^5 - 103595r^4 - 551337r^3 - 1357653r^2 - 1517298r - 598304}{3125r^5 + 43750r^4 + 229375r^3 + 554750r^2 + 605400r + 229824}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2671}{1026}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{5r+4}$	$-\frac{1}{4}$
b_2	$\frac{-49r^2-127r-70}{25r^2+65r+36}$	$-\frac{35}{18}$
b_3	$\frac{99r^3+554r^2+933r+462}{125r^3+675r^2+1090r+504}$	$\frac{11}{12}$
b_4	$\frac{2351r^4+21570r^3+68329r^2+86470r+35392}{625r^4+5750r^3+18275r^2+23230r+9576}$	$\frac{632}{171}$
b_5	$\frac{-7301r^5-103595r^4-551337r^3-1357653r^2-1517298r-598304}{3125r^5+43750r^4+229375r^3+554750r^2+605400r+229824}$	$-\frac{2671}{1026}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{x}{4} - \frac{35x^2}{18} + \frac{11x^3}{12} + \frac{632x^4}{171} - \frac{2671x^5}{1026} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{5}} \left(1 - \frac{6x}{25} - \frac{1217x^2}{625} + \frac{41972x^3}{46875} + \frac{1447799x^4}{390625} - \frac{375253322x^5}{146484375} + O(x^6)\right) \\ &\quad + c_2 \left(1 - \frac{x}{4} - \frac{35x^2}{18} + \frac{11x^3}{12} + \frac{632x^4}{171} - \frac{2671x^5}{1026} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{5}} \left(1 - \frac{6x}{25} - \frac{1217x^2}{625} + \frac{41972x^3}{46875} + \frac{1447799x^4}{390625} - \frac{375253322x^5}{146484375} + O(x^6)\right) \\ &\quad + c_2 \left(1 - \frac{x}{4} - \frac{35x^2}{18} + \frac{11x^3}{12} + \frac{632x^4}{171} - \frac{2671x^5}{1026} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{5}} \left(1 - \frac{6x}{25} - \frac{1217x^2}{625} + \frac{41972x^3}{46875} + \frac{1447799x^4}{390625} - \frac{375253322x^5}{146484375} + O(x^6)\right) + c_2 \left(1 - \frac{x}{4} - \frac{35x^2}{18} + \frac{11x^3}{12} + \frac{632x^4}{171} - \frac{2671x^5}{1026} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{5}} \left(1 - \frac{6x}{25} - \frac{1217x^2}{625} + \frac{41972x^3}{46875} + \frac{1447799x^4}{390625} - \frac{375253322x^5}{146484375} + O(x^6)\right) + c_2 \left(1 - \frac{x}{4} - \frac{35x^2}{18} + \frac{11x^3}{12} + \frac{632x^4}{171} - \frac{2671x^5}{1026} + O(x^6)\right)$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$x^2(10x^2 + x + 5)\, y'' + (48x^3 + 3x^2 + 4x)\, y' + (36x^2 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(36x+1)y}{x(10x^2+x+5)} - \frac{(48x^2+3x+4)y'}{x(10x^2+x+5)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(48x^2+3x+4)y'}{x(10x^2+x+5)} + \frac{(36x+1)y}{x(10x^2+x+5)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{48x^2+3x+4}{x(10x^2+x+5)}, P_3(x) = \frac{36x+1}{x(10x^2+x+5)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{4}{5}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $(36x+1)\, y + (48x^2+3x+4)\, y' + x(10x^2+x+5)\, y'' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+5r)\, x^{-1+r} + \left(a_1(1+r)\,(4+5r) + a_0(1+r)^2\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(5k+4+5r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+5r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{5}\right\}$

- Each term must be 0

$$a_1(1+r)(4+5r)+a_0(1+r)^2=0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(\left(a_k+10a_{k-1}+5a_{k+1}\right)k+\left(a_k+10a_{k-1}+5a_{k+1}\right)r+a_k+8a_{k-1}+4a_{k+1}\right)(k+r+1)=0$$

- Shift index using $k->k+1$

$$\left(\left(a_{k+1}+10a_k+5a_{k+2}\right)(k+1)+\left(a_{k+1}+10a_k+5a_{k+2}\right)r+a_{k+1}+8a_k+4a_{k+2}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+10ra_k+ra_{k+1}+18a_k+2a_{k+1}}{5k+5r+9}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}, 4a_1+a_0=0\right]$$

- Recursion relation for $r=\frac{1}{5}$

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}$$

- Solution for $r=\frac{1}{5}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{5}}, a_{k+2}=-\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}, 6a_1+\frac{36a_0}{25}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{5}}\right), a_{k+2}=-\frac{10ka_k+ka_{1+k}+18a_k+2a_{1+k}}{5k+9}, 4a_1+a_0=0, b_{k+2}=-\frac{10kb_k+}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(x^2*(5+x+10*x^2)*diff(y(x),x$2)+x*(4+3*x+48*x^2)*diff(y(x),x)+(x+36*x^2)*y(x)=0,y(x),
```

$$y(x) = c_1 x^{\frac{1}{5}} \left(1 - \frac{6}{25}x - \frac{1217}{625}x^2 + \frac{41972}{46875}x^3 + \frac{1447799}{390625}x^4 - \frac{375253322}{146484375}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 - \frac{1}{4}x - \frac{35}{18}x^2 + \frac{11}{12}x^3 + \frac{632}{171}x^4 - \frac{2671}{1026}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x^2*(5+x+10*x^2)*y''[x]+x*(4+3*x+48*x^2)*y'[x]+(x+36*x^2)*y[x]==0,y[x]
```

$$y(x) \to c_1 \sqrt[5]{x}\left(-\frac{375253322x^5}{146484375}+\frac{1447799x^4}{390625}+\frac{41972x^3}{46875}-\frac{1217x^2}{625}-\frac{6x}{25}+1\right) + c_2\left(-\frac{2671x^5}{1026}+\frac{632x^4}{171}+\frac{11x^3}{12}-\frac{35x^2}{18}-\frac{x}{4}+1\right)$$

14.10 problem 7

Internal problem ID [1301]
Internal file name [`OUTPUT/1302_Sunday_June_05_2022_02_09_10_AM_6451977/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2y'' - 2x\left(-x^2 - 4x + 3\right)y' + \left(x^2 + 6x + 3\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$8x^2y'' + \left(2x^3 + 8x^2 - 6x\right)y' + \left(x^2 + 6x + 3\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2 + 4x - 3}{4x}$$
$$q(x) = \frac{x^2 + 6x + 3}{8x^2}$$

Table 548: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+4x-3}{4x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{x^2+6x+3}{8x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$8x^2y'' + (2x^3 + 8x^2 - 6x)\, y' + (x^2 + 6x + 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$8x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + (2x^3 + 8x^2 - 6x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (x^2 + 6x + 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} x^{n+r+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^{n+r} \\
\sum_{n=0}^{\infty} 6x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$8x^{n+r}a_n(n+r)(n+r-1) - 6x^{n+r}a_n(n+r) + 3a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$8x^r a_0 r(-1+r) - 6x^r a_0 r + 3a_0 x^r = 0$$

Or

$$(8x^r r(-1+r) - 6x^r r + 3x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 - 14r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 - 14r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 - 14r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2}{2r - 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &8a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) \\ &+ 8a_{n-1}(n+r-1) - 6a_n(n+r) + a_{n-2} + 6a_{n-1} + 3a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-2} + 8na_{n-1} + 2ra_{n-2} + 8ra_{n-1} - 3a_{n-2} - 2a_{n-1}}{8n^2 + 16nr + 8r^2 - 14n - 14r + 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{(-a_{n-2} - 4a_{n-1})\, n - 5a_{n-1}}{4n^2 + 5n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r-1}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4r^2 + 16r + 29}{16r^3 + 28r^2 - 4r - 7}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{11}{26}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r-1}$	-1
a_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$\frac{11}{26}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64r^3 + 80r^2 - 448r - 596}{128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{109}{1326}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r-1}$	-1
a_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$\frac{11}{26}$
a_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$-\frac{109}{1326}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64r^5 - 336r^4 - 3808r^3 - 5064r^2 + 8724r + 13095}{(128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231)(8r^2 + 50r + 75)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{5}{12376}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r-1}$	-1
a_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$\frac{11}{26}$
a_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$-\frac{109}{1326}$
a_4	$\frac{64r^5-336r^4-3808r^3-5064r^2+8724r+13095}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)}$	$\frac{5}{12376}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1536r^6 - 11008r^5 + 5920r^4 + 191040r^3 + 397616r^2 + 96928r - 184710}{(128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231)(8r^2 + 50r + 75)(8r^2 + 66r + 133)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{229}{71400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r-1}$	-1
a_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$\frac{11}{26}$
a_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$-\frac{109}{1326}$
a_4	$\frac{64r^5-336r^4-3808r^3-5064r^2+8724r+13095}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)}$	$\frac{5}{12376}$
a_5	$\frac{-1536r^6-11008r^5+5920r^4+191040r^3+397616r^2+96928r-184710}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)(8r^2+66r+133)}$	$\frac{229}{71400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - x + \frac{11x^2}{26} - \frac{109x^3}{1326} + \frac{5x^4}{12376} + \frac{229x^5}{71400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{2}{2r-1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &8b_n(n+r)(n+r-1) + 2b_{n-2}(n+r-2) \\ &+ 8b_{n-1}(n+r-1) - 6b_n(n+r) + b_{n-2} + 6b_{n-1} + 3b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2nb_{n-2} + 8nb_{n-1} + 2rb_{n-2} + 8rb_{n-1} - 3b_{n-2} - 2b_{n-1}}{8n^2 + 16nr + 8r^2 - 14n - 14r + 3} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = \frac{(-4b_{n-2} - 16b_{n-1})\, n + 5b_{n-2}}{16n^2 - 20n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r-1}$	4

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-4r^2 + 16r + 29}{16r^3 + 28r^2 - 4r - 7}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = -\frac{131}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r-1}$	4
b_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$-\frac{131}{24}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{64r^3 + 80r^2 - 448r - 596}{128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = \frac{39}{14}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r-1}$	4
b_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$-\frac{131}{24}$
b_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$\frac{39}{14}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{64r^5 - 336r^4 - 3808r^3 - 5064r^2 + 8724r + 13095}{(128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231)(8r^2 + 50r + 75)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = -\frac{19865}{29568}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r-1}$	4
b_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$-\frac{131}{24}$
b_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$\frac{39}{14}$
b_4	$\frac{64r^5-336r^4-3808r^3-5064r^2+8724r+13095}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)}$	$-\frac{19865}{29568}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-1536r^6 - 11008r^5 + 5920r^4 + 191040r^3 + 397616r^2 + 96928r - 184710}{(128r^5 + 768r^4 + 1448r^3 + 732r^2 - 370r - 231)(8r^2 + 50r + 75)(8r^2 + 66r + 133)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = \frac{4421}{110880}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r-1}$	4
b_2	$\frac{-4r^2+16r+29}{16r^3+28r^2-4r-7}$	$-\frac{131}{24}$
b_3	$\frac{64r^3+80r^2-448r-596}{128r^5+768r^4+1448r^3+732r^2-370r-231}$	$\frac{39}{14}$
b_4	$\frac{64r^5-336r^4-3808r^3-5064r^2+8724r+13095}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)}$	$-\frac{19865}{29568}$
b_5	$\frac{-1536r^6-11008r^5+5920r^4+191040r^3+397616r^2+96928r-184710}{(128r^5+768r^4+1448r^3+732r^2-370r-231)(8r^2+50r+75)(8r^2+66r+133)}$	$\frac{4421}{110880}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{3}{2}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= x^{\frac{1}{4}}\left(1+4x-\frac{131x^2}{24}+\frac{39x^3}{14}-\frac{19865x^4}{29568}+\frac{4421x^5}{110880}+O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\frac{3}{2}}\left(1-x+\frac{11x^2}{26}-\frac{109x^3}{1326}+\frac{5x^4}{12376}+\frac{229x^5}{71400}+O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1+4x-\frac{131x^2}{24}+\frac{39x^3}{14}-\frac{19865x^4}{29568}+\frac{4421x^5}{110880}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{3}{2}}\left(1-x+\frac{11x^2}{26}-\frac{109x^3}{1326}+\frac{5x^4}{12376}+\frac{229x^5}{71400}+O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1+4x-\frac{131x^2}{24}+\frac{39x^3}{14}-\frac{19865x^4}{29568}+\frac{4421x^5}{110880}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1x^{\frac{3}{2}}\left(1-x+\frac{11x^2}{26}-\frac{109x^3}{1326}+\frac{5x^4}{12376}+\frac{229x^5}{71400}+O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1+4x-\frac{131x^2}{24}+\frac{39x^3}{14}-\frac{19865x^4}{29568}+\frac{4421x^5}{110880}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1x^{\frac{3}{2}}\left(1-x+\frac{11x^2}{26}-\frac{109x^3}{1326}+\frac{5x^4}{12376}+\frac{229x^5}{71400}+O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1+4x-\frac{131x^2}{24}+\frac{39x^3}{14}-\frac{19865x^4}{29568}+\frac{4421x^5}{110880}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

14.10.1 Maple step by step solution

Let's solve

$8x^2y'' + (2x^3 + 8x^2 - 6x)\, y' + (x^2 + 6x + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+6x+3)y}{8x^2} - \frac{(x^2+4x-3)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+4x-3)y'}{4x} + \frac{(x^2+6x+3)y}{8x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+4x-3}{4x}, P_3(x) = \frac{x^2+6x+3}{8x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{3}{4}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{8}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $8x^2y'' + 2x(x^2 + 4x - 3)\, y' + (x^2 + 6x + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+4r)\,(-3+2r)\, x^{r}+\left(a_{1}(3+4r)\,(-1+2r)+2a_{0}(3+4r)\right) x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(4k+4r-1)\,(\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)\,(-3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{4}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_{1}(3+4r)\,(-1+2r)+2a_{0}(3+4r)=0$
- Solve for the dependent coefficient(s)

  $a_{1}=-\frac{2a_{0}}{-1+2r}$
- Each term in the series must be 0, giving the recursion relation

  $8\left(k+r-\frac{1}{4}\right)\left(k+r-\frac{3}{2}\right) a_{k}+\left(2a_{k-2}+8a_{k-1}\right) k+\left(2a_{k-2}+8a_{k-1}\right) r-3a_{k-2}-2a_{k-1}=0$
- Shift index using $k->k+2$

  $8\left(k+\frac{7}{4}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2}+\left(2a_{k}+8a_{k+1}\right)(k+2)+\left(2a_{k}+8a_{k+1}\right) r-3a_{k}-2a_{k+1}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2ka_{k}+8ka_{k+1}+2ra_{k}+8ra_{k+1}+a_{k}+14a_{k+1}}{(4k+7+4r)(2k+2r+1)}$
- Recursion relation for $r=\frac{1}{4}$

  $a_{k+2}=-\frac{2ka_{k}+8ka_{k+1}+\frac{3}{2}a_{k}+16a_{k+1}}{(4k+8)\left(2k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2k a_k + 8k a_{k+1} + \frac{3}{2} a_k + 16 a_{k+1}}{(4k+8)\left(2k+\frac{3}{2}\right)}, a_1 = 4a_0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+2} = -\frac{2k a_k + 8k a_{k+1} + 4a_k + 26 a_{k+1}}{(4k+13)(2k+4)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{2k a_k + 8k a_{k+1} + 4a_k + 26 a_{k+1}}{(4k+13)(2k+4)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{2k a_k + 8k a_{1+k} + \frac{3}{2} a_k + 16 a_{1+k}}{(4k+8)\left(2k+\frac{3}{2}\right)}, a_1 = 4a_0, b_{k+2} = -\frac{2k b_k + 8k \cdots}{(4k \cdots}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(8*x^2*diff(y(x),x$2)-2*x*(3-4*x-x^2)*diff(y(x),x)+(3+6*x+x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 + 4x - \frac{131}{24}x^2 + \frac{39}{14}x^3 - \frac{19865}{29568}x^4 + \frac{4421}{110880}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{3}{2}} \left(1 - x + \frac{11}{26}x^2 - \frac{109}{1326}x^3 + \frac{5}{12376}x^4 + \frac{229}{71400}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
AsymptoticDSolveValue[8*x^2*y''[x]-2*x*(3-4*x-x^2)*y'[x]+(3+6*x+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{4421x^5}{110880} - \frac{19865x^4}{29568} + \frac{39x^3}{14} - \frac{131x^2}{24} + 4x + 1\right) \sqrt[4]{x} + c_1 \left(\frac{229x^5}{71400} + \frac{5x^4}{12376} - \frac{109x^3}{1326} + \frac{11x^2}{26} - x + 1\right) x^{3/2}$$

14.11 problem 8

Internal problem ID [1302]
Internal file name [OUTPUT/1303_Sunday_June_05_2022_02_09_14_AM_20394662/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{18x^2(x+1)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+11x+5}{6x\,(x+1)}$$
$$q(x)=\frac{5x^2+2x-1}{18x^2\,(x+1)}$$

Table 550: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+11x+5}{6x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{5x^2+2x-1}{18x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$18x^2(x+1)\,y'' + \left(3x^3 + 33x^2 + 15x\right) y' + \left(5x^2 + 2x - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$18x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^3+33x^2+15x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(5x^2+2x-1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 18x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 33x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 15x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 18a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 33x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 33a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 18 a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 18 x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=2}^{\infty} 3 a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 33 a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 15 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 5 a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$18 x^{n+r} a_n (n+r) (n+r-1) + 15 x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$18 x^r a_0 r(-1+r) + 15 x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(18 x^r r(-1+r) + 15 x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(18 r^2 - 3r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$18 r^2 - 3r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{6}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(18 r^2 - 3r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{6}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-1-6r}{6r+7}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 18a_{n-1}(n+r-1)(n+r-2) + 18a_n(n+r)(n+r-1) + 3a_{n-2}(n+r-2) \\ + 33a_{n-1}(n+r-1) + 15a_n(n+r) + 5a_{n-2} + 2a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{6na_{n-1} + 6ra_{n-1} + a_{n-2} - 5a_{n-1}}{6n + 6r + 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{(-6n+3)\,a_{n-1} - a_{n-2}}{6n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-6r}{6r+7}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{6r}{13 + 6r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-6r}{6r+7}$	$-\frac{1}{3}$
a_2	$\frac{6r}{13+6r}$	$\frac{2}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-36r^2 - 36r + 1}{(19 + 6r)(6r + 7)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{5}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-6r}{6r+7}$	$-\frac{1}{3}$
a_2	$\frac{6r}{13+6r}$	$\frac{2}{15}$
a_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$-\frac{5}{63}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{216r^3 + 648r^2 + 420r - 13}{(6r + 7)(13 + 6r)(25 + 6r)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{23}{405}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-6r}{6r+7}$	$-\frac{1}{3}$
a_2	$\frac{6r}{13+6r}$	$\frac{2}{15}$
a_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$-\frac{5}{63}$
a_4	$\frac{216r^3+648r^2+420r-13}{(6r+7)(13+6r)(25+6r)}$	$\frac{23}{405}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1296r^4 - 7776r^3 - 14148r^2 - 7440r + 234}{(6r + 7)(13 + 6r)(19 + 6r)(31 + 6r)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{458}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-6r}{6r+7}$	$-\frac{1}{3}$
a_2	$\frac{6r}{13+6r}$	$\frac{2}{15}$
a_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$-\frac{5}{63}$
a_4	$\frac{216r^3+648r^2+420r-13}{(6r+7)(13+6r)(25+6r)}$	$\frac{23}{405}$
a_5	$\frac{-1296r^4-7776r^3-14148r^2-7440r+234}{(6r+7)(13+6r)(19+6r)(31+6r)}$	$-\frac{458}{10395}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{3} + \frac{2x^2}{15} - \frac{5x^3}{63} + \frac{23x^4}{405} - \frac{458x^5}{10395} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-1 - 6r}{6r + 7}$$

For $2 \leq n$ the recursive equation is

$$18b_{n-1}(n + r - 1)(n + r - 2) + 18b_n(n + r)(n + r - 1) + 3b_{n-2}(n + r - 2) + 33b_{n-1}(n + r - 1) + 15b_n(n + r) + 5b_{n-2} + 2b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{6nb_{n-1} + 6rb_{n-1} + b_{n-2} - 5b_{n-1}}{6n + 6r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{6}$ becomes

$$b_n = \frac{(-6n + 6)\, b_{n-1} - b_{n-2}}{6n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{6}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-6r}{6r+7}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{6r}{13 + 6r}$$

Which for the root $r = -\frac{1}{6}$ becomes

$$b_2 = -\frac{1}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-6r}{6r+7}$	0
b_2	$\frac{6r}{13+6r}$	$-\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-36r^2 - 36r + 1}{(19 + 6r)(6r + 7)}$$

Which for the root $r = -\frac{1}{6}$ becomes

$$b_3 = \frac{1}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-6r}{6r+7}$	0
b_2	$\frac{6r}{13+6r}$	$-\frac{1}{12}$
b_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$\frac{1}{18}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{216r^3 + 648r^2 + 420r - 13}{(6r + 7)(13 + 6r)(25 + 6r)}$$

Which for the root $r = -\frac{1}{6}$ becomes

$$b_4 = -\frac{11}{288}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-6r}{6r+7}$	0
b_2	$\frac{6r}{13+6r}$	$-\frac{1}{12}$
b_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$\frac{1}{18}$
b_4	$\frac{216r^3+648r^2+420r-13}{(6r+7)(13+6r)(25+6r)}$	$-\frac{11}{288}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-1296r^4 - 7776r^3 - 14148r^2 - 7440r + 234}{(6r + 7)(13 + 6r)(19 + 6r)(31 + 6r)}$$

Which for the root $r = -\frac{1}{6}$ becomes

$$b_5 = \frac{31}{1080}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-6r}{6r+7}$	0
b_2	$\frac{6r}{13+6r}$	$-\frac{1}{12}$
b_3	$\frac{-36r^2-36r+1}{(19+6r)(6r+7)}$	$\frac{1}{18}$
b_4	$\frac{216r^3+648r^2+420r-13}{(6r+7)(13+6r)(25+6r)}$	$-\frac{11}{288}$
b_5	$\frac{-1296r^4-7776r^3-14148r^2-7440r+234}{(6r+7)(13+6r)(19+6r)(31+6r)}$	$\frac{31}{1080}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{12} + \frac{x^3}{18} - \frac{11x^4}{288} + \frac{31x^5}{1080} + O(x^6)}{x^{\frac{1}{6}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{3} + \frac{2x^2}{15} - \frac{5x^3}{63} + \frac{23x^4}{405} - \frac{458x^5}{10395} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - \frac{x^2}{12} + \frac{x^3}{18} - \frac{11x^4}{288} + \frac{31x^5}{1080} + O(x^6)\right)}{x^{\frac{1}{6}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{x}{3} + \frac{2x^2}{15} - \frac{5x^3}{63} + \frac{23x^4}{405} - \frac{458x^5}{10395} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - \frac{x^2}{12} + \frac{x^3}{18} - \frac{11x^4}{288} + \frac{31x^5}{1080} + O(x^6)\right)}{x^{\frac{1}{6}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{x}{3} + \frac{2x^2}{15} - \frac{5x^3}{63} + \frac{23x^4}{405} - \frac{458x^5}{10395} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{12} + \frac{x^3}{18} - \frac{11x^4}{288} + \frac{31x^5}{1080} + O(x^6)\right)}{x^{\frac{1}{6}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{x}{3} + \frac{2x^2}{15} - \frac{5x^3}{63} + \frac{23x^4}{405} - \frac{458x^5}{10395} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{12} + \frac{x^3}{18} - \frac{11x^4}{288} + \frac{31x^5}{1080} + O(x^6)\right)}{x^{\frac{1}{6}}}$$

Verified OK.

14.11.1 Maple step by step solution

Let's solve

$18x^2(x+1)\, y'' + (3x^3 + 33x^2 + 15x)\, y' + (5x^2 + 2x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+2x-1)y}{18x^2(x+1)} - \frac{(x^2+11x+5)y'}{6x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+11x+5)y'}{6x(x+1)} + \frac{(5x^2+2x-1)y}{18x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x^2+11x+5}{6x(x+1)}, P_3(x) = \frac{5x^2+2x-1}{18x^2(x+1)}\right]$

- ○ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{6}$

- ○ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $18x^2(x+1)\, y'' + 3x(x^2+11x+5)\, y' + (5x^2+2x-1)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(18u^3-36u^2+18u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^3+24u^2-42u+15)\left(\frac{d}{du}y(u)\right) + (5u^2-8u+2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r}+(3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r))\,u^r+(3a_2(2+r)\,(11+6r)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3r(-1+6r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,\frac{1}{6}\right\}$$

- The coefficients of each power of u must be 0

  $$[3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)=0,3a_2(2+r)\,(11+6r)-2a_1(4+3r)\,(5+6r)+$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1=\frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)},a_2=\frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$18(-2a_k+a_{k-1}+a_{k+1})\,k^2+3(12(-2a_k+a_{k-1}+a_{k+1})\,r-2a_k+a_{k-2}-10a_{k-1}+11a_{k+1})\,k+$$

- Shift index using $k->k+2$

  $$18(-2a_{k+2}+a_{k+1}+a_{k+3})\,(k+2)^2+3(12(-2a_{k+2}+a_{k+1}+a_{k+3})\,r-2a_{k+2}+a_k-10a_{k+1}+11$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+36kra_{k+1}-72kra_{k+2}+18r^2a_{k+1}-36r^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+3ra_k+42ra_{k+1}-150}{3(6k^2+12kr+6r^2+35k+35r+51)}$$

- Recursion relation for $r=0$

  $$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)},a_1=-\frac{2a_0}{15},a\right.$$

- Revert the change of variables $u=x+1$

  $$\left[y=\sum_{k=0}^{\infty}a_k(x+1)^k,a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)},a_1=-\frac{2a_0}{15}\right.$$

- Recursion relation for $r=\frac{1}{6}$

  $$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}$$

- Solution for $r=\frac{1}{6}$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{6}},a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)},a_1=0,a\right.$$

- Revert the change of variables $u=x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2 a_{k+1} - 36k^2 a_{k+2} + 3k a_k + 48k a_{k+1} - 162k a_{k+2} + \frac{11}{2} a_k + \frac{47}{2} a_{k+1} - 180 a_{k+2}}{3(6k^2+37k+57)}, a_1 = \right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{1}{6}}\right), a_{k+3} = -\frac{18k^2 a_{1+k} - 36k^2 a_{k+2} + 3k a_k + 42k a_{1+k} - 150k a_{k+2} + 5a}{3(6k^2+35k+51)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(18*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x+x^2)*diff(y(x),x)-(1-2*x-5*x^2)*y(x)=0,y(x),t
```

$$y(x) = \frac{c_2\sqrt{x}\left(1-\frac{1}{3}x+\frac{2}{15}x^2-\frac{5}{63}x^3+\frac{23}{405}x^4-\frac{458}{10395}x^5+O\left(x^6\right)\right)+c_1\left(1-\frac{1}{12}x^2+\frac{1}{18}x^3-\frac{11}{288}x^4+\frac{31}{1080}x^5+O\left(x\right.\right.}{x^{\frac{1}{6}}}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 85

```
AsymptoticDSolveValue[18*x^2*(1+x)*y''[x]+3*x*(5+11*x+x^2)*y'[x]-(1-2*x-5*x^2)*y[x]==0,y[x],
```

$$y(x)\to c_1\sqrt[3]{x}\left(-\frac{458x^5}{10395}+\frac{23x^4}{405}-\frac{5x^3}{63}+\frac{2x^2}{15}-\frac{x}{3}+1\right)+\frac{c_2\left(\frac{31x^5}{1080}-\frac{11x^4}{288}+\frac{x^3}{18}-\frac{x^2}{12}+1\right)}{\sqrt[6]{x}}$$

14.12 problem 9

Internal problem ID [1303]
Internal file name [OUTPUT/1304_Sunday_June_05_2022_02_09_17_AM_49582011/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x\left(x^2+x+3\right) y''+\left(-x^2+x+4\right) y'+yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2+3x\right) y''+\left(-x^2+x+4\right) y'+yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^2-x-4}{x\left(x^2+x+3\right)}$$
$$q(x)=\frac{1}{x^2+x+3}$$

Table 552: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-x-4}{x(x^2+x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{11}}{2} - \frac{1}{2}$	"regular"
$x = \frac{i\sqrt{11}}{2} - \frac{1}{2}$	"regular"

$q(x) = \frac{1}{x^2+x+3}$	
singularity	type
$x = -\frac{i\sqrt{11}}{2} - \frac{1}{2}$	"regular"
$x = \frac{i\sqrt{11}}{2} - \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{11}}{2} - \frac{1}{2}, \frac{i\sqrt{11}}{2} - \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x\left(x^2 + x + 3\right) y'' + \left(-x^2 + x + 4\right) y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(x^2 + x + 3\right) \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-x^2 + x + 4\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right)+\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1}a_n(n+r)\,(n+r-1)\right) + \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)\,x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4(n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1}a_n(n+r)\,(n+r-1) + 4(n+r)\,a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r}a_0 r(-1+r) + 4ra_0x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r}r(-1+r) + 4r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2+r\right)x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2+r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2+r\right)x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{r^2}{3r^2 + 7r + 4}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) \\ - a_{n-2}(n+r-2) + a_{n-1}(n+r-1) + 4a_n(n+r) + a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + n^2 a_{n-1} + 2nr a_{n-2} + 2nr a_{n-1} + r^2 a_{n-2} + r^2 a_{n-1} - 6n a_{n-2} - 2n a_{n-1} - 6r a_{n-2} - 2r a_{n-1}}{3n^2 + 6nr + 3r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-a_{n-2} - a_{n-1})\, n^2 + (6a_{n-2} + 2a_{n-1})\, n - 9a_{n-2} - a_{n-1}}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{3r^2+7r+4}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2r^3 + 3r^2 + 5r - 4}{9r^3 + 51r^2 + 94r + 56}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{3r^2+7r+4}$	0
a_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{5r^5 + 10r^4 - 10r^3 - 17r^2 + 2r + 8}{(3r^2 + 19r + 30)(9r^2 + 33r + 28)(1 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{3r^2+7r+4}$	0
a_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{1}{14}$
a_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{1}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^7 - 6r^6 - 64r^5 - 127r^4 - 55r^3 + 45r^2 + 30r - 8}{(3r^2 + 25r + 52)(1 + r)(9r^2 + 33r + 28)(3r + 10)(r + 2)}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{1}{3640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{3r^2+7r+4}$	0
a_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{1}{14}$
a_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{1}{105}$
a_4	$\frac{r^7-6r^6-64r^5-127r^4-55r^3+45r^2+30r-8}{(3r^2+25r+52)(1+r)(9r^2+33r+28)(3r+10)(r+2)}$	$-\frac{1}{3640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-16r^9 - 186r^8 - 756r^7 - 1032r^6 + 1053r^5 + 4476r^4 + 3939r^3 - 246r^2 - 1952r - 736}{(3r^2 + 31r + 80)(r + 2)(3r + 10)(3r + 13)(1 + r)(9r^2 + 33r + 28)(r + 3)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{23}{54600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{3r^2+7r+4}$	0
a_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{1}{14}$
a_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{1}{105}$
a_4	$\frac{r^7-6r^6-64r^5-127r^4-55r^3+45r^2+30r-8}{(3r^2+25r+52)(1+r)(9r^2+33r+28)(3r+10)(r+2)}$	$-\frac{1}{3640}$
a_5	$\frac{-16r^9-186r^8-756r^7-1032r^6+1053r^5+4476r^4+3939r^3-246r^2-1952r-736}{(3r^2+31r+80)(r+2)(3r+10)(3r+13)(1+r)(9r^2+33r+28)(r+3)}$	$-\frac{23}{54600}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{x^2}{14} + \frac{x^3}{105} - \frac{x^4}{3640} - \frac{23x^5}{54600} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{r^2}{3r^2 + 7r + 4}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-2}(n+r-2)(n-3+r) + b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ - b_{n-2}(n+r-2) + b_{n-1}(n+r-1) + 4(n+r) b_n + b_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-2} + n^2 b_{n-1} + 2nr b_{n-2} + 2nr b_{n-1} + r^2 b_{n-2} + r^2 b_{n-1} - 6n b_{n-2} - 2n b_{n-1} - 6r b_{n-2} - 2r b_{n-1} +}{3n^2 + 6nr + 3r^2 + n + r} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{(-9b_{n-2} - 9b_{n-1}) n^2 + (60b_{n-2} + 24b_{n-1}) n - 100b_{n-2} - 16b_{n-1}}{27n^2 - 9n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{3r^2+7r+4}$	$-\frac{1}{18}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-2r^3 + 3r^2 + 5r - 4}{9r^3 + 51r^2 + 94r + 56}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{71}{405}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{3r^2+7r+4}$	$-\frac{1}{18}$
b_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{71}{405}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{5r^5 + 10r^4 - 10r^3 - 17r^2 + 2r + 8}{(3r^2 + 19r + 30)(9r^2 + 33r + 28)(1 + r)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = \frac{719}{34992}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{3r^2+7r+4}$	$-\frac{1}{18}$
b_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{71}{405}$
b_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{719}{34992}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^7 - 6r^6 - 64r^5 - 127r^4 - 55r^3 + 45r^2 + 30r - 8}{(3r^2 + 25r + 52)(1 + r)(9r^2 + 33r + 28)(3r + 10)(r + 2)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = -\frac{1678}{1082565}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{3r^2+7r+4}$	$-\frac{1}{18}$
b_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{71}{405}$
b_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{719}{34992}$
b_4	$\frac{r^7-6r^6-64r^5-127r^4-55r^3+45r^2+30r-8}{(3r^2+25r+52)(1+r)(9r^2+33r+28)(3r+10)(r+2)}$	$-\frac{1678}{1082565}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-16r^9 - 186r^8 - 756r^7 - 1032r^6 + 1053r^5 + 4476r^4 + 3939r^3 - 246r^2 - 1952r - 736}{(3r^2 + 31r + 80)(r + 2)(3r + 10)(3r + 13)(1 + r)(9r^2 + 33r + 28)(r + 3)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = -\frac{513547}{992023200}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r^2}{3r^2+7r+4}$	$-\frac{1}{18}$
b_2	$\frac{-2r^3+3r^2+5r-4}{9r^3+51r^2+94r+56}$	$-\frac{71}{405}$
b_3	$\frac{5r^5+10r^4-10r^3-17r^2+2r+8}{(3r^2+19r+30)(9r^2+33r+28)(1+r)}$	$\frac{719}{34992}$
b_4	$\frac{r^7-6r^6-64r^5-127r^4-55r^3+45r^2+30r-8}{(3r^2+25r+52)(1+r)(9r^2+33r+28)(3r+10)(r+2)}$	$-\frac{1678}{1082565}$
b_5	$\frac{-16r^9-186r^8-756r^7-1032r^6+1053r^5+4476r^4+3939r^3-246r^2-1952r-736}{(3r^2+31r+80)(r+2)(3r+10)(3r+13)(1+r)(9r^2+33r+28)(r+3)}$	$-\frac{513547}{992023200}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x}{18} - \frac{71x^2}{405} + \frac{719x^3}{34992} - \frac{1678x^4}{1082565} - \frac{513547x^5}{992023200} + O(x^6)}{x^{\frac{1}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 - \frac{x^2}{14} + \frac{x^3}{105} - \frac{x^4}{3640} - \frac{23x^5}{54600} + O(x^6)\right) \\ &+ \frac{c_2 \left(1 - \frac{x}{18} - \frac{71x^2}{405} + \frac{719x^3}{34992} - \frac{1678x^4}{1082565} - \frac{513547x^5}{992023200} + O(x^6)\right)}{x^{\frac{1}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 - \frac{x^2}{14} + \frac{x^3}{105} - \frac{x^4}{3640} - \frac{23x^5}{54600} + O(x^6)\right) \\ &+ \frac{c_2 \left(1 - \frac{x}{18} - \frac{71x^2}{405} + \frac{719x^3}{34992} - \frac{1678x^4}{1082565} - \frac{513547x^5}{992023200} + O(x^6)\right)}{x^{\frac{1}{3}}} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\left(1-\frac{x^2}{14}+\frac{x^3}{105}-\frac{x^4}{3640}-\frac{23x^5}{54600}+O(x^6)\right) + \frac{c_2\left(1-\frac{x}{18}-\frac{71x^2}{405}+\frac{719x^3}{34992}-\frac{1678x^4}{1082565}-\frac{513547x^5}{992023200}+O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{14}+\frac{x^3}{105}-\frac{x^4}{3640}-\frac{23x^5}{54600}+O(x^6)\right) + \frac{c_2\left(1-\frac{x}{18}-\frac{71x^2}{405}+\frac{719x^3}{34992}-\frac{1678x^4}{1082565}-\frac{513547x^5}{992023200}+O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

14.12.1 Maple step by step solution

Let's solve

$x(x^2+x+3)\,y''+(-x^2+x+4)\,y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2+x+3}+\frac{(x^2-x-4)y'}{x(x^2+x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-x-4)y'}{x(x^2+x+3)}+\frac{y}{x^2+x+3}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-x-4}{x(x^2+x+3)},P_3(x)=\frac{1}{x^2+x+3}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{4}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+x+3)\,y'' + (-x^2+x+4)\,y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+3r)\,x^{-1+r} + \left(a_1(1+r)\,(4+3r) + a_0 r^2\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(3k+4+3r) + a_k(k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+3r)=0$

- Values of r that satisfy the indicial equation

$r \in \{0, -\frac{1}{3}\}$

- Each term must be 0

$a_1(1+r)(4+3r)+a_0 r^2=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(3k+4+3r)+a_k(k+r)^2+a_{k-1}(k-2+r)^2=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(3k+7+3r)+a_{k+1}(k+r+1)^2+a_k(k+r-1)^2=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{k^2a_k+k^2a_{k+1}+2kra_k+2kra_{k+1}+r^2a_k+r^2a_{k+1}-2ka_k+2ka_{k+1}-2ra_k+2ra_{k+1}+a_k+a_{k+1}}{(k+2+r)(3k+7+3r)}$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{k^2a_k+k^2a_{k+1}-2ka_k+2ka_{k+1}+a_k+a_{k+1}}{(k+2)(3k+7)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{k^2a_k+k^2a_{k+1}-2ka_k+2ka_{k+1}+a_k+a_{k+1}}{(k+2)(3k+7)}, 4a_1=0\right]$

- Recursion relation for $r=-\frac{1}{3}$

$a_{k+2}=-\frac{k^2a_k+k^2a_{k+1}-\frac{8}{3}ka_k+\frac{4}{3}ka_{k+1}+\frac{16}{9}a_k+\frac{4}{9}a_{k+1}}{(k+\frac{5}{3})(3k+6)}$

- Solution for $r=-\frac{1}{3}$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{k^2a_k+k^2a_{k+1}-\frac{8}{3}ka_k+\frac{4}{3}ka_{k+1}+\frac{16}{9}a_k+\frac{4}{9}a_{k+1}}{(k+\frac{5}{3})(3k+6)}, 2a_1+\frac{a_0}{9}=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), a_{k+2}=-\frac{k^2a_k+k^2a_{1+k}-2ka_k+2ka_{1+k}+a_k+a_{1+k}}{(k+2)(3k+7)}, 4a_1=0, b_{k+2}=-\frac{k^2}{}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
Order:=6;
dsolve(x*(3+x+x^2)*diff(y(x),x$2)+(4+x-x^2)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{1}{18}x-\frac{71}{405}x^2+\frac{719}{34992}x^3-\frac{1678}{1082565}x^4-\frac{513547}{992023200}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}} + c_2\left(1-\frac{1}{14}x^2+\frac{1}{105}x^3-\frac{1}{3640}x^4-\frac{23}{54600}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 80

```
AsymptoticDSolveValue[x*(3+x+x^2)*y''[x]+(4+x-x^2)*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{23x^5}{54600} - \frac{x^4}{3640} + \frac{x^3}{105} - \frac{x^2}{14} + 1\right) + \frac{c_2 \left(-\frac{513547x^5}{992023200} - \frac{1678x^4}{1082565} + \frac{719x^3}{34992} - \frac{71x^2}{405} - \frac{x}{18} + 1\right)}{\sqrt[3]{x}}$$

14.13 problem 10

Internal problem ID [1304]
Internal file name [`OUTPUT/1305_Sunday_June_05_2022_02_09_21_AM_89487669/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{10x^2\left(2x^2+x+1\right)y''+x\left(66x^2+13x+13\right)y'-\left(10x^2+4x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(20x^4+10x^3+10x^2\right)y''+\left(66x^3+13x^2+13x\right)y'+\left(-10x^2-4x-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{66x^2+13x+13}{10x\left(2x^2+x+1\right)}$$
$$q(x)=-\frac{10x^2+4x+1}{10x^2\left(2x^2+x+1\right)}$$

Table 554: Table $p(x), q(x)$ singularites.

$p(x) = \frac{66x^2+13x+13}{10x(2x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"
$x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"

$q(x) = -\frac{10x^2+4x+1}{10x^2(2x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"
$x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{7}}{4} - \frac{1}{4}, \frac{i\sqrt{7}}{4} - \frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$10x^2(2x^2 + x + 1)\, y'' + (66x^3 + 13x^2 + 13x)\, y' + (-10x^2 - 4x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &10x^2(2x^2 + x + 1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ (66x^3 + 13x^2 + 13x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) \\ &+ (-10x^2 - 4x - 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 20x^{n+r+2} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 10x^{1+n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 10x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 66x^{n+r+2} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 13x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 13x^{n+r} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-10x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-4x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 20x^{n+r+2} a_n(n+r)(n+r-1)=\sum_{n=2}^{\infty} 20a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 10x^{1+n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} 10a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 66x^{n+r+2} a_n(n+r)=\sum_{n=2}^{\infty} 66a_{n-2}(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 13x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} 13a_{n-1}(n+r-1)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty}\left(-10x^{n+r+2} a_n\right)=\sum_{n=2}^{\infty}\left(-10a_{n-2}x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty}\left(-4x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1}x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 20a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 10a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 10x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 66a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 13a_{n-1}(n+r-1)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 13x^{n+r} a_n(n+r)\right) \\
&+\sum_{n=2}^{\infty}\left(-10a_{n-2}x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-4a_{n-1}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$10x^{n+r}a_n(n+r)(n+r-1)+13x^{n+r}a_n(n+r)-a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$10x^r a_0 r(-1+r)+13x^r a_0 r-a_0x^r=0$$

Or

$$\left(10x^r r(-1+r)+13x^r r-x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(10r^2+3r-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$10r^2+3r-1=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}
r_1 &= \frac{1}{5} \\
r_2 &= -\frac{1}{2}
\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(10r^2+3r-1\right) x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{10}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{5}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1 - 2r}{2r + 3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &20a_{n-2}(n+r-2)(n-3+r) + 10a_{n-1}(n+r-1)(n+r-2) \\ &+ 10a_n(n+r)(n+r-1) + 66a_{n-2}(n+r-2) \\ &+ 13a_{n-1}(n+r-1) + 13a_n(n+r) - 10a_{n-2} - 4a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{20n^2a_{n-2} + 10n^2a_{n-1} + 40nra_{n-2} + 20nra_{n-1} + 20r^2a_{n-2} + 10r^2a_{n-1} - 34na_{n-2} - 17na_{n-1} - 34r}{10n^2 + 20nr + 10r^2 + 3n + 3r - 1} \tag{4}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_n = \frac{(-20a_{n-2} - 10a_{n-1})n^2 + (26a_{n-2} + 13a_{n-1})n + 28a_{n-2}}{10n^2 + 7n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{2r+3}$	$\frac{3}{17}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-20r^3 - 116r^2 - 123r + 21}{20r^3 + 116r^2 + 219r + 135}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_2 = -\frac{7}{153}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{2r+3}$	$\frac{3}{17}$
a_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$-\frac{7}{153}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{600r^5 + 5260r^4 + 15654r^3 + 17349r^2 + 3025r - 3402}{200r^5 + 2420r^4 + 11458r^3 + 26515r^2 + 29967r + 13230}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_3 = -\frac{547}{5661}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{2r+3}$	$\frac{3}{17}$
a_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$-\frac{7}{153}$
a_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$	$-\frac{547}{5661}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-2000r^7 - 16800r^6 + 2360r^5 + 433536r^4 + 1687211r^3 + 2569170r^2 + 1406065r - 10206}{2000r^7 + 40800r^6 + 349640r^5 + 1629984r^4 + 4459733r^3 + 7153626r^2 + 6222447r + 2262330}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_4 = \frac{26942}{266067}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-2r}{2r+3}$	$\frac{3}{17}$
a_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$-\frac{7}{153}$
a_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$	$-\frac{547}{5661}$
a_4	$\frac{-2000r^7-16800r^6+2360r^5+433536r^4+1687211r^3+2569170r^2+1406065r-10206}{2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330}$	$\frac{26942}{266067}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-100000r^9 - 2710000r^8 - 31054000r^7 - 196182200r^6 - 744265370r^5 - 1721296863r^4 - 232124294}{(10r^2 + 103r + 264)(2000r^7 + 40800r^6 + 349640r^5 + 1629984r^4 + 4459733r^3 + 715}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_5 = \frac{200432}{3991005}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{1-2r}{2r+3}$
a_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$
a_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$
a_4	$\frac{-2000r^7-16800r^6+2360r^5+433536r^4+1687211r^3+2569170r^2+1406065r-10206}{2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330}$
a_5	$\frac{-100000r^9-2710000r^8-31054000r^7-196182200r^6-744265370r^5-1721296863r^4-2321242940r^3-1556172021r^2-211154622r+1808911}{(10r^2+103r+264)(2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330)}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{5}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right) \\ &= x^{\frac{1}{5}}\left(1 + \frac{3x}{17} - \frac{7x^2}{153} - \frac{547x^3}{5661} + \frac{26942x^4}{266067} + \frac{200432x^5}{3991005} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{1-2r}{2r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &20b_{n-2}(n+r-2)(n-3+r) + 10b_{n-1}(n+r-1)(n+r-2) \\ &+ 10b_n(n+r)(n+r-1) + 66b_{n-2}(n+r-2) \\ &+ 13b_{n-1}(n+r-1) + 13b_n(n+r) - 10b_{n-2} - 4b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{20n^2b_{n-2} + 10n^2b_{n-1} + 40nrb_{n-2} + 20nrb_{n-1} + 20r^2b_{n-2} + 10r^2b_{n-1} - 34nb_{n-2} - 17nb_{n-1} - 34rb_{n}}{10n^2 + 20nr + 10r^2 + 3n + 3r - 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{(-20b_{n-2} - 10b_{n-1})\, n^2 + (54b_{n-2} + 27b_{n-1})\, n - 14b_{n-1}}{10n^2 - 7n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{2r+3}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-20r^3 - 116r^2 - 123r + 21}{20r^3 + 116r^2 + 219r + 135}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{14}{13}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{2r+3}$	1
b_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$\frac{14}{13}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{600r^5 + 5260r^4 + 15654r^3 + 17349r^2 + 3025r - 3402}{200r^5 + 2420r^4 + 11458r^3 + 26515r^2 + 29967r + 13230}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{556}{897}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{2r+3}$	1
b_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$\frac{14}{13}$
b_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$	$-\frac{556}{897}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-2000r^7 - 16800r^6 + 2360r^5 + 433536r^4 + 1687211r^3 + 2569170r^2 + 1406065r - 10206}{2000r^7 + 40800r^6 + 349640r^5 + 1629984r^4 + 4459733r^3 + 7153626r^2 + 6222447r + 2262330}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = -\frac{5314}{9867}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1-2r}{2r+3}$	1
b_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$	$\frac{14}{13}$
b_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$	$-\frac{556}{897}$
b_4	$\frac{-2000r^7-16800r^6+2360r^5+433536r^4+1687211r^3+2569170r^2+1406065r-10206}{2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330}$	$-\frac{5314}{9867}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-100000r^9 - 2710000r^8 - 31054000r^7 - 196182200r^6 - 744265370r^5 - 1721296863r^4 - 232124294}{(10r^2 + 103r + 264)(2000r^7 + 40800r^6 + 349640r^5 + 1629984r^4 + 4459733r^3 + 715}$$

Which for the root $r=-\frac{1}{2}$ becomes

$$b_5=\frac{2092186}{2121405}$$

And the table now becomes

n	$b_{n,r}$
b_0	1
b_1	$\frac{1-2r}{2r+3}$
b_2	$\frac{-20r^3-116r^2-123r+21}{20r^3+116r^2+219r+135}$
b_3	$\frac{600r^5+5260r^4+15654r^3+17349r^2+3025r-3402}{200r^5+2420r^4+11458r^3+26515r^2+29967r+13230}$
b_4	$\frac{-2000r^7-16800r^6+2360r^5+433536r^4+1687211r^3+2569170r^2+1406065r-10206}{2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330}$
b_5	$\frac{-100000r^9-2710000r^8-31054000r^7-196182200r^6-744265370r^5-1721296863r^4-2321242940r^3-1556172021r^2-211154622r+1808911}{(10r^2+103r+264)(2000r^7+40800r^6+349640r^5+1629984r^4+4459733r^3+7153626r^2+6222447r+2262330)}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x)&=x^{\frac{1}{5}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right)\\&=\frac{1+x+\frac{14x^2}{13}-\frac{556x^3}{897}-\frac{5314x^4}{9867}+\frac{2092186x^5}{2121405}+O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^{\frac{1}{5}}\left(1+\frac{3x}{17}-\frac{7x^2}{153}-\frac{547x^3}{5661}+\frac{26942x^4}{266067}+\frac{200432x^5}{3991005}+O(x^6)\right)\\&\quad+\frac{c_2\left(1+x+\frac{14x^2}{13}-\frac{556x^3}{897}-\frac{5314x^4}{9867}+\frac{2092186x^5}{2121405}+O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x^{\frac{1}{5}}\left(1+\frac{3x}{17}-\frac{7x^2}{153}-\frac{547x^3}{5661}+\frac{26942x^4}{266067}+\frac{200432x^5}{3991005}+O(x^6)\right)\\&\quad+\frac{c_2\left(1+x+\frac{14x^2}{13}-\frac{556x^3}{897}-\frac{5314x^4}{9867}+\frac{2092186x^5}{2121405}+O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{5}} \left(1+\frac{3x}{17}-\frac{7x^2}{153}-\frac{547x^3}{5661}+\frac{26942x^4}{266067}+\frac{200432x^5}{3991005}+O(x^6)\right) + \frac{c_2\left(1+x+\frac{14x^2}{13}-\frac{556x^3}{897}-\frac{5314x^4}{9867}+\frac{2092186x^5}{2121405}+O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{5}} \left(1+\frac{3x}{17}-\frac{7x^2}{153}-\frac{547x^3}{5661}+\frac{26942x^4}{266067}+\frac{200432x^5}{3991005}+O(x^6)\right) + \frac{c_2\left(1+x+\frac{14x^2}{13}-\frac{556x^3}{897}-\frac{5314x^4}{9867}+\frac{2092186x^5}{2121405}+O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

14.13.1 Maple step by step solution

Let's solve

$10x^2(2x^2+x+1)\,y''+(66x^3+13x^2+13x)\,y'+(-10x^2-4x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(10x^2+4x+1)y}{10x^2(2x^2+x+1)}-\frac{(66x^2+13x+13)y'}{10x(2x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(66x^2+13x+13)y'}{10x(2x^2+x+1)}-\frac{(10x^2+4x+1)y}{10x^2(2x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{66x^2+13x+13}{10x(2x^2+x+1)}, P_3(x)=-\frac{10x^2+4x+1}{10x^2(2x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{13}{10}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2 \cdot P_3(x)) \bigg|_{x=0} = -\frac{1}{10}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $10x^2(2x^2+x+1)\,y'' + x(66x^2+13x+13)\,y' + (-10x^2-4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+5r)\,x^r + (a_1(3+2r)\,(4+5r) + a_0(4+5r)\,(-1+2r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-1+5r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{5}\right\}$

- Each term must be 0

  $a_1(3+2r)(4+5r)+a_0(4+5r)(-1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=-\frac{(-1+2r)a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)(5k+5r-1)+a_{k-1}(5k+5r-1)(2k-3+2r)+2a_{k-2}(2k+2r+1)(5k-11$

- Shift index using $k->k+2$

  $a_{k+2}(2k+2r+5)(5k+9+5r)+a_{k+1}(5k+9+5r)(2k+2r+1)+2a_k(2k+2r+5)(5k+5r$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{20k^2a_k+10k^2a_{k+1}+40kra_k+20kra_{k+1}+20r^2a_k+10r^2a_{k+1}+46ka_k+23ka_{k+1}+46ra_k+23ra_{k+1}-10a_k+9a_{k+1}}{(2k+2r+5)(5k+9+5r)}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{20k^2a_k+10k^2a_{k+1}+26ka_k+13ka_{k+1}-28a_k}{(2k+4)\left(5k+\frac{13}{2}\right)}$

- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{20k^2a_k+10k^2a_{k+1}+26ka_k+13ka_{k+1}-28a_k}{(2k+4)\left(5k+\frac{13}{2}\right)}, a_1=a_0\right]$

- Recursion relation for $r=\frac{1}{5}$

  $a_{k+2}=-\frac{20k^2a_k+10k^2a_{k+1}+54ka_k+27ka_{k+1}+14a_{k+1}}{\left(2k+\frac{27}{5}\right)(5k+10)}$

- Solution for $r=\frac{1}{5}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{5}}, a_{k+2}=-\frac{20k^2a_k+10k^2a_{k+1}+54ka_k+27ka_{k+1}+14a_{k+1}}{\left(2k+\frac{27}{5}\right)(5k+10)}, a_1=\frac{3a_0}{17}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{5}}\right), a_{k+2}=-\frac{20k^2a_k+10k^2a_{1+k}+26ka_k+13ka_{1+k}-28a_k}{(2k+4)\left(5k+\frac{13}{2}\right)}, a_1=a_0, b_{k+2}=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 47

```
Order:=6;
dsolve(10*x^2*(1+x+2*x^2)*diff(y(x),x$2)+x*(13+13*x+66*x^2)*diff(y(x),x)-(1+4*x+10*x^2)*y(x)
```

$$y(x) = \frac{c_1\left(1+x+\frac{14}{13}x^2-\frac{556}{897}x^3-\frac{5314}{9867}x^4+\frac{2092186}{2121405}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + c_2 x^{\frac{1}{5}}\left(1+\frac{3}{17}x-\frac{7}{153}x^2-\frac{547}{5661}x^3+\frac{26942}{266067}x^4+\frac{200432}{3991005}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 86

```
AsymptoticDSolveValue[10*x^2*(1+x+2*x^2)*y''[x]+x*(13+13*x+66*x^2)*y'[x]-(1+4*x+10*x^2)*y[x]
```

$$y(x) \to c_1 \sqrt[5]{x}\left(\frac{200432x^5}{3991005}+\frac{26942x^4}{266067}-\frac{547x^3}{5661}-\frac{7x^2}{153}+\frac{3x}{17}+1\right)+\frac{c_2\left(\frac{2092186x^5}{2121405}-\frac{5314x^4}{9867}-\frac{556x^3}{897}+\frac{14x^2}{13}+x+1\right)}{\sqrt{x}}$$

14.14 problem 14

Internal problem ID [1305]
Internal file name [`OUTPUT/1306_Sunday_June_05_2022_02_09_25_AM_54010958/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + x(2x+3)\,y' - (1-x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(2x^2+3x\right)y' + y(x-1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{2x+3}{2x} \\ q(x) &= \frac{x-1}{2x^2} \end{aligned}$$

Table 556: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x+3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (2x^2 + 3x)\, y' + y(x - 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (2x^2 + 3x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x - 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + 3x^{n+r} a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 3x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + 3x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) + 3a_n(n+r) + a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}}{3+2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2+r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{4}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{2}{5}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{4}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{8}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{2}{5}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{4}{35}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{8}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{16}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{2}{5}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{4}{35}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{8}{315}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{16}{3465}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(2+r)(3+r)(4+r)(5+r)(6+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{32}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2+r}$	$-\frac{2}{5}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{4}{35}$
a_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{8}{315}$
a_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{16}{3465}$
a_5	$-\frac{1}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	$-\frac{32}{45045}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 - \frac{2x}{5} + \frac{4x^2}{35} - \frac{8x^3}{315} + \frac{16x^4}{3465} - \frac{32x^5}{45045} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) + 3b_n(n+r) + b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2+r}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(2+r)(3+r)(4+r)(5+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2+r}$	-1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(2+r)(3+r)(4+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{24}$
b_5	$-\frac{1}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{4x^2}{35} - \frac{8x^3}{315} + \frac{16x^4}{3465} - \frac{32x^5}{45045} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{4x^2}{35} - \frac{8x^3}{315} + \frac{16x^4}{3465} - \frac{32x^5}{45045} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 \sqrt{x}\left(1 - \frac{2x}{5} + \frac{4x^2}{35} - \frac{8x^3}{315} + \frac{16x^4}{3465} - \frac{32x^5}{45045} + O(x^6)\right) \\ + \frac{c_2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x} \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{2x}{5}+\frac{4x^2}{35}-\frac{8x^3}{315}+\frac{16x^4}{3465}-\frac{32x^5}{45045}+O(x^6)\right) + \frac{c_2\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)}{x}$$

Verified OK.

14.14.1 Maple step by step solution

Let's solve

$2x^2y''+(2x^2+3x)\,y'+y(x-1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x-1)y}{2x^2}-\frac{(2x+3)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x+3)y'}{2x}+\frac{(x-1)y}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x+3}{2x}, P_3(x)=\frac{x-1}{2x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{3}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{2}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2y''+x(2x+3)\,y'+y(x-1)=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)\left(a_k(k+r+1) + a_{k-1}\right) = 0$$

- Shift index using $k- >k + 1$

$$2\left(k+\frac{1}{2}+r\right)\left(a_{k+1}(k+2+r) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{a_k}{1+k}, b_{1+k} = -\frac{b_k}{k+\frac{5}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*(3+2*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}}\left(1-\frac{2}{5}x+\frac{4}{35}x^2-\frac{8}{315}x^3+\frac{16}{3465}x^4-\frac{32}{45045}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\mathrm{O}\left(\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*(3+2*x)*y'[x]-(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{32x^5}{45045}+\frac{16x^4}{3465}-\frac{8x^3}{315}+\frac{4x^2}{35}-\frac{2x}{5}+1\right)+\frac{c_2\left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)}{x}$$

14.15 problem 15

Internal problem ID [1306]
Internal file name [`OUTPUT/1307_Sunday_June_05_2022_02_09_29_AM_98679901/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(x+3)\,y''+x(4x+5)\,y'-(1-2x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+3x^2\right)y''+\left(4x^2+5x\right)y'+y(2x-1)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x+5}{x\,(x+3)}$$
$$q(x)=\frac{2x-1}{x^2\,(x+3)}$$

Table 558: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x+5}{x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{2x-1}{x^2(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+3)\, y'' + \left(4x^2 + 5x\right) y' + y(2x-1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2(x+3) & \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ & + \left(4x^2 + 5x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (2x-1) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 4a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r) (n+r-1) + 5x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 5x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) \\ + 4a_{n-1}(n+r-1) + 5a_n(n+r) + 2a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\, a_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(3n+1)\, a_{n-1}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{2+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{9r^2+21r+10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{14}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 6r^2 - 11r - 6}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{140}{2187}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{455}{19683}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{455}{19683}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{1456}{177147}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{455}{19683}$
a_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{1456}{177147}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ + 4b_{n-1}(n+r-1) + 5b_n(n+r) + 2b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(n+r)\, b_{n-1}}{3n + 3r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{(n-1)\, b_{n-1}}{3n - 4} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 - r}{2 + 3r}$$

Which for the root $r = -1$ becomes

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 + 3r + 2}{9r^2 + 21r + 10}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	0
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^3 - 6r^2 - 11r - 6}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	0
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	0
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	0
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	0
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	0
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	0
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	0
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	0
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	0
b_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
$$

Verified OK.

14.15.1 Maple step by step solution

Let's solve

$x^2(x+3)\,y''+(4x^2+5x)\,y'+y(2x-1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(2x-1)y}{x^2(x+3)}-\frac{(4x+5)y'}{x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x+5)y'}{x(x+3)}+\frac{(2x-1)y}{x^2(x+3)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{4x+5}{x(x+3)},P_3(x)=\frac{2x-1}{x^2(x+3)}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=\frac{7}{3}$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

  ◦ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $x^2(x+3)\,y''+x(4x+5)\,y'+y(2x-1)=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(u^3-6u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(4u^2-19u+21)\left(\frac{d}{du}y(u)\right)+(2u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_0 r(4+3r)\, u^{-1+r} + (3a_1(1+r)\,(7+3r) - a_0(7+6r)\,(1+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+r+1)\,(3k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3r(4+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{4}{3}\right\}$$

- Each term must be 0

  $$3a_1(1+r)\,(7+3r) - a_0(7+6r)\,(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-6\left(\left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right) k + \left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right) r + \frac{7a_k}{6} - \frac{7a_{k+1}}{2}\right) (k+r+1) = 0$$

- Shift index using $k- >k + 1$

  $$-6\left(\left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right) (k+1) + \left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right) r + \frac{7a_{k+1}}{6} - \frac{7a_{k+2}}{2}\right) (k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k a_k - 6k a_{k+1} + r a_k - 6r a_{k+1} + a_k - 13a_{k+1}}{3(3k+10+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k a_k - 6k a_{k+1} + a_k - 13a_{k+1}}{3(3k+10)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k a_k - 6k a_{k+1} + a_k - 13a_{k+1}}{3(3k+10)}, 21a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{k a_k - 6k a_{k+1} + a_k - 13a_{k+1}}{3(3k+10)}, 21a_1 - 7a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -3a_1 - \frac{a_0}{3} = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -3a_1 - \frac{a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k a_k - 6k a_{1+k} + a_k - 13a_{1+k}}{3(3k+10)}, 21a_1 - 7a_0 = 0, b_{k-}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*(3+x)*diff(y(x),x$2)+x*(5+4*x)*diff(y(x),x)-(1-2*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{4}{9}x + \frac{14}{81}x^2 - \frac{140}{2187}x^3 + \frac{455}{19683}x^4 - \frac{1456}{177147}x^5 + \mathrm{O}\left(x^6\right)\right) + c_1\left(1 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x^2*(3+x)*y''[x]+x*(5+4*x)*y'[x]-(1-2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{1456x^5}{177147} + \frac{455x^4}{19683} - \frac{140x^3}{2187} + \frac{14x^2}{81} - \frac{4x}{9} + 1\right) + \frac{c_2}{x}$$

14.16 problem 16

Internal problem ID [1307]
Internal file name [OUTPUT/1308_Sunday_June_05_2022_02_09_32_AM_98310107/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + x(x+5)\,y' - (-3x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+5}{2x}$$
$$q(x) = \frac{3x-2}{2x^2}$$

Table 560: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x-2}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (x^2 + 5x)\, y' + (3x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^2 + 5x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x - 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + 5x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 5a_n(n+r) + 3a_{n-1} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{1+2r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{1+2r}$	$-\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{1+2r}$	$-\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{1+2r}$	$-\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{3840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{1+2r}$	$-\frac{1}{2}$
a_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{8}$
a_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{48}$
a_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{384}$
a_5	$-\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{1}{3840}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + 5b_n(n+r) + 3b_{n-1} - 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n+2r-1} \tag{4}
$$

Which for the root $r = -2$ becomes

$$
b_n = -\frac{b_{n-1}}{2n-5} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{1 + 2r}$$

Which for the root $r = -2$ becomes

$$b_1 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{1+2r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^2 + 8r + 3}$$

Which for the root $r = -2$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{1+2r}$	$\frac{1}{3}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = -2$ becomes

$$b_3 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{1+2r}$	$\frac{1}{3}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{3}$
b_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{1}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{1+2r}$	$\frac{1}{3}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{3}$
b_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{3}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{9}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = -2$ becomes

$$b_5 = -\frac{1}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{1+2r}$	$\frac{1}{3}$
b_2	$\frac{1}{4r^2+8r+3}$	$\frac{1}{3}$
b_3	$-\frac{1}{8r^3+36r^2+46r+15}$	$-\frac{1}{3}$
b_4	$\frac{1}{16r^4+128r^3+344r^2+352r+105}$	$\frac{1}{9}$
b_5	$-\frac{1}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{1}{45}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3} + \frac{x^4}{9} - \frac{x^5}{45} + O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3} + \frac{x^4}{9} - \frac{x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3} + \frac{x^4}{9} - \frac{x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \sqrt{x}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + \frac{x}{3} + \frac{x^2}{3} - \frac{x^3}{3} + \frac{x^4}{9} - \frac{x^5}{45} + O(x^6)\right)}{x^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}-\frac{x^5}{3840}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{3}+\frac{x^2}{3}-\frac{x^3}{3}+\frac{x^4}{9}-\frac{x^5}{45}+O(x^6)\right)}{x^2}$$

Verified OK.

14.16.1 Maple step by step solution

Let's solve

$2x^2y''+(x^2+5x)\,y'+(3x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x-2)y}{2x^2}-\frac{(x+5)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+5)y'}{2x}+\frac{(3x-2)y}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+5}{2x}, P_3(x)=\frac{3x-2}{2x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2y''+x(x+5)\,y'+(3x-2)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(2k+2r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-2, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r+2)\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right) = 0$$

- Shift index using $k->k+1$

  $$2(k+r+3)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{2k+1+2r}$$

- Recursion relation for $r = -2$

$a_{k+1} = -\frac{a_k}{2k-3}$

- Solution for $r = -2$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{2k-3}\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = -\frac{a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{a_k}{2k-3}, b_{1+k} = -\frac{b_k}{2k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*(5+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{5}{2}} (1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 - \frac{1}{3840}x^5 + O(x^6)) + c_1 (1 + \frac{1}{3}x + \frac{1}{3}x^2 - \frac{1}{3}x^3 + \frac{1}{9}x^4 - \frac{1}{45}x^5 + O(x^6)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 88

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*(5+x)*y'[x]-(2-3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt{x} \left(-\frac{x^5}{3840} + \frac{x^4}{384} - \frac{x^3}{48} + \frac{x^2}{8} - \frac{x}{2} + 1\right) + \frac{c_2 \left(-\frac{x^5}{45} + \frac{x^4}{9} - \frac{x^3}{3} + \frac{x^2}{3} + \frac{x}{3} + 1\right)}{x^2}$$

14.17 problem 17

Internal problem ID [1308]
Internal file name [`OUTPUT/1309_Sunday_June_05_2022_02_09_35_AM_11476737/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' + x(x+1)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + \left(x^2 + x\right)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{3x}$$
$$q(x) = -\frac{1}{3x^2}$$

Table 562: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2 y'' + (x^2 + x) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (x^2 + x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right)+\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right)+\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1)+x^{n+r} a_n (n+r)-a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r)+x^r a_0 r-a_0 x^r=0$$

Or

$$\left(3x^r r(-1+r)+x^r r-x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2-2r-1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2-2r-1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 - 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{3n+3r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{3n+4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4 + 3r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4+3r}$	$-\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9r^2 + 33r + 28}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{70}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4+3r}$	$-\frac{1}{7}$
a_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{70}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{27r^3 + 189r^2 + 414r + 280}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{910}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4+3r}$	$-\frac{1}{7}$
a_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{70}$
a_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{910}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{81r^4 + 918r^3 + 3699r^2 + 6222r + 3640}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{14560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4+3r}$	$-\frac{1}{7}$
a_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{70}$
a_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{910}$
a_4	$\frac{1}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{14560}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{243r^5 + 4050r^4 + 25785r^3 + 77850r^2 + 110472r + 58240}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{276640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4+3r}$	$-\frac{1}{7}$
a_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{70}$
a_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{910}$
a_4	$\frac{1}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{14560}$
a_5	$-\frac{1}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{1}{276640}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{7} + \frac{x^2}{70} - \frac{x^3}{910} + \frac{x^4}{14560} - \frac{x^5}{276640} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{3n+3r+1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = -\frac{b_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{4+3r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4+3r}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{9r^2 + 33r + 28}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = \frac{1}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4+3r}$	$-\frac{1}{3}$
b_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{18}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{27r^3 + 189r^2 + 414r + 280}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = -\frac{1}{162}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4+3r}$	$-\frac{1}{3}$
b_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{18}$
b_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{162}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{81r^4 + 918r^3 + 3699r^2 + 6222r + 3640}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{1}{1944}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4+3r}$	$-\frac{1}{3}$
b_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{18}$
b_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{162}$
b_4	$\frac{1}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{1944}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{243r^5 + 4050r^4 + 25785r^3 + 77850r^2 + 110472r + 58240}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = -\frac{1}{29160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4+3r}$	$-\frac{1}{3}$
b_2	$\frac{1}{9r^2+33r+28}$	$\frac{1}{18}$
b_3	$-\frac{1}{27r^3+189r^2+414r+280}$	$-\frac{1}{162}$
b_4	$\frac{1}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{1}{1944}$
b_5	$-\frac{1}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{1}{29160}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{x}{7} + \frac{x^2}{70} - \frac{x^3}{910} + \frac{x^4}{14560} - \frac{x^5}{276640} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{x}{7} + \frac{x^2}{70} - \frac{x^3}{910} + \frac{x^4}{14560} - \frac{x^5}{276640} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x \left(1 - \frac{x}{7} + \frac{x^2}{70} - \frac{x^3}{910} + \frac{x^4}{14560} - \frac{x^5}{276640} + O(x^6)\right) \\
+ \frac{c_2 \left(1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 x \left(1 - \frac{x}{7} + \frac{x^2}{70} - \frac{x^3}{910} + \frac{x^4}{14560} - \frac{x^5}{276640} + O(x^6)\right) \\
+ \frac{c_2 \left(1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}$$

Verified OK.

14.17.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2 + x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{3x^2} - \frac{(x+1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x+1}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(x+1)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(\left(k+r+\frac{1}{3}\right) a_k + \frac{a_{k-1}}{3}\right)(k+r-1) = 0$

- Shift index using $k->k+1$

  $3\left(\left(k+\frac{4}{3}+r\right) a_{k+1} + \frac{a_k}{3}\right)(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{3k+4+3r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{3k+7}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{3k+7}\right]$

- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+1} = -\frac{a_k}{3k+3}$

- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k}{3k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), a_{1+k} = -\frac{a_k}{3k+7}, b_{1+k} = -\frac{b_k}{3k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{1}{3}x+\frac{1}{18}x^2-\frac{1}{162}x^3+\frac{1}{1944}x^4-\frac{1}{29160}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}} + c_2 x\left(1-\frac{1}{7}x+\frac{1}{70}x^2-\frac{1}{910}x^3+\frac{1}{14560}x^4-\frac{1}{276640}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[3*x^2*y''[x]+x*(1+x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(-\frac{x^5}{276640} + \frac{x^4}{14560} - \frac{x^3}{910} + \frac{x^2}{70} - \frac{x}{7} + 1\right) + \frac{c_2 \left(-\frac{x^5}{29160} + \frac{x^4}{1944} - \frac{x^3}{162} + \frac{x^2}{18} - \frac{x}{3} + 1\right)}{\sqrt[3]{x}}$$

14.18 problem 18

Internal problem ID [1309]
Internal file name [`OUTPUT/1310_Sunday_June_05_2022_02_09_38_AM_33333253/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - y'x + (1 - 2x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - y'x + (1 - 2x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{2x - 1}{2x^2}$$

Table 564: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2x-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - y'x + (1 - 2x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x + (1 - 2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) - a_n(n+r) + a_n - 2a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = \frac{2a_{n-1}}{2n^2 + n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2}{2r^2 + r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+r}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+r}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8}{r\,(4r^3 + 12r^2 + 11r + 3)\,(2r^2 + 9r + 10)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{4}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+r}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{15}$
a_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{r\left(4r^3+12r^2+11r+3\right)\left(2r^2+9r+10\right)\left(2r^2+13r+21\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{2}{2835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+r}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{15}$
a_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{315}$
a_4	$\frac{16}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{2}{2835}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32}{r\left(4r^3+12r^2+11r+3\right)\left(2r^2+9r+10\right)\left(2r^2+13r+21\right)\left(2r^2+17r+36\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{4}{155925}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{2r^2+r}$	$\frac{2}{3}$
a_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{15}$
a_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{315}$
a_4	$\frac{16}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{2}{2835}$
a_5	$\frac{32}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$\frac{4}{155925}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) + b_n - 2b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{2b_{n-1}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2}{2r^2 + r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+r}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+r}$	2
b_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8}{r\left(4r^3 + 12r^2 + 11r + 3\right)\left(2r^2 + 9r + 10\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{4}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+r}$	2
b_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{3}$
b_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{45}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{r\left(4r^3+12r^2+11r+3\right)\left(2r^2+9r+10\right)\left(2r^2+13r+21\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{2}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+r}$	2
b_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{3}$
b_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{45}$
b_4	$\frac{16}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{2}{315}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32}{r\left(4r^3+12r^2+11r+3\right)\left(2r^2+9r+10\right)\left(2r^2+13r+21\right)\left(2r^2+17r+36\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{4}{14175}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2}{2r^2+r}$	2
b_2	$\frac{4}{4r^4+12r^3+11r^2+3r}$	$\frac{2}{3}$
b_3	$\frac{8}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{4}{45}$
b_4	$\frac{16}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{2}{315}$
b_5	$\frac{32}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$\frac{4}{14175}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O\left(x^6\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O\left(x^6\right)\right) \\ &\quad + c_2\sqrt{x}\left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O\left(x^6\right)\right) \\ &\quad + c_2\sqrt{x}\left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x&\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O\left(x^6\right)\right) \\ &+ c_2\sqrt{x}\left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O\left(x^6\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = c_1 x&\left(1 + \frac{2x}{3} + \frac{2x^2}{15} + \frac{4x^3}{315} + \frac{2x^4}{2835} + \frac{4x^5}{155925} + O\left(x^6\right)\right) \\ &+ c_2\sqrt{x}\left(1 + 2x + \frac{2x^2}{3} + \frac{4x^3}{45} + \frac{2x^4}{315} + \frac{4x^5}{14175} + O\left(x^6\right)\right) \end{aligned}$$

Verified OK.

14.18.1 Maple step by step solution

Let's solve

$2x^2y'' - y'x + (1-2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} - \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - y'x + (1-2x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{1+k}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k}=\frac{2a_k}{(2k+3)(1+k)}, b_{1+k}=\frac{2b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-2*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1\sqrt{x}\left(1+2x+\frac{2}{3}x^2+\frac{4}{45}x^3+\frac{2}{315}x^4+\frac{4}{14175}x^5+\mathrm{O}\left(x^6\right)\right)+c_2x\left(1+\frac{2}{3}x+\frac{2}{15}x^2+\frac{4}{315}x^3+\frac{2}{2835}x^4+\frac{4}{155925}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(1-2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\rightarrow c_1x\left(\frac{4x^5}{155925}+\frac{2x^4}{2835}+\frac{4x^3}{315}+\frac{2x^2}{15}+\frac{2x}{3}+1\right)+c_2\sqrt{x}\left(\frac{4x^5}{14175}+\frac{2x^4}{315}+\frac{4x^3}{45}+\frac{2x^2}{3}+2x+1\right)$$

14.19 problem 19

Internal problem ID [1310]
Internal file name [`OUTPUT/1311_Sunday_June_05_2022_02_09_41_AM_13881759/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 9y'x - (3x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 9y'x + (-3x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{3x+1}{9x^2}$$

Table 566: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3x+1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 9y'x + (-3x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + 9\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x + (-3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
9x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) - a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
9x^r a_0 r(-1+r) + 9x^r a_0 r - a_0 x^r = 0
$$

Or

$$
\left(9x^r r(-1+r) + 9x^r r - x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(9r^2 - 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
9r^2 - 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(9r^2 - 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n + r)(n + r - 1) + 9a_n(n + r) - 3a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{3a_{n-1}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}}{n(3n + 2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3}{9r^2 + 18r + 8}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{9r^2+18r+8}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{9r^2+18r+8}$	$\frac{1}{5}$
a_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{80}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{1}{2640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{9r^2+18r+8}$	$\frac{1}{5}$
a_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{80}$
a_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{2640}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{147840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{9r^2+18r+8}$	$\frac{1}{5}$
a_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{80}$
a_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{2640}$
a_4	$\frac{81}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{1}{147840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2 + 90r + 224)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{1}{12566400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{9r^2+18r+8}$	$\frac{1}{5}$
a_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{80}$
a_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{2640}$
a_4	$\frac{81}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{1}{147840}$
a_5	$\frac{243}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$\frac{1}{12566400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \frac{x^5}{12566400} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 9b_n(n+r) - 3b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{3b_{n-1}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{b_{n-1}}{n(3n-2)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{3}{9r^2 + 18r + 8}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{9r^2+18r+8}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{9r^2+18r+8}$	1
b_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{27}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{9r^2+18r+8}$	1
b_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{8}$
b_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{168}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{1}{6720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{9r^2+18r+8}$	1
b_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{8}$
b_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{168}$
b_4	$\frac{81}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{1}{6720}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{243}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = \frac{1}{436800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{9r^2+18r+8}$	1
b_2	$\frac{9}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{1}{8}$
b_3	$\frac{27}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{168}$
b_4	$\frac{81}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{1}{6720}$
b_5	$\frac{243}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$\frac{1}{436800}$

Using the above table, then the solution $y_2(x)$ is

$$y_2(x) = x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right)$$
$$= \frac{1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \frac{x^5}{436800} + O(x^6)}{x^{\frac{1}{3}}}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \frac{x^5}{12566400} + O(x^6)\right)$$
$$+ \frac{c_2\left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \frac{x^5}{436800} + O(x^6)\right)}{x^{\frac{1}{3}}}$$

Hence the final solution is

$$y = y_h$$
$$= c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \frac{x^5}{12566400} + O(x^6)\right)$$
$$+ \frac{c_2\left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \frac{x^5}{436800} + O(x^6)\right)}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \frac{x^5}{12566400} + O(x^6)\right) + \frac{c_2\left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \frac{x^5}{436800} + O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \frac{x^5}{12566400} + O(x^6)\right) + \frac{c_2\left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \frac{x^5}{436800} + O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

14.19.1 Maple step by step solution

Let's solve

$9x^2y'' + 9y'x + (-3x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(3x+1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{3x+1}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 9y'x + (-3x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) - 3a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r+1)\,(3k+3r-1) - 3a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(3k+4+3r)\,(3k+2+3r) - 3a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{3a_k}{(3k+4+3r)(3k+2+3r)}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+1} = \frac{3a_k}{(3k+3)(3k+1)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = \frac{3a_k}{(3k+3)(3k+1)}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{3a_k}{(3k+5)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{3a_k}{(3k+5)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{1+k} = \frac{3a_k}{(3k+3)(3k+1)}, b_{1+k} = \frac{3b_k}{(3k+5)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+9*x*diff(y(x),x)-(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \frac{1}{2640}x^3 + \frac{1}{147840}x^4 + \frac{1}{12566400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \frac{1}{6720}x^4 + \frac{1}{436}\right.}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*x*y'[x]-(1+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(\frac{x^5}{12566400}+\frac{x^4}{147840}+\frac{x^3}{2640}+\frac{x^2}{80}+\frac{x}{5}+1\right) + \frac{c_2\left(\frac{x^5}{436800}+\frac{x^4}{6720}+\frac{x^3}{168}+\frac{x^2}{8}+x+1\right)}{\sqrt[3]{x}}$$

14.20 problem 20

Internal problem ID [1311]
Internal file name [`OUTPUT/1312_Sunday_June_05_2022_02_09_43_AM_25506030/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' + x(x+1)\,y' - (3x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + \left(x^2 + x\right)y' + (-3x - 1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{3x}$$
$$q(x) = -\frac{3x+1}{3x^2}$$

Table 568: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3x+1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + (x^2 + x) y' + (-3x - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (x^2 + x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (-3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 - 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -\frac{1}{3} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 - 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) - 3a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-4)}{3n^2 + 6nr + 3r^2 - 2n - 2r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n-3)}{n(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3 - r}{r (3r + 4)}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{2}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r(3r+4)}$	$\frac{2}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-3 + r) (-2 + r)}{9r^4 + 42r^3 + 61r^2 + 28r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{70}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r(3r+4)}$	$\frac{2}{7}$
a_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{70}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-3 + r) (-2 + r) (-1 + r)}{r (9r^3 + 42r^2 + 61r + 28) (3r^2 + 16r + 20)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r(3r+4)}$	$\frac{2}{7}$
a_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{70}$
a_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r(3r+4)}$	$\frac{2}{7}$
a_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{70}$
a_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	0
a_4	$\frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-3+r)(-2+r)(-1+r)}{(3r^2+28r+64)(3r^2+22r+39)(3r^2+16r+20)(9r^2+33r+28)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r(3r+4)}$	$\frac{2}{7}$
a_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$\frac{1}{70}$
a_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	0
a_4	$\frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$	0
a_5	$-\frac{(-3+r)(-2+r)(-1+r)}{(3r^2+28r+64)(3r^2+22r+39)(3r^2+16r+20)(9r^2+33r+28)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + \frac{2x}{7} + \frac{x^2}{70} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + b_n(n+r) - 3b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-4)}{3n^2 + 6nr + 3r^2 - 2n - 2r - 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{-3nb_{n-1} + 13b_{n-1}}{9n^2 - 12n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{3-r}{r(3r+4)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = -\frac{10}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-r}{r(3r+4)}$	$-\frac{10}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(-3+r)(-2+r)}{9r^4 + 42r^3 + 61r^2 + 28r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{35}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-r}{r(3r+4)}$	$-\frac{10}{3}$
b_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$-\frac{35}{18}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3 + 42r^2 + 61r + 28)(3r^2 + 16r + 20)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = -\frac{14}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-r}{r(3r+4)}$	$-\frac{10}{3}$
b_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$-\frac{35}{18}$
b_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	$-\frac{14}{81}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = -\frac{7}{3888}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-r}{r(3r+4)}$	$-\frac{10}{3}$
b_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$-\frac{35}{18}$
b_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	$-\frac{14}{81}$
b_4	$\frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$	$-\frac{7}{3888}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(-3+r)(-2+r)(-1+r)}{(3r^2+28r+64)(3r^2+22r+39)(3r^2+16r+20)(9r^2+33r+28)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = \frac{7}{320760}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-r}{r(3r+4)}$	$-\frac{10}{3}$
b_2	$\frac{(-3+r)(-2+r)}{9r^4+42r^3+61r^2+28r}$	$-\frac{35}{18}$
b_3	$-\frac{(-3+r)(-2+r)(-1+r)}{r(9r^3+42r^2+61r+28)(3r^2+16r+20)}$	$-\frac{14}{81}$
b_4	$\frac{(-3+r)(-2+r)(-1+r)}{(9r^3+42r^2+61r+28)(3r^2+16r+20)(3r^2+22r+39)}$	$-\frac{7}{3888}$
b_5	$-\frac{(-3+r)(-2+r)(-1+r)}{(3r^2+28r+64)(3r^2+22r+39)(3r^2+16r+20)(9r^2+33r+28)}$	$\frac{7}{320760}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{10x}{3} - \frac{35x^2}{18} - \frac{14x^3}{81} - \frac{7x^4}{3888} + \frac{7x^5}{320760} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{2x}{7} + \frac{x^2}{70} + O(x^6)\right) + \frac{c_2\left(1 - \frac{10x}{3} - \frac{35x^2}{18} - \frac{14x^3}{81} - \frac{7x^4}{3888} + \frac{7x^5}{320760} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 + \frac{2x}{7} + \frac{x^2}{70} + O(x^6)\right) + \frac{c_2\left(1 - \frac{10x}{3} - \frac{35x^2}{18} - \frac{14x^3}{81} - \frac{7x^4}{3888} + \frac{7x^5}{320760} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\left(1 + \frac{2x}{7} + \frac{x^2}{70} + O(x^6)\right) + \frac{c_2\left(1 - \frac{10x}{3} - \frac{35x^2}{18} - \frac{14x^3}{81} - \frac{7x^4}{3888} + \frac{7x^5}{320760} + O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1 + \frac{2x}{7} + \frac{x^2}{70} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{10x}{3} - \frac{35x^2}{18} - \frac{14x^3}{81} - \frac{7x^4}{3888} + \frac{7x^5}{320760} + O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

14.20.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2 + x)\, y' + (-3x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{3x^2} - \frac{(x+1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{3x} - \frac{(3x+1)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+1}{3x}, P_3(x) = -\frac{3x+1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(x + 1)\, y' + (-3x - 1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+3r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{1, -\frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3\left(k+r+\frac{1}{3}\right)(k+r-1)\, a_k + a_{k-1}(k-4+r) = 0$$

- Shift index using $k- >k + 1$

  $$3\left(k+\frac{4}{3}+r\right)(k+r)\, a_{k+1} + a_k(k+r-3) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-3)}{(3k+4+3r)(k+r)}$$

- Recursion relation for $r = 1$; series terminates at $k = 2$

$a_{k+1} = -\frac{a_k(k-2)}{(3k+7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{20}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{70}$

- Terminating series solution of the ODE for $r = 1$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$

- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}$

- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{1+k} = -\frac{b_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{10}{3}x-\frac{35}{18}x^2-\frac{14}{81}x^3-\frac{7}{3888}x^4+\frac{7}{320760}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}} + c_2 x\left(1+\frac{2}{7}x+\frac{1}{70}x^2+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
AsymptoticDSolveValue[3*x^2*y''[x]+x*(1+x)*y'[x]-(1+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^2}{70} + \frac{2x}{7} + 1\right) + \frac{c_2 \left(\frac{7x^5}{320760} - \frac{7x^4}{3888} - \frac{14x^3}{81} - \frac{35x^2}{18} - \frac{10x}{3} + 1\right)}{\sqrt[3]{x}}$$

14.21 problem 21

Internal problem ID [1312]
Internal file name [`OUTPUT/1313_Sunday_June_05_2022_02_09_46_AM_32578457/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+3)\,y''+x(1+5x)\,y'+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{1+5x}{2x\,(x+3)}\\q(x)&=\frac{x+1}{2x^2\,(x+3)}\end{aligned}$$

Table 570: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+5x}{2x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{x+1}{2x^2(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x+3)\, y'' + \left(5x^2 + x\right) y' + (x+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(5x^2 + x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$6x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$(6x^r r(-1+r) + x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(6r^2 - 5r + 1)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(6r^2 - 5r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n+r-1)(n+r-2) + 6a_n(n+r)(n+r-1) \\ &+ 5a_{n-1}(n+r-1) + a_n(n+r) + a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\, a_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(-1-2n)}{6n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{2+3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{3}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{9r^2+21r+10}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{15}{91}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{15}{91}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 6r^2 - 11r - 6}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{15}{247}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{15}{91}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{15}{247}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{27}{1235}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{15}{91}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{15}{247}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{27}{1235}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{297}{38285}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{2+3r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{15}{91}$
a_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{15}{247}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{27}{1235}$
a_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{297}{38285}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right) \\ &= \sqrt{x}\left(1 - \frac{3x}{7} + \frac{15x^2}{91} - \frac{15x^3}{247} + \frac{27x^4}{1235} - \frac{297x^5}{38285} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n+r-1)(n+r-2) + 6b_n(n+r)(n+r-1) \\ &+ 5b_{n-1}(n+r-1) + b_n(n+r) + b_{n-1} + b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(n+r)\,b_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -\frac{(3n+1)\,b_{n-1}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 - r}{2 + 3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = -\frac{4}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 + 3r + 2}{9r^2 + 21r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{14}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^3 - 6r^2 - 11r - 6}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{140}{2187}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{455}{19683}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{455}{19683}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{1456}{177147}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-r}{2+3r}$	$-\frac{4}{9}$
b_2	$\frac{r^2+3r+2}{9r^2+21r+10}$	$\frac{14}{81}$
b_3	$\frac{-r^3-6r^2-11r-6}{27r^3+135r^2+198r+80}$	$-\frac{140}{2187}$
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{455}{19683}$
b_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{1456}{177147}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{3x}{7} + \frac{15x^2}{91} - \frac{15x^3}{247} + \frac{27x^4}{1235} - \frac{297x^5}{38285} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{3x}{7} + \frac{15x^2}{91} - \frac{15x^3}{247} + \frac{27x^4}{1235} - \frac{297x^5}{38285} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1 - \frac{3x}{7} + \frac{15x^2}{91} - \frac{15x^3}{247} + \frac{27x^4}{1235} - \frac{297x^5}{38285} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{4x}{9} + \frac{14x^2}{81} - \frac{140x^3}{2187} + \frac{455x^4}{19683} - \frac{1456x^5}{177147} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{3x}{7}+\frac{15x^2}{91}-\frac{15x^3}{247}+\frac{27x^4}{1235}-\frac{297x^5}{38285}+O(x^6)\right) + c_2x^{\frac{1}{3}}\left(1-\frac{4x}{9}+\frac{14x^2}{81}-\frac{140x^3}{2187}+\frac{455x^4}{19683}-\frac{1456x^5}{177147}+O(x^6)\right)$$

Verified OK.

14.21.1 Maple step by step solution

Let's solve

$2x^2(x+3)\,y''+(5x^2+x)\,y'+(x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x+1)y}{2x^2(x+3)}-\frac{(1+5x)y'}{2x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+5x)y'}{2x(x+3)}+\frac{(x+1)y}{2x^2(x+3)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+5x}{2x(x+3)}, P_3(x)=\frac{x+1}{2x^2(x+3)}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=\frac{7}{3}$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

    $\left((x+3)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$

  - $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $2x^2(x+3)\,y''+x(1+5x)\,y'+(x+1)\,y=0$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

$$\left(2u^3 - 12u^2 + 18u\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(5u^2 - 29u + 42\right)\left(\frac{d}{du}y(u)\right) + (u - 2)\,y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,u^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)(k + r - 1)\,u^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)(k + 1 - m + r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0r(4 + 3r)\,u^{-1+r} + \left(6a_1(1 + r)(7 + 3r) - a_0\left(12r^2 + 17r + 2\right)\right)u^r + \left(\sum_{k=1}^{\infty}\left(6a_{k+1}(k + r + 1)(3k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$6r(4 + 3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{4}{3}\right\}$$

- Each term must be 0

$$6a_1(1+r)(7+3r) - a_0(12r^2+17r+2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-6a_k + a_{k-1} + 9a_{k+1})k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})r - 17a_k - a_{k-1} + 60a_{k+1})k + 2(-6a_k +$$

- Shift index using $k->k+1$

$$2(-6a_{k+1} + a_k + 9a_{k+2})(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})r - 17a_{k+1} - a_k + 60a_{k+2})(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2a_k - 12k^2a_{1+k} + 3ka_k - 41ka_{1+k} + a_k - 31a_{1+k}}{6(3k^2+16k+20)}, 42a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*(3+x)*diff(y(x),x$2)+x*(1+5*x)*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = c_1 x^{\frac{1}{3}}\left(1-\frac{4}{9}x+\frac{14}{81}x^2-\frac{140}{2187}x^3+\frac{455}{19683}x^4-\frac{1456}{177147}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1-\frac{3}{7}x+\frac{15}{91}x^2-\frac{15}{247}x^3+\frac{27}{1235}x^4-\frac{297}{38285}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 90

```
AsymptoticDSolveValue[2*x^2*(3+x)*y''[x]+x*(1+5*x)*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{297x^5}{38285}+\frac{27x^4}{1235}-\frac{15x^3}{247}+\frac{15x^2}{91}-\frac{3x}{7}+1\right) + c_2\sqrt[3]{x}\left(-\frac{1456x^5}{177147}+\frac{455x^4}{19683}-\frac{140x^3}{2187}+\frac{14x^2}{81}-\frac{4x}{9}+1\right)$$

14.22 problem 22

Internal problem ID [1313]
Internal file name [`OUTPUT/1314_Sunday_June_05_2022_02_09_49_AM_62221154/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+4)\,y''-x(-3x+1)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+4x^2\right)y''+\left(3x^2-x\right)y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x-1}{x\,(x+4)}$$
$$q(x)=\frac{1}{x^2\,(x+4)}$$

Table 572: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-1}{x(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x^2(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-4, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+4)\, y'' + \left(3x^2 - x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+4) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^2 - x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 5r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 5r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n + r - 1)(n + r - 2) + 4a_n(n + r)(n + r - 1) + 3a_{n-1}(n + r - 1) - a_n(n + r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(1 + n + r) a_{n-1}}{4n + 4r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{(2 + n) a_{n-1}}{4n + 3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2 - r}{3 + 4r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{3}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3+4r}$	$-\frac{3}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 + 5r + 6}{16r^2 + 40r + 21}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{12}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3+4r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{12}{77}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 9r^2 - 26r - 24}{64r^3 + 336r^2 + 524r + 231}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{4}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3+4r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{12}{77}$
a_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{4}{77}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{24}{1463}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3+4r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{12}{77}$
a_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{4}{77}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{24}{1463}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{24}{4807}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{3+4r}$	$-\frac{3}{7}$
a_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{12}{77}$
a_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{4}{77}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{24}{1463}$
a_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{24}{4807}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\big) \\ &= x\left(1 - \frac{3x}{7} + \frac{12x^2}{77} - \frac{4x^3}{77} + \frac{24x^4}{1463} - \frac{24x^5}{4807} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} &b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\ &+ 3b_{n-1}(n+r-1) - b_n(n+r) + b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(1+n+r)\, b_{n-1}}{4n+4r-1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = -\frac{(5+4n)\, b_{n-1}}{16n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2 - r}{3 + 4r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = -\frac{9}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3+4r}$	$-\frac{9}{16}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 + 5r + 6}{16r^2 + 40r + 21}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{117}{512}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3+4r}$	$-\frac{9}{16}$
b_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{117}{512}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^3 - 9r^2 - 26r - 24}{64r^3 + 336r^2 + 524r + 231}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = -\frac{663}{8192}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3+4r}$	$-\frac{9}{16}$
b_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{117}{512}$
b_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{663}{8192}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{13923}{524288}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3+4r}$	$-\frac{9}{16}$
b_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{117}{512}$
b_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{663}{8192}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{13923}{524288}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = -\frac{69615}{8388608}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{3+4r}$	$-\frac{9}{16}$
b_2	$\frac{r^2+5r+6}{16r^2+40r+21}$	$\frac{117}{512}$
b_3	$\frac{-r^3-9r^2-26r-24}{64r^3+336r^2+524r+231}$	$-\frac{663}{8192}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{13923}{524288}$
b_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{69615}{8388608}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= x^{\frac{1}{4}}\left(1-\frac{9x}{16}+\frac{117x^2}{512}-\frac{663x^3}{8192}+\frac{13923x^4}{524288}-\frac{69615x^5}{8388608}+O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x\left(1-\frac{3x}{7}+\frac{12x^2}{77}-\frac{4x^3}{77}+\frac{24x^4}{1463}-\frac{24x^5}{4807}+O\left(x^6\right)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1-\frac{9x}{16}+\frac{117x^2}{512}-\frac{663x^3}{8192}+\frac{13923x^4}{524288}-\frac{69615x^5}{8388608}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x\left(1-\frac{3x}{7}+\frac{12x^2}{77}-\frac{4x^3}{77}+\frac{24x^4}{1463}-\frac{24x^5}{4807}+O\left(x^6\right)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1-\frac{9x}{16}+\frac{117x^2}{512}-\frac{663x^3}{8192}+\frac{13923x^4}{524288}-\frac{69615x^5}{8388608}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1x\left(1-\frac{3x}{7}+\frac{12x^2}{77}-\frac{4x^3}{77}+\frac{24x^4}{1463}-\frac{24x^5}{4807}+O\left(x^6\right)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1-\frac{9x}{16}+\frac{117x^2}{512}-\frac{663x^3}{8192}+\frac{13923x^4}{524288}-\frac{69615x^5}{8388608}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{3x}{7} + \frac{12x^2}{77} - \frac{4x^3}{77} + \frac{24x^4}{1463} - \frac{24x^5}{4807} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 - \frac{9x}{16} + \frac{117x^2}{512} - \frac{663x^3}{8192} + \frac{13923x^4}{524288} - \frac{69615x^5}{8388608} + O(x^6)\right)$$

Verified OK.

14.22.1 Maple step by step solution

Let's solve

$x^2(x+4)\, y'' + (3x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x+4)} - \frac{(3x-1)y'}{x(x+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-1)y'}{x(x+4)} + \frac{y}{x^2(x+4)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x-1}{x(x+4)}, P_3(x) = \frac{1}{x^2(x+4)}\right]$

  - $(x+4) \cdot P_2(x)$ is analytic at $x = -4$

    $\left((x+4) \cdot P_2(x)\right)\bigg|_{x=-4} = \frac{13}{4}$

  - $(x+4)^2 \cdot P_3(x)$ is analytic at $x = -4$

    $\left((x+4)^2 \cdot P_3(x)\right)\bigg|_{x=-4} = 0$

  - $x = -4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(x+4)\, y'' + x(3x-1)\, y' + y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

  $(u^3 - 8u^2 + 16u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 25u + 52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r(9+4r)\, u^{-1+r} + (4a_1(1+r)(13+4r) - a_0(8r^2+17r-1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(4k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(9+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{9}{4}\right\}$

- Each term must be 0

  $4a_1(1+r)(13+4r) - a_0(8r^2+17r-1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-8a_k + a_{k-1} + 16a_{k+1})\, k^2 + (2(-8a_k + a_{k-1} + 16a_{k+1})\, r - 17a_k + 68a_{k+1})\, k + (-8a_k + a_{k-1} +$

- Shift index using $k- >k+1$

  $(-8a_{k+1} + a_k + 16a_{k+2})(k+1)^2 + (2(-8a_{k+1} + a_k + 16a_{k+2})\, r - 17a_{k+1} + 68a_{k+2})(k+1) + (-$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2kra_k - 16kra_{k+1} + r^2 a_k - 8r^2 a_{k+1} + 2ka_k - 33ka_{k+1} + 2ra_k - 33ra_{k+1} - 24a_{k+1}}{4(4k^2 + 8kr + 4r^2 + 25k + 25r + 34)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}, 52a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}, 52a_1 + a_0 = 0\right]$$

- Recursion relation for $r = -\frac{9}{4}$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}$$

- Solution for $r = -\frac{9}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+4)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+4)^{k-\frac{9}{4}}\right), a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{1+k} + 2ka_k - 33ka_{1+k} - 24a_{1+k}}{4(4k^2 + 25k + 34)}, 52a_1 +$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(4+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}}\left(1-\frac{9}{16}x+\frac{117}{512}x^2-\frac{663}{8192}x^3+\frac{13923}{524288}x^4-\frac{69615}{8388608}x^5+\mathrm{O}\left(x^6\right)\right) + c_2 x\left(1-\frac{3}{7}x+\frac{12}{77}x^2-\frac{4}{77}x^3+\frac{24}{1463}x^4-\frac{24}{4807}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 86

```
AsymptoticDSolveValue[x^2*(4+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(-\frac{24x^5}{4807}+\frac{24x^4}{1463}-\frac{4x^3}{77}+\frac{12x^2}{77}-\frac{3x}{7}+1\right) + c_2 \sqrt[4]{x} \left(-\frac{69615x^5}{8388608}+\frac{13923x^4}{524288}-\frac{663x^3}{8192}+\frac{117x^2}{512}-\frac{9x}{16}+1\right)$$

14.23 problem 23

Internal problem ID [1314]
Internal file name [`OUTPUT/1315_Sunday_June_05_2022_02_09_52_AM_93027207/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 5y'x + (x + 1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 5y'x + (x + 1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{5}{2x} \\ q(x) &= \frac{x + 1}{2x^2} \end{aligned}$$

Table 574: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 5y'x + (x+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 5\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) x + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r + a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 5x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) + 5a_n(n + r) + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)(2r^2 -}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$
a_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 5b_n(n+r) + b_{n-1} + b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)\,(2r^2 +$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$
b_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= \frac{1}{\sqrt{x}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)}{\sqrt{x}} \\
&\quad + \frac{c_2\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)}{\sqrt{x}} \\
&\quad + \frac{c_2\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)\right)}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} + \frac{c_2 \left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} + \frac{c_2 \left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x}$$

Verified OK.

14.23.1 Maple step by step solution

Let's solve

$2x^2y''+5y'x+(x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5y'}{2x}-\frac{(x+1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{2x}+\frac{(x+1)y}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5}{2x}, P_3(x)=\frac{x+1}{2x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 5y'x + (x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+r+1)\,a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $2\left(k+\frac{3}{2}+r\right)\left(k+2+r\right)a_{k+1}+a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_{k}}{\left(2k+3+2r\right)\left(k+2+r\right)}$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{a_{k}}{\left(2k+1\right)\left(k+1\right)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-1}, a_{k+1}=-\frac{a_{k}}{\left(2k+1\right)\left(k+1\right)}\right]$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{a_{k}}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-\frac{1}{2}}, a_{k+1}=-\frac{a_{k}}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_{k}x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k-\frac{1}{2}}\right), a_{1+k}=-\frac{a_{k}}{\left(2k+1\right)\left(1+k\right)}, b_{1+k}=-\frac{b_{k}}{\left(2k+2\right)\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 - \frac{1}{113400}x^5 + \mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 - \frac{1}{1247400}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[2*x^2*y''[x]+5*x*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^5}{1247400} + \frac{x^4}{22680} - \frac{x^3}{630} + \frac{x^2}{30} - \frac{x}{3} + 1\right)}{\sqrt{x}} + \frac{c_2\left(-\frac{x^5}{113400} + \frac{x^4}{2520} - \frac{x^3}{90} + \frac{x^2}{6} - x + 1\right)}{x}$$

14.24 problem 24

Internal problem ID [1315]
Internal file name [OUTPUT/1316_Sunday_June_05_2022_02_09_55_AM_82404606/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(4x+3)\,y''+x(5+18x)\,y'-(1-12x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+3x^2\right)y''+\left(18x^2+5x\right)y'+(12x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{5+18x}{x\,(4x+3)}\\ q(x)&=\frac{12x-1}{x^2\,(4x+3)}\end{aligned}$$

Table 576: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5+18x}{x(4x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{4}$	"regular"

$q(x) = \frac{12x-1}{x^2(4x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{3}{4}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(4x+3)\, y'' + \left(18x^2 + 5x\right) y' + (12x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(4x+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(18x^2 + 5x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (12x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 18x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 18a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{1+n+r} a_n = \sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 18a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 5x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) \\ &+ 18a_{n-1}(n+r-1) + 5a_n(n+r) + 12a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2(2n+2r+1)\,a_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{2(6n+5)\,a_{n-1}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4r-6}{2+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{22}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r-6}{2+3r}$	$-\frac{22}{9}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^2+64r+60}{9r^2+21r+10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{374}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r-6}{2+3r}$	$-\frac{22}{9}$
a_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	$\frac{374}{81}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-64r^3 - 480r^2 - 1136r - 840}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{17204}{2187}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r-6}{2+3r}$	$-\frac{22}{9}$
a_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	$\frac{374}{81}$
a_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	$-\frac{17204}{2187}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^4 + 3072r^3 + 13184r^2 + 23808r + 15120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{249458}{19683}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r-6}{2+3r}$	$-\frac{22}{9}$
a_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	$\frac{374}{81}$
a_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	$-\frac{17204}{2187}$
a_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{249458}{19683}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1024r^5 - 17920r^4 - 120320r^3 - 385280r^2 - 584256r - 332640}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{3492412}{177147}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r-6}{2+3r}$	$-\frac{22}{9}$
a_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	$\frac{374}{81}$
a_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	$-\frac{17204}{2187}$
a_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{249458}{19683}$
a_5	$\frac{-1024r^5-17920r^4-120320r^3-385280r^2-584256r-332640}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{3492412}{177147}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{22x}{9} + \frac{374x^2}{81} - \frac{17204x^3}{2187} + \frac{249458x^4}{19683} - \frac{3492412x^5}{177147} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ &+ 18b_{n-1}(n+r-1) + 5b_n(n+r) + 12b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2(2n+2r+1)\, b_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{(-4n+2)\, b_{n-1}}{3n-4} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4r - 6}{2 + 3r}$$

Which for the root $r = -1$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r-6}{2+3r}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16r^2 + 64r + 60}{9r^2 + 21r + 10}$$

Which for the root $r = -1$ becomes

$$b_2 = -6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r-6}{2+3r}$	2
b_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	-6

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-64r^3 - 480r^2 - 1136r - 840}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = -1$ becomes

$$b_3 = 12$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r-6}{2+3r}$	2
b_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	-6
b_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	12

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256r^4 + 3072r^3 + 13184r^2 + 23808r + 15120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = -1$ becomes

$$b_4 = -21$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r-6}{2+3r}$	2
b_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	-6
b_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	12
b_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{81r^4+702r^3+2079r^2+2418r+880}$	-21

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-1024r^5 - 17920r^4 - 120320r^3 - 385280r^2 - 584256r - 332640}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{378}{11}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r-6}{2+3r}$	2
b_2	$\frac{16r^2+64r+60}{9r^2+21r+10}$	-6
b_3	$\frac{-64r^3-480r^2-1136r-840}{27r^3+135r^2+198r+80}$	12
b_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{81r^4+702r^3+2079r^2+2418r+880}$	-21
b_5	$\frac{-1024r^5-17920r^4-120320r^3-385280r^2-584256r-332640}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{378}{11}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + 2x - 6x^2 + 12x^3 - 21x^4 + \frac{378x^5}{11} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{22x}{9} + \frac{374x^2}{81} - \frac{17204x^3}{2187} + \frac{249458x^4}{19683} - \frac{3492412x^5}{177147} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 6x^2 + 12x^3 - 21x^4 + \frac{378x^5}{11} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{22x}{9} + \frac{374x^2}{81} - \frac{17204x^3}{2187} + \frac{249458x^4}{19683} - \frac{3492412x^5}{177147} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + 2x - 6x^2 + 12x^3 - 21x^4 + \frac{378x^5}{11} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1-\frac{22x}{9}+\frac{374x^2}{81}-\frac{17204x^3}{2187}+\frac{249458x^4}{19683}-\frac{3492412x^5}{177147}+O\left(x^6\right)\right) + \frac{c_2 \left(1+2x-6x^2+12x^3-21x^4+\frac{378x^5}{11}+O\left(x^6\right)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1-\frac{22x}{9}+\frac{374x^2}{81}-\frac{17204x^3}{2187}+\frac{249458x^4}{19683}-\frac{3492412x^5}{177147}+O\left(x^6\right)\right) + \frac{c_2 \left(1+2x-6x^2+12x^3-21x^4+\frac{378x^5}{11}+O\left(x^6\right)\right)}{x}$$

Verified OK.

14.24.1 Maple step by step solution

Let's solve

$x^2(4x+3)\, y''+(18x^2+5x)\, y'+(12x-1)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(12x-1)y}{x^2(4x+3)}-\frac{(5+18x)y'}{x(4x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(5+18x)y'}{x(4x+3)}+\frac{(12x-1)y}{x^2(4x+3)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5+18x}{x(4x+3)}, P_3(x)=\frac{12x-1}{x^2(4x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0}=\frac{5}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{3}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4x+3)\, y'' + x(5+18x)\, y' + (12x-1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+1)\,(3k+3r-1) + 2a_{k-1}(k+r+1)\,(2k+1+2r))\, x^{k+\cdots}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$3\left(\frac{2a_{k-1}(2k+1+2r)}{3} + a_k\left(k+r-\frac{1}{3}\right)\right)(k+r+1) = 0$$

- Shift index using $k- > k+1$

$$3\left(\frac{2a_k(2k+2r+3)}{3} + a_{k+1}\left(k+\frac{2}{3}+r\right)\right)(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k(2k+2r+3)}{3k+2+3r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{2a_k(2k+1)}{3k-1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{2a_k(2k+1)}{3k-1}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{2a_k\left(2k+\frac{11}{3}\right)}{3k+3}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{2a_k\left(2k+\frac{11}{3}\right)}{3k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{1+k} = -\frac{2a_k(2k+1)}{3k-1}, b_{1+k} = -\frac{2b_k\left(2k+\frac{11}{3}\right)}{3k+3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*(3+4*x)*diff(y(x),x$2)+x*(5+18*x)*diff(y(x),x)-(1-12*x)*y(x)=0,y(x),type='series'
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{22}{9}x + \frac{374}{81}x^2 - \frac{17204}{2187}x^3 + \frac{249458}{19683}x^4 - \frac{3492412}{177147}x^5 + \mathrm{O}\left(x^6\right)\right) + c_1\left(1 + 2x - 6x^2 + 12x^3 - 21x^4 + \frac{37}{11}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 80

```
AsymptoticDSolveValue[x^2*(3+4*x)*y''[x]+x*(5+18*x)*y'[x]-(1-12*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{3492412x^5}{177147} + \frac{249458x^4}{19683} - \frac{17204x^3}{2187} + \frac{374x^2}{81} - \frac{22x}{9} + 1\right) + \frac{c_2\left(\frac{378x^5}{11} - 21x^4 + 12x^3 - 6x^2 + 2x + 1\right)}{x}$$

14.25 problem 25

Internal problem ID [1316]
Internal file name [`OUTPUT/1317_Sunday_June_05_2022_02_09_58_AM_51900119/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2y'' + x(10 - x)\,y' - (2 + x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$6x^2y'' + \left(-x^2 + 10x\right)y' + (-x - 2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x - 10}{6x}$$
$$q(x) = -\frac{2 + x}{6x^2}$$

Table 578: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-10}{6x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2+x}{6x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$6x^2y'' + (-x^2 + 10x)\, y' + (-x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$6x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + (-x^2 + 10x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (-x - 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$6x^{n+r} a_n (n+r)(n+r-1) + 10x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$6x^r a_0 r(-1+r) + 10x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(6x^r r(-1+r) + 10x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2 + 4r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 + 4r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2 + 4r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$6a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 10a_n(n+r) - a_{n-1} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r)}{6n^2 + 12nr + 6r^2 + 4n + 4r - 2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-1}(3n+1)}{6n\,(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{6r^2 + 16r + 8}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{2}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{6r^2+16r+8}$	$\frac{2}{21}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1+r}{36r^3 + 192r^2 + 292r + 120}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{180}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{6r^2+16r+8}$	$\frac{2}{21}$
a_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	$\frac{1}{180}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1+r}{216r^4 + 1944r^3 + 5904r^2 + 6976r + 2560}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{1}{4212}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{6r^2+16r+8}$	$\frac{2}{21}$
a_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	$\frac{1}{180}$
a_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	$\frac{1}{4212}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1+r}{1296r^5 + 17712r^4 + 89424r^3 + 205008r^2 + 207520r + 70400}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{124416}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{6r^2+16r+8}$	$\frac{2}{21}$
a_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	$\frac{1}{180}$
a_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	$\frac{1}{4212}$
a_4	$\frac{1+r}{1296r^5+17712r^4+89424r^3+205008r^2+207520r+70400}$	$\frac{1}{124416}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1+r}{32\left(3r^2 + 32r + 84\right)\left(81r^4 + 702r^3 + 2079r^2 + 2418r + 880\right)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{1}{4432320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{6r^2+16r+8}$	$\frac{2}{21}$
a_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	$\frac{1}{180}$
a_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	$\frac{1}{4212}$
a_4	$\frac{1+r}{1296r^5+17712r^4+89424r^3+205008r^2+207520r+70400}$	$\frac{1}{124416}$
a_5	$\frac{1+r}{32(3r^2+32r+84)(81r^4+702r^3+2079r^2+2418r+880)}$	$\frac{1}{4432320}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 + \frac{2x}{21} + \frac{x^2}{180} + \frac{x^3}{4212} + \frac{x^4}{124416} + \frac{x^5}{4432320} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
6b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 10b_n(n+r) - b_{n-1} - 2b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(n+r)}{6n^2 + 12nr + 6r^2 + 4n + 4r - 2} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = \frac{b_{n-1}(n-1)}{6n^2 - 8n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1+r}{6r^2+16r+8}$$

Which for the root $r = -1$ becomes

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{6r^2+16r+8}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1+r}{36r^3+192r^2+292r+120}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{6r^2+16r+8}$	0
b_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{6r^2+16r+8}$	0
b_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	0
b_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1+r}{1296r^5 + 17712r^4 + 89424r^3 + 205008r^2 + 207520r + 70400}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{6r^2+16r+8}$	0
b_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	0
b_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	0
b_4	$\frac{1+r}{1296r^5+17712r^4+89424r^3+205008r^2+207520r+70400}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1+r}{32\,(3r^2 + 32r + 84)\,(81r^4 + 702r^3 + 2079r^2 + 2418r + 880)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+r}{6r^2+16r+8}$	0
b_2	$\frac{1+r}{36r^3+192r^2+292r+120}$	0
b_3	$\frac{1+r}{216r^4+1944r^3+5904r^2+6976r+2560}$	0
b_4	$\frac{1+r}{1296r^5+17712r^4+89424r^3+205008r^2+207520r+70400}$	0
b_5	$\frac{1+r}{32(3r^2+32r+84)(81r^4+702r^3+2079r^2+2418r+880)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{21} + \frac{x^2}{180} + \frac{x^3}{4212} + \frac{x^4}{124416} + \frac{x^5}{4432320} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{21} + \frac{x^2}{180} + \frac{x^3}{4212} + \frac{x^4}{124416} + \frac{x^5}{4432320} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{21} + \frac{x^2}{180} + \frac{x^3}{4212} + \frac{x^4}{124416} + \frac{x^5}{4432320} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{21} + \frac{x^2}{180} + \frac{x^3}{4212} + \frac{x^4}{124416} + \frac{x^5}{4432320} + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x}
$$

Verified OK.

14.25.1 Maple step by step solution

Let's solve

$6x^2y'' + (-x^2 + 10x)\, y' + (-x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+x)y}{6x^2} + \frac{(x-10)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-10)y'}{6x} - \frac{(2+x)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-10}{6x}, P_3(x) = -\frac{2+x}{6x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{3}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' - x(x - 10)\, y' + (-x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0(1+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(2a_k(k+r+1)\,(3k+3r-1) - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2(1+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6\left(k+r-\frac{1}{3}\right)(k+r+1)\, a_k - a_{k-1}(k+r) = 0$$

- Shift index using $k- > k+1$

$$6\left(k+\frac{2}{3}+r\right)(k+2+r)\, a_{k+1} - a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+1)}{2(3k+2+3r)(k+2+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{a_k\left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{1+k} = \frac{a_k k}{2(3k-1)(1+k)}, b_{1+k} = \frac{b_k \left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
Order:=6;
dsolve(6*x^2*diff(y(x),x$2)+x*(10-x)*diff(y(x),x)-(2+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1+\frac{2}{21}x+\frac{1}{180}x^2+\frac{1}{4212}x^3+\frac{1}{124416}x^4+\frac{1}{4432320}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 53

```
AsymptoticDSolveValue[6*x^2*y''[x]+x*(10-x)*y'[x]-(2+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(\frac{x^5}{4432320}+\frac{x^4}{124416}+\frac{x^3}{4212}+\frac{x^2}{180}+\frac{2x}{21}+1\right)+\frac{c_2}{x}$$

14.26 problem 28

Internal problem ID [1317]
Internal file name [`OUTPUT/1318_Sunday_June_05_2022_02_10_02_AM_83461413/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+8)\,y''+x(3x+2)\,y'+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+8x^2\right)y''+\left(3x^2+2x\right)y'+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x+2}{x\,(x+8)}$$
$$q(x)=\frac{x+1}{x^2\,(x+8)}$$

Table 580: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+2}{x(x+8)}$	
singularity	type
$x = -8$	"regular"
$x = 0$	"regular"

$q(x) = \frac{x+1}{x^2(x+8)}$	
singularity	type
$x = -8$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-8, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+8)\, y'' + \left(3x^2+2x\right) y' + (x+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+8) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^2+2x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r} a_n (n+r)(n+r-1) + 2x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$8x^r a_0 r(-1+r) + 2x^r a_0 r + a_0 x^r = 0$$

Or

$$(8x^r r(-1+r) + 2x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 - 6r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + 8a_n(n+r)(n+r-1) \\ + 3a_{n-1}(n+r-1) + 2a_n(n+r) + a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2)}{8n^2 + 16nr + 8r^2 - 6n - 6r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}(2n + 1)^2}{32n^2 + 8n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{(r + 1)^2}{8r^2 + 10r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{9}{40}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r + 1)^2 (r + 2)^2}{64r^4 + 288r^3 + 452r^2 + 288r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{5}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{9}{40}$
a_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{5}{128}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+1)^2 (r+2)^2 (r+3)^2}{512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{245}{39936}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{9}{40}$
a_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{5}{128}$
a_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{245}{39936}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}{(512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465)(8r^2 + 58r + 105)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{6615}{7241728}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{9}{40}$
a_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{5}{128}$
a_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{245}{39936}$
a_4	$\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)}$	$\frac{6615}{7241728}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}{(512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465) (8r^2 + 58r + 105) (8r^2 + 74r +$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{7623}{57933824}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{9}{40}$
a_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{5}{128}$
a_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{245}{39936}$
a_4	$\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)}$	$\frac{6615}{7241728}$
a_5	$-\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)(8r^2+74r+171)}$	$-\frac{7623}{57933824}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{9x}{40} + \frac{5x^2}{128} - \frac{245x^3}{39936} + \frac{6615x^4}{7241728} - \frac{7623x^5}{57933824} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + 8b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) + 2b_n(n+r) + b_{n-1} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2)}{8n^2 + 16nr + 8r^2 - 6n - 6r + 1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = -\frac{b_{n-1}(4n+1)^2}{128n^2 - 32n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{(r+1)^2}{8r^2 + 10r + 3}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = -\frac{25}{96}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{25}{96}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r+1)^2 (r+2)^2}{64r^4 + 288r^3 + 452r^2 + 288r + 63}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{675}{14336}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{25}{96}$
b_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{675}{14336}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r+1)^2 (r+2)^2 (r+3)^2}{512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = -\frac{38025}{5046272}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{25}{96}$
b_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{675}{14336}$
b_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{38025}{5046272}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}{(512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465)(8r^2 + 58r + 105)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{732615}{645922816}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{25}{96}$
b_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{675}{14336}$
b_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{38025}{5046272}$
b_4	$\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)}$	$\frac{732615}{645922816}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}{(512r^6 + 4992r^5 + 19232r^4 + 37128r^3 + 37460r^2 + 18486r + 3465) (8r^2 + 58r + 105) (8r^2 + 74r +}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = -\frac{9230949}{56103010304}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{8r^2+10r+3}$	$-\frac{25}{96}$
b_2	$\frac{(r+1)^2(r+2)^2}{64r^4+288r^3+452r^2+288r+63}$	$\frac{675}{14336}$
b_3	$-\frac{(r+1)^2(r+2)^2(r+3)^2}{512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465}$	$-\frac{38025}{5046272}$
b_4	$\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)}$	$\frac{732615}{645922816}$
b_5	$-\frac{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}{(512r^6+4992r^5+19232r^4+37128r^3+37460r^2+18486r+3465)(8r^2+58r+105)(8r^2+74r+171)}$	$-\frac{9230949}{56103010304}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 - \frac{25x}{96} + \frac{675x^2}{14336} - \frac{38025x^3}{5046272} + \frac{732615x^4}{645922816} - \frac{9230949x^5}{56103010304} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1-\frac{9x}{40}+\frac{5x^2}{128}-\frac{245x^3}{39936}+\frac{6615x^4}{7241728}-\frac{7623x^5}{57933824}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}}\left(1-\frac{25x}{96}+\frac{675x^2}{14336}-\frac{38025x^3}{5046272}+\frac{732615x^4}{645922816}-\frac{9230949x^5}{56103010304}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1-\frac{9x}{40}+\frac{5x^2}{128}-\frac{245x^3}{39936}+\frac{6615x^4}{7241728}-\frac{7623x^5}{57933824}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}}\left(1-\frac{25x}{96}+\frac{675x^2}{14336}-\frac{38025x^3}{5046272}+\frac{732615x^4}{645922816}-\frac{9230949x^5}{56103010304}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \sqrt{x}\left(1-\frac{9x}{40}+\frac{5x^2}{128}-\frac{245x^3}{39936}+\frac{6615x^4}{7241728}-\frac{7623x^5}{57933824}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}}\left(1-\frac{25x}{96}+\frac{675x^2}{14336}-\frac{38025x^3}{5046272}+\frac{732615x^4}{645922816}-\frac{9230949x^5}{56103010304}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \sqrt{x}\left(1-\frac{9x}{40}+\frac{5x^2}{128}-\frac{245x^3}{39936}+\frac{6615x^4}{7241728}-\frac{7623x^5}{57933824}+O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{4}}\left(1-\frac{25x}{96}+\frac{675x^2}{14336}-\frac{38025x^3}{5046272}+\frac{732615x^4}{645922816}-\frac{9230949x^5}{56103010304}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

14.26.1 Maple step by step solution

Let's solve

$x^2(x+8)\,y''+(3x^2+2x)\,y'+(x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x+1)y}{x^2(x+8)}-\frac{(3x+2)y'}{x(x+8)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x+2)y'}{x(x+8)}+\frac{(x+1)y}{x^2(x+8)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x+2}{x(x+8)},P_3(x)=\frac{x+1}{x^2(x+8)}\right]$

  - $(x+8)\cdot P_2(x)$ is analytic at $x=-8$

    $\left((x+8)\cdot P_2(x)\right)\Big|_{x=-8}=\frac{11}{4}$

  - $(x+8)^2\cdot P_3(x)$ is analytic at $x=-8$

    $\left((x+8)^2\cdot P_3(x)\right)\Big|_{x=-8}=0$

  - $x=-8$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-8$

- Multiply by denominators

  $x^2(x+8)\,y''+x(3x+2)\,y'+(x+1)\,y=0$

- Change variables using $x=u-8$ so that the regular singular point is at $u=0$

  $(u^3-16u^2+64u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-46u+176)\left(\frac{d}{du}y(u)\right)+(u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$16a_0r(7+4r)\, u^{-1+r} + (16a_1(1+r)\,(11+4r) - a_0(16r^2+30r+7))\, u^r + \left(\sum_{k=1}^{\infty} (16a_{k+1}(k+r+1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$16r(7+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{7}{4}\right\}$$

- Each term must be 0

  $$16a_1(1+r)\,(11+4r) - a_0(16r^2+30r+7) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$64\left(k+\frac{11}{4}+r\right)(k+r+1)\, a_{k+1} - a_k(16k^2+32kr+16r^2+30k+30r+7) + a_{k-1}(k+r)^2 = 0$$

- Shift index using $k- >k + 1$

  $$64\left(k+\frac{15}{4}+r\right)(k+2+r)\, a_{k+2} - a_{k+1}\left(16(k+1)^2+32(k+1)\, r+16r^2+30k+37+30r\right) + a_k($$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}+2 k r a_k -32 k r a_{k+1}+r^2 a_k -16 r^2 a_{k+1}+2 k a_k -62 k a_{k+1}+2 r a_k -62 r a_{k+1}+a_k -53 a_{k+1}}{16 (4 k +15+4 r) (k +2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}+2 k a_k -62 k a_{k+1}+a_k -53 a_{k+1}}{16 (4 k +15) (k +2)}$$

- Solution for $r = 0$

$$\left[y \left(u \right) = \sum_{k =0}^{\infty} a_k u^{k}, a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}+2 k a_k -62 k a_{k+1}+a_k -53 a_{k+1}}{16 (4 k +15) (k +2)}, 176 a_1 -7 a_0 = 0\right]$$

- Revert the change of variables $u = x +8$

$$\left[y = \sum_{k =0}^{\infty} a_k \left(x +8\right)^{k}, a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}+2 k a_k -62 k a_{k+1}+a_k -53 a_{k+1}}{16 (4 k +15) (k +2)}, 176 a_1 -7 a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{4}$

$$a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}-\frac{3}{2} k a_k -6 k a_{k+1}+\frac{9}{16} a_k +\frac{13}{2} a_{k+1}}{16 (4 k +8) \left(k +\frac{1}{4}\right)}$$

- Solution for $r = -\frac{7}{4}$

$$\left[y \left(u \right) = \sum_{k =0}^{\infty} a_k u^{k -\frac{7}{4}}, a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}-\frac{3}{2} k a_k -6 k a_{k+1}+\frac{9}{16} a_k +\frac{13}{2} a_{k+1}}{16 (4 k +8) \left(k +\frac{1}{4}\right)}, -48 a_1 -\frac{7 a_0}{2} = 0\right]$$

- Revert the change of variables $u = x +8$

$$\left[y = \sum_{k =0}^{\infty} a_k \left(x +8\right)^{k -\frac{7}{4}}, a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{k+1}-\frac{3}{2} k a_k -6 k a_{k+1}+\frac{9}{16} a_k +\frac{13}{2} a_{k+1}}{16 (4 k +8) \left(k +\frac{1}{4}\right)}, -48 a_1 -\frac{7 a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k =0}^{\infty} a_k \left(x +8\right)^{k}\right)+\left(\sum_{k =0}^{\infty} b_k \left(x +8\right)^{k -\frac{7}{4}}\right), a_{k+2} = -\frac{k^2 a_k -16 k^2 a_{1+k}+2 k a_k -62 k a_{1+k}+a_k -53 a_{1+k}}{16 (4 k +15) (k +2)}, 176\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*(8+x)*diff(y(x),x$2)+x*(2+3*x)*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 - \frac{25}{96}x + \frac{675}{14336}x^2 - \frac{38025}{5046272}x^3 + \frac{732615}{645922816}x^4 - \frac{9230949}{56103010304}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{9}{40}x + \frac{5}{128}x^2 - \frac{245}{39936}x^3 + \frac{6615}{7241728}x^4 - \frac{7623}{57933824}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*(8+x)*y''[x]+x*(2+3*x)*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{7623x^5}{57933824}+\frac{6615x^4}{7241728}-\frac{245x^3}{39936}+\frac{5x^2}{128}-\frac{9x}{40}+1\right)+c_2\sqrt[4]{x}\left(-\frac{9230949x^5}{56103010304}+\frac{732615x^4}{645922816}-\frac{38025x^3}{5046272}+\frac{675x^2}{14336}-\frac{25x}{96}+1\right)$$

14.27 problem 29

Internal problem ID [1318]
Internal file name [`OUTPUT/1319_Sunday_June_05_2022_02_10_05_AM_13927065/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(4x+3)\, y''+x(11+4x)\, y'-(4x+3)\, y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+3x^2\right) y''+\left(4x^2+11x\right) y'+(-4x-3)\, y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}
p(x) &= \frac{11+4x}{x\,(4x+3)}\\
q(x) &= -\frac{1}{x^2}
\end{aligned}$$

Table 582: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11+4x}{x(4x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{4}$	"regular"

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{3}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(4x+3)\, y'' + \left(4x^2 + 11x\right) y' + (-4x-3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(4x+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(4x^2+11x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-4x-3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right)\\&+\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 11x^{n+r} a_n(n+r)\right)\\&+\sum_{n=0}^{\infty}\left(-4x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-3a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-4x^{1+n+r} a_n\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)\\&+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}\right)\\&+\left(\sum_{n=0}^{\infty} 11x^{n+r} a_n(n+r)\right)+\sum_{n=1}^{\infty}\left(-4a_{n-1} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-3a_n x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n(n+r)(n+r-1)+11x^{n+r} a_n(n+r)-3a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 11x^r a_0 r - 3a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 11x^r r - 3x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 8r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 8r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 8r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{10}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-3}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) \\ &+ 4a_{n-1}(n+r-1) + 11a_n(n+r) - 4a_{n-1} - 3a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r)}{3n^2 + 6nr + 3r^2 + 8n + 8r - 3} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{4a_{n-1}(9n^2 - 12n - 5)}{27n^2 + 90n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4r^2 + 4}{3r^2 + 14r + 8}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{32}{117}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{117}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16(r^2 - 1) r(r + 2)}{(3r^2 + 14r + 8)(3r^2 + 20r + 25)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{28}{1053}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{117}$
a_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$-\frac{28}{1053}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64(r^2-1)\,r(r+2)\,(r^2+4r+3)}{(3r^2+14r+8)\,(3r^2+20r+25)\,(3r^2+26r+48)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{4480}{540189}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{117}$
a_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$-\frac{28}{1053}$
a_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	$\frac{4480}{540189}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256(r+2)^2\,(-1+r)\,(r+1)^2\,r(r+3)}{(3r^2+32r+77)\,(3r^2+26r+48)\,(3r^2+20r+25)\,(3r+2)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = -\frac{15680}{4113747}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{117}$
a_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$-\frac{28}{1053}$
a_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	$\frac{4480}{540189}$
a_4	$\frac{256(r+2)^2(-1+r)(r+1)^2r(r+3)}{(3r^2+32r+77)(3r^2+26r+48)(3r^2+20r+25)(3r+2)}$	$-\frac{15680}{4113747}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024(r+3)^2 (r+2)^2 (-1+r) (r+1)^2 r}{(3r^2+38r+112)(3r+2)(3r+5)(3r^2+26r+48)(3r^2+32r+77)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{401408}{185118615}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{117}$
a_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$-\frac{28}{1053}$
a_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	$\frac{4480}{540189}$
a_4	$\frac{256(r+2)^2(-1+r)(r+1)^2r(r+3)}{(3r^2+32r+77)(3r^2+26r+48)(3r^2+20r+25)(3r+2)}$	$-\frac{15680}{4113747}$
a_5	$-\frac{1024(r+3)^2(r+2)^2(-1+r)(r+1)^2r}{(3r^2+38r+112)(3r+2)(3r+5)(3r^2+26r+48)(3r^2+32r+77)}$	$\frac{401408}{185118615}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{32x}{117} - \frac{28x^2}{1053} + \frac{4480x^3}{540189} - \frac{15680x^4}{4113747} + \frac{401408x^5}{185118615} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) + 4b_{n-1}(n+r-1) + 11b_n(n+r) - 4b_{n-1} - 3b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4b_{n-1}(n^2+2nr+r^2-2n-2r)}{3n^2+6nr+3r^2+8n+8r-3} \tag{4}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{4b_{n-1}(n^2-8n+15)}{n(3n-10)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-4r^2+4}{3r^2+14r+8}$$

Which for the root $r = -3$ becomes

$$b_1 = \frac{32}{7}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{7}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16(r^2-1)\,r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$$

Which for the root $r = -3$ becomes

$$b_2 = \frac{48}{7}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{7}$
b_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$\frac{48}{7}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64(r^2-1)\,r(r+2)\,(r^2+4r+3)}{(3r^2+14r+8)\,(3r^2+20r+25)\,(3r^2+26r+48)}$$

Which for the root $r = -3$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{7}$
b_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$\frac{48}{7}$
b_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256(r+2)^2\,(-1+r)\,(r+1)^2\,r(r+3)}{(3r^2+32r+77)\,(3r^2+26r+48)\,(3r^2+20r+25)\,(3r+2)}$$

Which for the root $r = -3$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{7}$
b_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$\frac{48}{7}$
b_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	0
b_4	$\frac{256(r+2)^2(-1+r)(r+1)^2r(r+3)}{(3r^2+32r+77)(3r^2+26r+48)(3r^2+20r+25)(3r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024(r+3)^2(r+2)^2(-1+r)(r+1)^2 r}{(3r^2+38r+112)(3r+2)(3r+5)(3r^2+26r+48)(3r^2+32r+77)}$$

Which for the root $r = -3$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-4r^2+4}{3r^2+14r+8}$	$\frac{32}{7}$
b_2	$\frac{16(r^2-1)r(r+2)}{(3r^2+14r+8)(3r^2+20r+25)}$	$\frac{48}{7}$
b_3	$-\frac{64(r^2-1)r(r+2)(r^2+4r+3)}{(3r^2+14r+8)(3r^2+20r+25)(3r^2+26r+48)}$	0
b_4	$\frac{256(r+2)^2(-1+r)(r+1)^2r(r+3)}{(3r^2+32r+77)(3r^2+26r+48)(3r^2+20r+25)(3r+2)}$	0
b_5	$-\frac{1024(r+3)^2(r+2)^2(-1+r)(r+1)^2r}{(3r^2+38r+112)(3r+2)(3r+5)(3r^2+26r+48)(3r^2+32r+77)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{32x}{7} + \frac{48x^2}{7} + O(x^6)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{32x}{117} - \frac{28x^2}{1053} + \frac{4480x^3}{540189} - \frac{15680x^4}{4113747} + \frac{401408x^5}{185118615} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{32x}{7} + \frac{48x^2}{7} + O(x^6)\right)}{x^3}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{32x}{117} - \frac{28x^2}{1053} + \frac{4480x^3}{540189} - \frac{15680x^4}{4113747} + \frac{401408x^5}{185118615} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{32x}{7} + \frac{48x^2}{7} + O(x^6)\right)}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 + \frac{32x}{117} - \frac{28x^2}{1053} + \frac{4480x^3}{540189} - \frac{15680x^4}{4113747} + \frac{401408x^5}{185118615} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + \frac{32x}{7} + \frac{48x^2}{7} + O(x^6)\right)}{x^3}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{1}{3}} &\left(1 + \frac{32x}{117} - \frac{28x^2}{1053} + \frac{4480x^3}{540189} - \frac{15680x^4}{4113747} + \frac{401408x^5}{185118615} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + \frac{32x}{7} + \frac{48x^2}{7} + O(x^6)\right)}{x^3}
\end{aligned}
$$

Verified OK.

14.27.1 Maple step by step solution

Let's solve

$x^2(4x+3)\,y''+(4x^2+11x)\,y'+(-4x-3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x^2}-\frac{(11+4x)y'}{x(4x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(11+4x)y'}{x(4x+3)}-\frac{y}{x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{11+4x}{x(4x+3)},P_3(x)=-\frac{1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{11}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(4x+3)\,y''+x(11+4x)\,y'+(-4x-3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(3+r)\,(-1+3r)\, x^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k}(k+r+3)\,(3k+3r-1)+4a_{k-1}(k+r)\,(k-2+r)\right) x^{k+r}\right)=$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-3, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+3)\left(k+r-\frac{1}{3}\right) a_{k}+4a_{k-1}(k+r)\,(k-2+r)=0$

- Shift index using $k->k+1$

  $3(k+4+r)\left(k+\frac{2}{3}+r\right) a_{k+1}+4a_{k}(k+r+1)\,(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{4a_{k}(k+r+1)(k+r-1)}{(k+4+r)(3k+2+3r)}$

- Recursion relation for $r=-3$; series terminates at $k=2$

  $a_{k+1}=-\frac{4a_{k}(k-2)(k-4)}{(k+1)(3k-7)}$

- Apply recursion relation for $k=0$

$a_1 = \frac{32a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 + \frac{32}{7}x + \frac{48}{7}x^2\right)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{32}{7}x + \frac{48}{7}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{1+k} = -\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
Order:=6;
dsolve(x^2*(3+4*x)*diff(y(x),x$2)+x*(11+4*x)*diff(y(x),x)-(3+4*x)*y(x)=0,y(x),type='series',
```

$$y(x) = \frac{c_1\left(1+\frac{32}{7}x+\frac{48}{7}x^2+\mathrm{O}\left(x^6\right)\right)}{x^3} + c_2 x^{\frac{1}{3}}\left(1+\frac{32}{117}x-\frac{28}{1053}x^2+\frac{4480}{540189}x^3-\frac{15680}{4113747}x^4+\frac{401408}{185118615}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 67

```
AsymptoticDSolveValue[x^2*(3+4*x)*y''[x]+x*(11+4*x)*y'[x]-(3+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2\left(\frac{48x^2}{7}+\frac{32x}{7}+1\right)}{x^3}+c_1\sqrt[3]{x}\left(\frac{401408x^5}{185118615}-\frac{15680x^4}{4113747}+\frac{4480x^3}{540189}-\frac{28x^2}{1053}+\frac{32x}{117}+1\right)$$

14.28 problem 30

Internal problem ID [1319]
Internal file name [`OUTPUT/1320_Sunday_June_05_2022_02_10_08_AM_27791504/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(3x+2)\,y''+x(4+11x)\,y'-(1-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+y(x-1)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{4+11x}{2x\,(3x+2)}\\ q(x)&=\frac{x-1}{2x^2\,(3x+2)}\end{aligned}$$

Table 584: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4+11x}{2x(3x+2)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

$q(x) = \frac{x-1}{2x^2(3x+2)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{2}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(3x+2)\, y'' + \left(11x^2+4x\right) y' + y(x-1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(3x+2) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(11x^2+4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)(x-1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 11x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 11x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 4x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$6a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) + 11a_{n-1}(n+r-1) + 4a_n(n+r) + a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(3n+3r-2)\,a_{n-1}}{2n+2r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{-6na_{n-1} + a_{n-1}}{4+4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-3r}{3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{5}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-3r}{3+2r}$	$-\frac{5}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r^2+15r+4}{4r^2+16r+15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{55}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-3r}{3+2r}$	$-\frac{5}{8}$
a_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$\frac{55}{96}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^3 - 108r^2 - 117r - 28}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{935}{1536}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-3r}{3+2r}$	$-\frac{5}{8}$
a_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$\frac{55}{96}$
a_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$-\frac{935}{1536}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^4 + 594r^3 + 1431r^2 + 1254r + 280}{(4r^2 + 16r + 15)(7 + 2r)(9 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{4301}{6144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-3r}{3+2r}$	$-\frac{5}{8}$
a_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$\frac{55}{96}$
a_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$-\frac{935}{1536}$
a_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{(4r^2+16r+15)(7+2r)(9+2r)}$	$\frac{4301}{6144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-243r^5 - 2835r^4 - 12015r^3 - 22365r^2 - 17142r - 3640}{(4r^2 + 16r + 15)(7 + 2r)(11 + 2r)(9 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{124729}{147456}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-3r}{3+2r}$	$-\frac{5}{8}$
a_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$\frac{55}{96}$
a_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$-\frac{935}{1536}$
a_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{(4r^2+16r+15)(7+2r)(9+2r)}$	$\frac{4301}{6144}$
a_5	$\frac{-243r^5-2835r^4-12015r^3-22365r^2-17142r-3640}{(4r^2+16r+15)(7+2r)(11+2r)(9+2r)}$	$-\frac{124729}{147456}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 - \frac{5x}{8} + \frac{55x^2}{96} - \frac{935x^3}{1536} + \frac{4301x^4}{6144} - \frac{124729x^5}{147456} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1 - 3r}{3 + 2r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-1 - 3r}{3 + 2r} &= \lim_{r \to -\frac{1}{2}} \frac{-1 - 3r}{3 + 2r} \\ &= \frac{1}{4} \end{aligned}$$

The limit is $\frac{1}{4}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &6b_{n-1}(n + r - 1)(n + r - 2) + 4b_n(n + r)(n + r - 1) \\ &+ 11b_{n-1}(n + r - 1) + 4b_n(n + r) + b_{n-1} - b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$\begin{aligned} &6b_{n-1}\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right) + 4b_n\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \\ &+ 11b_{n-1}\left(n - \frac{3}{2}\right) + 4b_n\left(n - \frac{1}{2}\right) + b_{n-1} - b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(3n + 3r - 2)\, b_{n-1}}{2n + 2r + 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{\left(3n - \frac{7}{2}\right) b_{n-1}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1 + 3r}{3 + 2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-3r}{3+2r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r^2 + 15r + 4}{(3 + 2r)(5 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{5}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-3r}{3+2r}$	$\frac{1}{4}$
b_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$-\frac{5}{32}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27r^3 + 108r^2 + 117r + 28}{(3 + 2r)(7 + 2r)(5 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{55}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-3r}{3+2r}$	$\frac{1}{4}$
b_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$-\frac{5}{32}$
b_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$\frac{55}{384}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^4 + 594r^3 + 1431r^2 + 1254r + 280}{(3 + 2r)(5 + 2r)(7 + 2r)(9 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = -\frac{935}{6144}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-3r}{3+2r}$	$\frac{1}{4}$
b_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$-\frac{5}{32}$
b_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$\frac{55}{384}$
b_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{16r^4+192r^3+824r^2+1488r+945}$	$-\frac{935}{6144}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243r^5 + 2835r^4 + 12015r^3 + 22365r^2 + 17142r + 3640}{(3 + 2r)(7 + 2r)(5 + 2r)(11 + 2r)(9 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{4301}{24576}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1-3r}{3+2r}$	$\frac{1}{4}$
b_2	$\frac{9r^2+15r+4}{4r^2+16r+15}$	$-\frac{5}{32}$
b_3	$\frac{-27r^3-108r^2-117r-28}{8r^3+60r^2+142r+105}$	$\frac{55}{384}$
b_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{16r^4+192r^3+824r^2+1488r+945}$	$-\frac{935}{6144}$
b_5	$\frac{-243r^5-2835r^4-12015r^3-22365r^2-17142r-3640}{(3+2r)(7+2r)(5+2r)(11+2r)(9+2r)}$	$\frac{4301}{24576}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x}{4} - \frac{5x^2}{32} + \frac{55x^3}{384} - \frac{935x^4}{6144} + \frac{4301x^5}{24576} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{5x}{8} + \frac{55x^2}{96} - \frac{935x^3}{1536} + \frac{4301x^4}{6144} - \frac{124729x^5}{147456} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{4} - \frac{5x^2}{32} + \frac{55x^3}{384} - \frac{935x^4}{6144} + \frac{4301x^5}{24576} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - \frac{5x}{8} + \frac{55x^2}{96} - \frac{935x^3}{1536} + \frac{4301x^4}{6144} - \frac{124729x^5}{147456} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{4} - \frac{5x^2}{32} + \frac{55x^3}{384} - \frac{935x^4}{6144} + \frac{4301x^5}{24576} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1-\frac{5x}{8}+\frac{55x^2}{96}-\frac{935x^3}{1536}+\frac{4301x^4}{6144}-\frac{124729x^5}{147456}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{4}-\frac{5x^2}{32}+\frac{55x^3}{384}-\frac{935x^4}{6144}+\frac{4301x^5}{24576}+O(x^6)\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1-\frac{5x}{8}+\frac{55x^2}{96}-\frac{935x^3}{1536}+\frac{4301x^4}{6144}-\frac{124729x^5}{147456}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{4}-\frac{5x^2}{32}+\frac{55x^3}{384}-\frac{935x^4}{6144}+\frac{4301x^5}{24576}+O(x^6)\right)}{\sqrt{x}}
$$

Verified OK.

14.28.1 Maple step by step solution

Let's solve

$2x^2(3x+2)\,y''+(11x^2+4x)\,y'+y(x-1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y(x-1)}{2x^2(3x+2)}-\frac{(4+11x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4+11x)y'}{2x(3x+2)}+\frac{y(x-1)}{2x^2(3x+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{4+11x}{2x(3x+2)}, P_3(x)=\frac{x-1}{2x^2(3x+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$2x^2(3x+2)\,y'' + x(4+11x)\,y' + y(x-1) = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\,(3k-2+3r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{3r}{2}-1\right)a_{k-1}+a_k\left(k+r+\frac{1}{2}\right)\right)=0$
- Shift index using $k->k+1$

  $4\left(k+r+\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{1}{2}+\frac{3r}{2}\right)a_k+a_{k+1}\left(k+\frac{3}{2}+r\right)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(3k+3r+1)a_k}{2k+3+2r}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{1+k}=-\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x),type='series',
```

$$y(x) = \frac{c_1 x\left(1-\frac{5}{8}x+\frac{55}{96}x^2-\frac{935}{1536}x^3+\frac{4301}{6144}x^4-\frac{124729}{147456}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-\frac{5}{4}x+\frac{25}{32}x^2-\frac{275}{384}x^3+\frac{4675}{6144}x^4-\frac{21505}{24576}\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 94

```
AsymptoticDSolveValue[2*x^2*(2+3*x)*y''[x]+x*(4+11*x)*y'[x]-(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{935x^{7/2}}{6144}+\frac{55x^{5/2}}{384}-\frac{5x^{3/2}}{32}+\frac{\sqrt{x}}{4}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{4301x^{9/2}}{6144}-\frac{935x^{7/2}}{1536}+\frac{55x^{5/2}}{96}-\frac{5x^{3/2}}{8}+\sqrt{x}\right)$$

14.29 problem 31

Internal problem ID [1320]
Internal file name [OUTPUT/1321_Sunday_June_05_2022_02_10_11_AM_33254565/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2+x)\,y'' + 5x(1-x)\,y' - (-8x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3 + 2x^2\right) y'' + \left(-5x^2 + 5x\right) y' + (8x - 2)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5(x-1)}{x\,(2+x)}$$
$$q(x) = \frac{8x-2}{x^2\,(2+x)}$$

Table 586: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5(x-1)}{x(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = \frac{8x-2}{x^2(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(2+x)\,y'' + \left(-5x^2+5x\right)y' + (8x-2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(2+x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(-5x^2+5x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (8x-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 8x^{1+n+r} a_n = \sum_{n=1}^{\infty} 8a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r - 2a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 5x^r r - 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 + 3r - 2)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 + 3r - 2)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ -5a_{n-1}(n+r-1) + 5a_n(n+r) + 8a_{n-1} - 2a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 8n - 8r + 15)}{2n^2 + 4nr + 2r^2 + 3n + 3r - 2} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}(4n^2 - 28n + 45)}{8n^2 + 20n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 6r - 8}{2r^2 + 7r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 - 10r^3 + 35r^2 - 50r + 24}{(2r^2 + 7r + 3)(2r^2 + 11r + 12)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{5}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	$-\frac{3}{4}$
a_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	$\frac{5}{96}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{5}{4224}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	$-\frac{3}{4}$
a_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	$\frac{5}{96}$
a_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$\frac{5}{4224}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1+r)^2(r-2)^2(r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{5}{292864}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	$-\frac{3}{4}$
a_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	$\frac{5}{96}$
a_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$\frac{5}{4224}$
a_4	$\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$	$\frac{5}{292864}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-1+r)^2 (r-2)^2 (r-3)(r-4) r^2 (r+1)(r+2)}{(16r^8 + 416r^7 + 4552r^6 + 27248r^5 + 96913r^4 + 207506r^3 + 256719r^2 + 162630r + 37800)(2r^2 + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{3514368}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	$-\frac{3}{4}$
a_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	$\frac{5}{96}$
a_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$\frac{5}{4224}$
a_4	$\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$	$\frac{5}{292864}$
a_5	$-\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r^2(r+1)(r+2)}{(16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800)(2r^2+23r+63)}$	$-\frac{1}{3514368}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{3x}{4} + \frac{5x^2}{96} + \frac{5x^3}{4224} + \frac{5x^4}{292864} - \frac{x^5}{3514368} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)\,(n+r-2) + 2b_n(n+r)\,(n+r-1) \\ -5b_{n-1}(n+r-1) + 5b_n(n+r) + 8b_{n-1} - 2b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 8n - 8r + 15)}{2n^2 + 4nr + 2r^2 + 3n + 3r - 2} \tag{4}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-1}(n^2 - 12n + 35)}{n\,(2n - 5)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-r^2 + 6r - 8}{2r^2 + 7r + 3}$$

Which for the root $r = -2$ becomes

$$b_1 = 8$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	8

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^4 - 10r^3 + 35r^2 - 50r + 24}{(2r^2 + 7r + 3)\,(2r^2 + 11r + 12)}$$

Which for the root $r = -2$ becomes

$$b_2 = 60$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	8
b_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	60

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(-1+r)(r-2)^2 (r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$$

Which for the root $r = -2$ becomes

$$b_3 = -160$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	8
b_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	60
b_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	-160

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(-1+r)^2 (r-2)^2 (r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$$

Which for the root $r = -2$ becomes

$$b_4 = 40$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	8
b_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	60
b_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	-160
b_4	$\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$	40

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(-1+r)^2 (r-2)^2 (r-3) (r-4) r^2(r+1) (r+2)}{(16r^8 + 416r^7 + 4552r^6 + 27248r^5 + 96913r^4 + 207506r^3 + 256719r^2 + 162630r + 37800) (2r^2 + 2}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+6r-8}{2r^2+7r+3}$	8
b_2	$\frac{r^4-10r^3+35r^2-50r+24}{(2r^2+7r+3)(2r^2+11r+12)}$	60
b_3	$-\frac{(-1+r)(r-2)^2(r-3)(r-4)r}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	-160
b_4	$\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r(r+1)}{16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800}$	40
b_5	$-\frac{(-1+r)^2(r-2)^2(r-3)(r-4)r^2(r+1)(r+2)}{(16r^8+416r^7+4552r^6+27248r^5+96913r^4+207506r^3+256719r^2+162630r+37800)(2r^2+23r+63)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\big(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\big) \\ &= \frac{1 + 8x + 60x^2 - 160x^3 + 40x^4 + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1-\frac{3x}{4}+\frac{5x^2}{96}+\frac{5x^3}{4224}+\frac{5x^4}{292864}-\frac{x^5}{3514368}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2 (1+8x+60x^2-160x^3+40x^4+O(x^6))}{x^2}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1-\frac{3x}{4}+\frac{5x^2}{96}+\frac{5x^3}{4224}+\frac{5x^4}{292864}-\frac{x^5}{3514368}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2 (1+8x+60x^2-160x^3+40x^4+O(x^6))}{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 \sqrt{x} \left(1-\frac{3x}{4}+\frac{5x^2}{96}+\frac{5x^3}{4224}+\frac{5x^4}{292864}-\frac{x^5}{3514368}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2 (1+8x+60x^2-160x^3+40x^4+O(x^6))}{x^2}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 \sqrt{x} \left(1-\frac{3x}{4}+\frac{5x^2}{96}+\frac{5x^3}{4224}+\frac{5x^4}{292864}-\frac{x^5}{3514368}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2 (1+8x+60x^2-160x^3+40x^4+O(x^6))}{x^2}
\end{aligned}$$

Verified OK.

14.29.1 Maple step by step solution

Let's solve

$x^2 (2+x) y'' + (-5x^2+5x) y' + (8x-2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2(4x-1)y}{x^2(2+x)} + \frac{5(x-1)y'}{x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{5(x-1)y'}{x(2+x)} + \frac{2(4x-1)y}{x^2(2+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = -\frac{5(x-1)}{x(2+x)}, P_3(x) = \frac{2(4x-1)}{x^2(2+x)}\right]$

- ○ $(2+x) \cdot P_2(x)$ is analytic at $x = -2$

$\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = -\frac{15}{2}$

- ○ $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$

$\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

- ○ $x = -2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -2$

- Multiply by denominators

$x^2(2+x)\, y'' - 5x(x-1)\, y' + (8x-2)\, y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$(u^3 - 4u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-5u^2 + 25u - 30)\left(\frac{d}{du}y(u)\right) + (8u - 18)\, y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- > k - m$

$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}r(-17+2r)\, u^{-1+r}+\left(2a_{1}(1+r)\,(-15+2r)-a_{0}(4r^{2}-29r+18)\right)u^{r}+\left(\sum_{k=1}^{\infty}(2a_{k+1}(k+1+$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$2r(-17+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{17}{2}\right\}$$

- Each term must be 0

$$2a_{1}(1+r)\,(-15+2r)-a_{0}(4r^{2}-29r+18)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_{k}+a_{k-1}+4a_{k+1})\, k^{2}+\left((-8a_{k}+2a_{k-1}+8a_{k+1})\, r+29a_{k}-8a_{k-1}-26a_{k+1}\right)k+(-4a_{k}+a$$

- Shift index using $k->k+1$

$$(-4a_{k+1}+a_{k}+4a_{k+2})\,(k+1)^{2}+\left((-8a_{k+1}+2a_{k}+8a_{k+2})\, r+29a_{k+1}-8a_{k}-26a_{k+2}\right)(k+1)\,-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{k^{2}a_{k}-4k^{2}a_{k+1}+2kra_{k}-8kra_{k+1}+r^{2}a_{k}-4r^{2}a_{k+1}-6ka_{k}+21ka_{k+1}-6ra_{k}+21ra_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}+4kr+2r^{2}-9k-9r-26)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{k^{2}a_{k}-4k^{2}a_{k+1}-6ka_{k}+21ka_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}-9k-26)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_{k}u^{k}, a_{k+2}=-\frac{k^{2}a_{k}-4k^{2}a_{k+1}-6ka_{k}+21ka_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}-9k-26)}, -30a_{1}-18a_{0}=0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k , a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$$

- Recursion relation for $r = \frac{17}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}$$

- Solution for $r = \frac{17}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{17}{2}} , a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{17}{2}}\right), a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{1+k} - 6k a_k + 21k a_{1+k} + 8a_k + 7a_{1+k}}{2(2k^2 - 9k - 26)}, -30\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(2+x)*diff(y(x),x$2)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_2 x^{\frac{5}{2}}\left(1-\frac{3}{4}x+\frac{5}{96}x^2+\frac{5}{4224}x^3+\frac{5}{292864}x^4-\frac{1}{3514368}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+8x+60x^2-160x^3+40x^4+\mathrm{O}\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 73

```
AsymptoticDSolveValue[x^2*(2+x)*y''[x]+5*x*(1-x)*y'[x]-(2-8*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{x^2} + c_1\sqrt{x}\left(-\frac{x^5}{3514368} + \frac{5x^4}{292864} + \frac{5x^3}{4224} + \frac{5x^2}{96} - \frac{3x}{4} + 1\right)$$

14.30 problem 32

Internal problem ID [1321]
Internal file name [`OUTPUT/1322_Sunday_June_05_2022_02_10_15_AM_64338585/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+6)\,y''+x(11+4x)\,y'+(1+2x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+6x^2\right)y''+\left(4x^2+11x\right)y'+(1+2x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{11+4x}{x\,(x+6)}$$
$$q(x)=\frac{1+2x}{x^2\,(x+6)}$$

Table 588: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11+4x}{x(x+6)}$	
singularity	type
$x = -6$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1+2x}{x^2(x+6)}$	
singularity	type
$x = -6$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-6, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+6)\, y'' + \left(4x^2 + 11x\right) y' + (1+2x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+6) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(4x^2 + 11x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (1+2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 11x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 11x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
6x^{n+r} a_n (n+r)(n+r-1) + 11x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$6x^r a_0 r(-1+r) + 11x^r a_0 r + a_0 x^r = 0$$

Or

$$(6x^r r(-1+r) + 11x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(6r^2 + 5r + 1)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 + 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -\frac{1}{3} \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(6r^2 + 5r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)\,(n+r-2) + 6a_n(n+r)\,(n+r-1) \\ &+ 4a_{n-1}(n+r-1) + 11a_n(n+r) + a_n + 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 + n + r)}{6n^2 + 12nr + 6r^2 + 5n + 5r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_n = -\frac{a_{n-1}(9n^2 + 3n - 2)}{54n^2 + 9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 - 3r - 2}{6r^2 + 17r + 12}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_1 = -\frac{10}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{10}{63}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+3)(r+1)(r+2)^2}{36r^4 + 276r^3 + 775r^2 + 943r + 420}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_2 = \frac{200}{7371}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{10}{63}$
a_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{200}{7371}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+3)^2 (r+1) (r+2)^2 (r+4)}{216r^6 + 3132r^5 + 18486r^4 + 56753r^3 + 95433r^2 + 83230r + 29400}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_3 = -\frac{17600}{3781323}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{10}{63}$
a_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{200}{7371}$
a_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{17600}{3781323}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+3)^2 (r+1) (r+2)^2 (r+4)^2 (r+5)}{(216r^6 + 3132r^5 + 18486r^4 + 56753r^3 + 95433r^2 + 83230r + 29400) (6r^2 + 53r + 117)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_4 = \frac{3872}{4861701}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{10}{63}$
a_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{200}{7371}$
a_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{17600}{3781323}$
a_4	$\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)}$	$\frac{3872}{4861701}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)^2(6+r)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)(6r^2+65r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_5 = -\frac{921536}{6782072895}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{10}{63}$
a_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{200}{7371}$
a_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{17600}{3781323}$
a_4	$\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)}$	$\frac{3872}{4861701}$
a_5	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)^2(6+r)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)(6r^2+65r+176)}$	$-\frac{921536}{6782072895}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^{\frac{1}{3}}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{10x}{63} + \frac{200x^2}{7371} - \frac{17600x^3}{3781323} + \frac{3872x^4}{4861701} - \frac{921536x^5}{6782072895} + O(x^6)}{x^{\frac{1}{3}}} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)(n+r-2)+6b_n(n+r)(n+r-1) \\ +4b_{n-1}(n+r-1)+11b_n(n+r)+b_n+2b_{n-1}=0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2+2nr+r^2+n+r)}{6n^2+12nr+6r^2+5n+5r+1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{-4n^2b_{n-1}+b_{n-1}}{24n^2-4n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-r^2-3r-2}{6r^2+17r+12}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = -\frac{3}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{3}{20}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{9}{352}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{3}{20}$
b_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{9}{352}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r+3)^2 (r+1) (r+2)^2 (r+4)}{216r^6 + 3132r^5 + 18486r^4 + 56753r^3 + 95433r^2 + 83230r + 29400}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{105}{23936}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{3}{20}$
b_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{9}{352}$
b_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{105}{23936}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r+3)^2 (r+1) (r+2)^2 (r+4)^2 (r+5)}{(216r^6 + 3132r^5 + 18486r^4 + 56753r^3 + 95433r^2 + 83230r + 29400) (6r^2 + 53r + 117)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{6615}{8808448}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{3}{20}$
b_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{9}{352}$
b_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{105}{23936}$
b_4	$\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)}$	$\frac{6615}{8808448}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+3)^2 (r+1) (r+2)^2 (r+4)^2 (r+5)^2 (6+r)}{(216r^6 + 3132r^5 + 18486r^4 + 56753r^3 + 95433r^2 + 83230r + 29400) (6r^2 + 53r + 117) (6r^2 + 65r +}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{11907}{92889088}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2-3r-2}{6r^2+17r+12}$	$-\frac{3}{20}$
b_2	$\frac{(r+3)(r+1)(r+2)^2}{36r^4+276r^3+775r^2+943r+420}$	$\frac{9}{352}$
b_3	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)}{216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400}$	$-\frac{105}{23936}$
b_4	$\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)}$	$\frac{6615}{8808448}$
b_5	$-\frac{(r+3)^2(r+1)(r+2)^2(r+4)^2(r+5)^2(6+r)}{(216r^6+3132r^5+18486r^4+56753r^3+95433r^2+83230r+29400)(6r^2+53r+117)(6r^2+65r+176)}$	$-\frac{11907}{92889088}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x^{\frac{1}{3}}} \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \frac{1 - \frac{3x}{20} + \frac{9x^2}{352} - \frac{105x^3}{23936} + \frac{6615x^4}{8808448} - \frac{11907x^5}{92889088} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1-\frac{10x}{63}+\frac{200x^2}{7371}-\frac{17600x^3}{3781323}+\frac{3872x^4}{4861701}-\frac{921536x^5}{6782072895}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1-\frac{3x}{20}+\frac{9x^2}{352}-\frac{105x^3}{23936}+\frac{6615x^4}{8808448}-\frac{11907x^5}{92889088}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-\frac{10x}{63}+\frac{200x^2}{7371}-\frac{17600x^3}{3781323}+\frac{3872x^4}{4861701}-\frac{921536x^5}{6782072895}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1-\frac{3x}{20}+\frac{9x^2}{352}-\frac{105x^3}{23936}+\frac{6615x^4}{8808448}-\frac{11907x^5}{92889088}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1\left(1-\frac{10x}{63}+\frac{200x^2}{7371}-\frac{17600x^3}{3781323}+\frac{3872x^4}{4861701}-\frac{921536x^5}{6782072895}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1-\frac{3x}{20}+\frac{9x^2}{352}-\frac{105x^3}{23936}+\frac{6615x^4}{8808448}-\frac{11907x^5}{92889088}+O(x^6)\right)}{\sqrt{x}}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= \frac{c_1\left(1-\frac{10x}{63}+\frac{200x^2}{7371}-\frac{17600x^3}{3781323}+\frac{3872x^4}{4861701}-\frac{921536x^5}{6782072895}+O(x^6)\right)}{x^{\frac{1}{3}}} \\
&\quad + \frac{c_2\left(1-\frac{3x}{20}+\frac{9x^2}{352}-\frac{105x^3}{23936}+\frac{6615x^4}{8808448}-\frac{11907x^5}{92889088}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}$$

Verified OK.

14.30.1 Maple step by step solution

Let's solve

$x^2(x+6)\,y''+(4x^2+11x)\,y'+(1+2x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(1+2x)y}{x^2(x+6)}-\frac{(11+4x)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(11+4x)y'}{x(x+6)}+\frac{(1+2x)y}{x^2(x+6)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{11+4x}{x(x+6)}, P_3(x)=\frac{1+2x}{x^2(x+6)}\right]$

  - $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

    $\left((x+6)\cdot P_2(x)\right)\Big|_{x=-6}=\frac{13}{6}$

  - $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

    $\left((x+6)^2\cdot P_3(x)\right)\Big|_{x=-6}=0$

  - $x=-6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-6$

- Multiply by denominators

  $x^2(x+6)\,y''+x(11+4x)\,y'+(1+2x)\,y=0$

- Change variables using $x=u-6$ so that the regular singular point is at $u=0$

  $(u^3-12u^2+36u)\left(\frac{d^2}{du^2}y(u)\right)+(4u^2-37u+78)\left(\frac{d}{du}y(u)\right)+(-11+2u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$6a_0 r(7+6r)\, u^{-1+r} + \left(6a_1(1+r)\,(13+6r) - a_0(12r^2+25r+11)\right) u^r + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)\,(\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$6r(7+6r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{7}{6}\right\}$$

- Each term must be 0

  $$6a_1(1+r)\,(13+6r) - a_0(12r^2+25r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-12a_k + a_{k-1} + 36a_{k+1})\, k^2 + ((-24a_k + 2a_{k-1} + 72a_{k+1})\, r - 25a_k + a_{k-1} + 114a_{k+1})\, k + (-12$$

- Shift index using $k- >k + 1$

  $$(-12a_{k+1} + a_k + 36a_{k+2})\,(k+1)^2 + ((-24a_{k+1} + 2a_k + 72a_{k+2})\, r - 25a_{k+1} + a_k + 114a_{k+2})\,(k +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + 2kra_k - 24kra_{k+1} + r^2 a_k - 12r^2 a_{k+1} + 3ka_k - 49ka_{k+1} + 3ra_k - 49ra_{k+1} + 2a_k - 48a_{k+1}}{6(6k^2 + 12kr + 6r^2 + 31k + 31r + 38)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + 3ka_k - 49ka_{k+1} + 2a_k - 48a_{k+1}}{6(6k^2 + 31k + 38)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + 3ka_k - 49ka_{k+1} + 2a_k - 48a_{k+1}}{6(6k^2 + 31k + 38)}, 78a_1 - 11a_0 = 0\right]$$

- Revert the change of variables $u = x + 6$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+6)^k, a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + 3ka_k - 49ka_{k+1} + 2a_k - 48a_{k+1}}{6(6k^2 + 31k + 38)}, 78a_1 - 11a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{6}$

$$a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + \frac{2}{3}ka_k - 21ka_{k+1} - \frac{5}{36}a_k - \frac{43}{6}a_{k+1}}{6(6k^2 + 17k + 10)}$$

- Solution for $r = -\frac{7}{6}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{6}}, a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + \frac{2}{3}ka_k - 21ka_{k+1} - \frac{5}{36}a_k - \frac{43}{6}a_{k+1}}{6(6k^2 + 17k + 10)}, -6a_1 + \frac{11a_0}{6} = 0\right]$$

- Revert the change of variables $u = x + 6$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+6)^{k-\frac{7}{6}}, a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{k+1} + \frac{2}{3}ka_k - 21ka_{k+1} - \frac{5}{36}a_k - \frac{43}{6}a_{k+1}}{6(6k^2 + 17k + 10)}, -6a_1 + \frac{11a_0}{6} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+6)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+6)^{k-\frac{7}{6}}\right), a_{k+2} = -\frac{k^2 a_k - 12k^2 a_{1+k} + 3ka_k - 49ka_{1+k} + 2a_k - 48a_{1+k}}{6(6k^2 + 31k + 38)}, 78a\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*(6+x)*diff(y(x),x$2)+x*(11+4*x)*diff(y(x),x)+(1+2*x)*y(x)=0,y(x),type='series',x=
```

$$y(x) = \frac{c_2\left(1-\frac{10}{63}x+\frac{200}{7371}x^2-\frac{17600}{3781323}x^3+\frac{3872}{4861701}x^4-\frac{921536}{6782072895}x^5+\mathrm{O}\left(x^6\right)\right)x^{\frac{1}{6}}+c_1\left(1-\frac{3}{20}x+\frac{9}{352}x^2-\frac{105}{23936}x^3\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*(6+x)*y''[x]+x*(11+4*x)*y'[x]+(1+2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{921536x^5}{6782072895}+\frac{3872x^4}{4861701}-\frac{17600x^3}{3781323}+\frac{200x^2}{7371}-\frac{10x}{63}+1\right)}{\sqrt[3]{x}} + \frac{c_2\left(-\frac{11907x^5}{92889088}+\frac{6615x^4}{8808448}-\frac{105x^3}{23936}+\frac{9x^2}{352}-\frac{3x}{20}+1\right)}{\sqrt{x}}$$

14.31 problem 33

Internal problem ID [1322]
Internal file name [`OUTPUT/1323_Sunday_June_05_2022_02_10_18_AM_572790/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2y'' + x(x^2+2)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$8x^2y'' + (x^3+2x)\,y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2+2}{8x}$$
$$q(x) = \frac{1}{8x^2}$$

Table 590: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+2}{8x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{1}{8x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$8x^2y'' + (x^3 + 2x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$8x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^3 + 2x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r} a_n(n+r)(n+r-1) + 2x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$8x^r a_0 r(-1+r) + 2x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(8x^r r(-1+r) + 2x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 - 6r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$8a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) + 2a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{8n^2 + 16nr + 8r^2 - 6n - 6r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{-2na_{n-2} + 3a_{n-2}}{16n^2 + 4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r}{8r^2 + 26r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{72}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{72}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2 + r)}{(8r^2 + 26r + 21)(8r^2 + 58r + 105)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{5}{19584}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{72}$
a_3	0	0
a_4	$\frac{r(2+r)}{(8r^2+26r+21)(8r^2+58r+105)}$	$\frac{5}{19584}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{72}$
a_3	0	0
a_4	$\frac{r(2+r)}{(8r^2+26r+21)(8r^2+58r+105)}$	$\frac{5}{19584}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{72} + \frac{5x^4}{19584} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$8b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) + 2b_n(n+r) + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n+r-2)}{8n^2 + 16nr + 8r^2 - 6n - 6r + 1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = \frac{-4nb_{n-2} + 7b_{n-2}}{32n^2 - 8n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r}{8r^2 + 26r + 21}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = -\frac{1}{112}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{112}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{112}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(2+r)}{(8r^2+26r+21)(8r^2+58r+105)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{3}{17920}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{112}$
b_3	0	0
b_4	$\frac{r(2+r)}{(8r^2+26r+21)(8r^2+58r+105)}$	$\frac{3}{17920}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{8r^2+26r+21}$	$-\frac{1}{112}$
b_3	0	0
b_4	$\frac{r(2+r)}{(8r^2+26r+21)(8r^2+58r+105)}$	$\frac{3}{17920}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots) \\
&= x^{\frac{1}{4}}\left(1 - \frac{x^2}{112} + \frac{3x^4}{17920} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{x^2}{72} + \frac{5x^4}{19584} + O(x^6)\right) + c_2 x^{\frac{1}{4}}\left(1 - \frac{x^2}{112} + \frac{3x^4}{17920} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{x^2}{72} + \frac{5x^4}{19584} + O(x^6)\right) + c_2 x^{\frac{1}{4}}\left(1 - \frac{x^2}{112} + \frac{3x^4}{17920} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1 - \frac{x^2}{72} + \frac{5x^4}{19584} + O(x^6)\right) + c_2 x^{\frac{1}{4}}\left(1 - \frac{x^2}{112} + \frac{3x^4}{17920} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1 - \frac{x^2}{72} + \frac{5x^4}{19584} + O(x^6)\right) + c_2 x^{\frac{1}{4}}\left(1 - \frac{x^2}{112} + \frac{3x^4}{17920} + O(x^6)\right)
$$

Verified OK.

14.31.1 Maple step by step solution

Let's solve

$8x^2y'' + (x^3+2x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{8x^2} - \frac{(x^2+2)y'}{8x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+2)y'}{8x} + \frac{y}{8x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2+2}{8x}, P_3(x) = \frac{1}{8x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{8}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2y'' + x(x^2+2)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2\cdot y''$ to series expansion

  $$x^{2}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(-1+4r)\,(-1+2r)\,x^{r}+a_{1}(3+4r)\,(1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(4k+4r-1)\,(2k+2r-1)+a_{k-}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+4r)\,(-1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{\frac{1}{2},\frac{1}{4}\right\}$$

- Each term must be 0

  $$a_{1}(3+4r)\,(1+2r)=0$$

- Solve for the dependent coefficient(s)

  $$a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$8\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{1}{4}\right)a_{k}+a_{k-2}(k-2+r)=0$$

- Shift index using $k->k+2$

  $$8\left(k+\frac{3}{2}+r\right)\left(k+\frac{7}{4}+r\right)a_{k+2}+a_{k}(k+r)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{a_{k}(k+r)}{(2k+3+2r)(4k+7+4r)}$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+2}=-\frac{a_{k}\left(k+\frac{1}{2}\right)}{(2k+4)(4k+9)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_{k}\left(k+\frac{1}{2}\right)}{(2k+4)(4k+9)}, a_{1}=0\right]$$

- Recursion relation for $r=\frac{1}{4}$

  $$a_{k+2}=-\frac{a_{k}\left(k+\frac{1}{4}\right)}{\left(2k+\frac{7}{2}\right)(4k+8)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{a_k \left(k+\frac{1}{4}\right)}{\left(2k+\frac{7}{2}\right)\left(4k+8\right)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{a_k \left(k+\frac{1}{2}\right)}{\left(2k+4\right)\left(4k+9\right)}, a_1 = 0, b_{k+2} = -\frac{b_k \left(k+\frac{1}{4}\right)}{\left(2k+\frac{7}{2}\right)\left(4k+8\right)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(8*x^2*diff(y(x),x$2)+x*(2+x^2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 - \frac{1}{112}x^2 + \frac{3}{17920}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{1}{72}x^2 + \frac{5}{19584}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[8*x^2*y''[x]+x*(2+x^2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{5x^4}{19584} - \frac{x^2}{72} + 1\right) + c_2\sqrt[4]{x}\left(\frac{3x^4}{17920} - \frac{x^2}{112} + 1\right)$$

14.32 problem 34

Internal problem ID [1323]
Internal file name [`OUTPUT/1324_Sunday_June_05_2022_02_10_21_AM_82739945/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2(-x^2+1)\,y''+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-8x^4+8x^2)\,y''+(-26x^3+2x)\,y'+(-9x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{13x^2-1}{4x\,(x^2-1)}$$
$$q(x)=\frac{9x^2-1}{8x^2\,(x^2-1)}$$

Table 592: Table $p(x), q(x)$ singularites.

$p(x) = \frac{13x^2-1}{4x(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{9x^2-1}{8x^2(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-8y''x^2\left(x^2-1\right) + \left(-26x^3+2x\right)y' + \left(-9x^2+1\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-8\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2\left(x^2-1\right) + \left(-26x^3+2x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-9x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-26x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n (n+r)(n+r-1)\right) = \sum_{n=2}^{\infty} \left(-8a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-26x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-26a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-9a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty} \left(-8a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-26a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-9a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r} a_n (n+r)(n+r-1) + 2x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$8x^r a_0 r(-1+r) + 2x^r a_0 r + a_0 x^r = 0$$

Or

$$(8x^r r(-1+r) + 2x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 - 6r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 - 6r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
-8a_{n-2}(n+r-2)(n-3+r) + 8a_n(n+r)(n+r-1) \\
-26a_{n-2}(n+r-2) + 2a_n(n+r) - 9a_{n-2} + a_n = 0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{(4n+4r-5)\, a_{n-2}}{4n+4r-1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_n = \frac{(4n-3)\, a_{n-2}}{4n+1} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{3+4r}{7+4r}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_2 = \frac{5}{9}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{3+4r}{7+4r}$	$\frac{5}{9}$

For $n = 3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{3+4r}{7+4r}$	$\frac{5}{9}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^2 + 56r + 33}{16r^2 + 88r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{65}{153}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{3+4r}{7+4r}$	$\frac{5}{9}$
a_3	0	0
a_4	$\frac{16r^2+56r+33}{16r^2+88r+105}$	$\frac{65}{153}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{3+4r}{7+4r}$	$\frac{5}{9}$
a_3	0	0
a_4	$\frac{16r^2+56r+33}{16r^2+88r+105}$	$\frac{65}{153}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 + \frac{5x^2}{9} + \frac{65x^4}{153} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}&-8b_{n-2}(n+r-2)(n-3+r) + 8b_n(n+r)(n+r-1) \\ &- 26b_{n-2}(n+r-2) + 2b_n(n+r) - 9b_{n-2} + b_n = 0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{(4n+4r-5)\, b_{n-2}}{4n+4r-1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = \frac{(n-1)\, b_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3+4r}{7+4r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{3+4r}{7+4r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{3+4r}{7+4r}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^2 + 56r + 33}{16r^2 + 88r + 105}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{3+4r}{7+4r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16r^2+56r+33}{16r^2+88r+105}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{3+4r}{7+4r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{16r^2+56r+33}{16r^2+88r+105}$	$\frac{3}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\,(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots) \\ &= x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 + \frac{5x^2}{9} + \frac{65x^4}{153} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x} \left(1 + \frac{5x^2}{9} + \frac{65x^4}{153} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x} \left(1 + \frac{5x^2}{9} + \frac{65x^4}{153} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x} \left(1 + \frac{5x^2}{9} + \frac{65x^4}{153} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)$$

Verified OK.

14.32.1 Maple step by step solution

Let's solve

$-8y''x^2(x^2-1)+(-26x^3+2x)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{(9x^2-1)y}{8x^2(x^2-1)}-\frac{(13x^2-1)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(13x^2-1)y'}{4x(x^2-1)}+\frac{(9x^2-1)y}{8x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)},P_3(x)=\frac{9x^2-1}{8x^2(x^2-1)}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{3}{2}$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$8y''x^2(x^2-1)+2x(13x^2-1)\,y'+y(9x^2-1)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(8u^4-32u^3+40u^2-16u)\left(\frac{d^2}{du^2}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)\,y$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-8a_0 r(1+2r)\, u^{-1+r} + \left(-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r)\right) u^r + \left(-8a_2(2+r)\,(5+2r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-8r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

  $$[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) -$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a_0\left(10r^2+9r+2\right)}{2(2r^2+5r+3)}, a_2 = \frac{a_0\left(34r^3+76r^2+41r+5\right)}{4(2r^3+11r^2+19r+10)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 18a_k - 7a_{k-2} + 9a_{k-1}$$

- Shift index using $k->k+2$

  $8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})(k+2)^2+2(8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})r+18a_{k+2}-7a_k+9$

- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+16kra_k-64kra_{k+1}+80kra_{k+2}+8r^2a_k-32r^2a_{k+1}+40r^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+}}{8(2k^2+4kr+2r^2+13k+13r+21)}$

- Recursion relation for $r=0$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\frac{a_0}{3}\right.$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\right.$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$

- Solution for $r=-\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1=\right.$

- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x+1)^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{8k^2a_k-32k^2a_{1+k}+40k^2a_{k+2}+18ka_k-110ka_{1+k}+196ka}{8(2k^2+13k+21)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(8*x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-13*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x),type='
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{5}{9}x^2 + \frac{65}{153}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 52

```
AsymptoticDSolveValue[8*x^2*(1-x^2)*y''[x]+2*x*(1-13*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],{x,0,
```

$$y(x) \to c_1\sqrt{x}\left(\frac{65x^4}{153}+\frac{5x^2}{9}+1\right)+c_2\sqrt[4]{x}\left(\frac{3x^4}{8}+\frac{x^2}{2}+1\right)$$

14.33 problem 35

Internal problem ID [1324]
Internal file name [`OUTPUT/1325_Sunday_June_05_2022_02_10_24_AM_5215957/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(2x^3-4x)\,y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{2x^2-4}{x\,(x^2+1)} \\ q(x) &= \frac{4}{x^2\,(x^2+1)} \end{aligned}$$

Table 594: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2-4}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{4}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (2x^3-4x)\,y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3-4x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \tag{2B}$$
$$+ \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - 4x^{n+r} a_n(n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 4x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 4x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 4 \\ r_2 &= 1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r)+a_n(n+r)(n+r-1) \\ +2a_{n-2}(n+r-2)-4a_n(n+r)+4a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{n+r-4} \tag{4}$$

Which for the root $r=4$ becomes

$$a_n = -\frac{a_{n-2}(n+2)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2 = -\frac{r}{r-2}$$

Which for the root $r=4$ becomes

$$a_2 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r-2}$	-2

For $n=3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r-2}$	-2
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2+r}{r-2}$$

Which for the root $r = 4$ becomes

$$a_4 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r-2}$	-2
a_3	0	0
a_4	$\frac{2+r}{r-2}$	3

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r-2}$	-2
a_3	0	0
a_4	$\frac{2+r}{r-2}$	3
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^4 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= x^4 (1 - 2x^2 + 3x^4 + O(x^6))
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to 1} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&b_{n-2}(n + r - 2)(n - 3 + r) + b_n(n + r)(n + r - 1) \\
&+ 2b_{n-2}(n + r - 2) - 4b_n(n + r) + 4b_n = 0
\end{aligned} \tag{4}
$$

Which for for the root $r = 1$ becomes

$$
b_{n-2}(n - 1)(n - 2) + b_n(n + 1) n + 2b_{n-2}(n - 1) - 4b_n(n + 1) + 4b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n+r-2)}{n+r-4} \tag{5}$$

Which for the root $r = 1$ becomes

$$b_n = -\frac{b_{n-2}(n-1)}{n-3} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r}{r-2}$$

Which for the root $r = 1$ becomes

$$b_2 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r-2}$	1

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r-2}$	1
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{2+r}{r-2}$$

Which for the root $r = 1$ becomes

$$b_4 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r-2}$	1
b_3	0	0
b_4	$\frac{2+r}{r-2}$	-3

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{r-2}$	1
b_3	0	0
b_4	$\frac{2+r}{r-2}$	-3
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^4\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x\left(1 + x^2 - 3x^4 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4\left(1 - 2x^2 + 3x^4 + O(x^6)\right) + c_2 x\left(1 + x^2 - 3x^4 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^4 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) + c_2 x \left(1 + x^2 - 3x^4 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^4 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) + c_2 x \left(1 + x^2 - 3x^4 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^4 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) + c_2 x \left(1 + x^2 - 3x^4 + O(x^6)\right)$$

Verified OK.

14.33.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (2x^3 - 4x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{2(x^2-2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x^2-2)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2-2)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\,y''+2x(x^2-2)\,y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\,x^r+a_1r(-3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r-1)\,(k+r-4)+a_{k-2}(k-2+r)\,(\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{1,4\}$

- Each term must be 0

  $a_1r(-3+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-4)+a_{k-2}(k-2+r)\right)=0$
- Shift index using $k->k+2$

  $(k+r+1)\left(a_{k+2}(k-2+r)+a_k(k+r)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r)}{k-2+r}$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k(k+1)}{k-1}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k(k+1)}{k-1}, a_1=0\right]$
- Recursion relation for $r=4$

  $a_{k+2}=-\frac{a_k(k+4)}{k+2}$
- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{1+k}\right)+\left(\sum_{k=0}^{\infty} b_k x^{4+k}\right), a_{k+2}=-\frac{a_k(1+k)}{k-1}, a_1=0, b_{k+2}=-\frac{b_k(4+k)}{k+2}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-2*x*(2-x^2)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^4 \left(1 - 2x^2 + 3x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x \left(12 + 12x^2 - 36x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 34

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-2*x*(2-x^2)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-3x^5 + x^3 + x\right) + c_2 \left(3x^8 - 2x^6 + x^4\right)$$

14.34 problem 36

Internal problem ID [1325]
Internal file name [OUTPUT/1326_Sunday_June_05_2022_02_10_26_AM_97069197/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+3)\,y''+(-x^2+2)\,y'-8yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3+3x)\,y''+(-x^2+2)\,y'-8yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^2-2}{x\,(x^2+3)}$$
$$q(x)=-\frac{8}{x^2+3}$$

Table 596: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-2}{x(x^2+3)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

$q(x) = -\frac{8}{x^2+3}$	
singularity	type
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -i\sqrt{3}, i\sqrt{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2+3)\, y'' + (-x^2+2)\, y' - 8yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^2+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-x^2+2) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-8x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n(n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-8x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1} a_n(n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+3r)=0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2-r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{3}$$
$$r_2=0$$

Since $a_0\neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+3r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{3}}$$
$$y_2(x)=\sum_{n=0}^{\infty}b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. Substituting $n=1$ in Eq. (2B) gives

$$a_1=0$$

For $2\leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r)+3a_n(n+r)(n+r-1)-a_{n-2}(n+r-2)+2a_n(n+r)-8a_{n-2}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-6)\, a_{n-2}}{3n-1+3r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(3n-17)\, a_{n-2}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4-r}{5+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{11}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{5+3r}$	$\frac{11}{18}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{5+3r}$	$\frac{11}{18}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 - 6r + 8}{9r^2 + 48r + 55}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{55}{648}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{5+3r}$	$\frac{11}{18}$
a_3	0	0
a_4	$\frac{r^2-6r+8}{9r^2+48r+55}$	$\frac{55}{648}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{5+3r}$	$\frac{11}{18}$
a_3	0	0
a_4	$\frac{r^2-6r+8}{9r^2+48r+55}$	$\frac{55}{648}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}} \left(1 + \frac{11x^2}{18} + \frac{55x^4}{648} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-2}(n+r-2)(n-3+r) + 3b_n(n+r)(n+r-1) \\ - b_{n-2}(n+r-2) + 2(n+r)b_n - 8b_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(n+r-6)\,b_{n-2}}{3n-1+3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{(n-6)\,b_{n-2}}{3n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4-r}{5+3r}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{4}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4-r}{5+3r}$	$\frac{4}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4-r}{5+3r}$	$\frac{4}{5}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 - 6r + 8}{9r^2 + 48r + 55}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{8}{55}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4-r}{5+3r}$	$\frac{4}{5}$
b_3	0	0
b_4	$\frac{r^2-6r+8}{9r^2+48r+55}$	$\frac{8}{55}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4-r}{5+3r}$	$\frac{4}{5}$
b_3	0	0
b_4	$\frac{r^2-6r+8}{9r^2+48r+55}$	$\frac{8}{55}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{4x^2}{5} + \frac{8x^4}{55} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}} \left(1 + \frac{11x^2}{18} + \frac{55x^4}{648} + O(x^6)\right) + c_2 \left(1 + \frac{4x^2}{5} + \frac{8x^4}{55} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}} \left(1 + \frac{11x^2}{18} + \frac{55x^4}{648} + O(x^6)\right) + c_2 \left(1 + \frac{4x^2}{5} + \frac{8x^4}{55} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \left(1 + \frac{11x^2}{18} + \frac{55x^4}{648} + O(x^6)\right) + c_2 \left(1 + \frac{4x^2}{5} + \frac{8x^4}{55} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 + \frac{11x^2}{18} + \frac{55x^4}{648} + O(x^6)\right) + c_2 \left(1 + \frac{4x^2}{5} + \frac{8x^4}{55} + O(x^6)\right)$$

Verified OK.

14.34.1 Maple step by step solution

Let's solve

$x(x^2+3)\,y''+(-x^2+2)\,y'-8yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x^2-2)y'}{x(x^2+3)}+\frac{8y}{x^2+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-2)y'}{x(x^2+3)}-\frac{8y}{x^2+3}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x^2+3)}, P_3(x)=-\frac{8}{x^2+3}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+3)\,y''+(-x^2+2)\,y'-8yx=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x\cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-1+3r)\, x^{-1+r}+a_{1}(1+r)\,(2+3r)\, x^{r}+\left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(3k+2+3r)+a_{k-1}(k+r-$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_{1}(1+r)\,(2+3r)=0$

- Each term in the series must be 0, giving the recursion relation

  $\left(a_{k-1}(k-5+r)+3\left(k+\frac{2}{3}+r\right) a_{k+1}\right)(k+r+1)=0$

- Shift index using $k->k+1$

  $\left(a_{k}(k+r-4)+3\left(k+\frac{5}{3}+r\right) a_{k+2}\right)(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r-4)}{3k+5+3r}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+2}=-\frac{a_{k}(k-4)}{3k+5}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(-4+k)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*(3+x^2)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}}\left(1+\frac{11}{18}x^2+\frac{55}{648}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(1+\frac{4}{5}x^2+\frac{8}{55}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 47

```
AsymptoticDSolveValue[x*(3+x^2)*y''[x]+(2-x^2)*y'[x]-8*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{55x^4}{648} + \frac{11x^2}{18} + 1 \right) + c_2 \left(\frac{8x^4}{55} + \frac{4x^2}{5} + 1 \right)$$

14.35 problem 37

Internal problem ID [1326]
Internal file name [`OUTPUT/1327_Sunday_June_05_2022_02_10_28_AM_98618836/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(-x^2+1)\,y''+x(-19x^2+7)\,y'-(14x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-4x^4+4x^2)\,y''+(-19x^3+7x)\,y'+(-14x^2-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{19x^2-7}{4x\,(x^2-1)}\\ q(x)&=\frac{14x^2+1}{4x^2\,(x^2-1)}\end{aligned}$$

Table 598: Table $p(x), q(x)$ singularites.

$p(x) = \frac{19x^2-7}{4x(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{14x^2+1}{4x^2(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-4y''x^2(x^2-1) + (-19x^3+7x)y' + (-14x^2-1)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-4\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2(x^2-1) + (-19x^3+7x)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + (-14x^2-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-4x^{n+r+2}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-19x^{n+r+2}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}7x^{n+r}a_n(n+r)\right) \tag{2A}$$
$$+\sum_{n=0}^{\infty}\left(-14x^{n+r+2}a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-4x^{n+r+2}a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-19x^{n+r+2}a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-19a_{n-2}(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-14x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-14a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=2}^{\infty}\left(-19a_{n-2}(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}7x^{n+r}a_n(n+r)\right) \tag{2B}$$
$$+\sum_{n=2}^{\infty}\left(-14a_{n-2}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1)+7x^{n+r}a_n(n+r)-a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 7x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 7x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 3r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
-4a_{n-2}(n+r-2)(n-3+r)+4a_n(n+r)(n+r-1) \\
-19a_{n-2}(n+r-2)+7a_n(n+r)-14a_{n-2}-a_n=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{(n+r)\, a_{n-2}}{n+1+r} \tag{4}
$$

Which for the root $r = \frac{1}{4}$ becomes

$$
a_n = \frac{(4n+1)\, a_{n-2}}{4n+5} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{2+r}{3+r}
$$

Which for the root $r = \frac{1}{4}$ becomes

$$
a_2 = \frac{9}{13}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{3+r}$	$\frac{9}{13}$

For $n = 3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{3+r}$	$\frac{9}{13}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 6r + 8}{(3+r)(5+r)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{51}{91}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{3+r}$	$\frac{9}{13}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(3+r)(5+r)}$	$\frac{51}{91}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{3+r}$	$\frac{9}{13}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(3+r)(5+r)}$	$\frac{51}{91}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{4}}\left(1 + \frac{9x^2}{13} + \frac{51x^4}{91} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \le n$ the recursive equation is

$$
\begin{aligned}
&-4b_{n-2}(n+r-2)(n-3+r) + 4b_n(n+r)(n+r-1) \\
&- 19b_{n-2}(n+r-2) + 7b_n(n+r) - 14b_{n-2} - b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{(n+r)\, b_{n-2}}{n+1+r} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = \frac{(n-1)\, b_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{2+r}{3+r}
$$

Which for the root $r = -1$ becomes

$$
b_2 = \frac{1}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{3+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{3+r}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 6r + 8}{(3 + r)(5 + r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{3+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2+6r+8}{(3+r)(5+r)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{2+r}{3+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2+6r+8}{(3+r)(5+r)}$	$\frac{3}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{4}}\left(1 + \frac{9x^2}{13} + \frac{51x^4}{91} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{4}}\left(1 + \frac{9x^2}{13} + \frac{51x^4}{91} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(1 + \frac{9x^2}{13} + \frac{51x^4}{91} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 + \frac{9x^2}{13} + \frac{51x^4}{91} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x}$$

Verified OK.

14.35.1 Maple step by step solution

Let's solve

$-4y''x^2(x^2-1) + (-19x^3+7x)\,y' + (-14x^2-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(14x^2+1)y}{4x^2(x^2-1)} - \frac{(19x^2-7)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(19x^2-7)y'}{4x(x^2-1)} + \frac{(14x^2+1)y}{4x^2(x^2-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{19x^2-7}{4x(x^2-1)}, P_3(x) = \frac{14x^2+1}{4x^2(x^2-1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{3}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4y''x^2(x^2-1) + x(19x^2-7)\,y' + (14x^2+1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$$\left(4u^4-16u^3+20u^2-8u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(19u^3-57u^2+50u-12\right)\left(\frac{d}{du}y(u)\right)+\left(14u^2-28u+15\right)$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m=0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..4$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0r(1+2r)\,u^{-1+r}+\left(-4a_1(1+r)\,(3+2r)+5a_0(4r^2+6r+3)\right)u^r+\left(-4a_2(2+r)\,(5+2r)+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

$$\left[-4a_1(1+r)\,(3+2r)+5a_0(4r^2+6r+3)=0,-4a_2(2+r)\,(5+2r)+5a_1(4r^2+14r+13)-a_0\right.$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})r + 30a_k - a_{k-2} - 9a_{k-1} - 2$$

- Shift index using $k- >k+2$

$$4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})r + 30a_{k+2} - a_k - 9a_k$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 8kra_k - 32kra_{k+1} + 40kra_{k+2} + 4r^2a_k - 16r^2a_{k+1} + 20r^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}, a_1 = \frac{5a}{4}\right.$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}, a_1 =\right.$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}, a_1 =\right.$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}, a\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{4k^2a_k - 16k^2a_{1+k} + 20k^2a_{k+2} + 15ka_k - 73ka_{1+k} + 110ka_k}{4(2k^2+13k+21)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*(1-x^2)*diff(y(x),x$2)+x*(7-19*x^2)*diff(y(x),x)-(1+14*x^2)*y(x)=0,y(x),type='s
```

$$y(x) = \frac{c_2 x^{\frac{5}{4}}\left(1+\frac{9}{13}x^2+\frac{51}{91}x^4+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+\frac{1}{2}x^2+\frac{3}{8}x^4+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 50

```
AsymptoticDSolveValue[4*x^2*(1-x^2)*y''[x]+x*(7-19*x^2)*y'[x]-(1+14*x^2)*y[x]==0,y[x],{x,0,5
```

$$y(x) \to c_1 \sqrt[4]{x} \left(\frac{51x^4}{91} + \frac{9x^2}{13} + 1\right) + \frac{c_2 \left(\frac{3x^4}{8} + \frac{x^2}{2} + 1\right)}{x}$$

14.36 problem 38

Internal problem ID [1327]
Internal file name [`OUTPUT/1328_Sunday_June_05_2022_02_10_31_AM_25923185/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^{2}\left(-x^{2}+2\right) y''+x\left(-11x^{2}+1\right) y'+\left(-5x^{2}+1\right) y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-3x^{4}+6x^{2}\right) y''+\left(-11x^{3}+x\right) y'+\left(-5x^{2}+1\right) y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{11x^{2}-1}{3x\left(x^{2}-2\right)}$$
$$q(x)=\frac{5x^{2}-1}{3x^{2}\left(x^{2}-2\right)}$$

Table 600: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11x^2-1}{3x(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = \frac{5x^2-1}{3x^2(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \sqrt{2}, -\sqrt{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-3y''x^2(x^2-2) + (-11x^3 + x)\, y' + (-5x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-3\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x^2-2) + (-11x^3 + x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-5x^2 + 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-3x^{n+r+2}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}6x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-11x^{n+r+2}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right) \tag{2A}$$
$$+\sum_{n=0}^{\infty}\left(-5x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-3x^{n+r+2}a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)$$
$$\sum_{n=0}^{\infty}\left(-11x^{n+r+2}a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-11a_{n-2}(n+r-2)x^{n+r}\right)$$
$$\sum_{n=0}^{\infty}\left(-5x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-5a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)$$
$$+\left(\sum_{n=0}^{\infty}6x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-11a_{n-2}(n+r-2)x^{n+r}\right) \tag{2B}$$
$$+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-5a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r}a_n(n+r)(n+r-1)+x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$6x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(6x^r r(-1+r) + x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&-3a_{n-2}(n+r-2)(n-3+r)+6a_n(n+r)(n+r-1) \\
&-11a_{n-2}(n+r-2)+a_n(n+r)-5a_{n-2}+a_n=0
\end{aligned}
\tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{(n+r-1)\,a_{n-2}}{2n+2r-1} \tag{4}
$$

Which for the root $r=\frac{1}{2}$ becomes

$$
a_n = \frac{(2n-1)\,a_{n-2}}{4n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$
a_2 = \frac{1+r}{3+2r}
$$

Which for the root $r=\frac{1}{2}$ becomes

$$
a_2 = \frac{3}{8}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{3+2r}$	$\frac{3}{8}$

For $n=3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{3+2r}$	$\frac{3}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r + 3}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{21}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{3+2r}$	$\frac{3}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{21}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1+r}{3+2r}$	$\frac{3}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{21}{128}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= \sqrt{x}\left(1 + \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&-3b_{n-2}(n+r-2)\,(n-3+r) + 6b_n(n+r)\,(n+r-1) \\
&- 11b_{n-2}(n+r-2) + b_n(n+r) - 5b_{n-2} + b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{(n+r-1)\,b_{n-2}}{2n+2r-1} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_n = \frac{(3n-2)\,b_{n-2}}{6n-1} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{1+r}{3+2r}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
b_2 = \frac{4}{11}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1+r}{3+2r}$	$\frac{4}{11}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1+r}{3+2r}$	$\frac{4}{11}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 4r + 3}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{40}{253}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1+r}{3+2r}$	$\frac{4}{11}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{40}{253}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1+r}{3+2r}$	$\frac{4}{11}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{40}{253}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{4x^2}{11} + \frac{40x^4}{253} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 + \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 + \frac{4x^2}{11} + \frac{40x^4}{253} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 + \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 + \frac{4x^2}{11} + \frac{40x^4}{253} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1 + \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 + \frac{4x^2}{11} + \frac{40x^4}{253} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1 + \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 + \frac{4x^2}{11} + \frac{40x^4}{253} + O(x^6)\right)$$

Verified OK.

14.36.1 Maple step by step solution

Let's solve

$-3y''x^2(x^2-2)+(-11x^3+x)\,y'+(-5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{(5x^2-1)y}{3x^2(x^2-2)}-\frac{(11x^2-1)y'}{3x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(11x^2-1)y'}{3x(x^2-2)}+\frac{(5x^2-1)y}{3x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{11x^2-1}{3x(x^2-2)},P_3(x)=\frac{5x^2-1}{3x^2(x^2-2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{6}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{6}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$3y''x^2(x^2-2)+x(11x^2-1)\,y'+(5x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(-1+3r)\,(-1+2r)\,x^{r}-a_{1}(2+3r)\,(1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(-a_{k}(3k+3r-1)\,(2k+2r-1)+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+3r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{1}{3}\right\}$

- Each term must be 0

  $-a_{1}(2+3r)\,(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $-6\left(k+r-\frac{1}{3}\right)\left(\frac{(-k-r+1)a_{k-2}}{2}+a_{k}\left(k+r-\frac{1}{2}\right)\right)=0$

- Shift index using $k->k+2$

  $-6\left(k+\frac{5}{3}+r\right)\left(\frac{(-k-1-r)a_{k}}{2}+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

$a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$

- Solution for $r = \frac{1}{3}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0, b_{k+2} = \frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(3*x^2*(2-x^2)*diff(y(x),x$2)+x*(1-11*x^2)*diff(y(x),x)+(1-5*x^2)*y(x)=0,y(x),type='se
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 + \frac{4}{11}x^2 + \frac{40}{253}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{3}{8}x^2 + \frac{21}{128}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 52

```
AsymptoticDSolveValue[3*x^2*(2-x^2)*y''[x]+x*(1-11*x^2)*y'[x]+(1-5*x^2)*y[x]==0,y[x],{x,0,5}
```

$$y(x) \to c_1\sqrt{x}\left(\frac{21x^4}{128}+\frac{3x^2}{8}+1\right)+c_2\sqrt[3]{x}\left(\frac{40x^4}{253}+\frac{4x^2}{11}+1\right)$$

14.37 problem 39

Internal problem ID [1328]
Internal file name [OUTPUT/1329_Sunday_June_05_2022_02_10_33_AM_35309778/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+2)\,y''-x(-7x^2+12)\,y'+(3x^2+7)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+4x^2)\,y''+(7x^3-12x)\,y'+(3x^2+7)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2-12}{2\,(x^2+2)\,x}$$
$$q(x)=\frac{3x^2+7}{2x^2\,(x^2+2)}$$

Table 602: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2-12}{2(x^2+2)x}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = \frac{3x^2+7}{2x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i\sqrt{2}, i\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x^2+2)\,y'' + (7x^3-12x)\,y' + (3x^2+7)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (7x^3-12x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (3x^2+7)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-12x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-12x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - 12x^{n+r} a_n (n+r) + 7a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) - 12x^r a_0 r + 7a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) - 12x^r r + 7x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 16r + 7\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 16r + 7 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 16r + 7\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{7}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \sqrt{x} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) \\ &+ 7a_{n-2}(n+r-2) - 12a_n(n+r) + 3a_{n-2} + 7a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-1)\,a_{n-2}}{2n+2r-7} \tag{4}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_n = -\frac{(2n+5)\,a_{n-2}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1-r}{-3+2r}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_2 = -\frac{9}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-3+2r}$	$-\frac{9}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-3+2r}$	$-\frac{9}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r + 3}{4r^2 - 4r - 3}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_4 = \frac{117}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-3+2r}$	$-\frac{9}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2-4r-3}$	$\frac{117}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-3+2r}$	$-\frac{9}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2-4r-3}$	$\frac{117}{128}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{7}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{7}{2}}\left(1 - \frac{9x^2}{8} + \frac{117x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to \frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-2}(n+r-2)(n-3+r)+4b_n(n+r)(n+r-1) \\ &+7b_{n-2}(n+r-2)-12b_n(n+r)+3b_{n-2}+7b_n=0 \end{aligned} \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$\begin{aligned} &2b_{n-2}\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)+4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) \\ &+7b_{n-2}\left(n-\frac{3}{2}\right)-12b_n\left(n+\frac{1}{2}\right)+3b_{n-2}+7b_n=0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r-1)\, b_{n-2}}{2n+2r-7} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{\left(n-\frac{1}{2}\right) b_{n-2}}{2n-6} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1+r}{-3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{3}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{-3+2r}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{-3+2r}$	$\frac{3}{4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 4r + 3}{(-3 + 2r)(1 + 2r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = -\frac{21}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{-3+2r}$	$\frac{3}{4}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2-4r-3}$	$-\frac{21}{16}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{-3+2r}$	$\frac{3}{4}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2-4r-3}$	$-\frac{21}{16}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{7}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{3x^2}{4} - \frac{21x^4}{16} + O\left(x^6\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{7}{2}}\left(1 - \frac{9x^2}{8} + \frac{117x^4}{128} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{3x^2}{4} - \frac{21x^4}{16} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{7}{2}}\left(1 - \frac{9x^2}{8} + \frac{117x^4}{128} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{3x^2}{4} - \frac{21x^4}{16} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{7}{2}}\left(1 - \frac{9x^2}{8} + \frac{117x^4}{128} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{3x^2}{4} - \frac{21x^4}{16} + O\left(x^6\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x^{\frac{7}{2}}\left(1 - \frac{9x^2}{8} + \frac{117x^4}{128} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{3x^2}{4} - \frac{21x^4}{16} + O\left(x^6\right)\right)$$

Verified OK.

14.37.1 Maple step by step solution

Let's solve

$2x^2(x^2+2)\,y''+(7x^3-12x)\,y'+(3x^2+7)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+7)y}{2x^2(x^2+2)}-\frac{(7x^2-12)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2-12)y'}{2x(x^2+2)}+\frac{(3x^2+7)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x^2-12}{2(x^2+2)x}, P_3(x)=\frac{3x^2+7}{2x^2(x^2+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{7}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\,y''+x(7x^2-12)\,y'+(3x^2+7)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..3$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m=2..4$

  $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-7+2r)\, x^r+a_1(1+2r)\,(-5+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7)+a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-7+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-5+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2}+a_k\left(k+r-\frac{7}{2}\right)\right)\left(k+r-\frac{1}{2}\right)=0$

- Shift index using $k->k+2$

  $4\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k-\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r+1)}{2k-3+2r}$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{a_k(k+\frac{3}{2})}{2k-2}$

- Solution for $r = \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k(k+\frac{3}{2})}{2k-2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{7}{2}$

$a_{k+2} = -\frac{a_k(k+\frac{9}{2})}{2k+4}$

- Solution for $r = \frac{7}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+2} = -\frac{a_k(k+\frac{9}{2})}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{7}{2}}\right), a_{k+2} = -\frac{a_k(k+\frac{3}{2})}{2k-2}, a_1 = 0, b_{k+2} = -\frac{b_k(k+\frac{9}{2})}{2k+4}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)-x*(12-7*x^2)*diff(y(x),x)+(7+3*x^2)*y(x)=0,y(x),type='se
```

$$y(x) = \sqrt{x}\left(\left(1-\frac{9}{8}x^2+\frac{117}{128}x^4+\mathrm{O}\left(x^6\right)\right)x^3c_1+\left(12+9x^2-\frac{63}{4}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 58

```
AsymptoticDSolveValue[2*x^2*(2+x^2)*y''[x]-x*(12-7*x^2)*y'[x]+(7+3*x^2)*y[x]==0,y[x],{x,0,5}
```

$$y(x) \to c_1 \left(-\frac{21x^{9/2}}{16} + \frac{3x^{5/2}}{4} + \sqrt{x}\right) + c_2 \left(\frac{117x^{15/2}}{128} - \frac{9x^{11/2}}{8} + x^{7/2}\right)$$

14.38 problem 40

Internal problem ID [1329]
Internal file name [`OUTPUT/1330_Sunday_June_05_2022_02_10_36_AM_35752125/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^{2}\left(x^{2}+2\right)y''+x\left(7x^{2}+4\right)y'-\left(-3x^{2}+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^{4}+4x^{2}\right)y''+\left(7x^{3}+4x\right)y'+\left(3x^{2}-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^{2}+4}{2\left(x^{2}+2\right)x}$$
$$q(x)=\frac{3x^{2}-1}{2x^{2}\left(x^{2}+2\right)}$$

Table 604: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+4}{2(x^2+2)x}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = \frac{3x^2-1}{2x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i\sqrt{2}, i\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x^2+2)\, y'' + (7x^3+4x)\, y' + (3x^2-1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (7x^3+4x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x^2-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) (n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) (n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 7a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) (n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r) (n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 4x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) + 7a_{n-2}(n+r-2) + 4a_n(n+r) + 3a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-1)\, a_{n-2}}{2n+2r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{-2na_{n-2} + a_{n-2}}{4n+4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1-r}{5+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r + 3}{4r^2 + 28r + 45}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{7}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+28r+45}$	$\frac{7}{80}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+28r+45}$	$\frac{7}{80}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right)\\ &= \sqrt{x}\left(1 - \frac{x^2}{4} + \frac{7x^4}{80} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1\\ &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -\frac{1}{2}} 0\\ &= 0\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r}\\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-2}(n+r-2)(n-3+r)+4b_n(n+r)(n+r-1) \\ &+7b_{n-2}(n+r-2)+4b_n(n+r)+3b_{n-2}-b_n=0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$\begin{aligned} &2b_{n-2}\left(n-\frac{5}{2}\right)\left(n-\frac{7}{2}\right)+4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \\ &+7b_{n-2}\left(n-\frac{5}{2}\right)+4b_n\left(n-\frac{1}{2}\right)+3b_{n-2}-b_n=0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r-1)\, b_{n-2}}{2n+2r+1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{\left(n-\frac{3}{2}\right) b_{n-2}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1+r}{5+2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 4r + 3}{(5 + 2r)(9 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{5}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{8}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2+28r+45}$	$\frac{5}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-1-r}{5+2r}$	$-\frac{1}{8}$
b_3	0	0
b_4	$\frac{r^2+4r+3}{4r^2+28r+45}$	$\frac{5}{128}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{8} + \frac{5x^4}{128} + O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{4} + \frac{7x^4}{80} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{5x^4}{128} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{4} + \frac{7x^4}{80} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{5x^4}{128} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1 - \frac{x^2}{4} + \frac{7x^4}{80} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{8} + \frac{5x^4}{128} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x^2}{4}+\frac{7x^4}{80}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{8}+\frac{5x^4}{128}+O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

14.38.1 Maple step by step solution

Let's solve

$2x^2(x^2+2)\,y''+(7x^3+4x)\,y'+(3x^2-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2-1)y}{2x^2(x^2+2)}-\frac{(7x^2+4)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+4)y'}{2x(x^2+2)}+\frac{(3x^2-1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{2(x^2+2)x}, P_3(x)=\frac{3x^2-1}{2x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\,y''+x(7x^2+4)\,y'+(3x^2-1)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2}+a_k\left(k+r+\frac{1}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k+\frac{5}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+1)}{2k+5+2r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+x*(4+7*x^2)*diff(y(x),x)-(1-3*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{c_1 x\left(1 - \frac{1}{4}x^2 + \frac{7}{80}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - \frac{1}{8}x^2 + \frac{5}{128}x^4 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 58

```
AsymptoticDSolveValue[2*x^2*(2+x^2)*y''[x]+x*(4+7*x^2)*y'[x]-(1-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{5x^{7/2}}{128} - \frac{x^{3/2}}{8} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{7x^{9/2}}{80} - \frac{x^{5/2}}{4} + \sqrt{x}\right)$$

14.39 problem 41

Internal problem ID [1330]
Internal file name [`OUTPUT/1331_Sunday_June_05_2022_02_10_39_AM_77927004/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'-(-40x^2+2)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+2x^2)\,y''+(30x^3+5x)\,y'+(40x^2-2)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{15x^2+\frac{5}{2}}{x\,(2x^2+1)}$$
$$q(x)=\frac{20x^2-1}{x^2\,(2x^2+1)}$$

Table 606: Table $p(x), q(x)$ singularites.

$p(x) = \frac{15x^2+\frac{5}{2}}{x(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = \frac{20x^2-1}{x^2(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(2x^2+1)\,y'' + (30x^3+5x)\,y' + (40x^2-2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(2x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (30x^3+5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (40x^2-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 30x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 40x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)(n+r-1) &= \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\,x^{n+r} \\
\sum_{n=0}^{\infty} 30x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 30a_{n-2}(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty} 40x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 40a_{n-2}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 30a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right)+\left(\sum_{n=2}^{\infty} 40a_{n-2}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n(n+r)(n+r-1)+5x^{n+r} a_n(n+r)-2a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1 + r) + 5x^r a_0 r - 2a_0 x^r = 0$$

Or

$$(2x^r r(-1 + r) + 5x^r r - 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&4a_{n-2}(n+r-2)(n-3+r)+2a_n(n+r)(n+r-1) \\
&+30a_{n-2}(n+r-2)+5a_n(n+r)+40a_{n-2}-2a_n=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2(2n+2r+1)\, a_{n-2}}{2n+2r-1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_n = -\frac{2(n+1)\, a_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{-4r-10}{3+2r}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
a_2 = -3
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4r-10}{3+2r}$	-3

For $n = 3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4r-10}{3+2r}$	-3
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^2 + 112r + 180}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{15}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4r-10}{3+2r}$	-3
a_3	0	0
a_4	$\frac{16r^2+112r+180}{4r^2+20r+21}$	$\frac{15}{2}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4r-10}{3+2r}$	-3
a_3	0	0
a_4	$\frac{16r^2+112r+180}{4r^2+20r+21}$	$\frac{15}{2}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= \sqrt{x}\left(1 - 3x^2 + \frac{15x^4}{2} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&4b_{n-2}(n + r - 2)\,(n - 3 + r) + 2b_n(n + r)\,(n + r - 1) \\
&\quad + 30b_{n-2}(n + r - 2) + 5b_n(n + r) + 40b_{n-2} - 2b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2(2n + 2r + 1)\, b_{n-2}}{2n + 2r - 1} \tag{4}
$$

Which for the root $r = -2$ becomes

$$
b_n = \frac{(6 - 4n)\, b_{n-2}}{2n - 5} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{-4r - 10}{3 + 2r}
$$

Which for the root $r = -2$ becomes

$$
b_2 = 2
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4r-10}{3+2r}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4r-10}{3+2r}$	2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^2 + 112r + 180}{4r^2 + 20r + 21}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{20}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4r-10}{3+2r}$	2
b_3	0	0
b_4	$\frac{16r^2+112r+180}{4r^2+20r+21}$	$-\frac{20}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4r-10}{3+2r}$	2
b_3	0	0
b_4	$\frac{16r^2+112r+180}{4r^2+20r+21}$	$-\frac{20}{3}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + 2x^2 - \frac{20x^4}{3} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - 3x^2 + \frac{15x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 + 2x^2 - \frac{20x^4}{3} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - 3x^2 + \frac{15x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 + 2x^2 - \frac{20x^4}{3} + O(x^6)\right)}{x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1 - 3x^2 + \frac{15x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 + 2x^2 - \frac{20x^4}{3} + O(x^6)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-3x^2+\frac{15x^4}{2}+O\left(x^6\right)\right)+\frac{c_2\left(1+2x^2-\frac{20x^4}{3}+O(x^6)\right)}{x^2}$$

Verified OK.

14.39.1 Maple step by step solution

Let's solve

$2x^2(2x^2+1)\, y''+(30x^3+5x)\, y'+(40x^2-2)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(20x^2-1)y}{x^2(2x^2+1)}-\frac{5(6x^2+1)y'}{2x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5(6x^2+1)y'}{2x(2x^2+1)}+\frac{(20x^2-1)y}{x^2(2x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x)=\frac{20x^2-1}{x^2(2x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(2x^2+1)\, y''+5x(6x^2+1)\, y'+(40x^2-2)\, y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - ◦ Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+2r)\, x^r + a_1(3+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1) + 2a_{k-2}(k -$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-2, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(3+r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r+2)\left(a_{k-2}(2k+1+2r)+a_k\left(k+r-\frac{1}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$2(k+r+4)\left(a_k(2k+2r+5)+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2a_k(2k+2r+5)}{2k+3+2r}$$

- Recursion relation for $r=-2$

$$a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-2}, a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0, b_{k+2}=-\frac{2b_k(2k+6)}{2k+4}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
Order:=6;
dsolve(2*x^2*(1+2*x^2)*diff(y(x),x$2)+5*x*(1+6*x^2)*diff(y(x),x)-(2-40*x^2)*y(x)=0,y(x),type
```

$$y(x) = \frac{c_2 x^{\frac{5}{2}} \left(1 - 3x^2 + \frac{15}{2}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 + 2x^2 - \frac{20}{3}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 46

```
AsymptoticDSolveValue[2*x^2*(1+2*x^2)*y''[x]+5*x*(1+6*x^2)*y'[x]-(2-40*x^2)*y[x]==0,y[x],{x,
```

$$y(x) \to c_1\sqrt{x}\left(\frac{15x^4}{2} - 3x^2 + 1\right) + \frac{c_2\left(-\frac{20x^4}{3} + 2x^2 + 1\right)}{x^2}$$

14.40 problem 42

Internal problem ID [1331]
Internal file name [OUTPUT/1332_Sunday_June_05_2022_02_10_42_AM_62052714/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^{2}\left(x^{2}+1\right) y''+5x\left(x^{2}+1\right) y'-\left(-5x^{2}+1\right) y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^{4}+3x^{2}\right) y''+\left(5x^{3}+5x\right) y'+\left(5x^{2}-1\right) y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}
p(x) &= \frac{5}{3x}\\
q(x) &= \frac{5x^{2}-1}{3x^{2}\left(x^{2}+1\right)}
\end{aligned}$$

Table 608: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{5x^2-1}{3x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2(x^2+1)\,y'' + (5x^3+5x)\,y' + (5x^2-1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}
&3x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\
&+ (5x^3+5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (5x^2-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 5a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + 5x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&3a_{n-2}(n+r-2)\,(n-3+r)+3a_n(n+r)\,(n+r-1) \\
&+5a_{n-2}(n+r-2)+5a_n(n+r)+5a_{n-2}-a_n=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}(3n^2+6nr+3r^2-10n-10r+13)}{3n^2+6nr+3r^2+2n+2r-1} \tag{4}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_n = -\frac{a_{n-2}(3n^2-8n+10)}{3n^2+4n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{-3r^2-2r-5}{3r^2+14r+15}
$$

Which for the root $r = \frac{1}{3}$ becomes

$$
a_2 = -\frac{3}{10}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{10}$

For $n = 3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{9r^4 + 48r^3 + 106r^2 + 112r + 105}{9r^4 + 120r^3 + 574r^2 + 1160r + 825}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{39}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{10}$
a_3	0	0
a_4	$\frac{9r^4+48r^3+106r^2+112r+105}{9r^4+120r^3+574r^2+1160r+825}$	$\frac{39}{320}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{10}$
a_3	0	0
a_4	$\frac{9r^4+48r^3+106r^2+112r+105}{9r^4+120r^3+574r^2+1160r+825}$	$\frac{39}{320}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{3x^2}{10} + \frac{39x^4}{320} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&3b_{n-2}(n + r - 2)(n - 3 + r) + 3b_n(n + r)(n + r - 1) \\
&+ 5b_{n-2}(n + r - 2) + 5b_n(n + r) + 5b_{n-2} - b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}(3n^2 + 6nr + 3r^2 - 10n - 10r + 13)}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-2}(3n^2 - 16n + 26)}{3n^2 - 4n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{-3r^2 - 2r - 5}{3r^2 + 14r + 15}
$$

Which for the root $r = -1$ becomes

$$
b_2 = -\frac{3}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{9r^4 + 48r^3 + 106r^2 + 112r + 105}{9r^4 + 120r^3 + 574r^2 + 1160r + 825}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{15}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{2}$
b_3	0	0
b_4	$\frac{9r^4+48r^3+106r^2+112r+105}{9r^4+120r^3+574r^2+1160r+825}$	$\frac{15}{32}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3r^2-2r-5}{3r^2+14r+15}$	$-\frac{3}{2}$
b_3	0	0
b_4	$\frac{9r^4+48r^3+106r^2+112r+105}{9r^4+120r^3+574r^2+1160r+825}$	$\frac{15}{32}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^2}{2} + \frac{15x^4}{32} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{10} + \frac{39x^4}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{2} + \frac{15x^4}{32} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{10} + \frac{39x^4}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{2} + \frac{15x^4}{32} + O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{10} + \frac{39x^4}{320} + O(x^6)\right) + \frac{c_2\left(1 - \frac{3x^2}{2} + \frac{15x^4}{32} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{3x^2}{10} + \frac{39x^4}{320} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{3x^2}{2} + \frac{15x^4}{32} + O(x^6)\right)}{x}$$

Verified OK.

14.40.1 Maple step by step solution

Let's solve

$3x^2(x^2+1)\, y'' + (5x^3+5x)\, y' + (5x^2-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2-1)y}{3x^2(x^2+1)} - \frac{5y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{3x} + \frac{(5x^2-1)y}{3x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5}{3x}, P_3(x) = \frac{5x^2-1}{3x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3x^2(x^2+1)\, y'' + 5x(x^2+1)\, y' + (5x^2-1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+3r)\, x^r + a_1(2+r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + a_{k-2}(3(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(2+r)\,(2+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(3k^2+\left(6r-10\right)k+3r^2-10r+13\right)a_{k-2}+3(k+r+1)\left(k+r-\frac{1}{3}\right)a_k=0$$

- Shift index using $k->k+2$

$$\left(3(k+2)^2+\left(6r-10\right)\left(k+2\right)+3r^2-10r+13\right)a_k+3(k+3+r)\left(k+\frac{5}{3}+r\right)a_{k+2}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{\left(3k^2+6kr+3r^2+2k+2r+5\right)a_k}{\left(k+3+r\right)\left(3k+5+3r\right)}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{\left(3k^2-4k+6\right)a_k}{\left(k+2\right)\left(3k+2\right)}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-1},a_{k+2}=-\frac{\left(3k^2-4k+6\right)a_k}{\left(k+2\right)\left(3k+2\right)},a_1=0\right]$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+2}=-\frac{\left(3k^2+4k+6\right)a_k}{\left(k+\frac{10}{3}\right)\left(3k+6\right)}$$

- Solution for $r=\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}},a_{k+2}=-\frac{\left(3k^2+4k+6\right)a_k}{\left(k+\frac{10}{3}\right)\left(3k+6\right)},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right),a_{k+2}=-\frac{\left(3k^2-4k+6\right)a_k}{\left(k+2\right)\left(3k+2\right)},a_1=0,b_{k+2}=-\frac{\left(3k^2+4k+6\right)b_k}{\left(k+\frac{10}{3}\right)\left(3k+6\right)},b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(3*x^2*(1+x^2)*diff(y(x),x$2)+5*x*(1+x^2)*diff(y(x),x)-(1-5*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1 - \frac{3}{10} x^2 + \frac{39}{320} x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{3}{2} x^2 + \frac{15}{32} x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 50

```
AsymptoticDSolveValue[3*x^2*(1+x^2)*y''[x]+5*x*(1+x^2)*y'[x]-(1-5*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(\frac{39x^4}{320} - \frac{3x^2}{10} + 1\right) + \frac{c_2\left(\frac{15x^4}{32} - \frac{3x^2}{2} + 1\right)}{x}$$

14.41 problem 43

Internal problem ID [1332]
Internal file name [`OUTPUT/1333_Sunday_June_05_2022_02_10_44_AM_72635037/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+1)\,y''+(7x^2+4)\,y'+8yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3+x)\,y''+(7x^2+4)\,y'+8yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{7x^2+4}{x\,(x^2+1)}\\q(x)&=\frac{8}{x^2+1}\end{aligned}$$

Table 610: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+4}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{8}{x^2+1}$	
singularity	type
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2+1)\,y'' + (7x^2+4)\,y' + 8yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ (7x^2+4)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 7 x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n\right) \\
&=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1} \\
\sum_{n=0}^{\infty} 7 x^{1+n+r} a_n(n+r) &= \sum_{n=2}^{\infty} 7 a_{n-2}(n+r-2) x^{n+r-1} \\
\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n &= \sum_{n=2}^{\infty} 8 a_{n-2} x^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 7 a_{n-2}(n+r-2) x^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} 8 a_{n-2} x^{n+r-1}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r-1} a_n(n+r)(n+r-1)+4(n+r) a_n x^{n+r-1}=0
$$

When $n=0$ the above becomes

$$
x^{-1+r} a_0 r(-1+r)+4 r a_0 x^{-1+r}=0
$$

Or

$$\left(x^{-1+r} r(-1+r)+4 r\, x^{-1+r}\right) a_{0}=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(3+r)=0$$

Solving for r gives the roots of the indicial equation as

$$r_{1}=0$$
$$r_{2}=-3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3+r)=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_{1}(x)=x^{r_{1}}\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+x^{r_{2}}\left(\sum_{n=0}^{\infty} b_{n} x^{n}\right)$$

Or

$$y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+\frac{\sum_{n=0}^{\infty} b_{n} x^{n}}{x^{3}}$$

Or

$$y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n}$$
$$y_{2}(x)=C y_{1}(x) \ln (x)+\left(\sum_{n=0}^{\infty} b_{n} x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ + 7a_{n-2}(n+r-2) + 4a_n(n+r) + 8a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2+n+r)\,a_{n-2}}{n+3+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{(2+n)\,a_{n-2}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4-r}{5+r}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{4}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{5+r}$	$-\frac{4}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{5+r}$	$-\frac{4}{5}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 10r + 24}{(5 + r)(7 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{24}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{5+r}$	$-\frac{4}{5}$
a_3	0	0
a_4	$\frac{r^2+10r+24}{(5+r)(7+r)}$	$\frac{24}{35}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{5+r}$	$-\frac{4}{5}$
a_3	0	0
a_4	$\frac{r^2+10r+24}{(5+r)(7+r)}$	$\frac{24}{35}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{4x^2}{5} + \frac{24x^4}{35} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -3} 0 \\
&= 0
\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-3}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-2}(n+r-2)\,(n-3+r) + b_n(n+r)\,(n+r-1) \\ + 7b_{n-2}(n+r-2) + 4(n+r)\,b_n + 8b_{n-2} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_{n-2}(n-5)\,(n-6) + b_n(n-3)\,(n-4) + 7b_{n-2}(n-5) + 4(n-3)\,b_n + 8b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(2+n+r)\,b_{n-2}}{n+3+r} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{(-1+n)\,b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4+r}{5+r}$$

Which for the root $r = -3$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{5+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{5+r}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 10r + 24}{(5 + r)(7 + r)}$$

Which for the root $r = -3$ becomes

$$b_4 = \frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{5+r}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2+10r+24}{(5+r)(7+r)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{5+r}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2+10r+24}{(5+r)(7+r)}$	$\frac{3}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{4x^2}{5} + \frac{24x^4}{35} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^3} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - \frac{4x^2}{5} + \frac{24x^4}{35} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^3} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1 - \frac{4x^2}{5} + \frac{24x^4}{35} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^3} \tag{1}$$

Verification of solutions

$$y = c_1\left(1 - \frac{4x^2}{5} + \frac{24x^4}{35} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^3}$$

Verified OK.

14.41.1 Maple step by step solution

Let's solve

$x(x^2+1)\,y''+(7x^2+4)\,y'+8yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{x^2+1}-\frac{(7x^2+4)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+4)y'}{x(x^2+1)}+\frac{8y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{x(x^2+1)}, P_3(x)=\frac{8}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=4$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+1)\,y''+(7x^2+4)\,y'+8yx=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^{m} \cdot y'$ to series expansion for $m=0..2$

  $x^{m} \cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m} \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$

- Convert $x^{m} \cdot y''$ to series expansion for $m=1..3$

  $x^{m} \cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m} \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0} r(3+r) x^{-1+r}+a_{1}(1+r)(4+r) x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)(k+r+4)+a_{k-1}(k+r+3)(k\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in\{-3,0\}$

- Each term must be 0

  $a_{1}(1+r)(4+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(a_{k+1}(k+r+4)+a_{k-1}(k+r+3)\right)=0$

- Shift index using $k->k+1$

  $(k+r+2)\left(a_{k+2}(k+5+r)+a_{k}(k+r+4)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r+4)}{k+5+r}$

- Recursion relation for $r=-3$

  $a_{k+2}=-\frac{a_{k}(k+1)}{k+2}$

- Solution for $r = -3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k(k+4)}{k+5}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k(1+k)}{k+2}, -2a_1 = 0, b_{k+2} = -\frac{b_k(4+k)}{5+k}, 4b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*(1+x^2)*diff(y(x),x$2)+(4+7*x^2)*diff(y(x),x)+8*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{4}{5}x^2 + \frac{24}{35}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 - 6x^2 + \frac{9}{2}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*(1+x^2)*y''[x]+(4+7*x^2)*y'[x]+8*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{x^3} + \frac{3x}{8} - \frac{1}{2x} \right) + c_2 \left(\frac{24x^4}{35} - \frac{4x^2}{5} + 1 \right)$$

14.42 problem 44

Internal problem ID [1333]
Internal file name [`OUTPUT/1334_Sunday_June_05_2022_02_10_47_AM_37706204/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+2)\,y''+x(x^2+3)\,y'-y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+2x^2)\,y''+(x^3+3x)\,y'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+3}{(x^2+2)\,x}$$
$$q(x)=-\frac{1}{x^2\,(x^2+2)}$$

Table 612: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+3}{(x^2+2)x}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = -\frac{1}{x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -i\sqrt{2}, i\sqrt{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+2)\,y'' + (x^3+3x)\,y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\ + (x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r} \\
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} a_{n-2} (n+r-2) x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) - a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) + 3x^r a_0 r - a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) + 3x^r r - x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 + r - 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} & a_{n-2}(n + r - 2)\,(n - 3 + r) + 2a_n(n + r)\,(n + r - 1) \\ & + a_{n-2}(n + r - 2) + 3a_n(n + r) - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}(-3+2n)^2}{8n^2+12n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r^2}{2r^2+9r+9}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{56}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 (r+2)^2}{4r^4 + 52r^3 + 241r^2 + 468r + 315}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{25}{9856}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{56}$
a_3	0	0
a_4	$\frac{r^2(r+2)^2}{4r^4+52r^3+241r^2+468r+315}$	$\frac{25}{9856}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{56}$
a_3	0	0
a_4	$\frac{r^2(r+2)^2}{4r^4+52r^3+241r^2+468r+315}$	$\frac{25}{9856}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{56} + \frac{25x^4}{9856} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} & b_{n-2}(n+r-2)(n-3+r) + 2b_n(n+r)(n+r-1) \\ & + b_{n-2}(n+r-2) + 3b_n(n+r) - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{2n^2 + 4nr + 2r^2 + n + r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-2}(n-3)^2}{2n^2 - 3n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r^2}{2r^2 + 9r + 9}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2(r+2)^2}{4r^4 + 52r^3 + 241r^2 + 468r + 315}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2(r+2)^2}{4r^4+52r^3+241r^2+468r+315}$	$\frac{1}{40}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r^2}{2r^2+9r+9}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{r^2(r+2)^2}{4r^4+52r^3+241r^2+468r+315}$	$\frac{1}{40}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= \sqrt{x}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right)\\ &= \frac{1-\frac{x^2}{2}+\frac{x^4}{40}+O(x^6)}{x}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x)+c_2y_2(x)\\ &= c_1\sqrt{x}\left(1-\frac{x^2}{56}+\frac{25x^4}{9856}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}+O(x^6)\right)}{x}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1\sqrt{x}\left(1-\frac{x^2}{56}+\frac{25x^4}{9856}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}+O(x^6)\right)}{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{x^2}{56}+\frac{25x^4}{9856}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x^2}{56}+\frac{25x^4}{9856}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{40}+O(x^6)\right)}{x}$$

Verified OK.

14.42.1 Maple step by step solution

Let's solve

$x^2(x^2+2)\,y''+(x^3+3x)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{y}{x^2(x^2+2)}-\frac{(x^2+3)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(x^2+3)y'}{x(x^2+2)}-\frac{y}{x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x^2+3}{(x^2+2)x}, P_3(x)=-\frac{1}{x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{2}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(x^2+2)\,y''+x(x^2+3)\,y'-y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+2r)\, x^r + a_1(2+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-2}(k-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(2+r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(2k+2r-1) + a_{k-2}(k-2+r)^2 = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+3+r)\,(2k+3+2r) + a_k(k+r)^2 = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)^2}{(k+3+r)(2k+3+2r)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{a_k(k-1)^2}{(k+2)(2k+1)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k-1)^2}{(k+2)(2k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k+\frac{7}{2}\right)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k+\frac{7}{2}\right)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k (k-1)^2}{(k+2)(2k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k \left(k+\frac{1}{2}\right)^2}{\left(k+\frac{7}{2}\right)(2k+4)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(2+x^2)*diff(y(x),x$2)+x*(3+x^2)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}} \left(1 - \frac{1}{56}x^2 + \frac{25}{9856}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x^2*(2+x^2)*y''[x]+x*(3+x^2)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt{x} \left(\frac{25x^4}{9856} - \frac{x^2}{56} + 1\right) + \frac{c_2 \left(\frac{x^4}{40} - \frac{x^2}{2} + 1\right)}{x}$$

14.43 problem 45

Internal problem ID [1334]
Internal file name [OUTPUT/1335_Sunday_June_05_2022_02_10_50_AM_29886199/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+2x^2)\,y''+(8x^3+3x)\,y'+(4x^2-3)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{8x^2+3}{2x\,(x^2+1)}$$
$$q(x)=\frac{4x^2-3}{2x^2\,(x^2+1)}$$

Table 614: Table $p(x), q(x)$ singularites.

$p(x) = \frac{8x^2+3}{2x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{4x^2-3}{2x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x^2+1)\,y'' + (8x^3+3x)\,y' + (4x^2-3)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ (8x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (4x^2-3)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-3a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 8a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 8a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right)+\left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-3a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) - 3a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1 + r) + 3x^r a_0 r - 3a_0 x^r = 0$$

Or

$$\left(2x^r r(-1 + r) + 3x^r r - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_{n-2}(n+r-2)(n-3+r)+2a_n(n+r)(n+r-1) \\ +8a_{n-2}(n+r-2)+3a_n(n+r)+4a_{n-2}-3a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2(n+r)\,a_{n-2}}{2n+2r+3} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{2(n+1)\,a_{n-2}}{2n+5} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4-2r}{7+2r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-2r}{7+2r}$	$-\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-2r}{7+2r}$	$-\frac{2}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 24r + 32}{4r^2 + 36r + 77}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{20}{39}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-2r}{7+2r}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{4r^2+24r+32}{4r^2+36r+77}$	$\frac{20}{39}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-2r}{7+2r}$	$-\frac{2}{3}$
a_3	0	0
a_4	$\frac{4r^2+24r+32}{4r^2+36r+77}$	$\frac{20}{39}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{2x^2}{3} + \frac{20x^4}{39} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{gathered} 2b_{n-2}(n+r-2)(n-3+r) + 2b_n(n+r)(n+r-1) \\ + 8b_{n-2}(n+r-2) + 3b_n(n+r) + 4b_{n-2} - 3b_n = 0 \end{gathered} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2(n+r)\, b_{n-2}}{2n+2r+3} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{(2n-3)\, b_{n-2}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-4-2r}{7+2r}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = -\frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-2r}{7+2r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-2r}{7+2r}$	$-\frac{1}{4}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 24r + 32}{4r^2 + 36r + 77}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = \frac{5}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-2r}{7+2r}$	$-\frac{1}{4}$
b_3	0	0
b_4	$\frac{4r^2+24r+32}{4r^2+36r+77}$	$\frac{5}{32}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-2r}{7+2r}$	$-\frac{1}{4}$
b_3	0	0
b_4	$\frac{4r^2+24r+32}{4r^2+36r+77}$	$\frac{5}{32}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{4} + \frac{5x^4}{32} + O(x^6)}{x^{\frac{3}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x \left(1 - \frac{2x^2}{3} + \frac{20x^4}{39} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{4} + \frac{5x^4}{32} + O(x^6)\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x \left(1 - \frac{2x^2}{3} + \frac{20x^4}{39} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{4} + \frac{5x^4}{32} + O(x^6)\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1 - \frac{2x^2}{3} + \frac{20x^4}{39} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{4} + \frac{5x^4}{32} + O(x^6)\right)}{x^{\frac{3}{2}}} \quad (1)$$

Verification of solutions

$$y = c_1 x\left(1 - \frac{2x^2}{3} + \frac{20x^4}{39} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{4} + \frac{5x^4}{32} + O(x^6)\right)}{x^{\frac{3}{2}}}$$

Verified OK.

14.43.1 Maple step by step solution

Let's solve

$2x^2(x^2+1)\,y'' + (8x^3+3x)\,y' + (4x^2-3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-3)y}{2x^2(x^2+1)} - \frac{(8x^2+3)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(8x^2+3)y'}{2x(x^2+1)} + \frac{(4x^2-3)y}{2x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{8x^2+3}{2x(x^2+1)}, P_3(x) = \frac{4x^2-3}{2x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{3}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+1)\,y'' + x(8x^2+3)\,y' + (4x^2-3)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2r+3)\,(-1+r)\, x^r + a_1(5+2r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)\,(k+r-1) + 2a_{k-2}(k+r)\,(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2r+3)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(5+2r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{3}{2}\right) a_{k}+a_{k-2}(k+r)\right) (k+r-1)=0$$

- Shift index using $k->k+2$

$$2\left(\left(k+\frac{7}{2}+r\right) a_{k+2}+a_{k}(k+r+2)\right) (k+r+1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2a_{k}(k+r+2)}{2k+7+2r}$$

- Recursion relation for $r=1$

$$a_{k+2}=-\frac{2a_{k}(k+3)}{2k+9}$$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+2}=-\frac{2a_{k}(k+3)}{2k+9}, a_{1}=0\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+2}=-\frac{2a_{k}\left(k+\frac{1}{2}\right)}{2k+4}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{3}{2}}, a_{k+2}=-\frac{2a_{k}\left(k+\frac{1}{2}\right)}{2k+4}, a_{1}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{1+k}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{2a_{k}(k+3)}{2k+9}, a_{1}=0, b_{k+2}=-\frac{2b_{k}\left(k+\frac{1}{2}\right)}{2k+4}, b_{1}=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
Order:=6;
dsolve(2*x^2*(1+x^2)*diff(y(x),x$2)+x*(3+8*x^2)*diff(y(x),x)-(3-4*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{c_1\left(1-\frac{1}{4}x^2+\frac{5}{32}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}+c_2x\left(1-\frac{2}{3}x^2+\frac{20}{39}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*(1+x^2)*y''[x]+x*(3+8*x^2)*y'[x]-(3-4*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{20x^4}{39} - \frac{2x^2}{3} + 1\right) + \frac{c_2 \left(\frac{5x^4}{32} - \frac{x^2}{4} + 1\right)}{x^{3/2}}$$

14.44 problem 46

Internal problem ID [1335]
Internal file name [OUTPUT/1336_Sunday_June_05_2022_02_10_53_AM_36193417/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 3x(x^2+3)\,y' - (-5x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + (3x^3+9x)\,y' + (5x^2-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2+3}{3x}$$
$$q(x) = \frac{5x^2-1}{9x^2}$$

Table 616: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+3}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{5x^2-1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (3x^3 + 9x)\, y' + (5x^2 - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (3x^3 + 9x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (5x^2 - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{3} \\ r_2 &= -\frac{1}{3} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(9r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9a_n(n + r)(n + r - 1) + 3a_{n-2}(n + r - 2) + 9a_n(n + r) + 5a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{3n + 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{a_{n-2}}{3n+2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{7+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{7+3r}$	$-\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{7+3r}$	$-\frac{1}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{9r^2 + 60r + 91}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{112}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{7+3r}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{9r^2+60r+91}$	$\frac{1}{112}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{7+3r}$	$-\frac{1}{8}$
a_3	0	0
a_4	$\frac{1}{9r^2+60r+91}$	$\frac{1}{112}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x^2}{8} + \frac{x^4}{112} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 3b_{n-2}(n+r-2) + 9b_n(n+r) + 5b_{n-2} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{3n+3r+1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = -\frac{b_{n-2}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{7+3r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{7+3r}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{7+3r}$	$-\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{9r^2 + 60r + 91}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{1}{72}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{7+3r}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{9r^2+60r+91}$	$\frac{1}{72}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{7+3r}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{9r^2+60r+91}$	$\frac{1}{72}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{6} + \frac{x^4}{72} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{8} + \frac{x^4}{112} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{6} + \frac{x^4}{72} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{8} + \frac{x^4}{112} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{6} + \frac{x^4}{72} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{8} + \frac{x^4}{112} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{6} + \frac{x^4}{72} + O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{8} + \frac{x^4}{112} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{6} + \frac{x^4}{72} + O(x^6)\right)}{x^{\frac{1}{3}}}
$$

Verified OK.

14.44.1 Maple step by step solution

Let's solve

$9x^2y'' + (3x^3 + 9x)\,y' + (5x^2 - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+3)y'}{3x} - \frac{(5x^2-1)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+3)y'}{3x} + \frac{(5x^2-1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{9}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + 3x(x^2 + 3)\,y' + (5x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+3r)\,(-1+3r)\, x^{r}+a_{1}(4+3r)\,(2+3r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(3k+3r+1)\,(3k+3r-1)+a_{k-2}\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_{1}(4+3r)\,(2+3r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)\,(3a_{k}k+3a_{k}r+a_{k}+a_{k-2})=0$

- Shift index using $k->k+2$

  $(3k+3r+5)\,(3a_{k+2}(k+2)+3a_{k+2}r+a_{k+2}+a_{k})=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}}{3k+7+3r}$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+2}=-\frac{a_{k}}{3k+6}$

- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k}{3k+8}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{b_k}{3k+8}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+3*x*(3+x^2)*diff(y(x),x)-(1-5*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 - \frac{1}{8}x^2 + \frac{1}{112}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{1}{6}x^2 + \frac{1}{72}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[9*x^2*y''[x]+3*x*(3+x^2)*y'[x]-(1-5*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(\frac{x^4}{112}-\frac{x^2}{8}+1\right)+\frac{c_2\left(\frac{x^4}{72}-\frac{x^2}{6}+1\right)}{\sqrt[3]{x}}$$

14.45 problem 47

Internal problem ID [1336]
Internal file name [`OUTPUT/1337_Sunday_June_05_2022_02_10_55_AM_9283402/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2y'' + x(6x^2+1)\,y' + (9x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$6x^2y'' + (6x^3+x)\,y' + (9x^2+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{6x^2+1}{6x}$$
$$q(x) = \frac{9x^2+1}{6x^2}$$

Table 618: Table $p(x), q(x)$ singularites.

$p(x) = \frac{6x^2+1}{6x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{9x^2+1}{6x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$6x^2y'' + (6x^3 + x)\,y' + (9x^2 + 1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$6x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (6x^3 + x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (9x^2 + 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 6x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 6a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 9x^{n+r+2} a_n = \sum_{n=2}^{\infty} 9a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 6a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 9a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$6x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$(6x^r r(-1+r) + x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(6r^2 - 5r + 1)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(6r^2 - 5r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$6a_n(n+r)(n+r-1) + 6a_{n-2}(n+r-2) + a_n(n+r) + 9a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-2}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{6a_{n-2}}{6n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{3}{5+3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{6}{13}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{3}{5+3r}$	$-\frac{6}{13}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{3}{5+3r}$	$-\frac{6}{13}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{9}{9r^2 + 48r + 55}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{36}{325}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{3}{5+3r}$	$-\frac{6}{13}$
a_3	0	0
a_4	$\frac{9}{9r^2+48r+55}$	$\frac{36}{325}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{3}{5+3r}$	$-\frac{6}{13}$
a_3	0	0
a_4	$\frac{9}{9r^2+48r+55}$	$\frac{36}{325}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{6x^2}{13} + \frac{36x^4}{325} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$6b_n(n+r)(n+r-1) + 6b_{n-2}(n+r-2) + b_n(n+r) + 9b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-2}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -\frac{b_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{3}{5+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{3}{5+3r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{3}{5+3r}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{9}{9r^2 + 48r + 55}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{3}{5+3r}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{9}{9r^2+48r+55}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{3}{5+3r}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{9}{9r^2+48r+55}$	$\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O\left(x^6\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{6x^2}{13} + \frac{36x^4}{325} + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - \frac{6x^2}{13} + \frac{36x^4}{325} + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O\left(x^6\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1 - \frac{6x^2}{13} + \frac{36x^4}{325} + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1 - \frac{6x^2}{13} + \frac{36x^4}{325} + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O\left(x^6\right)\right)$$

Verified OK.

14.45.1 Maple step by step solution

Let's solve

$6x^2y'' + (6x^3 + x)\,y' + (9x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+1)y}{6x^2} - \frac{(6x^2+1)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(6x^2+1)y'}{6x} + \frac{(9x^2+1)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{6}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{6}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $6x^2y'' + x(6x^2 + 1)\,y' + (9x^2 + 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(-1+3r)\,(-1+2r)\, x^{r}+a_{1}(2+3r)\,(1+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(3k+3r-1)\,(2k+2r-1)+3a$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)\,(-1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term must be 0

  $$a_{1}(2+3r)\,(1+2r)=0$$

- Solve for the dependent coefficient(s)

  $$a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$6\left(k+r-\frac{1}{2}\right)\left(\left(k+r-\frac{1}{3}\right) a_{k}+a_{k-2}\right)=0$$

- Shift index using $k->k+2$

  $$6\left(k+\frac{3}{2}+r\right)\left(\left(k+\frac{5}{3}+r\right) a_{k+2}+a_{k}\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{3a_{k}}{3k+5+3r}$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+2}=-\frac{3a_{k}}{3k+\frac{13}{2}}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+2} = -\frac{3a_k}{3k+6}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(6*x^2*diff(y(x),x$2)+x*(1+6*x^2)*diff(y(x),x)+(1+9*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\sqrt{x}\left(1 - \frac{6}{13}x^2 + \frac{36}{325}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[6*x^2*y''[x]+x*(1+6*x^2)*y'[x]+(1+9*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{36x^4}{325} - \frac{6x^2}{13} + 1\right) + c_2\sqrt[3]{x}\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

14.46 problem 48

Internal problem ID [1337]
Internal file name [OUTPUT/1338_Sunday_June_05_2022_02_10_58_AM_50909819/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(x^2+8)\,y''+7x(x^2+2)\,y'-(-9x^2+2)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+8x^2)\,y''+(7x^3+14x)\,y'+(9x^2-2)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2+14}{x\,(x^2+8)}$$
$$q(x)=\frac{9x^2-2}{x^2\,(x^2+8)}$$

Table 620: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+14}{x(x^2+8)}$	
singularity	type
$x = 0$	"regular"
$x = -2i\sqrt{2}$	"regular"
$x = 2i\sqrt{2}$	"regular"

$q(x) = \frac{9x^2-2}{x^2(x^2+8)}$	
singularity	type
$x = 0$	"regular"
$x = -2i\sqrt{2}$	"regular"
$x = 2i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -2i\sqrt{2}, 2i\sqrt{2}, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+8)\,y'' + (7x^3+14x)\,y' + (9x^2-2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+8)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (7x^3+14x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (9x^2-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 8 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7 x^{n+r+2} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 14 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9 x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)=\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 7 x^{n+r+2} a_n (n+r)=\sum_{n=2}^{\infty} 7 a_{n-2}(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 9 x^{n+r+2} a_n=\sum_{n=2}^{\infty} 9 a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 8 x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7 a_{n-2}(n+r-2) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 14 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 9 a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8 x^{n+r} a_n (n+r)(n+r-1)+14 x^{n+r} a_n (n+r)-2 a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$8x^r a_0 r(-1+r) + 14x^r a_0 r - 2a_0 x^r = 0$$

Or

$$(8x^r r(-1+r) + 14x^r r - 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 + 6r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 + 6r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 + 6r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r)+8a_n(n+r)(n+r-1) \\ +7a_{n-2}(n+r-2)+14a_n(n+r)+9a_{n-2}-2a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(1+n+r)\,a_{n-2}}{2\,(4n+4r-1)} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{(5+4n)\,a_{n-2}}{32n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-3-r}{14+8r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -\frac{13}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-r}{14+8r}$	$-\frac{13}{64}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-r}{14+8r}$	$-\frac{13}{64}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 8r + 15}{64r^2 + 352r + 420}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{273}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-r}{14+8r}$	$-\frac{13}{64}$
a_3	0	0
a_4	$\frac{r^2+8r+15}{64r^2+352r+420}$	$\frac{273}{8192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-r}{14+8r}$	$-\frac{13}{64}$
a_3	0	0
a_4	$\frac{r^2+8r+15}{64r^2+352r+420}$	$\frac{273}{8192}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{4}}\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6\ldots\right) \\ &= x^{\frac{1}{4}}\left(1-\frac{13x^2}{64}+\frac{273x^4}{8192}+O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0=1$. Substituting $n=1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-2}(n+r-2)\,(n-3+r)+8b_n(n+r)\,(n+r-1) \\ &+7b_{n-2}(n+r-2)+14b_n(n+r)+9b_{n-2}-2b_n=0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(1+n+r)\,b_{n-2}}{2\,(4n+4r-1)} \tag{4}$$

Which for the root $r=-1$ becomes

$$b_n = -\frac{nb_{n-2}}{8n-10} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r=-1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n=2$, using the above recursive equation gives

$$b_2 = \frac{-3-r}{14+8r}$$

Which for the root $r=-1$ becomes

$$b_2 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3-r}{14+8r}$	$-\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3-r}{14+8r}$	$-\frac{1}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^2 + 8r + 15}{64r^2 + 352r + 420}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{2}{33}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3-r}{14+8r}$	$-\frac{1}{3}$
b_3	0	0
b_4	$\frac{r^2+8r+15}{64r^2+352r+420}$	$\frac{2}{33}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-3-r}{14+8r}$	$-\frac{1}{3}$
b_3	0	0
b_4	$\frac{r^2+8r+15}{64r^2+352r+420}$	$\frac{2}{33}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{3} + \frac{2x^4}{33} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{4}}\left(1 - \frac{13x^2}{64} + \frac{273x^4}{8192} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{2x^4}{33} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{4}}\left(1 - \frac{13x^2}{64} + \frac{273x^4}{8192} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{2x^4}{33} + O(x^6)\right)}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(1 - \frac{13x^2}{64} + \frac{273x^4}{8192} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{3} + \frac{2x^4}{33} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 - \frac{13x^2}{64} + \frac{273x^4}{8192} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{3} + \frac{2x^4}{33} + O(x^6)\right)}{x}$$

Verified OK.

14.46.1 Maple step by step solution

Let's solve

$x^2(x^2+8)\, y'' + (7x^3 + 14x)\, y' + (9x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-2)y}{x^2(x^2+8)} - \frac{7(x^2+2)y'}{x(x^2+8)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7(x^2+2)y'}{x(x^2+8)} + \frac{(9x^2-2)y}{x^2(x^2+8)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{7(x^2+2)}{x(x^2+8)}, P_3(x) = \frac{9x^2-2}{x^2(x^2+8)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{7}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+8)\, y'' + 7x(x^2+2)\, y' + (9x^2-2)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0(1+r)\,(-1+4r)\, x^r + 2a_1(2+r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+1)\,(4k+4r-1) + a_{k-2}(\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2(1+r)\,(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{4}\right\}$$

- Each term must be 0

$$2a_1(2+r)\,(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(\frac{a_{k-2}(k+r+1)}{8}+a_k\left(k+r-\frac{1}{4}\right)\right)(k+r+1)=0$$

- Shift index using $k->k+2$

$$8\left(\frac{a_k(k+r+3)}{8}+a_{k+2}\left(k+\frac{7}{4}+r\right)\right)(k+r+3)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+3)}{2(4k+7+4r)}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{a_k(k+2)}{2(4k+3)}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+2)}{2(4k+3)}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{13}{4}\right)}{2(4k+8)}$$

- Solution for $r=\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{a_k\left(k+\frac{13}{4}\right)}{2(4k+8)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{a_k(k+2)}{2(4k+3)}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{13}{4}\right)}{2(4k+8)}, b_1=0\right]$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(8+x^2)*diff(y(x),x$2)+7*x*(2+x^2)*diff(y(x),x)-(2-9*x^2)*y(x)=0,y(x),type='serie
```

$$y(x)=\frac{c_2 x^{\frac{5}{4}}\left(1-\frac{13}{64}x^2+\frac{273}{8192}x^4+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-\frac{1}{3}x^2+\frac{2}{33}x^4+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x^2*(8+x^2)*y''[x]+7*x*(2+x^2)*y'[x]-(2-9*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[4]{x}\left(\frac{273x^4}{8192} - \frac{13x^2}{64} + 1\right) + \frac{c_2\left(\frac{2x^4}{33} - \frac{x^2}{3} + 1\right)}{x}$$

14.47 problem 49

Internal problem ID [1338]
Internal file name [OUTPUT/1339_Sunday_June_05_2022_02_11_52_AM_625352/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(9x^4+9x^2)\,y''+(39x^3+9x)\,y'+(25x^2-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{13x^2+3}{3x\,(x^2+1)}\\ q(x)&=\frac{25x^2-1}{9x^2\,(x^2+1)}\end{aligned}$$

Table 622: Table $p(x), q(x)$ singularites.

$p(x) = \frac{13x^2+3}{3x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{25x^2-1}{9x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2(x^2+1)\, y'' + (39x^3+9x)\, y' + (25x^2-1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (39x^3+9x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (25x^2-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 39x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 25x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} 9a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 39x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 39a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 25x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 9a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 39a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
9x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) - a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9a_{n-2}(n+r-2)(n-3+r)+9a_n(n+r)(n+r-1) \\ +39a_{n-2}(n+r-2)+9a_n(n+r)+25a_{n-2}-a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(-1+3n+3r)\, a_{n-2}}{3n+3r+1} \tag{4}$$

Which for the root $r=\frac{1}{3}$ becomes

$$a_n = -\frac{3n a_{n-2}}{3n+2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2 = \frac{-5-3r}{7+3r}$$

Which for the root $r=\frac{1}{3}$ becomes

$$a_2 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-3r}{7+3r}$	$-\frac{3}{4}$

For $n=3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-3r}{7+3r}$	$-\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{9r^2 + 48r + 55}{9r^2 + 60r + 91}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{9}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-3r}{7+3r}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{9r^2+48r+55}{9r^2+60r+91}$	$\frac{9}{14}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-3r}{7+3r}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{9r^2+48r+55}{9r^2+60r+91}$	$\frac{9}{14}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{3x^2}{4} + \frac{9x^4}{14} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&9b_{n-2}(n+r-2)(n-3+r) + 9b_n(n+r)(n+r-1) \\
&+ 39b_{n-2}(n+r-2) + 9b_n(n+r) + 25b_{n-2} - b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{(-1+3n+3r)\, b_{n-2}}{3n+3r+1} \tag{4}
$$

Which for the root $r = -\frac{1}{3}$ becomes

$$
b_n = -\frac{(-2+3n)\, b_{n-2}}{3n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{-5-3r}{7+3r}
$$

Which for the root $r = -\frac{1}{3}$ becomes

$$
b_2 = -\frac{2}{3}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-3r}{7+3r}$	$-\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-3r}{7+3r}$	$-\frac{2}{3}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{9r^2 + 48r + 55}{9r^2 + 60r + 91}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{5}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-3r}{7+3r}$	$-\frac{2}{3}$
b_3	0	0
b_4	$\frac{9r^2+48r+55}{9r^2+60r+91}$	$\frac{5}{9}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-3r}{7+3r}$	$-\frac{2}{3}$
b_3	0	0
b_4	$\frac{9r^2+48r+55}{9r^2+60r+91}$	$\frac{5}{9}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{2x^2}{3} + \frac{5x^4}{9} + O(x^6)}{x^{\frac{1}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{4} + \frac{9x^4}{14} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{3} + \frac{5x^4}{9} + O(x^6)\right)}{x^{\frac{1}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{4} + \frac{9x^4}{14} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{3} + \frac{5x^4}{9} + O(x^6)\right)}{x^{\frac{1}{3}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{4} + \frac{9x^4}{14} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{3} + \frac{5x^4}{9} + O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{3x^2}{4} + \frac{9x^4}{14} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{2x^2}{3} + \frac{5x^4}{9} + O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

14.47.1 Maple step by step solution

Let's solve

$9x^2(x^2+1)\,y'' + (39x^3+9x)\,y' + (25x^2-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2-1)y}{9x^2(x^2+1)} - \frac{(13x^2+3)y'}{3x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(13x^2+3)y'}{3x(x^2+1)} + \frac{(25x^2-1)y}{9x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{13x^2+3}{3x(x^2+1)}, P_3(x) = \frac{25x^2-1}{9x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{9}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2(x^2+1)\,y'' + 3x(13x^2+3)\,y' + (25x^2-1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(4+3r)\,(2+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(\left(k+r-\frac{1}{3}\right) a_{k-2}+a_k\left(k+r+\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right)=0$$

- Shift index using $k->k+2$

$$9\left(\left(k+\frac{5}{3}+r\right) a_k+a_{k+2}\left(k+\frac{7}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(3k+3r+5)a_k}{3k+7+3r}$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+2}=-\frac{(3k+4)a_k}{3k+6}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{(3k+4)a_k}{3k+6}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+2}=-\frac{(3k+6)a_k}{3k+8}$$

- Solution for $r=\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2}=-\frac{(3k+6)a_k}{3k+8}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-\frac{(3k+4)a_k}{3k+6}, a_1=0, b_{k+2}=-\frac{(3k+6)b_k}{3k+8}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(9*x^2*(1+x^2)*diff(y(x),x$2)+3*x*(3+13*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x),type=
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 - \frac{3}{4} x^2 + \frac{9}{14} x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{2}{3} x^2 + \frac{5}{9} x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 52

```
AsymptoticDSolveValue[9*x^2*(1+x^2)*y''[x]+3*x*(3+13*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],{x,0
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{9x^4}{14} - \frac{3x^2}{4} + 1 \right) + \frac{c_2 \left(\frac{5x^4}{9} - \frac{2x^2}{3} + 1 \right)}{\sqrt[3]{x}}$$

14.48 problem 50

Internal problem ID [1339]
Internal file name [OUTPUT/1340_Sunday_June_05_2022_02_11_55_AM_4748891/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+4x^2)\,y''+(24x^3+4x)\,y'+(25x^2-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{6x^2+1}{x\,(x^2+1)}$$
$$q(x)=\frac{25x^2-1}{4x^2\,(x^2+1)}$$

Table 624: Table $p(x), q(x)$ singularites.

$p(x) = \frac{6x^2+1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{25x^2-1}{4x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x^2+1)\, y'' + (24x^3+4x)\, y' + (25x^2-1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & 4x^2(x^2+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ & + (24x^3+4x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (25x^2-1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 24x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 25x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 24x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 24a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 25x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 24a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2-1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 4a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) \\ + 24a_{n-2}(n+r-2) + 4a_n(n+r) + 25a_{n-2} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(1+2n+2r)\, a_{n-2}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(1+n)\, a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-5-2r}{3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-2r}{3+2r}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-2r}{3+2r}$	$-\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 28r + 45}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{15}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-2r}{3+2r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{4r^2+28r+45}{4r^2+20r+21}$	$\frac{15}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-2r}{3+2r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{4r^2+28r+45}{4r^2+20r+21}$	$\frac{15}{8}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O\left(x^6\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\
&= 0
\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4b_{n-2}(n+r-2)\,(n-3+r) + 4b_n(n+r)\,(n+r-1) \\ &+ 24b_{n-2}(n+r-2) + 4b_n(n+r) + 25b_{n-2} - b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$\begin{aligned} &4b_{n-2}\left(n-\frac{5}{2}\right)\left(n-\frac{7}{2}\right) + 4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \\ &+ 24b_{n-2}\left(n-\frac{5}{2}\right) + 4b_n\left(n-\frac{1}{2}\right) + 25b_{n-2} - b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(1+2n+2r)\,b_{n-2}}{2n+2r-1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{2nb_{n-2}}{2n-2} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{5+2r}{3+2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-2r}{3+2r}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-2r}{3+2r}$	-2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 28r + 45}{(3 + 2r)(7 + 2r)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{8}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-2r}{3+2r}$	-2
b_3	0	0
b_4	$\frac{4r^2+28r+45}{4r^2+20r+21}$	$\frac{8}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-2r}{3+2r}$	-2
b_3	0	0
b_4	$\frac{4r^2+28r+45}{4r^2+20r+21}$	$\frac{8}{3}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - 2x^2 + \frac{8x^4}{3} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) + \frac{c_2\left(1 - 2x^2 + \frac{8x^4}{3} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) + \frac{c_2\left(1 - 2x^2 + \frac{8x^4}{3} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) + \frac{c_2\left(1 - 2x^2 + \frac{8x^4}{3} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)+\frac{c_2\left(1-2x^2+\frac{8x^4}{3}+O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

14.48.1 Maple step by step solution

Let's solve

$4x^2(x^2+1)\,y''+(24x^3+4x)\,y'+(25x^2-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(25x^2-1)y}{4x^2(x^2+1)}-\frac{(6x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(6x^2+1)y'}{x(x^2+1)}+\frac{(25x^2-1)y}{4x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{6x^2+1}{x(x^2+1)}, P_3(x)=\frac{25x^2-1}{4x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'+(25x^2-1)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{5}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_k+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r+5)a_k}{2k+3+2r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+4)a_k}{2k+2}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{(2k+4)a_k}{2k+2}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+6)a_k}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{(2k+6)a_k}{2k+4}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{(2k+4)a_k}{2k+2}, a_1=0, b_{k+2}=-\frac{(2k+6)b_k}{2k+4}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(1+6*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x),type='
```

$$y(x) = \frac{c_1 x \left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 - 2x^2 + \frac{8}{3}x^4 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 56

```
AsymptoticDSolveValue[4*x^2*(1+x^2)*y''[x]+4*x*(1+6*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],{x,0,
```

$$y(x) \to c_1\left(\frac{8x^{7/2}}{3} - 2x^{3/2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{15x^{9/2}}{8} - \frac{3x^{5/2}}{2} + \sqrt{x}\right)$$

14.49 problem 51

Internal problem ID [1340]
Internal file name [OUTPUT/1341_Sunday_June_05_2022_02_11_57_AM_26476703/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(16x^4+8x^2)\,y''+(68x^3+10x)\,y'+(30x^2-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{34x^2+5}{4x\,(2x^2+1)}$$
$$q(x)=\frac{30x^2-1}{8x^2\,(2x^2+1)}$$

Table 626: Table $p(x), q(x)$ singularites.

$p(x) = \frac{34x^2+5}{4x(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = \frac{30x^2-1}{8x^2(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$8x^2(2x^2+1)\,y'' + (68x^3+10x)\,y' + (30x^2-1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &8x^2(2x^2+1)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ (68x^3+10x)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + (30x^2-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 68x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 30x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 16a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 68x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 68a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 30x^{n+r+2} a_n = \sum_{n=2}^{\infty} 30a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 16a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 68a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 30a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r} a_n (n+r)(n+r-1) + 10x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$8x^r a_0 r(-1+r) + 10x^r a_0 r - a_0 x^r = 0$$

Or

$$(8x^r r(-1+r) + 10x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$16a_{n-2}(n+r-2)(n-3+r)+8a_n(n+r)(n+r-1) \\ +68a_{n-2}(n+r-2)+10a_n(n+r)+30a_{n-2}-a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2(4n+4r-5)\,a_{n-2}}{4n+4r-1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{2(n-1)\,a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-8r-6}{7+4r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-8r-6}{7+4r}$	-1

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-8r-6}{7+4r}$	-1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64r^2 + 224r + 132}{16r^2 + 88r + 105}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-8r-6}{7+4r}$	-1
a_3	0	0
a_4	$\frac{64r^2+224r+132}{16r^2+88r+105}$	$\frac{3}{2}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-8r-6}{7+4r}$	-1
a_3	0	0
a_4	$\frac{64r^2+224r+132}{16r^2+88r+105}$	$\frac{3}{2}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\
&= x^{\frac{1}{4}}\left(1 - x^2 + \frac{3x^4}{2} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&16b_{n-2}(n+r-2)\,(n-3+r) + 8b_n(n+r)\,(n+r-1) \\
&+ 68b_{n-2}(n+r-2) + 10b_n(n+r) + 30b_{n-2} - b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2(4n+4r-5)\,b_{n-2}}{4n+4r-1} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = \frac{(-8n+14)\,b_{n-2}}{4n-3} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{-8r-6}{7+4r}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_2 = -\frac{2}{5}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-8r-6}{7+4r}$	$-\frac{2}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-8r-6}{7+4r}$	$-\frac{2}{5}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{64r^2 + 224r + 132}{16r^2 + 88r + 105}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{36}{65}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-8r-6}{7+4r}$	$-\frac{2}{5}$
b_3	0	0
b_4	$\frac{64r^2+224r+132}{16r^2+88r+105}$	$\frac{36}{65}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-8r-6}{7+4r}$	$-\frac{2}{5}$
b_3	0	0
b_4	$\frac{64r^2+224r+132}{16r^2+88r+105}$	$\frac{36}{65}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{2x^2}{5} + \frac{36x^4}{65} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{4}}\left(1 - x^2 + \frac{3x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{5} + \frac{36x^4}{65} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{4}}\left(1 - x^2 + \frac{3x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{5} + \frac{36x^4}{65} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(1 - x^2 + \frac{3x^4}{2} + O(x^6)\right) + \frac{c_2\left(1 - \frac{2x^2}{5} + \frac{36x^4}{65} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 - x^2 + \frac{3x^4}{2} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{2x^2}{5} + \frac{36x^4}{65} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

14.49.1 Maple step by step solution

Let's solve

$8x^2(2x^2+1)\, y'' + (68x^3+10x)\, y' + (30x^2-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(30x^2-1)y}{8x^2(2x^2+1)} - \frac{(34x^2+5)y'}{4x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(34x^2+5)y'}{4x(2x^2+1)} + \frac{(30x^2-1)y}{8x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{34x^2+5}{4x(2x^2+1)}, P_3(x) = \frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{8}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2(2x^2+1)\, y'' + 2x(34x^2+5)\, y' + (30x^2-1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+4r)\, x^r + a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(4k+4r-1) + 2a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(\left(2k+2r-\frac{5}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{4}\right)\right)\left(k+r+\frac{1}{2}\right)=0$$

- Shift index using $k->k+2$

$$8\left(\left(2k+\frac{3}{2}+2r\right)a_k+a_{k+2}\left(k+\frac{7}{4}+r\right)\right)\left(k+\frac{5}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2(4k+4r+3)a_k}{4k+7+4r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}$$

- Solution for $r=\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0, b_{k+2}=-\frac{2(4k+4)b_k}{4k+8}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(8*x^2*(1+2*x^2)*diff(y(x),x$2)+2*x*(5+34*x^2)*diff(y(x),x)-(1-30*x^2)*y(x)=0,y(x),typ
```

$$y(x) = \frac{c_2 x^{\frac{3}{4}} \left(1 - x^2 + \frac{3}{2} x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 - \frac{2}{5} x^2 + \frac{36}{65} x^4 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 50

```
AsymptoticDSolveValue[8*x^2*(1+2*x^2)*y''[x]+2*x*(5+34*x^2)*y'[x]-(1-30*x^2)*y[x]==0,y[x],{x
```

$$y(x) \to c_1 \sqrt[4]{x}\left(\frac{3x^4}{2} - x^2 + 1\right) + \frac{c_2\left(\frac{36x^4}{65} - \frac{2x^2}{5} + 1\right)}{\sqrt{x}}$$

14.50 problem 61

Internal problem ID [1341]
Internal file name [`OUTPUT/1342_Sunday_June_05_2022_02_12_00_AM_46222525/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 61.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+1)\,y''-x(-3x+1)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x-1}{2x\,(x+1)}$$
$$q(x)=\frac{1}{2\,(x+1)\,x^2}$$

Table 628: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-1}{2x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{2(x+1)x^2}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x+1)\, y'' + (3x^2 - x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (3x^2 - x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r) (n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n + r - 1)(n + r - 2) + 2a_n(n + r)(n + r - 1) \\ &+ 3a_{n-1}(n + r - 1) - a_n(n + r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -1$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = 1$$

Which for the root $r = 1$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -1$$

Which for the root $r = 1$ becomes

$$a_3 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$a_4 = 1$$

Which for the root $r = 1$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1

For $n = 5$, using the above recursive equation gives

$$a_5 = -1$$

Which for the root $r = 1$ becomes

$$a_5 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1
a_5	-1	-1

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\
&+ 3b_{n-1}(n+r-1) - b_n(n+r) + b_n = 0
\end{aligned}
\tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -b_{n-1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = -b_{n-1} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -1
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_1 = -1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$
b_2 = 1
$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1

For $n = 3$, using the above recursive equation gives

$$b_3 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$b_4 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1

For $n = 5$, using the above recursive equation gives

$$b_5 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1
b_5	-1	-1

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) + c_2\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) + c_2\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) + c_2\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) + c_2\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)$$

Verified OK.

14.50.1 Maple step by step solution

Let's solve

$2x^2(x+1)\,y''+(3x^2-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{2x^2(x+1)}-\frac{(3x-1)y'}{2x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x-1)y'}{2x(x+1)}+\frac{y}{2x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x-1}{2x(x+1)},P_3(x)=\frac{1}{2(x+1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $2x^2(x+1)\,y''+x(3x-1)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-7u+4)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(1+r)\, u^{-1+r} + (2a_1(1+r)\,(2+r) - a_0(1+r)\,(-1+4r))\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)\,(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $2a_1(1+r)\,(2+r) - a_0(1+r)\,(-1+4r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 3a_k - 3a_{k-1} + 6a_{k+1})\, k + (-4a_k + 2a$
- Shift index using $k- >k+1$

  $(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 3a_{k+1} - 3a_k + 6a_{k+2})\,(k+1) +$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2+2kr+r^2+5k+5r+6)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}$
- Solution for $r = -1$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)}, 0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)}, 0=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k}, a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)}, 4a_1+a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k}, a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)}, 4a_1+a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k}\right), a_{k+2}=-\frac{2k^2a_k-4k^2a_{1+k}-3ka_k-3ka_{1+k}+a_k+a_{1+k}}{2(k^2+3k+2)}, 0=0,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(-x^5 + x^4 - x^3 + x^2 - x + 1\right)\left(c_1\sqrt{x} + c_2 x\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 58

```
AsymptoticDSolveValue[2*x^2*(1+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(-x^5 + x^4 - x^3 + x^2 - x + 1\right) + c_2\sqrt{x}\left(-x^5 + x^4 - x^3 + x^2 - x + 1\right)$$

14.51 problem 62

Internal problem ID [1342]
Internal file name [`OUTPUT/1343_Sunday_June_05_2022_02_12_03_AM_85998779/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 62.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(12x^4+6x^2)\,y''+(50x^3+x)\,y'+(30x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{50x^2+1}{6x\,(2x^2+1)}$$
$$q(x)=\frac{30x^2+1}{6x^2\,(2x^2+1)}$$

Table 630: Table $p(x), q(x)$ singularites.

$p(x) = \frac{50x^2+1}{6x(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = \frac{30x^2+1}{6x^2(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$6x^2(2x^2+1)\,y'' + (50x^3+x)\,y' + (30x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$6x^2(2x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (50x^3+x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (30x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 12x^{n+r+2} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 50x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 30x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 12x^{n+r+2} a_n(n+r)(n+r-1) &= \sum_{n=2}^{\infty} 12a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 50x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 50a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 30x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 30a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 12a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 6x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 50a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\left(\sum_{n=2}^{\infty} 30a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
6x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)+a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$6x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$(6x^r r(-1+r) + x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2 - 5r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{6}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$12a_{n-2}(n+r-2)(n-3+r)+6a_n(n+r)(n+r-1) \\ +50a_{n-2}(n+r-2)+a_n(n+r)+30a_{n-2}+a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -2a_{n-2} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -2a_{n-2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -2$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-2	-2

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-2	-2
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 4$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-2	-2
a_3	0	0
a_4	4	4

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-2	-2
a_3	0	0
a_4	4	4
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - 2x^2 + 4x^4 + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}&12b_{n-2}(n+r-2)(n-3+r) + 6b_n(n+r)(n+r-1) \\ &+ 50b_{n-2}(n+r-2) + b_n(n+r) + 30b_{n-2} + b_n = 0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -2b_{n-2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -2b_{n-2} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -2$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	-2	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	-2	-2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 4$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = 4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	-2	-2
b_3	0	0
b_4	4	4

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	-2	-2
b_3	0	0
b_4	4	4
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= \sqrt{x}(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots) \\
&= x^{\frac{1}{3}}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right) + c_2 x^{\frac{1}{3}}\left(1 - 2x^2 + 4x^4 + O\left(x^6\right)\right)$$

Verified OK.

14.51.1 Maple step by step solution

Let's solve

$6x^2(2x^2+1)\, y''+(50x^3+x)\, y'+(30x^2+1)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(30x^2+1)y}{6x^2(2x^2+1)}-\frac{(50x^2+1)y'}{6x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(50x^2+1)y'}{6x(2x^2+1)}+\frac{(30x^2+1)y}{6x^2(2x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{50x^2+1}{6x(2x^2+1)}, P_3(x)=\frac{30x^2+1}{6x^2(2x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{1}{6}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{6}$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $6x^2(2x^2+1)\, y''+x(50x^2+1)\, y'+(30x^2+1)\, y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)\,(-1+2r)\, x^r + a_1(2+3r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)\,(2k+2r-1) + 2a_{\cdot}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term must be 0

  $$a_1(2+3r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(3k+3r-1)\,(2k+2r-1)\,(a_k + 2a_{k-2}) = 0$$

- Shift index using $k->k+2$

  $$(3k+3r+5)\,(2k+2r+3)\,(a_{k+2} + 2a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -2a_k$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(6*x^2*(1+2*x^2)*diff(y(x),x$2)+x*(1+50*x^2)*diff(y(x),x)+(1+30*x^2)*y(x)=0,y(x),type=
```

$$y(x) = (4x^4 - 2x^2 + 1)\, x^{\frac{1}{3}} \left(c_2 x^{\frac{1}{6}} + c_1\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 44

```
AsymptoticDSolveValue[6*x^2*(1+2*x^2)*y''[x]+x*(1+50*x^2)*y'[x]+(1+30*x^2)*y[x]==0,y[x],{x,0
```

$$y(x) \to c_1\sqrt{x}\left(4x^4-2x^2+1\right)+c_2\sqrt[3]{x}\left(4x^4-2x^2+1\right)$$

14.52 problem 63

Internal problem ID [1343]
Internal file name [`OUTPUT/1344_Sunday_June_05_2022_02_12_05_AM_33849928/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 63.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{28x^2(-3x+1)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-84x^3+28x^2\right)y''+\left(-63x^2-35x\right)y'+(63x+14)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5+9x}{4x\,(3x-1)}$$
$$q(x)=-\frac{2+9x}{4x^2\,(3x-1)}$$

Table 632: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5+9x}{4x(3x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{3}$	"regular"

$q(x) = -\frac{2+9x}{4x^2(3x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-28y''x^2(3x-1) + \left(-63x^2 - 35x\right) y' + (63x + 14)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-28\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(3x-1) + \left(-63x^2 - 35x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (63x + 14)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-84x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}28x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-63x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-35x^{n+r}a_n(n+r)\right) \tag{2A}$$
$$+\left(\sum_{n=0}^{\infty}63x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}14a_nx^{n+r}\right)=0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-84x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-84a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-63x^{1+n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-63a_{n-1}(n+r-1)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}63x^{1+n+r}a_n=\sum_{n=1}^{\infty}63a_{n-1}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-84a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$
$$+\left(\sum_{n=0}^{\infty}28x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-63a_{n-1}(n+r-1)\,x^{n+r}\right) \tag{2B}$$
$$+\sum_{n=0}^{\infty}\left(-35x^{n+r}a_n(n+r)\right)+\left(\sum_{n=1}^{\infty}63a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}14a_nx^{n+r}\right)=0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$28x^{n+r}a_n(n+r)(n+r-1)-35x^{n+r}a_n(n+r)+14a_nx^{n+r}=0$$

When $n = 0$ the above becomes

$$28x^r a_0 r(-1+r) - 35x^r a_0 r + 14a_0 x^r = 0$$

Or

$$\left(28x^r r(-1+r) - 35x^r r + 14x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(28r^2 - 63r + 14\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$28r^2 - 63r + 14 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(28r^2 - 63r + 14\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -84a_{n-1}(n+r-1)(n+r-2) + 28a_n(n+r)(n+r-1) \\ - 63a_{n-1}(n+r-1) - 35a_n(n+r) + 63a_{n-1} + 14a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 3a_{n-1} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = 3a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 3$$

Which for the root $r = 2$ becomes

$$a_1 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	3	3

For $n = 2$, using the above recursive equation gives

$$a_2 = 9$$

Which for the root $r = 2$ becomes

$$a_2 = 9$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	3	3
a_2	9	9

For $n = 3$, using the above recursive equation gives

$$a_3 = 27$$

Which for the root $r = 2$ becomes

$$a_3 = 27$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	3	3
a_2	9	9
a_3	27	27

For $n = 4$, using the above recursive equation gives

$$a_4 = 81$$

Which for the root $r = 2$ becomes

$$a_4 = 81$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	3	3
a_2	9	9
a_3	27	27
a_4	81	81

For $n = 5$, using the above recursive equation gives

$$a_5 = 243$$

Which for the root $r = 2$ becomes

$$a_5 = 243$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	3	3
a_2	9	9
a_3	27	27
a_4	81	81
a_5	243	243

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right) \\ &= x^2\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-84b_{n-1}(n+r-1)(n+r-2) + 28b_n(n+r)(n+r-1) \\ &- 63b_{n-1}(n+r-1) - 35b_n(n+r) + 63b_{n-1} + 14b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 3b_{n-1} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = 3b_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 3$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	3	3

For $n = 2$, using the above recursive equation gives

$$b_2 = 9$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = 9$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	3	3
b_2	9	9

For $n = 3$, using the above recursive equation gives

$$b_3 = 27$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = 27$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	3	3
b_2	9	9
b_3	27	27

For $n = 4$, using the above recursive equation gives

$$b_4 = 81$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = 81$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	3	3
b_2	9	9
b_3	27	27
b_4	81	81

For $n = 5$, using the above recursive equation gives

$$b_5 = 243$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = 243$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	3	3
b_2	9	9
b_3	27	27
b_4	81	81
b_5	243	243

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}} \left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x^2\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\\ &\quad + c_2x^{\frac{1}{4}}\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^2\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\\ &\quad + c_2x^{\frac{1}{4}}\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1x^2\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\\ &\quad + c_2x^{\frac{1}{4}}\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1x^2\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\\ &\quad + c_2x^{\frac{1}{4}}\left(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + O(x^6)\right)\end{aligned}$$

Verified OK.

14.52.1 Maple step by step solution

Let's solve

$-28y''x^2(3x-1) + (-63x^2 - 35x)\,y' + (63x+14)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+9x)y}{4x^2(3x-1)} - \frac{(5+9x)y'}{4x(3x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(5+9x)y'}{4x(3x-1)} - \frac{(2+9x)y}{4x^2(3x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{5+9x}{4x(3x-1)}, P_3(x)=-\frac{2+9x}{4x^2(3x-1)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-\frac{5}{4}$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{2}$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

• Multiply by denominators

  $4y''x^2(3x-1)+x(5+9x)\,y'+(-9x-2)\,y=0$

• Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=2..3$

  $x^m\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_{0}(-1+4r)\,(-2+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(4k+4r-1)\,(k+r-2)+3a_{k-1}(4k+4r-1)\,(k+r-\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-(-1+4r)\,(-2+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{2,\frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-4\left(k+r-\frac{1}{4}\right)(a_{k}-3a_{k-1})\,(k+r-2)=0$$

- Shift index using $k->k+1$

  $$-4\left(k+\frac{3}{4}+r\right)(a_{k+1}-3a_{k})\,(k+r-1)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=3a_{k}$$

- Recursion relation for $r=2$

  $$a_{k+1}=3a_{k}$$

- Solution for $r=2$

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k+2}, a_{k+1}=3a_{k}\right]$$

- Recursion relation for $r=\frac{1}{4}$

  $$a_{k+1}=3a_{k}$$

- Solution for $r=\frac{1}{4}$

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k+\frac{1}{4}}, a_{k+1}=3a_{k}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_{k}x^{k+2}\right)+\left(\sum_{k=0}^{\infty} b_{k}x^{k+\frac{1}{4}}\right), a_{1+k}=3a_{k}, b_{1+k}=3b_{k}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 47

```
Order:=6;
dsolve(28*x^2*(1-3*x)*diff(y(x),x$2)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x)=0,y(x),type='se
```

$$y(x) = \left(243x^5 + 81x^4 + 27x^3 + 9x^2 + 3x + 1\right)\left(c_1 x^{\frac{1}{4}} + c_2 x^2\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 68

```
AsymptoticDSolveValue[28*x^2*(1-3*x)*y''[x]-7*x*(5+9*x)*y'[x]+7*(2+9*x)*y[x]==0,y[x],{x,0,5}
```

$$\begin{aligned} y(x) \to\ & c_1 \left(243x^5 + 81x^4 + 27x^3 + 9x^2 + 3x + 1\right) x^2 \\ & + c_2 \left(243x^5 + 81x^4 + 27x^3 + 9x^2 + 3x + 1\right) \sqrt[4]{x} \end{aligned}$$

14.53 problem 64

Internal problem ID [1344]
Internal file name [`OUTPUT/1345_Sunday_June_05_2022_02_12_07_AM_78276812/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 64.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x+5)\,y''+9x(5+9x)\,y'-(5-8x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(9x^3+45x^2\right)y''+\left(81x^2+45x\right)y'+(8x-5)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{5+9x}{x\,(x+5)}\\ q(x)&=\frac{8x-5}{9x^2\,(x+5)}\end{aligned}$$

Table 634: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5+9x}{x(x+5)}$	
singularity	type
$x = -5$	"regular"
$x = 0$	"regular"

$q(x) = \frac{8x-5}{9x^2(x+5)}$	
singularity	type
$x = -5$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-5, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2(x+5)\,y'' + \left(81x^2+45x\right) y' + (8x-5)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2(x+5)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(81x^2+45x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (8x-5)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9 x^{1+n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 45 x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 81 x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 45 x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-5 a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 9 x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 9 a_{n-1} (n+r-1)(n+r-2) x^{n+r} \\
\sum_{n=0}^{\infty} 81 x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 81 a_{n-1} (n+r-1) x^{n+r} \\
\sum_{n=0}^{\infty} 8 x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 8 a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 9 a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 45 x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 81 a_{n-1} (n+r-1) x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 45 x^{n+r} a_n (n+r)\right)+\left(\sum_{n=1}^{\infty} 8 a_{n-1} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-5 a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
45 x^{n+r} a_n (n+r)(n+r-1)+45 x^{n+r} a_n (n+r)-5 a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$45x^r a_0 r(-1+r) + 45x^r a_0 r - 5a_0 x^r = 0$$

Or

$$(45x^r r(-1+r) + 45x^r r - 5x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(45r^2 - 5\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$45r^2 - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(45r^2 - 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 9a_{n-1}(n+r-1)\,(n+r-2) + 45a_n(n+r)\,(n+r-1) \\ + 81a_{n-1}(n+r-1) + 45a_n(n+r) + 8a_{n-1} - 5a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(9n^2 + 18nr + 9r^2 + 54n + 54r - 55)}{5\,(9n^2 + 18nr + 9r^2 - 1)} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{3a_{n-1}\left(n^2 + \frac{20}{3}n - 4\right)}{15n^2 + 10n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-9r^2 - 72r - 8}{45r^2 + 90r + 40}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{11}{25}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	$-\frac{11}{25}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81r^4 + 1458r^3 + 7353r^2 + 7128r + 712}{25\,(9r^2 + 18r + 8)\,(9r^2 + 36r + 35)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{11}{50}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	$-\frac{11}{25}$
a_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{50}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(9r^2 + 90r + 89)(9r^2 + 72r + 8)(9r^2 + 108r + 188)}{125(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	$-\frac{11}{25}$
a_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{50}$
a_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{1}{10}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(9r^2 + 90r + 89)(9r^2 + 72r + 8)(9r^2 + 108r + 188)(9r^2 + 126r + 305)}{625(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{29}{700}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	$-\frac{11}{25}$
a_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{50}$
a_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{1}{10}$
a_4	$\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)}{625(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{29}{700}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)(9r^2+144r+440)}{3125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{4727}{297500}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	$-\frac{11}{25}$
a_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{50}$
a_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{1}{10}$
a_4	$\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)}{625(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{29}{700}$
a_5	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)(9r^2+144r+440)}{3125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$-\frac{4727}{297500}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{11x}{25} + \frac{11x^2}{50} - \frac{x^3}{10} + \frac{29x^4}{700} - \frac{4727x^5}{297500} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_{n-1}(n+r-1)(n+r-2) + 45b_n(n+r)(n+r-1) + 81b_{n-1}(n+r-1) + 45b_n(n+r) + 8b_{n-1} - 5b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(9n^2+18nr+9r^2+54n+54r-55)}{5(9n^2+18nr+9r^2-1)} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = -\frac{3\left(n^2+\frac{16}{3}n-8\right)b_{n-1}}{15n^2-10n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-9r^2-72r-8}{45r^2+90r+40}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	1
b_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = \frac{17}{70}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	1
b_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{1}{2}$
b_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{17}{70}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)}{625(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = -\frac{187}{1750}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	1
b_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{1}{2}$
b_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{17}{70}$
b_4	$\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)}{625(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$-\frac{187}{1750}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(9r^2 + 90r + 89)(9r^2 + 72r + 8)(9r^2 + 108r + 188)(9r^2 + 126r + 305)(9r^2 + 144r + 440)}{3125(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2 + 90r + 224)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = \frac{24497}{568750}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2-72r-8}{45r^2+90r+40}$	1
b_2	$\frac{81r^4+1458r^3+7353r^2+7128r+712}{25(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{1}{2}$
b_3	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)}{125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{17}{70}$
b_4	$\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)}{625(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$-\frac{187}{1750}$
b_5	$-\frac{(9r^2+90r+89)(9r^2+72r+8)(9r^2+108r+188)(9r^2+126r+305)(9r^2+144r+440)}{3125(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$\frac{24497}{568750}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x - \frac{x^2}{2} + \frac{17x^3}{70} - \frac{187x^4}{1750} + \frac{24497x^5}{568750} + O(x^6)}{x^{\frac{1}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 - \frac{11x}{25} + \frac{11x^2}{50} - \frac{x^3}{10} + \frac{29x^4}{700} - \frac{4727x^5}{297500} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + x - \frac{x^2}{2} + \frac{17x^3}{70} - \frac{187x^4}{1750} + \frac{24497x^5}{568750} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(1 - \frac{11x}{25} + \frac{11x^2}{50} - \frac{x^3}{10} + \frac{29x^4}{700} - \frac{4727x^5}{297500} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + x - \frac{x^2}{2} + \frac{17x^3}{70} - \frac{187x^4}{1750} + \frac{24497x^5}{568750} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1 - \frac{11x}{25} + \frac{11x^2}{50} - \frac{x^3}{10} + \frac{29x^4}{700} - \frac{4727x^5}{297500} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + x - \frac{x^2}{2} + \frac{17x^3}{70} - \frac{187x^4}{1750} + \frac{24497x^5}{568750} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1 - \frac{11x}{25} + \frac{11x^2}{50} - \frac{x^3}{10} + \frac{29x^4}{700} - \frac{4727x^5}{297500} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 + x - \frac{x^2}{2} + \frac{17x^3}{70} - \frac{187x^4}{1750} + \frac{24497x^5}{568750} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Verified OK.

14.53.1 Maple step by step solution

Let's solve

$9x^2(x+5)\,y''+(81x^2+45x)\,y'+(8x-5)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(8x-5)y}{9x^2(x+5)}-\frac{(5+9x)y'}{x(x+5)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(5+9x)y'}{x(x+5)}+\frac{(8x-5)y}{9x^2(x+5)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5+9x}{x(x+5)}, P_3(x)=\frac{8x-5}{9x^2(x+5)}\right]$

  ◦ $(x+5)\cdot P_2(x)$ is analytic at $x=-5$

  $\left((x+5)\cdot P_2(x)\right)\Big|_{x=-5}=8$

  ◦ $(x+5)^2\cdot P_3(x)$ is analytic at $x=-5$

  $\left((x+5)^2\cdot P_3(x)\right)\Big|_{x=-5}=0$

  ◦ $x=-5$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-5$

- Multiply by denominators

  $9x^2(x+5)\,y''+9x(5+9x)\,y'+(8x-5)\,y=0$

- Change variables using $x=u-5$ so that the regular singular point is at $u=0$

  $(9u^3-90u^2+225u)\left(\frac{d^2}{du^2}y(u)\right)+(81u^2-765u+1800)\left(\frac{d}{du}y(u)\right)+(8u-45)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$225a_0 r(7+r)\, u^{-1+r} + \left(225a_1(1+r)\,(8+r) - 45a_0(2r^2+15r+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(225a_{k+1}(k+1+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$225r(7+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-7, 0\}$$

- Each term must be 0

  $$225a_1(1+r)\,(8+r) - 45a_0(2r^2+15r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$9(-10a_k + a_{k-1} + 25a_{k+1})\, k^2 + 9(2(-10a_k + a_{k-1} + 25a_{k+1})\, r - 75a_k + 6a_{k-1} + 225a_{k+1})\, k + 9($$

- Shift index using $k- >k + 1$

  $$9(-10a_{k+1} + a_k + 25a_{k+2})\,(k+1)^2 + 9(2(-10a_{k+1} + a_k + 25a_{k+2})\, r - 75a_{k+1} + 6a_k + 225a_{k+2})$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} + 18kra_k - 180kra_{k+1} + 9r^2 a_k - 90r^2 a_{k+1} + 72ka_k - 855ka_{k+1} + 72ra_k - 855ra_{k+1} + 8a_k - 810a_{k+1}}{225(k^2 + 2kr + r^2 + 11k + 11r + 18)}$$

- Recursion relation for $r = -7$

$$a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} - 54ka_k + 405ka_{k+1} - 55a_k + 765a_{k+1}}{225(k^2 - 3k - 10)}$$

- Series not valid for $r = -7$, division by 0 in the recursion relation at $k = 5$

$$a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} - 54ka_k + 405ka_{k+1} - 55a_k + 765a_{k+1}}{225(k^2 - 3k - 10)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} + 72ka_k - 855ka_{k+1} + 8a_k - 810a_{k+1}}{225(k^2 + 11k + 18)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} + 72ka_k - 855ka_{k+1} + 8a_k - 810a_{k+1}}{225(k^2 + 11k + 18)}, 1800a_1 - 45a_0 = 0\right]$$

- Revert the change of variables $u = x + 5$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+5)^k, a_{k+2} = -\frac{9k^2 a_k - 90k^2 a_{k+1} + 72ka_k - 855ka_{k+1} + 8a_k - 810a_{k+1}}{225(k^2 + 11k + 18)}, 1800a_1 - 45a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*(5+x)*diff(y(x),x$2)+9*x*(5+9*x)*diff(y(x),x)-(5-8*x)*y(x)=0,y(x),type='series'
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}} \left(1 - \frac{11}{25} x + \frac{11}{50} x^2 - \frac{1}{10} x^3 + \frac{29}{700} x^4 - \frac{4727}{297500} x^5 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 + x - \frac{1}{2} x^2 + \frac{17}{70} x^3 - \frac{187}{1750} x^4 + \frac{24497}{568750} x^5}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 86

```
AsymptoticDSolveValue[9*x^2*(5+x)*y''[x]+9*x*(5+9*x)*y'[x]-(5-8*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{4727x^5}{297500}+\frac{29x^4}{700}-\frac{x^3}{10}+\frac{11x^2}{50}-\frac{11x}{25}+1\right) + \frac{c_2\left(\frac{24497x^5}{568750}-\frac{187x^4}{1750}+\frac{17x^3}{70}-\frac{x^2}{2}+x+1\right)}{\sqrt[3]{x}}$$

14.54 problem 65

Internal problem ID [1345]
Internal file name [`OUTPUT/1346_Sunday_June_05_2022_02_12_10_AM_76298910/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 65.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{8x^2\left(-x^2+2\right)y''+2x\left(-21x^2+10\right)y'-\left(35x^2+2\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-8x^4+16x^2\right)y''+\left(-42x^3+20x\right)y'+\left(-35x^2-2\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{21x^2-10}{4x\left(x^2-2\right)}$$
$$q(x)=\frac{35x^2+2}{8x^2\left(x^2-2\right)}$$

Table 636: Table $p(x), q(x)$ singularites.

$p(x) = \frac{21x^2-10}{4x(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = \frac{35x^2+2}{8x^2(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \sqrt{2}, -\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-8y''x^2(x^2-2) + (-42x^3+20x)\,y' + (-35x^2-2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-8\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(x^2-2) + (-42x^3+20x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-35x^2-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-8x^{n+r+2}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}16x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-42x^{n+r+2}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}20x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-35x^{n+r+2}a_n\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-8x^{n+r+2}a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-8a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-42x^{n+r+2}a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-42a_{n-2}(n+r-2)x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-35x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-35a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty}\left(-8a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}16x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-42a_{n-2}(n+r-2)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}20x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-35a_{n-2}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$16x^{n+r}a_n(n+r)(n+r-1)+20x^{n+r}a_n(n+r)-2a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$16x^r a_0 r(-1+r) + 20x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(16x^r r(-1+r) + 20x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(16r^2 + 4r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$16r^2 + 4r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(16r^2 + 4r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$-8a_{n-2}(n+r-2)(n-3+r)+16a_n(n+r)(n+r-1) \\ -42a_{n-2}(n+r-2)+20a_n(n+r)-35a_{n-2}-2a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = \frac{a_{n-2}}{2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{4}$	$\frac{1}{4}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{4}$	$\frac{1}{4}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\
&= x^{\frac{1}{4}}\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&-8b_{n-2}(n+r-2)(n-3+r) + 16b_n(n+r)(n+r-1) \\
&- 42b_{n-2}(n+r-2) + 20b_n(n+r) - 35b_{n-2} - 2b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-2}}{2} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = \frac{b_{n-2}}{2} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{1}{2}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_2 = \frac{1}{2}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4}$	$\frac{1}{4}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4}$	$\frac{1}{4}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{4}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{4}}\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{4}}\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

14.54.1 Maple step by step solution

Let's solve

$-8y''x^2(x^2-2) + (-42x^3+20x)\,y' + (-35x^2-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(35x^2+2)y}{8x^2(x^2-2)} - \frac{(21x^2-10)y'}{4x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(21x^2-10)y'}{4x(x^2-2)} + \frac{(35x^2+2)y}{8x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{21x^2-10}{4x(x^2-2)}, P_3(x) = \frac{35x^2+2}{8x^2(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{8}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8y''x^2(x^2-2) + 2x(21x^2-10)\,y' + (35x^2+2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(1+2r)\,(-1+4r)\, x^r - 2a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)\,(4k+4r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(1+2r)\,(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

$$-2a_1(3+2r)\,(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$

- Shift index using $k->k+2$

  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{2}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(8*x^2*(2-x^2)*diff(y(x),x$2)+2*x*(10-21*x^2)*diff(y(x),x)-(2+35*x^2)*y(x)=0,y(x),type
```

$$y(x) = \frac{\left(1+\frac{1}{2}x^2+\frac{1}{4}x^4\right)\left(x^{\frac{3}{4}}c_2+c_1\right)}{\sqrt{x}} + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 52

```
AsymptoticDSolveValue[8*x^2*(2-x^2)*y''[x]+2*x*(10-21*x^2)*y'[x]-(2+35*x^2)*y[x]==0,y[x],{x,
```

$$y(x) \rightarrow c_1\sqrt[4]{x}\left(\frac{x^4}{4}+\frac{x^2}{2}+1\right) + \frac{c_2\left(\frac{x^4}{4}+\frac{x^2}{2}+1\right)}{\sqrt{x}}$$

14.55 problem 66

Internal problem ID [1346]
Internal file name [OUTPUT/1347_Sunday_June_05_2022_02_12_13_AM_27586300/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 66.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+3x+1)\,y''-4x(-3x^2-3x+1)\,y'+3(x^2-x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+12x^3+4x^2)\,y''+(12x^3+12x^2-4x)\,y'+(3x^2-3x+3)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{3x^2+3x-1}{x\,(x^2+3x+1)}\\q(x)&=\frac{\frac{3}{4}x^2-\frac{3}{4}x+\frac{3}{4}}{x^2\,(x^2+3x+1)}\end{aligned}$$

Table 638: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2+3x-1}{x(x^2+3x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{2} - \frac{\sqrt{5}}{2}$	"regular"
$x = \frac{\sqrt{5}}{2} - \frac{3}{2}$	"regular"

$q(x) = \frac{\frac{3}{4}x^2-\frac{3}{4}x+\frac{3}{4}}{x^2(x^2+3x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{2} - \frac{\sqrt{5}}{2}$	"regular"
$x = \frac{\sqrt{5}}{2} - \frac{3}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2} - \frac{3}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2\left(x^2 + 3x + 1\right) y'' + \left(12x^3 + 12x^2 - 4x\right) y' + \left(3x^2 - 3x + 3\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(x^2 + 3x + 1\right)\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(12x^3+12x^2-4x\right)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(3x^2-3x+3\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 4a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 12a_{n-1} (n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 12a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 12a_{n-1} (n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 12a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\,x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-4x^{n+r}a_n(n+r)\right) \\
&\quad+\left(\sum_{n=2}^{\infty} 3a_{n-2}x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-3a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)\,(n+r-1)-4x^{n+r}a_n(n+r)+3a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r)-4x^r a_0 r+3a_0x^r=0$$

Or

$$(4x^r r(-1+r)-4x^r r+3x^r)\,a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(4r^2-8r+3\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2-8r+3=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}
r_1&=\frac{3}{2}\\
r_2&=\frac{1}{2}
\end{aligned}$$

Since $a_0\neq 0$ then the indicial equation becomes

$$\left(4r^2-8r+3\right)x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + \sqrt{x}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -3$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-2}(n+r-2)(n-3+r)+12a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1) \\ &+12a_{n-2}(n+r-2)+12a_{n-1}(n+r-1)-4a_n(n+r)+3a_{n-2}-3a_{n-1}+3a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} - 3a_{n-1} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -a_{n-2} - 3a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = 8$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = 8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8

For $n = 3$, using the above recursive equation gives

$$a_3 = -21$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -21$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21

For $n = 4$, using the above recursive equation gives

$$a_4 = 55$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = 55$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21
a_4	55	55

For $n = 5$, using the above recursive equation gives

$$a_5 = -144$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -144$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21
a_4	55	55
a_5	-144	-144

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -3 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} -3 &= \lim_{r\rightarrow \frac{1}{2}} -3 \\ &= -3 \end{aligned}$$

The limit is -3. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = -3$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4b_{n-2}(n+r-2)(n-3+r)+12b_{n-1}(n+r-1)(n+r-2)+4b_n(n+r)(n+r-1) \\ &+12b_{n-2}(n+r-2)+12b_{n-1}(n+r-1)-4b_n(n+r)+3b_{n-2}-3b_{n-1}+3b_n=0 \end{aligned} \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$\begin{aligned} &4b_{n-2}\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)+12b_{n-1}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) \\ &+12b_{n-2}\left(n-\frac{3}{2}\right)+12b_{n-1}\left(n-\frac{1}{2}\right)-4b_n\left(n+\frac{1}{2}\right)+3b_{n-2}-3b_{n-1}+3b_n=0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -b_{n-2} - 3b_{n-1} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -b_{n-2} - 3b_{n-1} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = 8$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 8$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	8	8

For $n = 3$, using the above recursive equation gives

$$b_3 = -21$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -21$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	8	8
b_3	-21	-21

For $n = 4$, using the above recursive equation gives

$$b_4 = 55$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 55$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	8	8
b_3	-21	-21
b_4	55	55

For $n = 5$, using the above recursive equation gives

$$b_5 = -144$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -144$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-3	-3
b_2	8	8
b_3	-21	-21
b_4	55	55
b_5	-144	-144

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}} \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\, \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\, \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}} \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\, \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}} \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2 \sqrt{x}\, \left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)
\end{aligned}
$$

Verified OK.

14.55.1 Maple step by step solution

Let's solve

$4x^2(x^2 + 3x + 1)\, y'' + (12x^3 + 12x^2 - 4x)\, y' + (3x^2 - 3x + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} - \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)} + \frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x)=\frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{3}{4}$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\,y''+4x(3x^2+3x-1)\,y'+(3x^2-3x+3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=2..4$

  $x^m\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)(-3+2r)\,x^r + (a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = -3a_0$$

- Each term in the series must be 0, giving the recursion relation

  $$(2k+2r-1)(2k+2r-3)(a_k + 3a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

  $$(2k+2r+3)(2k+2r+1)(a_{k+2} + 3a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -3a_{k+1} - a_k$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -3a_{k+1} - a_k$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+2} = -3a_{k+1} - a_k$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{1+k} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{1+k} - b_k, b_1 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)-4*x*(1-3*x-3*x^2)*diff(y(x),x)+3*(1-x+x^2)*y(x)=0,y(
```

$$y(x) = \sqrt{x}\left(x\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + \mathrm{O}\left(x^6\right)\right) c_1 + \left(1 - 6x + 17x^2 - 45x^3 + 118x^4 - 309x^5 + \mathrm{O}\left(x^6\right)\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 78

```
AsymptoticDSolveValue[4*x^2*(1+3*x+x^2)*y''[x]-4*x*(1-3*x-3*x^2)*y'[x]+3*(1-x+x^2)*y[x]==0,y
```

$$y(x) \to c_1\left(55x^{9/2} - 21x^{7/2} + 8x^{5/2} - 3x^{3/2} + \sqrt{x}\right) + c_2\left(55x^{11/2} - 21x^{9/2} + 8x^{7/2} - 3x^{5/2} + x^{3/2}\right)$$

14.56 problem 67

Internal problem ID [1347]
Internal file name [`OUTPUT/1348_Sunday_June_05_2022_02_12_15_AM_28508719/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 67.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2(x+1)^2 y'' - x\left(-11x^2 - 10x + 1\right) y' + \left(5x^2 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^4 + 6x^3 + 3x^2\right) y'' + \left(11x^3 + 10x^2 - x\right) y' + \left(5x^2 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{11x - 1}{3x\,(x + 1)}$$
$$q(x) = \frac{5x^2 + 1}{3x^2\,(x + 1)^2}$$

Table 640: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11x-1}{3x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{5x^2+1}{3x^2(x+1)^2}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2(x^2+2x+1)\,y'' + (11x^3+10x^2-x)\,y' + (5x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2(x^2+2x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (11x^3+10x^2-x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (5x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 10x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 10x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 10a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} 5x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 10a_{n-1}(n+r-1)\, x^{n+r}\right)+\sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2-4r+1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2-4r+1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2-4r+1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -2$$

For $2 \leq n$ the recursive equation is

$$3a_{n-2}(n+r-2)(n-3+r)+6a_{n-1}(n+r-1)(n+r-2)+3a_n(n+r)(n+r-1) + 11a_{n-2}(n+r-2)+10a_{n-1}(n+r-1)-a_n(n+r)+5a_{n-2}+a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} - 2a_{n-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -a_{n-2} - 2a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = 3$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	3	3

For $n = 3$, using the above recursive equation gives

$$a_3 = -4$$

Which for the root $r = 1$ becomes

$$a_3 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	3	3
a_3	-4	-4

For $n = 4$, using the above recursive equation gives

$$a_4 = 5$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	3	3
a_3	-4	-4
a_4	5	5

For $n = 5$, using the above recursive equation gives

$$a_5 = -6$$

Which for the root $r = 1$ becomes

$$a_5 = -6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	3	3
a_3	-4	-4
a_4	5	5
a_5	-6	-6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -2$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &3b_{n-2}(n+r-2)(n-3+r) + 6b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ &+ 11b_{n-2}(n+r-2) + 10b_{n-1}(n+r-1) - b_n(n+r) + 5b_{n-2} + b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -b_{n-2} - 2b_{n-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -b_{n-2} - 2b_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-2	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = 3$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-2	-2
b_2	3	3

For $n = 3$, using the above recursive equation gives

$$b_3 = -4$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-2	-2
b_2	3	3
b_3	-4	-4

For $n = 4$, using the above recursive equation gives

$$b_4 = 5$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = 5$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-2	-2
b_2	3	3
b_3	-4	-4
b_4	5	5

For $n = 5$, using the above recursive equation gives

$$b_5 = -6$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -6$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-2	-2
b_2	3	3
b_3	-4	-4
b_4	5	5
b_5	-6	-6

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1-2x+3x^2-4x^3+5x^4-6x^5+O(x^6)\right)
\end{aligned}
$$

Verified OK.

14.56.1 Maple step by step solution

Let's solve

$3x^2(x^2+2x+1)\, y'' + (11x^3+10x^2-x)\, y' + (5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+1)y}{3x^2(x^2+2x+1)} - \frac{y'(11x-1)}{3x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'(11x-1)}{3x(x+1)} + \frac{(5x^2+1)y}{3x^2(x^2+2x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{11x-1}{3x(x+1)}, P_3(x) = \frac{5x^2+1}{3x^2(x^2+2x+1)}\right]$
- $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $((x+1)\cdot P_2(x))\Big|_{x=-1} = 4$
- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $((x+1)^2\cdot P_3(x))\Big|_{x=-1} = 2$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $3x^2(x+1)(x^2+2x+1)y'' + (11x-1)x(x^2+2x+1)y' + (5x^2+1)(x+1)y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(3u^5-6u^4+3u^3)\left(\frac{d^2}{du^2}y(u)\right) + (11u^4-23u^3+12u^2)\left(\frac{d}{du}y(u)\right) + (5u^3-10u^2+6u)y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m = 1..3$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 3..5$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_0(2+r)(1+r)\,u^{1+r} + (3a_1(3+r)(2+r) - a_0(2+r)(5+6r))\,u^{2+r} + \left(\sum_{k=3}^{\infty} (3a_{k-1}(k+r+1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3(2+r)(1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, -1\}$$

- Each term must be 0

  $$3a_1(3+r)(2+r) - a_0(2+r)(5+6r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(5+6r)}{3(3+r)}$$

- Each term in the series must be 0, giving the recursion relation

  $$3a_{k-1}(k+r+1)(k+r) - 6(k+r)\,a_{k-2}\left(k-\frac{7}{6}+r\right) + a_{k-3}(3k-4+3r)(k-2+r) = 0$$

- Shift index using $k- >k+3$

  $$3a_{k+2}(k+4+r)(k+3+r) - 6(k+3+r)\,a_{k+1}\left(k+\frac{11}{6}+r\right) + a_k(3k+3r+5)(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 6kra_k - 12kra_{k+1} + 3r^2a_k - 6r^2a_{k+1} + 8ka_k - 29ka_{k+1} + 8ra_k - 29ra_{k+1} + 5a_k - 33a_{k+1}}{3(k+4+r)(k+3+r)}$$

- Recursion relation for $r = -2$

  $$a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k+2)(k+1)}$$

- Solution for $r = -2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k+2)(k+1)}, a_1 = -\frac{7a_0}{3}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-2}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} - 4k a_k - 5k a_{k+1} + a_k + a_{k+1}}{3(k+2)(k+1)}, a_1 = -\frac{7a_0}{3}\right]$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k+3)(k+2)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k+3)(k+2)}, a_1 = -\frac{a_0}{6}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k+3)(k+2)}, a_1 = -\frac{a_0}{6}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-1}\right), a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{1+k} - 4k a_k - 5k a_{1+k} + a_k + a_{1+k}}{3(k+2)(1+k)}, a_1 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(3*x^2*(1+x)^2*diff(y(x),x$2)-x*(1-10*x-11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x),typ
```

$$y(x) = (-6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1)\left(c_1 x^{\frac{1}{3}} + c_2 x\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 66

```
AsymptoticDSolveValue[3*x^2*(1+x)^2*y''[x]-x*(1-10*x-11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],{x
```

$$y(x) \to c_1 x\left(-6x^5+5x^4-4x^3+3x^2-2x+1\right)+c_2\sqrt[3]{x}\left(-6x^5+5x^4-4x^3+3x^2-2x+1\right)$$

14.57 problem 68

Internal problem ID [1348]
Internal file name [`OUTPUT/1349_Sunday_June_05_2022_02_12_18_AM_35802698/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.5 THE METHOD OF FROBENIUS I. Exercises 7.5. Page 358
Problem number: 68.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2\left(x^2+2x+3\right)y''-x\left(-15x^2-14x+3\right)y'+\left(7x^2+3\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{15x^2+14x-3}{4x\left(x^2+2x+3\right)} \\ q(x) &= \frac{7x^2+3}{4x^2\left(x^2+2x+3\right)} \end{aligned}$$

Table 642: Table $p(x), q(x)$ singularites.

$p(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -1 - i\sqrt{2}$	"regular"
$x = -1 + i\sqrt{2}$	"regular"

$q(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}$	
singularity	type
$x = 0$	"regular"
$x = -1 - i\sqrt{2}$	"regular"
$x = -1 + i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -1 - i\sqrt{2}, -1 + i\sqrt{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2\left(x^2 + 2x + 3\right) y'' + \left(15x^3 + 14x^2 - 3x\right) y' + \left(7x^2 + 3\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(x^2 + 2x + 3\right) \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(15x^3 + 14x^2 - 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(7x^2 + 3\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n(n+r)\,(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 12x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 15x^{n+r+2} a_n(n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 14x^{1+n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n(n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)\,(n+r-1) &= \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r} \\
\sum_{n=0}^{\infty} 8x^{1+n+r} a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty} 15x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 15a_{n-2}(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty} 14x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} 14a_{n-1}(n+r-1)\,x^{n+r} \\
\sum_{n=0}^{\infty} 7x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 15a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 14a_{n-1}(n+r-1)\,x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$12x^{n+r}a_n(n+r)(n+r-1) - 3x^{n+r}a_n(n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$12x^r a_0 r(-1+r) - 3x^r a_0 r + 3a_0 x^r = 0$$

Or

$$(12x^r r(-1+r) - 3x^r r + 3x^r)\,a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(12r^2 - 15r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$12r^2 - 15r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(12r^2 - 15r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2}{3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-2}(n+r-2)(n-3+r)+8a_{n-1}(n+r-1)(n+r-2)+12a_n(n+r)(n+r-1) \\ &+15a_{n-2}(n+r-2)+14a_{n-1}(n+r-1)-3a_n(n+r)+7a_{n-2}+3a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{3} - \frac{2a_{n-1}}{3} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{3} - \frac{2a_{n-1}}{3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3}$	$-\frac{2}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4}{27}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{4}{27}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3}$	$-\frac{2}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$\frac{4}{27}$	$\frac{4}{27}$

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{11}{81}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{11}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3}$	$-\frac{2}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$\frac{4}{27}$	$\frac{4}{27}$
a_4	$-\frac{11}{81}$	$-\frac{11}{81}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{10}{243}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{10}{243}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{3}$	$-\frac{2}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$\frac{4}{27}$	$\frac{4}{27}$
a_4	$-\frac{11}{81}$	$-\frac{11}{81}$
a_5	$\frac{10}{243}$	$\frac{10}{243}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{2}{3}$$

For $2 \leq n$ the recursive equation is

$$4b_{n-2}(n+r-2)(n-3+r)+8b_{n-1}(n+r-1)(n+r-2)+12b_n(n+r)(n+r-1) \\ +15b_{n-2}(n+r-2)+14b_{n-1}(n+r-1)-3b_n(n+r)+7b_{n-2}+3b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{3} - \frac{2b_{n-1}}{3} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = -\frac{b_{n-2}}{3} - \frac{2b_{n-1}}{3} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{9}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{1}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3}$	$-\frac{2}{3}$
b_2	$\frac{1}{9}$	$\frac{1}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4}{27}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = \frac{4}{27}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3}$	$-\frac{2}{3}$
b_2	$\frac{1}{9}$	$\frac{1}{9}$
b_3	$\frac{4}{27}$	$\frac{4}{27}$

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{11}{81}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = -\frac{11}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3}$	$-\frac{2}{3}$
b_2	$\frac{1}{9}$	$\frac{1}{9}$
b_3	$\frac{4}{27}$	$\frac{4}{27}$
b_4	$-\frac{11}{81}$	$-\frac{11}{81}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{10}{243}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = \frac{10}{243}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{3}$	$-\frac{2}{3}$
b_2	$\frac{1}{9}$	$\frac{1}{9}$
b_3	$\frac{4}{27}$	$\frac{4}{27}$
b_4	$-\frac{11}{81}$	$-\frac{11}{81}$
b_5	$\frac{10}{243}$	$\frac{10}{243}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \\ &+ c_2 x^{\frac{1}{4}}\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \\ &+ c_2 x^{\frac{1}{4}}\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \\ + c_2 x^{\frac{1}{4}}\left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right) + c_2 x^{\frac{1}{4}} \left(1 - \frac{2x}{3} + \frac{x^2}{9} + \frac{4x^3}{27} - \frac{11x^4}{81} + \frac{10x^5}{243} + O(x^6)\right)$$

Verified OK.

14.57.1 Maple step by step solution

Let's solve

$4x^2(x^2 + 2x + 3) y'' + (15x^3 + 14x^2 - 3x) y' + (7x^2 + 3) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+3)y}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)} + \frac{(7x^2+3)y}{4x^2(x^2+2x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2 + 2x + 3) y'' + x(15x^2 + 14x - 3) y' + (7x^2 + 3) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  ◦ Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$3a_0(-1+4r)\,(-1+r)\, x^r + (3a_1(3+4r)\, r + 2a_0 r(3+4r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)\,(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3(-1+4r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{4}\right\}$$

- Each term must be 0

$$3a_1(3+4r)\, r + 2a_0 r(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$a_1 = -\frac{2a_0}{3}$

- Each term in the series must be 0, giving the recursion relation

  $(4k+4r-1)(k+r-1)(3a_k+2a_{k-1}+a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $(4k+4r+7)(k+r+1)(3a_{k+2}+2a_{k+1}+a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2a_{1+k}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}, b_{k+2} = -\frac{2b_{1+k}}{3} - \frac{b_k}{3}, b_1 = -\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(4*x^2*(3+2*x+x^2)*diff(y(x),x$2)-x*(3-14*x-15*x^2)*diff(y(x),x)+(3+7*x^2)*y(x)=0,y(x)
```

$$y(x) = \left(1 - \frac{2}{3}x + \frac{1}{9}x^2 + \frac{4}{27}x^3 - \frac{11}{81}x^4 + \frac{10}{243}x^5\right)\left(c_1 x^{\frac{1}{4}} + c_2 x\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*(3+2*x+x^2)*y''[x]-x*(3-14*x-15*x^2)*y'[x]+(3+7*x^2)*y[x]==0,y[x
```

$$y(x) \to c_1 x\left(\frac{10x^5}{243} - \frac{11x^4}{81} + \frac{4x^3}{27} + \frac{x^2}{9} - \frac{2x}{3} + 1\right) + c_2 \sqrt[4]{x}\left(\frac{10x^5}{243} - \frac{11x^4}{81} + \frac{4x^3}{27} + \frac{x^2}{9} - \frac{2x}{3} + 1\right)$$

15 Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374

15.1 problem Example 7.6.1 page 367

Internal problem ID [1349]
Internal file name [`OUTPUT/1350_Sunday_June_05_2022_02_12_20_AM_74780778/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: Example 7.6.1 page 367.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(x+4)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4-2x^3+x^2)\,y''+(-x^2-3x)\,y'+(x+4)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x+3}{x\,(x-1)^2}$$
$$q(x)=\frac{x+4}{x^2\,(x-1)^2}$$

Table 644: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+3}{x(x-1)^2}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"irregular"

$q(x) = \frac{x+4}{x^2(x-1)^2}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 (x^2 - 2x + 1) y'' + (-x^2 - 3x) y' + (x + 4) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 (x^2 - 2x + 1) \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (-x^2 - 3x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+ \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 3x^r r + 4x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r+1}{-1+r}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) - 2a_{n-1}(n+r-1)(n+r-2) \\ &+ a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 3a_n(n+r) + a_{n-1} + 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-2} - 2na_{n-1} + ra_{n-2} - 2ra_{n-1} - 3a_{n-2} + a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(-a_{n-2} + 2a_{n-1})\,n + a_{n-2} + 3a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2 + 10r + 2}{r(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = 17$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^3 + 34r^2 + 54r + 10}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{143}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 + 80r^3 + 321r^2 + 384r + 68}{(2 + r) r (r^2 - 1)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{355}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$
a_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6r^5 + 155r^4 + 1156r^3 + 3295r^2 + 3336r + 572}{r^5 + 5r^4 + 5r^3 - 5r^2 - 6r}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{4043}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$
a_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$
a_5	$\frac{6r^5+155r^4+1156r^3+3295r^2+3336r+572}{r^5+5r^4+5r^3-5r^2-6r}$	$\frac{4043}{15}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{2r+1}{-1+r}$	5	$-\frac{3}{(-1+r)^2}$
b_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17	$\frac{-13r^2-4r+2}{r^2(-1+r)^2}$
b_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$	$\frac{-34r^4-116r^3-64r^2+10}{r^2(r^2-1)^2}$
b_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$	$\frac{-70r^6-652r^5-1904r^4-2128r^3-666r^2+136r+136}{(2+r)^2r^2(r^2-1)^2}$
b_5	$\frac{6r^5+155r^4+1156r^3+3295r^2+3336r+572}{r^5+5r^4+5r^3-5r^2-6r}$	$\frac{4043}{15}$	$\frac{-125r^8-2252r^7-14980r^6-47988r^5-77945r^4-58672r^3-11670r^2+5720r+3432}{r^2(r^4+5r^3+5r^2-5r-6)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \ldots \\
&= x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + x^2 \left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + x^2 \left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + x^2 \left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.1.1 Maple step by step solution

Let's solve

$x^2(x^2 - 2x + 1)\, y'' + (-x^2 - 3x)\, y' + (x + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+4)y}{x^2(x^2-2x+1)} + \frac{(x+3)y'}{x(x^2-2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x(x^2-2x+1)} + \frac{(x+4)y}{x^2(x^2-2x+1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x(x^2-2x+1)}, P_3(x) = \frac{x+4}{x^2(x^2-2x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\,y'' - x(x+3)\,y' + (x+4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k - \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term must be 0

  $$a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(1+2r)}{-1+r}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1})\, k + (a_k + a_{k-2} - 2a_{k-1})\, r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k->k+2$

  $$(k+r)\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})\, r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k a_k - 2k a_{k+1} + r a_k - 2r a_{k+1} - a_k - 3a_{k+1}}{k+r}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(1-2*x+x^2)*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4+x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \left((c_2 \ln(x) + c_1) \left(1 + 5x + 17x^2 + \frac{143}{3}x^3 + \frac{355}{3}x^4 + \frac{4043}{15}x^5 + \mathrm{O}\left(x^6\right)\right) + \left((-3)\,x - \frac{29}{2}x^2 - \frac{859}{18}x^3 - \frac{4693}{36}x^4 - \frac{285181}{900}x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \right) x^2$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 118

```
AsymptoticDSolveValue[x^2*(1-2*x+x^2)*y''[x]-x*(3+x)*y'[x]+(4+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{4043x^5}{15} + \frac{355x^4}{3} + \frac{143x^3}{3} + 17x^2 + 5x + 1\right) x^2 + c_2 \left(\left(-\frac{285181x^5}{900} - \frac{4693x^4}{36} - \frac{859x^3}{18} - \frac{29x^2}{2} - 3x\right) x^2 + \left(\frac{4043x^5}{15} + \frac{355x^4}{3} + \frac{143x^3}{3} + 17x^2 + 5x + 1\right) x^2 \log(x) \right)$$

15.2 problem Example 7.6.2 page 369

Internal problem ID [1350]
Internal file name [OUTPUT/1351_Sunday_June_05_2022_02_12_22_AM_98132168/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: Example 7.6.2 page 369.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(2+x)\,y''+5y'x^2+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+4x^2\right)y''+5y'x^2+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5}{2\,(2+x)}$$
$$q(x)=\frac{x+1}{2x^2\,(2+x)}$$

Table 646: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2(2+x)}$	
singularity	type
$x = -2$	"regular"

$q(x) = \frac{x+1}{2x^2(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(2+x)\,y'' + 5y'x^2 + (x+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(2+x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 5\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (2r-1)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r (2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right) \end{aligned}$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)+5a_{n-1}(n+r-1)+a_{n-1}+a_n=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\,a_{n-1}}{-1+2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(2n+1)\,a_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{4r^2+8r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{15}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{35}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{315}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{693}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$
a_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{693}{8192}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$	$\frac{1}{(1+2r)^2}$
b_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$	$\frac{-4r^2-10r-7}{(4r^2+8r+3)^2}$
b_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$	$\frac{12r^4+84r^3+219r^2+252r+111}{(8r^3+36r^2+46r+15)^2}$
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$	$\frac{-32r^6-432r^5-2384r^4-6876r^3-10946r^2-9162r-3198}{(16r^4+128r^3+344r^2+352r+105)^2}$
b_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{693}{8192}$	$\frac{80r^8+1760r^7+16600r^6+87560r^5+282265r^4+569360r^3+702575r^2+486750r+1}{(32r^5+400r^4+1840r^3+3800r^2+3378r+945)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(\frac{x}{4} - \frac{13x^2}{64} + \frac{101x^3}{768} - \frac{641x^4}{8192} + \frac{7303x^5}{163840} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + \sqrt{x}\left(\frac{x}{4} - \frac{13x^2}{64} + \frac{101x^3}{768} - \frac{641x^4}{8192} + \frac{7303x^5}{163840} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\ &\quad \left. + \sqrt{x}\left(\frac{x}{4}-\frac{13x^2}{64}+\frac{101x^3}{768}-\frac{641x^4}{8192}+\frac{7303x^5}{163840}+O\left(x^6\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\ &\quad \left. + \sqrt{x}\left(\frac{x}{4}-\frac{13x^2}{64}+\frac{101x^3}{768}-\frac{641x^4}{8192}+\frac{7303x^5}{163840}+O\left(x^6\right)\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1-\frac{3x}{4}+\frac{15x^2}{32}-\frac{35x^3}{128}+\frac{315x^4}{2048}-\frac{693x^5}{8192}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\ &\quad \left. + \sqrt{x}\left(\frac{x}{4}-\frac{13x^2}{64}+\frac{101x^3}{768}-\frac{641x^4}{8192}+\frac{7303x^5}{163840}+O\left(x^6\right)\right)\right) \end{aligned}$$

Verified OK.

15.2.1 Maple step by step solution

Let's solve

$2x^2(2+x)\,y''+5y'x^2+(x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x+1)y}{2x^2(2+x)}-\frac{5y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{2(2+x)}+\frac{(x+1)y}{2x^2(2+x)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5}{2(2+x)}, P_3(x)=\frac{x+1}{2x^2(2+x)}\right]$

  - $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{5}{2}$

  - $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  - $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-2$

- Multiply by denominators

  $2x^2(2+x)\,y''+5y'x^2+(x+1)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..3$

  $$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$4 a_{0} r \left(3+2 r \right) u^{-1+r}+\left(4 a_{1} \left(1+r \right) \left(5+2 r \right)-a_{0} \left(8 r^{2}+12 r +1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(4 a_{k +1} \left(k +r +1\right) \left(2 k \right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$4 r \left(3+2 r \right)=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

  $$4 a_{1} \left(1+r \right) \left(5+2 r \right)-a_{0} \left(8 r^{2}+12 r +1\right)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$2 \left(-4 a_{k}+a_{k -1}+4 a_{k +1}\right) k^{2}+\left(4 \left(-4 a_{k}+a_{k -1}+4 a_{k +1}\right) r -12 a_{k}-a_{k -1}+28 a_{k +1}\right) k +2 \left(-4 a_{k}+\right.$$

- Shift index using $k->k+1$

  $$2 \left(-4 a_{k +1}+a_{k}+4 a_{k +2}\right) \left(k +1\right)^{2}+\left(4 \left(-4 a_{k +1}+a_{k}+4 a_{k +2}\right) r -12 a_{k +1}-a_{k}+28 a_{k +2}\right) \left(k +1\right)$$

- Recursion relation that defines series solution to ODE

  $$a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k r a_{k}-16 k r a_{k +1}+2 r^{2} a_{k}-8 r^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+3 r a_{k}-28 r a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+4 k r +2 r^{2}+11 k +11 r +14\right)}$$

- Recursion relation for $r =0$

  $$a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+11 k +14\right)}$$

- Solution for $r =0$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+11 k +14\right)}, 20 a_{1}-a_{0}=0\right]$$

- Revert the change of variables $u =2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{1+k} + 3k a_k - 28k a_{1+k} + a_k - 21 a_{1+k}}{4(2k^2+11k+14)}, 20a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
Order:=6;
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+5*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left((c_2\ln(x)+c_1)\left(1-\frac{3}{4}x+\frac{15}{32}x^2-\frac{35}{128}x^3+\frac{315}{2048}x^4-\frac{693}{8192}x^5+\mathrm{O}\left(x^6\right)\right)+\left(\frac{1}{4}x-\frac{13}{64}x^2+\frac{101}{768}x^3-\frac{641}{8192}x^4+\frac{7303}{163840}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 134

```
AsymptoticDSolveValue[2*x^2*(2+x)*y''[x]+5*x^2*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt{x}\left(-\frac{693x^5}{8192}+\frac{315x^4}{2048}-\frac{35x^3}{128}+\frac{15x^2}{32}-\frac{3x}{4}+1\right)+c_2\left(\sqrt{x}\left(\frac{7303x^5}{163840}-\frac{641x^4}{8192}+\frac{101x^3}{768}-\frac{13x^2}{64}+\frac{x}{4}\right)+\sqrt{x}\left(-\frac{693x^5}{8192}+\frac{315x^4}{2048}-\frac{35x^3}{128}+\frac{15x^2}{32}-\frac{3x}{4}+1\right)\log(x)\right)$$

15.3 problem Example 7.6.3 page 370

Internal problem ID [1351]
Internal file name [OUTPUT/1352_Sunday_June_05_2022_02_12_25_AM_30929406/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: Example 7.6.3 page 370.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2\left(-x^2+2\right)y''-2x\left(2x^2+1\right)y'+\left(-2x^2+2\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-2x\right)y'+\left(-2x^2+2\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x^2+2}{x\left(x^2-2\right)}$$
$$q(x)=\frac{2x^2-2}{x^2\left(x^2-2\right)}$$

Table 648: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2+2}{x(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = \frac{2x^2-2}{x^2(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \sqrt{2}, -\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x^2-2) + (-4x^3-2x)\, y' + (-2x^2+2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 (x^2-2) + (-4x^3-2x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-2x^2+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-4x^{n+r+2} a_n (n+r)\right)+\sum_{n=0}^{\infty}\left(-2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty}\left(-2x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n (n+r)(n+r-1)\right) = \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-4x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)\, x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - 2x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - 2x^r a_0 r + 2a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) - 2x^r r + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r (-1+r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r (-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-a_{n-2}(n+r-2)(n-3+r) + 2a_n(n+r)(n+r-1) \\ &- 4a_{n-2}(n+r-2) - 2a_n(n+r) - 2a_{n-2} + 2a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r)\, a_{n-2}}{2n+2r-2} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(n+1)\, a_{n-2}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{2+2r}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{2+2r}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{2+2r}$	$\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 6r + 8}{4\left(3 + r\right)\left(1 + r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{15}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{2+2r}$	$\frac{3}{4}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{4(3+r)(1+r)}$	$\frac{15}{32}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2+r}{2+2r}$	$\frac{3}{4}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{4(3+r)(1+r)}$	$\frac{15}{32}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{3x^2}{4} + \frac{15x^4}{32} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{2+r}{2+2r}$	$\frac{3}{4}$	$-\frac{1}{2(1+r)^2}$	$-\frac{1}{8}$
b_3	0	0	0	0
b_4	$\frac{r^2+6r+8}{4(3+r)(1+r)}$	$\frac{15}{32}$	$\frac{-r^2-5r-7}{2(1+r)^2(3+r)^2}$	$-\frac{13}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{8}-\frac{13x^4}{128}+O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{8}-\frac{13x^4}{128}+O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{8}-\frac{13x^4}{128}+O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{8}-\frac{13x^4}{128}+O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{3x^2}{4}+\frac{15x^4}{32}+O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{8}-\frac{13x^4}{128}+O(x^6)\right)\right)
\end{aligned}$$

Verified OK.

15.3.1 Maple step by step solution

Let's solve

$-y''x^2(x^2-2)+(-4x^3-2x)\,y'+(-2x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(x^2-1)y}{x^2(x^2-2)}-\frac{2(2x^2+1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(2x^2+1)y'}{x(x^2-2)}+\frac{2(x^2-1)y}{x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(2x^2+1)}{x(x^2-2)},P_3(x)=\frac{2(x^2-1)}{x^2(x^2-2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(x^2-2)+2x(2x^2+1)\,y'+(2x^2-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $-2a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1) = 0$

- Shift index using $k->k+2$

  $-2a_{k+2}(k+r+1)^2 + a_k(k+r+2)\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+r+2)}{2(k+r+1)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-2*x*(1+2*x^2)*diff(y(x),x)+(2-2*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + \frac{3}{4}x^2 + \frac{15}{32}x^4 + \mathrm{O}(x^6)\right) + \left(-\frac{1}{8}x^2 - \frac{13}{128}x^4 + \mathrm{O}(x^6)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*(2-x^2)*y''[x]-2*x*(1+2*x^2)*y'[x]+(2-2*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{15x^4}{32} + \frac{3x^2}{4} + 1\right) + c_2\left(x\left(-\frac{13x^4}{128} - \frac{x^2}{8}\right) + x\left(\frac{15x^4}{32} + \frac{3x^2}{4} + 1\right)\log(x)\right)$$

15.4 problem Example 7.6.4 page 372

Internal problem ID [1352]
Internal file name [`OUTPUT/1353_Sunday_June_05_2022_02_12_28_AM_25208340/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: Example 7.6.4 page 372.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - 5x\right)y' + (9-4x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-5}{x}$$
$$q(x) = -\frac{4x-9}{x^2}$$

Table 650: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{4x-9}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - 5x)\, y' + (9 - 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^2 - 5x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (9 - 4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-4a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 5x^{n+r} a_n (n+r) + 9a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 5x^r a_0 r + 9a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 5x^r r + 9x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-3)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r - 3)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r - 3)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) - 5a_n(n + r) + 9a_n - 4a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-5)}{n^2+2nr+r^2-6n-6r+9} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}(n-2)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{4-r}{(r-2)^2}$$

Which for the root $r = 3$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4-r}{(r-2)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$$

Which for the root $r = 3$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4-r}{(r-2)^2}$	1
a_2	$\frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-4+r)(r-3)}{(r-2)(-1+r)^2 r^2}$$

Which for the root $r = 3$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4-r}{(r-2)^2}$	1
a_2	$\frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$	0
a_3	$-\frac{(-4+r)(r-3)}{(r-2)(-1+r)^2r^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-4+r)(r-3)}{(-1+r)(r-2)r^2(r+1)^2}$$

Which for the root $r = 3$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4-r}{(r-2)^2}$	1
a_2	$\frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$	0
a_3	$-\frac{(-4+r)(r-3)}{(r-2)(-1+r)^2r^2}$	0
a_4	$\frac{(-4+r)(r-3)}{(-1+r)(r-2)r^2(r+1)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-4+r)(r-3)}{(-1+r)(r-2)r(r+1)^2(r+2)^2}$$

Which for the root $r = 3$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4-r}{(r-2)^2}$	1
a_2	$\frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$	0
a_3	$-\frac{(-4+r)(r-3)}{(r-2)(-1+r)^2 r^2}$	0
a_4	$\frac{(-4+r)(r-3)}{(-1+r)(r-2)r^2(r+1)^2}$	0
a_5	$-\frac{(-4+r)(r-3)}{(-1+r)(r-2)r(r+1)^2(r+2)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^3\left(x + 1 + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=3)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{4-r}{(r-2)^2}$	1	$\frac{r-6}{(r-2)^3}$	-3
b_2	$\frac{(-4+r)(r-3)}{(r-2)^2(-1+r)^2}$	0	$\frac{-2r^3+21r^2-65r+58}{(r-2)^3(-1+r)^3}$	$-\frac{1}{4}$
b_3	$-\frac{(-4+r)(r-3)}{(r-2)(-1+r)^2r^2}$	0	$\frac{3r^4-33r^3+116r^2-146r+48}{(r-2)^2(-1+r)^3r^3}$	$\frac{1}{36}$
b_4	$\frac{(-4+r)(r-3)}{(-1+r)(r-2)r^2(r+1)^2}$	0	$\frac{-4r^5+42r^4-136r^3+132r^2+26r-48}{(-1+r)^2(r-2)^2r^3(r+1)^3}$	$-\frac{1}{288}$
b_5	$-\frac{(-4+r)(r-3)}{(-1+r)(r-2)r(r+1)^2(r+2)^2}$	0	$\frac{5r^6-45r^5+95r^4+75r^3-286r^2+72r+48}{(-1+r)^2(r-2)^2r^2(r+1)^3(r+2)^3}$	$\frac{1}{2400}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x^3(x+1+O(x^6))\ln(x) + x^3\left(-3x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{288} + \frac{x^5}{2400} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^3(x+1+O(x^6)) \\
&\quad + c_2\left(x^3(x+1+O(x^6))\ln(x) + x^3\left(-3x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{288} + \frac{x^5}{2400} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^3(x+1+O(x^6)) \\
&\quad + c_2\left(x^3(x+1+O(x^6))\ln(x) + x^3\left(-3x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{288} + \frac{x^5}{2400} + O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x^3(x+1+O(x^6)) \\
&\quad + c_2\left(x^3(x+1+O(x^6))\ln(x) + x^3\left(-3x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{288} + \frac{x^5}{2400} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \left(x + 1 + O(x^6)\right) + c_2 \left(x^3 \left(x + 1 + O(x^6)\right) \ln(x) + x^3 \left(-3x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{288} + \frac{x^5}{2400} + O(x^6)\right)\right)$$

Verified OK.

15.4.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 5x) y' + (9 - 4x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x-9)y}{x^2} - \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-5)y'}{x} - \frac{(4x-9)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-5}{x}, P_3(x) = -\frac{4x-9}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(x - 5) y' + (9 - 4x) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-3+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 3$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-3)^2 + a_{k-1}(k-5+r) = 0$$

- Shift index using $k->k+1$

  $$a_{k+1}(k-2+r)^2 + a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-4)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$; series terminates at $k = 1$

$a_{k+1} = -\frac{a_k(k-1)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$

- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

  $y = a_0 \cdot (x+1)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(5-x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + x + \mathrm{O}\left(x^6\right)\right) + \left((-3)\,x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{288}x^4 + \frac{1}{2400}x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \right) x^3$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 62

```
AsymptoticDSolveValue[x^2*y''[x]-x*(5-x)*y'[x]+(9-4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(x+1)x^3 + c_2\left((x+1)x^3\log(x) + \left(\frac{x^5}{2400} - \frac{x^4}{288} + \frac{x^3}{36} - \frac{x^2}{4} - 3x\right)x^3\right)$$

15.5 problem 1

Internal problem ID [1353]
Internal file name [`OUTPUT/1354_Sunday_June_05_2022_02_12_30_AM_89799754/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(1-x)\,y' + \left(-x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 - x\right)y' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-1}{x}$$
$$q(x) = -\frac{x^2-1}{x^2}$$

Table 652: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - x)\, y' + (-x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (-1+r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r (-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-1} + r a_{n-1} - a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-1} + a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r+1}{r(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-3r^2 - 7r - 3}{r(1+r)^2(r+2)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{13}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$
a_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^3 + 25r^2 + 36r + 13}{r(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{79}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$
a_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$
a_4	$\frac{5r^3+25r^2+36r+13}{r(1+r)^2(r+2)^2(r+3)^2}$	$\frac{79}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-8r^4 - 70r^3 - 208r^2 - 238r - 79}{r(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{67}{1600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$
a_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$
a_4	$\frac{5r^3+25r^2+36r+13}{r(1+r)^2(r+2)^2(r+3)^2}$	$\frac{79}{576}$
a_5	$\frac{-8r^4-70r^3-208r^2-238r-79}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{67}{1600}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{13r^5 + 175r^4 + 874r^3 + 1979r^2 + 1949r + 603}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{5593}{518400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$
a_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$
a_4	$\frac{5r^3+25r^2+36r+13}{r(1+r)^2(r+2)^2(r+3)^2}$	$\frac{79}{576}$
a_5	$\frac{-8r^4-70r^3-208r^2-238r-79}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{67}{1600}$
a_6	$\frac{13r^5+175r^4+874r^3+1979r^2+1949r+603}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{5593}{518400}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-21r^6 - 403r^5 - 3032r^4 - 11291r^3 - 21482r^2 - 19037r - 5593}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{60859}{25401600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$
a_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$
a_4	$\frac{5r^3+25r^2+36r+13}{r(1+r)^2(r+2)^2(r+3)^2}$	$\frac{79}{576}$
a_5	$\frac{-8r^4-70r^3-208r^2-238r-79}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{67}{1600}$
a_6	$\frac{13r^5+175r^4+874r^3+1979r^2+1949r+603}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{5593}{518400}$
a_7	$\frac{-21r^6-403r^5-3032r^4-11291r^3-21482r^2-19037r-5593}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{60859}{25401600}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)\\ &= x\left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{1}{r}$	-1	$\frac{1}{r^2}$
b_2	$\frac{2r+1}{r(1+r)^2}$	$\frac{3}{4}$	$\frac{-4r^2-3r-1}{r^2(1+r)^3}$
b_3	$\frac{-3r^2-7r-3}{r(1+r)^2(r+2)^2}$	$-\frac{13}{36}$	$\frac{9r^4+37r^3+51r^2+27r+6}{r^2(1+r)^3(r+2)^3}$
b_4	$\frac{5r^3+25r^2+36r+13}{r(1+r)^2(r+2)^2(r+3)^2}$	$\frac{79}{576}$	$\frac{-20r^6-185r^5-666r^4-1170r^3-1032r^2-429r-78}{r^2(1+r)^3(r+2)^3(r+3)^3}$
b_5	$\frac{-8r^4-70r^3-208r^2-238r-79}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{67}{1600}$	$\frac{40r^8+660r^7+4536r^6+16804r^5+36255r^4+45890r^3+32633r}{r^2(1+r)^3(r+2)^3(r+3)^3(r+4)^3}$
b_6	$\frac{13r^5+175r^4+874r^3+1979r^2+1949r+603}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{5593}{518400}$	$\frac{-78r^{10}-2005r^9-22327r^8-141096r^7-556696r^6-1420698r}{r^2(1+r)^3(r+2)^3(r+}$
b_7	$\frac{-21r^6-403r^5-3032r^4-11291r^3-21482r^2-19037r-5593}{r(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{60859}{25401600}$	$\frac{147r^{12}+5429r^{11}+89091r^{10}+856149r^9+5344496r^8+227185}{r^2(}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= x\left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)\ln(x) \\
&\quad + x\left(-x^2 + x + \frac{65x^3}{108} - \frac{895x^4}{3456} + \frac{12547x^5}{144000} - \frac{41729x^6}{1728000} + \frac{10121677x^7}{1778112000} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right) \\
&\quad + c_2\left(x\left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)\ln(x)\right. \\
&\quad \left. + x\left(-x^2 + x + \frac{65x^3}{108} - \frac{895x^4}{3456} + \frac{12547x^5}{144000} - \frac{41729x^6}{1728000} + \frac{10121677x^7}{1778112000} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= c_1 x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)$$
$$+ c_2 \left(x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right) \ln(x)\right.$$
$$\left. + x \left(-x^2 + x + \frac{65x^3}{108} - \frac{895x^4}{3456} + \frac{12547x^5}{144000} - \frac{41729x^6}{1728000} + \frac{10121677x^7}{1778112000} + O(x^8)\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)$$
$$+ c_2 \left(x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right) \ln(x)\right.$$
$$\left. + x \left(-x^2 + x + \frac{65x^3}{108} - \frac{895x^4}{3456} + \frac{12547x^5}{144000} - \frac{41729x^6}{1728000} + \frac{10121677x^7}{1778112000} + O(x^8)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right)$$
$$+ c_2 \left(x \left(1 - x + \frac{3x^2}{4} - \frac{13x^3}{36} + \frac{79x^4}{576} - \frac{67x^5}{1600} + \frac{5593x^6}{518400} - \frac{60859x^7}{25401600} + O(x^8)\right) \ln(x)\right.$$
$$\left. + x \left(-x^2 + x + \frac{65x^3}{108} - \frac{895x^4}{3456} + \frac{12547x^5}{144000} - \frac{41729x^6}{1728000} + \frac{10121677x^7}{1778112000} + O(x^8)\right)\right)$$

Verified OK.

15.5.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - x) y' + (-x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x} - \frac{(x^2-1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x^2-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2 y'' + x(x-1)\, y' + (-x^2+1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (a_1 r^2 + a_0 r) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)^2 + a_{k-1}(k+r-1) - a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 + a_0 r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0}{r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r-1)^2 + a_{k-1}(k+r-1) - a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(k+1+r)^2 + a_{k+1}(k+1+r) - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k a_{k+1} + r a_{k+1} - a_k + a_{k+1}}{(k+1+r)^2}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k a_{k+1} - a_k + 2 a_{k+1}}{(k+2)^2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k a_{k+1} - a_k + 2 a_{k+1}}{(k+2)^2}, a_1 = -a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)-x*(1-x)*diff(y(x),x)+(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - x + \frac{3}{4}x^2 - \frac{13}{36}x^3 + \frac{79}{576}x^4 - \frac{67}{1600}x^5 + \frac{5593}{518400}x^6 - \frac{60859}{25401600}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(x - x^2 + \frac{65}{108}x^3 - \frac{895}{3456}x^4 + \frac{12547}{144000}x^5 - \frac{41729}{1728000}x^6 + \frac{10121677}{1778112000}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 154

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1-x)*y'[x]+(1-x^2)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(-\frac{60859x^7}{25401600}+\frac{5593x^6}{518400}-\frac{67x^5}{1600}+\frac{79x^4}{576}-\frac{13x^3}{36}+\frac{3x^2}{4}-x+1\right) + c_2\left(x\left(\frac{10121677x^7}{1778112000}-\frac{41729x^6}{1728000}+\frac{12547x^5}{144000}-\frac{895x^4}{3456}+\frac{65x^3}{108}-x^2+x\right) + x\left(-\frac{60859x^7}{25401600}+\frac{5593x^6}{518400}-\frac{67x^5}{1600}+\frac{79x^4}{576}-\frac{13x^3}{36}+\frac{3x^2}{4}-x+1\right)\log(x)\right)$$

15.6 problem 2

Internal problem ID [1354]
Internal file name [OUTPUT/1355_Sunday_June_05_2022_02_12_36_AM_42745091/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(2x^2+x+1)\,y''+x(7x^2+6x+3)\,y'+(-3x^2+6x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+x^3+x^2)\,y''+(7x^3+6x^2+3x)\,y'+(-3x^2+6x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2+6x+3}{x\,(2x^2+x+1)}$$
$$q(x)=-\frac{3x^2-6x-1}{x^2\,(2x^2+x+1)}$$

Table 654: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+6x+3}{x(2x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"
$x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"

$q(x) = -\frac{3x^2-6x-1}{x^2(2x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"
$x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{7}}{4} - \frac{1}{4}, \frac{i\sqrt{7}}{4} - \frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(2x^2 + x + 1)\, y'' + (7x^3 + 6x^2 + 3x)\, y' + (-3x^2 + 6x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(2x^2 + x + 1)\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (7x^3+6x^2+3x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-3x^2+6x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)\right)\\&+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n(n+r)\right)\\&+\left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right)\\&+\sum_{n=0}^{\infty}\left(-3x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)(n+r-1)=\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n(n+r)=\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n(n+r)=\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-3x^{n+r+2} a_n\right)=\sum_{n=2}^{\infty}\left(-3a_{n-2} x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n=\sum_{n=1}^{\infty} 6a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r}a_n(n+r)\right) + \sum_{n=2}^{\infty}\left(-3a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 6a_{n-1}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)\,(n+r-1) + 3x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r-3}{r+2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-2}(n+r-2)(n-3+r) + a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 7a_{n-2}(n+r-2) + 6a_{n-1}(n+r-1) + 3a_n(n+r) - 3a_{n-2} + 6a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-2} + na_{n-1} + 2ra_{n-2} + ra_{n-1} - 5a_{n-2} + 2a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{(-2a_{n-2} - a_{n-1})\, n + 7a_{n-2} - a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^2 + 4r + 14}{(r+2)(r+3)}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{9}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3r^3 + 14r^2 - 10r - 61}{(4+r)(r+2)(r+3)}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{20}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$
a_3	$\frac{3r^3+14r^2-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^4 - 29r^3 - 134r^2 - 81r + 198}{(r+2)(r+3)(4+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{173}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$
a_3	$\frac{3r^3+14r^2-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$
a_4	$\frac{-r^4-29r^3-134r^2-81r+198}{(r+2)(r+3)(4+r)(5+r)}$	$\frac{173}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-5r^5 - 37r^4 + 72r^3 + 941r^2 + 1534r + 139}{(r+2)(r+3)(4+r)(5+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{93}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$
a_3	$\frac{3r^3+14r^2-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$
a_4	$\frac{-r^4-29r^3-134r^2-81r+198}{(r+2)(r+3)(4+r)(5+r)}$	$\frac{173}{24}$
a_5	$\frac{-5r^5-37r^4+72r^3+941r^2+1534r+139}{(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{93}{20}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7r^6 + 154r^5 + 1085r^4 + 2409r^3 - 2291r^2 - 12771r - 9428}{(7 + r)(r + 2)(r + 3)(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = -1$ becomes

$$a_6 = -\frac{419}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$
a_3	$\frac{3r^3+14r^2-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$
a_4	$\frac{-r^4-29r^3-134r^2-81r+198}{(r+2)(r+3)(4+r)(5+r)}$	$\frac{173}{24}$
a_5	$\frac{-5r^5-37r^4+72r^3+941r^2+1534r+139}{(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{93}{20}$
a_6	$\frac{7r^6+154r^5+1085r^4+2409r^3-2291r^2-12771r-9428}{(7+r)(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{419}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{3r^7 - 28r^6 - 1449r^5 - 13381r^4 - 48637r^3 - 61453r^2 + 24528r + 76095}{(8 + r)(r + 2)(r + 3)(4 + r)(5 + r)(6 + r)(7 + r)}$$

Which for the root $r = -1$ becomes

$$a_7 = \frac{6697}{1260}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r+2}$	-2
a_2	$\frac{-r^2+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$
a_3	$\frac{3r^3+14r^2-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$
a_4	$\frac{-r^4-29r^3-134r^2-81r+198}{(r+2)(r+3)(4+r)(5+r)}$	$\frac{173}{24}$
a_5	$\frac{-5r^5-37r^4+72r^3+941r^2+1534r+139}{(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{93}{20}$
a_6	$\frac{7r^6+154r^5+1085r^4+2409r^3-2291r^2-12771r-9428}{(7+r)(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{419}{720}$
a_7	$\frac{3r^7-28r^6-1449r^5-13381r^4-48637r^3-61453r^2+24528r+76095}{(8+r)(r+2)(r+3)(4+r)(5+r)(6+r)(7+r)}$	$\frac{6697}{1260}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{1 - 2x + \frac{9x^2}{2} - \frac{20x^3}{3} + \frac{173x^4}{24} - \frac{93x^5}{20} - \frac{419x^6}{720} + \frac{6697x^7}{1260} + O(x^8)}{x}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-r-3}{r+2}$	-2	$\frac{1}{(r+2)^{2}}$
b_2	$\frac{-r^{2}+4r+14}{(r+2)(r+3)}$	$\frac{9}{2}$	$\frac{-9r^{2}-40r-46}{(r+2)^{2}(r+3)^{2}}$
b_3	$\frac{3r^{3}+14r^{2}-10r-61}{(4+r)(r+2)(r+3)}$	$-\frac{20}{3}$	$\frac{13r^{4}+176r^{3}+853r^{2}+1770r+1346}{(r+2)^{2}(r+3)^{2}(4+r)^{2}}$
b_4	$\frac{-r^{4}-29r^{3}-134r^{2}-81r+198}{(r+2)(r+3)(4+r)(5+r)}$	$\frac{173}{24}$	$\frac{15r^{6}+126r^{5}-402r^{4}-7936r^{3}-33641r^{2}-60276r-40212}{(r+2)^{2}(r+3)^{2}(4+r)^{2}(5+r)^{2}}$
b_5	$\frac{-5r^{5}-37r^{4}+72r^{3}+941r^{2}+1534r+139}{(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{93}{20}$	$\frac{-63r^{8}-1694r^{7}-18698r^{6}-107576r^{5}-330714r^{4}-442884r^{3}}{(r+2)^{2}(r+3)^{2}(4+r)^{2}(5+r)^{2}(6}$
b_6	$\frac{7r^{6}+154r^{5}+1085r^{4}+2409r^{3}-2291r^{2}-12771r-9428}{(7+r)(r+2)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{419}{720}$	$\frac{35r^{10}+1960r^{9}+43873r^{8}+534810r^{7}+3984324r^{6}+1902013}{(7+r)^{2}(r+2)^{2}(}$
b_7	$\frac{3r^{7}-28r^{6}-1449r^{5}-13381r^{4}-48637r^{3}-61453r^{2}+24528r+76095}{(8+r)(r+2)(r+3)(4+r)(5+r)(6+r)(7+r)}$	$\frac{6697}{1260}$	$\frac{133r^{12}+5964r^{11}+112775r^{10}+1126906r^{9}+5604900r^{8}+544}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \frac{\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)\ln(x)}{x} \\
&+ \frac{x-\frac{15x^2}{4}+\frac{133x^3}{18}-\frac{3077x^4}{288}+\frac{4217x^5}{400}-\frac{70949x^6}{14400}-\frac{125221x^7}{29400}+O(x^8)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)}{x} \\
&+ c_2\left(\frac{\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)\ln(x)}{x}\right. \\
&\left. + \frac{x-\frac{15x^2}{4}+\frac{133x^3}{18}-\frac{3077x^4}{288}+\frac{4217x^5}{400}-\frac{70949x^6}{14400}-\frac{125221x^7}{29400}+O(x^8)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)}{x} \\ &\quad + c_2\left(\frac{\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)\ln(x)}{x}\right. \\ &\quad \left.+\frac{x-\frac{15x^2}{4}+\frac{133x^3}{18}-\frac{3077x^4}{288}+\frac{4217x^5}{400}-\frac{70949x^6}{14400}-\frac{125221x^7}{29400}+O(x^8)}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{c_1\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)}{x} \\ &\quad + c_2\left(\frac{\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)\ln(x)}{x}\right. \\ &\quad \left.+\frac{x-\frac{15x^2}{4}+\frac{133x^3}{18}-\frac{3077x^4}{288}+\frac{4217x^5}{400}-\frac{70949x^6}{14400}-\frac{125221x^7}{29400}+O(x^8)}{x}\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \frac{c_1\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)}{x} \\ &\quad + c_2\left(\frac{\left(1-2x+\frac{9x^2}{2}-\frac{20x^3}{3}+\frac{173x^4}{24}-\frac{93x^5}{20}-\frac{419x^6}{720}+\frac{6697x^7}{1260}+O(x^8)\right)\ln(x)}{x}\right. \\ &\quad \left.+\frac{x-\frac{15x^2}{4}+\frac{133x^3}{18}-\frac{3077x^4}{288}+\frac{4217x^5}{400}-\frac{70949x^6}{14400}-\frac{125221x^7}{29400}+O(x^8)}{x}\right) \end{aligned}$$

Verified OK.

15.6.1 Maple step by step solution

Let's solve

$x^2(2x^2+x+1)\,y''+(7x^3+6x^2+3x)\,y'+(-3x^2+6x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(3x^2-6x-1)y}{x^2(2x^2+x+1)}-\frac{(7x^2+6x+3)y'}{x(2x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+6x+3)y'}{x(2x^2+x+1)}-\frac{(3x^2-6x-1)y}{x^2(2x^2+x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{7x^2+6x+3}{x(2x^2+x+1)}, P_3(x)=-\frac{3x^2-6x-1}{x^2(2x^2+x+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x^2+x+1)\,y''+x(7x^2+6x+3)\,y'+(-3x^2+6x+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)^{2} x^{r}+\left(a_{1}(2+r)^{2}+a_{0}(3+r)\,(2+r)\right) x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+1)^{2}+a_{k-1}(k+r+2)\,(k\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-1$

- Each term must be 0

  $a_{1}(2+r)^{2}+a_{0}(3+r)\,(2+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=-\frac{(3+r)a_{0}}{2+r}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(\left(a_{k}+2a_{k-2}+a_{k-1}\right) k+\left(a_{k}+2a_{k-2}+a_{k-1}\right) r+a_{k}-5a_{k-2}+2a_{k-1}\right)=0$

- Shift index using $k->k+2$

  $(k+r+3)\left(\left(a_{k+2}+2a_{k}+a_{k+1}\right) (k+2)+\left(a_{k+2}+2a_{k}+a_{k+1}\right) r+a_{k+2}-5a_{k}+2a_{k+1}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2ka_{k}+ka_{k+1}+2ra_{k}+ra_{k+1}-a_{k}+4a_{k+1}}{k+r+3}$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2ka_k + ka_{k+1} - 3a_k + 3a_{k+1}}{k+2}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2ka_k + ka_{k+1} - 3a_k + 3a_{k+1}}{k+2}, a_1 = -2a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*(1+x+2*x^2)*diff(y(x),x$2)+x*(3+6*x+7*x^2)*diff(y(x),x)+(1+6*x-3*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - 2x + \frac{9}{2}x^2 - \frac{20}{3}x^3 + \frac{173}{24}x^4 - \frac{93}{20}x^5 - \frac{419}{720}x^6 + \frac{6697}{1260}x^7 + \mathrm{O}(x^8)\right) + \left(x - \frac{15}{4}x^2 + \frac{133}{18}x^3 - \right.}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 162

```
AsymptoticDSolveValue[x^2*(1+x+2*x^2)*y''[x]+x*(3+6*x+7*x^2)*y'[x]+(1+6*x-3*x^2)*y[x]==0,y[x
```

$$y(x) \to \frac{c_1\left(\frac{6697x^7}{1260} - \frac{419x^6}{720} - \frac{93x^5}{20} + \frac{173x^4}{24} - \frac{20x^3}{3} + \frac{9x^2}{2} - 2x + 1\right)}{x} + c_2\left(\frac{-\frac{125221x^7}{29400} - \frac{70949x^6}{14400} + \frac{4217x^5}{400} - \frac{3077x^4}{288} + \frac{133x^3}{18} - \frac{15x^2}{4} + x}{x} + \frac{\left(\frac{6697x^7}{1260} - \frac{419x^6}{720} - \frac{93x^5}{20} + \frac{173x^4}{24} - \frac{20x^3}{3} + \frac{9x^2}{2} - 2x + 1\right)\log(x)}{x}\right)$$

15.7 problem 3

Internal problem ID [1355]
Internal file name [`OUTPUT/1356_Sunday_June_05_2022_02_12_41_AM_68323968/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2(x^2+2x+1)\,y''+x(4x^2+3x+1)\,y'-x(1-2x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$x^2(x+1)^2\,y''+\left(4x^3+3x^2+x\right)y'+\left(2x^2-x\right)y=0$$

Or

$$x\left(y''x^3+4y'x^2+2x^2y''+2yx+3y'x+y''x-y+y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$y(2x-1)+\left(4x^2+3x+1\right)y'+x(x+1)^2\,y''=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4+2x^3+x^2\right)y''+\left(4x^3+3x^2+x\right)y'+\left(2x^2-x\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4x^2 + 3x + 1}{x (x + 1)^2}$$
$$q(x) = \frac{2x - 1}{x (x + 1)^2}$$

Table 656: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2+3x+1}{x(x+1)^2}$	
singularity	type
$x = -1$	"irregular"
$x = 0$	"regular"

$q(x) = \frac{2x-1}{x(x+1)^2}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[-1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 (x^2 + 2x + 1) y'' + (4x^3 + 3x^2 + x) y' + (2x^2 - x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^2\left(x^2+2x+1\right)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\
&+\left(4x^3+3x^2+x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(2x^2-x\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}4x^{n+r+2}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}2x^{n+r+2}a_n\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)\,(n+r-1)=\sum_{n=2}^{\infty}a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r)\,(n+r-1)=\sum_{n=1}^{\infty}2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty}4x^{n+r+2}a_n(n+r)=\sum_{n=2}^{\infty}4a_{n-2}(n+r-2)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)=\sum_{n=1}^{\infty}3a_{n-1}(n+r-1)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty}2x^{n+r+2}a_n=\sum_{n=2}^{\infty}2a_{n-2}x^{n+r}
$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2r+1}{1+r}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + 2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 4a_{n-2}(n+r-2) + 3a_{n-1}(n+r-1) + a_n(n+r) + 2a_{n-2} - a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-2} + 2n a_{n-1} + r a_{n-2} + 2r a_{n-1} - a_{n-2} - 3a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-a_{n-2} - 2a_{n-1}) n + a_{n-2} + 3a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2 - 2r - 2}{(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4r^3 + 2r^2 + 14r + 2}{(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1
a_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 - 51r^2 - 44r + 8}{(4+r)(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1
a_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_4	$\frac{5r^4-51r^2-44r+8}{(4+r)(1+r)(2+r)(3+r)}$	$\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-6r^5 - 5r^4 + 136r^3 + 299r^2 + 52r - 88}{(1+r)(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{11}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1
a_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_4	$\frac{5r^4-51r^2-44r+8}{(4+r)(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_5	$\frac{-6r^5-5r^4+136r^3+299r^2+52r-88}{(1+r)(2+r)(3+r)(4+r)(5+r)}$	$-\frac{11}{15}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7r^6 + 14r^5 - 301r^4 - 1268r^3 - 1088r^2 + 728r + 592}{(6+r)(1+r)(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{37}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1
a_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_4	$\frac{5r^4-51r^2-44r+8}{(4+r)(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_5	$\frac{-6r^5-5r^4+136r^3+299r^2+52r-88}{(1+r)(2+r)(3+r)(4+r)(5+r)}$	$-\frac{11}{15}$
a_6	$\frac{7r^6+14r^5-301r^4-1268r^3-1088r^2+728r+592}{(6+r)(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{37}{45}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-8r^7 - 28r^6 + 588r^5 + 4096r^4 + 7588r^3 - 788r^2 - 10008r - 3344}{(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{209}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+r}$	1
a_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1
a_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_4	$\frac{5r^4-51r^2-44r+8}{(4+r)(1+r)(2+r)(3+r)}$	$\frac{1}{3}$
a_5	$\frac{-6r^5-5r^4+136r^3+299r^2+52r-88}{(1+r)(2+r)(3+r)(4+r)(5+r)}$	$-\frac{11}{15}$
a_6	$\frac{7r^6+14r^5-301r^4-1268r^3-1088r^2+728r+592}{(6+r)(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{37}{45}$
a_7	$\frac{-8r^7-28r^6+588r^5+4096r^4+7588r^3-788r^2-10008r-3344}{(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{209}{315}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= -x^2 + x + 1 + \frac{x^3}{3} + \frac{x^4}{3} - \frac{11x^5}{15} + \frac{37x^6}{45} - \frac{209x^7}{315} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-2r+1}{1+r}$	1	$-\frac{3}{(1+r)^2}$
b_2	$\frac{3r^2-2r-2}{(2+r)(1+r)}$	-1	$\frac{11r^2+16r+2}{(2+r)^2(1+r)^2}$
b_3	$\frac{-4r^3+2r^2+14r+2}{(1+r)(2+r)(3+r)}$	$\frac{1}{3}$	$\frac{-26r^4-116r^3-140r^2+62}{(1+r)^2(2+r)^2(3+r)^2}$
b_4	$\frac{5r^4-51r^2-44r+8}{(4+r)(1+r)(2+r)(3+r)}$	$\frac{1}{3}$	$\frac{50r^6+452r^5+1392r^4+1328r^3-1250r^2-3008r-1456}{(1+r)^2(2+r)^2(3+r)^2(4+r)^2}$
b_5	$\frac{-6r^5-5r^4+136r^3+299r^2+52r-88}{(1+r)(2+r)(3+r)(4+r)(5+r)}$	$-\frac{11}{15}$	$\frac{-85r^8-1292r^7-7412r^6-18004r^5-4425r^4+68568r^3+141626r^2 \ldots}{(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)^2 \ldots}$
b_6	$\frac{7r^6+14r^5-301r^4-1268r^3-1088r^2+728r+592}{(6+r)(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{37}{45}$	$\frac{133r^{10}+3052r^9+28010r^8+123660r^7+195517r^6-532528r^5-32 \ldots}{(1+r)^2(2+r)^2(3+r)^2(4+r) \ldots}$
b_7	$\frac{-8r^7-28r^6+588r^5+4096r^4+7588r^3-788r^2-10008r-3344}{(1+r)(2+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{209}{315}$	$-\frac{4(49r^{12}+1582r^{11}+21202r^{10}+146524r^9+473440r^8-280982r \ldots}{(1+r)^2(2 \ldots}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \left(-x^2 + x + 1 + \frac{x^3}{3} + \frac{x^4}{3} - \frac{11x^5}{15} + \frac{37x^6}{45} - \frac{209x^7}{315} + O(x^8)\right)\ln(x) \\
&\quad - 3x + \frac{x^2}{2} + \frac{31x^3}{18} - \frac{91x^4}{36} + \frac{1897x^5}{900} - \frac{301x^6}{300} - \frac{3901x^7}{14700} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(-x^2 + x + 1 + \frac{x^3}{3} + \frac{x^4}{3} - \frac{11x^5}{15} + \frac{37x^6}{45} - \frac{209x^7}{315} + O(x^8)\right) \\
&\quad + c_2\left(\left(-x^2 + x + 1 + \frac{x^3}{3} + \frac{x^4}{3} - \frac{11x^5}{15} + \frac{37x^6}{45} - \frac{209x^7}{315} + O(x^8)\right)\ln(x) - 3x\right. \\
&\qquad \left. + \frac{x^2}{2} + \frac{31x^3}{18} - \frac{91x^4}{36} + \frac{1897x^5}{900} - \frac{301x^6}{300} - \frac{3901x^7}{14700} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$= c_1\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right) + c_2\left(\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right)\ln(x)-3x+\frac{x^2}{2} + \frac{31x^3}{18}-\frac{91x^4}{36}+\frac{1897x^5}{900}-\frac{301x^6}{300}-\frac{3901x^7}{14700}+O(x^8)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right) + c_2\left(\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right)\ln(x)-3x + \frac{x^2}{2}+\frac{31x^3}{18}-\frac{91x^4}{36}+\frac{1897x^5}{900}-\frac{301x^6}{300}-\frac{3901x^7}{14700}+O(x^8)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right) + c_2\left(\left(-x^2+x+1+\frac{x^3}{3}+\frac{x^4}{3}-\frac{11x^5}{15}+\frac{37x^6}{45}-\frac{209x^7}{315}+O(x^8)\right)\ln(x)-3x+\frac{x^2}{2} + \frac{31x^3}{18}-\frac{91x^4}{36}+\frac{1897x^5}{900}-\frac{301x^6}{300}-\frac{3901x^7}{14700}+O(x^8)\right)$$

Verified OK.

15.7.1 Maple step by step solution

Let's solve

$x^2(x^2+2x+1)\,y''+(4x^3+3x^2+x)\,y'+(2x^2-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x-1)y}{x(x^2+2x+1)}-\frac{(4x^2+3x+1)y'}{x(x^2+2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x^2+3x+1)y'}{x(x^2+2x+1)}+\frac{(2x-1)y}{x(x^2+2x+1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{4x^2+3x+1}{x(x^2+2x+1)}, P_3(x)=\frac{2x-1}{x(x^2+2x+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$
  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$
- Multiply by denominators

  $(x^2+2x+1)\,y''x+(4x^2+3x+1)\,y'+y(2x-1)=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=0..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot y''$ to series expansion for $m=1..3$

$$x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}r^{2}x^{-1+r}+\left(a_{1}(1+r)^{2}+a_{0}(1+r)\,(-1+2r)\right)x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)^{2}+a_{k}(k+r+1)\,(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^{2}=0$$

- Values of r that satisfy the indicial equation

$$r=0$$

- Each term must be 0

$$a_{1}(1+r)^{2}+a_{0}(1+r)\,(-1+2r)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+1)\,(2ka_{k}+a_{k-1}k+ka_{k+1}-a_{k}+a_{k+1})=0$$

- Shift index using $k->k+1$

$$(k+2)\,(2a_{k+1}(k+1)+a_{k}(k+1)+(k+1)\,a_{k+2}-a_{k+1}+a_{k+2})=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_{k}+2ka_{k+1}+a_{k}+a_{k+1}}{k+2}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{ka_{k}+2ka_{k+1}+a_{k}+a_{k+1}}{k+2}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+2}=-\frac{ka_{k}+2ka_{k+1}+a_{k}+a_{k+1}}{k+2},a_{1}-a_{0}=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 71

```
Order:=8;
dsolve(x^2*(1+2*x+x^2)*diff(y(x),x$2)+x*(1+3*x+4*x^2)*diff(y(x),x)-x*(1-2*x)*y(x)=0,y(x),typ
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + x - x^2 + \frac{1}{3}x^3 + \frac{1}{3}x^4 - \frac{11}{15}x^5 + \frac{37}{45}x^6 - \frac{209}{315}x^7 + \mathrm{O}\left(x^8\right)\right) + \left((-3)\,x + \frac{1}{2}x^2 + \frac{31}{18}x^3 - \frac{91}{36}x^4 + \frac{1897}{900}x^5 - \frac{301}{300}x^6 - \frac{3901}{14700}x^7 + \mathrm{O}\left(x^8\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 145

```
AsymptoticDSolveValue[x^2*(1+2*x+x^2)*y''[x]+x*(1+3*x+4*x^2)*y'[x]-x*(1-2*x)*y[x]==0,y[x],{x
```

$$y(x) \to c_1\left(-\frac{209x^7}{315} + \frac{37x^6}{45} - \frac{11x^5}{15} + \frac{x^4}{3} + \frac{x^3}{3} - x^2 + x + 1\right) + c_2\left(-\frac{3901x^7}{14700} - \frac{301x^6}{300} + \frac{1897x^5}{900} - \frac{91x^4}{36} + \frac{31x^3}{18} + \frac{x^2}{2} + \left(-\frac{209x^7}{315} + \frac{37x^6}{45} - \frac{11x^5}{15} + \frac{x^4}{3} + \frac{x^3}{3} - x^2 + x + 1\right)\log(x) - 3x\right)$$

15.8 problem 4

Internal problem ID [1356]
Internal file name [`OUTPUT/1357_Sunday_June_05_2022_02_12_43_AM_60657887/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+x+1)\,y''+12x^2(x+1)\,y'+(3x^2+3x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+4x^3+4x^2)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3+3x}{x^2+x+1}$$
$$q(x)=\frac{3x^2+3x+1}{4x^2\,(x^2+x+1)}$$

Table 658: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+3x}{x^2+x+1}$	
singularity	type
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

$q(x) = \frac{3x^2+3x+1}{4x^2(x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[-\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, 0, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2\left(x^2+x+1\right)y'' + \left(12x^3+12x^2\right)y' + \left(3x^2+3x+1\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(x^2+x+1\right)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(12x^3+12x^2\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(3x^2+3x+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 12x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 12a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r} \\
\sum_{n=0}^{\infty} 3x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right)$$
$$+\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 12a_{n-2}(n+r-2)\,x^{n+r}\right)$$
$$+\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\,x^{n+r}\right)+\left(\sum_{n=2}^{\infty} 3a_{n-2}x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 3a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)$$
$$=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)\,(n+r-1)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r)+a_0 x^r=0$$

Or

$$(4x^r r(-1+r)+x^r)\,a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2\,x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{2}$$
$$r_2=\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2\,x^r=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-3 - 2r}{2r + 1}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 4a_{n-2}(n+r-2)(n-3+r) + 4a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + 12a_{n-2}(n+r-2) + 12a_{n-1}(n+r-1) + 3a_{n-2} + 3a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-2} + 2na_{n-1} + 2ra_{n-2} + 2ra_{n-1} - 3a_{n-2} + a_{n-1}}{-1 + 2n + 2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{(-a_{n-2} - a_{n-1})\, n + a_{n-2} - a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{12r + 14}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{5}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 12r^2 - 58r - 71}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$
a_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-16r^4 - 144r^3 - 288r^2 + 84r + 289}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{5}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$
a_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2
a_4	$\frac{-16r^4-144r^3-288r^2+84r+289}{16r^4+128r^3+344r^2+352r+105}$	$\frac{5}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{192r^4 + 1664r^3 + 4320r^2 + 3328r + 300}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{17}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$
a_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2
a_4	$\frac{-16r^4-144r^3-288r^2+84r+289}{16r^4+128r^3+344r^2+352r+105}$	$\frac{5}{8}$
a_5	$\frac{192r^4+1664r^3+4320r^2+3328r+300}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{17}{20}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^6 + 768r^5 + 1808r^4 - 8576r^3 - 43668r^2 - 61072r - 27309}{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_6 = -\frac{121}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$
a_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2
a_4	$\frac{-16r^4-144r^3-288r^2+84r+289}{16r^4+128r^3+344r^2+352r+105}$	$\frac{5}{8}$
a_5	$\frac{192r^4+1664r^3+4320r^2+3328r+300}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{17}{20}$
a_6	$\frac{64r^6+768r^5+1808r^4-8576r^3-43668r^2-61072r-27309}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$-\frac{121}{80}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-128r^7 - 3264r^6 - 30240r^5 - 123696r^4 - 188760r^3 + 106812r^2 + 554810r + 373335}{128r^7 + 3136r^6 + 31136r^5 + 160720r^4 + 459032r^3 + 709324r^2 + 528414r + 135135}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_7 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{2r+1}$	-2
a_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$
a_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2
a_4	$\frac{-16r^4-144r^3-288r^2+84r+289}{16r^4+128r^3+344r^2+352r+105}$	$\frac{5}{8}$
a_5	$\frac{192r^4+1664r^3+4320r^2+3328r+300}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{17}{20}$
a_6	$\frac{64r^6+768r^5+1808r^4-8576r^3-43668r^2-61072r-27309}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$-\frac{121}{80}$
a_7	$\frac{-128r^7-3264r^6-30240r^5-123696r^4-188760r^3+106812r^2+554810r+373335}{128r^7+3136r^6+31136r^5+160720r^4+459032r^3+709324r^2+528414r+135135}$	1

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\
&= \sqrt{x}\left(1 - 2x + \frac{5x^2}{2} - 2x^3 + \frac{5x^4}{8} + \frac{17x^5}{20} - \frac{121x^6}{80} + x^7 + O(x^8)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-3-2r}{2r+1}$	-2	$\frac{4}{(2r+1)^2}$
b_2	$\frac{12r+14}{4r^2+8r+3}$	$\frac{5}{2}$	$\frac{-48r^2-112r-76}{(4r^2+8r+3)^2}$
b_3	$\frac{8r^3+12r^2-58r-71}{8r^3+36r^2+46r+15}$	-2	$\frac{192r^4+1664r^3+4704r^2+5472r+2396}{(8r^3+36r^2+46r+15)^2}$
b_4	$\frac{-16r^4-144r^3-288r^2+84r+289}{16r^4+128r^3+344r^2+352r+105}$	$\frac{5}{8}$	$256r^6-1792r^5-33600r^4-148096r^3-286608\ldots$ / $(16r^4+128r^3+344r^2+352r+\ldots$
b_5	$\frac{192r^4+1664r^3+4320r^2+3328r+300}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{17}{20}$	$-\frac{8(768r^8+13312r^7+90880r^6+302848r^5+4\ldots}{(32r^5+400r^4+1840\ldots}$
b_6	$\frac{64r^6+768r^5+1808r^4-8576r^3-43668r^2-61072r-27309}{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}$	$-\frac{121}{80}$	$24576r^{10}+802816r^9+11114496r^8+86130688\ldots$
b_7	$\frac{-128r^7-3264r^6-30240r^5-123696r^4-188760r^3+106812r^2+554810r+373335}{128r^7+3136r^6+31136r^5+160720r^4+459032r^3+709324r^2+528414r+135135}$	1	$16384r^{12}-229376r^{11}-21012480r^{10}-411738\ldots$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \sqrt{x}\left(1 - 2x + \frac{5x^2}{2} - 2x^3 + \frac{5x^4}{8} + \frac{17x^5}{20} - \frac{121x^6}{80} + x^7 + O(x^8)\right)\ln(x) \\
&\quad + \sqrt{x}\left(x - \frac{9x^2}{4} + \frac{17x^3}{6} - \frac{205x^4}{96} + \frac{481x^5}{1200} + \frac{2109x^6}{1600} - \frac{1063x^7}{560} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(x-\frac{9x^2}{4}+\frac{17x^3}{6}-\frac{205x^4}{96}+\frac{481x^5}{1200}+\frac{2109x^6}{1600}-\frac{1063x^7}{560}+O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(x-\frac{9x^2}{4}+\frac{17x^3}{6}-\frac{205x^4}{96}+\frac{481x^5}{1200}+\frac{2109x^6}{1600}-\frac{1063x^7}{560}+O(x^8)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(x-\frac{9x^2}{4}+\frac{17x^3}{6}-\frac{205x^4}{96}+\frac{481x^5}{1200}+\frac{2109x^6}{1600}-\frac{1063x^7}{560}+O(x^8)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-2x+\frac{5x^2}{2}-2x^3+\frac{5x^4}{8}+\frac{17x^5}{20}-\frac{121x^6}{80}+x^7+O(x^8)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(x-\frac{9x^2}{4}+\frac{17x^3}{6}-\frac{205x^4}{96}+\frac{481x^5}{1200}+\frac{2109x^6}{1600}-\frac{1063x^7}{560}+O(x^8)\right)\right)
\end{aligned}
$$

Verified OK.

15.8.1 Maple step by step solution

Let's solve

$4x^2(x^2+x+1)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}-\frac{3(x+1)y'}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(x+1)y'}{x^2+x+1}+\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3(x+1)}{x^2+x+1},P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+x+1)\,y''+12x^2(x+1)\,y'+(3x^2+3x+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-1+2r)^2 x^r + \left(a_1 (1+2r)^2 + a_0 (3+2r)\,(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (2k+2r-1)^2 + a_{k-1} (2k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1 (1+2r)^2 + a_0 (3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(3+2r) a_0}{1+2r}$

- Each term in the series must be 0, giving the recursion relation

  $4\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r - \frac{a_k}{2} - \frac{3a_{k-2}}{2} + \frac{a_{k-1}}{2}\right) \left(k + r - \frac{1}{2}\right) = 0$

- Shift index using $k->k+2$

  $4\left((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r - \frac{a_{k+2}}{2} - \frac{3a_k}{2} + \frac{a_{k+1}}{2}\right) \left(k + \frac{3}{2} + r\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}, a_1 = -2a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 81

```
Order:=8;
dsolve(4*x^2*(1+x+x^2)*diff(y(x),x$2)+12*x^2*(1+x)*diff(y(x),x)+(1+3*x+3*x^2)*y(x)=0,y(x),ty
```

$$y(x) = \sqrt{x}\left((c_2\ln(x)+c_1)\left(1-2x+\frac{5}{2}x^2-2x^3+\frac{5}{8}x^4+\frac{17}{20}x^5-\frac{121}{80}x^6+x^7+\mathrm{O}\left(x^8\right)\right)\right.$$
$$\left.+\left(x-\frac{9}{4}x^2+\frac{17}{6}x^3-\frac{205}{96}x^4+\frac{481}{1200}x^5+\frac{2109}{1600}x^6-\frac{1063}{560}x^7+\mathrm{O}\left(x^8\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 156

```
AsymptoticDSolveValue[4*x^2*(1+x+x^2)*y''[x]+12*x^2*(1+x)*y'[x]+(1+3*x+3*x^2)*y[x]==0,y[x],{
```

$$y(x) \to c_1\sqrt{x}\left(x^7-\frac{121x^6}{80}+\frac{17x^5}{20}+\frac{5x^4}{8}-2x^3+\frac{5x^2}{2}-2x+1\right)$$
$$+c_2\left(\sqrt{x}\left(-\frac{1063x^7}{560}+\frac{2109x^6}{1600}+\frac{481x^5}{1200}-\frac{205x^4}{96}+\frac{17x^3}{6}-\frac{9x^2}{4}+x\right)\right.$$
$$\left.+\sqrt{x}\left(x^7-\frac{121x^6}{80}+\frac{17x^5}{20}+\frac{5x^4}{8}-2x^3+\frac{5x^2}{2}-2x+1\right)\log(x)\right)$$

15.9 problem 5

Internal problem ID [1357]
Internal file name [`OUTPUT/1358_Sunday_June_05_2022_02_12_49_AM_75467743/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^3+x^2)\,y''+(2x^3+4x^2-x)\,y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2+4x-1}{(x^2+x+1)\,x}$$
$$q(x)=\frac{1}{x^2\,(x^2+x+1)}$$

Table 660: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+4x-1}{(x^2+x+1)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

$q(x) = \frac{1}{x^2(x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2 + x + 1)\, y'' + (2x^3 + 4x^2 - x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2 + x + 1)\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (2x^3 + 4x^2 - x)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) (n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2) (n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 4a_{n-1} (n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2) (n-3+r) x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1) - x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = \frac{-r - 3}{r}
$$

For $2 \le n$ the recursive equation is

$$
\begin{aligned}
&a_{n-2}(n+r-2)\,(n-3+r) + a_{n-1}(n+r-1)\,(n+r-2) + a_n(n+r)\,(n+r-1) \\
&+ 2a_{n-2}(n+r-2) + 4a_{n-1}(n+r-1) - a_n(n+r) + a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{na_{n-2} + na_{n-1} + ra_{n-2} + ra_{n-1} - 2a_{n-2} + 2a_{n-1}}{n + r - 1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = \frac{(-a_{n-2} - a_{n-1})\,n + a_{n-2} - 3a_{n-1}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{7r + 12}{(1 + r)\,r}
$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{19}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^3 - 2r^2 - 40r - 57}{(2+r)\, r\, (1+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{49}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$
a_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^4 - 11r^3 + 12r^2 + 221r + 294}{r\,(1+r)\,(2+r)\,(r+3)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{515}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$
a_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$
a_4	$\frac{-r^4-11r^3+12r^2+221r+294}{r(1+r)(2+r)(r+3)}$	$\frac{515}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{14r^4 + 108r^3 + 10r^2 - 1139r - 1545}{(4+r)\, r\, (1+r)\, (2+r)\, (r+3)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{319}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$
a_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$
a_4	$\frac{-r^4-11r^3+12r^2+221r+294}{r(1+r)(2+r)(r+3)}$	$\frac{515}{24}$
a_5	$\frac{14r^4+108r^3+10r^2-1139r-1545}{(4+r)r(1+r)(2+r)(r+3)}$	$-\frac{319}{15}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{r^6 + 5r^5 - 128r^4 - 1015r^3 - 1195r^2 + 4769r + 7656}{(5+r)\, r\, (1+r)\, (2+r)\, (r+3)\, (4+r)}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{10093}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$
a_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$
a_4	$\frac{-r^4-11r^3+12r^2+221r+294}{r(1+r)(2+r)(r+3)}$	$\frac{515}{24}$
a_5	$\frac{14r^4+108r^3+10r^2-1139r-1545}{(4+r)r(1+r)(2+r)(r+3)}$	$-\frac{319}{15}$
a_6	$\frac{r^6+5r^5-128r^4-1015r^3-1195r^2+4769r+7656}{(5+r)r(1+r)(2+r)(r+3)(4+r)}$	$\frac{10093}{720}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-r^7 - 28r^6 - 165r^5 + 727r^4 + 8669r^3 + 18671r^2 - 6652r - 30279}{r(1+r)(2+r)(r+3)(4+r)(5+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{647}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-3}{r}$	-4
a_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$
a_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$
a_4	$\frac{-r^4-11r^3+12r^2+221r+294}{r(1+r)(2+r)(r+3)}$	$\frac{515}{24}$
a_5	$\frac{14r^4+108r^3+10r^2-1139r-1545}{(4+r)r(1+r)(2+r)(r+3)}$	$-\frac{319}{15}$
a_6	$\frac{r^6+5r^5-128r^4-1015r^3-1195r^2+4769r+7656}{(5+r)r(1+r)(2+r)(r+3)(4+r)}$	$\frac{10093}{720}$
a_7	$\frac{-r^7-28r^6-165r^5+727r^4+8669r^3+18671r^2-6652r-30279}{r(1+r)(2+r)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{647}{360}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x\left(-4x + 1 + \frac{19x^2}{2} - \frac{49x^3}{3} + \frac{515x^4}{24} - \frac{319x^5}{15} + \frac{10093x^6}{720} - \frac{647x^7}{360} + O(x^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-r-3}{r}$	-4	$\frac{3}{r^2}$
b_2	$\frac{7r+12}{(1+r)r}$	$\frac{19}{2}$	$\frac{-7r^2-24r-12}{(1+r)^2 r^2}$
b_3	$\frac{r^3-2r^2-40r-57}{(2+r)r(1+r)}$	$-\frac{49}{3}$	$\frac{5r^4+84r^3+287r^2+342r+114}{(2+r)^2 r^2 (1+r)^2}$
b_4	$\frac{-r^4-11r^3+12r^2+221r+294}{r(1+r)(2+r)(r+3)}$	$\frac{515}{24}$	$\frac{5r^6-46r^5-874r^4-3960r^3-7651r^2-6468r-1764}{r^2(1+r)^2(2+r)^2(r+3)^2}$
b_5	$\frac{14r^4+108r^3+10r^2-1139r-1545}{(4+r)r(1+r)(2+r)(r+3)}$	$-\frac{319}{15}$	$\frac{-14r^8-216r^7-620r^6+5756r^5+47953r^4+146714r^3+219415r^2}{r^2(1+r)^2(2+r)^2(r+3)^2(4+r)^2}$
b_6	$\frac{r^6+5r^5-128r^4-1015r^3-1195r^2+4769r+7656}{(5+r)r(1+r)(2+r)(r+3)(4+r)}$	$\frac{10093}{720}$	$\frac{10r^{10}+426r^9+6065r^8+38576r^7+92115r^6-196670r^5-1845610}{r^2(1+r)^2(2+r)^2(r+3)^2(4+}$
b_7	$\frac{-r^7-28r^6-165r^5+727r^4+8669r^3+18671r^2-6652r-30279}{r(1+r)(2+r)(r+3)(4+r)(5+r)(6+r)}$	$-\frac{647}{360}$	$\frac{7r^{12}-20r^{11}-5821r^{10}-112866r^9-1033238r^8-5300410r^7-1505}{r^2(1+r)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= x\left(-4x + 1 + \frac{19x^2}{2} - \frac{49x^3}{3} + \frac{515x^4}{24} - \frac{319x^5}{15} + \frac{10093x^6}{720} - \frac{647x^7}{360} + O\left(x^8\right)\right)\ln(x) \\
&\quad + x\left(3x - \frac{43x^2}{4} + \frac{208x^3}{9} - \frac{10379x^4}{288} + \frac{76321x^5}{1800} - \frac{172499x^6}{4800} + \frac{39091x^7}{2400} + O\left(x^8\right)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= c_1 x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right) \\
&+c_2\left(x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right)\ln\left(x\right)\right. \\
&\left.+x\left(3x-\frac{43x^2}{4}+\frac{208x^3}{9}-\frac{10379x^4}{288}+\frac{76321x^5}{1800}-\frac{172499x^6}{4800}+\frac{39091x^7}{2400}+O\left(x^8\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right) \\
&+c_2\left(x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right)\ln\left(x\right)\right. \\
&\left.+x\left(3x-\frac{43x^2}{4}+\frac{208x^3}{9}-\frac{10379x^4}{288}+\frac{76321x^5}{1800}-\frac{172499x^6}{4800}+\frac{39091x^7}{2400}+O\left(x^8\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right) \\
&+c_2\left(x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}\right.\right. \\
&\left.+O\left(x^8\right)\right)\ln\left(x\right) \\
&\left.+x\left(3x-\frac{43x^2}{4}+\frac{208x^3}{9}-\frac{10379x^4}{288}+\frac{76321x^5}{1800}-\frac{172499x^6}{4800}+\frac{39091x^7}{2400}+O\left(x^8\right)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right) \\
&+c_2\left(x\left(-4x+1+\frac{19x^2}{2}-\frac{49x^3}{3}+\frac{515x^4}{24}-\frac{319x^5}{15}+\frac{10093x^6}{720}-\frac{647x^7}{360}+O\left(x^8\right)\right)\ln\left(x\right)\right. \\
&\left.+x\left(3x-\frac{43x^2}{4}+\frac{208x^3}{9}-\frac{10379x^4}{288}+\frac{76321x^5}{1800}-\frac{172499x^6}{4800}+\frac{39091x^7}{2400}+O\left(x^8\right)\right)\right)
\end{aligned}
$$

Verified OK.

15.9.1 Maple step by step solution

Let's solve

$x^2(x^2+x+1)\,y''+(2x^3+4x^2-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2(x^2+x+1)}-\frac{(2x^2+4x-1)y'}{x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x^2+4x-1)y'}{x(x^2+x+1)}+\frac{y}{x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{2x^2+4x-1}{(x^2+x+1)x}, P_3(x)=\frac{1}{x^2(x^2+x+1)}\right]$

  ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

  ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

  ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+x+1)\,y''+x(2x^2+4x-1)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r} + (a_{1}r^{2} + a_{0}r(3+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k}(k+r-1)^{2} + a_{k-1}(k+r-1)\,(k+2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2} = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_{1}r^{2} + a_{0}r(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_{1} = -\frac{(3+r)a_{0}}{r}$

- Each term in the series must be 0, giving the recursion relation

  $((a_{k}+a_{k-2}+a_{k-1})\,k + (a_{k}+a_{k-2}+a_{k-1})\,r - a_{k} - 2a_{k-2} + 2a_{k-1})\,(k+r-1) = 0$

- Shift index using $k->k+2$

  $((a_{k+2}+a_{k}+a_{k+1})\,(k+2) + (a_{k+2}+a_{k}+a_{k+1})\,r - a_{k+2} - 2a_{k} + 2a_{k+1})\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_{k}+ka_{k+1}+ra_{k}+ra_{k+1}+4a_{k+1}}{k+r+1}$

- Recursion relation for $r=1$

  $a_{k+2} = -\frac{ka_{k}+ka_{k+1}+a_{k}+5a_{k+1}}{k+2}$

- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_{k}x^{k+1}, a_{k+2} = -\frac{ka_{k}+ka_{k+1}+a_{k}+5a_{k+1}}{k+2}, a_{1} = -4a_{0}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*(1+x+x^2)*diff(y(x),x$2)-x*(1-4*x-2*x^2)*diff(y(x),x)+y(x)=0,y(x),type='series',x
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 4x + \frac{19}{2}x^2 - \frac{49}{3}x^3 + \frac{515}{24}x^4 - \frac{319}{15}x^5 + \frac{10093}{720}x^6 - \frac{647}{360}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(3x - \frac{43}{4}x^2 + \frac{208}{9}x^3 - \frac{10379}{288}x^4 + \frac{76321}{1800}x^5 - \frac{172499}{4800}x^6 + \frac{39091}{2400}x^7 + \mathrm{O}\left(x^8\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 158

```
AsymptoticDSolveValue[x^2*(1+x+x^2)*y''[x]-x*(1-4*x-2*x^2)*y'[x]+y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(-\frac{647x^7}{360} + \frac{10093x^6}{720} - \frac{319x^5}{15} + \frac{515x^4}{24} - \frac{49x^3}{3} + \frac{19x^2}{2} - 4x + 1\right) + c_2\left(x\left(\frac{39091x^7}{2400} - \frac{172499x^6}{4800} + \frac{76321x^5}{1800} - \frac{10379x^4}{288} + \frac{208x^3}{9} - \frac{43x^2}{4} + 3x\right) + x\left(-\frac{647x^7}{360} + \frac{10093x^6}{720} - \frac{319x^5}{15} + \frac{515x^4}{24} - \frac{49x^3}{3} + \frac{19x^2}{2} - 4x + 1\right)\log(x)\right)$$

15.10 problem 6

Internal problem ID [1358]
Internal file name [`OUTPUT/1359_Sunday_June_05_2022_02_12_59_AM_1175149/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 3x\left(-2x^2 + 3x + 5\right)y' + \left(-14x^2 + 12x + 1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + \left(-6x^3 + 9x^2 + 15x\right)y' + \left(-14x^2 + 12x + 1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2 - 3x - 5}{3x}$$
$$q(x) = -\frac{14x^2 - 12x - 1}{9x^2}$$

Table 662: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2-3x-5}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{14x^2-12x-1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (-6x^3 + 9x^2 + 15x)\, y' + (-14x^2 + 12x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &9x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ (-6x^3 + 9x^2 + 15x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) \\ &+ (-14x^2 + 12x + 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 15x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-14x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n (n+r)\right) &= \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} \left(-14x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty} \left(-14a_{n-2} x^{n+r}\right) \\
\sum_{n=0}^{\infty} 12x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 15x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-14a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 12a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$9x^{n+r}a_n(n+r)(n+r-1) + 15x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) + 15x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 15x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r+1\right)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$\left(3r+1\right)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r+1\right)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{3}{3r+4}$$

For $2 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) - 6a_{n-2}(n+r-2) + 9a_{n-1}(n+r-1) + 15a_n(n+r) - 14a_{n-2} + 12a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2} - 3a_{n-1}}{1+3n+3r} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_n = \frac{2a_{n-2} - 3a_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{6r+17}{9r^2+33r+28}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_2 = \frac{5}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-36r - 93}{27r^3 + 189r^2 + 414r + 280}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_3 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$
a_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{36r^2 + 330r + 619}{81r^4 + 918r^3 + 3699r^2 + 6222r + 3640}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_4 = \frac{19}{72}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$
a_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$
a_4	$\frac{36r^2+330r+619}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{19}{72}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-324r^2 - 2484r - 4275}{243r^5 + 4050r^4 + 25785r^3 + 77850r^2 + 110472r + 58240}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_5 = -\frac{43}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$
a_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$
a_4	$\frac{36r^2+330r+619}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{19}{72}$
a_5	$\frac{-324r^2-2484r-4275}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{43}{360}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{216r^3 + 4104r^2 + 21726r + 32633}{729r^6 + 16767r^5 + 154305r^4 + 723465r^3 + 1810566r^2 + 2273688r + 1106560}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_6 = \frac{319}{6480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$
a_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$
a_4	$\frac{36r^2+330r+619}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{19}{72}$
a_5	$\frac{-324r^2-2484r-4275}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{43}{360}$
a_6	$\frac{216r^3+4104r^2+21726r+32633}{729r^6+16767r^5+154305r^4+723465r^3+1810566r^2+2273688r+1106560}$	$\frac{319}{6480}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-2592r^3 - 39528r^2 - 185220r - 260349}{2187r^7 + 66339r^6 + 831789r^5 + 5565105r^4 + 21347928r^3 + 46653516r^2 + 53340816r + 24344320}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_7 = -\frac{167}{9072}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3r+4}$	-1
a_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$
a_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$
a_4	$\frac{36r^2+330r+619}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{19}{72}$
a_5	$\frac{-324r^2-2484r-4275}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{43}{360}$
a_6	$\frac{216r^3+4104r^2+21726r+32633}{729r^6+16767r^5+154305r^4+723465r^3+1810566r^2+2273688r+1106560}$	$\frac{319}{6480}$
a_7	$\frac{-2592r^3-39528r^2-185220r-260349}{2187r^7+66339r^6+831789r^5+5565105r^4+21347928r^3+46653516r^2+53340816r+24344320}$	$-\frac{167}{9072}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^{\frac{1}{3}}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= \frac{1 - x + \frac{5x^2}{6} - \frac{x^3}{2} + \frac{19x^4}{72} - \frac{43x^5}{360} + \frac{319x^6}{6480} - \frac{167x^7}{9072} + O(x^8)}{x^{\frac{1}{3}}} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -\frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts fr
b_1	$-\frac{3}{3r+4}$	-1	$\frac{9}{(3r+4)^2}$
b_2	$\frac{6r+17}{9r^2+33r+28}$	$\frac{5}{6}$	$\frac{-54r^2-306r-393}{(9r^2+33r+28)^2}$
b_3	$\frac{-36r-93}{27r^3+189r^2+414r+280}$	$-\frac{1}{2}$	$\frac{1944r^3+14337r^2+35154r+284}{(27r^3+189r^2+414r+280)^2}$
b_4	$\frac{36r^2+330r+619}{81r^4+918r^3+3699r^2+6222r+3640}$	$\frac{19}{72}$	$-\frac{6(972r^5+18873r^4+134406r^3}{(81r^4+918r^3+36}$
b_5	$\frac{-324r^2-2484r-4275}{243r^5+4050r^4+25785r^3+77850r^2+110472r+58240}$	$-\frac{43}{360}$	$\frac{236196r^6+5038848r^5+4372906}{(243r^5+4050}$
b_6	$\frac{216r^3+4104r^2+21726r+32633}{729r^6+16767r^5+154305r^4+723465r^3+1810566r^2+2273688r+1106560}$	$\frac{319}{6480}$	$-\frac{9(52488r^8+2134512r^7+3543}{(}$
b_7	$\frac{-2592r^3-39528r^2-185220r-260349}{2187r^7+66339r^6+831789r^5+5565105r^4+21347928r^3+46653516r^2+53340816r+24344320}$	$-\frac{167}{9072}$	$22674816r^9+948090744r^8+172$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\ &= \frac{\left(1 - x + \frac{5x^2}{6} - \frac{x^3}{2} + \frac{19x^4}{72} - \frac{43x^5}{360} + \frac{319x^6}{6480} - \frac{167x^7}{9072} + O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}} \\ &\quad + \frac{x - \frac{11x^2}{12} + \frac{25x^3}{36} - \frac{113x^4}{288} + \frac{4211x^5}{21600} - \frac{32773x^6}{388800} + \frac{126647x^7}{3810240} + O(x^8)}{x^{\frac{1}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= \frac{c_1\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)}{x^{\frac{1}{3}}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}}\right. \\
&\quad \left. + \frac{x-\frac{11x^2}{12}+\frac{25x^3}{36}-\frac{113x^4}{288}+\frac{4211x^5}{21600}-\frac{32773x^6}{388800}+\frac{126647x^7}{3810240}+O(x^8)}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)}{x^{\frac{1}{3}}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}}\right. \\
&\quad \left. + \frac{x-\frac{11x^2}{12}+\frac{25x^3}{36}-\frac{113x^4}{288}+\frac{4211x^5}{21600}-\frac{32773x^6}{388800}+\frac{126647x^7}{3810240}+O(x^8)}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)}{x^{\frac{1}{3}}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}}\right. \\
&\quad \left. + \frac{x-\frac{11x^2}{12}+\frac{25x^3}{36}-\frac{113x^4}{288}+\frac{4211x^5}{21600}-\frac{32773x^6}{388800}+\frac{126647x^7}{3810240}+O(x^8)}{x^{\frac{1}{3}}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)}{x^{\frac{1}{3}}} + c_2 \left(\frac{\left(1-x+\frac{5x^2}{6}-\frac{x^3}{2}+\frac{19x^4}{72}-\frac{43x^5}{360}+\frac{319x^6}{6480}-\frac{167x^7}{9072}+O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}} + \frac{x-\frac{11x^2}{12}+\frac{25x^3}{36}-\frac{113x^4}{288}+\frac{4211x^5}{21600}-\frac{32773x^6}{388800}+\frac{126647x^7}{3810240}+O(x^8)}{x^{\frac{1}{3}}} \right)$$

Verified OK.

15.10.1 Maple step by step solution

Let's solve

$9x^2y'' + (-6x^3+9x^2+15x)\, y' + (-14x^2+12x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(14x^2-12x-1)y}{9x^2} + \frac{(2x^2-3x-5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-3x-5)y'}{3x} - \frac{(14x^2-12x-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$9x^2y'' - 3x(2x^2 - 3x - 5)\,y' + (-14x^2 + 12x + 1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1 + 3r)^2\, x^r + \left(a_1(4 + 3r)^2 + 3a_0(4 + 3r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k + 3r + 1)^2 + 3a_{k-1}(3k + 3r + 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1 + 3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{3}$$

- Each term must be 0

$$a_1(4 + 3r)^2 + 3a_0(4 + 3r) = 0$$

- Solve for the dependent coefficient(s)

$a_1 = -\frac{3a_0}{4+3r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+1)^2 + (3k+3r+1)(-2a_{k-2}+3a_{k-1}) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(3k+3r+7)^2 + (3k+3r+7)(-2a_k+3a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k-3a_{k+1}}{3k+3r+7}$
- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+2} = \frac{2a_k-3a_{k+1}}{3k+6}$
- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k-3a_{k+1}}{3k+6}, a_1 = -a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 81

```
Order:=8;
dsolve(9*x^2*diff(y(x),x$2)+3*x*(5+3*x-2*x^2)*diff(y(x),x)+(1+12*x-14*x^2)*y(x)=0,y(x),type=
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{5}{6}x^2 - \frac{1}{2}x^3 + \frac{19}{72}x^4 - \frac{43}{360}x^5 + \frac{319}{6480}x^6 - \frac{167}{9072}x^7 + O(x^8)\right) + \left(x - \frac{11}{12}x^2 + \frac{25}{36}x^3 - \frac{1}{2}\right.}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 168

```
AsymptoticDSolveValue[9*x^2*y''[x]+3*x*(5+3*x-2*x^2)*y'[x]+(1+12*x-14*x^2)*y[x]==0,y[x],{x,0
```

$$y(x) \to \frac{c_1\left(-\frac{167x^7}{9072}+\frac{319x^6}{6480}-\frac{43x^5}{360}+\frac{19x^4}{72}-\frac{x^3}{2}+\frac{5x^2}{6}-x+1\right)}{\sqrt[3]{x}} + c_2\left(\frac{\frac{126647x^7}{3810240}-\frac{32773x^6}{388800}+\frac{4211x^5}{21600}-\frac{113x^4}{288}+\frac{25x^3}{36}-\frac{11x^2}{12}+x}{\sqrt[3]{x}} + \frac{\left(-\frac{167x^7}{9072}+\frac{319x^6}{6480}-\frac{43x^5}{360}+\frac{19x^4}{72}-\frac{x^3}{2}+\frac{5x^2}{6}-x+1\right)\log(x)}{\sqrt[3]{x}}\right)$$

15.11 problem 7

Internal problem ID [1359]
Internal file name [OUTPUT/1360_Sunday_June_05_2022_02_13_02_AM_25472708/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x^2 + x + 1)\,y' + x(2 - x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$x^2y'' + (x^3 + x^2 + x)\,y' + (-x^2 + 2x)\,y = 0$$

Or

$$x(y'x^2 + y'x - yx + y''x + y' + 2y) = 0$$

For $x \neq 0$ the above simplifies to

$$y''x + (x^2 + x + 1)\,y' - (-2 + x)\,y = 0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 + x^2 + x)\,y' + (-x^2 + 2x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2 + x + 1}{x}$$
$$q(x) = -\frac{-2 + x}{x}$$

Table 664: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+x+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{-2+x}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 + x^2 + x) y' + (-x^2 + 2x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
x^{2}\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}\right) \\
+\left(x^{3}+x^{2}+x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}\right)+\left(-x^{2}+2x\right)\left(\sum_{n=0}^{\infty}a_{n}x^{n+r}\right)=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r+2}a_{n}(n+r)\right) \\
+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_{n}(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\right) \\
+\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_{n}\right)+\left(\sum_{n=0}^{\infty}2x^{1+n+r}a_{n}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}x^{n+r+2}a_{n}(n+r) &= \sum_{n=2}^{\infty}a_{n-2}(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}x^{1+n+r}a_{n}(n+r) &= \sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_{n}\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right) \\
\sum_{n=0}^{\infty}2x^{1+n+r}a_{n} &= \sum_{n=1}^{\infty}2a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n + r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) + x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2-r}{(1+r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_{n-2}(n+r-2)+a_{n-1}(n+r-1)+a_n(n+r)-a_{n-2}+2a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-2} + na_{n-1} + ra_{n-2} + ra_{n-1} - 3a_{n-2} + a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(-a_{n-2} - a_{n-1})\,n + 3a_{n-2} - a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^3 + 6r + 7}{(1+r)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{7}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2r^4 + 10r^3 + 6r^2 - 23r - 28}{(1+r)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{7}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$
a_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^6 + 5r^5 - 11r^4 - 96r^3 - 146r^2 - 16r + 77}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{77}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$
a_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$
a_4	$\frac{r^6+5r^5-11r^4-96r^3-146r^2-16r+77}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{77}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-3r^7 - 41r^6 - 189r^5 - 259r^4 + 468r^3 + 1716r^2 + 1651r + 434}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{217}{7200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$
a_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$
a_4	$\frac{r^6+5r^5-11r^4-96r^3-146r^2-16r+77}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{77}{576}$
a_5	$\frac{-3r^7-41r^6-189r^5-259r^4+468r^3+1716r^2+1651r+434}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{217}{7200}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-r^9 - 15r^8 - 47r^7 + 365r^6 + 3206r^5 + 9364r^4 + 10369r^3 - 2834r^2 - 15026r - 8813}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = -\frac{8813}{518400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$
a_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$
a_4	$\frac{r^6+5r^5-11r^4-96r^3-146r^2-16r+77}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{77}{576}$
a_5	$\frac{-3r^7-41r^6-189r^5-259r^4+468r^3+1716r^2+1651r+434}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{217}{7200}$
a_6	$\frac{-r^9-15r^8-47r^7+365r^6+3206r^5+9364r^4+10369r^3-2834r^2-15026r-8813}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{8813}{518400}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4r^{10} + 112r^9 + 1264r^8 + 7170r^7 + 19330r^6 + 4756r^5 - 116404r^4 - 318504r^3 - 355034r^2 - 145179r}{(1+r)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2 (r+6)^2 (r+7)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = \frac{143}{453600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(1+r)^2}$	-2
a_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$
a_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$
a_4	$\frac{r^6+5r^5-11r^4-96r^3-146r^2-16r+77}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{77}{576}$
a_5	$\frac{-3r^7-41r^6-189r^5-259r^4+468r^3+1716r^2+1651r+434}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{217}{7200}$
a_6	$\frac{-r^9-15r^8-47r^7+365r^6+3206r^5+9364r^4+10369r^3-2834r^2-15026r-8813}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{8813}{518400}$
a_7	$\frac{4r^{10}+112r^9+1264r^8+7170r^7+19330r^6+4756r^5-116404r^4-318504r^3-355034r^2-145179r+8008}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{143}{453600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 - 2x + \frac{7x^2}{4} - \frac{7x^3}{9} + \frac{77x^4}{576} + \frac{217x^5}{7200} - \frac{8813x^6}{518400} + \frac{143x^7}{453600} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n st
b_1	$\frac{-2-r}{(1+r)^2}$	-2	$\frac{r+3}{(1+r)^3}$
b_2	$\frac{-r^3+6r+7}{(1+r)^2(r+2)^2}$	$\frac{7}{4}$	$\frac{r^4-3r^3-24r^2-46r-}{(1+r)^3(r+2)^3}$
b_3	$\frac{2r^4+10r^3+6r^2-23r-28}{(1+r)^2(r+2)^2(r+3)^2}$	$-\frac{7}{9}$	$\frac{-4r^6-30r^5-40r^4+2}{(1+r)^3(r}$
b_4	$\frac{r^6+5r^5-11r^4-96r^3-146r^2-16r+77}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{77}{576}$	$-2r^9-15r^8+64r^7+1$
b_5	$\frac{-3r^7-41r^6-189r^5-259r^4+468r^3+1716r^2+1651r+434}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{217}{7200}$	$9r^{11}+209r^{10}+1920r$
b_6	$\frac{-r^9-15r^8-47r^7+365r^6+3206r^5+9364r^4+10369r^3-2834r^2-15026r-8813}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2}$	$-\frac{8813}{518400}$	$3r^{14}+81r^{13}+690r^{12}$
b_7	$\frac{4r^{10}+112r^9+1264r^8+7170r^7+19330r^6+4756r^5-116404r^4-318504r^3-355034r^2-145179r+8008}{(1+r)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(r+6)^2(r+7)^2}$	$\frac{143}{453600}$	$-16r^{16}-784r^{15}-169$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \left(1 - 2x + \frac{7x^2}{4} - \frac{7x^3}{9} + \frac{77x^4}{576} + \frac{217x^5}{7200} - \frac{8813x^6}{518400} + \frac{143x^7}{453600} + O(x^8)\right)\ln(x) \\
&\quad + 3x - \frac{15x^2}{4} + \frac{239x^3}{108} - \frac{2021x^4}{3456} - \frac{1241x^5}{54000} + \frac{93859x^6}{1728000} - \frac{311177x^7}{42336000} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right)\ln(x)\right. \\
&\quad \left.+3x-\frac{15x^2}{4}+\frac{239x^3}{108}-\frac{2021x^4}{3456}-\frac{1241x^5}{54000}+\frac{93859x^6}{1728000}-\frac{311177x^7}{42336000}+O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right)\ln(x)\right. \\
&\quad \left.+3x-\frac{15x^2}{4}+\frac{239x^3}{108}-\frac{2021x^4}{3456}-\frac{1241x^5}{54000}+\frac{93859x^6}{1728000}-\frac{311177x^7}{42336000}+O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right)\ln(x)\right. \\
&\quad \left.+3x-\frac{15x^2}{4}+\frac{239x^3}{108}-\frac{2021x^4}{3456}-\frac{1241x^5}{54000}+\frac{93859x^6}{1728000}-\frac{311177x^7}{42336000}+O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right) \\
&\quad + c_2\left(\left(1-2x+\frac{7x^2}{4}-\frac{7x^3}{9}+\frac{77x^4}{576}+\frac{217x^5}{7200}-\frac{8813x^6}{518400}+\frac{143x^7}{453600}+O(x^8)\right)\ln(x)\right. \\
&\quad \left.+3x-\frac{15x^2}{4}+\frac{239x^3}{108}-\frac{2021x^4}{3456}-\frac{1241x^5}{54000}+\frac{93859x^6}{1728000}-\frac{311177x^7}{42336000}+O(x^8)\right)
\end{aligned}
$$

Verified OK.

15.11.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 + x^2 + x)\, y' + (-x^2 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(-2+x)y}{x} - \frac{(x^2+x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+x+1)y'}{x} - \frac{(-2+x)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2+x+1}{x}, P_3(x) = -\frac{-2+x}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (x^2 + x + 1)\, y' + (2 - x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) \, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) \, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) \, (k+r-1) \, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) \, (k+r) \, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(a_1 (1+r)^2 + a_0 (2+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+1+r)^2 + a_k (k+r+2) + a_{k-1} (k-2\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1 (1+r)^2 + a_0 (2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+1} + (a_k + a_{k-1} + 2a_{k+1}) \, k + 2a_k - 2a_{k-1} + a_{k+1} = 0$

- Shift index using $k->k+1$

  $(k+1)^2 a_{k+2} + (a_{k+1} + a_k + 2a_{k+2}) \, (k+1) + 2a_{k+1} - 2a_k + a_{k+2} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k k + k a_{k+1} - a_k + 3a_{k+1}}{k^2+4k+4}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k k + k a_{k+1} - a_k + 3a_{k+1}}{k^2+4k+4}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k k + k a_{k+1} - a_k + 3a_{k+1}}{k^2 + 4k + 4}, a_1 + 2a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
Order:=8;
dsolve(x^2*diff(y(x),x$2)+x*(1+x+x^2)*diff(y(x),x)+x*(2-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - 2x + \frac{7}{4}x^2 - \frac{7}{9}x^3 + \frac{77}{576}x^4 + \frac{217}{7200}x^5 - \frac{8813}{518400}x^6 + \frac{143}{453600}x^7 + \mathrm{O}\left(x^8\right)\right) + \left(3x - \frac{15}{4}x^2 + \frac{239}{108}x^3 - \frac{2021}{3456}x^4 - \frac{1241}{54000}x^5 + \frac{93859}{1728000}x^6 - \frac{311177}{42336000}x^7 + \mathrm{O}\left(x^8\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 153

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1+x+x^2)*y'[x]+x*(2-x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(\frac{143x^7}{453600} - \frac{8813x^6}{518400} + \frac{217x^5}{7200} + \frac{77x^4}{576} - \frac{7x^3}{9} + \frac{7x^2}{4} - 2x + 1\right) + c_2\left(-\frac{311177x^7}{42336000} + \frac{93859x^6}{1728000} - \frac{1241x^5}{54000} - \frac{2021x^4}{3456} + \frac{239x^3}{108} - \frac{15x^2}{4} + \left(\frac{143x^7}{453600} - \frac{8813x^6}{518400} + \frac{217x^5}{7200} + \frac{77x^4}{576} - \frac{7x^3}{9} + \frac{7x^2}{4} - 2x + 1\right)\log(x) + 3x\right)$$

15.12 problem 8

Internal problem ID [1360]
Internal file name [`OUTPUT/1361_Sunday_June_05_2022_02_13_06_AM_50960082/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+2x)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}
p(x) &= \frac{3x^2+14x+5}{x\,(1+2x)}\\
q(x) &= \frac{12x^2+18x+4}{x^2\,(1+2x)}
\end{aligned}$$

Table 666: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2+14x+5}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = \frac{12x^2+18x+4}{x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}\right]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\,y'' + \left(3x^3 + 14x^2 + 5x\right) y' + \left(12x^2 + 18x + 4\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(1+2x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^3 + 14x^2 + 5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(12x^2 + 18x + 4\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right)\\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 14x^{1+n+r} a_n(n+r)\right)\\
&+ \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r+2} a_n\right)\\
&+ \left(\sum_{n=0}^{\infty} 18x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\\
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r}\\
\sum_{n=0}^{\infty} 14x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} 14a_{n-1}(n+r-1)\,x^{n+r}\\
\sum_{n=0}^{\infty} 12x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 12a_{n-2}x^{n+r}\\
\sum_{n=0}^{\infty} 18x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 18a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 14a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 12a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 18a_{n-1}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) + 5x^{n+r} a_n(n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 5x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 5x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r+2\right)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$\left(r+2\right)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r+2\right)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -2$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + 3a_{n-2}(n+r-2) \\ &+ 14a_{n-1}(n+r-1) + 5a_n(n+r) + 12a_{n-2} + 18a_{n-1} + 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-1} + 2ra_{n-1} + 3a_{n-2} + 4a_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = \frac{-2na_{n-1} - 3a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r + 13}{4 + r}$$

Which for the root $r = -2$ becomes

$$a_2 = \frac{5}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^2 - 60r - 106}{(4 + r)(5 + r)}$$

Which for the root $r = -2$ becomes

$$a_3 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$
a_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^3 + 204r^2 + 833r + 1077}{(6+r)(4+r)(5+r)}$$

Which for the root $r = -2$ becomes

$$a_4 = \frac{33}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$
a_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3
a_4	$\frac{16r^3+204r^2+833r+1077}{(6+r)(4+r)(5+r)}$	$\frac{33}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^4 - 608r^3 - 4198r^2 - 12418r - 13170}{(4+r)(5+r)(6+r)(7+r)}$$

Which for the root $r = -2$ becomes

$$a_5 = -\frac{129}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$
a_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3
a_4	$\frac{16r^3+204r^2+833r+1077}{(6+r)(4+r)(5+r)}$	$\frac{33}{8}$
a_5	$\frac{-32r^4-608r^3-4198r^2-12418r-13170}{(4+r)(5+r)(6+r)(7+r)}$	$-\frac{129}{20}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^5 + 1680r^4 + 17176r^3 + 85221r^2 + 204304r + 188103}{(8 + r)(4 + r)(5 + r)(6 + r)(7 + r)}$$

Which for the root $r = -2$ becomes

$$a_6 = \frac{867}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$
a_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3
a_4	$\frac{16r^3+204r^2+833r+1077}{(6+r)(4+r)(5+r)}$	$\frac{33}{8}$
a_5	$\frac{-32r^4-608r^3-4198r^2-12418r-13170}{(4+r)(5+r)(6+r)(7+r)}$	$-\frac{129}{20}$
a_6	$\frac{64r^5+1680r^4+17176r^3+85221r^2+204304r+188103}{(8+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{867}{80}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-128r^6 - 4416r^5 - 62000r^4 - 452424r^3 - 1804580r^2 - 3716136r - 3069774}{(4 + r)(5 + r)(6 + r)(7 + r)(8 + r)(9 + r)}$$

Which for the root $r = -2$ becomes

$$a_7 = -\frac{1059}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-2	-2
a_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$
a_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3
a_4	$\frac{16r^3+204r^2+833r+1077}{(6+r)(4+r)(5+r)}$	$\frac{33}{8}$
a_5	$\frac{-32r^4-608r^3-4198r^2-12418r-13170}{(4+r)(5+r)(6+r)(7+r)}$	$-\frac{129}{20}$
a_6	$\frac{64r^5+1680r^4+17176r^3+85221r^2+204304r+188103}{(8+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{867}{80}$
a_7	$\frac{-128r^6-4416r^5-62000r^4-452424r^3-1804580r^2-3716136r-3069774}{(4+r)(5+r)(6+r)(7+r)(8+r)(9+r)}$	$-\frac{1059}{56}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{1 - 2x + \frac{5x^2}{2} - 3x^3 + \frac{33x^4}{8} - \frac{129x^5}{20} + \frac{867x^6}{80} - \frac{1059x^7}{56} + O(x^8)}{x^2}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	-2	-2	0
b_2	$\frac{4r+13}{4+r}$	$\frac{5}{2}$	$\frac{3}{(4+r)^2}$
b_3	$\frac{-8r^2-60r-106}{(4+r)(5+r)}$	-3	$\frac{-12r^2-108r-246}{(4+r)^2(5+r)^2}$
b_4	$\frac{16r^3+204r^2+833r+1077}{(6+r)(4+r)(5+r)}$	$\frac{33}{8}$	$\frac{36r^4+702r^3+5130r^2+16650r+20262}{(6+r)^2(4+r)^2(5+r)^2}$
b_5	$\frac{-32r^4-608r^3-4198r^2-12418r-13170}{(4+r)(5+r)(6+r)(7+r)}$	$-\frac{129}{20}$	$-\frac{6(16r^6+510r^5+6745r^4+47376r^3+186407r^2+38}{(4+r)^2(5+r)^2(6+r)^2(7+r)^2}$
b_6	$\frac{64r^5+1680r^4+17176r^3+85221r^2+204304r+188103}{(8+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{867}{80}$	$\frac{240r^8+11088r^7+222897r^6+2546388r^5+18081150r}{(8+r)^2(4+r)^2(}$
b_7	$\frac{-128r^6-4416r^5-62000r^4-452424r^3-1804580r^2-3716136r-3069774}{(4+r)(5+r)(6+r)(7+r)(8+r)(9+r)}$	$-\frac{1059}{56}$	$-\frac{6(96r^{10}+6000r^9+167748r^8+2762496r^7+296738}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \frac{\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)\ln(x)}{x^2} \\
&\quad + \frac{\frac{3x^2}{4}-\frac{13x^3}{6}+\frac{407x^4}{96}-\frac{9047x^5}{1200}+\frac{63851x^6}{4800}-\frac{559033x^7}{23520}+O(x^8)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{\frac{3x^2}{4}-\frac{13x^3}{6}+\frac{407x^4}{96}-\frac{9047x^5}{1200}+\frac{63851x^6}{4800}-\frac{559033x^7}{23520}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{\frac{3x^2}{4}-\frac{13x^3}{6}+\frac{407x^4}{96}-\frac{9047x^5}{1200}+\frac{63851x^6}{4800}-\frac{559033x^7}{23520}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{\frac{3x^2}{4}-\frac{13x^3}{6}+\frac{407x^4}{96}-\frac{9047x^5}{1200}+\frac{63851x^6}{4800}-\frac{559033x^7}{23520}+O(x^8)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(1-2x+\frac{5x^2}{2}-3x^3+\frac{33x^4}{8}-\frac{129x^5}{20}+\frac{867x^6}{80}-\frac{1059x^7}{56}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{\frac{3x^2}{4}-\frac{13x^3}{6}+\frac{407x^4}{96}-\frac{9047x^5}{1200}+\frac{63851x^6}{4800}-\frac{559033x^7}{23520}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Verified OK.

15.12.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y''+(3x^3+14x^2+5x)\,y'+(12x^2+18x+4)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2(6x^2+9x+2)y}{x^2(1+2x)}-\frac{(3x^2+14x+5)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(3x^2+14x+5)y'}{x(1+2x)}+\frac{2(6x^2+9x+2)y}{x^2(1+2x)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{3x^2+14x+5}{x(1+2x)},P_3(x)=\frac{2(6x^2+9x+2)}{x^2(1+2x)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=5$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(1+2x)\,y''+x(3x^2+14x+5)\,y'+(12x^2+18x+4)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2+r)^{2}\,x^{r}+\left(a_{1}(3+r)^{2}+2a_{0}(3+r)^{2}\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r+2)^{2}+2a_{k-1}(k+r+2)^{2}+3a\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-2$

- Each term must be 0

  $a_{1}(3+r)^{2}+2a_{0}(3+r)^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=-2a_{0}$

- Each term in the series must be 0, giving the recursion relation

  $((2k+2r+4)\,a_{k-1}+a_{k}(k+r+2)+3a_{k-2})\,(k+r+2)=0$

- Shift index using $k->k+2$

  $((2k+8+2r)\,a_{k+1}+a_{k+2}(k+r+4)+3a_{k})\,(k+r+4)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2ka_{k+1}+2ra_{k+1}+3a_{k}+8a_{k+1}}{k+r+4}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2k a_{k+1}+3a_k+4a_{k+1}}{k+2}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2k a_{k+1}+3a_k+4a_{k+1}}{k+2}, a_1 = -2a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 79

```
Order:=8;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+14*x+3*x^2)*diff(y(x),x)+(4+18*x+12*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - 2x + \frac{5}{2}x^2 - 3x^3 + \frac{33}{8}x^4 - \frac{129}{20}x^5 + \frac{867}{80}x^6 - \frac{1059}{56}x^7 + \mathrm{O}(x^8)\right) + \left(\frac{3}{4}x^2 - \frac{13}{6}x^3 + \frac{407}{96}x^4 \cdots\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 157

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]+x*(5+14*x+3*x^2)*y'[x]+(4+18*x+12*x^2)*y[x]==0,y[x]
```

$$y(x) \to \frac{c_1\left(-\frac{1059x^7}{56} + \frac{867x^6}{80} - \frac{129x^5}{20} + \frac{33x^4}{8} - 3x^3 + \frac{5x^2}{2} - 2x + 1\right)}{x^2} + c_2\left(\frac{-\frac{559033x^7}{23520} + \frac{63851x^6}{4800} - \frac{9047x^5}{1200} + \frac{407x^4}{96} - \frac{13x^3}{6} + \frac{3x^2}{4}}{x^2} + \frac{\left(-\frac{1059x^7}{56} + \frac{867x^6}{80} - \frac{129x^5}{20} + \frac{33x^4}{8} - 3x^3 + \frac{5x^2}{2} - 2x + 1\right)\log(x)}{x^2}\right)$$

15.13 problem 9

Internal problem ID [1361]
Internal file name [`OUTPUT/1362_Sunday_June_05_2022_02_13_10_AM_25547940/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x(x^2+x+4)\,y' + (3x^2+5x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (2x^3+2x^2+8x)\,y' + (3x^2+5x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{x^2+x+4}{2x} \\ q(x) &= \frac{3x^2+5x+1}{4x^2} \end{aligned}$$

Table 668: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+x+4}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{3x^2+5x+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (2x^3 + 2x^2 + 8x)\, y' + (3x^2 + 5x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + (2x^3+2x^2+8x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) + (3x^2+5x+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} 3x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r} \\
\sum_{n=0}^{\infty} 5x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) + 8x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 8x^r a_0 r + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 8x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2r-5}{(2r+3)^2}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &4a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) \\ &+ 2a_{n-1}(n+r-1) + 8a_n(n+r) + 3a_{n-2} + 5a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-2} + 2na_{n-1} + 2ra_{n-2} + 2ra_{n-1} - a_{n-2} + 3a_{n-1}}{4n^2 + 8nr + 4r^2 + 4n + 4r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = \frac{(-a_{n-2} - a_{n-1})\,n + a_{n-2} - a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4r^3 - 16r^2 - 15r + 4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1
a_2	$\frac{-4r^3-16r^2-15r+4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{32r^4 + 296r^3 + 948r^2 + 1254r + 553}{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{7}{2}\right)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_3 = \frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1
a_2	$\frac{-4r^3-16r^2-15r+4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$	$\frac{1}{4}$
a_3	$\frac{32r^4+296r^3+948r^2+1254r+553}{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{7}{2}\right)^2}$	$\frac{1}{18}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64r^6 + 864r^5 + 4336r^4 + 9456r^3 + 6188r^2 - 6962r - 8827}{(2r+3)^2(2r+5)^2(2r+7)^2(4r^2+36r+81)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = -\frac{37}{1152}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1
a_2	$\frac{-4r^3-16r^2-15r+4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$	$\frac{1}{4}$
a_3	$\frac{32r^4+296r^3+948r^2+1254r+553}{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{7}{2}\right)^2}$	$\frac{1}{18}$
a_4	$\frac{64r^6+864r^5+4336r^4+9456r^3+6188r^2-6962r-8827}{(2r+3)^2(2r+5)^2(2r+7)^2(4r^2+36r+81)}$	$-\frac{37}{1152}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-192r^7 - 4192r^6 - 37504r^5 - 177440r^4 - 475836r^3 - 713390r^2 - 537382r - 144193}{512\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(\frac{11}{2}+r\right)^2\left(r+\frac{7}{2}\right)^2\left(r+\frac{9}{2}\right)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_5 = -\frac{17}{28800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1
a_2	$\frac{-4r^3-16r^2-15r+4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$	$\frac{1}{4}$
a_3	$\frac{32r^4+296r^3+948r^2+1254r+553}{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{7}{2}\right)^2}$	$\frac{1}{18}$
a_4	$\frac{64r^6+864r^5+4336r^4+9456r^3+6188r^2-6962r-8827}{(2r+3)^2(2r+5)^2(2r+7)^2(4r^2+36r+81)}$	$-\frac{37}{1152}$
a_5	$\frac{-192r^7-4192r^6-37504r^5-177440r^4-475836r^3-713390r^2-537382r-144193}{512\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(\frac{11}{2}+r\right)^2\left(r+\frac{7}{2}\right)^2\left(r+\frac{9}{2}\right)^2}$	$-\frac{17}{28800}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-512r^9 - 14592r^8 - 172672r^7 - 1084672r^6 - 3760736r^5 - 6170848r^4 + 1039816r^3 + 21534576r^2 +}{\left(4r^2+52r+169\right)(2r+3)^2(2r+5)^2(11+2r)^2(2r+7)^2(2r+9)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_6 = \frac{593}{259200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r-5}{(2r+3)^2}$	-1
a_2	$\frac{-4r^3-16r^2-15r+4}{8(r+\frac{3}{2})^2(r+\frac{5}{2})^2}$	$\frac{1}{4}$
a_3	$\frac{32r^4+296r^3+948r^2+1254r+553}{64(r+\frac{3}{2})^2(r+\frac{5}{2})^2(r+\frac{7}{2})^2}$	$\frac{1}{18}$
a_4	$\frac{64r^6+864r^5+4336r^4+9456r^3+6188r^2-6962r-8827}{(2r+3)^2(2r+5)^2(2r+7)^2(4r^2+36r+81)}$	$-\frac{37}{1152}$
a_5	$\frac{-192r^7-4192r^6-37504r^5-177440r^4-475836r^3-713390r^2-537382r-144193}{512(r+\frac{3}{2})^2(r+\frac{5}{2})^2(\frac{11}{2}+r)^2(r+\frac{7}{2})^2(r+\frac{9}{2})^2}$	$-\frac{17}{28800}$
a_6	$\frac{-512r^9-14592r^8-172672r^7-1084672r^6-3760736r^5-6170848r^4+1039816r^3+21534576r^2+32373056r+16074527}{(4r^2+52r+169)(2r+3)^2(2r+5)^2(11+2r)^2(2r+7)^2(2r+9)^2}$	$\frac{593}{259200}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{4096r^{10} + 164864r^9 + 2890752r^8 + 28990080r^7 + 183413312r^6 + 760790176r^5 + 2078667344r^4 + 36}{16384\left(r+\frac{3}{2}\right)^2\left(r+\frac{15}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(\frac{11}{2}+r\right)^2\left(r+\frac{7}{2}\right)^2(r+}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_7 = -\frac{1913}{12700800}$$

And the table now becomes

n	$a_{n,r}$
a_0	1
a_1	$\frac{-2r-5}{(2r+3)^{2}}$
a_2	$\frac{-4r^{3}-16r^{2}-15r+4}{8\left(r+\frac{3}{2}\right)^{2}\left(r+\frac{5}{2}\right)^{2}}$
a_3	$\frac{32r^{4}+296r^{3}+948r^{2}+1254r+553}{64\left(r+\frac{3}{2}\right)^{2}\left(r+\frac{5}{2}\right)^{2}\left(r+\frac{7}{2}\right)^{2}}$
a_4	$\frac{64r^{6}+864r^{5}+4336r^{4}+9456r^{3}+6188r^{2}-6962r-8827}{(2r+3)^{2}(2r+5)^{2}(2r+7)^{2}(4r^{2}+36r+81)}$
a_5	$\frac{-192r^{7}-4192r^{6}-37504r^{5}-177440r^{4}-475836r^{3}-713390r^{2}-537382r-144193}{512\left(r+\frac{3}{2}\right)^{2}\left(r+\frac{5}{2}\right)^{2}\left(\frac{11}{2}+r\right)^{2}\left(r+\frac{7}{2}\right)^{2}\left(r+\frac{9}{2}\right)^{2}}$
a_6	$\frac{-512r^{9}-14592r^{8}-172672r^{7}-1084672r^{6}-3760736r^{5}-6170848r^{4}+1039816r^{3}+21534576r^{2}+32373056r+16074527}{(4r^{2}+52r+169)(2r+3)^{2}(2r+5)^{2}(11+2r)^{2}(2r+7)^{2}(2r+9)^{2}}$
a_7	$\frac{4096r^{10}+164864r^{9}+2890752r^{8}+28990080r^{7}+183413312r^{6}+760790176r^{5}+2078667344r^{4}+3646802552r^{3}+3838600668r^{2}+2071188906r-}{16384\left(r+\frac{3}{2}\right)^{2}\left(r+\frac{15}{2}\right)^{2}\left(r+\frac{5}{2}\right)^{2}\left(\frac{11}{2}+r\right)^{2}\left(r+\frac{7}{2}\right)^{2}\left(r+\frac{13}{2}\right)^{2}\left(r+\frac{9}{2}\right)^{2}}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{4} + \frac{x^3}{18} - \frac{37x^4}{1152} - \frac{17x^5}{28800} + \frac{593x^6}{259200} - \frac{1913x^7}{12700800} + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -\frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$
b_0	1
b_1	$\frac{-2r-5}{(2r+3)^2}$
b_2	$\frac{-4r^3-16r^2-15r+4}{8\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2}$
b_3	$\frac{32r^4+296r^3+948r^2+1254r+553}{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{7}{2}\right)^2}$
b_4	$\frac{64r^6+864r^5+4336r^4+9456r^3+6188r^2-6962r-8827}{(2r+3)^2(2r+5)^2(2r+7)^2(4r^2+36r+81)}$
b_5	$\frac{-192r^7-4192r^6-37504r^5-177440r^4-475836r^3-713390r^2-537382r-144193}{512\left(r+\frac{3}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(\frac{11}{2}+r\right)^2\left(r+\frac{7}{2}\right)^2\left(r+\frac{9}{2}\right)^2}$
b_6	$\frac{-512r^9-14592r^8-172672r^7-1084672r^6-3760736r^5-6170848r^4+1039816r^3+21534576r^2+32373056r+16074527}{(4r^2+52r+169)(2r+3)^2(2r+5)^2(11+2r)^2(2r+7)^2(2r+9)^2}$
b_7	$\frac{4096r^{10}+164864r^9+2890752r^8+28990080r^7+183413312r^6+760790176r^5+2078667344r^4+3646802552r^3+3838600668r^2+2071188906r+}{16384\left(r+\frac{3}{2}\right)^2\left(r+\frac{15}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(\frac{11}{2}+r\right)^2\left(r+\frac{7}{2}\right)^2\left(r+\frac{13}{2}\right)^2\left(r+\frac{9}{2}\right)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \frac{\left(1 - x + \frac{x^2}{4} + \frac{x^3}{18} - \frac{37x^4}{1152} - \frac{17x^5}{28800} + \frac{593x^6}{259200} - \frac{1913x^7}{12700800} + O(x^8)\right)\ln(x)}{\sqrt{x}} \\
&\quad + \frac{\frac{3x}{2} - \frac{13x^2}{16} + \frac{x^3}{54} + \frac{1103x^4}{13824} - \frac{19507x^5}{1728000} - \frac{98531x^6}{20736000} + \frac{982189x^7}{889056000} + O(x^8)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - x + \frac{x^2}{4} + \frac{x^3}{18} - \frac{37x^4}{1152} - \frac{17x^5}{28800} + \frac{593x^6}{259200} - \frac{1913x^7}{12700800} + O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{x^2}{4} + \frac{x^3}{18} - \frac{37x^4}{1152} - \frac{17x^5}{28800} + \frac{593x^6}{259200} - \frac{1913x^7}{12700800} + O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\quad \left. + \frac{\frac{3x}{2} - \frac{13x^2}{16} + \frac{x^3}{54} + \frac{1103x^4}{13824} - \frac{19507x^5}{1728000} - \frac{98531x^6}{20736000} + \frac{982189x^7}{889056000} + O(x^8)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\quad \left. + \frac{\frac{3x}{2}-\frac{13x^2}{16}+\frac{x^3}{54}+\frac{1103x^4}{13824}-\frac{19507x^5}{1728000}-\frac{98531x^6}{20736000}+\frac{982189x^7}{889056000}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\quad \left. + \frac{\frac{3x}{2}-\frac{13x^2}{16}+\frac{x^3}{54}+\frac{1103x^4}{13824}-\frac{19507x^5}{1728000}-\frac{98531x^6}{20736000}+\frac{982189x^7}{889056000}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)}{\sqrt{x}} \\
&\quad + c_2\left(\frac{\left(1-x+\frac{x^2}{4}+\frac{x^3}{18}-\frac{37x^4}{1152}-\frac{17x^5}{28800}+\frac{593x^6}{259200}-\frac{1913x^7}{12700800}+O(x^8)\right)\ln(x)}{\sqrt{x}}\right. \\
&\quad \left. + \frac{\frac{3x}{2}-\frac{13x^2}{16}+\frac{x^3}{54}+\frac{1103x^4}{13824}-\frac{19507x^5}{1728000}-\frac{98531x^6}{20736000}+\frac{982189x^7}{889056000}+O(x^8)}{\sqrt{x}}\right)
\end{aligned}
$$

Verified OK.

15.13.1 Maple step by step solution

Let's solve

$4x^2y'' + (2x^3 + 2x^2 + 8x)\, y' + (3x^2 + 5x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2+5x+1)y}{4x^2} - \frac{(x^2+x+4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+x+4)y'}{2x} + \frac{(3x^2+5x+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2+x+4}{2x}, P_3(x) = \frac{3x^2+5x+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 2x(x^2 + x + 4)\, y' + (3x^2 + 5x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+2r)^{2}\,x^{r}+\left(a_{1}(3+2r)^{2}+a_{0}(5+2r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(2k+2r+1)^{2}+a_{k-1}(2k+2r+3)\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+2r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-\frac{1}{2}$

- Each term must be 0

  $a_{1}(3+2r)^{2}+a_{0}(5+2r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=-\frac{a_{0}(5+2r)}{4r^{2}+12r+9}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(2k+2r+1)^{2}+a_{k-1}(2k+2r+3)+a_{k-2}(2k-1+2r)=0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+2r+5)^{2}+a_{k+1}(2k+7+2r)+a_{k}(2k+2r+3)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2ka_{k}+2ka_{k+1}+2ra_{k}+2ra_{k+1}+3a_{k}+7a_{k+1}}{(2k+2r+5)^{2}}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{2ka_{k}+2ka_{k+1}+2a_{k}+6a_{k+1}}{(2k+4)^{2}}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2k a_k + 2k a_{k+1} + 2a_k + 6a_{k+1}}{(2k+4)^2}, a_1 = -a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 81

```
Order:=8;
dsolve(4*x^2*diff(y(x),x$2)+2*x*(4+x+x^2)*diff(y(x),x)+(1+5*x+3*x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{1}{4}x^2 + \frac{1}{18}x^3 - \frac{37}{1152}x^4 - \frac{17}{28800}x^5 + \frac{593}{259200}x^6 - \frac{1913}{12700800}x^7 + \mathrm{O}(x^8)\right) + \left(\frac{3}{2}x - \frac{13}{16}x^2 \right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 172

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x*(4+x+x^2)*y'[x]+(1+5*x+3*x^2)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{1913x^7}{12700800}+\frac{593x^6}{259200}-\frac{17x^5}{28800}-\frac{37x^4}{1152}+\frac{x^3}{18}+\frac{x^2}{4}-x+1\right)}{\sqrt{x}} + c_2\left(\frac{\frac{982189x^7}{889056000}-\frac{98531x^6}{20736000}-\frac{19507x^5}{1728000}+\frac{1103x^4}{13824}+\frac{x^3}{54}-\frac{13x^2}{16}+\frac{3x}{2}}{\sqrt{x}} + \frac{\left(-\frac{1913x^7}{12700800}+\frac{593x^6}{259200}-\frac{17x^5}{28800}-\frac{37x^4}{1152}+\frac{x^3}{18}+\frac{x^2}{4}-x+1\right)\log(x)}{\sqrt{x}}\right)$$

15.14 problem 10

Internal problem ID [1362]
Internal file name [`OUTPUT/1363_Sunday_June_05_2022_02_13_13_AM_74302315/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$16x^2y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{2x^2 + x + 6}{4x} \\ q(x) &= \frac{18x^2 + 5x + 1}{16x^2} \end{aligned}$$

Table 670: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+x+6}{4x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{18x^2+5x+1}{16x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$16x^2y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$16x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (8x^3+4x^2+24x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (18x^2+5x+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 24x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} 18x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 8a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 4a_{n-1} (n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 18x^{n+r+2} a_n = \sum_{n=2}^{\infty} 18a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n = \sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=2}^{\infty} 8a_{n-2} (n+r-2)\, x^{n+r}\right) \\ + \left(\sum_{n=1}^{\infty} 4a_{n-1} (n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 24x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=2}^{\infty} 18a_{n-2} x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$16x^{n+r} a_n (n+r)(n+r-1) + 24x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$16x^r a_0 r(-1+r) + 24x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(16x^r r(-1+r) + 24x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(4r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(4r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{4}$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{4}$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{4}} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{4}}\right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = -\frac{1}{4r+5}
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&16a_n(n+r)(n+r-1) + 8a_{n-2}(n+r-2) \\
&+ 4a_{n-1}(n+r-1) + 24a_n(n+r) + 18a_{n-2} + 5a_{n-1} + a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2a_{n-2} + a_{n-1}}{4n + 4r + 1} \tag{4}
$$

Which for the root $r = -\frac{1}{4}$ becomes

$$
a_n = \frac{-2a_{n-2} - a_{n-1}}{4n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{-9 - 8r}{16r^2 + 56r + 45}
$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_2 = -\frac{7}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{16r + 27}{64r^3 + 432r^2 + 908r + 585}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_3 = \frac{23}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$
a_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64r^2 + 264r + 207}{256r^4 + 2816r^3 + 10976r^2 + 17776r + 9945}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_4 = \frac{145}{6144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$
a_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$
a_4	$\frac{64r^2+264r+207}{256r^4+2816r^3+10976r^2+17776r+9945}$	$\frac{145}{6144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-192r^2 - 1024r - 1125}{1024r^5 + 16640r^4 + 103040r^3 + 301600r^2 + 413076r + 208845}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_5 = -\frac{881}{122880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$
a_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$
a_4	$\frac{64r^2+264r+207}{256r^4+2816r^3+10976r^2+17776r+9945}$	$\frac{145}{6144}$
a_5	$\frac{-192r^2-1024r-1125}{1024r^5+16640r^4+103040r^3+301600r^2+413076r+208845}$	$-\frac{881}{122880}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-512r^3 - 4608r^2 - 11720r - 7569}{4096r^6 + 92160r^5 + 828160r^4 + 3782400r^3 + 9192304r^2 + 11162280r + 5221125}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_6 = -\frac{4919}{2949120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$
a_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$
a_4	$\frac{64r^2+264r+207}{256r^4+2816r^3+10976r^2+17776r+9945}$	$\frac{145}{6144}$
a_5	$\frac{-192r^2-1024r-1125}{1024r^5+16640r^4+103040r^3+301600r^2+413076r+208845}$	$-\frac{881}{122880}$
a_6	$\frac{-512r^3-4608r^2-11720r-7569}{4096r^6+92160r^5+828160r^4+3782400r^3+9192304r^2+11162280r+5221125}$	$-\frac{4919}{2949120}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{2048r^3 + 22400r^2 + 71920r + 63819}{16384r^7 + 487424r^6 + 5985280r^5 + 39146240r^4 + 146458816r^3 + 311225936r^2 + 344590620r + 151}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_7 = \frac{47207}{82575360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$
a_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$
a_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$
a_4	$\frac{64r^2+264r+207}{256r^4+2816r^3+10976r^2+17776r+9945}$	$\frac{145}{6144}$
a_5	$\frac{-192r^2-1024r-1125}{1024r^5+16640r^4+103040r^3+301600r^2+413076r+208845}$	$-\frac{881}{122880}$
a_6	$\frac{-512r^3-4608r^2-11720r-7569}{4096r^6+92160r^5+828160r^4+3782400r^3+9192304r^2+11162280r+5221125}$	$-\frac{4919}{2949120}$
a_7	$\frac{2048r^3+22400r^2+71920r+63819}{16384r^7+487424r^6+5985280r^5+39146240r^4+146458816r^3+311225936r^2+344590620r+151412625}$	$\frac{47207}{82575360}$

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = \frac{1}{x^{\frac{1}{4}}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right)$$

$$= \frac{1 - \frac{x}{4} - \frac{7x^2}{32} + \frac{23x^3}{384} + \frac{145x^4}{6144} - \frac{881x^5}{122880} - \frac{4919x^6}{2949120} + \frac{47207x^7}{82575360} + O(x^8)}{x^{\frac{1}{4}}}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -\frac{1}{4}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n
b_1	$-\frac{1}{4r+5}$	$-\frac{1}{4}$	$\frac{4}{(4r+5)^2}$
b_2	$\frac{-9-8r}{16r^2+56r+45}$	$-\frac{7}{32}$	$\frac{128r^2+288r+144}{(16r^2+56r+45)^2}$
b_3	$\frac{16r+27}{64r^3+432r^2+908r+585}$	$\frac{23}{384}$	$\frac{-2048r^3-12096r^2}{(64r^3+432r^2+}$
b_4	$\frac{64r^2+264r+207}{256r^4+2816r^3+10976r^2+17776r+9945}$	$\frac{145}{6144}$	$-\frac{8(4096r^5+4787}{(256r^4}$
b_5	$\frac{-192r^2-1024r-1125}{1024r^5+16640r^4+103040r^3+301600r^2+413076r+208845}$	$-\frac{881}{122880}$	$\frac{589824r^6+105840}{(10}$
b_6	$\frac{-512r^3-4608r^2-11720r-7569}{4096r^6+92160r^5+828160r^4+3782400r^3+9192304r^2+11162280r+5221125}$	$-\frac{4919}{2949120}$	$6291456r^8+16986$
b_7	$\frac{2048r^3+22400r^2+71920r+63819}{16384r^7+487424r^6+5985280r^5+39146240r^4+146458816r^3+311225936r^2+344590620r+151412625}$	$\frac{47207}{82575360}$	$-\frac{4(33554432r^9+}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \frac{\left(1 - \frac{x}{4} - \frac{7x^2}{32} + \frac{23x^3}{384} + \frac{145x^4}{6144} - \frac{881x^5}{122880} - \frac{4919x^6}{2949120} + \frac{47207x^7}{82575360} + O(x^8)\right)\ln(x)}{x^{\frac{1}{4}}} \\
&\quad + \frac{\frac{x}{4} + \frac{5x^2}{64} - \frac{157x^3}{2304} - \frac{841x^4}{73728} + \frac{65017x^5}{7372800} + \frac{50791x^6}{58982400} - \frac{953509x^7}{1284505600} + O(x^8)}{x^{\frac{1}{4}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= \frac{c_1\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)}{x^{\frac{1}{4}}} + c_2\left(\frac{\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{\frac{x}{4}+\frac{5x^2}{64}-\frac{157x^3}{2304}-\frac{841x^4}{73728}+\frac{65017x^5}{7372800}+\frac{50791x^6}{58982400}-\frac{953509x^7}{1284505600}+O(x^8)}{x^{\frac{1}{4}}}\right)$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)}{x^{\frac{1}{4}}} + c_2\left(\frac{\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{\frac{x}{4}+\frac{5x^2}{64}-\frac{157x^3}{2304}-\frac{841x^4}{73728}+\frac{65017x^5}{7372800}+\frac{50791x^6}{58982400}-\frac{953509x^7}{1284505600}+O(x^8)}{x^{\frac{1}{4}}}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)}{x^{\frac{1}{4}}} + c_2\left(\frac{\left(1-\frac{x}{4}-\frac{7x^2}{32}+\frac{23x^3}{384}+\frac{145x^4}{6144}-\frac{881x^5}{122880}-\frac{4919x^6}{2949120}+\frac{47207x^7}{82575360}+O(x^8)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{\frac{x}{4}+\frac{5x^2}{64}-\frac{157x^3}{2304}-\frac{841x^4}{73728}+\frac{65017x^5}{7372800}+\frac{50791x^6}{58982400}-\frac{953509x^7}{1284505600}+O(x^8)}{x^{\frac{1}{4}}}\right) \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \left(1-\frac{x}{4}-\frac{7 x^{2}}{32}+\frac{23 x^{3}}{384}+\frac{145 x^{4}}{6144}-\frac{881 x^{5}}{122880}-\frac{4919 x^{6}}{2949120}+\frac{47207 x^{7}}{82575360}+O\left(x^{8}\right)\right)}{x^{\frac{1}{4}}} + c_2 \left(\frac{\left(1-\frac{x}{4}-\frac{7 x^{2}}{32}+\frac{23 x^{3}}{384}+\frac{145 x^{4}}{6144}-\frac{881 x^{5}}{122880}-\frac{4919 x^{6}}{2949120}+\frac{47207 x^{7}}{82575360}+O\left(x^{8}\right)\right) \ln \left(x \right)}{x^{\frac{1}{4}}} + \frac{\frac{x}{4}+\frac{5 x^{2}}{64}-\frac{157 x^{3}}{2304}-\frac{841 x^{4}}{73728}+\frac{65017 x^{5}}{7372800}+\frac{50791 x^{6}}{58982400}-\frac{953509 x^{7}}{1284505600}+O\left(x^{8}\right)}{x^{\frac{1}{4}}}\right)$$

Verified OK.

15.14.1 Maple step by step solution

Let's solve

$16x^2 y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(18x^2+5x+1)y}{16x^2} - \frac{(2x^2+x+6)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+x+6)y'}{4x} + \frac{(18x^2+5x+1)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{3}{2}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{16}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$16x^2 y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 4r)^2 x^r + \left(a_1(5 + 4r)^2 + a_0(5 + 4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k + 4r + 1)^2 + a_{k-1}(4k + 4r + 1)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1 + 4r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = -\frac{1}{4}$

- Each term must be 0

$a_1(5 + 4r)^2 + a_0(5 + 4r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = -\frac{a_0}{5+4r}$

- Each term in the series must be 0, giving the recursion relation

$a_k(4k+4r+1)^2 + (4k+4r+1)(2a_{k-2} + a_{k-1}) = 0$

- Shift index using $k- > k+2$

$a_{k+2}(4k+4r+9)^2 + (4k+4r+9)(2a_k + a_{k+1}) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$

- Recursion relation for $r = -\frac{1}{4}$

$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$

- Solution for $r = -\frac{1}{4}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 81

```
Order:=8;
dsolve(16*x^2*diff(y(x),x$2)+4*x*(6+x+2*x^2)*diff(y(x),x)+(1+5*x+18*x^2)*y(x)=0,y(x),type='s
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x - \frac{7}{32}x^2 + \frac{23}{384}x^3 + \frac{145}{6144}x^4 - \frac{881}{122880}x^5 - \frac{4919}{2949120}x^6 + \frac{47207}{82575360}x^7 + O(x^8)\right) + \left(\frac{1}{4}x + \right.}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 176

```
AsymptoticDSolveValue[16*x^2*y''[x]+4*x*(6+x+2*x^2)*y'[x]+(1+5*x+18*x^2)*y[x]==0,y[x],{x,0,7
```

$$y(x) \to \frac{c_1\left(\frac{47207x^7}{82575360} - \frac{4919x^6}{2949120} - \frac{881x^5}{122880} + \frac{145x^4}{6144} + \frac{23x^3}{384} - \frac{7x^2}{32} - \frac{x}{4} + 1\right)}{\sqrt[4]{x}} + c_2\left(\frac{-\frac{953509x^7}{1284505600} + \frac{50791x^6}{58982400} + \frac{65017x^5}{7372800} - \frac{841x^4}{73728} - \frac{157x^3}{2304} + \frac{5x^2}{64} + \frac{x}{4}}{\sqrt[4]{x}} + \frac{\left(\frac{47207x^7}{82575360} - \frac{4919x^6}{2949120} - \frac{881x^5}{122880} + \frac{145x^4}{6144} + \frac{23x^3}{384} - \frac{7x^2}{32} - \frac{x}{4} + 1\right)\log(x)}{\sqrt[4]{x}}\right)$$

15.15 problem 11

Internal problem ID [1363]
Internal file name [OUTPUT/1364_Sunday_June_05_2022_02_13_17_AM_2731518/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x+1)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^2-11x-5}{3x\,(x+1)}$$
$$q(x)=-\frac{7x^2-16x-1}{9x^2\,(x+1)}$$

Table 672: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-11x-5}{3x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{7x^2-16x-1}{9x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2(x+1)\, y'' + \left(-3x^3 + 33x^2 + 15x\right) y' + \left(-7x^2 + 16x + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-3x^3 + 33x^2 + 15x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-7x^2 + 16x + 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)\,(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-3x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 33x^{1+n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 15x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-7x^{n+r+2} a_n\right) \\
&+\left(\sum_{n=0}^{\infty} 16x^{1+n+r} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 9x^{1+n+r} a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-3x^{n+r+2} a_n(n+r)\right) &= \sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty} 33x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} 33a_{n-1}(n+r-1)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-7x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty}\left(-7a_{n-2}x^{n+r}\right) \\
\sum_{n=0}^{\infty} 16x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 16a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\left(n+r-2\right)x^{n+r}\right)\\&+\left(\sum_{n=0}^{\infty} 9x^{n+r}a_n(n+r)\left(n+r-1\right)\right)+\sum_{n=2}^{\infty}\left(-3a_{n-2}(n+r-2)\,x^{n+r}\right)\\&+\left(\sum_{n=1}^{\infty} 33a_{n-1}(n+r-1)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 15x^{n+r}a_n(n+r)\right)\\&+\sum_{n=2}^{\infty}\left(-7a_{n-2}x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 16a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r}a_n(n+r)\left(n+r-1\right)+15x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r)+15x^r a_0 r+a_0 x^r=0$$

Or

$$\left(9x^r r(-1+r)+15x^r r+x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$x^r(3r+1)^2=0$$

Since the above is true for all x then the indicial equation becomes

$$(3r+1)^2=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=-\frac{1}{3}\\r_2&=-\frac{1}{3}\end{aligned}$$

Since $a_0\neq 0$ then the indicial equation becomes

$$x^r(3r+1)^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -1$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 9a_{n-1}(n+r-1)(n+r-2) + 9a_n(n+r)(n+r-1) - 3a_{n-2}(n+r-2) \\ + 33a_{n-1}(n+r-1) + 15a_n(n+r) - 7a_{n-2} + 16a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3na_{n-1} + 3ra_{n-1} - a_{n-2} + a_{n-1}}{1 + 3n + 3r} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_n = \frac{-3na_{n-1} + a_{n-2}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r + 8}{7 + 3r}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_2 = \frac{7}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-9r^2 - 57r - 87}{9r^2 + 51r + 70}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_3 = -\frac{23}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$
a_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{27r^3 + 297r^2 + 1056r + 1211}{27r^3 + 270r^2 + 873r + 910}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_4 = \frac{11}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$
a_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$
a_4	$\frac{27r^3+297r^2+1056r+1211}{27r^3+270r^2+873r+910}$	$\frac{11}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-81r^4 - 1350r^3 - 8208r^2 - 21531r - 20507}{(16 + 3r)(9r^2 + 51r + 70)(13 + 3r)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_5 = -\frac{1577}{1080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$
a_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$
a_4	$\frac{27r^3+297r^2+1056r+1211}{27r^3+270r^2+873r+910}$	$\frac{11}{8}$
a_5	$\frac{-81r^4-1350r^3-8208r^2-21531r-20507}{(16+3r)(9r^2+51r+70)(13+3r)}$	$-\frac{1577}{1080}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{243r^5 + 5670r^4 + 51597r^3 + 228465r^2 + 491139r + 409009}{243r^5 + 5265r^4 + 44415r^3 + 181935r^2 + 360942r + 276640}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_6 = \frac{3319}{2160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$
a_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$
a_4	$\frac{27r^3+297r^2+1056r+1211}{27r^3+270r^2+873r+910}$	$\frac{11}{8}$
a_5	$\frac{-81r^4-1350r^3-8208r^2-21531r-20507}{(16+3r)(9r^2+51r+70)(13+3r)}$	$-\frac{1577}{1080}$
a_6	$\frac{243r^5+5670r^4+51597r^3+228465r^2+491139r+409009}{243r^5+5265r^4+44415r^3+181935r^2+360942r+276640}$	$\frac{3319}{2160}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-729r^6 - 22599r^5 - 285120r^4 - 1870803r^3 - 6720192r^2 - 12502695r - 9387831}{(22 + 3r)(13 + 3r)(9r^2 + 51r + 70)(16 + 3r)(19 + 3r)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$a_7 = -\frac{72853}{45360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$
a_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$
a_4	$\frac{27r^3+297r^2+1056r+1211}{27r^3+270r^2+873r+910}$	$\frac{11}{8}$
a_5	$\frac{-81r^4-1350r^3-8208r^2-21531r-20507}{(16+3r)(9r^2+51r+70)(13+3r)}$	$-\frac{1577}{1080}$
a_6	$\frac{243r^5+5670r^4+51597r^3+228465r^2+491139r+409009}{243r^5+5265r^4+44415r^3+181935r^2+360942r+276640}$	$\frac{3319}{2160}$
a_7	$\frac{-729r^6-22599r^5-285120r^4-1870803r^3-6720192r^2-12502695r-9387831}{(22+3r)(13+3r)(9r^2+51r+70)(16+3r)(19+3r)}$	$-\frac{72853}{45360}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x^{\frac{1}{3}}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -\frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	-1	-1	0
b_2	$\frac{3r+8}{7+3r}$	$\frac{7}{6}$	$-\frac{3}{(7+3r)^2}$
b_3	$\frac{-9r^2-57r-87}{9r^2+51r+70}$	$-\frac{23}{18}$	$\frac{54r^2+306r+447}{(9r^2+51r+70)^2}$
b_4	$\frac{27r^3+297r^2+1056r+1211}{27r^3+270r^2+873r+910}$	$\frac{11}{8}$	$\frac{-729r^4-9882r^3-50220r^2-113400r-96243}{(27r^3+270r^2+873r+910)^2}$
b_5	$\frac{-81r^4-1350r^3-8208r^2-21531r-20507}{(16+3r)(9r^2+51r+70)(13+3r)}$	$-\frac{1577}{1080}$	$\frac{8748r^6+205578r^5+2001105r^4+10325232r^3+}{(16+3r)^2(9r^2+51r+7}$
b_6	$\frac{243r^5+5670r^4+51597r^3+228465r^2+491139r+409009}{243r^5+5265r^4+44415r^3+181935r^2+360942r+276640}$	$\frac{3319}{2160}$	$\frac{-98415r^8-3490452r^7-53745525r^6-469145}{(243r^5+5}$
b_7	$\frac{-729r^6-22599r^5-285120r^4-1870803r^3-6720192r^2-12502695r-9387831}{(22+3r)(13+3r)(9r^2+51r+70)(16+3r)(19+3r)}$	$-\frac{72853}{45360}$	$\frac{1062882r^{10}+52553610r^9+1159663311r^8+15}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \frac{\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}} \\
&\quad + \frac{-\frac{x^2}{12} + \frac{13x^3}{108} - \frac{131x^4}{864} + \frac{11449x^5}{64800} - \frac{76919x^6}{388800} + \frac{4118557x^7}{19051200} + O(x^8)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)}{x^{\frac{1}{3}}} \\
&\quad + c_2\left(\frac{\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)\ln(x)}{x^{\frac{1}{3}}}\right. \\
&\quad \left. + \frac{-\frac{x^2}{12} + \frac{13x^3}{108} - \frac{131x^4}{864} + \frac{11449x^5}{64800} - \frac{76919x^6}{388800} + \frac{4118557x^7}{19051200} + O(x^8)}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$
$$= \frac{c_1 \left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)}{x^{\frac{1}{3}}} + c_2 \left(\frac{\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right) \ln(x)}{x^{\frac{1}{3}}} + \frac{-\frac{x^2}{12} + \frac{13x^3}{108} - \frac{131x^4}{864} + \frac{11449x^5}{64800} - \frac{76919x^6}{388800} + \frac{4118557x^7}{19051200} + O(x^8)}{x^{\frac{1}{3}}} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)}{x^{\frac{1}{3}}} + c_2 \left(\frac{\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right) \ln(x)}{x^{\frac{1}{3}}} + \frac{-\frac{x^2}{12} + \frac{13x^3}{108} - \frac{131x^4}{864} + \frac{11449x^5}{64800} - \frac{76919x^6}{388800} + \frac{4118557x^7}{19051200} + O(x^8)}{x^{\frac{1}{3}}} \right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right)}{x^{\frac{1}{3}}} + c_2 \left(\frac{\left(1 - x + \frac{7x^2}{6} - \frac{23x^3}{18} + \frac{11x^4}{8} - \frac{1577x^5}{1080} + \frac{3319x^6}{2160} - \frac{72853x^7}{45360} + O(x^8)\right) \ln(x)}{x^{\frac{1}{3}}} + \frac{-\frac{x^2}{12} + \frac{13x^3}{108} - \frac{131x^4}{864} + \frac{11449x^5}{64800} - \frac{76919x^6}{388800} + \frac{4118557x^7}{19051200} + O(x^8)}{x^{\frac{1}{3}}} \right)$$

Verified OK.

15.15.1 Maple step by step solution

Let's solve

$9x^2(x+1)\,y''+(-3x^3+33x^2+15x)\,y'+(-7x^2+16x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(7x^2-16x-1)y}{9x^2(x+1)}+\frac{(x^2-11x-5)y'}{3x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-11x-5)y'}{3x(x+1)}-\frac{(7x^2-16x-1)y}{9x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-11x-5}{3x(x+1)},P_3(x)=-\frac{7x^2-16x-1}{9x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{7}{3}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $9x^2(x+1)\,y''-3x(x^2-11x-5)\,y'+(-7x^2+16x+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(9u^3-18u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right)+(-7u^2+30u-22)\,y(u)=$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_0r(4+3r)\, u^{-1+r} + (3a_1(1+r)\,(7+3r) - 2a_0(9r^2+21r+11))\, u^r + (3a_2(2+r)\,(10+3r) - 2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3r(4+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{4}{3}\right\}$$

- The coefficients of each power of u must be 0

  $$[3a_1(1+r)\,(7+3r) - 2a_0(9r^2+21r+11) = 0, 3a_2(2+r)\,(10+3r) - 2a_1(9r^2+39r+41) + 3a$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{2a_0\,(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0\,(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$9(-2a_k+a_{k-1}+a_{k+1})\, k^2 + 3(6(-2a_k+a_{k-1}+a_{k+1})\, r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})\, k + 9($$

- Shift index using $k- >k+2$

$$9(-2a_{k+2}+a_{k+1}+a_{k+3})\left(k+2\right)^{2}+3(6(-2a_{k+2}+a_{k+1}+a_{k+3})r-14a_{k+2}-a_{k}+5a_{k+1}+10a_{k+}$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^{2}a_{k+1}-18r^{2}a_{k+2}-3ka_{k}+51ka_{k+1}-114ka_{k+2}-3ra_{k}+51ra_{k+1}-114ra}{3(3k^{2}+6kr+3r^{2}+22k+22r+39)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+51ka_{k+1}-114ka_{k+2}-7a_{k}+72a_{k+1}-178a_{k+2}}{3(3k^{2}+22k+39)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+51ka_{k+1}-114ka_{k+2}-7a_{k}+72a_{k+1}-178a_{k+2}}{3(3k^{2}+22k+39)},a_{1}=\frac{22a_{0}}{21},a_{2}=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty}a_{k}(x+1)^{k},a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+51ka_{k+1}-114ka_{k+2}-7a_{k}+72a_{k+1}-178a_{k+2}}{3(3k^{2}+22k+39)},a_{1}=\frac{22a_{0}}{21},\right.$$

- Recursion relation for $r=-\frac{4}{3}$

$$a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+27ka_{k+1}-66ka_{k+2}-3a_{k}+20a_{k+1}-58a_{k+2}}{3(3k^{2}+14k+15)}$$

- Solution for $r=-\frac{4}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k-\frac{4}{3}},a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+27ka_{k+1}-66ka_{k+2}-3a_{k}+20a_{k+1}-58a_{k+2}}{3(3k^{2}+14k+15)},a_{1}=\frac{2a_{0}}{3},a_{2}=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty}a_{k}(x+1)^{k-\frac{4}{3}},a_{k+3}=-\frac{9k^{2}a_{k+1}-18k^{2}a_{k+2}-3ka_{k}+27ka_{k+1}-66ka_{k+2}-3a_{k}+20a_{k+1}-58a_{k+2}}{3(3k^{2}+14k+15)},a_{1}=\frac{2a_{0}}{3},\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_{k}(x+1)^{k}\right)+\left(\sum_{k=0}^{\infty}b_{k}(x+1)^{k-\frac{4}{3}}\right),a_{k+3}=-\frac{9k^{2}a_{1+k}-18k^{2}a_{k+2}-3ka_{k}+51ka_{1+k}-114ka_{k+2}-7a_{k}}{3(3k^{2}+22k+39)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 79

```
Order:=8;
dsolve(9*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x-x^2)*diff(y(x),x)+(1+16*x-7*x^2)*y(x)=0,y(x),t
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - x + \frac{7}{6}x^2 - \frac{23}{18}x^3 + \frac{11}{8}x^4 - \frac{1577}{1080}x^5 + \frac{3319}{2160}x^6 - \frac{72853}{45360}x^7 + O(x^8)\right) + \left(-\frac{1}{12}x^2 + \frac{13}{108}x^3 - \right.}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 167

```
AsymptoticDSolveValue[9*x^2*(1+x)*y''[x]+3*x*(5+11*x-x^2)*y'[x]+(1+16*x-7*x^2)*y[x]==0,y[x],
```

$$y(x) \to \frac{c_1\left(-\frac{72853x^7}{45360}+\frac{3319x^6}{2160}-\frac{1577x^5}{1080}+\frac{11x^4}{8}-\frac{23x^3}{18}+\frac{7x^2}{6}-x+1\right)}{\sqrt[3]{x}} + c_2\left(\frac{\frac{4118557x^7}{19051200}-\frac{76919x^6}{388800}+\frac{11449x^5}{64800}-\frac{131x^4}{864}+\frac{13x^3}{108}-\frac{x^2}{12}}{\sqrt[3]{x}} + \frac{\left(-\frac{72853x^7}{45360}+\frac{3319x^6}{2160}-\frac{1577x^5}{1080}+\frac{11x^4}{8}-\frac{23x^3}{18}+\frac{7x^2}{6}-x+1\right)\log(x)}{\sqrt[3]{x}}\right)$$

15.16 problem 12

Internal problem ID [1364]
Internal file name [OUTPUT/1365_Sunday_June_05_2022_02_13_21_AM_36205429/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (4x+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (4x+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{4x+1}{4x^2} \end{aligned}$$

Table 674: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{4x+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (4x + 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (4x + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n + r) (n + r - 1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 4a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(2r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(2r+1)^2 (2r+3)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(2r+1)^2}$	-1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(2r+1)^2 (2r+3)^2 (2r+5)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(2r+1)^2}$	-1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$-\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(2r+1)^2 (2r+3)^2 (2r+5)^2 (2r+7)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(2r+1)^2}$	-1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$-\frac{1}{36}$
a_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(2r+1)^2 (2r+3)^2 (2r+5)^2 (2r+7)^2 (2r+9)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(2r+1)^2}$	-1
a_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$-\frac{1}{36}$
a_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$
a_5	$-\frac{1024}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$-\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{4}{(2r+1)^2}$	-1	$\frac{16}{(2r+1)^3}$	2
b_2	$\frac{16}{(2r+1)^2(2r+3)^2}$	$\frac{1}{4}$	$\frac{-256r-256}{(2r+1)^3(2r+3)^3}$	$-\frac{3}{4}$
b_3	$-\frac{64}{(2r+1)^2(2r+3)^2(2r+5)^2}$	$-\frac{1}{36}$	$\frac{3072r^2+9216r+5888}{(2r+1)^3(2r+3)^3(2r+5)^3}$	$\frac{11}{108}$
b_4	$\frac{256}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2}$	$\frac{1}{576}$	$-\frac{32768(2+r)\left(r^2+4r+\frac{11}{4}\right)}{(2r+1)^3(2r+3)^3(2r+5)^3(2r+7)^3}$	$-\frac{25}{3456}$
b_5	$-\frac{1024}{(2r+1)^2(2r+3)^2(2r+5)^2(2r+7)^2(2r+9)^2}$	$-\frac{1}{14400}$	$\frac{327680r^4+3276800r^3+11304960r^2+15564800r+6918144}{(2r+1)^3(2r+3)^3(2r+5)^3(2r+7)^3(2r+9)^3}$	$\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x) \\ &\quad + \sqrt{x}\left(2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + \sqrt{x}\left(2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + \sqrt{x}\left(2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \frac{x^4}{576} - \frac{x^5}{14400} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + \sqrt{x}\left(2x - \frac{3x^2}{4} + \frac{11x^3}{108} - \frac{25x^4}{3456} + \frac{137x^5}{432000} + O(x^6)\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+O(x^6)\right) \\
& + c_2\left(\sqrt{x}\left(1-x+\frac{x^2}{4}-\frac{x^3}{36}+\frac{x^4}{576}-\frac{x^5}{14400}+O(x^6)\right)\ln(x)\right. \\
& \left. + \sqrt{x}\left(2x-\frac{3x^2}{4}+\frac{11x^3}{108}-\frac{25x^4}{3456}+\frac{137x^5}{432000}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.16.1 Maple step by step solution

Let's solve

$4x^2y''+(4x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x+1)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{4x+1}{4x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$$4x^2y'' + (4x+1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)^2 a_k + 4a_{k-1} = 0$$

- Shift index using $k- > k + 1$

$$4\left(k+\frac{1}{2}+r\right)^2 a_{k+1} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{4a_k}{(2k+2)^2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{4a_k}{(2k+2)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 69

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+(1+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\sqrt{x}\left((c_2\ln(x)+c_1)\left(1-x+\frac{1}{4}x^2-\frac{1}{36}x^3+\frac{1}{576}x^4-\frac{1}{14400}x^5+\mathrm{O}\left(x^6\right)\right)+\left(2x-\frac{3}{4}x^2+\frac{11}{108}x^3-\frac{25}{3456}x^4+\frac{137}{432000}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 128

```
AsymptoticDSolveValue[4*x^2*y''[x]+(1+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1\right) + c_2\left(\sqrt{x}\left(\frac{137x^5}{432000}-\frac{25x^4}{3456}+\frac{11x^3}{108}-\frac{3x^2}{4}+2x\right) + \sqrt{x}\left(-\frac{x^5}{14400}+\frac{x^4}{576}-\frac{x^3}{36}+\frac{x^2}{4}-x+1\right)\log(x)\right)$$

15.17 problem 13

Internal problem ID [1365]
Internal file name [`OUTPUT/1366_Sunday_June_05_2022_02_13_23_AM_50001526/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{36x^2(1-2x)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-72x^3+36x^2\right)y''+\left(-216x^2+24x\right)y'+(1-70x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{6x-\frac{2}{3}}{x\,(2x-1)}\\ q(x)&=\frac{70x-1}{36x^2\,(2x-1)}\end{aligned}$$

Table 676: Table $p(x), q(x)$ singularites.

$p(x) = \frac{6x-\frac{2}{3}}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = \frac{70x-1}{36x^2(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-36y''x^2(2x-1) + \left(-216x^2 + 24x\right) y' + (1 - 70x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-36\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) x^2(2x - 1) + \left(-216x^2 + 24x\right)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (1 - 70x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-72x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}36x^{n+r}a_n(n+r)(n+r-1)\right)\\+\sum_{n=0}^{\infty}\left(-216x^{1+n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}24x^{n+r}a_n(n+r)\right)\\+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-70x^{1+n+r}a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-72x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-72a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-216x^{1+n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-216a_{n-1}(n+r-1)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-70x^{1+n+r}a_n\right)=\sum_{n=1}^{\infty}\left(-70a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-72a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)\\+\left(\sum_{n=0}^{\infty}36x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-216a_{n-1}(n+r-1)\,x^{n+r}\right)\\+\left(\sum_{n=0}^{\infty}24x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-70a_{n-1}x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$36x^{n+r}a_n(n+r)(n+r-1)+24x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$36x^r a_0 r(-1+r) + 24x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(36x^r r(-1+r) + 24x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(6r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(6r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{6}$$
$$r_2 = \frac{1}{6}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(6r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{6}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{6}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{6}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -72a_{n-1}(n+r-1)(n+r-2) + 36a_n(n+r)(n+r-1) \\ -216a_{n-1}(n+r-1) + 24a_n(n+r) + a_n - 70a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(6n+6r+1)\, a_{n-1}}{-1+6n+6r} \tag{4}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_n = \frac{2(3n+1)\, a_{n-1}}{3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{6}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{12r+14}{5+6r}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_1 = \frac{8}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{144r^2 + 480r + 364}{36r^2 + 96r + 55}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_2 = \frac{56}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$
a_2	$\frac{144r^2+480r+364}{36r^2+96r+55}$	$\frac{56}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1728r^3 + 11232r^2 + 22608r + 13832}{216r^3 + 1188r^2 + 1962r + 935}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_3 = \frac{1120}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$
a_2	$\frac{144r^2+480r+364}{36r^2+96r+55}$	$\frac{56}{9}$
a_3	$\frac{1728r^3+11232r^2+22608r+13832}{216r^3+1188r^2+1962r+935}$	$\frac{1120}{81}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{20736r^4 + 221184r^3 + 832896r^2 + 1296384r + 691600}{1296r^4 + 12096r^3 + 39096r^2 + 50736r + 21505}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_4 = \frac{7280}{243}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$
a_2	$\frac{144r^2+480r+364}{36r^2+96r+55}$	$\frac{56}{9}$
a_3	$\frac{1728r^3+11232r^2+22608r+13832}{216r^3+1188r^2+1962r+935}$	$\frac{1120}{81}$
a_4	$\frac{20736r^4+221184r^3+832896r^2+1296384r+691600}{1296r^4+12096r^3+39096r^2+50736r+21505}$	$\frac{7280}{243}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{248832r^5 + 3939840r^4 + 23708160r^3 + 67196160r^2 + 88675008r + 42879200}{7776r^5 + 110160r^4 + 585360r^3 + 1438200r^2 + 1600374r + 623645}$$

Which for the root $r = \frac{1}{6}$ becomes

$$a_5 = \frac{46592}{729}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$
a_2	$\frac{144r^2+480r+364}{36r^2+96r+55}$	$\frac{56}{9}$
a_3	$\frac{1728r^3+11232r^2+22608r+13832}{216r^3+1188r^2+1962r+935}$	$\frac{1120}{81}$
a_4	$\frac{20736r^4+221184r^3+832896r^2+1296384r+691600}{1296r^4+12096r^3+39096r^2+50736r+21505}$	$\frac{7280}{243}$
a_5	$\frac{248832r^5+3939840r^4+23708160r^3+67196160r^2+88675008r+42879200}{7776r^5+110160r^4+585360r^3+1438200r^2+1600374r+623645}$	$\frac{46592}{729}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{6}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{6}}\left(1 + \frac{8x}{3} + \frac{56x^2}{9} + \frac{1120x^3}{81} + \frac{7280x^4}{243} + \frac{46592x^5}{729} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{6}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{12r+14}{5+6r}$	$\frac{8}{3}$	$-\frac{24}{(5+6r)^2}$
b_2	$\frac{144r^2+480r+364}{36r^2+96r+55}$	$\frac{56}{9}$	$-\frac{96(36r^2+108r+89)}{(36r^2+96r+55)^2}$
b_3	$\frac{1728r^3+11232r^2+22608r+13832}{216r^3+1188r^2+1962r+935}$	$\frac{1120}{81}$	$-\frac{288(1296r^4+10368r^3+31032r^2+41184r+20833)}{(216r^3+1188r^2+1962r+935)^2}$
b_4	$\frac{20736r^4+221184r^3+832896r^2+1296384r+691600}{1296r^4+12096r^3+39096r^2+50736r+21505}$	$\frac{7280}{243}$	$-\frac{768(46656r^6+699840r^5+4311792r^4+13957920r}{(1296r^4+12096r^3+39096r^2-}$
b_5	$\frac{248832r^5+3939840r^4+23708160r^3+67196160r^2+88675008r+42879200}{7776r^5+110160r^4+585360r^3+1438200r^2+1600374r+623645}$	$\frac{46592}{729}$	$-\frac{1920(1679616r^8+40310784r^7+416358144r^6+24}{(7776r^5+1}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^{\frac{1}{6}} \left(1 + \frac{8x}{3} + \frac{56x^2}{9} + \frac{1120x^3}{81} + \frac{7280x^4}{243} + \frac{46592x^5}{729} + O(x^6)\right) \ln(x) \\
&\quad + x^{\frac{1}{6}} \left(-\frac{2x}{3} - 2x^2 - \frac{1192x^3}{243} - \frac{8168x^4}{729} - \frac{270112x^5}{10935} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{6}} \left(1 + \frac{8x}{3} + \frac{56x^2}{9} + \frac{1120x^3}{81} + \frac{7280x^4}{243} + \frac{46592x^5}{729} + O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{1}{6}} \left(1 + \frac{8x}{3} + \frac{56x^2}{9} + \frac{1120x^3}{81} + \frac{7280x^4}{243} + \frac{46592x^5}{729} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + x^{\frac{1}{6}} \left(-\frac{2x}{3} - 2x^2 - \frac{1192x^3}{243} - \frac{8168x^4}{729} - \frac{270112x^5}{10935} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right) \\
&\quad + c_2\left(x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
&\quad \left. + x^{\frac{1}{6}}\left(-\frac{2x}{3}-2x^2-\frac{1192x^3}{243}-\frac{8168x^4}{729}-\frac{270112x^5}{10935}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right) \\
&\quad + c_2\left(x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
&\quad \left. + x^{\frac{1}{6}}\left(-\frac{2x}{3}-2x^2-\frac{1192x^3}{243}-\frac{8168x^4}{729}-\frac{270112x^5}{10935}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right) \\
&\quad + c_2\left(x^{\frac{1}{6}}\left(1+\frac{8x}{3}+\frac{56x^2}{9}+\frac{1120x^3}{81}+\frac{7280x^4}{243}+\frac{46592x^5}{729}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
&\quad \left. + x^{\frac{1}{6}}\left(-\frac{2x}{3}-2x^2-\frac{1192x^3}{243}-\frac{8168x^4}{729}-\frac{270112x^5}{10935}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

15.17.1 Maple step by step solution

Let's solve

$-36y''x^2(2x-1)+\left(-216x^2+24x\right)y'+\left(1-70x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(70x-1)y}{36x^2(2x-1)}-\frac{2(9x-1)y'}{3x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(9x-1)y'}{3x(2x-1)}+\frac{(70x-1)y}{36x^2(2x-1)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{2(9x-1)}{3x(2x-1)}, P_3(x)=\frac{70x-1}{36x^2(2x-1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{2}{3}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{36}$
  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $36y''x^2(2x-1)+24x(9x-1)y'+y(70x-1)=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-1+6r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(6k+6r-1)^2 + 2a_{k-1}(6k+1+6r)(6k+6r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-1+6r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = \frac{1}{6}$$

- Each term in the series must be 0, giving the recursion relation

  $$-36\left(\left(-2k-2r-\frac{1}{3}\right) a_{k-1} + a_k\left(k+r-\frac{1}{6}\right)\right)\left(k+r-\frac{1}{6}\right) = 0$$

- Shift index using $k- >k+1$

  $$-36\left(\left(-2k-\frac{7}{3}-2r\right) a_k + a_{k+1}\left(k+\frac{5}{6}+r\right)\right)\left(k+\frac{5}{6}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2(6k+6r+7)a_k}{6k+6r+5}$$

- Recursion relation for $r = \frac{1}{6}$

  $$a_{k+1} = \frac{2(6k+8)a_k}{6k+6}$$

- Solution for $r = \frac{1}{6}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{6}}, a_{k+1} = \frac{2(6k+8)a_k}{6k+6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 69

```
Order:=6;
dsolve(36*x^2*(1-2*x)*diff(y(x),x$2)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x)=0,y(x),type='se
```

$$y(x) = x^{\frac{1}{6}}\left((c_2 \ln(x) + c_1)\left(1 + \frac{8}{3}x + \frac{56}{9}x^2 + \frac{1120}{81}x^3 + \frac{7280}{243}x^4 + \frac{46592}{729}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{2}{3}x - 2x^2 - \frac{1192}{243}x^3 - \frac{8168}{729}x^4 - \frac{270112}{10935}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 132

```
AsymptoticDSolveValue[36*x^2*(1-2*x)*y''[x]+24*x*(1-9*x)*y'[x]+(1-70*x)*y[x]==0,y[x],{x,0,5}
```

$$y(x) \to c_1\sqrt[6]{x}\left(\frac{46592x^5}{729} + \frac{7280x^4}{243} + \frac{1120x^3}{81} + \frac{56x^2}{9} + \frac{8x}{3} + 1\right) + c_2\left(\sqrt[6]{x}\left(-\frac{270112x^5}{10935} - \frac{8168x^4}{729} - \frac{1192x^3}{243} - 2x^2 - \frac{2x}{3}\right) + \sqrt[6]{x}\left(\frac{46592x^5}{729} + \frac{7280x^4}{243} + \frac{1120x^3}{81} + \frac{56x^2}{9} + \frac{8x}{3} + 1\right)\log(x)\right)$$

15.18 problem 14

Internal problem ID [1366]
Internal file name [OUTPUT/1367_Sunday_June_05_2022_02_13_26_AM_29279783/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''-x(3-x)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3+x^2)\,y''+(x^2-3x)\,y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-3}{x\,(x+1)}$$
$$q(x)=\frac{4}{(x+1)\,x^2}$$

Table 678: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-3}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4}{(x+1)x^2}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + \left(x^2-3x\right)y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(x^2-3x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 1)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{a_{n-1}(1+n)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r^2}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{(-1+r)^2}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+1)^2}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 9$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{(-1+r)^2}$	-4
a_2	$\frac{(r+1)^2}{(-1+r)^2}$	9

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+2)^2}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -16$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{(-1+r)^2}$	-4
a_2	$\frac{(r+1)^2}{(-1+r)^2}$	9
a_3	$-\frac{(r+2)^2}{(-1+r)^2}$	-16

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+3)^2}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = 25$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{(-1+r)^2}$	-4
a_2	$\frac{(r+1)^2}{(-1+r)^2}$	9
a_3	$-\frac{(r+2)^2}{(-1+r)^2}$	-16
a_4	$\frac{(r+3)^2}{(-1+r)^2}$	25

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+4)^2}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -36$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{(-1+r)^2}$	-4
a_2	$\frac{(r+1)^2}{(-1+r)^2}$	9
a_3	$-\frac{(r+2)^2}{(-1+r)^2}$	-16
a_4	$\frac{(r+3)^2}{(-1+r)^2}$	25
a_5	$-\frac{(r+4)^2}{(-1+r)^2}$	-36

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{r^2}{(-1+r)^2}$	-4	$\frac{2r}{(-1+r)^3}$	4
b_2	$\frac{(r+1)^2}{(-1+r)^2}$	9	$\frac{-4r-4}{(-1+r)^3}$	-12
b_3	$-\frac{(r+2)^2}{(-1+r)^2}$	-16	$\frac{6r+12}{(-1+r)^3}$	24
b_4	$\frac{(r+3)^2}{(-1+r)^2}$	25	$\frac{-8r-24}{(-1+r)^3}$	-40
b_5	$-\frac{(r+4)^2}{(-1+r)^2}$	-36	$\frac{10r+40}{(-1+r)^3}$	60

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right)\ln(x) \\ &\quad + x^2\left(60x^5 - 40x^4 + 24x^3 - 12x^2 + 4x + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right) \\ &\quad + c_2\Big(x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right)\ln(x) \\ &\quad + x^2\left(60x^5 - 40x^4 + 24x^3 - 12x^2 + 4x + O(x^6)\right)\Big) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right) \\ &\quad + c_2\Big(x^2\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1 + O(x^6)\right)\ln(x) \\ &\quad + x^2\left(60x^5 - 40x^4 + 24x^3 - 12x^2 + 4x + O(x^6)\right)\Big) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2\left(-36x^5+25x^4-16x^3+9x^2-4x+1+O\left(x^6\right)\right) \\
& + c_2\left(x^2\left(-36x^5+25x^4-16x^3+9x^2-4x+1+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
& \left. + x^2\left(60x^5-40x^4+24x^3-12x^2+4x+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^2\left(-36x^5+25x^4-16x^3+9x^2-4x+1+O\left(x^6\right)\right) \\
& + c_2\left(x^2\left(-36x^5+25x^4-16x^3+9x^2-4x+1+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
& \left. + x^2\left(60x^5-40x^4+24x^3-12x^2+4x+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

15.18.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y''+\left(x^2-3x\right)y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2(x+1)}-\frac{(x-3)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x-3)y'}{x(x+1)}+\frac{4y}{x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x-3}{x(x+1)}, P_3(x)=\frac{4}{(x+1)x^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.\left((x+1)\cdot P_2(x)\right)\right|_{x=-1}=4$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.\left((x+1)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$x^2(x+1)\, y'' + x(x-3)\, y' + 4y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

$(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 5u + 4)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(3+r)\, u^{-1+r} + \left(a_1(1+r)\,(4+r) - a_0(2r^2+3r-4)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+4+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, 0\}$

- Each term must be 0

$a_1(1+r)\,(4+r) - a_0(2r^2+3r-4) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+4+r) - a_k(2k^2+4kr+2r^2+3k+3r-4) + a_{k-1}(k+r-1)^2 = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+5+r) - a_{k+1}\left(2(k+1)^2+4(k+1)r+2r^2+3k-1+3r\right) + a_k(k+r)^2 =$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-7ka_{k+1}-7ra_{k+1}-a_{k+1}}{(k+2+r)(k+5+r)}$
- Recursion relation for $r=-3$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=1$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$
- Recursion relation for $r=0$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0=0\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-7ka_{k+1}-a_{k+1}}{(k+2)(k+5)}, 4a_1+4a_0=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3-x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^2\left((c_2 \ln(x) + c_1)\left(1 - 4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \mathrm{O}\left(x^6\right)\right) + \left(4x - 12x^2 + 24x^3 - 40x^4 + 60x^5 + \mathrm{O}\left(x^6\right)\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 98

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-x*(3-x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1\right)x^2 + c_2\left(\left(60x^5 - 40x^4 + 24x^3 - 12x^2 + 4x\right)x^2 + \left(-36x^5 + 25x^4 - 16x^3 + 9x^2 - 4x + 1\right)x^2 \log(x)\right)$$

15.19 problem 15

Internal problem ID [1367]
Internal file name [`OUTPUT/1368_Sunday_June_05_2022_02_13_29_AM_77264433/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''-x(5-4x)\,y'+(9-4x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+x^2\right)y''+\left(4x^2-5x\right)y'+(9-4x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{4x-5}{x\,(2x-1)}$$
$$q(x)=\frac{4x-9}{x^2\,(2x-1)}$$

Table 680: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{4x-5}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = \frac{4x-9}{x^2(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x-1) + \left(4x^2 - 5x\right) y' + (9 - 4x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(2x-1) + \left(4x^2 - 5x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (9 - 4x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}4x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)+\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}4x^{1+n+r}a_n(n+r)=\sum_{n=1}^{\infty}4a_{n-1}(n+r-1)x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty}4a_{n-1}(n+r-1)x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)+\sum_{n=1}^{\infty}\left(-4a_{n-1}x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1)-5x^{n+r}a_n(n+r)+9a_nx^{n+r}=0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 5x^r a_0 r + 9a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 5x^r r + 9x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-3)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-3)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 3 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-3)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} -2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 4a_{n-1}(n+r-1) - 5a_n(n+r) + 9a_n - 4a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(n+r-2)\,a_{n-1}}{-3+n+r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{2(1+n)\,a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+2r}{r-2}$$

Which for the root $r = 3$ becomes

$$a_1 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r-2}$	4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r}{r-2}$$

Which for the root $r = 3$ becomes

$$a_2 = 12$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r-2}$	4
a_2	$\frac{4r}{r-2}$	12

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8+8r}{r-2}$$

Which for the root $r = 3$ becomes

$$a_3 = 32$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r-2}$	4
a_2	$\frac{4r}{r-2}$	12
a_3	$\frac{8+8r}{r-2}$	32

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{32+16r}{r-2}$$

Which for the root $r = 3$ becomes

$$a_4 = 80$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r-2}$	4
a_2	$\frac{4r}{r-2}$	12
a_3	$\frac{8+8r}{r-2}$	32
a_4	$\frac{32+16r}{r-2}$	80

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{96 + 32r}{r - 2}$$

Which for the root $r = 3$ becomes

$$a_5 = 192$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r-2}$	4
a_2	$\frac{4r}{r-2}$	12
a_3	$\frac{8+8r}{r-2}$	32
a_4	$\frac{32+16r}{r-2}$	80
a_5	$\frac{96+32r}{r-2}$	192

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 3)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-2+2r}{r-2}$	4	$-\frac{2}{(r-2)^2}$	-2
b_2	$\frac{4r}{r-2}$	12	$-\frac{8}{(r-2)^2}$	-8
b_3	$\frac{8+8r}{r-2}$	32	$-\frac{24}{(r-2)^2}$	-24
b_4	$\frac{32+16r}{r-2}$	80	$-\frac{64}{(r-2)^2}$	-64
b_5	$\frac{96+32r}{r-2}$	192	$-\frac{160}{(r-2)^2}$	-160

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \ln(x) \\ &\quad + x^3 \left(-160x^5 - 64x^4 - 24x^3 - 8x^2 - 2x + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \\ &\quad + c_2 \left(x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \ln(x) \right. \\ &\qquad \left. + x^3 \left(-160x^5 - 64x^4 - 24x^3 - 8x^2 - 2x + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \\ &\quad + c_2 \left(x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \ln(x) \right. \\ &\qquad \left. + x^3 \left(-160x^5 - 64x^4 - 24x^3 - 8x^2 - 2x + O(x^6)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \\
+ c_2 \left(x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \ln(x)\right. \\
\left. + x^3 \left(-160x^5 - 64x^4 - 24x^3 - 8x^2 - 2x + O(x^6)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \\
+ c_2 \left(x^3 \left(192x^5 + 80x^4 + 32x^3 + 12x^2 + 4x + 1 + O(x^6)\right) \ln(x)\right. \\
\left. + x^3 \left(-160x^5 - 64x^4 - 24x^3 - 8x^2 - 2x + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.19.1 Maple step by step solution

Let's solve

$-y''x^2(2x-1) + (4x^2 - 5x)\, y' + (9 - 4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-9)y}{x^2(2x-1)} + \frac{(4x-5)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4x-5)y'}{x(2x-1)} + \frac{(4x-9)y}{x^2(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{4x-5}{x(2x-1)}, P_3(x) = \frac{4x-9}{x^2(2x-1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -5$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 9$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $y'' x^2 (2x-1) - x(4x-5)\, y' + (4x-9)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- > k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 (-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k (k+r-3)^2 + 2a_{k-1} (k+r-2)\,(k+r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 3$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$

- Shift index using $k- > k+1$

  $-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{2a_k(k+2)}{k+1}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = x^3\big((c_2 \ln(x) + c_1)\left(1 + 4x + 12x^2 + 32x^3 + 80x^4 + 192x^5 + O\left(x^6\right)\right) \\ + \left((-2)x - 8x^2 - 24x^3 - 64x^4 - 160x^5 + O\left(x^6\right)\right)c_2\big)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 98

```
AsymptoticDSolveValue[x^2*(1-2*x)*y''[x]-x*(5-4*x)*y'[x]+(9-4*x)*y[x]==0,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(192x^5+80x^4+32x^3+12x^2+4x+1\right)x^3 \\
& + c_2\left(\left(-160x^5-64x^4-24x^3-8x^2-2x\right)x^3\right. \\
& \left.+\left(192x^5+80x^4+32x^3+12x^2+4x+1\right)x^3\log(x)\right)
\end{aligned}
$$

15.20 problem 16

Internal problem ID [1368]
Internal file name [`OUTPUT/1369_Sunday_June_05_2022_02_13_32_AM_16321906/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{25x^2y'' + x(15 + x)\, y' + (x + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$25x^2y'' + \left(x^2 + 15x\right) y' + (x + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{15 + x}{25x}$$
$$q(x) = \frac{x + 1}{25x^2}$$

Table 682: Table $p(x), q(x)$ singularites.

$p(x) = \frac{15+x}{25x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+1}{25x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$25x^2y'' + (x^2 + 15x)\, y' + (x + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$25x^2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 + 15x)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (x + 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 25x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 15x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 25x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 15x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$25x^{n+r} a_n(n+r)(n+r-1) + 15x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$25x^r a_0 r(-1+r) + 15x^r a_0 r + a_0 x^r = 0$$

Or

$$(25x^r r(-1+r) + 15x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(5r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(5r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{5}$$
$$r_2 = \frac{1}{5}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(5r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{5}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{5}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{5}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$25a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 15a_n(n+r) + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{25n^2 + 50nr + 25r^2 - 10n - 10r + 1} \tag{4}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_n = -\frac{a_{n-1}(5n+1)}{125n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{5}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{(5r+4)^2}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_1 = -\frac{6}{125}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(2+r)(1+r)}{(5r+4)^2 (5r+9)^2}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_2 = \frac{33}{31250}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$
a_2	$\frac{(2+r)(1+r)}{(5r+4)^2(5r+9)^2}$	$\frac{33}{31250}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(3+r)(2+r)(1+r)}{(5r+4)^2(5r+9)^2(5r+14)^2}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_3 = -\frac{88}{5859375}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$
a_2	$\frac{(2+r)(1+r)}{(5r+4)^2(5r+9)^2}$	$\frac{33}{31250}$
a_3	$-\frac{(3+r)(2+r)(1+r)}{(5r+4)^2(5r+9)^2(5r+14)^2}$	$-\frac{88}{5859375}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3+r)(2+r)(1+r)(4+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_4 = \frac{77}{488281250}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$
a_2	$\frac{(2+r)(1+r)}{(5r+4)^2(5r+9)^2}$	$\frac{33}{31250}$
a_3	$-\frac{(3+r)(2+r)(1+r)}{(5r+4)^2(5r+9)^2(5r+14)^2}$	$-\frac{88}{5859375}$
a_4	$\frac{(3+r)(2+r)(1+r)(4+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2}$	$\frac{77}{488281250}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(3+r)(2+r)(1+r)(4+r)(5+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2(5r+24)^2}$$

Which for the root $r = \frac{1}{5}$ becomes

$$a_5 = -\frac{1001}{762939453125}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$
a_2	$\frac{(2+r)(1+r)}{(5r+4)^2(5r+9)^2}$	$\frac{33}{31250}$
a_3	$-\frac{(3+r)(2+r)(1+r)}{(5r+4)^2(5r+9)^2(5r+14)^2}$	$-\frac{88}{5859375}$
a_4	$\frac{(3+r)(2+r)(1+r)(4+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2}$	$\frac{77}{488281250}$
a_5	$-\frac{(3+r)(2+r)(1+r)(4+r)(5+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2(5r+24)^2}$	$-\frac{1001}{762939453125}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{5}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{5}}\left(1 - \frac{6x}{125} + \frac{33x^2}{31250} - \frac{88x^3}{5859375} + \frac{77x^4}{488281250} - \frac{1001x^5}{762939453125} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{5}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-1-r}{(5r+4)^2}$	$-\frac{6}{125}$	$\frac{5r+6}{(5r+4)^3}$
b_2	$\frac{(2+r)(1+r)}{(5r+4)^2(5r+9)^2}$	$\frac{33}{31250}$	$\frac{-50r^3-225r^2-323r-152}{(5r+9)^3(5r+4)^3}$
b_3	$-\frac{(3+r)(2+r)(1+r)}{(5r+4)^2(5r+9)^2(5r+14)^2}$	$-\frac{88}{5859375}$	$\frac{375r^5+3675r^4+13885r^3+25263r^2+22142r+7536}{(5r+4)^3(5r+9)^3(5r+14)^3}$
b_4	$\frac{(3+r)(2+r)(1+r)(4+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2}$	$\frac{77}{488281250}$	$\frac{-2500r^7-42750r^6-303750r^5-1160040r^4-2566146r^3-3281970r}{(5r+9)^3(5r+14)^3(5r+19)^3(5r+4)^3}$
b_5	$-\frac{(3+r)(2+r)(1+r)(4+r)(5+r)}{(5r+4)^2(5r+9)^2(5r+14)^2(5r+19)^2(5r+24)^2}$	$-\frac{1001}{762939453125}$	$\frac{15625r^9+412500r^8+4713750r^7+30545250r^6+123443175r^5+321}{(5r+4)^3(5r+9)^3(5r+14}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= x^{\frac{1}{5}}\left(1 - \frac{6x}{125} + \frac{33x^2}{31250} - \frac{88x^3}{5859375} + \frac{77x^4}{488281250} - \frac{1001x^5}{762939453125} + O(x^6)\right)\ln(x) \\ &\quad + x^{\frac{1}{5}}\left(\frac{7x}{125} - \frac{113x^2}{62500} + \frac{1091x^3}{35156250} - \frac{1721x^4}{4687500000} + \frac{609221x^5}{183105468750000} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \\
&+ c_2 \left(x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125} + O\left(x^6\right)\right) \ln\left(x\right) \right. \\
&\left. + x^{\frac{1}{5}} \left(\frac{7x}{125}-\frac{113x^2}{62500}+\frac{1091x^3}{35156250}-\frac{1721x^4}{4687500000}+\frac{609221x^5}{183105468750000}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \\
&+ c_2 \left(x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \ln\left(x\right) \right. \\
&\left. + x^{\frac{1}{5}} \left(\frac{7x}{125}-\frac{113x^2}{62500}+\frac{1091x^3}{35156250}-\frac{1721x^4}{4687500000}+\frac{609221x^5}{183105468750000}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \\
&+ c_2 \left(x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125} + O\left(x^6\right)\right) \ln\left(x\right) \right. \\
&\left. + x^{\frac{1}{5}} \left(\frac{7x}{125}-\frac{113x^2}{62500}+\frac{1091x^3}{35156250}-\frac{1721x^4}{4687500000}+\frac{609221x^5}{183105468750000}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \\
&+ c_2 \left(x^{\frac{1}{5}} \left(1-\frac{6x}{125}+\frac{33x^2}{31250}-\frac{88x^3}{5859375}+\frac{77x^4}{488281250}-\frac{1001x^5}{762939453125}+O\left(x^6\right)\right) \ln\left(x\right) \right. \\
&\left. + x^{\frac{1}{5}} \left(\frac{7x}{125}-\frac{113x^2}{62500}+\frac{1091x^3}{35156250}-\frac{1721x^4}{4687500000}+\frac{609221x^5}{183105468750000}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

15.20.1 Maple step by step solution

Let's solve

$25x^2y'' + (x^2+15x)\,y' + (x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+1)y}{25x^2} - \frac{(15+x)y'}{25x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(15+x)y'}{25x} + \frac{(x+1)y}{25x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{15+x}{25x}, P_3(x) = \frac{x+1}{25x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{5}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{25}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $25x^2y'' + x(15+x)\,y' + (x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (-1+5r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (5k+5r-1)^2 + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+5r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = \frac{1}{5}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k (5k+5r-1)^2 + a_{k-1}(k+r) = 0$$

- Shift index using $k- > k+1$

  $$a_{k+1} (5k+4+5r)^2 + a_k (k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k (k+r+1)}{(5k+4+5r)^2}$$

- Recursion relation for $r = \frac{1}{5}$

  $$a_{k+1} = -\frac{a_k \left(k+\frac{6}{5}\right)}{(5k+5)^2}$$

- Solution for $r = \frac{1}{5}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{5}}, a_{k+1} = -\frac{a_k \left(k+\frac{6}{5}\right)}{(5k+5)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(25*x^2*diff(y(x),x$2)+x*(15+x)*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{1}{5}}\left((c_2\ln(x)+c_1)\left(1-\frac{6}{125}x+\frac{33}{31250}x^2-\frac{88}{5859375}x^3+\frac{77}{488281250}x^4-\frac{1001}{762939453125}x^5+\mathrm{O}\left(x^6\right)\right)+\left(\frac{7}{125}x-\frac{113}{62500}x^2+\frac{1091}{35156250}x^3-\frac{1721}{4687500000}x^4+\frac{609221}{183105468750000}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 134

```
AsymptoticDSolveValue[25*x^2*y''[x]+x*(15+x)*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[5]{x}\left(-\frac{1001x^5}{762939453125} + \frac{77x^4}{488281250} - \frac{88x^3}{5859375} + \frac{33x^2}{31250} - \frac{6x}{125} + 1\right) + c_2\left(\sqrt[5]{x}\left(\frac{609221x^5}{183105468750000} - \frac{1721x^4}{4687500000} + \frac{1091x^3}{35156250} - \frac{113x^2}{62500} + \frac{7x}{125}\right) + \sqrt[5]{x}\left(-\frac{1001x^5}{762939453125} + \frac{77x^4}{488281250} - \frac{88x^3}{5859375} + \frac{33x^2}{31250} - \frac{6x}{125} + 1\right)\log(x)\right)$$

15.21 problem 17

Internal problem ID [1369]
Internal file name [`OUTPUT/1370_Sunday_June_05_2022_02_13_34_AM_11378560/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^{2}(2+x)\,y''+y'x^{2}+(1-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^{3}+4x^{2}\right)y''+y'x^{2}+(1-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{2x+4}$$
$$q(x)=-\frac{x-1}{2x^{2}\left(2+x\right)}$$

Table 684: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x+4}$	
singularity	type
$x = -2$	"regular"

$q(x) = -\frac{x-1}{2x^2(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(2+x)\, y'' + y'x^2 + (1-x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(2+x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + (1-x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) \\
+ \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) \\
+ \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r) (n+r-1) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) + a_0 x^r = 0
$$

Or

$$
\left(4x^r r(-1+r) + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)+a_{n-1}(n+r-1)+a_n-a_{n-1}=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-2)\,a_{n-1}}{-1+2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(2n-3)\,a_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1-r}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1+r)\,r}{4r^2+8r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{(-1+r)r}{4r^2+8r+3}$	$-\frac{1}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 + r}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{(-1+r)r}{4r^2+8r+3}$	$-\frac{1}{32}$
a_3	$\frac{-r^3+r}{8r^3+36r^2+46r+15}$	$\frac{1}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r^2 - 1)(2 + r)}{(7 + 2r)(8r^3 + 36r^2 + 46r + 15)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{5}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{(-1+r)r}{4r^2+8r+3}$	$-\frac{1}{32}$
a_3	$\frac{-r^3+r}{8r^3+36r^2+46r+15}$	$\frac{1}{128}$
a_4	$\frac{r(r^2-1)(2+r)}{(7+2r)(8r^3+36r^2+46r+15)}$	$-\frac{5}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 5r^4 - 5r^3 + 5r^2 + 6r}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{7}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$
a_2	$\frac{(-1+r)r}{4r^2+8r+3}$	$-\frac{1}{32}$
a_3	$\frac{-r^3+r}{8r^3+36r^2+46r+15}$	$\frac{1}{128}$
a_4	$\frac{r(r^2-1)(2+r)}{(7+2r)(8r^3+36r^2+46r+15)}$	$-\frac{5}{2048}$
a_5	$\frac{-r^5-5r^4-5r^3+5r^2+6r}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{7}{8192}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} - \frac{5x^4}{2048} + \frac{7x^5}{8192} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{1-r}{1+2r}$	$\frac{1}{4}$	$-\frac{3}{(1+2r)^2}$
b_2	$\frac{(-1+r)r}{4r^2+8r+3}$	$-\frac{1}{32}$	$\frac{12r^2+6r-3}{(4r^2+8r+3)^2}$
b_3	$\frac{-r^3+r}{8r^3+36r^2+46r+15}$	$\frac{1}{128}$	$\frac{-36r^4-108r^3-81r^2+15}{(8r^3+36r^2+46r+15)^2}$
b_4	$\frac{r(r^2-1)(2+r)}{(7+2r)(8r^3+36r^2+46r+15)}$	$-\frac{5}{2048}$	$\frac{96r^6+720r^5+1968r^4+2340r^3+966r^2-210r-210}{(8r^3+36r^2+46r+15)^2(7+2r)^2}$
b_5	$\frac{-r^5-5r^4-5r^3+5r^2+6r}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{7}{8192}$	$-\frac{15(16r^8+224r^7+1272r^6+3752r^5+6053r^4+4984r^3+1339r^2-630r-378)}{(32r^5+400r^4+1840r^3+3800r^2+3378r+945)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \sqrt{x}\left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} - \frac{5x^4}{2048} + \frac{7x^5}{8192} + O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(-\frac{3x}{4} + \frac{3x^2}{64} - \frac{7x^3}{768} + \frac{61x^4}{24576} - \frac{391x^5}{491520} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} - \frac{5x^4}{2048} + \frac{7x^5}{8192} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} - \frac{5x^4}{2048} + \frac{7x^5}{8192} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(-\frac{3x}{4} + \frac{3x^2}{64} - \frac{7x^3}{768} + \frac{61x^4}{24576} - \frac{391x^5}{491520} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(-\frac{3x}{4}+\frac{3x^2}{64}-\frac{7x^3}{768}+\frac{61x^4}{24576}-\frac{391x^5}{491520}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(-\frac{3x}{4}+\frac{3x^2}{64}-\frac{7x^3}{768}+\frac{61x^4}{24576}-\frac{391x^5}{491520}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}-\frac{5x^4}{2048}+\frac{7x^5}{8192}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(-\frac{3x}{4}+\frac{3x^2}{64}-\frac{7x^3}{768}+\frac{61x^4}{24576}-\frac{391x^5}{491520}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.21.1 Maple step by step solution

Let's solve

$2x^2(2+x)\,y''+y'x^2+(1-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x-1)y}{2x^2(2+x)}-\frac{y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{2(2+x)}-\frac{(x-1)y}{2x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{2(2+x)}, P_3(x)=-\frac{x-1}{2x^2(2+x)}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{1}{2}$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-2$

- Multiply by denominators

  $2x^2(2+x)\,y''+y'x^2+(1-x)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-4u+4)\left(\frac{d}{du}y(u)\right)+(3-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(-1+2r)\,u^{-1+r}+\left(4a_1(1+r)\,(1+2r)-a_0(8r^2-4r-3)\right)u^r+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)\,(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(1+2r)-a_0(8r^2-4r-3)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k+a_{k-1}+4a_{k+1})\,k^2+\left(4(-4a_k+a_{k-1}+4a_{k+1})\,r+4a_k-5a_{k-1}+12a_{k+1}\right)k+2(-4a_k+$$

- Shift index using $k->k+1$

$$2(-4a_{k+1}+a_k+4a_{k+2})\,(k+1)^2+\left(4(-4a_{k+1}+a_k+4a_{k+2})\,r+4a_{k+1}-5a_k+12a_{k+2}\right)(k+1)-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+4kra_k-16kra_{k+1}+2r^2a_k-8r^2a_{k+1}-ka_k-12ka_{k+1}-ra_k-12ra_{k+1}-a_k-a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}-ka_k-12ka_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)}, 4a_1+3a_0=0\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}-k a_k-12k a_{k+1}-a_k-a_{k+1}}{4(2k^2+7k+6)}, 4a_1+3a_0=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}+k a_k-20k a_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}+k a_k-20k a_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)}, 12a_1+3a_0=0\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{1}{2}}, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}+k a_k-20k a_{k+1}-a_k-9a_{k+1}}{4(2k^2+9k+10)}, 12a_1+3a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (2+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{1+k}-k a_k-12k a_{1+k}-a_k-a_{1+k}}{4(2k^2+7k+6)}, 4a_1+\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
Order:=6;
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+x^2*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left((c_2 \ln(x) + c_1)\left(1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 - \frac{5}{2048}x^4 + \frac{7}{8192}x^5 + \mathrm{O}(x^6)\right) + \left(-\frac{3}{4}x + \frac{3}{64}x^2 - \frac{7}{768}x^3 + \frac{61}{24576}x^4 - \frac{391}{491520}x^5 + \mathrm{O}(x^6)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 134

```
AsymptoticDSolveValue[2*x^2*(2+x)*y''[x]+x^2*y'[x]+(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{7x^5}{8192} - \frac{5x^4}{2048} + \frac{x^3}{128} - \frac{x^2}{32} + \frac{x}{4} + 1\right) + c_2\left(\sqrt{x}\left(-\frac{391x^5}{491520} + \frac{61x^4}{24576} - \frac{7x^3}{768} + \frac{3x^2}{64} - \frac{3x}{4}\right) + \sqrt{x}\left(\frac{7x^5}{8192} - \frac{5x^4}{2048} + \frac{x^3}{128} - \frac{x^2}{32} + \frac{x}{4} + 1\right)\log(x)\right)$$

15.22 problem 18

Internal problem ID [1370]
Internal file name [`OUTPUT/1371_Sunday_June_05_2022_02_13_37_AM_28562163/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(4x+9)\,y''+3y'x+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+9x^2\right)y''+3y'x+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x\,(4x+9)}$$
$$q(x)=\frac{x+1}{x^2\,(4x+9)}$$

Table 686: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(4x+9)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{9}{4}$	"regular"

$q(x) = \frac{x+1}{x^2(4x+9)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{9}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{9}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(4x+9)\, y'' + 3y'x + (x+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(4x+9) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + (x+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2B}$$
$$+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3r-1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(3r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_{n-1}(n+r-1)(n+r-2)+9a_n(n+r)(n+r-1)+3a_n(n+r)+a_{n-1}+a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(4n^2+8nr+4r^2-12n-12r+9)}{9n^2+18nr+9r^2-6n-6r+1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(6n-7)^2\, a_{n-1}}{81n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{(2r-1)^2}{(3r+2)^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{1}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{25}{26244}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$
a_2	$\frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{25}{26244}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{(3r+2)^2(3r+5)^2(9r^2+48r+64)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{3025}{19131876}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$
a_2	$\frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{25}{26244}$
a_3	$-\frac{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{(3r+2)^2(3r+5)^2(9r^2+48r+64)}$	$-\frac{3025}{19131876}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(2r+3)^2(2r+1)^2(2r-1)^2(4r^2+20r+25)}{6561\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)^2\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{874225}{24794911296}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$
a_2	$\frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{25}{26244}$
a_3	$-\frac{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{(3r+2)^2(3r+5)^2(9r^2+48r+64)}$	$-\frac{3025}{19131876}$
a_4	$\frac{(2r+3)^2(2r+1)^2(2r-1)^2(4r^2+20r+25)}{6561\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)^2\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{874225}{24794911296}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(2r+3)^2(2r+5)^2(7+2r)^2(2r+1)^2(2r-1)^2}{(3r+8)^2(3r+11)^2(3r+2)^2(3r+5)^2(9r^2+84r+196)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{18498601}{2008387814976}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$
a_2	$\frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{25}{26244}$
a_3	$-\frac{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{(3r+2)^2(3r+5)^2(9r^2+48r+64)}$	$-\frac{3025}{19131876}$
a_4	$\frac{(2r+3)^2(2r+1)^2(2r-1)^2(4r^2+20r+25)}{6561\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)^2\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{874225}{24794911296}$
a_5	$-\frac{(2r+3)^2(2r+5)^2(7+2r)^2(2r+1)^2(2r-1)^2}{(3r+8)^2(3r+11)^2(3r+2)^2(3r+5)^2(9r^2+84r+196)}$	$-\frac{18498601}{2008387814976}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{81} + \frac{25x^2}{26244} - \frac{3025x^3}{19131876} + \frac{874225x^4}{24794911296} - \frac{18498601x^5}{2008387814976} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{(2r-1)^2}{(3r+2)^2}$	$-\frac{1}{81}$	$\frac{-28r+14}{(3r+2)^3}$
b_2	$\frac{16\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{81\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{25}{26244}$	$\frac{672r^4+784r^3-196r-42}{(3r+5)^3(3r+2)^3}$
b_3	$-\frac{64\left(r+\frac{3}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r-\frac{1}{2}\right)^2}{(3r+2)^2(3r+5)^2(9r^2+48r+64)}$	$-\frac{3025}{19131876}$	$-\frac{14(108r^4+468r^3+687r^2+390r+62)(2r+3)(4r^2-1)}{(3r+8)^3(3r+2)^3(3r+5)^3}$
b_4	$\frac{(2r+3)^2(2r+1)^2(2r-1)^2(4r^2+20r+25)}{6561\left(r+\frac{8}{3}\right)^2\left(r+\frac{11}{3}\right)^2\left(r+\frac{2}{3}\right)^2\left(r+\frac{5}{3}\right)^2}$	$\frac{874225}{24794911296}$	$28(2r+3)(864r^7+10368r^6+51048r^5+132484r^4+193156r^3 \ldots$ / $(3r+5)^3(3r+2)^3(3r+11)^3(3 \ldots$
b_5	$-\frac{(2r+3)^2(2r+5)^2(7+2r)^2(2r+1)^2(2r-1)^2}{(3r+8)^2(3r+11)^2(3r+2)^2(3r+5)^2(9r^2+84r+196)}$	$-\frac{18498601}{2008387814976}$	$-$ $70(1296r^8+21600r^7+150552r^6+569400r^5+1266613r^4+ \ldots$ / $14348907\left(r+\frac{14}{3}\right) \ldots$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= x^{\frac{1}{3}}\left(1 - \frac{x}{81} + \frac{25x^2}{26244} - \frac{3025x^3}{19131876} + \frac{874225x^4}{24794911296} - \frac{18498601x^5}{2008387814976} + O(x^6)\right)\ln(x) \\
&\quad + x^{\frac{1}{3}}\left(\frac{14x}{81} - \frac{35x^2}{2916} + \frac{110495x^3}{57395628} - \frac{62786185x^4}{148769467776} + \frac{1315043653x^5}{12050326889856} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}+O\left(x^6\right)\right) \\
&+ c_2\left(x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}\right.\right. \\
&\left.+O\left(x^6\right)\right)\ln\left(x\right) \\
&\left.+x^{\frac{1}{3}}\left(\frac{14x}{81}-\frac{35x^2}{2916}+\frac{110495x^3}{57395628}-\frac{62786185x^4}{148769467776}+\frac{1315043653x^5}{12050326889856}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}+O\left(x^6\right)\right) \\
&+ c_2\left(x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}\right.\right. \\
&\left.+O\left(x^6\right)\right)\ln\left(x\right) \\
&\left.+x^{\frac{1}{3}}\left(\frac{14x}{81}-\frac{35x^2}{2916}+\frac{110495x^3}{57395628}-\frac{62786185x^4}{148769467776}+\frac{1315043653x^5}{12050326889856}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}+O\left(x^6\right)\right) \\
&+ c_2\left(x^{\frac{1}{3}}\left(1-\frac{x}{81}+\frac{25x^2}{26244}-\frac{3025x^3}{19131876}+\frac{874225x^4}{24794911296}-\frac{18498601x^5}{2008387814976}\right.\right. \\
&\left.+O\left(x^6\right)\right)\ln\left(x\right) \\
&\left.+x^{\frac{1}{3}}\left(\frac{14x}{81}-\frac{35x^2}{2916}+\frac{110495x^3}{57395628}-\frac{62786185x^4}{148769467776}+\frac{1315043653x^5}{12050326889856}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(1 - \frac{x}{81} + \frac{25x^2}{26244} - \frac{3025x^3}{19131876} + \frac{874225x^4}{24794911296} - \frac{18498601x^5}{2008387814976} + O(x^6)\right)$$
$$+ c_2 \left(x^{\frac{1}{3}} \left(1 - \frac{x}{81} + \frac{25x^2}{26244} - \frac{3025x^3}{19131876} + \frac{874225x^4}{24794911296} - \frac{18498601x^5}{2008387814976} + O(x^6)\right) \ln(x)\right.$$
$$\left. + x^{\frac{1}{3}} \left(\frac{14x}{81} - \frac{35x^2}{2916} + \frac{110495x^3}{57395628} - \frac{62786185x^4}{148769467776} + \frac{1315043653x^5}{12050326889856} + O(x^6)\right)\right)$$

Verified OK.

15.22.1 Maple step by step solution

Let's solve

$x^2(4x+9)y'' + 3y'x + (x+1)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+1)y}{x^2(4x+9)} - \frac{3y'}{x(4x+9)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x(4x+9)} + \frac{(x+1)y}{x^2(4x+9)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{x(4x+9)}, P_3(x) = \frac{x+1}{x^2(4x+9)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2(4x+9)\, y'' + 3y'x + (x+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(2k-3+2r)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-1)^2 + a_{k-1}(2k-3+2r)^2 = 0$

- Shift index using $k- >k + 1$

$$a_{k+1}(3k+2+3r)^2 + a_k(2k+2r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r-1)^2}{(3k+2+3r)^2}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{a_k\left(2k-\frac{1}{3}\right)^2}{(3k+3)^2}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(2k-\frac{1}{3}\right)^2}{(3k+3)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(9+4*x)*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{1}{3}}\left((c_2\ln(x)+c_1)\left(1-\frac{1}{81}x+\frac{25}{26244}x^2-\frac{3025}{19131876}x^3+\frac{874225}{24794911296}x^4-\frac{18498601}{2008387814976}x^5+\mathrm{O}\left(x^6\right)\right)+\left(\frac{14}{81}x-\frac{35}{2916}x^2+\frac{110495}{57395628}x^3-\frac{62786185}{148769467776}x^4+\frac{1315043653}{12050326889856}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 134

```
AsymptoticDSolveValue[x^2*(9+4*x)*y''[x]+3*x*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(-\frac{18498601x^5}{2008387814976}+\frac{874225x^4}{24794911296}-\frac{3025x^3}{19131876}+\frac{25x^2}{26244}-\frac{x}{81}+1\right)+c_2\left(\sqrt[3]{x}\left(\frac{1315043653x^5}{12050326889856}-\frac{62786185x^4}{148769467776}+\frac{110495x^3}{57395628}-\frac{35x^2}{2916}+\frac{14x}{81}\right)+\sqrt[3]{x}\left(-\frac{18498601x^5}{2008387814976}+\frac{874225x^4}{24794911296}-\frac{3025x^3}{19131876}+\frac{25x^2}{26244}-\frac{x}{81}+1\right)\log(x)\right)$$

15.23 problem 19

Internal problem ID [1371]
Internal file name [OUTPUT/1372_Sunday_June_05_2022_02_13_40_AM_86649237/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(-2x+3)\,y' + (3x+4)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(2x^2 - 3x\right)y' + (3x+4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{-3+2x}{x}$$
$$q(x) = \frac{3x+4}{x^2}$$

Table 688: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-3+2x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(2x^2 - 3x\right) y' + (3x + 4) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(2x^2 - 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (3x + 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - 3a_n(n+r) + 3a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(2n+2r+1)}{n^2+2nr+r^2-4n-4r+4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}(-2n-5)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3-2r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -7$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{(-1+r)^2}$	-7

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+16r+15}{(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{63}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{(-1+r)^2}$	-7
a_2	$\frac{4r^2+16r+15}{(-1+r)^2 r^2}$	$\frac{63}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 60r^2 - 142r - 105}{(-1+r)^2 r^2 (r+1)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{77}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{(-1+r)^2}$	-7
a_2	$\frac{4r^2+16r+15}{(-1+r)^2 r^2}$	$\frac{63}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{77}{4}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 192r^3 + 824r^2 + 1488r + 945}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1001}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{(-1+r)^2}$	-7
a_2	$\frac{4r^2+16r+15}{(-1+r)^2 r^2}$	$\frac{63}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{77}{4}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{1001}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^5 - 560r^4 - 3760r^3 - 12040r^2 - 18258r - 10395}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{3003}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-2r}{(-1+r)^2}$	-7
a_2	$\frac{4r^2+16r+15}{(-1+r)^2 r^2}$	$\frac{63}{4}$
a_3	$\frac{-8r^3-60r^2-142r-105}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{77}{4}$
a_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{1001}{64}$
a_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{3003}{320}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-3-2r}{(-1+r)^2}$	-7	$\frac{8+2r}{(-1+r)^3}$
b_2	$\frac{4r^2+16r+15}{(-1+r)^2 r^2}$	$\frac{63}{4}$	$\frac{-8r^3-48r^2-44r+30}{(-1+r)^3 r^3}$
b_3	$\frac{-8r^3-60r^2-142r-105}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{77}{4}$	$\frac{24r^5+240r^4+718r^3+630r^2-142r-210}{(-1+r)^3 r^3 (r+1)^3}$
b_4	$\frac{16r^4+192r^3+824r^2+1488r+945}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}$	$\frac{1001}{64}$	$\frac{-64r^7-1024r^6-6096r^5-16880r^4-21176r^3-6876r^2+6756r+3780}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3}$
b_5	$\frac{-32r^5-560r^4-3760r^3-12040r^2-18258r-10395}{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}$	$-\frac{3003}{320}$	$\frac{160r^9+3840r^8+37680r^7+196080r^6+582498r^5+971700r^4+774410r^3+37}{(-1+r)^3 r^3 (r+1)^3 (r+2)^3 (r+3)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^2\left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right)\ln(x) \\ &\quad + x^2\left(12x - \frac{157x^2}{4} + \frac{2063x^3}{36} - \frac{59875x^4}{1152} + \frac{323399x^5}{9600} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right) \\ &\quad + c_2\left(x^2\left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(12x - \frac{157x^2}{4} + \frac{2063x^3}{36} - \frac{59875x^4}{1152} + \frac{323399x^5}{9600} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2\left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right) \\ &\quad + c_2\left(x^2\left(1 - 7x + \frac{63x^2}{4} - \frac{77x^3}{4} + \frac{1001x^4}{64} - \frac{3003x^5}{320} + O(x^6)\right)\ln(x)\right. \\ &\quad \left. + x^2\left(12x - \frac{157x^2}{4} + \frac{2063x^3}{36} - \frac{59875x^4}{1152} + \frac{323399x^5}{9600} + O(x^6)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 \left(1-7x+\frac{63x^2}{4}-\frac{77x^3}{4}+\frac{1001x^4}{64}-\frac{3003x^5}{320}+O\left(x^6\right)\right) \\
+ c_2 \left(x^2 \left(1-7x+\frac{63x^2}{4}-\frac{77x^3}{4}+\frac{1001x^4}{64}-\frac{3003x^5}{320}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
\left. + x^2 \left(12x-\frac{157x^2}{4}+\frac{2063x^3}{36}-\frac{59875x^4}{1152}+\frac{323399x^5}{9600}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 \left(1-7x+\frac{63x^2}{4}-\frac{77x^3}{4}+\frac{1001x^4}{64}-\frac{3003x^5}{320}+O\left(x^6\right)\right) \\
+ c_2 \left(x^2 \left(1-7x+\frac{63x^2}{4}-\frac{77x^3}{4}+\frac{1001x^4}{64}-\frac{3003x^5}{320}+O\left(x^6\right)\right)\ln\left(x\right)\right. \\
\left. + x^2 \left(12x-\frac{157x^2}{4}+\frac{2063x^3}{36}-\frac{59875x^4}{1152}+\frac{323399x^5}{9600}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

15.23.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(2x^2-3x\right) y' + \left(3x+4\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x+4)y}{x^2} - \frac{(-3+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-3+2x)y'}{x} + \frac{(3x+4)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-3+2x}{x}, P_3(x) = \frac{3x+4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(-3+2x)\, y' + (3x+4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 + a_{k-1}(2k+1+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = 2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + a_{k-1}(2k+1+2r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 + a_k(2k+2r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(2k+2r+3)}{(k+r-1)^2}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(2k+7)}{(k+1)^2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(2k+7)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(3-2*x)*diff(y(x),x)+(4+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 7x + \frac{63}{4}x^2 - \frac{77}{4}x^3 + \frac{1001}{64}x^4 - \frac{3003}{320}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(12x - \frac{157}{4}x^2 + \frac{2063}{36}x^3 - \frac{59875}{1152}x^4 + \frac{323399}{9600}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 122

```
AsymptoticDSolveValue[x^2*y''[x]-x*(3-2*x)*y'[x]+(4+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{3003x^5}{320} + \frac{1001x^4}{64} - \frac{77x^3}{4} + \frac{63x^2}{4} - 7x + 1\right)x^2 + c_2\left(\left(\frac{323399x^5}{9600} - \frac{59875x^4}{1152} + \frac{2063x^3}{36} - \frac{157x^2}{4} + 12x\right)x^2 + \left(-\frac{3003x^5}{320} + \frac{1001x^4}{64} - \frac{77x^3}{4} + \frac{63x^2}{4} - 7x + 1\right)x^2 \log(x)\right)$$

15.24 problem 20

Internal problem ID [1372]
Internal file name [`OUTPUT/1373_Sunday_June_05_2022_02_13_43_AM_55688922/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(-4x+1)\,y''+3x(1-6x)\,y'+(1-12x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-4x^3+x^2\right)y''+\left(-18x^2+3x\right)y'+(1-12x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{-3+18x}{x\,(4x-1)}\\ q(x)&=\frac{12x-1}{x^2\,(4x-1)}\end{aligned}$$

Table 690: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-3+18x}{x(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

$q(x) = \frac{12x-1}{x^2(4x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(4x-1) + \left(-18x^2 + 3x\right) y' + (1 - 12x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(4x-1) + \left(-18x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (1-12x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-18x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-12x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-4x^{1+n+r} a_n (n+r) (n+r-1)\right) = \sum_{n=1}^{\infty} \left(-4a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-18x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-18a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-12x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-12a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-4a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-18a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-12a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r+1\right)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$\left(r+1\right)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r+1\right)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -4a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ -18a_{n-1}(n+r-1) + 3a_n(n+r) + a_n - 12a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(2n+2r+1)\, a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{(4n-2)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{4r+6}{2+r}$$

Which for the root $r = -1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r+6}{2+r}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^2 + 64r + 60}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$a_2 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r+6}{2+r}$	2
a_2	$\frac{16r^2+64r+60}{(2+r)(3+r)}$	6

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64r^3 + 480r^2 + 1136r + 840}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$a_3 = 20$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r+6}{2+r}$	2
a_2	$\frac{16r^2+64r+60}{(2+r)(3+r)}$	6
a_3	$\frac{64r^3+480r^2+1136r+840}{(2+r)(3+r)(4+r)}$	20

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^4 + 3072r^3 + 13184r^2 + 23808r + 15120}{(2 + r)(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = 70$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r+6}{2+r}$	2
a_2	$\frac{16r^2+64r+60}{(2+r)(3+r)}$	6
a_3	$\frac{64r^3+480r^2+1136r+840}{(2+r)(3+r)(4+r)}$	20
a_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{(2+r)(3+r)(4+r)(5+r)}$	70

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1024r^5 + 17920r^4 + 120320r^3 + 385280r^2 + 584256r + 332640}{(2 + r)(3 + r)(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = -1$ becomes

$$a_5 = 252$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r+6}{2+r}$	2
a_2	$\frac{16r^2+64r+60}{(2+r)(3+r)}$	6
a_3	$\frac{64r^3+480r^2+1136r+840}{(2+r)(3+r)(4+r)}$	20
a_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{(2+r)(3+r)(4+r)(5+r)}$	70
a_5	$\frac{1024r^5+17920r^4+120320r^3+385280r^2+584256r+332640}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	252

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6)}{x}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{4r+6}{2+r}$	2	$\frac{2}{(2+r)^2}$
b_2	$\frac{16r^2+64r+60}{(2+r)(3+r)}$	6	$\frac{16r^2+72r+84}{(2+r)^2(3+r)^2}$
b_3	$\frac{64r^3+480r^2+1136r+840}{(2+r)(3+r)(4+r)}$	20	$\frac{96r^4+1056r^3+4344r^2+7920r+5424}{(2+r)^2(3+r)^2(4+r)^2}$
b_4	$\frac{256r^4+3072r^3+13184r^2+23808r+15120}{(2+r)(3+r)(4+r)(5+r)}$	70	$\frac{512r^6+9984r^5+80384r^4+341952r^3+810848r^2+1017120r+528480}{(5+r)^2(2+r)^2(3+r)^2(4+r)^2}$
b_5	$\frac{1024r^5+17920r^4+120320r^3+385280r^2+584256r+332640}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	252	$\frac{2560r^8+76800r^7+997120r^6+7315200r^5+33160480r^4+95107200r^3}{(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \frac{(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6))\ln(x)}{x} \\
&\quad + \frac{7x^2 + 2x + \frac{74x^3}{3} + \frac{533x^4}{6} + \frac{1627x^5}{5} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6))}{x} \\ &\quad + c_2 \left(\frac{(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6)) \ln (x)}{x} \right. \\ &\quad \left. + \frac{7x^2 + 2x + \frac{74x^3}{3} + \frac{533x^4}{6} + \frac{1627x^5}{5} + O(x^6)}{x} \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6))}{x} \\ &\quad + c_2 \left(\frac{(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6)) \ln (x)}{x} \right. \\ &\quad \left. + \frac{7x^2 + 2x + \frac{74x^3}{3} + \frac{533x^4}{6} + \frac{1627x^5}{5} + O(x^6)}{x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{c_1(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6))}{x} \\ &\quad + c_2 \left(\frac{(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1 + O(x^6)) \ln (x)}{x} \right. \\ &\quad \left. + \frac{7x^2 + 2x + \frac{74x^3}{3} + \frac{533x^4}{6} + \frac{1627x^5}{5} + O(x^6)}{x} \right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(252x^5+70x^4+20x^3+6x^2+2x+1+O(x^6))}{x} + c_2\left(\frac{(252x^5+70x^4+20x^3+6x^2+2x+1+O(x^6))\ln(x)}{x} + \frac{7x^2+2x+\frac{74x^3}{3}+\frac{533x^4}{6}+\frac{1627x^5}{5}+O(x^6)}{x}\right)$$

Verified OK.

15.24.1 Maple step by step solution

Let's solve

$-y''x^2(4x-1)+(-18x^2+3x)\,y'+(1-12x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(12x-1)y}{x^2(4x-1)}-\frac{3(-1+6x)y'}{x(4x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(-1+6x)y'}{x(4x-1)}+\frac{(12x-1)y}{x^2(4x-1)}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3(-1+6x)}{x(4x-1)}, P_3(x)=\frac{12x-1}{x^2(4x-1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=3$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(4x-1)+3x(-1+6x)\,y'+(12x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m=1..2$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m=2..3$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)^2+2a_{k-1}(k+r+1)\,(2k+1+2r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=-1$

- Each term in the series must be 0, giving the recursion relation

  $-((-4k-4r-2)\,a_{k-1}+a_k(k+r+1))\,(k+r+1)=0$

- Shift index using $k->k+1$

  $-((-4k-6-4r)\, a_k + a_{k+1}(k+r+2))\, (k+r+2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2(2k+2r+3)a_k}{k+r+2}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2(2k+1)a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2(2k+1)a_k}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(1-4*x)*diff(y(x),x$2)+3*x*(1-6*x)*diff(y(x),x)+(1-12*x)*y(x)=0,y(x),type='series
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)(1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + \mathrm{O}(x^6)) + \left(2x + 7x^2 + \frac{74}{3}x^3 + \frac{533}{6}x^4 + \frac{1627}{5}x^5 +\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 104

```
AsymptoticDSolveValue[x^2*(1-4*x)*y''[x]+3*x*(1-6*x)*y'[x]+(1-12*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1)}{x} + c_2\left(\frac{\frac{1627x^5}{5} + \frac{533x^4}{6} + \frac{74x^3}{3} + 7x^2 + 2x}{x} + \frac{(252x^5 + 70x^4 + 20x^3 + 6x^2 + 2x + 1)\log(x)}{x}\right)$$

15.25 problem 21

Internal problem ID [1373]
Internal file name [OUTPUT/1374_Sunday_June_05_2022_02_13_46_AM_31841646/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(1+2x)\,y''+x(3+5x)\,y'+(1-2x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(5x^2+3x\right)y'+(1-2x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3+5x}{x\,(1+2x)}$$
$$q(x)=-\frac{2x-1}{x^2\,(1+2x)}$$

Table 692: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+5x}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = -\frac{2x-1}{x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\,y'' + \left(5x^2+3x\right)y' + (1-2x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(1+2x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(5x^2+3x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (1-2x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ &+ 5a_{n-1}(n+r-1) + 3a_n(n+r) + a_n - 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-3)\, a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-1}(5-2n)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2r+1}{2+r}$$

Which for the root $r = -1$ becomes

$$a_1 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{2+r}$	3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 1}{(3 + r)(2 + r)}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{2+r}$	3
a_2	$\frac{4r^2-1}{(3+r)(2+r)}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 12r^2 + 2r + 3}{(2 + r)(3 + r)(4 + r)}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{2+r}$	3
a_2	$\frac{4r^2-1}{(3+r)(2+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-12r^2+2r+3}{(2+r)(3+r)(4+r)}$	$-\frac{1}{2}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 64r^3 + 56r^2 - 16r - 15}{(5 + r)(2 + r)(3 + r)(4 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{2+r}$	3
a_2	$\frac{4r^2-1}{(3+r)(2+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-12r^2+2r+3}{(2+r)(3+r)(4+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+64r^3+56r^2-16r-15}{(5+r)(2+r)(3+r)(4+r)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^5 - 240r^4 - 560r^3 - 360r^2 + 142r + 105}{(2 + r)(3 + r)(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{2+r}$	3
a_2	$\frac{4r^2-1}{(3+r)(2+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-12r^2+2r+3}{(2+r)(3+r)(4+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+64r^3+56r^2-16r-15}{(5+r)(2+r)(3+r)(4+r)}$	$\frac{3}{8}$
a_5	$\frac{-32r^5-240r^4-560r^3-360r^2+142r+105}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	$-\frac{3}{8}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + O(x^6)}{x}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-2r+1}{2+r}$	3	$-\frac{5}{(2+r)^2}$
b_2	$\frac{4r^2-1}{(3+r)(2+r)}$	$\frac{3}{2}$	$\frac{20r^2+50r+5}{(2+r)^2(3+r)^2}$
b_3	$\frac{-8r^3-12r^2+2r+3}{(2+r)(3+r)(4+r)}$	$-\frac{1}{2}$	$\frac{-60r^4-420r^3-915r^2-630r-30}{(2+r)^2(3+r)^2(4+r)^2}$
b_4	$\frac{16r^4+64r^3+56r^2-16r-15}{(5+r)(2+r)(3+r)(4+r)}$	$\frac{3}{8}$	$\frac{160r^6+2160r^5+11200r^4+27900r^3+33430r^2+15570r+390}{(5+r)^2(2+r)^2(3+r)^2(4+r)^2}$
b_5	$\frac{-32r^5-240r^4-560r^3-360r^2+142r+105}{(2+r)(3+r)(4+r)(5+r)(6+r)}$	$-\frac{3}{8}$	$-\frac{5(80r^8+1760r^7+16120r^6+79640r^5+228985r^4+382580r^3+343325r^2+128040r+1}{(2+r)^2(3+r)^2(4+r)^2(5+r)^2(6+r)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \frac{\left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + O(x^6)\right)\ln(x)}{x} \\
&\quad + \frac{-5x - \frac{25x^2}{4} + \frac{5x^3}{4} - \frac{25x^4}{32} + \frac{113x^5}{160} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}+O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}+O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+O(x^6)\right)\ln(x)}{x}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}+O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + O(x^6)\right)}{x} + c_2 \left(\frac{\left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + O(x^6)\right) \ln(x)}{x} + \frac{-5x - \frac{25x^2}{4} + \frac{5x^3}{4} - \frac{25x^4}{32} + \frac{113x^5}{160} + O(x^6)}{x} \right)$$

Verified OK.

15.25.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y'' + (5x^2+3x)\,y' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{x^2(1+2x)} - \frac{(3+5x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+5x)y'}{x(1+2x)} - \frac{(2x-1)y}{x^2(1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3+5x}{x(1+2x)}, P_3(x) = -\frac{2x-1}{x^2(1+2x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2(1+2x)\, y'' + x(3+5x)\, y' + (1-2x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 + a_{k-1}(k+r+1)\,(2k-3+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+1)\left(a_k(k+r+1)+a_{k-1}(2k-3+2r)\right)=0$$

- Shift index using $k->k+1$

$$(k+r+2)\left(a_{k+1}(k+r+2)+a_k(2k+2r-1)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k(2k+2r-1)}{k+r+2}$$

- Recursion relation for $r=-1$

$$a_{k+1}=-\frac{a_k(2k-3)}{k+1}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(3+5*x)*diff(y(x),x)+(1-2*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \frac{(c_2\ln(x)+c_1)\left(1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\frac{3}{8}x^4-\frac{3}{8}x^5+\mathrm{O}(x^6)\right)+\left((-5)x-\frac{25}{4}x^2+\frac{5}{4}x^3-\frac{25}{32}x^4+\frac{113}{160}x^5+\mathrm{O}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 122

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]+x*(3+5*x)*y'[x]+(1-2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{3x^5}{8}+\frac{3x^4}{8}-\frac{x^3}{2}+\frac{3x^2}{2}+3x+1\right)}{x}+c_2\left(\frac{\frac{113x^5}{160}-\frac{25x^4}{32}+\frac{5x^3}{4}-\frac{25x^2}{4}-5x}{x}+\frac{\left(-\frac{3x^5}{8}+\frac{3x^4}{8}-\frac{x^3}{2}+\frac{3x^2}{2}+3x+1\right)\log(x)}{x}\right)$$

15.26 problem 22

Internal problem ID [1374]
Internal file name [`OUTPUT/1375_Sunday_June_05_2022_02_13_49_AM_51768753/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+1)\,y''-x(6-x)\,y'+(8-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+2x^2\right)y''+\left(x^2-6x\right)y'+(8-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-6}{2x\,(x+1)}$$
$$q(x)=-\frac{x-8}{2x^2\,(x+1)}$$

Table 694: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-6}{2x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{x-8}{2x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x+1)\, y'' + \left(x^2 - 6x\right) y' + (8 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(x^2 - 6x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (8 - x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\,(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-6x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r}\right) \\
&+\sum_{n=0}^{\infty}\left(-6x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 8a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n(n+r)\,(n+r-1)-6x^{n+r} a_n(n+r)+8a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - 6x^r a_0 r + 8a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - 6x^r r + 8x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r (r-2)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r (r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\
&+ a_{n-1}(n+r-1) - 6a_n(n+r) + 8a_n - a_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(2n+2r-1)\, a_{n-1}}{2(n+r-2)} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = -\frac{(2n+3)\, a_{n-1}}{2n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{-1-2r}{-2+2r}
$$

Which for the root $r = 2$ becomes

$$
a_1 = -\frac{5}{2}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 + 8r + 3}{4r\,(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{35}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$
a_2	$\frac{4r^2+8r+3}{4r(-1+r)}$	$\frac{35}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 36r^2 - 46r - 15}{8r^3 - 8r}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{105}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$
a_2	$\frac{4r^2+8r+3}{4r(-1+r)}$	$\frac{35}{8}$
a_3	$\frac{-8r^3-36r^2-46r-15}{8r^3-8r}$	$-\frac{105}{16}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{16r^4 + 32r^3 - 16r^2 - 32r}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1155}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$
a_2	$\frac{4r^2+8r+3}{4r(-1+r)}$	$\frac{35}{8}$
a_3	$\frac{-8r^3-36r^2-46r-15}{8r^3-8r}$	$-\frac{105}{16}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{16r^4+32r^3-16r^2-32r}$	$\frac{1155}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^5 - 400r^4 - 1840r^3 - 3800r^2 - 3378r - 945}{32r^5 + 160r^4 + 160r^3 - 160r^2 - 192r}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{3003}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$
a_2	$\frac{4r^2+8r+3}{4r(-1+r)}$	$\frac{35}{8}$
a_3	$\frac{-8r^3-36r^2-46r-15}{8r^3-8r}$	$-\frac{105}{16}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{16r^4+32r^3-16r^2-32r}$	$\frac{1155}{128}$
a_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{32r^5+160r^4+160r^3-160r^2-192r}$	$-\frac{3003}{256}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	b
b_0	1	1	N/A since b_n starts from 1	[illegible]
b_1	$\frac{-1-2r}{-2+2r}$	$-\frac{5}{2}$	$\frac{3}{2(-1+r)^2}$	[illegible]
b_2	$\frac{4r^2+8r+3}{4r(-1+r)}$	$\frac{35}{8}$	$\frac{-3r^2-\frac{3}{2}r+\frac{3}{4}}{r^2(-1+r)^2}$	[illegible]
b_3	$\frac{-8r^3-36r^2-46r-15}{8r^3-8r}$	$-\frac{105}{16}$	$\frac{\frac{9}{2}r^4+\frac{27}{2}r^3+\frac{81}{8}r^2-\frac{15}{8}}{r^2(r^2-1)^2}$	[illegible]
b_4	$\frac{16r^4+128r^3+344r^2+352r+105}{16r^4+32r^3-16r^2-32r}$	$\frac{1155}{128}$	$\frac{-6r^6-45r^5-123r^4-\frac{585}{4}r^3-\frac{483}{8}r^2+\frac{105}{8}r+\frac{105}{8}}{r^2(r^3+2r^2-r-2)^2}$	[illegible]
b_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{32r^5+160r^4+160r^3-160r^2-192r}$	$-\frac{3003}{256}$	$\frac{\frac{15}{2}r^8+105r^7+\frac{2385}{4}r^6+\frac{7035}{4}r^5+\frac{90795}{32}r^4+\frac{9345}{4}r^3+\frac{20085}{32}r^2-\frac{4725}{16}r-\frac{2835}{16}}{r^2(r^4+5r^3+5r^2-5r-6)^2}$	[illegible]

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= x^2\left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right)\ln(x) \\ &\quad + x^2\left(\frac{3x}{2} - \frac{57x^2}{16} + \frac{583x^3}{96} - \frac{13771x^4}{1536} + \frac{187339x^5}{15360} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \\ &\quad + c_2\left(x^2\left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right)\ln(x)\right. \\ &\qquad \left. + x^2\left(\frac{3x}{2} - \frac{57x^2}{16} + \frac{583x^3}{96} - \frac{13771x^4}{1536} + \frac{187339x^5}{15360} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \ln(x) \right. \\
&\qquad \left. + x^2 \left(\frac{3x}{2} - \frac{57x^2}{16} + \frac{583x^3}{96} - \frac{13771x^4}{1536} + \frac{187339x^5}{15360} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \ln(x) \right. \\
&\qquad \left. + x^2 \left(\frac{3x}{2} - \frac{57x^2}{16} + \frac{583x^3}{96} - \frac{13771x^4}{1536} + \frac{187339x^5}{15360} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - \frac{5x}{2} + \frac{35x^2}{8} - \frac{105x^3}{16} + \frac{1155x^4}{128} - \frac{3003x^5}{256} + O(x^6)\right) \ln(x) \right. \\
&\qquad \left. + x^2 \left(\frac{3x}{2} - \frac{57x^2}{16} + \frac{583x^3}{96} - \frac{13771x^4}{1536} + \frac{187339x^5}{15360} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.26.1 Maple step by step solution

Let's solve

$2x^2(x+1)\, y'' + (x^2 - 6x)\, y' + (8 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-8)y}{2x^2(x+1)} - \frac{(x-6)y'}{2x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-6)y'}{2x(x+1)} - \frac{(x-8)y}{2x^2(x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x-6}{2x(x+1)}, P_3(x) = -\frac{x-8}{2x^2(x+1)}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = \frac{7}{2}$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $2x^2(x+1)\,y'' + x(x-6)\,y' + (8-x)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 8u + 7)\left(\frac{d}{du}y(u)\right) + (9-u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r(5+2r)\,u^{-1+r}+\left(a_{1}(1+r)\,(7+2r)-a_{0}(4r^{2}+4r-9)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(5+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,-\frac{5}{2}\right\}$$

- Each term must be 0

  $$a_{1}(1+r)\,(7+2r)-a_{0}(4r^{2}+4r-9)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-4a_{k}+2a_{k-1}+2a_{k+1})\,k^{2}+\left((-8a_{k}+4a_{k-1}+4a_{k+1})\,r-4a_{k}-5a_{k-1}+9a_{k+1}\right)k+(-4a_{k}+2a$$

- Shift index using $k->k+1$

  $$(-4a_{k+1}+2a_{k}+2a_{k+2})\,(k+1)^{2}+\left((-8a_{k+1}+4a_{k}+4a_{k+2})\,r-4a_{k+1}-5a_{k}+9a_{k+2}\right)(k+1)+$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{2k^{2}a_{k}-4k^{2}a_{k+1}+4kra_{k}-8kra_{k+1}+2r^{2}a_{k}-4r^{2}a_{k+1}-ka_{k}-12ka_{k+1}-ra_{k}-12ra_{k+1}-a_{k}+a_{k+1}}{2k^{2}+4kr+2r^{2}+13k+13r+18}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{2k^{2}a_{k}-4k^{2}a_{k+1}-ka_{k}-12ka_{k+1}-a_{k}+a_{k+1}}{2k^{2}+13k+18}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{2k^{2}a_{k}-4k^{2}a_{k+1}-ka_{k}-12ka_{k+1}-a_{k}+a_{k+1}}{2k^{2}+13k+18},7a_{1}+9a_{0}=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-k a_k-12k a_{k+1}-a_k+a_{k+1}}{2k^2+13k+18}, 7a_1+9a_0=0\right]$$

- Recursion relation for $r=-\frac{5}{2}$

$$a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}$$

- Solution for $r=-\frac{5}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}, -3a_1-6a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{5}{2}}, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}, -3a_1-6a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{5}{2}}\right), a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{1+k}-k a_k-12k a_{1+k}-a_k+a_{1+k}}{2k^2+13k+18}, 7a_1+\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
Order:=6;
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(6-x)*diff(y(x),x)+(8-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1) \left(1 - \frac{5}{2}x + \frac{35}{8}x^2 - \frac{105}{16}x^3 + \frac{1155}{128}x^4 - \frac{3003}{256}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{3}{2}x - \frac{57}{16}x^2 + \frac{583}{96}x^3 - \frac{13771}{1536}x^4 + \frac{187339}{15360}x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \right) x^2$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 128

```
AsymptoticDSolveValue[2*x^2*(1+x)*y''[x]-x*(6-x)*y'[x]+(8-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{3003x^5}{256} + \frac{1155x^4}{128} - \frac{105x^3}{16} + \frac{35x^2}{8} - \frac{5x}{2} + 1\right) x^2 + c_2 \left(\left(\frac{187339x^5}{15360} - \frac{13771x^4}{1536} + \frac{583x^3}{96} - \frac{57x^2}{16} + \frac{3x}{2}\right) x^2 + \left(-\frac{3003x^5}{256} + \frac{1155x^4}{128} - \frac{105x^3}{16} + \frac{35x^2}{8} - \frac{5x}{2} + 1\right) x^2 \log(x) \right)$$

15.27 problem 23

Internal problem ID [1375]
Internal file name [`OUTPUT/1376_Sunday_June_05_2022_02_13_51_AM_71262174/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+2x)\,y''+x(5+9x)\,y'+(3x+4)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(3x+4)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5+9x}{x\,(1+2x)}$$
$$q(x)=\frac{3x+4}{x^2\,(1+2x)}$$

Table 696: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5+9x}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = \frac{3x+4}{x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\, y'' + \left(9x^2+5x\right) y' + (3x+4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(1+2x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(9x^2+5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x+4)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 5x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 5x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + 9a_{n-1}(n+r-1) + 5a_n(n+r) + 3a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-1)\,a_{n-1}}{2+n+r} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = \frac{a_{n-1}(5-2n)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-2r}{3+r}$$

Which for the root $r = -2$ becomes

$$a_1 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 + 8r + 3}{(4 + r)(3 + r)}$$

Which for the root $r = -2$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 36r^2 - 46r - 15}{(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = -2$ becomes

$$a_3 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{(6 + r)(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = -2$ becomes

$$a_4 = \frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(6+r)(3+r)(4+r)(5+r)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^5 - 400r^4 - 1840r^3 - 3800r^2 - 3378r - 945}{(3 + r)(4 + r)(5 + r)(6 + r)(7 + r)}$$

Which for the root $r = -2$ becomes

$$a_5 = -\frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(6+r)(3+r)(4+r)(5+r)}$	$\frac{3}{8}$
a_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{3}{8}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{(8 + r)(3 + r)(4 + r)(5 + r)(6 + r)(7 + r)}$$

Which for the root $r = -2$ becomes

$$a_6 = \frac{7}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(6+r)(3+r)(4+r)(5+r)}$	$\frac{3}{8}$
a_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{3}{8}$
a_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{(8+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{7}{16}$

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-128r^7 - 3136r^6 - 31136r^5 - 160720r^4 - 459032r^3 - 709324r^2 - 528414r - 135135}{(3 + r)(4 + r)(5 + r)(6 + r)(7 + r)(8 + r)(9 + r)}$$

Which for the root $r = -2$ becomes

$$a_7 = -\frac{9}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-2r}{3+r}$	3
a_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$
a_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(6+r)(3+r)(4+r)(5+r)}$	$\frac{3}{8}$
a_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{3}{8}$
a_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{(8+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{7}{16}$
a_7	$\frac{-128r^7-3136r^6-31136r^5-160720r^4-459032r^3-709324r^2-528414r-135135}{(3+r)(4+r)(5+r)(6+r)(7+r)(8+r)(9+r)}$	$-\frac{9}{16}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\
&= \frac{3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + \frac{7x^6}{16} - \frac{9x^7}{16} + O(x^8)}{x^2}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-1-2r}{3+r}$	3	$-\frac{5}{(3+r)^2}$
b_2	$\frac{4r^2+8r+3}{(4+r)(3+r)}$	$\frac{3}{2}$	$\frac{20r^2+90r+75}{(3+r)^2(4+r)^2}$
b_3	$\frac{-8r^3-36r^2-46r-15}{(3+r)(4+r)(5+r)}$	$-\frac{1}{2}$	$\frac{-60r^4-660r^3-2535r^2-3960r-2055}{(3+r)^2(4+r)^2(5+r)^2}$
b_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(6+r)(3+r)(4+r)(5+r)}$	$\frac{3}{8}$	$\frac{160r^6+3120r^5+24400r^4+97500r^3+208330r^2 \cdots}{(3+r)^2(4+r)^2(5+r)^2(6+r)^2 \cdots}$
b_5	$\frac{-32r^5-400r^4-1840r^3-3800r^2-3378r-945}{(3+r)(4+r)(5+r)(6+r)(7+r)}$	$-\frac{3}{8}$	$-\frac{5(80r^8+2400r^7+30680r^6+217800r^5+9361 \cdots}{(3+r)^2(4+r)^2 \cdots}$
b_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{(8+r)(3+r)(4+r)(5+r)(6+r)(7+r)}$	$\frac{7}{16}$	$\frac{960r^{10}+40800r^9+764400r^8+8302800r^7+578 \cdots}{\cdots}$
b_7	$\frac{-128r^7-3136r^6-31136r^5-160720r^4-459032r^3-709324r^2-528414r-135135}{(3+r)(4+r)(5+r)(6+r)(7+r)(8+r)(9+r)}$	$-\frac{9}{16}$	$-\frac{35(64r^{12}+3648r^{11}+93680r^{10}+1431840r^9 \cdots}{\cdots}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= \frac{\left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + \frac{7x^6}{16} - \frac{9x^7}{16} + O(x^8)\right)\ln(x)}{x^2} \\
&\quad + \frac{-5x - \frac{25x^2}{4} + \frac{5x^3}{4} - \frac{25x^4}{32} + \frac{113x^5}{160} - \frac{247x^6}{320} + \frac{2123x^7}{2240} + O(x^8)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + \frac{7x^6}{16} - \frac{9x^7}{16} + O(x^8)\right)}{x^2} \\
&\quad + c_2\left(\frac{\left(3x + 1 + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} - \frac{3x^5}{8} + \frac{7x^6}{16} - \frac{9x^7}{16} + O(x^8)\right)\ln(x)}{x^2}\right. \\
&\qquad \left. + \frac{-5x - \frac{25x^2}{4} + \frac{5x^3}{4} - \frac{25x^4}{32} + \frac{113x^5}{160} - \frac{247x^6}{320} + \frac{2123x^7}{2240} + O(x^8)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)}{x^2} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}-\frac{247x^6}{320}+\frac{2123x^7}{2240}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)}{x^2} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}-\frac{247x^6}{320}+\frac{2123x^7}{2240}+O(x^8)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 \left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)}{x^2} \\
&\quad + c_2 \left(\frac{\left(3x+1+\frac{3x^2}{2}-\frac{x^3}{2}+\frac{3x^4}{8}-\frac{3x^5}{8}+\frac{7x^6}{16}-\frac{9x^7}{16}+O(x^8)\right)\ln(x)}{x^2}\right. \\
&\quad \left. + \frac{-5x-\frac{25x^2}{4}+\frac{5x^3}{4}-\frac{25x^4}{32}+\frac{113x^5}{160}-\frac{247x^6}{320}+\frac{2123x^7}{2240}+O(x^8)}{x^2}\right)
\end{aligned}
$$

Verified OK.

15.27.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y''+(9x^2+5x)\,y'+(3x+4)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{(3x+4)y}{x^2(1+2x)}-\frac{(5+9x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(5+9x)y'}{x(1+2x)}+\frac{(3x+4)y}{x^2(1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5+9x}{x(1+2x)}, P_3(x)=\frac{3x+4}{x^2(1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=4$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(1+2x)\,y''+x(5+9x)\,y'+(3x+4)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)^2\, x^{r}+\left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2+a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=-2$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\left(a_k(k+r+2)+a_{k-1}(2k-1+2r)\right)=0$

- Shift index using $k->k+1$

  $(k+r+3)\left(a_{k+1}(k+r+3)+a_k(2k+2r+1)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(2k+2r+1)}{k+r+3}$

- Recursion relation for $r=-2$

  $a_{k+1}=-\frac{a_k(2k-3)}{k+1}$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=-\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+9*x)*diff(y(x),x)+(4+3*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \frac{(c_2\ln(x)+c_1)\left(1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\frac{3}{8}x^4-\frac{3}{8}x^5+\frac{7}{16}x^6-\frac{9}{16}x^7+\mathrm{O}(x^8)\right)+\left((-5)x-\frac{25}{4}x^2+\frac{5}{4}x^3-\frac{25}{32}\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 164

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]+x*(5+9*x)*y'[x]+(4+3*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1\left(-\frac{9x^7}{16}+\frac{7x^6}{16}-\frac{3x^5}{8}+\frac{3x^4}{8}-\frac{x^3}{2}+\frac{3x^2}{2}+3x+1\right)}{x^2} + c_2\left(\frac{\frac{2123x^7}{2240}-\frac{247x^6}{320}+\frac{113x^5}{160}-\frac{25x^4}{32}+\frac{5x^3}{4}-\frac{25x^2}{4}-5x}{x^2} + \frac{\left(-\frac{9x^7}{16}+\frac{7x^6}{16}-\frac{3x^5}{8}+\frac{3x^4}{8}-\frac{x^3}{2}+\frac{3x^2}{2}+3x+1\right)\log(x)}{x^2}\right)$$

15.28 problem 24

Internal problem ID [1376]
Internal file name [OUTPUT/1377_Sunday_June_05_2022_02_13_55_AM_89476925/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''-x(4x+5)\,y'+(4x+9)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(4x+9)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x+5}{x\,(2x-1)}$$
$$q(x)=-\frac{4x+9}{x^2\,(2x-1)}$$

Table 698: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x+5}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = -\frac{4x+9}{x^2(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x-1) + \left(-4x^2 - 5x\right) y' + (4x+9)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(2x-1) + \left(-4x^2 - 5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x+9)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)\\
&+\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)\\
&+\left(\sum_{n=0}^{\infty}4x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)&=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)\\
\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)\right)&=\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)\,x^{n+r}\right)\\
\sum_{n=0}^{\infty}4x^{1+n+r}a_n&=\sum_{n=1}^{\infty}4a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)\\
&+\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)\,x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)\\
&+\left(\sum_{n=1}^{\infty}4a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r}a_n(n+r)(n+r-1)-5x^{n+r}a_n(n+r)+9a_nx^{n+r}=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 5x^r a_0 r + 9a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 5x^r r + 9x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-3)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-3)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-3)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&-2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1)\\&- 4a_{n-1}(n+r-1) - 5a_n(n+r) + 4a_{n-1} + 9a_n = 0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n^2 + 2nr + r^2 - n - r - 2)}{n^2 + 2nr + r^2 - 6n - 6r + 9} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{2a_{n-1}(n^2 + 5n + 4)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 + 2r - 4}{(r-2)^2}$$

Which for the root $r = 3$ becomes

$$a_1 = 20$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4(r+3)\, r(r+2)}{(-1+r)\,(r-2)^2}$$

Which for the root $r = 3$ becomes

$$a_2 = 180$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8(r+4)\,(r+1)\,(r+3)\,(r+2)}{r\,(-1+r)\,(r-2)^2}$$

Which for the root $r = 3$ becomes

$$a_3 = 1120$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180
a_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_4 = 5600$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180
a_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120
a_4	$\frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$	5600

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(6+r)(r+3)^2(r+4)(r+5)}{r(r-2)^2(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_5 = 24192$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180
a_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120
a_4	$\frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$	5600
a_5	$\frac{32(6+r)(r+3)^2(r+4)(r+5)}{r(r-2)^2(r^2-1)}$	24192

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64(7+r)(r+4)^2(r+5)(6+r)}{r(r-2)^2(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_6 = 94080$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180
a_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120
a_4	$\frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$	5600
a_5	$\frac{32(6+r)(r+3)^2(r+4)(r+5)}{r(r-2)^2(r^2-1)}$	24192
a_6	$\frac{64(7+r)(r+4)^2(r+5)(6+r)}{r(r-2)^2(r^2-1)}$	94080

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{128(8+r)(r+5)^2(6+r)(7+r)}{r(r-2)^2(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_7 = 337920$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20
a_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180
a_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120
a_4	$\frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$	5600
a_5	$\frac{32(6+r)(r+3)^2(r+4)(r+5)}{r(r-2)^2(r^2-1)}$	24192
a_6	$\frac{64(7+r)(r+4)^2(r+5)(6+r)}{r(r-2)^2(r^2-1)}$	94080
a_7	$\frac{128(8+r)(r+5)^2(6+r)(7+r)}{r(r-2)^2(r^2-1)}$	337920

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\ &= x^3\left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 + O(x^8)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=3)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r^2+2r-4}{(r-2)^2}$	20	$\frac{-10r+4}{(r-2)^3}$	-26
b_2	$\frac{4(r+3)r(r+2)}{(-1+r)(r-2)^2}$	180	$\frac{-40r^3-64r^2+104r+48}{(-1+r)^2(r-2)^3}$	-324
b_3	$\frac{8(r+4)(r+1)(r+3)(r+2)}{r(-1+r)(r-2)^2}$	1120	$-\frac{24(5r^5+28r^4+25r^3-58r^2-56r+16)}{r^2(-1+r)^2(r-2)^3}$	$-\frac{6968}{3}$
b_4	$\frac{16(r+5)(r+2)^2(r+3)(r+4)}{r(r-2)^2(r^2-1)}$	5600	$-\frac{64(r+2)(5r^6+48r^5+128r^4-301r^2-120r+60)}{r^2(r-2)^3(r^2-1)^2}$	$-\frac{37780}{3}$
b_5	$\frac{32(6+r)(r+3)^2(r+4)(r+5)}{r(r-2)^2(r^2-1)}$	24192	$-\frac{160(r+3)(5r^6+63r^5+233r^4+105r^3-622r^2-264r+144)}{r^2(r-2)^3(r^2-1)^2}$	-57360
b_6	$\frac{64(7+r)(r+4)^2(r+5)(6+r)}{r(r-2)^2(r^2-1)}$	94080	$-\frac{384(r+4)(5r^6+78r^5+368r^4+312r^3-1113r^2-490r+280)}{r^2(r-2)^3(r^2-1)^2}$	$-\frac{694736}{3}$
b_7	$\frac{128(8+r)(r+5)^2(6+r)(7+r)}{r(r-2)^2(r^2-1)}$	337920	$-\frac{896(r+5)(5r^6+93r^5+533r^4+651r^3-1810r^2-816r+480)}{r^2(r-2)^3(r^2-1)^2}$	$-\frac{2566144}{3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= x^3\left(337920x^7+94080x^6+24192x^5+5600x^4+1120x^3+180x^2+20x+1+O(x^8)\right)\ln(x) \\
&\quad + x^3\left(-324x^2-26x-\frac{6968x^3}{3}-\frac{37780x^4}{3}-57360x^5-\frac{694736x^6}{3}-\frac{2566144x^7}{3} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3\left(337920x^7+94080x^6+24192x^5+5600x^4+1120x^3+180x^2+20x+1+O(x^8)\right) \\
&\quad + c_2\Bigg(x^3\left(337920x^7+94080x^6+24192x^5+5600x^4+1120x^3+180x^2+20x+1 + O(x^8)\right)\ln(x) \\
&\quad + x^3\left(-324x^2-26x-\frac{6968x^3}{3}-\frac{37780x^4}{3}-57360x^5 - \frac{694736x^6}{3}-\frac{2566144x^7}{3}+O(x^8)\right)\Bigg)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 + O(x^8)\right) \\
&\quad + c_2 \left(x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 \right.\right. \\
&\quad \left. + O(x^8)\right) \ln(x) + x^3 \left(-324x^2 - 26x - \frac{6968x^3}{3} - \frac{37780x^4}{3} - 57360x^5 - \frac{694736x^6}{3} \right. \\
&\quad \left.\left. - \frac{2566144x^7}{3} + O(x^8) \right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 + O(x^8)\right) \\
&\quad + c_2 \left(x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 \right.\right. \\
&\quad \left. + O(x^8)\right) \ln(x) + x^3 \left(-324x^2 - 26x - \frac{6968x^3}{3} - \frac{37780x^4}{3} - 57360x^5 \right. \\
&\quad \left.\left. - \frac{694736x^6}{3} - \frac{2566144x^7}{3} + O(x^8) \right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 + O(x^8)\right) \\
&\quad + c_2 \left(x^3 \left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1 \right.\right. \\
&\quad \left. + O(x^8)\right) \ln(x) + x^3 \left(-324x^2 - 26x - \frac{6968x^3}{3} - \frac{37780x^4}{3} - 57360x^5 - \frac{694736x^6}{3} \right. \\
&\quad \left.\left. - \frac{2566144x^7}{3} + O(x^8) \right)\right)
\end{aligned}
$$

Verified OK.

15.28.1 Maple step by step solution

Let's solve

$-y''x^2(2x-1)+(-4x^2-5x)\,y'+(4x+9)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(4x+9)y}{x^2(2x-1)}-\frac{(4x+5)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x+5)y'}{x(2x-1)}-\frac{(4x+9)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{4x+5}{x(2x-1)}, P_3(x)=-\frac{4x+9}{x^2(2x-1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-5$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=9$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(4x+5)\,y'+(-4x-9)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 3$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$

- Shift index using $k->k+1$

  $-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 81

```
Order:=8;
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5+4*x)*diff(y(x),x)+(9+4*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + 20x + 180x^2 + 1120x^3 + 5600x^4 + 24192x^5 + 94080x^6 + 337920x^7 + \mathrm{O}(x^8)\right) + \left((-26)x - 324x^2 - \frac{6968}{3}x^3 - \frac{37780}{3}x^4 - 57360x^5 - \frac{694736}{3}x^6 - \frac{2566144}{3}x^7 + \mathrm{O}(x^8)\right)c_2\right)x^3$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 136

```
AsymptoticDSolveValue[x^2*(1-2*x)*y''[x]-x*(5+4*x)*y'[x]+(9+4*x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1\left(337920x^7 + 94080x^6 + 24192x^5 + 5600x^4 + 1120x^3 + 180x^2 + 20x + 1\right)x^3 + c_2\left(\left(-\frac{2566144x^7}{3} - \frac{694736x^6}{3} - 57360x^5 - \frac{37780x^4}{3} - \frac{6968x^3}{3} - 324x^2 - 26x\right)x^3 + \left(337920x^7+94080x^6+24192x^5+5600x^4+1120x^3+180x^2+20x+1\right)x^3\log(x)\right)$$

15.29 problem 25

Internal problem ID [1377]
Internal file name [`OUTPUT/1378_Sunday_June_05_2022_02_13_58_AM_79581135/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(4x+1)\,y''-x(-4x+1)\,y'+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+x^2\right)y''+\left(4x^2-x\right)y'+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x-1}{x\,(4x+1)}$$
$$q(x)=\frac{x+1}{x^2\,(4x+1)}$$

Table 700: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x-1}{x(4x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{4}$	"regular"

$q(x) = \frac{x+1}{x^2(4x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{4}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{4}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(4x+1)\, y'' + \left(4x^2 - x\right) y' + (x+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(4x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(4x^2 - x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ &+ 4a_{n-1}(n+r-1) - a_n(n+r) + a_{n-1} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(4n^2 + 8nr + 4r^2 - 8n - 8r + 5)}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(-4n^2 - 1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4r^2 - 1}{r^2}$$

Which for the root $r = 1$ becomes

$$a_1 = -5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^4 + 32r^3 + 24r^2 + 8r + 5}{r^2 (r+1)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{85}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-64r^6 - 384r^5 - 880r^4 - 960r^3 - 556r^2 - 216r - 85}{r^2 (r+1)^2 (r+2)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{3145}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$
a_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{204425}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$
a_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$
a_4	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$	$\frac{204425}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{825877}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$
a_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$
a_4	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$	$\frac{204425}{576}$
a_5	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{825877}{576}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$$

Which for the root $r = 1$ becomes

$$a_6 = \frac{119752165}{20736}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$
a_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$
a_4	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$	$\frac{204425}{576}$
a_5	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{825877}{576}$
a_6	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{119752165}{20736}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)(4r^2}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$$

Which for the root $r = 1$ becomes

$$a_7 = -\frac{23591176505}{1016064}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2-1}{r^2}$	-5
a_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$
a_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$
a_4	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$	$\frac{204425}{576}$
a_5	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{825877}{576}$
a_6	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{119752165}{20736}$
a_7	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)(4r^2+48r+145)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$	$-\frac{23591176505}{1016064}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 \dots\right) \\
&= x\left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} \right. \\
&\qquad \left. - \frac{23591176505x^7}{1016064} + O(x^8)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}$
b_0	1	1	N/A sinc
b_1	$\frac{-4r^2-1}{r^2}$	-5	$\frac{2}{r^3}$
b_2	$\frac{16r^4+32r^3+24r^2+8r+5}{r^2(r+1)^2}$	$\frac{85}{4}$	$\frac{-16r^3-24r}{r^3(r+}$
b_3	$\frac{-64r^6-384r^5-880r^4-960r^3-556r^2-216r-85}{r^2(r+1)^2(r+2)^2}$	$-\frac{3145}{36}$	$96r^6+576r^5$
b_4	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)}{r^2(r+1)^2(r+2)^2(r+3)^2}$	$\frac{204425}{576}$	$-512r^9-69$
b_5	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$-\frac{825877}{576}$	$2560r^{12}+61$
b_6	$\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{119752165}{20736}$	$-12288r^{15}-$
b_7	$-\frac{(4r^2+24r+37)(4r^2+1)(4r^2+8r+5)(4r^2+16r+17)(4r^2+32r+65)(4r^2+40r+101)(4r^2+48r+145)}{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$	$-\frac{23591176505}{1016064}$	$57344r^{18}+3$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \ldots \\
&= x\left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} \right.\\
&\quad \left. - \frac{23591176505x^7}{1016064} + O(x^8)\right)\ln(x) + x\left(2x - \frac{39x^2}{4} + \frac{4499x^3}{108} - \frac{594305x^4}{3456} \right.\\
&\quad \left. + \frac{2420617x^5}{3456} - \frac{117547073x^6}{41472} + \frac{162576422327x^7}{14224896} + O(x^8)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} \right.\\
&\quad \left. - \frac{23591176505x^7}{1016064} + O(x^8)\right) \\
&\quad + c_2\left(x\left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} \right.\right.\\
&\quad \left. - \frac{23591176505x^7}{1016064} + O(x^8)\right)\ln(x) + x\left(2x - \frac{39x^2}{4} + \frac{4499x^3}{108} - \frac{594305x^4}{3456} \right.\\
&\quad \left.\left. + \frac{2420617x^5}{3456} - \frac{117547073x^6}{41472} + \frac{162576422327x^7}{14224896} + O(x^8)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$
$$= c_1 x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) + c_2 \left(x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) \ln(x) + x \left(2x - \frac{39x^2}{4} + \frac{4499x^3}{108} - \frac{594305x^4}{3456} + \frac{2420617x^5}{3456} - \frac{117547073x^6}{41472} + \frac{162576422327x^7}{14224896} + O(x^8)\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) + c_2 \left(x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) \ln(x) + x \left(2x - \frac{39x^2}{4} + \frac{4499x^3}{108} - \frac{594305x^4}{3456} + \frac{2420617x^5}{3456} - \frac{117547073x^6}{41472} + \frac{162576422327x^7}{14224896} + O(x^8)\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) + c_2 \left(x \left(1 - 5x + \frac{85x^2}{4} - \frac{3145x^3}{36} + \frac{204425x^4}{576} - \frac{825877x^5}{576} + \frac{119752165x^6}{20736} - \frac{23591176505x^7}{1016064} + O(x^8)\right) \ln(x) + x \left(2x - \frac{39x^2}{4} + \frac{4499x^3}{108} - \frac{594305x^4}{3456} + \frac{2420617x^5}{3456} - \frac{117547073x^6}{41472} + \frac{162576422327x^7}{14224896} + O(x^8)\right)\right)$$

Verified OK.

15.29.1 Maple step by step solution

Let's solve

$x^2(4x+1)\,y''+(4x^2-x)\,y'+(x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{(x+1)y}{x^2(4x+1)}-\frac{(4x-1)y'}{x(4x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(4x-1)y'}{x(4x+1)}+\frac{(x+1)y}{x^2(4x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4x-1}{x(4x+1)},P_3(x)=\frac{x+1}{x^2(4x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(4x+1)\,y''+x(4x-1)\,y'+(x+1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2}\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r-1)^{2}+a_{k-1}\left(4(k-1)^{2}+8(k-1)\,r+4r^{2}+1\right)\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r-1)^{2}+a_{k-1}(4k^{2}+8kr+4r^{2}-8k-8r+5)=0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r)^{2}+a_{k}\left(4(k+1)^{2}+8(k+1)\,r+4r^{2}-8k-3-8r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_{k}\left(4k^{2}+8kr+4r^{2}+1\right)}{(k+r)^{2}}$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_{k}\left(4k^{2}+8k+5\right)}{(k+1)^{2}}$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k(4k^2+8k+5)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
Order:=8;
dsolve(x^2*(1+4*x)*diff(y(x),x$2)-x*(1-4*x)*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0
```

$$y(x)=\left((c_2\ln(x)+c_1)\left(1-5x+\frac{85}{4}x^2-\frac{3145}{36}x^3+\frac{204425}{576}x^4-\frac{825877}{576}x^5+\frac{119752165}{20736}x^6-\frac{23591176505}{1016064}x^7+\mathrm{O}\left(x^8\right)\right)+\left(2x-\frac{39}{4}x^2+\frac{4499}{108}x^3-\frac{594305}{3456}x^4+\frac{2420617}{3456}x^5-\frac{117547073}{41472}x^6+\frac{162576422327}{14224896}x^7+\mathrm{O}\left(x^8\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 158

```
AsymptoticDSolveValue[x^2*(1+4*x)*y''[x]-x*(1-4*x)*y'[x]+(1+x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to c_1 x\left(-\frac{23591176505x^7}{1016064}+\frac{119752165x^6}{20736}-\frac{825877x^5}{576}+\frac{204425x^4}{576}-\frac{3145x^3}{36}+\frac{85x^2}{4}-5x+1\right)+c_2\left(x\left(\frac{162576422327x^7}{14224896}-\frac{117547073x^6}{41472}+\frac{2420617x^5}{3456}-\frac{594305x^4}{3456}+\frac{4499x^3}{108}-\frac{39x^2}{4}+2x\right)+x\left(-\frac{23591176505x^7}{1016064}+\frac{119752165x^6}{20736}-\frac{825877x^5}{576}+\frac{204425x^4}{576}-\frac{3145x^3}{36}+\frac{85x^2}{4}-5x+1\right)\log(x)\right)$$

15.30 problem 26

Internal problem ID [1378]
Internal file name [`OUTPUT/1379_Sunday_June_05_2022_02_14_01_AM_5965303/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''+x(1+2x)\,y'+yx=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$x^2(x+1)\,y''+\left(2x^2+x\right)y'+yx=0$$

Or

$$x\left(x^2y''+2y'x+y''x+y+y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(x^2+x\right)y''+(1+2x)\,y'+y=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(2x^2+x\right)y'+yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = \frac{1+2x}{x(x+1)}$$
$$q(x) = \frac{1}{x(x+1)}$$

Table 702: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+2x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + \left(2x^2 + x\right) y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^{2}(x+1)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}\right)\\
&+\left(2x^{2}+x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}a_{n}x^{n+r}\right)x=0
\end{aligned}\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{1+n+r}a_{n}(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty}2x^{1+n+r}a_{n}(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_{n}\right)=0
\end{aligned}\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}x^{1+n+r}a_{n}(n+r)\,(n+r-1)&=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\\
\sum_{n=0}^{\infty}2x^{1+n+r}a_{n}(n+r)&=\sum_{n=1}^{\infty}2a_{n-1}(n+r-1)\,x^{n+r}\\
\sum_{n=0}^{\infty}x^{1+n+r}a_{n}&=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=1}^{\infty}2a_{n-1}(n+r-1)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_{n}(n+r)\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r}\right)=0
\end{aligned}\tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1) + x^{n+r}a_n(n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) + x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ & + 2a_{n-1}(n+r-1) + a_n(n+r) + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - n - r + 1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - n + 1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 - r - 1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 + 4r^3 + 7r^2 + 6r + 3}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^6 - 9r^5 - 34r^4 - 69r^3 - 82r^2 - 57r - 21}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{7}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$
a_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^2 + r + 1)(r^2 + 3r + 3)(r^2 + 5r + 7)(r^2 + 7r + 13)}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{91}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$
a_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$
a_4	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{91}{192}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{637}{1600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$
a_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$
a_4	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{91}{192}$
a_5	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{637}{1600}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{19747}{57600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$
a_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$
a_4	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{91}{192}$
a_5	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{637}{1600}$
a_6	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$	$\frac{19747}{57600}$

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)(r^2+13r+4}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2(7+r)^2}$$

Which for the root $r = 0$ becomes

$$a_7 = -\frac{17329}{57600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1
a_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$
a_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$
a_4	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{91}{192}$
a_5	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{637}{1600}$
a_6	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$	$\frac{19747}{57600}$
a_7	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)(r^2+13r+43)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2(7+r)^2}$	$-\frac{17329}{57600}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots \\
&= 1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts fro
b_1	$\frac{-r^2-r-1}{(r+1)^2}$	-1	$\frac{1-r}{(r+1)^3}$
b_2	$\frac{r^4+4r^3+7r^2+6r+3}{(r+1)^2(r+2)^2}$	$\frac{3}{4}$	$\frac{2r^4+6r^3+6r^2-2r-6}{(r+1)^3(r+2)^3}$
b_3	$\frac{-r^6-9r^5-34r^4-69r^3-82r^2-57r-21}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{7}{12}$	$\frac{-3r^7-30r^6-126r^5-276r^4-30}{(r+1)^3(r+2)^3(}$
b_4	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{91}{192}$	$\frac{4r^{10}+80r^9+708r^8+3624r^7+11}{(}$
b_5	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{637}{1600}$	$\frac{-5r^{13}-165r^{12}-2475r^{11}-2228}{}$
b_6	$\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2}$	$\frac{19747}{57600}$	$\frac{6r^{16}+294r^{15}+6660r^{14}+92460r}{}$
b_7	$-\frac{(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)(r^2+11r+31)(r^2+13r+43)}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2(6+r)^2(7+r)^2}$	$-\frac{17329}{57600}$	$-\frac{7(r^{19}+68r^{18}+2163r^{17}+4274}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \ln(x) \\
&\quad + x - \frac{3x^2}{4} + \frac{5x^3}{9} - \frac{499x^4}{1152} + \frac{16919x^5}{48000} - \frac{56861x^6}{192000} + \frac{1027717x^7}{4032000} + O(x^8)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \\
&+ c_2 \left(\left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \ln(x) \right. \\
&\left. + x - \frac{3x^2}{4} + \frac{5x^3}{9} - \frac{499x^4}{1152} + \frac{16919x^5}{48000} - \frac{56861x^6}{192000} + \frac{1027717x^7}{4032000} + O(x^8)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \\
&+ c_2 \left(\left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \ln(x) \right. \\
&\left. + x - \frac{3x^2}{4} + \frac{5x^3}{9} - \frac{499x^4}{1152} + \frac{16919x^5}{48000} - \frac{56861x^6}{192000} + \frac{1027717x^7}{4032000} + O(x^8)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \\
&+ c_2 \left(\left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \ln(x) \right. \\
&\left. + x - \frac{3x^2}{4} + \frac{5x^3}{9} - \frac{499x^4}{1152} + \frac{16919x^5}{48000} - \frac{56861x^6}{192000} + \frac{1027717x^7}{4032000} + O(x^8)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \\
&+ c_2 \left(\left(1 - x + \frac{3x^2}{4} - \frac{7x^3}{12} + \frac{91x^4}{192} - \frac{637x^5}{1600} + \frac{19747x^6}{57600} - \frac{17329x^7}{57600} + O(x^8)\right) \ln(x) \right. \\
&\left. + x - \frac{3x^2}{4} + \frac{5x^3}{9} - \frac{499x^4}{1152} + \frac{16919x^5}{48000} - \frac{56861x^6}{192000} + \frac{1027717x^7}{4032000} + O(x^8)\right)
\end{aligned}
$$

Verified OK.

15.30.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y''+(2x^2+x)\,y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x+1)}-\frac{(1+2x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+2x)y'}{x(x+1)}+\frac{y}{x(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1+2x}{x(x+1)},P_3(x)=\frac{1}{x(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(x+1)\,y''+(1+2x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(-1+2u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)^2 + a_k(k^2+2kr+r^2+k+r+1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+1)^2 + a_k(k^2+k+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k^2+k+1)}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k^2+k+1)}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+k+1)}{(k+1)^2}\right]$$

- Revert the change of variables $u = x+1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(k^2+k+1)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
Order:=8;
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1+2*x)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - x + \frac{3}{4}x^2 - \frac{7}{12}x^3 + \frac{91}{192}x^4 - \frac{637}{1600}x^5 + \frac{19747}{57600}x^6 - \frac{17329}{57600}x^7 + O(x^8)\right) + \left(x - \frac{3}{4}x^2 + \frac{5}{9}x^3 - \frac{499}{1152}x^4 + \frac{16919}{48000}x^5 - \frac{56861}{192000}x^6 + \frac{1027717}{4032000}x^7 + O(x^8)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 151

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]+x*(1+2*x)*y'[x]+x*y[x]==0,y[x],{x,0,7}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\left(-\frac{17329x^7}{57600}+\frac{19747x^6}{57600}-\frac{637x^5}{1600}+\frac{91x^4}{192}-\frac{7x^3}{12}+\frac{3x^2}{4}-x+1\right) \\
& + c_2\left(\frac{1027717x^7}{4032000}-\frac{56861x^6}{192000}+\frac{16919x^5}{48000}-\frac{499x^4}{1152}+\frac{5x^3}{9}-\frac{3x^2}{4}\right. \\
& \left.+\left(-\frac{17329x^7}{57600}+\frac{19747x^6}{57600}-\frac{637x^5}{1600}+\frac{91x^4}{192}-\frac{7x^3}{12}+\frac{3x^2}{4}-x+1\right)\log(x)+x\right)
\end{aligned}
$$

15.31 problem 27

Internal problem ID [1379]
Internal file name [OUTPUT/1380_Sunday_June_05_2022_02_14_05_AM_84000659/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{7+x}{x\,(x-1)}$$
$$q(x)=\frac{-9+x}{x^2\,(x-1)}$$

Table 704: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{7+x}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{-9+x}{x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x-1) + (x^2 + 7x)\, y' + (9 - x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + (x^2 + 7x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (9 - x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 7x^{n+r} a_n (n+r) + 9a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 7x^r a_0 r + 9a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 7x^r r + 9x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+3)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+3)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -3$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+3)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-3}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) + 7a_n(n+r) + 9a_n - a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{n^2 + 2nr + r^2 + 6n + 6r + 9} \tag{4}$$

Which for the root $r = -3$ becomes

$$a_n = \frac{a_{n-1}(n-5)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(-1+r)^2}{(r+4)^2}$$

Which for the root $r = -3$ becomes

$$a_1 = 16$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$$

Which for the root $r = -3$ becomes

$$a_2 = 36$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$$

Which for the root $r = -3$ becomes

$$a_3 = 16$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36
a_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$$

Which for the root $r = -3$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36
a_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16
a_4	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$	1

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2 (8+r)^2}$$

Which for the root $r = -3$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36
a_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16
a_4	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$	1
a_5	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2 (8+r)^2}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(9+r)^2 (8+r)^2 (7+r)^2 (6+r)^2 (r+5)^2}$$

Which for the root $r = -3$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36
a_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16
a_4	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$	1
a_5	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2 (8+r)^2}$	0
a_6	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(9+r)^2 (8+r)^2 (7+r)^2 (6+r)^2 (r+5)^2}$	0

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+10)^2 (6+r)^2 (7+r)^2 (8+r)^2 (9+r)^2}$$

Which for the root $r = -3$ becomes

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(r+4)^2}$	16
a_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36
a_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16
a_4	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$	1
a_5	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2 (8+r)^2}$	0
a_6	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(9+r)^2 (8+r)^2 (7+r)^2 (6+r)^2 (r+5)^2}$	0
a_7	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+10)^2 (6+r)^2 (7+r)^2 (8+r)^2 (9+r)^2}$	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x^3}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots\right) \\
&= \frac{x^4 + 16x^3 + 36x^2 + 16x + 1 + O(x^8)}{x^3}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{(-1+r)^2}{(r+4)^2}$	16	$\frac{-10+10r}{(r+4)^3}$
b_2	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+5)^2}$	36	$\frac{20r(-1+r)(r^2+4r-2)}{(r+4)^3 (r+5)^3}$
b_3	$\frac{(-1+r)^2 r^2 (r+1)^2}{(r+4)^2 (r+5)^2 (6+r)^2}$	16	$\frac{30r(r^4+10r^3+25r^2-8)(r^2-1)}{(r+4)^3 (r+5)^3 (6+r)^3}$
b_4	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2}$	1	$\frac{40r(r+2)(r^6+18r^5+115r^4+300r^3+238r^2-84r-84)(r^2-1)}{(r+4)^3 (r+5)^3 (6+r)^3 (7+r)^3}$
b_5	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+4)^2 (r+5)^2 (6+r)^2 (7+r)^2 (8+r)^2}$	0	$\frac{50(r+2)(r^8+28r^7+314r^6+1792r^5+5417r^4+7924r^3+3340r^2-2688r-\frac{8064}{5})(r+3)(r+1)r(}{(r+4)^3 (r+5)^3 (6+r)^3 (7+r)^3 (8+r)^3}$
b_6	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(9+r)^2 (8+r)^2 (7+r)^2 (6+r)^2 (r+5)^2}$	0	$\frac{60r(r+2)(r+3)(r^8+32r^7+408r^6+2624r^5+8813r^4+14032r^3+6346r^2-5040r-3024)(r^2-}{(9+r)^3 (8+r)^3 (7+r)^3 (6+r)^3 (r+5)^3}$
b_7	$\frac{(-1+r)^2 r^2 (r+1)^2 (r+2)^2 (r+3)^2}{(r+10)^2 (6+r)^2 (7+r)^2 (8+r)^2 (9+r)^2}$	0	$\frac{70r(r+2)(r+3)(r^2+9r+12)(r^6+27r^5+259r^4+1017r^3+1252r^2-396r-432)(-1+r)(r+1)}{(r+10)^3 (6+r)^3 (7+r)^3 (8+r)^3 (9+r)^3}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots \\
&= \frac{(x^4 + 16x^3 + 36x^2 + 16x + 1 + O(x^8)) \ln(x)}{x^3} + \frac{-150x^2 - 40x - \frac{280x^3}{3} - \frac{25x^4}{3} + O(x^8)}{x^3}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1 (x^4 + 16x^3 + 36x^2 + 16x + 1 + O(x^8))}{x^3} \\
&\quad + c_2 \left(\frac{(x^4 + 16x^3 + 36x^2 + 16x + 1 + O(x^8)) \ln(x)}{x^3} \right. \\
&\quad \left. + \frac{-150x^2 - 40x - \frac{280x^3}{3} - \frac{25x^4}{3} + O(x^8)}{x^3} \right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= \frac{c_1(x^4+16x^3+36x^2+16x+1+O(x^8))}{x^3} \\
&\quad + c_2\left(\frac{(x^4+16x^3+36x^2+16x+1+O(x^8))\ln(x)}{x^3}\right. \\
&\quad \left. + \frac{-150x^2-40x-\frac{280x^3}{3}-\frac{25x^4}{3}+O(x^8)}{x^3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1(x^4+16x^3+36x^2+16x+1+O(x^8))}{x^3} \\
&\quad + c_2\left(\frac{(x^4+16x^3+36x^2+16x+1+O(x^8))\ln(x)}{x^3}\right. \\
&\quad \left. + \frac{-150x^2-40x-\frac{280x^3}{3}-\frac{25x^4}{3}+O(x^8)}{x^3}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1(x^4+16x^3+36x^2+16x+1+O(x^8))}{x^3} \\
&\quad + c_2\left(\frac{(x^4+16x^3+36x^2+16x+1+O(x^8))\ln(x)}{x^3}\right. \\
&\quad \left. + \frac{-150x^2-40x-\frac{280x^3}{3}-\frac{25x^4}{3}+O(x^8)}{x^3}\right)
\end{aligned}
$$

Verified OK.

15.31.1 Maple step by step solution

Let's solve

$-y''x^2(x-1)+(x^2+7x)\,y'+(9-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-9+x)y}{x^2(x-1)}+\frac{(7+x)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(7+x)y'}{x(x-1)} + \frac{(-9+x)y}{x^2(x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{7+x}{x(x-1)}, P_3(x) = \frac{-9+x}{x^2(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 7$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 9$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1) - x(7+x)\,y' + y(-9+x) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = -3$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_k(k+r+3)^2 + a_{k-1}(k-2+r)^2 = 0$$

- Shift index using $k->k+1$

  $$-a_{k+1}(k+4+r)^2 + a_k(k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r-1)^2}{(k+4+r)^2}$$

- Recursion relation for $r=-3$; series terminates at $k=4$

  $$a_{k+1} = \frac{a_k(k-4)^2}{(k+1)^2}$$

- Apply recursion relation for $k=0$

  $$a_1 = 16a_0$$

- Apply recursion relation for $k=1$

  $$a_2 = \frac{9a_1}{4}$$

- Express in terms of a_0

  $$a_2 = 36a_0$$

- Apply recursion relation for $k=2$

  $$a_3 = \frac{4a_2}{9}$$

- Express in terms of a_0

  $$a_3 = 16a_0$$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$
- Express in terms of a_0

  $a_4 = a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 63

```
Order:=8;
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(7+x)*diff(y(x),x)+(9-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)(1 + 16x + 36x^2 + 16x^3 + x^4 + O(x^8)) + ((-40)x - 150x^2 - \frac{280}{3}x^3 - \frac{25}{3}x^4 + O(x^8))c_2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 83

```
AsymptoticDSolveValue[x^2*(1-x)*y''[x]+x*(7+x)*y'[x]+(9-x)*y[x]==0,y[x],{x,0,7}]
```

$$y(x) \to \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} + c_2\left(\frac{-\frac{25x^4}{3} - \frac{280x^3}{3} - 150x^2 - 40x}{x^3} + \frac{(x^4 + 16x^3 + 36x^2 + 16x + 1)\log(x)}{x^3}\right)$$

15.32 problem 28

Internal problem ID [1380]
Internal file name [`OUTPUT/1381_Sunday_June_05_2022_02_14_08_AM_55793503/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(-x^2+1)\,y' + y(x^2+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 - x)\,y' + y(x^2+1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2-1}{x}$$
$$q(x) = \frac{x^2+1}{x^2}$$

Table 706: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{x^2+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 - x)\, y' + y(x^2 + 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (x^3 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x^2 + 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - a_n(n+r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{1+r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(1+r)(3+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(1+r)(3+r)}$	$\frac{1}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$	$\frac{1}{(1+r)^2}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(1+r)(3+r)}$	$\frac{1}{8}$	$\frac{-4-2r}{(1+r)^2(3+r)^2}$	$-\frac{3}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.32.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 - x) y' + y(x^2 + 1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2 y'' + x(x^2-1)\, y' + y(x^2+1) = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right) + c_2\left(x\left(\frac{x^2}{4} - \frac{3x^4}{32}\right) + x\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)\log(x)\right)$$

15.33 problem 29

Internal problem ID [1381]
Internal file name [OUTPUT/1382_Sunday_June_05_2022_02_14_10_AM_64284537/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-3x(-x^2+1)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(3x^3-3x)\,y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x^2-3}{x\,(x^2+1)}$$
$$q(x)=\frac{4}{x^2\,(x^2+1)}$$

Table 708: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2-3}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{4}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 (x^2 + 1) y'' + (3x^3 - 3x) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 (x^2 + 1) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + (3x^3 - 3x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + 4 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 3a_{n-2} (n+r-2) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=2}^{\infty} 3a_{n-2} (n+r-2) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-2}(n+r-2)(n-3+r)+a_n(n+r)(n+r-1) \\
&+3a_{n-2}(n+r-2)-3a_n(n+r)+4a_n=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r)\, a_{n-2}}{n+r-2} \tag{4}
$$

Which for the root $r=2$ becomes

$$
a_n = -\frac{(n+2)\, a_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$
a_2 = \frac{-2-r}{r}
$$

Which for the root $r=2$ becomes

$$
a_2 = -2
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{r}$	-2

For $n=3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{r}$	-2
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4+r}{r}$$

Which for the root $r = 2$ becomes

$$a_4 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{r}$	-2
a_3	0	0
a_4	$\frac{4+r}{r}$	3

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{r}$	-2
a_3	0	0
a_4	$\frac{4+r}{r}$	3
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-2-r}{r}$	-2	$\frac{2}{r^2}$	$\frac{1}{2}$
b_3	0	0	0	0
b_4	$\frac{4+r}{r}$	3	$-\frac{4}{r^2}$	-1
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \ln(x) + x^2 \left(\frac{x^2}{2} - x^4 + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \ln(x) + x^2 \left(\frac{x^2}{2} - x^4 + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \ln(x) + x^2 \left(\frac{x^2}{2} - x^4 + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \ln(x) + x^2 \left(\frac{x^2}{2} - x^4 + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \\
&\quad + c_2 \left(x^2 \left(1 - 2x^2 + 3x^4 + O(x^6)\right) \ln(x) + x^2 \left(\frac{x^2}{2} - x^4 + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.33.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\,y'' + (3x^3 - 3x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{3(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x^2-1)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3(x^2-1)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2(x^2+1)\,y'' + 3x(x^2-1)\,y' + 4y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + a_1(-1+r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\,(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$r = 2$

- Each term must be 0

  $a_1(-1+r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r-2)+a_{k-2}(k+r)\right) = 0$

- Shift index using $k->k+2$

  $(k+r)\left(a_{k+2}(k+r)+a_k(k+r+2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-3*x*(1-x^2)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 2x^2 + 3x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{2}x^2 - x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 61

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-3*x*(1-x^2)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(3x^4 - 2x^2 + 1)x^2 + c_2\left(\left(\frac{x^2}{2} - x^4\right)x^2 + (3x^4 - 2x^2 + 1)x^2 \log(x)\right)$$

15.34 problem 30

Internal problem ID [1382]
Internal file name [OUTPUT/1383_Sunday_June_05_2022_02_14_12_AM_83402995/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x^3y' + (3x^2 + 1) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 2x^3y' + (3x^2 + 1) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x}{2}$$
$$q(x) = \frac{3x^2 + 1}{4x^2}$$

Table 710: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x}{2}$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{3x^2+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 2x^3y' + (3x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2x^3 \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x^2+1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{2+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{2+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{2+n+r} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{2+n+r} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) + 3a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3+2r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3+2r}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{4r^2+20r+21}$	$\frac{1}{32}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{4r^2+20r+21}$	$\frac{1}{32}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right)\\ &= \sqrt{x}\left(1 - \frac{x^2}{4} + \frac{x^4}{32} + O\left(x^6\right)\right)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{3+2r}$	$-\frac{1}{4}$	$\frac{2}{(3+2r)^2}$	$\frac{1}{8}$
b_3	0	0	0	0
b_4	$\frac{1}{4r^2+20r+21}$	$\frac{1}{32}$	$\frac{-20-8r}{(4r^2+20r+21)^2}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots\\ &= \sqrt{x}\left(1 - \frac{x^2}{4} + \frac{x^4}{32} + O\left(x^6\right)\right)\ln(x) + \sqrt{x}\left(\frac{x^2}{8} - \frac{3x^4}{128} + O\left(x^6\right)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{4} + \frac{x^4}{32} + O\left(x^6\right)\right)\\ &\quad + c_2\left(\sqrt{x}\left(1 - \frac{x^2}{4} + \frac{x^4}{32} + O\left(x^6\right)\right)\ln(x) + \sqrt{x}\left(\frac{x^2}{8} - \frac{3x^4}{128} + O\left(x^6\right)\right)\right)\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\ln(x)+\sqrt{x}\left(\frac{x^2}{8}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\ln(x)+\sqrt{x}\left(\frac{x^2}{8}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\ln(x)+\sqrt{x}\left(\frac{x^2}{8}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.34.1 Maple step by step solution

Let's solve

$4x^2y''+2x^3y'+(3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'x}{2}-\frac{(3x^2+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'x}{2}+\frac{(3x^2+1)y}{4x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 2x^3y' + (3x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2} (k - 2 + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$
- Each term must be 0

  $a_1(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{2k+2r+3}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+2*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(\left(c_2 \ln(x) + c_1\right)\left(1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{8}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 77

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^4}{32} - \frac{x^2}{4} + 1\right) + c_2\left(\sqrt{x}\left(\frac{x^2}{8} - \frac{3x^4}{128}\right) + \sqrt{x}\left(\frac{x^4}{32} - \frac{x^2}{4} + 1\right)\log(x)\right)$$

15.35 problem 31

Internal problem ID [1383]
Internal file name [`OUTPUT/1384_Sunday_June_05_2022_02_14_15_AM_14988136/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-x(-2x^2+1)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(2x^3-x)\,y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2-1}{x\,(x^2+1)}$$
$$q(x)=\frac{1}{x^2\,(x^2+1)}$$

Table 712: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2-1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{1}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (2x^3 - x)\,y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3 - x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)\,(n-3+r)+a_n(n+r)\,(n+r-1) \\ +2a_{n-2}(n+r-2)-a_n(n+r)+a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}(n-1)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r}{1+r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{1+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{1+r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{(3+r)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(2+r)}{(3+r)(1+r)}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(2+r)}{(3+r)(1+r)}$	$\frac{3}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)\\ &= x\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{r}{1+r}$	$-\frac{1}{2}$	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{r(2+r)}{(3+r)(1+r)}$	$\frac{3}{8}$	$\frac{2r^2+6r+6}{(1+r)^2(3+r)^2}$	$\frac{7}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\\ &= x\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{4} + \frac{7x^4}{32} + O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)\\ &\quad + c_2\left(x\left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{4} + \frac{7x^4}{32} + O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \ln(x) + x \left(-\frac{x^2}{4} + \frac{7x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \ln(x) + x \left(-\frac{x^2}{4} + \frac{7x^4}{32} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(x \left(1 - \frac{x^2}{2} + \frac{3x^4}{8} + O(x^6)\right) \ln(x) + x \left(-\frac{x^2}{4} + \frac{7x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.35.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (2x^3 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x^2+1)} - \frac{(2x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2-1)y'}{x(x^2+1)} + \frac{y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

$\left[P_2(x) = \frac{2x^2-1}{x(x^2+1)}, P_3(x) = \frac{1}{x^2(x^2+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(2x^2-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-2+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r-1) + a_{k-2}(k-2+r)) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)(a_{k+2}(k+r+1) + a_k(k+r)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+1)}{k+2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-2*x^2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{1}{4}x^2 + \frac{7}{32}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-x*(1-2*x^2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{3x^4}{8} - \frac{x^2}{2} + 1\right) + c_2\left(x\left(\frac{7x^4}{32} - \frac{x^2}{4}\right) + x\left(\frac{3x^4}{8} - \frac{x^2}{2} + 1\right)\log(x)\right)$$

15.36 problem 32

Internal problem ID [1384]
Internal file name [`OUTPUT/1385_Sunday_June_05_2022_02_14_17_AM_88013605/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+2)\,y''+7x^3y'+(3x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+4x^2)\,y''+7x^3y'+(3x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x}{2(x^2+2)}$$
$$q(x)=\frac{3x^2+1}{2x^2(x^2+2)}$$

Table 714: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x}{2(x^2+2)}$	
singularity	type
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = \frac{3x^2+1}{2x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-i\sqrt{2}, i\sqrt{2}, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(x^2+2)y'' + 7x^3y' + (3x^2+1)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 7x^3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + (3x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) + 7a_{n-2}(n+r-2) + 3a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-1)\,a_{n-2}}{-1+2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(2n-1)\,a_{n-2}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1-r}{3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{3+2r}$	$-\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{3+2r}$	$-\frac{3}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r + 3}{4r^2 + 20r + 21}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{21}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{3+2r}$	$-\frac{3}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{21}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{3+2r}$	$-\frac{3}{8}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{21}{128}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-1-r}{3+2r}$	$-\frac{3}{8}$	$-\frac{1}{(3+2r)^2}$	$-\frac{1}{16}$
b_3	0	0	0	0
b_4	$\frac{r^2+4r+3}{4r^2+20r+21}$	$\frac{21}{128}$	$\frac{4r^2+18r+24}{(4r^2+20r+21)^2}$	$\frac{17}{512}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{16} + \frac{17x^4}{512} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{16} + \frac{17x^4}{512} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{16} + \frac{17x^4}{512} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{16} + \frac{17x^4}{512} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{8} + \frac{21x^4}{128} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{x^2}{16} + \frac{17x^4}{512} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.36.1 Maple step by step solution

Let's solve

$2x^2(x^2+2)\,y''+7x^3y'+(3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+1)y}{2x^2(x^2+2)}-\frac{7xy'}{2(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{7xy'}{2(x^2+2)}+\frac{(3x^2+1)y}{2x^2(x^2+2)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x}{2(x^2+2)},P_3(x)=\frac{3x^2+1}{2x^2(x^2+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\,y''+7x^3y'+(3x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{3}\cdot y'$ to series expansion

  $x^{3}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r+2}$

- Shift index using $k->k-2$

  $x^{3}\cdot y'=\sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+2r)^{2} x^{r}+a_{1}(1+2r)^{2} x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(2k+2r-1)^{2}+a_{k-2}(2k+2r-1)\,(k+r-1)\right) x$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=\frac{1}{2}$

- Each term must be 0

  $a_{1}(1+2r)^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2}+a_{k}\left(k+r-\frac{1}{2}\right)\right)=0$

- Shift index using $k->k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(\frac{a_{k}(k+r+1)}{2}+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+7*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x),type='series',x
```

$$y(x) = \sqrt{x}\left((c_2 \ln(x) + c_1)\left(1 - \frac{3}{8}x^2 + \frac{21}{128}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{1}{16}x^2 + \frac{17}{512}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 77

```
AsymptoticDSolveValue[2*x^2*(2+x^2)*y''[x]+7*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{21x^4}{128} - \frac{3x^2}{8} + 1\right) + c_2\left(\sqrt{x}\left(\frac{17x^4}{512} - \frac{x^2}{16}\right) + \sqrt{x}\left(\frac{21x^4}{128} - \frac{3x^2}{8} + 1\right)\log(x)\right)$$

15.37 problem 33

Internal problem ID [1385]
Internal file name [`OUTPUT/1386_Sunday_June_05_2022_02_14_20_AM_27250862/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+y(2x^2+1)=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+y(2x^2+1)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{4x^2-1}{x\,(x^2+1)}\\ q(x)&=\frac{2x^2+1}{x^2\,(x^2+1)}\end{aligned}$$

Table 716: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x^2-1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{2x^2+1}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (4x^3 - x)\,y' + y(2x^2+1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}\right) + (4x^3 - x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)(2x^2+1) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 2x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n (n+r)\right) \\
&+\left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\
&+ 4a_{n-2}(n+r-2) - a_n(n+r) + 2a_{n-2} + a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r)\, a_{n-2}}{n+r-1} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = -\frac{(n+1)\, a_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{-2-r}{1+r}
$$

Which for the root $r = 1$ becomes

$$
a_2 = -\frac{3}{2}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{1+r}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{1+r}$	$-\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 6r + 8}{(3 + r)(1 + r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{15}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{1+r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(3+r)(1+r)}$	$\frac{15}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{1+r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{(3+r)(1+r)}$	$\frac{15}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-2-r}{1+r}$	$-\frac{3}{2}$	$\frac{1}{(1+r)^2}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{r^2+6r+8}{(3+r)(1+r)}$	$\frac{15}{8}$	$\frac{-2r^2-10r-14}{(1+r)^2(3+r)^2}$	$-\frac{13}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.37.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\,y''+(4x^3-x)\,y'+y(2x^2+1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y(2x^2+1)}{x^2(x^2+1)}-\frac{(4x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x^2-1)y'}{x(x^2+1)}+\frac{y(2x^2+1)}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{4x^2-1}{x(x^2+1)}, P_3(x)=\frac{2x^2+1}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=1$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\,y''+x(4x^2-1)\,y'+y(2x^2+1)=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r}+a_{1} r^{2} x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r-1)^{2}+a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $a_{1} r^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k}(k+r-1)+a_{k-2}(k+r)\right)=0$

- Shift index using $k->k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1)+a_{k}(k+r+2)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r+2)}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-4*x^2)*diff(y(x),x)+(1+2*x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 + \mathrm{O}(x^6)\right) + \left(\frac{1}{4}x^2 - \frac{13}{32}x^4 + \mathrm{O}(x^6)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-x*(1-4*x^2)*y'[x]+(1+2*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{15x^4}{8} - \frac{3x^2}{2} + 1\right) + c_2\left(x\left(\frac{x^2}{4} - \frac{13x^4}{32}\right) + x\left(\frac{15x^4}{8} - \frac{3x^2}{2} + 1\right)\log(x)\right)$$

15.38 problem 34

Internal problem ID [1386]
Internal file name [`OUTPUT/1387_Sunday_June_05_2022_02_14_22_AM_47979777/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+4)\,y''+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+16x^2)\,y''+(9x^3+24x)\,y'+(-9x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{\frac{9x^2}{4}+6}{x\,(x^2+4)}$$
$$q(x)=-\frac{9x^2-1}{4x^2\,(x^2+4)}$$

Table 718: Table $p(x), q(x)$ singularites.

$p(x) = \frac{\frac{9x^2}{4}+6}{x(x^2+4)}$	
singularity	type
$x = 0$	"regular"
$x = -2i$	"regular"
$x = 2i$	"regular"

$q(x) = -\frac{9x^2-1}{4x^2(x^2+4)}$	
singularity	type
$x = 0$	"regular"
$x = -2i$	"regular"
$x = 2i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -2i, 2i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x^2+4)\, y'' + (9x^3+24x)\, y' + (-9x^2+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x^2+4)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (9x^3+24x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-9x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right)$$
$$+ \left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 24x^{n+r} a_n (n+r)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) (n+r-1) = \sum_{n=2}^{\infty} 4a_{n-2} (n+r-2) (n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 9a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-9a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2} (n+r-2) (n-3+r) x^{n+r}\right)$$
$$+ \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=2}^{\infty} 9a_{n-2} (n+r-2) x^{n+r}\right) \tag{2B}$$
$$+ \left(\sum_{n=0}^{\infty} 24x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-9a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$16x^{n+r} a_n (n+r) (n+r-1) + 24x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$16x^r a_0 r(-1+r) + 24x^r a_0 r + a_0 x^r = 0$$

Or

$$(16x^r r(-1+r) + 24x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(4r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(4r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{4}$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{4}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{4}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{4}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 4a_{n-2}(n+r-2)(n-3+r) + 16a_n(n+r)(n+r-1) \\ + 9a_{n-2}(n+r-2) + 24a_n(n+r) - 9a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n-3+r)\, a_{n-2}}{1+4n+4r} \tag{4}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_n = -\frac{(4n-13)\, a_{n-2}}{16n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1-r}{9+4r}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_2 = \frac{5}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{9+4r}$	$\frac{5}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{9+4r}$	$\frac{5}{32}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 - 1}{(17 + 4r)(9 + 4r)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$a_4 = -\frac{15}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{9+4r}$	$\frac{5}{32}$
a_3	0	0
a_4	$\frac{r^2-1}{(17+4r)(9+4r)}$	$-\frac{15}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{9+4r}$	$\frac{5}{32}$
a_3	0	0
a_4	$\frac{r^2-1}{(17+4r)(9+4r)}$	$-\frac{15}{2048}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^{\frac{1}{4}}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)}{x^{\frac{1}{4}}} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -\frac{1}{4}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = -\frac{1}{4})$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1-r}{9+4r}$	$\frac{5}{32}$	$-\frac{13}{(9+4r)^2}$	$-\frac{13}{64}$
b_3	0	0	0	0
b_4	$\frac{r^2-1}{(17+4r)(9+4r)}$	$-\frac{15}{2048}$	$\frac{104r^2+338r+104}{(17+4r)^2(9+4r)^2}$	$\frac{13}{8192}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \frac{\left(1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{-\frac{13x^2}{64} + \frac{13x^4}{8192} + O(x^6)}{x^{\frac{1}{4}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)\right)}{x^{\frac{1}{4}}} \\
&\quad + c_2\left(\frac{\left(1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{-\frac{13x^2}{64} + \frac{13x^4}{8192} + O(x^6)}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)\right)}{x^{\frac{1}{4}}} \\
&\quad + c_2\left(\frac{\left(1 + \frac{5x^2}{32} - \frac{15x^4}{2048} + O(x^6)\right)\ln(x)}{x^{\frac{1}{4}}} + \frac{-\frac{13x^2}{64} + \frac{13x^4}{8192} + O(x^6)}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\left(1+\frac{5x^2}{32}-\frac{15x^4}{2048}+O(x^6)\right)}{x^{\frac{1}{4}}} + c_2\left(\frac{\left(1+\frac{5x^2}{32}-\frac{15x^4}{2048}+O(x^6)\right)\ln(x)}{x^{\frac{1}{4}}}+\frac{-\frac{13x^2}{64}+\frac{13x^4}{8192}+O(x^6)}{x^{\frac{1}{4}}}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1+\frac{5x^2}{32}-\frac{15x^4}{2048}+O(x^6)\right)}{x^{\frac{1}{4}}} + c_2\left(\frac{\left(1+\frac{5x^2}{32}-\frac{15x^4}{2048}+O(x^6)\right)\ln(x)}{x^{\frac{1}{4}}}+\frac{-\frac{13x^2}{64}+\frac{13x^4}{8192}+O(x^6)}{x^{\frac{1}{4}}}\right)$$

Verified OK.

15.38.1 Maple step by step solution

Let's solve

$4x^2(x^2+4)\,y''+(9x^3+24x)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(9x^2-1)y}{4x^2(x^2+4)}-\frac{3(3x^2+8)y'}{4x(x^2+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(3x^2+8)y'}{4x(x^2+4)}-\frac{(9x^2-1)y}{4x^2(x^2+4)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3(3x^2+8)}{4x(x^2+4)}, P_3(x)=-\frac{9x^2-1}{4x^2(x^2+4)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+4)\, y'' + 3x(3x^2+8)\, y' + (-9x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+4r)^2 x^r + a_1(5+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+4r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -\frac{1}{4}$
- Each term must be 0

  $a_1(5+4r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $16\left(\frac{a_{k-2}(k-3+r)}{4} + a_k\left(k+r+\frac{1}{4}\right)\right)\left(k+r+\frac{1}{4}\right) = 0$
- Shift index using $k->k+2$

  $16\left(\frac{a_k(k+r-1)}{4} + a_{k+2}\left(k+\frac{9}{4}+r\right)\right)\left(k+\frac{9}{4}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-1)}{4k+4r+9}$
- Recursion relation for $r = -\frac{1}{4}$

  $a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$
- Solution for $r = -\frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*(4+x^2)*diff(y(x),x$2)+3*x*(8+3*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x),type='s
```

$$y(x)=\frac{(c_2\ln(x)+c_1)\left(1+\frac{5}{32}x^2-\frac{15}{2048}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-\frac{13}{64}x^2+\frac{13}{8192}x^4+\mathrm{O}\left(x^6\right)\right)c_2}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 77

```
AsymptoticDSolveValue[4*x^2*(4+x^2)*y''[x]+3*x*(8+3*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],{x,0,5
```

$$y(x) \to \frac{c_1\left(-\frac{15x^4}{2048}+\frac{5x^2}{32}+1\right)}{\sqrt[4]{x}}+c_2\left(\frac{\frac{13x^4}{8192}-\frac{13x^2}{64}}{\sqrt[4]{x}}+\frac{\left(-\frac{15x^4}{2048}+\frac{5x^2}{32}+1\right)\log(x)}{\sqrt[4]{x}}\right)$$

15.39 problem 35

Internal problem ID [1387]
Internal file name [`OUTPUT/1388_Sunday_June_05_2022_02_14_25_AM_93920342/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^{2}\left(x^{2}+3\right) y''+x\left(11x^{2}+3\right) y'+\left(5x^{2}+1\right) y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^{4}+9x^{2}\right) y''+\left(11x^{3}+3x\right) y'+\left(5x^{2}+1\right) y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{11x^{2}+3}{3x\left(x^{2}+3\right)}$$
$$q(x)=\frac{5x^{2}+1}{3x^{2}\left(x^{2}+3\right)}$$

Table 720: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11x^2+3}{3x(x^2+3)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

$q(x) = \frac{5x^2+1}{3x^2(x^2+3)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i\sqrt{3}, i\sqrt{3}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2(x^2+3)\, y'' + (11x^3+3x)\, y' + (5x^2+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2(x^2+3)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (11x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (5x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 11x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{n+r+2} a_n = \sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 5a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(9x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(3r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 3a_{n-2}(n+r-2)(n-3+r) + 9a_n(n+r)(n+r-1) \\ + 11a_{n-2}(n+r-2) + 3a_n(n+r) + 5a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-1)\, a_{n-2}}{-1+3n+3r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(3n-2)\, a_{n-2}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1-r}{5+3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+3r}$	$-\frac{2}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+3r}$	$-\frac{2}{9}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 4r + 3}{9r^2 + 48r + 55}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{5}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+3r}$	$-\frac{2}{9}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{9r^2+48r+55}$	$\frac{5}{81}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{5+3r}$	$-\frac{2}{9}$
a_3	0	0
a_4	$\frac{r^2+4r+3}{9r^2+48r+55}$	$\frac{5}{81}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{3}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-1-r}{5+3r}$	$-\frac{2}{9}$	$-\frac{2}{(5+3r)^2}$	$-\frac{1}{18}$
b_3	0	0	0	0
b_4	$\frac{r^2+4r+3}{9r^2+48r+55}$	$\frac{5}{81}$	$\frac{12r^2+56r+76}{(9r^2+48r+55)^2}$	$\frac{1}{54}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(-\frac{x^2}{18} + \frac{x^4}{54} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right) \\
&\quad + c_2\left(x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(-\frac{x^2}{18} + \frac{x^4}{54} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right) \\
&\quad + c_2\left(x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(-\frac{x^2}{18} + \frac{x^4}{54} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right) \\
&\quad + c_2\left(x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(-\frac{x^2}{18} + \frac{x^4}{54} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right) \\
&\quad + c_2\left(x^{\frac{1}{3}}\left(1 - \frac{2x^2}{9} + \frac{5x^4}{81} + O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(-\frac{x^2}{18} + \frac{x^4}{54} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.39.1 Maple step by step solution

Let's solve

$3x^2(x^2+3)\, y'' + (11x^3+3x)\, y' + (5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{(5x^2+1)y}{3x^2(x^2+3)} - \frac{(11x^2+3)y'}{3x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(11x^2+3)y'}{3x(x^2+3)} + \frac{(5x^2+1)y}{3x^2(x^2+3)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{11x^2+3}{3x(x^2+3)}, P_3(x) = \frac{5x^2+1}{3x^2(x^2+3)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{9}$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3x^2(x^2+3)\, y'' + x(11x^2+3)\, y' + (5x^2+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+3r)^{2} x^{r}+a_{1}(2+3r)^{2} x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(3k+3r-1)^{2}+a_{k-2}(3k+3r-1)\,(k+r-1)\right) x\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=\frac{1}{3}$

- Each term must be 0

  $a_{1}(2+3r)^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(\frac{a_{k-2}(k+r-1)}{3}+a_{k}\left(k+r-\frac{1}{3}\right)\right)=0$

- Shift index using $k->k+2$

  $9\left(k+\frac{5}{3}+r\right)\left(\frac{a_{k}(k+r+1)}{3}+a_{k+2}\left(k+\frac{5}{3}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(3*x^2*(3+x^2)*diff(y(x),x$2)+x*(3+11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x),type='se
```

$$y(x) = x^{\frac{1}{3}}\left((c_2\ln(x)+c_1)\left(1-\frac{2}{9}x^2+\frac{5}{81}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-\frac{1}{18}x^2+\frac{1}{54}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
AsymptoticDSolveValue[3*x^2*(3+x^2)*y'[x]+x*(3+11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{1139x^5}{405}+\frac{53x^4}{81}+\frac{7x^3}{9}-\frac{11x^2}{9}+x+1\right)}{\sqrt[3]{x}}$$

15.40 problem 36

Internal problem ID [1388]
Internal file name [`OUTPUT/1389_Sunday_June_05_2022_02_14_30_AM_97257421/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2\left(4x^2+1\right)y''+32x^3y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(16x^4+4x^2\right)y''+32x^3y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{8x}{4x^2+1}$$
$$q(x)=\frac{1}{4x^2\left(4x^2+1\right)}$$

Table 722: Table $p(x), q(x)$ singularites.

$p(x) = \frac{8x}{4x^2+1}$	
singularity	type
$x = -\frac{i}{2}$	"regular"
$x = \frac{i}{2}$	"regular"

$q(x) = \frac{1}{4x^2(4x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i}{2}$	"regular"
$x = \frac{i}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[-\frac{i}{2}, \frac{i}{2}, 0, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(4x^2+1)\, y'' + 32x^3 y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &4x^2(4x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 32x^3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 32x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 16a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 32x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 32a_{n-2}(n+r-2)\,x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=2}^{\infty} 16a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 32a_{n-2}(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) + a_0 x^r = 0
$$

Or

$$
(4x^r r(-1+r) + x^r)\,a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
(2r-1)^2 x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$16a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) + 32a_{n-2}(n+r-2) + a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{16a_{n-2}(n^2 + 2nr + r^2 - 3n - 3r + 2)}{4n^2 + 8nr + 4r^2 - 4n - 4r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}(4n^2 - 8n + 3)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{16(1+r)\,r}{(2r+3)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16(1+r)r}{(2r+3)^2}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16(1+r)r}{(2r+3)^2}$	$-\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256(1+r)\,r(r+3)\,(r+2)}{(2r+3)^2\,(7+2r)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{105}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16(1+r)r}{(2r+3)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{256(1+r)r(r+3)(r+2)}{(2r+3)^2(7+2r)^2}$	$\frac{105}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{16(1+r)r}{(2r+3)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{256(1+r)r(r+3)(r+2)}{(2r+3)^2(7+2r)^2}$	$\frac{105}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= \sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O\left(x^6\right)\right)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{16(1+r)r}{(2r+3)^2}$	$-\frac{3}{4}$	$\frac{-64r-48}{(2r+3)^3}$	$-\frac{5}{4}$
b_3	0	0	0	0
b_4	$\frac{256(1+r)r(r+3)(r+2)}{(2r+3)^2(7+2r)^2}$	$\frac{105}{64}$	$\frac{4096r^4+29696r^3+78336r^2+87552r+32256}{(2r+3)^3(7+2r)^3}$	$\frac{389}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= \sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{5x^2}{4} + \frac{389x^4}{128} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{5x^2}{4} + \frac{389x^4}{128} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{5x^2}{4} + \frac{389x^4}{128} + O(x^6)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{5x^2}{4} + \frac{389x^4}{128} + O(x^6)\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x^2}{4} + \frac{105x^4}{64} + O(x^6)\right)\ln(x) + \sqrt{x}\left(-\frac{5x^2}{4} + \frac{389x^4}{128} + O(x^6)\right)\right) \end{aligned}$$

Verified OK.

15.40.1 Maple step by step solution

Let's solve

$4x^2(4x^2+1)\,y''+32x^3y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{4x^2(4x^2+1)}-\frac{8xy'}{4x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{8xy'}{4x^2+1}+\frac{y}{4x^2(4x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{8x}{4x^2+1}, P_3(x)=\frac{1}{4x^2(4x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2(4x^2+1)\,y''+32x^3y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^3\cdot y'$ to series expansion

    $x^3\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

- Shift index using $k->k-2$

  $$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 16a_{k-2}(k-2+r)\,(k+r-1)\right) x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = \frac{1}{2}$$

- Each term must be 0

  $$a_1(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r-1)^2 + 16a_{k-2}(k-2+r)\,(k+r-1) = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(2k+3+2r)^2 + 16a_k(k+r)\,(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{16a_k(k+r)(k+r+1)}{(2k+3+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{16a_k\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)^2}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{16a_k\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)^2}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*(1+4*x^2)*diff(y(x),x$2)+32*x^3*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\sqrt{x}\left((c_2\ln(x)+c_1)\left(1-\frac{3}{4}x^2+\frac{105}{64}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-\frac{5}{4}x^2+\frac{389}{128}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 77

```
AsymptoticDSolveValue[4*x^2*(1+4*x^2)*y''[x]+32*x^3*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{105x^4}{64} - \frac{3x^2}{4} + 1\right) + c_2\left(\sqrt{x}\left(\frac{389x^4}{128} - \frac{5x^2}{4}\right) + \sqrt{x}\left(\frac{105x^4}{64} - \frac{3x^2}{4} + 1\right)\log(x)\right)$$

15.41 problem 37

Internal problem ID [1389]
Internal file name [OUTPUT/1390_Sunday_June_05_2022_02_14_33_AM_15941403/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' - 3x(-2x^2 + 7)\,y' + (2x^2 + 25)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + (6x^3 - 21x)\,y' + (2x^2 + 25)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x^2 - 7}{3x}$$
$$q(x) = \frac{2x^2 + 25}{9x^2}$$

Table 724: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2-7}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{2x^2+25}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (6x^3 - 21x) y' + (2x^2 + 25) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) + (6x^3 - 21x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (2x^2 + 25) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-21x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 25 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 6x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 6a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 6a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-21x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 25 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) - 21x^{n+r} a_n (n+r) + 25 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) - 21x^r a_0 r + 25 a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) - 21x^r r + 25x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3r-5)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(3r-5)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{3}$$
$$r_2 = \frac{5}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3r-5)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{5}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{5}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 6a_{n-2}(n+r-2) - 21a_n(n+r) + 2a_{n-2} + 25a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}}{3n+3r-5} \quad (4)$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_n = -\frac{2a_{n-2}}{3n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{2}{1+3r}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_2 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{1+3r}$	$-\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{1+3r}$	$-\frac{1}{3}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{9r^2 + 24r + 7}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_4 = \frac{1}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{1+3r}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{4}{9r^2+24r+7}$	$\frac{1}{18}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{1+3r}$	$-\frac{1}{3}$
a_3	0	0
a_4	$\frac{4}{9r^2+24r+7}$	$\frac{1}{18}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = x^{\frac{5}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= x^{\frac{5}{3}}\left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{5}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{5}{3}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{2}{1+3r}$	$-\frac{1}{3}$	$\frac{6}{(1+3r)^2}$	$\frac{1}{6}$
b_3	0	0	0	0
b_4	$\frac{4}{9r^2+24r+7}$	$\frac{1}{18}$	$\frac{-96-72r}{(9r^2+24r+7)^2}$	$-\frac{1}{24}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots$$
$$= x^{\frac{5}{3}}\left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right)\ln(x) + x^{\frac{5}{3}}\left(\frac{x^2}{6} - \frac{x^4}{24} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 x^{\frac{5}{3}}\left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right)$$
$$+ c_2\left(x^{\frac{5}{3}}\left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right)\ln(x) + x^{\frac{5}{3}}\left(\frac{x^2}{6} - \frac{x^4}{24} + O(x^6)\right)\right)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \ln(x) + x^{\frac{5}{3}} \left(\frac{x^2}{6} - \frac{x^4}{24} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \ln(x) + x^{\frac{5}{3}} \left(\frac{x^2}{6} - \frac{x^4}{24} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{5}{3}} \left(1 - \frac{x^2}{3} + \frac{x^4}{18} + O(x^6)\right) \ln(x) + x^{\frac{5}{3}} \left(\frac{x^2}{6} - \frac{x^4}{24} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.41.1 Maple step by step solution

Let's solve

$9x^2y'' + (6x^3 - 21x)\,y' + (2x^2 + 25)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x^2+25)y}{9x^2} - \frac{(2x^2-7)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2-7)y'}{3x} + \frac{(2x^2+25)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{2x^2-7}{3x}, P_3(x)=\frac{2x^2+25}{9x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-\frac{7}{3}$
- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{25}{9}$
- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9x^2y''+3x(2x^2-7)\,y'+(2x^2+25)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-5+3r)^2 x^r + a_1(-2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-5+3r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{5}{3}$
- Each term must be 0

  $a_1(-2+3r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$
- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{2a_k}{3k+6}$
- Solution for $r = \frac{5}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)-3*x*(7-2*x^2)*diff(y(x),x)+(25+2*x^2)*y(x)=0,y(x),type='series',
```

$$y(x) = x^{\frac{5}{3}}\left((c_2\ln(x)+c_1)\left(1-\frac{1}{3}x^2+\frac{1}{18}x^4+\mathrm{O}\left(x^6\right)\right)+\left(\frac{1}{6}x^2-\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 77

```
AsymptoticDSolveValue[9*x^2*y''[x]-3*x*(7-2*x^2)*y'[x]+(25+2*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{18}-\frac{x^2}{3}+1\right)x^{5/3}+c_2\left(\left(\frac{x^2}{6}-\frac{x^4}{24}\right)x^{5/3}+\left(\frac{x^4}{18}-\frac{x^2}{3}+1\right)x^{5/3}\log(x)\right)$$

15.42 problem 38

Internal problem ID [1390]
Internal file name [`OUTPUT/1391_Sunday_June_05_2022_02_14_37_AM_82072267/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(2x^2+1)\,y''+x(7x^2+3)\,y'+(-3x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+x^2)\,y''+(7x^3+3x)\,y'+(-3x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2+3}{x\,(2x^2+1)}$$
$$q(x)=-\frac{3x^2-1}{x^2\,(2x^2+1)}$$

Table 726: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+3}{x(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = -\frac{3x^2-1}{x^2(2x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(2x^2+1)\,y'' + (7x^3+3x)\,y' + (-3x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(2x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (7x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-3x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ + 7a_{n-2}(n+r-2) + 3a_n(n+r) - 3a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-5)\, a_{n-2}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-2}(7-2n)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1-2r}{3+r}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{3+r}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{3+r}$	$\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 4r - 3}{(5 + r)(3 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = -\frac{3}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{3+r}$	$\frac{3}{2}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{(5+r)(3+r)}$	$-\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{3+r}$	$\frac{3}{2}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{(5+r)(3+r)}$	$-\frac{3}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 + \frac{3x^2}{2} - \frac{3x^4}{8} + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1-2r}{3+r}$	$\frac{3}{2}$	$-\frac{7}{(3+r)^2}$	$-\frac{7}{4}$
b_3	0	0	0	0
b_4	$\frac{4r^2+4r-3}{(5+r)(3+r)}$	$-\frac{3}{8}$	$\frac{28r^2+126r+84}{(5+r)^2(3+r)^2}$	$-\frac{7}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\ldots \\
&= \frac{\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{-\frac{7x^2}{4}-\frac{7x^4}{32}+O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{-\frac{7x^2}{4}-\frac{7x^4}{32}+O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{-\frac{7x^2}{4}-\frac{7x^4}{32}+O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{-\frac{7x^2}{4}-\frac{7x^4}{32}+O(x^6)}{x}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)}{x} + c_2\left(\frac{\left(1+\frac{3x^2}{2}-\frac{3x^4}{8}+O(x^6)\right)\ln(x)}{x}+\frac{-\frac{7x^2}{4}-\frac{7x^4}{32}+O(x^6)}{x}\right)$$

Verified OK.

15.42.1 Maple step by step solution

Let's solve

$x^2(2x^2+1)\,y''+(7x^3+3x)\,y'+(-3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(3x^2-1)y}{x^2(2x^2+1)}-\frac{(7x^2+3)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+3)y'}{x(2x^2+1)}-\frac{(3x^2-1)y}{x^2(2x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x^2+3}{x(2x^2+1)}, P_3(x)=-\frac{3x^2-1}{x^2(2x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

$$x^2(2x^2+1)\, y''+x(7x^2+3)\, y'+(-3x^2+1)\, y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m=0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^m \cdot y''$ to series expansion for $m=2..4$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)^2\, x^r+a_1(2+r)^2\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)^2+a_{k-2}(k+r+1)\,(2k-5+2r)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=-1$$

- Each term must be 0

$$a_1(2+r)^2=0$$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(a_k(k+r+1)+a_{k-2}(2k-5+2r)\right)=0$

- Shift index using $k->k+2$

  $(k+r+3)\left(a_{k+2}(k+r+3)+a_k(2k+2r-1)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(2k+2r-1)}{k+r+3}$

- Recursion relation for $r=-1$

  $a_{k+2} = -\frac{a_k(2k-3)}{k+2}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k(2k-3)}{k+2}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1+2*x^2)*diff(y(x),x$2)+x*(3+7*x^2)*diff(y(x),x)+(1-3*x^2)*y(x)=0,y(x),type='ser
```

$$y(x)=\frac{(c_2\ln(x)+c_1)\left(1+\frac{3}{2}x^2-\frac{3}{8}x^4+\mathrm{O}(x^6)\right)+\left(-\frac{7}{4}x^2-\frac{7}{32}x^4+\mathrm{O}(x^6)\right)c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1+2*x^2)*y''[x]+x*(3+7*x^2)*y'[x]+(1-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{3x^4}{8}+\frac{3x^2}{2}+1\right)}{x}+c_2\left(\frac{-\frac{7x^4}{32}-\frac{7x^2}{4}}{x}+\frac{\left(-\frac{3x^4}{8}+\frac{3x^2}{2}+1\right)\log(x)}{x}\right)$$

15.43 problem 39

Internal problem ID [1391]
Internal file name [`OUTPUT/1392_Sunday_June_05_2022_02_14_40_AM_93462567/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(x^2+1)\,y''+x(8x^2+3)\,y'+(12x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(8x^3+3x)\,y'+(12x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{8x^2+3}{x\,(x^2+1)}$$
$$q(x)=\frac{12x^2+1}{x^2\,(x^2+1)}$$

Table 728: Table $p(x), q(x)$ singularites.

$p(x) = \frac{8x^2+3}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{12x^2+1}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (8x^3+3x)\,y' + (12x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (8x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (12x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 8x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 8a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 12x^{n+r+2} a_n = \sum_{n=2}^{\infty} 12a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 8a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 12a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1 + r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1 + r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r + 1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r + 1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r + 1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-1} \\
y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\
&+ 8a_{n-2}(n+r-2) + 3a_n(n+r) + 12a_{n-2} + a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r+2)\, a_{n-2}}{1+n+r} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
a_n = -\frac{(n+1)\, a_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{-4-r}{3+r}
$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{3+r}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{3+r}$	$-\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 10r + 24}{(5 + r)(3 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{15}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{3+r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{r^2+10r+24}{(5+r)(3+r)}$	$\frac{15}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{3+r}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{r^2+10r+24}{(5+r)(3+r)}$	$\frac{15}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-4-r}{3+r}$	$-\frac{3}{2}$	$\frac{1}{(3+r)^2}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{r^2+10r+24}{(5+r)(3+r)}$	$\frac{15}{8}$	$\frac{-2r^2-18r-42}{(5+r)^2(3+r)^2}$	$-\frac{13}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \frac{\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)}{x}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)}{x} + c_2 \left(\frac{\left(1-\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)}{x} + \frac{\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)}{x}\right)$$

Verified OK.

15.43.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (8x^3+3x)\, y' + (12x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(12x^2+1)y}{x^2(x^2+1)} - \frac{(8x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(8x^2+3)y'}{x(x^2+1)} + \frac{(12x^2+1)y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{8x^2+3}{x(x^2+1)}, P_3(x) = \frac{12x^2+1}{x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$x^2(x^2+1)\,y''+x(8x^2+3)\,y'+(12x^2+1)\,y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..4$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)^2\,x^r+a_1(2+r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+1)^2+a_{k-2}(k+2+r)\,(k+r+1)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=-1$$

- Each term must be 0

$$a_1(2+r)^2=0$$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\left(a_k(k+r+1)+a_{k-2}(k+2+r)\right)=0$
- Shift index using $k->k+2$

  $(k+r+3)\left(a_{k+2}(k+r+3)+a_k(k+r+4)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+4)}{k+r+3}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k(k+3)}{k+2}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+3)}{k+2}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(3+8*x^2)*diff(y(x),x)+(1+12*x^2)*y(x)=0,y(x),type='seri
```

$$y(x)=\frac{(c_2\ln(x)+c_1)\left(1-\frac{3}{2}x^2+\frac{15}{8}x^4+\mathrm{O}(x^6)\right)+\left(\frac{1}{4}x^2-\frac{13}{32}x^4+\mathrm{O}(x^6)\right)c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*(3+8*x^2)*y'[x]+(1+12*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{15x^4}{8} - \frac{3x^2}{2} + 1\right)}{x} + c_2\left(\frac{\frac{x^2}{4} - \frac{13x^4}{32}}{x} + \frac{\left(\frac{15x^4}{8} - \frac{3x^2}{2} + 1\right)\log(x)}{x}\right)$$

15.44 problem 40

Internal problem ID [1392]
Internal file name [`OUTPUT/1393_Sunday_June_05_2022_02_14_43_AM_10284365/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(-x^2+1)\,y' + y(x^2+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 - x)\,y' + y(x^2+1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2-1}{x}$$
$$q(x) = \frac{x^2+1}{x^2}$$

Table 730: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{x^2+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^3 - x)\, y' + y(x^2 + 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^3 - x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)(x^2 + 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}\right) \tag{2B}$$
$$+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - a_n(n+r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{1+r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(1+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr}a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$	$\frac{1}{(1+r)^2}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(3+r)(1+r)}$	$\frac{1}{8}$	$\frac{-4-2r}{(3+r)^2(1+r)^2}$	$-\frac{3}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right)\ln(x)+x\left(\frac{x^2}{4}-\frac{3x^4}{32}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right)\ln(x)+x\left(\frac{x^2}{4}-\frac{3x^4}{32}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1-\frac{x^2}{2}+\frac{x^4}{8}+O(x^6)\right)\ln(x)+x\left(\frac{x^2}{4}-\frac{3x^4}{32}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.44.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 - x)\,y' + y(x^2+1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-1)\, y' + y(x^2+1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right) + c_2\left(x\left(\frac{x^2}{4} - \frac{3x^4}{32}\right) + x\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)\log(x)\right)$$

15.45 problem 41

Internal problem ID [1393]
Internal file name [`OUTPUT/1394_Sunday_June_05_2022_02_14_46_AM_10385741/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2\left(-2x^2+1\right)y''+x\left(-9x^2+5\right)y'+\left(-3x^2+4\right)y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^4+x^2\right)y''+\left(-9x^3+5x\right)y'+\left(-3x^2+4\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{9x^2-5}{x\left(2x^2-1\right)}\\ q(x)&=\frac{3x^2-4}{x^2\left(2x^2-1\right)}\end{aligned}$$

Table 732: Table $p(x), q(x)$ singularites.

$p(x) = \frac{9x^2-5}{x(2x^2-1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{\sqrt{2}}{2}$	"regular"
$x = \frac{\sqrt{2}}{2}$	"regular"

$q(x) = \frac{3x^2-4}{x^2(2x^2-1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{\sqrt{2}}{2}$	"regular"
$x = \frac{\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x^2-1) + (-9x^3+5x)\,y' + (-3x^2+4)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(2x^2-1) + (-9x^3+5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-3x^2+4)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)(n+r-1)\right) &= \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n (n+r)\right) &= \sum_{n=2}^{\infty} \left(-9a_{n-2}(n+r-2)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right)
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \sum_{n=2}^{\infty} \left(-9a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 5x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 5x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-2a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ &- 9a_{n-2}(n+r-2) + 5a_n(n+r) - 3a_{n-2} + 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(2n^2 + 4nr + 2r^2 - n - r - 3)}{n^2 + 2nr + r^2 + 4n + 4r + 4} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = \frac{a_{n-2}(2n^2 - 9n + 7)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r^2 + 7r + 3}{(r+4)^2}$$

Which for the root $r = -2$ becomes

$$a_2 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2+7r+3}{(r+4)^2}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2+7r+3}{(r+4)^2}$	$-\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^4 + 44r^3 + 161r^2 + 220r + 75}{(r+4)^2 (r+6)^2}$$

Which for the root $r = -2$ becomes

$$a_4 = -\frac{9}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2+7r+3}{(r+4)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{4r^4+44r^3+161r^2+220r+75}{(r+4)^2(r+6)^2}$	$-\frac{9}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2r^2+7r+3}{(r+4)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{4r^4+44r^3+161r^2+220r+75}{(r+4)^2(r+6)^2}$	$-\frac{9}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^2}{4} - \frac{9x^4}{64} + O(x^6)}{x^2}\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = -2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{2r^2+7r+3}{(r+4)^2}$	$-\frac{3}{4}$	$\frac{9r+22}{(r+4)^3}$	$\frac{1}{2}$
b_3	0	0	0	0
b_4	$\frac{4r^4+44r^3+161r^2+220r+75}{(r+4)^2(r+6)^2}$	$-\frac{9}{64}$	$\frac{36r^4+502r^3+2508r^2+5228r+3780}{(r+4)^3(r+6)^3}$	$-\frac{21}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots\\ &= \frac{\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)\ln(x)}{x^2} + \frac{\frac{x^2}{2}-\frac{21x^4}{128}+O(x^6)}{x^2}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= \frac{c_1\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)}{x^2} + c_2\left(\frac{\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)\ln(x)}{x^2} + \frac{\frac{x^2}{2}-\frac{21x^4}{128}+O(x^6)}{x^2}\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= \frac{c_1\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)}{x^2} + c_2\left(\frac{\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)\ln(x)}{x^2} + \frac{\frac{x^2}{2}-\frac{21x^4}{128}+O(x^6)}{x^2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = &\frac{c_1\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)}{x^2}\\ &+ c_2\left(\frac{\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)\ln(x)}{x^2} + \frac{\frac{x^2}{2}-\frac{21x^4}{128}+O(x^6)}{x^2}\right)\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)}{x^2} + c_2\left(\frac{\left(1-\frac{3x^2}{4}-\frac{9x^4}{64}+O(x^6)\right)\ln(x)}{x^2}+\frac{\frac{x^2}{2}-\frac{21x^4}{128}+O(x^6)}{x^2}\right)$$

Verified OK.

15.45.1 Maple step by step solution

Let's solve

$-y''x^2(2x^2-1)+(-9x^3+5x)\,y'+(-3x^2+4)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2-4)y}{x^2(2x^2-1)}-\frac{(9x^2-5)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(9x^2-5)y'}{x(2x^2-1)}+\frac{(3x^2-4)y}{x^2(2x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{9x^2-5}{x(2x^2-1)}, P_3(x)=\frac{3x^2-4}{x^2(2x^2-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$$y''x^2(2x^2-1)+x(9x^2-5)\,y'+(3x^2-4)\,y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..4$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(2+r)^2\,x^r-a_1(3+r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(-a_k(k+r+2)^2+a_{k-2}(k+1+r)\,(2k-3+2r)\right)x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(2+r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=-2$$

- Each term must be 0

$$-a_1(3+r)^2=0$$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+2)^2 + a_{k-2}(k+1+r)(2k-3+2r) = 0$

- Shift index using $k->k+2$

  $-a_{k+2}(k+4+r)^2 + a_k(k+r+3)(2k+2r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+r+3)(2k+2r+1)}{(k+4+r)^2}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a_k(k+1)(2k-3)}{(k+2)^2}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a_k(k+1)(2k-3)}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1-2*x^2)*diff(y(x),x$2)+x*(5-9*x^2)*diff(y(x),x)+(4-3*x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = \frac{(c_2 \ln (x) + c_1) \left(1 - \frac{3}{4}x^2 - \frac{9}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{2}x^2 - \frac{21}{128}x^4 + \mathrm{O}\left(x^6\right)\right) c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1-2*x^2)*y''[x]+x*(5-9*x^2)*y'[x]+(4-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{9x^4}{64}-\frac{3x^2}{4}+1\right)}{x^2}+c_2\left(\frac{\frac{x^2}{2}-\frac{21x^4}{128}}{x^2}+\frac{\left(-\frac{9x^4}{64}-\frac{3x^2}{4}+1\right)\log(x)}{x^2}\right)$$

15.46 problem 42

Internal problem ID [1394]
Internal file name [`OUTPUT/1395_Sunday_June_05_2022_02_14_51_AM_85027384/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+2)\,y''+x(-x^2+14)\,y'+2(x^2+9)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+2x^2)\,y''+(-x^3+14x)\,y'+(2x^2+18)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^2-14}{(x^2+2)\,x}$$
$$q(x)=\frac{2x^2+18}{x^2\,(x^2+2)}$$

Table 734: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-14}{(x^2+2)x}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

$q(x) = \frac{2x^2+18}{x^2(x^2+2)}$	
singularity	type
$x = 0$	"regular"
$x = -i\sqrt{2}$	"regular"
$x = i\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i\sqrt{2}, i\sqrt{2}, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+2)\,y'' + (-x^3+14x)\,y' + (2x^2+18)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (-x^3+14x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (2x^2+18)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 14x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 18 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)=\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n=\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 14x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 18 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1)+14x^{n+r} a_n(n+r)+18 a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 14x^r a_0 r + 18a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 14x^r r + 18x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r (r+3)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2(r+3)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -3$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r (r+3)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-3}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) + 2a_n(n+r)(n+r-1) \\ &- a_{n-2}(n+r-2) + 14a_n(n+r) + 2a_{n-2} + 18a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2 + 2nr + r^2 - 6n - 6r + 10)}{2(n^2 + 2nr + r^2 + 6n + 6r + 9)} \tag{4}$$

Which for the root $r = -3$ becomes

$$a_n = -\frac{a_{n-2}(n^2 - 12n + 37)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^2 + 2r - 2}{2(5+r)^2}$$

Which for the root $r = -3$ becomes

$$a_2 = -\frac{17}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2+2r-2}{2(5+r)^2}$	$-\frac{17}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2+2r-2}{2(5+r)^2}$	$-\frac{17}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 4}{4 (5 + r)^2 (r + 7)^2}$$

Which for the root $r = -3$ becomes

$$a_4 = \frac{85}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2+2r-2}{2(5+r)^2}$	$-\frac{17}{8}$
a_3	0	0
a_4	$\frac{r^4+4}{4(5+r)^2(r+7)^2}$	$\frac{85}{256}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2+2r-2}{2(5+r)^2}$	$-\frac{17}{8}$
a_3	0	0
a_4	$\frac{r^4+4}{4(5+r)^2(r+7)^2}$	$\frac{85}{256}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^3}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{17x^2}{8} + \frac{85x^4}{256} + O(x^6)}{x^3} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = -3)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-r^2+2r-2}{2(5+r)^2}$	$-\frac{17}{8}$	$\frac{-6r+7}{(5+r)^3}$	$\frac{25}{8}$
b_3	0	0	0	0
b_4	$\frac{r^4+4}{4(5+r)^2(r+7)^2}$	$\frac{85}{256}$	$\frac{6r^4+35r^3-4r-24}{(5+r)^3(r+7)^3}$	$-\frac{471}{512}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \frac{\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)\ln(x)}{x^3} + \frac{\frac{25x^2}{8}-\frac{471x^4}{512}+O(x^6)}{x^3}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)}{x^3} \\
&\quad + c_2\left(\frac{\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)\ln(x)}{x^3} + \frac{\frac{25x^2}{8}-\frac{471x^4}{512}+O(x^6)}{x^3}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)}{x^3} \\
&\quad + c_2\left(\frac{\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)\ln(x)}{x^3} + \frac{\frac{25x^2}{8}-\frac{471x^4}{512}+O(x^6)}{x^3}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)}{x^3} \\
&+ c_2 \left(\frac{\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)\ln(x)}{x^3}+\frac{\frac{25x^2}{8}-\frac{471x^4}{512}+O(x^6)}{x^3}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 \left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)}{x^3} \\
&+ c_2 \left(\frac{\left(1-\frac{17x^2}{8}+\frac{85x^4}{256}+O(x^6)\right)\ln(x)}{x^3}+\frac{\frac{25x^2}{8}-\frac{471x^4}{512}+O(x^6)}{x^3}\right)
\end{aligned}
$$

Verified OK.

15.46.1 Maple step by step solution

Let's solve

$x^2(x^2+2)\,y''+(-x^3+14x)\,y'+(2x^2+18)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(x^2+9)y}{x^2(x^2+2)}+\frac{(x^2-14)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-14)y'}{x(x^2+2)}+\frac{2(x^2+9)y}{x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-14}{(x^2+2)x}, P_3(x)=\frac{2(x^2+9)}{x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=7$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^{2} \cdot P_{3}(x)\right)\Big|_{x=0} = 9$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^{2}(x^{2}+2)\, y'' - x(x^{2}-14)\, y' + (2x^{2}+18)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0(3+r)^{2} x^{r} + 2a_1(4+r)^{2} x^{1+r} + \left(\sum_{k=2}^{\infty} \left(2a_k(k+r+3)^{2} + a_{k-2}\left((k-2)^{2} + 2(k-2)\, r + r^{2} - 2\right.\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2(3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -3$

- Each term must be 0

  $2a_1(4+r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)^2 + (k^2 + (2r-6)k + r^2 - 6r + 10)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $2a_{k+2}(k+5+r)^2 + \left((k+2)^2 + (2r-6)(k+2) + r^2 - 6r + 10\right) a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(k^2+2kr+r^2-2k-2r+2)a_k}{2(k+5+r)^2}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{(k^2-8k+17)a_k}{2(k+2)^2}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{(k^2-8k+17)a_k}{2(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(2+x^2)*diff(y(x),x$2)+x*(14-x^2)*diff(y(x),x)+2*(9+x^2)*y(x)=0,y(x),type='series
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - \frac{17}{8}x^2 + \frac{85}{256}x^4 + \mathrm{O}(x^6)\right) + \left(\frac{25}{8}x^2 - \frac{471}{512}x^4 + \mathrm{O}(x^6)\right)c_2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(2+x^2)*y''[x]+x*(14-x^2)*y'[x]+2*(9+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{85x^4}{256} - \frac{17x^2}{8} + 1\right)}{x^3} + c_2\left(\frac{\frac{25x^2}{8} - \frac{471x^4}{512}}{x^3} + \frac{\left(\frac{85x^4}{256} - \frac{17x^2}{8} + 1\right)\log(x)}{x^3}\right)$$

15.47 problem 43

Internal problem ID [1395]
Internal file name [OUTPUT/1396_Sunday_June_05_2022_02_14_54_AM_48327790/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$x^2(x^2+1)\,y''+x(7x^2+3)\,y'+(8x^2+1)\,y=0$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(7x^3+3x)\,y'+(8x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7x^2+3}{x\,(x^2+1)}$$
$$q(x)=\frac{8x^2+1}{x^2\,(x^2+1)}$$

Table 736: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x^2+3}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{8x^2+1}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (7x^3+3x)\,y' + (8x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (7x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (8x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 8x^{n+r+2} a_n = \sum_{n=2}^{\infty} 8a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 8a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) + 7a_{n-2}(n+r-2) + 3a_n(n+r) + 8a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2 + 2nr + r^2 + 2n + 2r)}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-2}(-n^2 + 1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^2 - 6r - 8}{(r+3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-6r-8}{(r+3)^2}$	$-\frac{3}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-6r-8}{(r+3)^2}$	$-\frac{3}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+6)(r+2)(r+4)^2}{(r+3)^2(5+r)^2}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{45}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-6r-8}{(r+3)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{(r+6)(r+2)(r+4)^2}{(r+3)^2(5+r)^2}$	$\frac{45}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-r^2-6r-8}{(r+3)^2}$	$-\frac{3}{4}$
a_3	0	0
a_4	$\frac{(r+6)(r+2)(r+4)^2}{(r+3)^2(5+r)^2}$	$\frac{45}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-r^2-6r-8}{(r+3)^2}$	$-\frac{3}{4}$	$-\frac{2}{(r+3)^3}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{(r+6)(r+2)(r+4)^2}{(r+3)^2(5+r)^2}$	$\frac{45}{64}$	$\frac{4(r+4)(r^2+8r+18)}{(r+3)^3(5+r)^3}$	$\frac{33}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \frac{\left(1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)\right)\ln(x)}{x} + \frac{-\frac{x^2}{4} + \frac{33x^4}{128} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)\right)\ln(x)}{x} + \frac{-\frac{x^2}{4} + \frac{33x^4}{128} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)\right)}{x} \\
&\quad + c_2\left(\frac{\left(1 - \frac{3x^2}{4} + \frac{45x^4}{64} + O(x^6)\right)\ln(x)}{x} + \frac{-\frac{x^2}{4} + \frac{33x^4}{128} + O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1-\frac{3x^2}{4}+\frac{45x^4}{64}+O(x^6)\right)}{x} + c_2\left(\frac{\left(1-\frac{3x^2}{4}+\frac{45x^4}{64}+O(x^6)\right)\ln(x)}{x}+\frac{-\frac{x^2}{4}+\frac{33x^4}{128}+O(x^6)}{x}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1-\frac{3x^2}{4}+\frac{45x^4}{64}+O(x^6)\right)}{x} + c_2\left(\frac{\left(1-\frac{3x^2}{4}+\frac{45x^4}{64}+O(x^6)\right)\ln(x)}{x}+\frac{-\frac{x^2}{4}+\frac{33x^4}{128}+O(x^6)}{x}\right)$$

Verified OK.

15.47.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\,y''+(7x^3+3x)\,y'+(8x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(8x^2+1)y}{x^2(x^2+1)}-\frac{(7x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+3)y'}{x(x^2+1)}+\frac{(8x^2+1)y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{7x^2+3}{x(x^2+1)}, P_3(x)=\frac{8x^2+1}{x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+1)\, y'' + x(7x^2+3)\, y' + (8x^2+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)^2\, x^r + a_1(2+r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)^2 + a_{k-2}(k+r+2)\,(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)^{2} = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term must be 0

  $a_{1}(2+r)^{2} = 0$

- Solve for the dependent coefficient(s)

  $a_{1} = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)^{2} + a_{k-2}(k+r+2)(k+r) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)^{2} + a_{k}(k+r+4)(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_{k}(k+r+4)(k+r+2)}{(k+3+r)^{2}}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_{k}(k+3)(k+1)}{(k+2)^{2}}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k-1}, a_{k+2} = -\frac{a_{k}(k+3)(k+1)}{(k+2)^{2}}, a_{1} = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   <- elliptic successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(3+7*x^2)*diff(y(x),x)+(1+8*x^2)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - \frac{3}{4}x^2 + \frac{45}{64}x^4 + \mathrm{O}(x^6)\right) + \left(-\frac{1}{4}x^2 + \frac{33}{128}x^4 + \mathrm{O}(x^6)\right)c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*(3+7*x^2)*y'[x]+(1+8*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{45x^4}{64} - \frac{3x^2}{4} + 1\right)}{x} + c_2\left(\frac{\frac{33x^4}{128} - \frac{x^2}{4}}{x} + \frac{\left(\frac{45x^4}{64} - \frac{3x^2}{4} + 1\right)\log(x)}{x}\right)$$

15.48 problem 44

Internal problem ID [1396]
Internal file name [`OUTPUT/1397_Sunday_June_05_2022_02_14_58_AM_76846185/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''+3y'x+(4x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+x^2\right)y''+3y'x+(4x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{3}{x\,(2x-1)}$$
$$q(x)=-\frac{4x+1}{x^2\,(2x-1)}$$

Table 738: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = -\frac{4x+1}{x^2(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x-1) + 3y'x + (4x+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(2x-1) + 3\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) x + (4x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}4x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty}4x^{1+n+r}a_n=\sum_{n=1}^{\infty}4a_{n-1}x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=1}^{\infty}4a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r}a_n(n+r)(n+r-1)+3x^{n+r}a_n(n+r)+a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+3x^r a_0 r+a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)+3x^r r+x^r\right)a_0=0
$$

Since $a_0\neq 0$ then the above simplifies to

$$
(r+1)^2 x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + 3a_n(n+r) + 4a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n^2 + 2nr + r^2 - 3n - 3r)}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{2a_{n-1}(n^2 - 5n + 4)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 - 2r - 4}{(r+2)^2}$$

Which for the root $r = -1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 - 8r^2 - 4r + 8}{(r+3)^2 (r+2)}$$

Which for the root $r = -1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0
a_2	$\frac{4r^3-8r^2-4r+8}{(r+3)^2(r+2)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^4 - 16r^3 - 8r^2 + 16r}{(r+4)^2 (r+2)(r+3)}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0
a_2	$\frac{4r^3-8r^2-4r+8}{(r+3)^2(r+2)}$	0
a_3	$\frac{8r^4-16r^3-8r^2+16r}{(r+4)^2(r+2)(r+3)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16(r+1)^2 (-1+r)(r-2) r}{(5+r)^2 (r+3)(r+2)(r+4)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0
a_2	$\frac{4r^3-8r^2-4r+8}{(r+3)^2(r+2)}$	0
a_3	$\frac{8r^4-16r^3-8r^2+16r}{(r+4)^2(r+2)(r+3)}$	0
a_4	$\frac{16(r+1)^2(-1+r)(r-2)r}{(5+r)^2(r+3)(r+2)(r+4)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32(r+1)^2 (-1+r)(r-2)r}{(r+6)^2 (r+4)(r+3)(5+r)}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0
a_2	$\frac{4r^3-8r^2-4r+8}{(r+3)^2(r+2)}$	0
a_3	$\frac{8r^4-16r^3-8r^2+16r}{(r+4)^2(r+2)(r+3)}$	0
a_4	$\frac{16(r+1)^2(-1+r)(r-2)r}{(5+r)^2(r+3)(r+2)(r+4)}$	0
a_5	$\frac{32(r+1)^2(-1+r)(r-2)r}{(r+6)^2(r+4)(r+3)(5+r)}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r^2-2r-4}{(r+2)^2}$	0	$\frac{10r+4}{(r+2)^3}$	-6
b_2	$\frac{4r^3-8r^2-4r+8}{(r+3)^2(r+2)}$	0	$\frac{40r^3+56r^2-112r-80}{(r+3)^3(r+2)^2}$	6
b_3	$\frac{8r^4-16r^3-8r^2+16r}{(r+4)^2(r+2)(r+3)}$	0	$\frac{120r^5+528r^4+24r^3-1536r^2-480r+384}{(r+4)^3(r+2)^2(r+3)^2}$	$-\frac{8}{3}$
b_4	$\frac{16(r+1)^2(-1+r)(r-2)r}{(5+r)^2(r+3)(r+2)(r+4)}$	0	$\frac{64(r+1)(5r^6+42r^5+83r^4-84r^3-274r^2-12r+60)}{(5+r)^3(r+3)^2(r+2)^2(r+4)^2}$	0
b_5	$\frac{32(r+1)^2(-1+r)(r-2)r}{(r+6)^2(r+4)(r+3)(5+r)}$	0	$\frac{160(r+1)(5r^6+57r^5+173r^4-57r^3-634r^2-24r+144)}{(r+6)^3(r+4)^2(r+3)^2(5+r)^2}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \frac{(1 + O(x^6))\ln(x)}{x} + \frac{6x^2 - 6x - \frac{8x^3}{3} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= \frac{c_1(1 + O(x^6))}{x} + c_2\left(\frac{(1 + O(x^6))\ln(x)}{x} + \frac{6x^2 - 6x - \frac{8x^3}{3} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= \frac{c_1(1 + O(x^6))}{x} + c_2\left(\frac{(1 + O(x^6))\ln(x)}{x} + \frac{6x^2 - 6x - \frac{8x^3}{3} + O(x^6)}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{6x^2-6x-\frac{8x^3}{3}+O(x^6)}{x}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(1+O(x^6))}{x} + c_2\left(\frac{(1+O(x^6))\ln(x)}{x} + \frac{6x^2-6x-\frac{8x^3}{3}+O(x^6)}{x}\right)$$

Verified OK.

15.48.1 Maple step by step solution

Let's solve

$-y''x^2(2x-1)+3y'x+(4x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(4x+1)y}{x^2(2x-1)}+\frac{3y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3y'}{x(2x-1)}-\frac{(4x+1)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{3}{x(2x-1)}, P_3(x)=-\frac{4x+1}{x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)-3y'x+(-4x-1)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m=2..3$

    $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)^2+2a_{k-1}(k+r)\,(k-3+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)^2=0$
- Values of r that satisfy the indicial equation

  $r=-1$
- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+1)^2+2a_{k-1}(k+r)\,(k-3+r)=0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+2+r)^2+2a_k(k+r+1)\,(k+r-2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^2}$

- Recursion relation for $r = -1$; series terminates at $k = 3$

  $a_{k+1} = \frac{2a_k k(k-3)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = 0$

- Apply recursion relation for $k = 1$

  $a_2 = -a_1$

- Express in terms of a_0

  $a_2 = 0$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{9}$

- Express in terms of a_0

  $a_3 = 0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot 0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)(1 + \mathrm{O}(x^6)) + \left((-6)x + 6x^2 - \frac{8}{3}x^3 + \mathrm{O}(x^6)\right)c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x^2*(1-2*x)*y''[x]+3*x*y'[x]+(1+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{-\frac{8x^3}{3} + 6x^2 - 6x}{x} + \frac{\log(x)}{x}\right) + \frac{c_1}{x}$$

15.49 problem 45

Internal problem ID [1397]
Internal file name [`OUTPUT/1398_Sunday_June_05_2022_02_15_01_AM_81066771/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+1)\,y''+(1-x)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2+x\right)y''+(1-x)\,y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x-1}{x\,(x+1)}$$
$$q(x)=\frac{1}{x\,(x+1)}$$

Table 740: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+1)\,y'' + (1-x)\,y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (1-x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r}=\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1) x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1)+(n+r) a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(x^{-1+r} r(-1+r)+r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}r_1&=0\\ r_2&=0\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x)=y_1(x) \ln (x)+\left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y=c_1 y_1(x)+c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0=1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\ &\quad-a_{n-1}(n+r-1)+a_n(n+r)+a_{n-1}=0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n-2)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{(-1+r)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1
a_2	$\frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(-1+r)^2 r^2}{(r+3)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1
a_2	$\frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$	0
a_3	$-\frac{(-1+r)^2 r^2}{(r+3)^2 (r+2)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1+r)^2 r^2}{(r+4)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1
a_2	$\frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$	0
a_3	$-\frac{(-1+r)^2 r^2}{(r+3)^2 (r+2)^2}$	0
a_4	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+3)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(-1+r)^2 r^2}{(5+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1
a_2	$\frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$	0
a_3	$-\frac{(-1+r)^2 r^2}{(r+3)^2 (r+2)^2}$	0
a_4	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+3)^2}$	0
a_5	$-\frac{(-1+r)^2 r^2}{(5+r)^2 (r+4)^2}$	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - x + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{(-1+r)^2}{(r+1)^2}$	-1	$\frac{4-4r}{(r+1)^3}$	4
b_2	$\frac{(-1+r)^2 r^2}{(r+1)^2 (r+2)^2}$	0	$\frac{8\left(r^2+r-\frac{1}{2}\right) r(-1+r)}{(r+1)^3 (r+2)^3}$	0
b_3	$-\frac{(-1+r)^2 r^2}{(r+3)^2 (r+2)^2}$	0	$-\frac{12r(-1+r)(r^2+2r-1)}{(r+3)^3 (r+2)^3}$	0
b_4	$\frac{(-1+r)^2 r^2}{(r+4)^2 (r+3)^2}$	0	$\frac{16r^4+32r^3-72r^2+24r}{(r+4)^3 (r+3)^3}$	0
b_5	$-\frac{(-1+r)^2 r^2}{(5+r)^2 (r+4)^2}$	0	$-\frac{20r(-1+r)(r^2+4r-2)}{(5+r)^3 (r+4)^3}$	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= \left(1 - x + O(x^6)\right) \ln(x) + 4x + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - x + O(x^6)\right) + c_2\left(\left(1 - x + O(x^6)\right) \ln(x) + 4x + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - x + O(x^6)\right) + c_2\left(\left(1 - x + O(x^6)\right) \ln(x) + 4x + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(1 - x + O(x^6)\right) + c_2\left(\left(1 - x + O(x^6)\right) \ln(x) + 4x + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(1 - x + O(x^6)\right) + c_2\left(\left(1 - x + O(x^6)\right) \ln(x) + 4x + O(x^6)\right)$$

Verified OK.

15.49.1 Maple step by step solution

Let's solve

$x(x+1)\,y''+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{y}{x(x+1)}+\frac{(x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(x-1)y'}{x(x+1)}+\frac{y}{x(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x-1}{x(x+1)}, P_3(x)=\frac{1}{x(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-2$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(x+1)\,y''+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k (k+r-1)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)(k-2+r) + a_k (k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k (k+r-1)^2}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k (k-1)^2}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = a_0\left(\frac{1}{2} - \frac{x}{2}\right)\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{1}{2} - \frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+3}\right), b_{1+k} = \frac{b_k (k+2)^2}{(4+k)(1+k)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x*(1+x)*diff(y(x),x$2)+(1-x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - x + O\left(x^6\right)\right) + \left(4x + O\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 27

```
AsymptoticDSolveValue[x*(1+x)*y''[x]+(1-x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(1-x) + c_2(4x + (1-x)\log(x))$$

15.50 problem 46

Internal problem ID [1398]
Internal file name [OUTPUT/1399_Sunday_June_05_2022_02_15_06_AM_33641452/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2(1-x)\, y'' + x(-2x+3)\, y' + (1+2x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+x^2\right) y'' + \left(-2x^2+3x\right) y' + (1+2x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{-3+2x}{x\,(x-1)}$$
$$q(x) = -\frac{1+2x}{x^2\,(x-1)}$$

Table 742: Table $p(x), q(x)$ singularites.

$p(x) = \frac{-3+2x}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{1+2x}{x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x-1) + \left(-2x^2 + 3x\right) y' + (1 + 2x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + \left(-2x^2 + 3x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (1+2x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2 x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1) x^{n+r}\right) \\
\sum_{n=0}^{\infty} 2 x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 2 a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)+3 x^{n+r} a_n(n+r)+a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ - 2a_{n-1}(n+r-1) + 3a_n(n+r) + a_n + 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-2)\, a_{n-1}}{1+n+r} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{(n-3)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+r}{2+r}$$

Which for the root $r = -1$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{2+r}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1+r)\, r}{(2+r)\,(3+r)}$$

Which for the root $r = -1$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{2+r}$	-2
a_2	$\frac{(-1+r)r}{(2+r)(3+r)}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^3 - r}{(2+r)\,(3+r)\,(4+r)}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{2+r}$	-2
a_2	$\frac{(-1+r)r}{(2+r)(3+r)}$	1
a_3	$\frac{r^3-r}{(2+r)(3+r)(4+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^3 - r}{(3 + r)(4 + r)(5 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{2+r}$	-2
a_2	$\frac{(-1+r)r}{(2+r)(3+r)}$	1
a_3	$\frac{r^3-r}{(2+r)(3+r)(4+r)}$	0
a_4	$\frac{r^3-r}{(3+r)(4+r)(5+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^3 - r}{(4 + r)(5 + r)(6 + r)}$$

Which for the root $r = -1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{2+r}$	-2
a_2	$\frac{(-1+r)r}{(2+r)(3+r)}$	1
a_3	$\frac{r^3-r}{(2+r)(3+r)(4+r)}$	0
a_4	$\frac{r^3-r}{(3+r)(4+r)(5+r)}$	0
a_5	$\frac{r^3-r}{(4+r)(5+r)(6+r)}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{x^2 - 2x + 1 + O(x^6)}{x} \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = -1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-1+r}{2+r}$	-2	$\frac{3}{(2+r)^2}$	3
b_2	$\frac{(-1+r)r}{(2+r)(3+r)}$	1	$\frac{6r^2+12r-6}{(2+r)^2(3+r)^2}$	-3
b_3	$\frac{r^3-r}{(2+r)(3+r)(4+r)}$	0	$\frac{9r^4+54r^3+81r^2-24}{(2+r)^2(3+r)^2(4+r)^2}$	$\frac{1}{3}$
b_4	$\frac{r^3-r}{(3+r)(4+r)(5+r)}$	0	$\frac{12r^4+96r^3+192r^2-60}{(3+r)^2(4+r)^2(5+r)^2}$	$\frac{1}{12}$
b_5	$\frac{r^3-r}{(4+r)(5+r)(6+r)}$	0	$\frac{15r^4+150r^3+375r^2-120}{(4+r)^2(5+r)^2(6+r)^2}$	$\frac{1}{30}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \frac{(x^2 - 2x + 1 + O(x^6))\ln(x)}{x} + \frac{-3x^2 + 3x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1(x^2 - 2x + 1 + O(x^6))}{x} \\
&\quad + c_2 \left(\frac{(x^2 - 2x + 1 + O(x^6)) \ln(x)}{x} + \frac{-3x^2 + 3x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} + O(x^6)}{x} \right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1(x^2 - 2x + 1 + O(x^6))}{x} \\
&\quad + c_2 \left(\frac{(x^2 - 2x + 1 + O(x^6)) \ln(x)}{x} + \frac{-3x^2 + 3x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} + O(x^6)}{x} \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1(x^2 - 2x + 1 + O(x^6))}{x} \\
&\quad + c_2 \left(\frac{(x^2 - 2x + 1 + O(x^6)) \ln(x)}{x} + \frac{-3x^2 + 3x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} + O(x^6)}{x} \right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1(x^2 - 2x + 1 + O(x^6))}{x} \\
&\quad + c_2 \left(\frac{(x^2 - 2x + 1 + O(x^6)) \ln(x)}{x} + \frac{-3x^2 + 3x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} + O(x^6)}{x} \right)
\end{aligned}
$$

Verified OK.

15.50.1 Maple step by step solution

Let's solve

$-y''x^2(x-1)+(-2x^2+3x)\,y'+(1+2x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(1+2x)y}{x^2(x-1)}-\frac{(-3+2x)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-3+2x)y'}{x(x-1)}-\frac{(1+2x)y}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ∘ Define functions

  $\left[P_2(x)=\frac{-3+2x}{x(x-1)},P_3(x)=-\frac{1+2x}{x^2(x-1)}\right]$

  ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=3$

  ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

  ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(x-1)+x(-3+2x)\,y'+(-1-2x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m} \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m} \cdot y'$ to series expansion for $m=1..2$

  $x^{m} \cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m} \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$

- Convert $x^{m} \cdot y''$ to series expansion for $m=2..3$

  $x^{m} \cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1) x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m} \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(1+r)^{2} x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+1)^{2}+a_{k-1}(k+r+1)(k-2+r)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-(1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-1$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r+1)^{2}+a_{k-1}(k+r+1)(k-2+r)=0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+r+2)^{2}+a_{k}(k+r+2)(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-1)}{k+r+2}$

- Recursion relation for $r=-1$; series terminates at $k=2$

  $a_{k+1}=\frac{a_{k}(k-2)}{k+1}$

- Apply recursion relation for $k=0$

  $a_{1}=-2 a_{0}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = a_0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (x^2 - 2x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 57

```
Order:=6;
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(3-2*x)*diff(y(x),x)+(1+2*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)(1 - 2x + x^2 + O(x^6)) + \left(3x - 3x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + O(x^6)\right) c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 70

```
AsymptoticDSolveValue[x^2*(1-x)*y''[x]+x*(3-2*x)*y'[x]+(1+2*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1(x^2 - 2x + 1)}{x} + c_2\left(\frac{(x^2 - 2x + 1)\log(x)}{x} + \frac{\frac{x^5}{30} + \frac{x^4}{12} + \frac{x^3}{3} - 3x^2 + 3x}{x}\right)$$

15.51 problem 47

Internal problem ID [1399]
Internal file name [`OUTPUT/1400_Sunday_June_05_2022_02_15_10_AM_71217187/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x+1)\,y''+4y'x^2+(1-5x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''+4y'x^2+(1-5x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x+1} \\ q(x) &= -\frac{5x-1}{4x^2\,(x+1)} \end{aligned}$$

Table 744: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x+1}$	
singularity	type
$x = -1$	"regular"

$q(x) = -\frac{5x-1}{4x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\,y'' + 4y'x^2 + (1-5x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 4\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) x^2 + (1-5x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) \tag{2B}$$
$$+ \left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)+4a_{n-1}(n+r-1)+a_n-5a_{n-1}=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(4n^2+8nr+4r^2-8n-8r-1)}{4n^2+8nr+4r^2-4n-4r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}(n^2-n-1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-4r^2+5}{(2r+1)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+5}{(2r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^4+32r^3-24r^2-40r+5}{16\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+5}{(2r+1)^2}$	1
a_2	$\frac{16r^4+32r^3-24r^2-40r+5}{16\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-64r^6 - 384r^5 - 592r^4 + 192r^3 + 884r^2 + 360r - 55}{64\left(r + \frac{5}{2}\right)^2 \left(r + \frac{1}{2}\right)^2 \left(r + \frac{3}{2}\right)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{5}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+5}{(2r+1)^2}$	1
a_2	$\frac{16r^4+32r^3-24r^2-40r+5}{16\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$-\frac{1}{4}$
a_3	$\frac{-64r^6-384r^5-592r^4+192r^3+884r^2+360r-55}{64\left(r+\frac{5}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$\frac{5}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(4r^2 + 24r + 31)(4r^2 - 5)(4r^2 + 8r - 1)(4r^2 + 16r + 11)}{(2r + 5)^2 (2r + 1)^2 (2r + 3)^2 (4r^2 + 28r + 49)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{55}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+5}{(2r+1)^2}$	1
a_2	$\frac{16r^4+32r^3-24r^2-40r+5}{16\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$-\frac{1}{4}$
a_3	$\frac{-64r^6-384r^5-592r^4+192r^3+884r^2+360r-55}{64\left(r+\frac{5}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$\frac{5}{36}$
a_4	$\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)}{(2r+5)^2(2r+1)^2(2r+3)^2(4r^2+28r+49)}$	$-\frac{55}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)(4r^2+32r+59)}{1024\left(r+\frac{7}{2}\right)^2\left(r+\frac{9}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{209}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-4r^2+5}{(2r+1)^2}$	1
a_2	$\frac{16r^4+32r^3-24r^2-40r+5}{16\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$-\frac{1}{4}$
a_3	$\frac{-64r^6-384r^5-592r^4+192r^3+884r^2+360r-55}{64\left(r+\frac{5}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$\frac{5}{36}$
a_4	$\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)}{(2r+5)^2(2r+1)^2(2r+3)^2(4r^2+28r+49)}$	$-\frac{55}{576}$
a_5	$-\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)(4r^2+32r+59)}{1024\left(r+\frac{7}{2}\right)^2\left(r+\frac{9}{2}\right)^2\left(r+\frac{5}{2}\right)^2\left(r+\frac{1}{2}\right)^2\left(r+\frac{3}{2}\right)^2}$	$\frac{209}{2880}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(x + 1 - \frac{x^2}{4} + \frac{5x^3}{36} - \frac{55x^4}{576} + \frac{209x^5}{2880} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-4r^2+5}{(2r+1)^2}$	1	$\frac{-8r-20}{(2r+1)^3}$
b_2	$\frac{16r^4+32r^3-24r^2-40r+5}{16(r+\frac{1}{2})^2(r+\frac{3}{2})^2}$	$-\frac{1}{4}$	$\frac{128r^4+640r^3+768r^2+96r-200}{(2r+3)^3(2r+1)^3}$
b_3	$\frac{-64r^6-384r^5-592r^4+192r^3+884r^2+360r-55}{64(r+\frac{5}{2})^2(r+\frac{1}{2})^2(r+\frac{3}{2})^2}$	$\frac{5}{36}$	$-\frac{4(384r^7+4032r^6+15840r^5+29616r^4+26184r^3+690}{(2r+3)^3(2r+1)^3(2r+5)^3}$
b_4	$\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)}{(2r+5)^2(2r+1)^2(2r+3)^2(4r^2+28r+49)}$	$-\frac{55}{576}$	$\frac{16384r^{10}+311296r^9+2494464r^8+11059200r^7+29755}{(2r+}$
b_5	$-\frac{(4r^2+24r+31)(4r^2-5)(4r^2+8r-1)(4r^2+16r+11)(4r^2+32r+59)}{1024(r+\frac{7}{2})^2(r+\frac{9}{2})^2(r+\frac{5}{2})^2(r+\frac{1}{2})^2(r+\frac{3}{2})^2}$	$\frac{209}{2880}$	$-\frac{20(8192r^{13}+249856r^{12}+3379200r^{11}+26781696r^{10}}{}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= \sqrt{x}\left(x + 1 - \frac{x^2}{4} + \frac{5x^3}{36} - \frac{55x^4}{576} + \frac{209x^5}{2880} + O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(-3x + \frac{x^2}{4} - \frac{5x^3}{54} + \frac{175x^4}{3456} - \frac{2863x^5}{86400} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(x + 1 - \frac{x^2}{4} + \frac{5x^3}{36} - \frac{55x^4}{576} + \frac{209x^5}{2880} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x + 1 - \frac{x^2}{4} + \frac{5x^3}{36} - \frac{55x^4}{576} + \frac{209x^5}{2880} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + \sqrt{x}\left(-3x + \frac{x^2}{4} - \frac{5x^3}{54} + \frac{175x^4}{3456} - \frac{2863x^5}{86400} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+\sqrt{x}\left(-3x+\frac{x^2}{4}-\frac{5x^3}{54}+\frac{175x^4}{3456}-\frac{2863x^5}{86400}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+\sqrt{x}\left(-3x+\frac{x^2}{4}-\frac{5x^3}{54}+\frac{175x^4}{3456}-\frac{2863x^5}{86400}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(x+1-\frac{x^2}{4}+\frac{5x^3}{36}-\frac{55x^4}{576}+\frac{209x^5}{2880}+O(x^6)\right)\ln(x)\right. \\
&\qquad \left.+\sqrt{x}\left(-3x+\frac{x^2}{4}-\frac{5x^3}{54}+\frac{175x^4}{3456}-\frac{2863x^5}{86400}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.51.1 Maple step by step solution

Let's solve

$4x^2(x+1)\,y''+4y'x^2+(1-5x)\,y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{(5x-1)y}{4x^2(x+1)}-\frac{y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x+1} - \frac{(5x-1)y}{4x^2(x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x+1}, P_3(x) = -\frac{5x-1}{4x^2(x+1)}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4x^2(x+1) y'' + 4y'x^2 + (1-5x) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u) \left(\frac{d^2}{du^2} y(u)\right) + (4u^2 - 8u + 4) \left(\frac{d}{du} y(u)\right) + (6 - 5u) y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- > k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_{0}r^{2}u^{-1+r}+\left(4a_{1}(1+r)^{2}-2a_{0}(4r^{2}-3)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)^{2}-2a_{k}(4k^{2}+8kr+4r^{2}\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$4r^{2}=0$$

- Values of r that satisfy the indicial equation

$$r=0$$

- Each term must be 0

$$4a_{1}(1+r)^{2}-2a_{0}(4r^{2}-3)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(-8a_{k}+4a_{k-1}+4a_{k+1})\,k^{2}+(-8a_{k-1}+8a_{k+1})\,k+6a_{k}-a_{k-1}+4a_{k+1}=0$$

- Shift index using $k->k+1$

$$(-8a_{k+1}+4a_{k}+4a_{k+2})\,(k+1)^{2}+(-8a_{k}+8a_{k+2})\,(k+1)+6a_{k+1}-a_{k}+4a_{k+2}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-16ka_{k+1}-5a_{k}-2a_{k+1}}{4(k^{2}+4k+4)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-16ka_{k+1}-5a_{k}-2a_{k+1}}{4(k^{2}+4k+4)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-16ka_{k+1}-5a_{k}-2a_{k+1}}{4(k^{2}+4k+4)},4a_{1}+6a_{0}=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^k\,, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-16ka_{k+1}-5a_k-2a_{k+1}}{4(k^2+4k+4)}, 4a_1+6a_0=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x^2*diff(y(x),x)+(1-5*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\sqrt{x}\left((c_2\ln(x)+c_1)\left(1+x-\frac{1}{4}x^2+\frac{5}{36}x^3-\frac{55}{576}x^4+\frac{209}{2880}x^5+\mathrm{O}\left(x^6\right)\right)+\left((-3)\,x+\frac{1}{4}x^2-\frac{5}{54}x^3+\frac{175}{3456}x^4-\frac{2863}{86400}x^5+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 124

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]+4*x^2*y'[x]+(1-5*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{209x^5}{2880} - \frac{55x^4}{576} + \frac{5x^3}{36} - \frac{x^2}{4} + x + 1\right) + c_2\left(\sqrt{x}\left(-\frac{2863x^5}{86400} + \frac{175x^4}{3456} - \frac{5x^3}{54} + \frac{x^2}{4} - 3x\right) + \sqrt{x}\left(\frac{209x^5}{2880} - \frac{55x^4}{576} + \frac{5x^3}{36} - \frac{x^2}{4} + x + 1\right)\log(x)\right)$$

15.52 problem 48

Internal problem ID [1400]
Internal file name [`OUTPUT/1401_Sunday_June_05_2022_02_15_14_AM_70027925/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{5x-3}{x\,(x-1)}\\q(x)&=\frac{5x-4}{x^2\,(x-1)}\end{aligned}$$

Table 746: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5x-3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{5x-4}{x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x-1) + (5x^2 - 3x)\, y' + (4 - 5x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{gathered} -\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) \\ + (5x^2 - 3x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4-5x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{gathered} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 5 x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-5 x^{1+n+r} a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{1+n+r} a_n(n+r)(n+r-1)\right) &=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right) \\
\sum_{n=0}^{\infty} 5 x^{1+n+r} a_n(n+r) &=\sum_{n=1}^{\infty} 5 a_{n-1}(n+r-1) x^{n+r} \\
\sum_{n=0}^{\infty}\left(-5 x^{1+n+r} a_n\right) &=\sum_{n=1}^{\infty}\left(-5 a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=1}^{\infty} 5 a_{n-1}(n+r-1) x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-3 x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} 4 a_n x^{n+r}\right)+\sum_{n=1}^{\infty}\left(-5 a_{n-1} x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)-3 x^{n+r} a_n(n+r)+4 a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 5a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n - 5a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-6)\, a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(n-4)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-5+r}{-1+r}$$

Which for the root $r = 2$ becomes

$$a_1 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-5+r}{-1+r}$	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 9r + 20}{r(-1 + r)}$$

Which for the root $r = 2$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-5+r}{-1+r}$	-3
a_2	$\frac{r^2-9r+20}{r(-1+r)}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^3 - 12r^2 + 47r - 60}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_3 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-5+r}{-1+r}$	-3
a_2	$\frac{r^2-9r+20}{r(-1+r)}$	3
a_3	$\frac{r^3-12r^2+47r-60}{r^3-r}$	-1

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 - 14r^3 + 71r^2 - 154r + 120}{r(r^2 - 1)(2 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-5+r}{-1+r}$	-3
a_2	$\frac{r^2-9r+20}{r(-1+r)}$	3
a_3	$\frac{r^3-12r^2+47r-60}{r^3-r}$	-1
a_4	$\frac{r^4-14r^3+71r^2-154r+120}{r(r^2-1)(2+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^4 - 14r^3 + 71r^2 - 154r + 120}{(3 + r)(2 + r)(1 + r)r}$$

Which for the root $r = 2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-5+r}{-1+r}$	-3
a_2	$\frac{r^2-9r+20}{r(-1+r)}$	3
a_3	$\frac{r^3-12r^2+47r-60}{r^3-r}$	-1
a_4	$\frac{r^4-14r^3+71r^2-154r+120}{r(r^2-1)(2+r)}$	0
a_5	$\frac{r^4-14r^3+71r^2-154r+120}{(3+r)(2+r)(1+r)r}$	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(-x^3 + 3x^2 - 3x + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-5+r}{-1+r}$	-3	$\frac{4}{(-1+r)^2}$	4
b_2	$\frac{r^2-9r+20}{r(-1+r)}$	3	$\frac{8r^2-40r+20}{r^2(-1+r)^2}$	-7
b_3	$\frac{r^3-12r^2+47r-60}{r^3-r}$	-1	$\frac{12r^4-96r^3+192r^2-60}{r^2(r^2-1)^2}$	$\frac{11}{3}$
b_4	$\frac{r^4-14r^3+71r^2-154r+120}{r(r^2-1)(2+r)}$	0	$\frac{16r^6-144r^5+328r^4+192r^3-1016r^2+240r+240}{r^2(r^2-1)^2(2+r)^2}$	$-\frac{1}{4}$
b_5	$\frac{r^4-14r^3+71r^2-154r+120}{(3+r)(2+r)(1+r)r}$	0	$\frac{20r^6-120r^5-100r^4+1200r^3-40r^2-2640r-720}{(3+r)^2(2+r)^2(1+r)^2r^2}$	$-\frac{1}{20}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= x^2(-x^3+3x^2-3x+1+O(x^6))\ln(x) + x^2\left(-7x^2+4x+\frac{11x^3}{3}-\frac{x^4}{4}-\frac{x^5}{20}+O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^2(-x^3+3x^2-3x+1+O(x^6)) + c_2\Bigg(x^2(-x^3+3x^2-3x+1+O(x^6))\ln(x) \\ &\quad + x^2\left(-7x^2+4x+\frac{11x^3}{3}-\frac{x^4}{4}-\frac{x^5}{20}+O(x^6)\right)\Bigg) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^2(-x^3+3x^2-3x+1+O(x^6)) + c_2\Bigg(x^2(-x^3+3x^2-3x+1+O(x^6))\ln(x) \\ &\quad + x^2\left(-7x^2+4x+\frac{11x^3}{3}-\frac{x^4}{4}-\frac{x^5}{20}+O(x^6)\right)\Bigg) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \left(-x^3+3x^2-3x+1+O(x^6)\right) + c_2 \left(x^2 \left(-x^3+3x^2-3x+1+O(x^6)\right) \ln(x) + x^2 \left(-7x^2+4x+\frac{11x^3}{3}-\frac{x^4}{4}-\frac{x^5}{20}+O(x^6)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(-x^3+3x^2-3x+1+O(x^6)\right) + c_2 \left(x^2 \left(-x^3+3x^2-3x+1+O(x^6)\right) \ln(x) + x^2 \left(-7x^2+4x+\frac{11x^3}{3}-\frac{x^4}{4}-\frac{x^5}{20}+O(x^6)\right)\right)$$

Verified OK.

15.52.1 Maple step by step solution

Let's solve

$-y''x^2(x-1)+(5x^2-3x)\,y'+(4-5x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5x-4)y}{x^2(x-1)}+\frac{(5x-3)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(5x-3)y'}{x(x-1)}+\frac{(5x-4)y}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{5x-3}{x(x-1)}, P_3(x)=\frac{5x-4}{x^2(x-1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-3$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1) - x(5x-3)\,y' + y(5x-4) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)\,(k-6+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r=2$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-2)^2+a_{k-1}(k+r-2)(k-6+r)=0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+r-1)^2+a_k(k+r-1)(k+r-5)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-5)}{k+r-1}$

- Recursion relation for $r=2$; series terminates at $k=3$

  $a_{k+1}=\frac{a_k(k-3)}{k+1}$

- Apply recursion relation for $k=0$

  $a_1=-3a_0$

- Apply recursion relation for $k=1$

  $a_2=-a_1$

- Express in terms of a_0

  $a_2=3a_0$

- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$

- Express in terms of a_0

  $a_3=-a_0$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second li

  $y=a_0\cdot(-x^3+3x^2-3x+1)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*(1-x)*diff(y(x),x$2)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 3x + 3x^2 - x^3 + \mathrm{O}\left(x^6\right)\right) + \left(4x - 7x^2 + \frac{11}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*(1-x)*y''[x]-x*(3-5*x)*y'[x]+(4-5*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-x^3 + 3x^2 - 3x + 1\right)x^2 + c_2\left(\left(-x^3 + 3x^2 - 3x + 1\right)x^2 \log(x) + \left(-\frac{x^5}{20} - \frac{x^4}{4} + \frac{11x^3}{3} - 7x^2 + 4x\right)x^2\right)$$

15.53 problem 49

Internal problem ID [1401]
Internal file name [`OUTPUT/1402_Sunday_June_05_2022_02_15_18_AM_53006219/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-x(9x^2+1)\,y'+(25x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(-9x^3-x)\,y'+(25x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{9x^2+1}{x\,(x^2+1)}$$
$$q(x)=\frac{25x^2+1}{x^2\,(x^2+1)}$$

Table 748: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{9x^2+1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{25x^2+1}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (-9x^3 - x)\,y' + (25x^2+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (-9x^3 - x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (25x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 25x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-9x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-9a_{n-2} (n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 25x^{n+r+2} a_n = \sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-9a_{n-2} (n+r-2) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ - 9a_{n-2}(n+r-2) - a_n(n+r) + 25a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2 + 2nr + r^2 - 14n - 14r + 49)}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}(n-6)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{(r-5)^2}{(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{(r-5)^2}{(1+r)^2}$	-4

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{(r-5)^2}{(1+r)^2}$	-4
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-5)^2 (r-3)^2}{(1+r)^2 (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{(r-5)^2}{(1+r)^2}$	-4
a_3	0	0
a_4	$\frac{(r-5)^2(r-3)^2}{(1+r)^2(r+3)^2}$	1

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{(r-5)^2}{(1+r)^2}$	-4
a_3	0	0
a_4	$\frac{(r-5)^2(r-3)^2}{(1+r)^2(r+3)^2}$	1
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \ldots\right) \\ &= x\left(x^4 - 4x^2 + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{(r-5)^2}{(1+r)^2}$	-4	$\frac{-12r+60}{(1+r)^3}$	6
b_3	0	0	0	0
b_4	$\frac{(r-5)^2(r-3)^2}{(1+r)^2(r+3)^2}$	1	$\frac{24(r-5)(r-3)(r^2-2r-7)}{(1+r)^3(r+3)^3}$	-3
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots\\ &= x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\ln(x) + x\left(-3x^4 + 6x^2 + O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x\left(x^4 - 4x^2 + 1 + O(x^6)\right) + c_2\left(x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\ln(x) + x\left(-3x^4 + 6x^2 + O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x\left(x^4-4x^2+1+O(x^6)\right)+c_2\left(x\left(x^4-4x^2+1+O(x^6)\right)\ln(x)+x\left(-3x^4+6x^2+O(x^6)\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\\ &\quad + c_2\left(x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\ln(x) + x\left(-3x^4 + 6x^2 + O(x^6)\right)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\\ &\quad + c_2\left(x\left(x^4 - 4x^2 + 1 + O(x^6)\right)\ln(x) + x\left(-3x^4 + 6x^2 + O(x^6)\right)\right)\end{aligned}$$

Verified OK.

15.53.1 Maple step by step solution

Let's solve

$x^2(x^2 + 1)\,y'' + (-9x^3 - x)\,y' + (25x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2+1)y}{x^2(x^2+1)} + \frac{(9x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(9x^2+1)y'}{x(x^2+1)} + \frac{(25x^2+1)y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' - x(9x^2+1)\, y' + (25x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term must be 0

  $$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2 = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+1+r)^2 + a_k(k+r-5)^2 = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r-5)^2}{(k+1+r)^2}$$

- Recursion relation for $r = 1$; series terminates at $k = 4$

  $$a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1+9*x^2)*diff(y(x),x)+(1+25*x^2)*y(x)=0,y(x),type='seri
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - 4x^2 + x^4 + \mathrm{O}\left(x^6\right)\right) + \left(6x^2 - 3x^4 + \mathrm{O}\left(x^6\right)\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 49

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-x*(1+9*x^2)*y'[x]+(1+25*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(x^4 - 4x^2 + 1\right) + c_2\left(x\left(6x^2 - 3x^4\right) + x\left(x^4 - 4x^2 + 1\right)\log(x)\right)$$

15.54 problem 50

Internal problem ID [1402]
Internal file name [`OUTPUT/1403_Sunday_June_05_2022_02_15_21_AM_22366952/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 3x(-x^2+1)\,y' + (7x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + (-3x^3+3x)\,y' + (7x^2+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x^2-1}{3x} \\ q(x) &= \frac{7x^2+1}{9x^2} \end{aligned}$$

Table 750: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-1}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{7x^2+1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (-3x^3 + 3x)\, y' + (7x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-3x^3 + 3x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (7x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n = \sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-3a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 3x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 3x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(3r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) - 3a_{n-2}(n+r-2) + 3a_n(n+r) + 7a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(3n+3r-13)}{9n^2+18nr+9r^2-6n-6r+1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{a_{n-2}(n-4)}{3n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-7+3r}{(3r+5)^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+3r}{(3r+5)^2}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+3r}{(3r+5)^2}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{9r^2 - 24r + 7}{(3r+5)^2 (3r+11)^2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+3r}{(3r+5)^2}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{9r^2-24r+7}{(3r+5)^2(3r+11)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+3r}{(3r+5)^2}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{9r^2-24r+7}{(3r+5)^2(3r+11)^2}$	0
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{x^2}{6} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = \frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{3}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-7+3r}{(3r+5)^2}$	$-\frac{1}{6}$	$\frac{-9r+57}{(3r+5)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{9r^2-24r+7}{(3r+5)^2(3r+11)^2}$	0	$\frac{-162r^3+648r^2+1890r-1992}{(3r+5)^3(3r+11)^3}$	$-\frac{1}{288}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\ldots\\ &= x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\ln(x) + x^{\frac{1}{3}}\left(\frac{x^2}{4}-\frac{x^4}{288}+O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)+c_2\left(x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\ln(x)+x^{\frac{1}{3}}\left(\frac{x^2}{4}-\frac{x^4}{288}+O(x^6)\right)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)+c_2\left(x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\ln(x)+x^{\frac{1}{3}}\left(\frac{x^2}{4}-\frac{x^4}{288}+O(x^6)\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\\ &\quad+c_2\left(x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\ln(x)+x^{\frac{1}{3}}\left(\frac{x^2}{4}-\frac{x^4}{288}+O(x^6)\right)\right)\end{aligned}\tag{1}$$

Verification of solutions

$$y = c_1x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)+c_2\left(x^{\frac{1}{3}}\left(1-\frac{x^2}{6}+O(x^6)\right)\ln(x)+x^{\frac{1}{3}}\left(\frac{x^2}{4}-\frac{x^4}{288}+O(x^6)\right)\right)$$

Verified OK.

15.54.1 Maple step by step solution

Let's solve

$9x^2y'' + (-3x^3 + 3x)\, y' + (7x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{9x^2} + \frac{(x^2-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{3x} + \frac{(7x^2+1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2-1}{3x}, P_3(x) = \frac{7x^2+1}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{9}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' - 3x(x^2 - 1)\, y' + (7x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 - a_{k-2}(3k-13+3r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-1)^2 + (-3k+13-3r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+5+3r)^2 + a_k(-3k-3r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(3k+3r-7)}{(3k+5+3r)^2}$

- Recursion relation for $r = \frac{1}{3}$; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+3*x*(1-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = x^{\frac{1}{3}} \left((c_2 \ln(x) + c_1) \left(1 - \frac{1}{6}x^2 + \mathrm{O}(x^6)\right) + \left(\frac{1}{4}x^2 - \frac{1}{288}x^4 + \mathrm{O}(x^6)\right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 63

```
AsymptoticDSolveValue[9*x^2*y''[x]+3*x*(1-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(1-\frac{x^2}{6}\right)+c_2\left(\sqrt[3]{x}\left(1-\frac{x^2}{6}\right)\log(x)+\sqrt[3]{x}\left(\frac{x^2}{4}-\frac{x^4}{288}\right)\right)$$

15.55 problem 51

Internal problem ID [1403]
Internal file name [OUTPUT/1404_Sunday_June_05_2022_02_15_24_AM_97671373/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+1)\,y''+\left(-x^2+1\right)y'-8yx=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x\right)y''+\left(-x^2+1\right)y'-8yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x^2-1}{x\,(x^2+1)}$$
$$q(x)=-\frac{8}{x^2+1}$$

Table 752: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = -\frac{8}{x^2+1}$	
singularity	type
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x^2+1)\, y'' + (-x^2+1)\, y' - 8yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 8\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-8x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-8x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r)+r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x)=y_1(x) \ln(x)+\left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y=c_1 y_1(x)+c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0=1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. Substituting $n=1$ in Eq. (2B) gives

$$a_1=0$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&a_{n-2}(n+r-2)\,(n-3+r)+a_n(n+r)\,(n+r-1) \\
&-a_{n-2}(n+r-2)+a_n(n+r)-8a_{n-2}=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r-6)\,a_{n-2}}{n+r} \tag{4}
$$

Which for the root $r=0$ becomes

$$
a_n = -\frac{(n-6)\,a_{n-2}}{n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$
a_2 = \frac{4-r}{2+r}
$$

Which for the root $r=0$ becomes

$$
a_2 = 2
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{2+r}$	2

For $n=3$, using the above recursive equation gives

$$
a_3 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{2+r}$	2
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 - 6r + 8}{(4 + r)(2 + r)}$$

Which for the root $r = 0$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{2+r}$	2
a_3	0	0
a_4	$\frac{r^2-6r+8}{(4+r)(2+r)}$	1

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4-r}{2+r}$	2
a_3	0	0
a_4	$\frac{r^2-6r+8}{(4+r)(2+r)}$	1
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= x^4 + 2x^2 + 1 + O(x^6)\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{4-r}{2+r}$	2	$-\frac{6}{(2+r)^2}$	$-\frac{3}{2}$
b_3	0	0	0	0
b_4	$\frac{r^2-6r+8}{(4+r)(2+r)}$	1	$\frac{12r^2-96}{(2+r)^2(4+r)^2}$	$-\frac{3}{2}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \left(x^4 + 2x^2 + 1 + O(x^6)\right) \ln(x) - \frac{3x^2}{2} - \frac{3x^4}{2} + O(x^6)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(x^4+2x^2+1+O(x^6)\right) + c_2 \left(\left(x^4+2x^2+1+O(x^6)\right) \ln(x) - \frac{3x^2}{2} - \frac{3x^4}{2} + O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$= c_1\left(x^4+2x^2+1+O(x^6)\right)+c_2\left(\left(x^4+2x^2+1+O(x^6)\right)\ln(x)-\frac{3x^2}{2}-\frac{3x^4}{2}+O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\left(x^4+2x^2+1+O(x^6)\right) + c_2\left(\left(x^4+2x^2+1+O(x^6)\right)\ln(x)-\frac{3x^2}{2}-\frac{3x^4}{2}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(x^4+2x^2+1+O(x^6)\right)+c_2\left(\left(x^4+2x^2+1+O(x^6)\right)\ln(x)-\frac{3x^2}{2}-\frac{3x^4}{2}+O(x^6)\right)$$

Verified OK.

15.55.1 Maple step by step solution

Let's solve

$x(x^2+1)\,y''+(-x^2+1)\,y'-8yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{8y}{x^2+1}+\frac{(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-1)y'}{x(x^2+1)}-\frac{8y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x^2-1}{x(x^2+1)}, P_3(x)=-\frac{8}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x(x^2+1)\,y'' + (-x^2+1)\,y' - 8yx = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1(1 + r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1})\,(k + 1) = 0$
- Shift index using $k- >k + 1$

  $((a_k + a_{k+2})\,(k + 1) - 5a_k + a_{k+2})\,(k + 2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)}{k+2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*(1+x^2)*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + 2x^2 + x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{3}{2}x^2 - \frac{3}{2}x^4 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 48

```
AsymptoticDSolveValue[x*(1+x^2)*y''[x]+(1-x^2)*y'[x]-8*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(x^4 + 2x^2 + 1\right) + c_2\left(-\frac{3x^4}{2} - \frac{3x^2}{2} + \left(x^4 + 2x^2 + 1\right)\log(x)\right)$$

15.56 problem 52

Internal problem ID [1404]
Internal file name [`OUTPUT/1405_Sunday_June_05_2022_02_15_26_AM_35527558/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x\left(-x^2+4\right)y' + \left(7x^2+1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + \left(-2x^3+8x\right)y' + \left(7x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2-4}{2x}$$
$$q(x) = \frac{7x^2+1}{4x^2}$$

Table 754: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-4}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{7x^2+1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(-2x^3 + 8x\right) y' + \left(7x^2 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-2x^3 + 8x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(7x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 7x^{n+r+2} a_n = \sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 7a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 8x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 8x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 8x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -\frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-2}(n+r-2) + 8a_n(n+r) + 7a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(2n+2r-11)}{4n^2+8nr+4r^2+4n+4r+1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = \frac{a_{n-2}(n-6)}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-7+2r}{(2r+5)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+2r}{(2r+5)^2}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+2r}{(2r+5)^2}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 - 20r + 21}{(2r+5)^2 (2r+9)^2}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = \frac{1}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+2r}{(2r+5)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4r^2-20r+21}{(2r+5)^2(2r+9)^2}$	$\frac{1}{32}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-7+2r}{(2r+5)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4r^2-20r+21}{(2r+5)^2(2r+9)^2}$	$\frac{1}{32}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -\frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = -\frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{-7+2r}{(2r+5)^2}$	$-\frac{1}{2}$	$\frac{-4r+38}{(2r+5)^3}$	$\frac{5}{8}$
b_3	0	0	0	0
b_4	$\frac{4r^2-20r+21}{(2r+5)^2(2r+9)^2}$	$\frac{1}{32}$	$\frac{-32r^3+240r^2+584r-2076}{(2r+5)^3(2r+9)^3}$	$-\frac{9}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= \frac{\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)\ln(x)}{\sqrt{x}} + \frac{\frac{5x^2}{8} - \frac{9x^4}{128} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)}{\sqrt{x}} + c_2\left(\frac{\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)\ln(x)}{\sqrt{x}} + \frac{\frac{5x^2}{8} - \frac{9x^4}{128} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)}{\sqrt{x}} + c_2\left(\frac{\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)\ln(x)}{\sqrt{x}} + \frac{\frac{5x^2}{8} - \frac{9x^4}{128} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)}{\sqrt{x}} \\
& + c_2\left(\frac{\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)\ln(x)}{\sqrt{x}} + \frac{\frac{5x^2}{8} - \frac{9x^4}{128} + O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & \frac{c_1\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)}{\sqrt{x}} \\
& + c_2\left(\frac{\left(-\frac{x^2}{2} + 1 + \frac{x^4}{32} + O(x^6)\right)\ln(x)}{\sqrt{x}} + \frac{\frac{5x^2}{8} - \frac{9x^4}{128} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Verified OK.

15.56.1 Maple step by step solution

Let's solve

$4x^2y'' + (-2x^3 + 8x)\, y' + (7x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{4x^2} + \frac{(x^2-4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-4)y'}{2x} + \frac{(7x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-4}{2x}, P_3(x) = \frac{7x^2+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 2x(x^2 - 4)\, y' + (7x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(1+2r)^{2}\,x^{r}+a_{1}(3+2r)^{2}\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(2k+2r+1)^{2}-a_{k-2}(2k-11+2r)\right)x^{k+r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)^{2}=0$$

- Values of r that satisfy the indicial equation

  $$r=-\frac{1}{2}$$

- Each term must be 0

  $$a_{1}(3+2r)^{2}=0$$

- Solve for the dependent coefficient(s)

  $$a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k}(2k+2r+1)^{2}+(-2k+11-2r)\,a_{k-2}=0$$

- Shift index using $k->k+2$

  $$a_{k+2}(2k+5+2r)^{2}+a_{k}(-2k-2r+7)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{a_{k}(2k+2r-7)}{(2k+5+2r)^{2}}$$

- Recursion relation for $r=-\frac{1}{2}$; series terminates at $k=4$

  $$a_{k+2}=\frac{a_{k}(2k-8)}{(2k+4)^{2}}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+2*x*(4-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{(c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{1}{32}x^4 + \mathrm{O}(x^6)\right) + \left(\frac{5}{8}x^2 - \frac{9}{128}x^4 + \mathrm{O}(x^6)\right)c_2}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 77

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x*(4-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{x^4}{32}-\frac{x^2}{2}+1\right)}{\sqrt{x}}+c_2\left(\frac{\frac{5x^2}{8}-\frac{9x^4}{128}}{\sqrt{x}}+\frac{\left(\frac{x^4}{32}-\frac{x^2}{2}+1\right)\log(x)}{\sqrt{x}}\right)$$

15.57 problem 58

Internal problem ID [1405]
Internal file name [`OUTPUT/1406_Sunday_June_05_2022_02_15_29_AM_73252834/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 58.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x+1)\,y''+8y'x^2+(x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''+8y'x^2+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2}{x+1}$$
$$q(x)=\frac{1}{4x^2}$$

Table 756: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x+1}$	
singularity	type
$x = -1$	"regular"

$q(x) = \frac{1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\, y'' + 8y'x^2 + (x+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 8\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right) \end{aligned}$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)+8a_{n-1}(n+r-1)+a_{n-1}+a_n=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -a_{n-1} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$a_4 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1

For $n = 5$, using the above recursive equation gives

$$a_5 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1
a_5	-1	-1

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	-1	-1	0	0
b_2	1	1	0	0
b_3	-1	-1	0	0
b_4	1	1	0	0
b_5	-1	-1	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \\ &\quad + c_2\left(\sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right) \end{aligned}$$

Verified OK.

15.57.1 Maple step by step solution

Let's solve

$4x^2(x+1)\, y'' + 8y'x^2 + (x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y'}{x+1} - \frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x+1} + \frac{y}{4x^2} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2}{x+1}, P_3(x) = \frac{1}{4x^2}\right]$

- ○ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 2$

- ○ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- ○ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(x+1)\,y'' + 8y'x^2 + (x+1)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (8u^2 - 16u + 8)\left(\frac{d}{du}y(u)\right) + uy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u \cdot y(u)$ to series expansion

  $u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$

- ○ Shift index using $k- >k-1$

  $u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^{m}\cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_{0}r(1+r)\, u^{-1+r} + \left(4a_{1}(1+r)\,(2+r) - 8a_{0}r(1+r)\right) u^{r} + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+r+1)\,(k+2+r) -\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$4r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$4a_{1}(1+r)\,(2+r) - 8a_{0}r(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-1}(2k-1+2r)^{2} - 8\left(\left(-\frac{k}{2}-\frac{r}{2}-1\right) a_{k+1} + a_{k}(k+r)\right)(k+r+1) = 0$$

- Shift index using $k->k+1$

$$a_{k}(2k+2r+1)^{2} - 8\left(\left(-\frac{k}{2}-\frac{3}{2}-\frac{r}{2}\right) a_{k+2} + a_{k+1}(k+r+1)\right)(k+2+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+8kra_{k}-16kra_{k+1}+4r^{2}a_{k}-8r^{2}a_{k+1}+4ka_{k}-24ka_{k+1}+4ra_{k}-24ra_{k+1}+a_{k}-16a_{k+1}}{4(k+3+r)(k+2+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}-8ka_{k+1}+a_{k}}{4(k+2)(k+1)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k-1}, a_{k+2} = -\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}-8ka_{k+1}+a_{k}}{4(k+2)(k+1)}, 0 = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-1}, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} - 4k a_k - 8k a_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 24k a_{k+1} + a_k - 16 a_{k+1}}{4(k+3)(k+2)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 24k a_{k+1} + a_k - 16 a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 24k a_{k+1} + a_k - 16 a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^k\right), a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{1+k} - 4k a_k - 8k a_{1+k} + a_k}{4(k+2)(1+k)}, 0 = 0, b_{k+2}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+8*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\,(-x^5 + x^4 - x^3 + x^2 - x + 1)\,(c_2 \ln (x) + c_1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 64

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]+8*x^2*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}(-x^5 + x^4 - x^3 + x^2 - x + 1) + c_2\sqrt{x}(-x^5 + x^4 - x^3 + x^2 - x + 1)\log(x)$$

15.58 problem 59

Internal problem ID [1406]
Internal file name [OUTPUT/1407_Sunday_June_05_2022_02_15_31_AM_36669901/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 59.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x+3)\,y''+3x(3+7x)\,y'+(4x+3)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(4x+3)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3+7x}{3x\,(x+3)}$$
$$q(x)=\frac{4x+3}{9x^2\,(x+3)}$$

Table 758: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+7x}{3x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4x+3}{9x^2(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2(x+3)\, y'' + \left(21x^2 + 9x\right) y' + (4x+3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2(x+3) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(21x^2 + 9x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x+3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 27x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 21a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 27x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 21a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$27x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$27x^r a_0 r(-1+r) + 9x^r a_0 r + 3a_0 x^r = 0$$

Or

$$(27x^r r(-1+r) + 9x^r r + 3x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$3(3r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3(3r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$3(3r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} 9a_{n-1}(n+r-1)(n+r-2) + 27a_n(n+r)(n+r-1) \\ + 21a_{n-1}(n+r-1) + 9a_n(n+r) + 4a_{n-1} + 3a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{3} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{a_{n-1}}{3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{3}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{9}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{1}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3}$	$-\frac{1}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{27}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{1}{27}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3}$	$-\frac{1}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$-\frac{1}{27}$	$-\frac{1}{27}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{81}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3}$	$-\frac{1}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$-\frac{1}{27}$	$-\frac{1}{27}$
a_4	$\frac{1}{81}$	$\frac{1}{81}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{243}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{1}{243}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{3}$	$-\frac{1}{3}$
a_2	$\frac{1}{9}$	$\frac{1}{9}$
a_3	$-\frac{1}{27}$	$-\frac{1}{27}$
a_4	$\frac{1}{81}$	$\frac{1}{81}$
a_5	$-\frac{1}{243}$	$-\frac{1}{243}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(-\frac{x}{3} + 1 + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{81} - \frac{x^5}{243} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{3}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0
b_2	$\frac{1}{9}$	$\frac{1}{9}$	0	0
b_3	$-\frac{1}{27}$	$-\frac{1}{27}$	0	0
b_4	$\frac{1}{81}$	$\frac{1}{81}$	0	0
b_5	$-\frac{1}{243}$	$-\frac{1}{243}$	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^{\frac{1}{3}}\left(-\frac{x}{3} + 1 + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{81} - \frac{x^5}{243} + O\left(x^6\right)\right)\ln(x) + x^{\frac{1}{3}} O\left(x^6\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}}\left(-\frac{x}{3} + 1 + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{81} - \frac{x^5}{243} + O\left(x^6\right)\right) \\ &\quad + c_2\left(x^{\frac{1}{3}}\left(-\frac{x}{3} + 1 + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{81} - \frac{x^5}{243} + O\left(x^6\right)\right)\ln(x) + x^{\frac{1}{3}} O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \\
&\quad + c_2 \left(x^{\frac{1}{3}} \left(-\frac{x}{3}+1+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{81}-\frac{x^5}{243}+O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right)
\end{aligned}
$$

Verified OK.

15.58.1 Maple step by step solution

Let's solve

$9x^2(x+3)\,y''+(21x^2+9x)\,y'+(4x+3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x+3)y}{9x^2(x+3)}-\frac{(3+7x)y'}{3x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3+7x)y'}{3x(x+3)}+\frac{(4x+3)y}{9x^2(x+3)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{3+7x}{3x(x+3)}, P_3(x)=\frac{4x+3}{9x^2(x+3)}\right]$

- $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $$\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=2$$

- $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $$\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$$

- $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $$9x^2(x+3)\,y''+3x(3+7x)\,y'+(4x+3)\,y=0$$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $$(9u^3-54u^2+81u)\left(\frac{d^2}{du^2}y(u)\right)+(21u^2-117u+162)\left(\frac{d}{du}y(u)\right)+(4u-9)\,y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

    $$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$81 a_{0} r \left(1+r \right) u^{-1+r}+\left(81 a_{1} \left(1+r \right) \left(2+r \right)-9 a_{0} \left(1+r \right) \left(1+6 r \right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(81 a_{k +1} \left(k +r +1\right) \left(k\right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$81 r \left(1+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, 0\right\}$$

- Each term must be 0

$$81 a_{1} \left(1+r \right) \left(2+r \right)-9 a_{0} \left(1+r \right) \left(1+6 r \right)=0$$

- Each term in the series must be 0, giving the recursion relation

$$81 a_{k +1} \left(k +r +1\right) \left(k +2+r \right)-54 \left(k +r +\frac{1}{6}\right) \left(k +r +1\right) a_{k}+a_{k -1} \left(3 k -1+3 r \right)^{2}=0$$

- Shift index using $k->k +1$

$$81 a_{k +2} \left(k +2+r \right) \left(k +3+r \right)-54 \left(k +\frac{7}{6}+r \right) \left(k +2+r \right) a_{k +1}+a_{k} \left(3 k +3 r +2\right)^{2}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +2}=-\frac{9 k^{2} a_{k}-54 k^{2} a_{k +1}+18 k r a_{k}-108 k r a_{k +1}+9 r^{2} a_{k}-54 r^{2} a_{k +1}+12 k a_{k}-171 k a_{k +1}+12 r a_{k}-171 r a_{k +1}+4 a_{k}-126 a_{k +1}}{81 \left(k +2+r \right) \left(k +3+r \right)}$$

- Recursion relation for $r =-1$

$$a_{k +2}=-\frac{9 k^{2} a_{k}-54 k^{2} a_{k +1}-6 k a_{k}-63 k a_{k +1}+a_{k}-9 a_{k +1}}{81 \left(k +1\right) \left(k +2\right)}$$

- Solution for $r =-1$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k -1}, a_{k +2}=-\frac{9 k^{2} a_{k}-54 k^{2} a_{k +1}-6 k a_{k}-63 k a_{k +1}+a_{k}-9 a_{k +1}}{81 \left(k +1\right) \left(k +2\right)}, 0=0\right]$$

- Revert the change of variables $u =x +3$

$$\left[y =\sum_{k =0}^{\infty}a_{k} \left(x +3\right)^{k -1}, a_{k +2}=-\frac{9 k^{2} a_{k}-54 k^{2} a_{k +1}-6 k a_{k}-63 k a_{k +1}+a_{k}-9 a_{k +1}}{81 \left(k +1\right) \left(k +2\right)}, 0=0\right]$$

- Recursion relation for $r =0$

$$a_{k +2}=-\frac{9 k^{2} a_{k}-54 k^{2} a_{k +1}+12 k a_{k}-171 k a_{k +1}+4 a_{k}-126 a_{k +1}}{81 \left(k +2\right) \left(k +3\right)}$$

- Solution for $r =0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}, 162a_1-9a_0=0\right]$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}, 162a_1-9a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+3)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k(x+3)^k\right), a_{k+2}=-\frac{9k^2a_k-54k^2a_{1+k}-6ka_k-63ka_{1+k}+a_k-9a_{1+k}}{81(1+k)(k+2)}, 0=\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(9*x^2*(3+x)*diff(y(x),x$2)+3*x*(3+7*x)*diff(y(x),x)+(3+4*x)*y(x)=0,y(x),type='series'
```

$$y(x)=x^{\frac{1}{3}}\left(1-\frac{1}{3}x+\frac{1}{9}x^2-\frac{1}{27}x^3+\frac{1}{81}x^4-\frac{1}{243}x^5\right)(c_2\ln(x)+c_1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 92

```
AsymptoticDSolveValue[9*x^2*(3+x)*y''[x]+3*x*(3+7*x)*y'[x]+(3+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(-\frac{x^5}{243}+\frac{x^4}{81}-\frac{x^3}{27}+\frac{x^2}{9}-\frac{x}{3}+1\right) + c_2 \sqrt[3]{x}\left(-\frac{x^5}{243}+\frac{x^4}{81}-\frac{x^3}{27}+\frac{x^2}{9}-\frac{x}{3}+1\right)\log(x)$$

15.59 problem 60

Internal problem ID [1407]
Internal file name [OUTPUT/1408_Sunday_June_05_2022_02_15_34_AM_76674236/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 60.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x^2+2)\, y'' - x(3x^2+2)\, y' + (-x^2+2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^4+2x^2)\, y'' + (-3x^3-2x)\, y' + (-x^2+2)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{3x^2+2}{x\,(x^2-2)} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Table 760: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x^2+2}{x(x^2-2)}$	
singularity	type
$x = 0$	"regular"
$x = \sqrt{2}$	"regular"
$x = -\sqrt{2}$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \sqrt{2}, -\sqrt{2}, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x^2-2) + (-3x^3-2x)\,y' + (-x^2+2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x^2-2) + (-3x^3-2x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x^2+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2 x^{n+r} a_n(n+r)(n+r-1)\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-3 x^{n+r+2} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right) \\
&\quad+\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} 2 a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)(n+r-1)\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-3 x^{n+r+2} a_n(n+r)\right) &= \sum_{n=2}^{\infty}\left(-3 a_{n-2}(n+r-2) x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right) &= \sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2 x^{n+r} a_n(n+r)(n+r-1)\right) \\
&\quad+\sum_{n=2}^{\infty}\left(-3 a_{n-2}(n+r-2) x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-2 x^{n+r} a_n(n+r)\right) \\
&\quad+\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2 a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2 x^{n+r} a_n(n+r)(n+r-1)-2 x^{n+r} a_n(n+r)+2 a_n x^{n+r}=0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - 2x^r a_0 r + 2a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) - 2x^r r + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r (-1+r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r (-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}-a_{n-2}(n+r-2)(n-3+r)+2a_n(n+r)(n+r-1)\\ -3a_{n-2}(n+r-2)-2a_n(n+r)-a_{n-2}+2a_n=0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{4}$	$\frac{1}{4}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{4}$	$\frac{1}{4}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1}{2}$	$\frac{1}{2}$	0	0
b_3	0	0	0	0
b_4	$\frac{1}{4}$	$\frac{1}{4}$	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)\ln(x) + xO(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + c_2\left(x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)\ln(x) + xO(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + c_2\left(x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)\ln(x) + xO(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + c_2\left(x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)\ln(x) + xO(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right) + c_2\left(x\left(1 + \frac{x^2}{2} + \frac{x^4}{4} + O(x^6)\right)\ln(x) + xO(x^6)\right)$$

Verified OK.

15.59.1 Maple step by step solution

Let's solve

$-y''x^2(x^2-2) + (-3x^3-2x)\,y' + (-x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x^2} - \frac{(3x^2+2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+2)y'}{x(x^2-2)} + \frac{y}{x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{3x^2+2}{x(x^2-2)}, P_3(x) = \frac{1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x^2-2) + x(3x^2+2)\, y' + y(x^2-2) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

$$x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m} \cdot y''$ to series expansion for $m = 2..4$

$$x^{m} \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^{m} \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(-1+r)^2\, x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$$

- Shift index using $k- >k+2$

$$-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k}{2}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{a_k}{2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 33

```
Order:=6;
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-x*(2+3*x^2)*diff(y(x),x)+(2-x^2)*y(x)=0,y(x),type='series'
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{4}x^4\right) x(c_2 \ln(x) + c_1) + O(x^6)$$

✓ Solution by Mathematica
Time used: 0.006 (sec). Leaf size: 46

```
AsymptoticDSolveValue[x^2*(2-x^2)*y''[x]-x*(2+3*x^2)*y'[x]+(2-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^4}{4} + \frac{x^2}{2} + 1\right) + c_2 x \left(\frac{x^4}{4} + \frac{x^2}{2} + 1\right) \log(x)$$

15.60 problem 61

Internal problem ID [1408]
Internal file name [`OUTPUT/1409_Sunday_June_05_2022_02_15_36_AM_41322562/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 61.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(16x^4+16x^2)\,y''+(72x^3+8x)\,y'+(49x^2+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{9x^2+1}{2x\,(x^2+1)} \\ q(x) &= \frac{49x^2+1}{16x^2\,(x^2+1)} \end{aligned}$$

Table 762: Table $p(x), q(x)$ singularites.

$p(x) = \frac{9x^2+1}{2x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{49x^2+1}{16x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$16x^2(x^2+1)\, y'' + (72x^3+8x)\, y' + (49x^2+1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$16x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}\right) + (72x^3+8x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (49x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 16 x^{n+r+2} a_n (n+r) (n+r-1)\right)+\left(\sum_{n=0}^{\infty} 16 x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 72 x^{n+r+2} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 8 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 49 x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 16 x^{n+r+2} a_n (n+r) (n+r-1) = \sum_{n=2}^{\infty} 16 a_{n-2} (n+r-2) (n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 72 x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 72 a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 49 x^{n+r+2} a_n = \sum_{n=2}^{\infty} 49 a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 16 a_{n-2} (n+r-2) (n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 16 x^{n+r} a_n (n+r) (n+r-1)\right)+\left(\sum_{n=2}^{\infty} 72 a_{n-2} (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8 x^{n+r} a_n (n+r)\right)+\left(\sum_{n=2}^{\infty} 49 a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$16 x^{n+r} a_n (n+r) (n+r-1)+8 x^{n+r} a_n (n+r)+a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$16x^r a_0 r(-1+r) + 8x^r a_0 r + a_0 x^r = 0$$

Or

$$(16x^r r(-1+r) + 8x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(4r-1)^2\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(4r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r-1)^2\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of

integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{4}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{4}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &16a_{n-2}(n+r-2)(n-3+r) + 16a_n(n+r)(n+r-1) \\ &+ 72a_{n-2}(n+r-2) + 8a_n(n+r) + 49a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} \tag{4}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -a_{n-2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -1$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-1	-1

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-1	-1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 1$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-1	-1
a_3	0	0
a_4	1	1

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	-1	-1
a_3	0	0
a_4	1	1
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{4}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{4}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	-1	-1	0	0
b_3	0	0	0	0
b_4	1	1	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right)\ln(x) + x^{\frac{1}{4}} O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right) + c_2\left(x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right)\ln(x) + x^{\frac{1}{4}} O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right) + c_2\left(x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right)\ln(x) + x^{\frac{1}{4}} O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right) + c_2\left(x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right)\ln(x) + x^{\frac{1}{4}} O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right) + c_2\left(x^{\frac{1}{4}}\left(x^4 - x^2 + 1 + O(x^6)\right)\ln(x) + x^{\frac{1}{4}} O(x^6)\right)$$

Verified OK.

15.60.1 Maple step by step solution

Let's solve

$16x^2(x^2+1)\, y'' + (72x^3 + 8x)\, y' + (49x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x^2+1)y}{16x^2(x^2+1)} - \frac{(9x^2+1)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(9x^2+1)y'}{2x(x^2+1)} + \frac{(49x^2+1)y}{16x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{9x^2+1}{2x(x^2+1)}, P_3(x) = \frac{49x^2+1}{16x^2(x^2+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

• Multiply by denominators

  $16x^2(x^2+1)\,y'' + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$

• Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$$

- ○ Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (-1+4r)^2 x^r + a_1 (3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (4k+4r-1)^2 + a_{k-2} (4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{4}$$

- Each term must be 0

$$a_1 (3+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(4k+4r-1)^2 (a_k + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(4k+4r+7)^2 (a_{k+2} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -a_k$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -a_k$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 37

```
Order:=6;
dsolve(16*x^2*(1+x^2)*diff(y(x),x$2)+8*x*(1+9*x^2)*diff(y(x),x)+(1+49*x^2)*y(x)=0,y(x),type=
```

$$y(x) = x^{\frac{1}{4}} \left(x^4 - x^2 + 1\right) \left(c_2 \ln \left(x\right) + c_1\right) + O\left(x^6\right)$$

✓ Solution by Mathematica
Time used: 0.006 (sec). Leaf size: 42

```
AsymptoticDSolveValue[16*x^2*(1+x^2)*y''[x]+8*x*(1+9*x^2)*y'[x]+(1+49*x^2)*y[x]==0,y[x],{x,0
```

$$y(x) \to c_1 \sqrt[4]{x} \left(x^4 - x^2 + 1\right) + c_2 \sqrt[4]{x} \left(x^4 - x^2 + 1\right) \log(x)$$

15.61 problem 62

Internal problem ID [1409]
Internal file name [`OUTPUT/1410_Sunday_June_05_2022_02_15_38_AM_38663203/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 62.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(3x+4)\,y''-x(4-3x)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3x-4}{x\,(3x+4)}$$
$$q(x)=\frac{4}{x^2\,(3x+4)}$$

Table 764: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-4}{x(3x+4)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{4}{3}$	"regular"

$q(x) = \frac{4}{x^2(3x+4)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{4}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{4}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(3x+4)\, y'' + \left(3x^2 - 4x\right) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(3x+4) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(3x^2 - 4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) (n+r-1) = \sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r) (n+r-1) - 4x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - 4x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 4x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$4x^r (-1+r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$4(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$4x^r(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) - 4a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{4} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{3a_{n-1}}{4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{4}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{16}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{9}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4}$	$-\frac{3}{4}$
a_2	$\frac{9}{16}$	$\frac{9}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{64}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{27}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4}$	$-\frac{3}{4}$
a_2	$\frac{9}{16}$	$\frac{9}{16}$
a_3	$-\frac{27}{64}$	$-\frac{27}{64}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{256}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{81}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4}$	$-\frac{3}{4}$
a_2	$\frac{9}{16}$	$\frac{9}{16}$
a_3	$-\frac{27}{64}$	$-\frac{27}{64}$
a_4	$\frac{81}{256}$	$\frac{81}{256}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{1024}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{243}{1024}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{4}$	$-\frac{3}{4}$
a_2	$\frac{9}{16}$	$\frac{9}{16}$
a_3	$-\frac{27}{64}$	$-\frac{27}{64}$
a_4	$\frac{81}{256}$	$\frac{81}{256}$
a_5	$-\frac{243}{1024}$	$-\frac{243}{1024}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{3}{4}$	$-\frac{3}{4}$	0	0
b_2	$\frac{9}{16}$	$\frac{9}{16}$	0	0
b_3	$-\frac{27}{64}$	$-\frac{27}{64}$	0	0
b_4	$\frac{81}{256}$	$\frac{81}{256}$	0	0
b_5	$-\frac{243}{1024}$	$-\frac{243}{1024}$	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right)\ln(x) + xO(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right)\ln(x) + xO(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right)\ln(x) + xO(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{3x}{4} + \frac{9x^2}{16} - \frac{27x^3}{64} + \frac{81x^4}{256} - \frac{243x^5}{1024} + O(x^6)\right)\ln(x) + xO(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x\left(1-\frac{3x}{4}+\frac{9x^2}{16}-\frac{27x^3}{64}+\frac{81x^4}{256}-\frac{243x^5}{1024}+O(x^6)\right)$$
$$+ c_2\left(x\left(1-\frac{3x}{4}+\frac{9x^2}{16}-\frac{27x^3}{64}+\frac{81x^4}{256}-\frac{243x^5}{1024}+O(x^6)\right)\ln(x)+xO(x^6)\right)$$

Verified OK.

15.61.1 Maple step by step solution

Let's solve

$x^2(3x+4)\, y'' + (3x^2-4x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(3x+4)} - \frac{(3x-4)y'}{x(3x+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-4)y'}{x(3x+4)} + \frac{4y}{x^2(3x+4)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x-4}{x(3x+4)}, P_3(x) = \frac{4}{x^2(3x+4)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(3x+4)\, y'' + x(3x-4)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(-1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + 3a_{k-1}(k+r-1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2 (4a_k + 3a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r)^2 (4a_{k+1} + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{4}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{4}$

- Solution for $r = 1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{3a_k}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(4+3*x)*diff(y(x),x$2)-x*(4-3*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{3}{4}x+\frac{9}{16}x^2-\frac{27}{64}x^3+\frac{81}{256}x^4-\frac{243}{1024}x^5\right)x(c_2\ln(x)+c_1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*(4+3*x)*y''[x]-x*(4-3*x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1x\left(-\frac{243x^5}{1024}+\frac{81x^4}{256}-\frac{27x^3}{64}+\frac{9x^2}{16}-\frac{3x}{4}+1\right)+c_2x\left(-\frac{243x^5}{1024}+\frac{81x^4}{256}-\frac{27x^3}{64}+\frac{9x^2}{16}-\frac{3x}{4}+1\right)\log(x)$$

15.62 problem 63

Internal problem ID [1410]
Internal file name [OUTPUT/1411_Sunday_June_05_2022_02_15_40_AM_61161214/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 63.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+3x+1)\,y''+8x^2(2x+3)\,y'+(9x^2+3x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+12x^3+4x^2)\,y''+(16x^3+24x^2)\,y'+(9x^2+3x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{6+4x}{x^2+3x+1}\\ q(x)&=\frac{9x^2+3x+1}{4x^2\,(x^2+3x+1)}\end{aligned}$$

Table 766: Table $p(x), q(x)$ singularites.

$p(x) = \frac{6+4x}{x^2+3x+1}$	
singularity	type
$x = -\frac{3}{2} - \frac{\sqrt{5}}{2}$	"regular"
$x = \frac{\sqrt{5}}{2} - \frac{3}{2}$	"regular"

$q(x) = \frac{9x^2+3x+1}{4x^2(x^2+3x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{3}{2} - \frac{\sqrt{5}}{2}$	"regular"
$x = \frac{\sqrt{5}}{2} - \frac{3}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[-\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2} - \frac{3}{2}, 0, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x^2+3x+1)\,y'' + (16x^3+24x^2)\,y' + (9x^2+3x+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x^2+3x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (16x^3+24x^2)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (9x^2+3x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) (n+r-1)\right)+\left(\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r) (n+r-1)\right) \\
+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r) (n+r-1)\right)+\left(\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)\right) \\
+\left(\sum_{n=0}^{\infty} 24x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n\right) \\
+\left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r) (n+r-1)=\sum_{n=2}^{\infty} 4a_{n-2} (n+r-2) (n-3+r) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 12x^{1+n+r} a_n (n+r) (n+r-1)=\sum_{n=1}^{\infty} 12a_{n-1} (n+r-1) (n+r-2) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 16x^{n+r+2} a_n (n+r)=\sum_{n=2}^{\infty} 16a_{n-2} (n+r-2) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 24x^{1+n+r} a_n (n+r)=\sum_{n=1}^{\infty} 24a_{n-1} (n+r-1) x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 9x^{n+r+2} a_n=\sum_{n=2}^{\infty} 9a_{n-2} x^{n+r}
$$

$$
\sum_{n=0}^{\infty} 3x^{1+n+r} a_n=\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 12a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 16a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 24a_{n-1}(n+r-1)\,x^{n+r}\right)+\left(\sum_{n=2}^{\infty} 9a_{n-2}x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 3a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)\,(n+r-1)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r)+a_0 x^r=0$$

Or

$$(4x^r r(-1+r)+x^r)\,a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$(2r-1)^2\,x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{2}$$
$$r_2=\frac{1}{2}$$

Since $a_0\neq 0$ then the indicial equation becomes

$$(2r-1)^2\,x^r=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -3$$

For $2 \leq n$ the recursive equation is

$$4a_{n-2}(n+r-2)(n-3+r) + 12a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) + 16a_{n-2}(n+r-2) + 24a_{n-1}(n+r-1) + 9a_{n-2} + 3a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} - 3a_{n-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -a_{n-2} - 3a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = 8$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8

For $n = 3$, using the above recursive equation gives

$$a_3 = -21$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -21$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21

For $n = 4$, using the above recursive equation gives

$$a_4 = 55$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 55$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21
a_4	55	55

For $n = 5$, using the above recursive equation gives

$$a_5 = -144$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -144$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-3	-3
a_2	8	8
a_3	-21	-21
a_4	55	55
a_5	-144	-144

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{2}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	-3	-3	0	0
b_2	8	8	0	0
b_3	-21	-21	0	0
b_4	55	55	0	0
b_5	-144	-144	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - 3x + 8x^2 - 21x^3 + 55x^4 - 144x^5 + O(x^6)\right)\ln(x) + \sqrt{x}\, O(x^6)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1-3x+8x^2-21x^3+55x^4-144x^5+O(x^6)\right) \\
& + c_2\left(\sqrt{x}\left(1-3x+8x^2-21x^3+55x^4-144x^5+O(x^6)\right)\ln(x)+\sqrt{x}\,O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1-3x+8x^2-21x^3+55x^4-144x^5+O(x^6)\right) \\
& + c_2\left(\sqrt{x}\left(1-3x+8x^2-21x^3+55x^4-144x^5+O(x^6)\right)\ln(x)+\sqrt{x}\,O(x^6)\right)
\end{aligned}
$$

Verified OK.

15.62.1 Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\,y''+(16x^3+24x^2)\,y'+(9x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)}-\frac{2(2x+3)y'}{x^2+3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(2x+3)y'}{x^2+3x+1}+\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(2x+3)}{x^2+3x+1}, P_3(x)=\frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\, y''+8x^2(2x+3)\, y'+(9x^2+3x+1)\, y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m=2..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m=2..4$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2 x^r+\left(a_1(1+2r)^2+3a_0(1+2r)^2\right) x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2+3a_{k-1}(2k+2r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2=0$

- Values of r that satisfy the indicial equation

  $r=\frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2+3a_0(1+2r)^2=0$

- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)+8*x^2*(3+2*x)*diff(y(x),x)+(1+3*x+9*x^2)*y(x)=0,y(x)
```

$$y(x) = \sqrt{x}\,(-144x^5 + 55x^4 - 21x^3 + 8x^2 - 3x + 1)\,(c_2 \ln(x) + c_1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 72

```
AsymptoticDSolveValue[4*x^2*(1+3*x+x^2)*y''[x]+8*x^2*(3+2*x)*y'[x]+(1+3*x+9*x^2)*y[x]==0,y[x
```

$$y(x) \to c_1\sqrt{x}(-144x^5 + 55x^4 - 21x^3 + 8x^2 - 3x + 1) + c_2\sqrt{x}(-144x^5 + 55x^4 - 21x^3 + 8x^2 - 3x + 1)\log(x)$$

15.63 problem 64

Internal problem ID [1411]
Internal file name [`OUTPUT/1412_Sunday_June_05_2022_02_15_42_AM_81537289/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 64.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)^2 y'' - x\left(-3x^2+2x+1\right) y' + y\left(x^2+1\right) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4-2x^3+x^2\right) y'' + \left(3x^3-2x^2-x\right) y' + y\left(x^2+1\right) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x+1}{x\,(x-1)}$$
$$q(x) = \frac{x^2+1}{x^2\,(x-1)^2}$$

Table 768: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{x^2+1}{x^2(x-1)^2}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 (x^2 - 2x + 1) y'' + (3x^3 - 2x^2 - x) y' + y(x^2 + 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 (x^2 - 2x + 1) \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (3x^3 - 2x^2 - x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x^2 + 1) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 3 x^{n+r+2} a_n(n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)(n+r-1)\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} 3 x^{n+r+2} a_n(n+r) &= \sum_{n=2}^{\infty} 3 a_{n-2}(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty}\left(-2 x^{1+n+r} a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-2 a_{n-1}(n+r-1)\, x^{n+r}\right) \\
\sum_{n=0}^{\infty} x^{n+r+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2}(n+r-2)\,x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\,x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r (-1+r)^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r (-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 2$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) - 2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ &+ 3a_{n-2}(n+r-2) - 2a_{n-1}(n+r-1) - a_n(n+r) + a_{n-2} + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} + 2a_{n-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -a_{n-2} + 2a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	2	2

For $n = 2$, using the above recursive equation gives

$$a_2 = 3$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	2	2
a_2	3	3

For $n = 3$, using the above recursive equation gives

$$a_3 = 4$$

Which for the root $r = 1$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	2	2
a_2	3	3
a_3	4	4

For $n = 4$, using the above recursive equation gives

$$a_4 = 5$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	2	2
a_2	3	3
a_3	4	4
a_4	5	5

For $n = 5$, using the above recursive equation gives

$$a_5 = 6$$

Which for the root $r = 1$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	2	2
a_2	3	3
a_3	4	4
a_4	5	5
a_5	6	6

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	2	2	0	0
b_2	3	3	0	0
b_3	4	4	0	0
b_4	5	5	0	0
b_5	6	6	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \ln(x) + x O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \\ &\quad + c_2\left(x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \\ &\quad + c_2\left(x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \\ &\quad + c_2\left(x\left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x\left(6x^5+5x^4+4x^3+3x^2+2x+1+O\left(x^6\right)\right) + c_2\left(x\left(6x^5+5x^4+4x^3+3x^2+2x+1+O\left(x^6\right)\right)\ln\left(x\right)+xO\left(x^6\right)\right)$$

Verified OK.

15.63.1 Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\,y''+(3x^3-2x^2-x)\,y'+y(x^2+1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2+1)y}{x^2(x^2-2x+1)}-\frac{y'(3x+1)}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'(3x+1)}{x(x-1)}+\frac{(x^2+1)y}{x^2(x^2-2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3x+1}{x(x-1)}, P_3(x)=\frac{x^2+1}{x^2(x^2-2x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $(x-1)\,y''x^2(x^2-2x+1)+(3x+1)\,x(x^2-2x+1)\,y'+(x^2+1)\,(x-1)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..3$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..4$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- > k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..5$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- > k + 2 - m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-1+r)^2 x^r + \left(-r^2 a_1 + a_0(3r^2 - 2r + 1)\right) x^{1+r} + \left(-a_2(1+r)^2 + a_1(3r^2 + 4r + 2) - a_0(3r^2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- The coefficients of each power of x must be 0

  $$\left[-r^2 a_1 + a_0(3r^2 - 2r + 1) = 0, -a_2(1+r)^2 + a_1(3r^2 + 4r + 2) - a_0(3r^2 + 2r + 1) = 0\right]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a_0(3r^2 - 2r + 1)}{r^2}, a_2 = \frac{2a_0(3r^4 + 2r^3 + 1)}{r^2(r^2 + 2r + 1)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-1)^2 + a_{k-3}(k-2+r)^2 + (-3k^2 + (-6r+10)k - 3r^2 + 10r - 9)a_{k-2} + 3(k^2 + (2r$

- Shift index using $k- >k+3$

  $-a_{k+3}(k+2+r)^2 + a_k(k+r+1)^2 + (-3(k+3)^2 + (-6r+10)(k+3) - 3r^2 + 10r - 9)a_{k+1}$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = \frac{k^2a_k - 3k^2a_{k+1} + 3k^2a_{k+2} + 2kra_k - 6kra_{k+1} + 6kra_{k+2} + r^2a_k - 3r^2a_{k+1} + 3r^2a_{k+2} + 2ka_k - 8ka_{k+1} + 10ka_{k+2} + 2ra_k - 8ra_k}{(k+2+r)^2}$

- Recursion relation for $r = 1$

  $a_{k+3} = \frac{k^2a_k - 3k^2a_{k+1} + 3k^2a_{k+2} + 4ka_k - 14ka_{k+1} + 16ka_{k+2} + 4a_k - 17a_{k+1} + 22a_{k+2}}{(k+3)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+3} = \frac{k^2a_k - 3k^2a_{k+1} + 3k^2a_{k+2} + 4ka_k - 14ka_{k+1} + 16ka_{k+2} + 4a_k - 17a_{k+1} + 22a_{k+2}}{(k+3)^2}, a_1 = 2a_0, a_2 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1-x)^2*diff(y(x),x$2)-x*(1+2*x-3*x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x),type='ser
```

$$y(x) = (6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1)\,x(c_2\ln(x) + c_1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*(1-x)^2*y''[x]-x*(1+2*x-3*x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) + c_2 x \left(6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1\right) \log(x)$$

15.64 problem 65

Internal problem ID [1412]
Internal file name [OUTPUT/1413_Sunday_June_05_2022_02_15_45_AM_32072313/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS II. Exercises 7.6. Page 374
Problem number: 65.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x^2+x+1)\,y''+3x(13x^2+7x+1)\,y'+(25x^2+4x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(9x^4+9x^3+9x^2)\,y''+(39x^3+21x^2+3x)\,y'+(25x^2+4x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{13x^2+7x+1}{3\,(x^2+x+1)\,x}$$
$$q(x)=\frac{25x^2+4x+1}{9x^2\,(x^2+x+1)}$$

Table 770: Table $p(x), q(x)$ singularites.

$p(x) = \frac{13x^2+7x+1}{3(x^2+x+1)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

$q(x) = \frac{25x^2+4x+1}{9x^2(x^2+x+1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	"regular"
$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2\left(x^2+x+1\right)y'' + \left(39x^3+21x^2+3x\right)y' + \left(25x^2+4x+1\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2\left(x^2+x+1\right)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(39x^3+21x^2+3x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(25x^2+4x+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 39x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 25x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 9a_{n-2}(n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 39x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 39a_{n-2}(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 21a_{n-1}(n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} 25x^{n+r+2} a_n = \sum_{n=2}^{\infty} 25a_{n-2} x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 9a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 39a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 21a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 25a_{n-2}x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1}x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n(n+r)(n+r-1) + 3x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(9x^r r(-1+r) + 3x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(3r-1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(3r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(3r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{3}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -1$$

For $2 \leq n$ the recursive equation is

$$9a_{n-2}(n+r-2)(n-3+r)+9a_{n-1}(n+r-1)(n+r-2)+9a_n(n+r)(n+r-1) \\ +39a_{n-2}(n+r-2)+21a_{n-1}(n+r-1)+3a_n(n+r)+25a_{n-2}+4a_{n-1}+a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-2} - a_{n-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -a_{n-2} - a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 1$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	0	0
a_3	1	1

For $n = 4$, using the above recursive equation gives

$$a_4 = -1$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	0	0
a_3	1	1
a_4	-1	-1

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	0	0
a_3	1	1
a_4	-1	-1
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(-x^4 + x^3 - x + 1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{3}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n\left(r = \frac{1}{3}\right)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	-1	-1	0	0
b_2	0	0	0	0
b_3	1	1	0	0
b_4	-1	-1	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\ &= x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) + c_2 \left(x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) + c_2 \left(x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) \\ & + c_2 \left(x^{\frac{1}{3}} \left(-x^4 + x^3 - x + 1 + O(x^6)\right) \ln(x) + x^{\frac{1}{3}} O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \left(-x^4+x^3-x+1+O(x^6)\right)+c_2 \left(x^{\frac{1}{3}} \left(-x^4+x^3-x+1+O(x^6)\right) \ln(x)+x^{\frac{1}{3}} O(x^6)\right)$$

Verified OK.

15.64.1 Maple step by step solution

Let's solve

$9x^2(x^2+x+1)\, y''+(39x^3+21x^2+3x)\, y'+(25x^2+4x+1)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)}-\frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}+\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{13x^2+7x+1}{3(x^2+x+1)x}, P_3(x)=\frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\bigg|_{x=0}=\frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0}=\frac{1}{9}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9x^2(x^2+x+1)\, y''+3x(13x^2+7x+1)\, y'+(25x^2+4x+1)\, y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$
- Each term must be 0

  $a_1(2+3r)^2 + a_0(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -a_0$
- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$

- Shift index using $k->k+2$

  $$\left(3k+3r+5\right)^{2} \left(a_{k+2}+a_{k+1}+a_{k}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -a_{k+1}-a_{k}$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+2} = -a_{k+1}-a_{k}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1}-a_{k}, a_{1} = -a_{0}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(9*x^2*(1+x+x^2)*diff(y(x),x$2)+3*x*(1+7*x+13*x^2)*diff(y(x),x)+(1+4*x+25*x^2)*y(x)=0,
```

$$y(x) = x^{\frac{1}{3}} \left(-x^{4}+x^{3}-x+1\right) \left(c_{2} \ln \left(x\right)+c_{1}\right)+O\left(x^{6}\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 48

```
AsymptoticDSolveValue[9*x^2*(1+x+x^2)*y''[x]+3*x*(1+7*x+13*x^2)*y'[x]+(1+4*x+25*x^2)*y[x]==0
```

$$y(x) \to c_1 \sqrt[3]{x}(-x^4 + x^3 - x + 1) + c_2 \sqrt[3]{x}(-x^4 + x^3 - x + 1) \log(x)$$

16 Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389

16.1 problem Example 7.7.1 page 381

Internal problem ID [1413]
Internal file name [`OUTPUT/1414_Sunday_June_05_2022_02_15_47_AM_30723664/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: Example 7.7.1 page 381.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(2+x)\,y'' - x(4-7x)\,y' - (5-3x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+4x^2\right)y'' + \left(7x^2-4x\right)y' + (3x-5)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{7x-4}{2x\,(2+x)}$$
$$q(x) = \frac{3x-5}{2x^2\,(2+x)}$$

Table 772: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7x-4}{2x(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = \frac{3x-5}{2x^2(2+x)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(2+x)\, y'' + \left(7x^2 - 4x\right) y' + (3x-5)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(2+x) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(7x^2 - 4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x-5) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 7x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 7a_{n-1} (n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} 3x^{1+n+r} a_n &= \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 7a_{n-1} (n+r-1)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) - 5a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1 + r) - 4x^r a_0 r - 5a_0 x^r = 0$$

Or

$$(4x^r r(-1 + r) - 4x^r r - 5x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 8r - 5\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 8r - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 8r - 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{5}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n + \frac{5}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n - \frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + 7a_{n-1}(n+r-1) - 4a_n(n+r) + 3a_{n-1} - 5a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\, a_{n-1}}{2n+2r-5} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = -\frac{(2n+5)\, a_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{-3+2r}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_1 = -\frac{7}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{-3+2r}$	$-\frac{7}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{4r^2-8r+3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = \frac{63}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{-3+2r}$	$-\frac{7}{4}$
a_2	$\frac{r^2+3r+2}{4r^2-8r+3}$	$\frac{63}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 - 12r^2 - 2r + 3}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_3 = -\frac{231}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{-3+2r}$	$-\frac{7}{4}$
a_2	$\frac{r^2+3r+2}{4r^2-8r+3}$	$\frac{63}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3-12r^2-2r+3}$	$-\frac{231}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{16r^4 - 40r^2 + 9}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{3003}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{-3+2r}$	$-\frac{7}{4}$
a_2	$\frac{r^2+3r+2}{4r^2-8r+3}$	$\frac{63}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3-12r^2-2r+3}$	$-\frac{231}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4-40r^2+9}$	$\frac{3003}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{(16r^4 - 40r^2 + 9)(5 + 2r)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_5 = -\frac{9009}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{-3+2r}$	$-\frac{7}{4}$
a_2	$\frac{r^2+3r+2}{4r^2-8r+3}$	$\frac{63}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3-12r^2-2r+3}$	$-\frac{231}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4-40r^2+9}$	$\frac{3003}{2048}$
a_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{(16r^4-40r^2+9)(5+2r)}$	$-\frac{9009}{8192}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{5}{2}}\left(1 - \frac{7x}{4} + \frac{63x^2}{32} - \frac{231x^3}{128} + \frac{3003x^4}{2048} - \frac{9009x^5}{8192} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 - 12r^2 - 2r + 3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 - 12r^2 - 2r + 3} &= \lim_{r\to -\frac{1}{2}} \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 - 12r^2 - 2r + 3} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $2x^2(2 + x)\, y'' + (7x^2 - 4x)\, y' + (3x - 5)\, y = 0$

gives

$$
\begin{aligned}
&2x^2(2+x)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(7x^2-4x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(3x-5)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(2x^2(2+x)\,y_1''(x)+\left(7x^2-4x\right)y_1'(x)+(3x-5)\,y_1(x)\right)\ln(x)\right.\\
&\left.+2x^2(2+x)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(7x^2-4x\right)y_1(x)}{x}\right)C\\
&+2x^2(2+x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(7x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(3x-5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
2x^2(2+x)\,y_1''(x)+\left(7x^2-4x\right)y_1'(x)+(3x-5)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(2x^2(2+x)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(7x^2-4x\right)y_1(x)}{x}\right)C\\
&+2x^2(2+x)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(7x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(3x-5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(4x(2+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (5x-8)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\ + 2\left(x^3+2x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \\ + \left(7x^2-4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (3x-5)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = \frac{5}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$\left(4x(2+x)\left(\sum_{n=0}^{\infty} x^{\frac{3}{2}+n} a_n \left(n+\frac{5}{2}\right)\right) + (5x-8)\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}\right)\right) C \\ + 2\left(x^3+2x^2\right)\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right) \\ + \left(7x^2-4x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n \left(n-\frac{1}{2}\right)\right) + (3x-5)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right) = 0 \tag{10}$$

Expanding $4C\,x^{\frac{5}{2}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned} 4C\,x^{\frac{5}{2}} &= 4C\,x^{\frac{5}{2}} + \dots \\ &= 4C\,x^{\frac{5}{2}} \end{aligned}$$

Expanding $5C\,x^{\frac{7}{2}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned} 5C\,x^{\frac{7}{2}} &= 5C\,x^{\frac{7}{2}} + \dots \\ &= 5C\,x^{\frac{7}{2}} \end{aligned}$$

Expanding $-8C\,x^{\frac{5}{2}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned} -8C\,x^{\frac{5}{2}} &= -8C\,x^{\frac{5}{2}} + \dots \\ &= -8C\,x^{\frac{5}{2}} \end{aligned}$$

Expanding $\frac{\sqrt{x}}{2}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{\sqrt{x}}{2} &= \frac{\sqrt{x}}{2} + \dots \\ &= \frac{\sqrt{x}}{2}\end{aligned}$$

Expanding $\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}} + \dots \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

Expanding $\frac{7\sqrt{x}}{2}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{7\sqrt{x}}{2} &= \frac{7\sqrt{x}}{2} + \dots \\ &= \frac{7\sqrt{x}}{2}\end{aligned}$$

Expanding $-\frac{2}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{2}{\sqrt{x}} &= -\frac{2}{\sqrt{x}} + \dots \\ &= -\frac{2}{\sqrt{x}}\end{aligned}$$

Expanding $3\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}3\sqrt{x} &= 3\sqrt{x} + \dots \\ &= 3\sqrt{x}\end{aligned}$$

Expanding $-\frac{5}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{5}{\sqrt{x}} &= -\frac{5}{\sqrt{x}} + \dots \\ &= -\frac{5}{\sqrt{x}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} (8n+20)\, C a_n x^{n+\frac{5}{2}}\right) + \left(\sum_{n=0}^{\infty} (4n+10)\, C a_n x^{\frac{7}{2}+n}\right) \\
&+ \left(\sum_{n=0}^{\infty} 5C\, x^{\frac{7}{2}+n} a_n\right) + \sum_{n=0}^{\infty} \left(-8C\, x^{n+\frac{5}{2}} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}} b_n (4n^2-8n+3)}{2}\right) + \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n (4n^2-8n+3)\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{7x^{n+\frac{1}{2}} b_n (2n-1)}{2}\right) + \left(\sum_{n=0}^{\infty} (-4n+2)\, b_n x^{n-\frac{1}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{n+\frac{1}{2}} b_n\right) + \sum_{n=0}^{\infty} \left(-5b_n x^{n-\frac{1}{2}}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-\frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} (8n+20)\, C a_n x^{n+\frac{5}{2}} = \sum_{n=3}^{\infty} C a_{n-3} (8n-4)\, x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} (4n+10)\, C a_n x^{\frac{7}{2}+n} = \sum_{n=4}^{\infty} C a_{n-4} (4n-6)\, x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} 5C\, x^{\frac{7}{2}+n} a_n = \sum_{n=4}^{\infty} 5C a_{n-4} x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \left(-8C\, x^{n+\frac{5}{2}} a_n\right) = \sum_{n=3}^{\infty} \left(-8C a_{n-3} x^{n-\frac{1}{2}}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}} b_n (4n^2-8n+3)}{2} = \sum_{n=1}^{\infty} \frac{b_{n-1} \left(4(n-1)^2-8n+11\right) x^{n-\frac{1}{2}}}{2}$$

$$\sum_{n=0}^{\infty} \frac{7x^{n+\frac{1}{2}} b_n (2n-1)}{2} = \sum_{n=1}^{\infty} \frac{7b_{n-1}(-3+2n)\, x^{n-\frac{1}{2}}}{2}$$

$$\sum_{n=0}^{\infty} 3x^{n+\frac{1}{2}} b_n = \sum_{n=1}^{\infty} 3b_{n-1} x^{n-\frac{1}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$\begin{aligned}
&\left(\sum_{n=3}^{\infty} C a_{n-3}(8n-4)\, x^{n-\frac{1}{2}}\right)+\left(\sum_{n=4}^{\infty} C a_{n-4}(4n-6)\, x^{n-\frac{1}{2}}\right)\\
&+\left(\sum_{n=4}^{\infty} 5 C a_{n-4} x^{n-\frac{1}{2}}\right)+\sum_{n=3}^{\infty}\left(-8 C a_{n-3} x^{n-\frac{1}{2}}\right)\\
&\quad+\left(\sum_{n=1}^{\infty} \frac{b_{n-1}\left(4(n-1)^{2}-8n+11\right) x^{n-\frac{1}{2}}}{2}\right)\\
&\quad+\left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_{n}\left(4n^{2}-8n+3\right)\right)+\left(\sum_{n=1}^{\infty} \frac{7 b_{n-1}(-3+2n)\, x^{n-\frac{1}{2}}}{2}\right)\\
&\quad+\left(\sum_{n=0}^{\infty}(-4n+2)\, b_{n} x^{n-\frac{1}{2}}\right)+\left(\sum_{n=1}^{\infty} 3 b_{n-1} x^{n-\frac{1}{2}}\right)+\sum_{n=0}^{\infty}\left(-5 b_{n} x^{n-\frac{1}{2}}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$b_0 - 8b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 8b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{8}$$

For $n=2$, Eq (2B) gives

$$6b_1 - 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{3}{4} - 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{32}$$

For $n=N$, where $N=3$ which is the difference between the two roots, we are free to choose $b_3=0$. Hence for $n=3$, Eq (2B) gives

$$12C + \frac{45}{32} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{15}{128}$$

For $n = 4$, Eq (2B) gives

$$(15a_0 + 20a_1) C + 28b_3 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{75}{32} + 16b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{75}{512}$$

For $n = 5$, Eq (2B) gives

$$(19a_1 + 28a_2) C + 45b_4 + 40b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{9375}{1024} + 40b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1875}{8192}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{15}{128}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{15}{128}\left(x^{\frac{5}{2}}\left(1 - \frac{7x}{4} + \frac{63x^2}{32} - \frac{231x^3}{128} + \frac{3003x^4}{2048} - \frac{9009x^5}{8192} + O(x^6)\right)\right) \ln(x) + \frac{1 + \frac{x}{8} + \frac{3x^2}{32} - \frac{75x^4}{512} + \frac{1875x^5}{8192} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{2}} \left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{15}{128}\left(x^{\frac{5}{2}}\left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right)\right)\ln(x) \right. \\
&\quad \left. + \frac{1+\frac{x}{8}+\frac{3x^2}{32}-\frac{75x^4}{512}+\frac{1875x^5}{8192}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{2}} \left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{15x^{\frac{5}{2}}\left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right)\ln(x)}{128} \right. \\
&\quad \left. + \frac{1+\frac{x}{8}+\frac{3x^2}{32}-\frac{75x^4}{512}+\frac{1875x^5}{8192}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{5}{2}} \left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{15x^{\frac{5}{2}}\left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right)\ln(x)}{128} \right. \\
&\quad \left. + \frac{1+\frac{x}{8}+\frac{3x^2}{32}-\frac{75x^4}{512}+\frac{1875x^5}{8192}+O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{5}{2}} \left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right) + c_2 \left(-\frac{15x^{\frac{5}{2}} \left(1-\frac{7x}{4}+\frac{63x^2}{32}-\frac{231x^3}{128}+\frac{3003x^4}{2048}-\frac{9009x^5}{8192}+O(x^6)\right) \ln(x)}{128} + \frac{1+\frac{x}{8}+\frac{3x^2}{32}-\frac{75x^4}{512}+\frac{1875x^5}{8192}+O(x^6)}{\sqrt{x}}\right)$$

Verified OK.

16.1.1 Maple step by step solution

Let's solve

$2x^2(2+x)\,y''+(7x^2-4x)\,y'+(3x-5)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x-5)y}{2x^2(2+x)}-\frac{(7x-4)y'}{2x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x-4)y'}{2x(2+x)}+\frac{(3x-5)y}{2x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{7x-4}{2x(2+x)}, P_3(x)=\frac{3x-5}{2x^2(2+x)}\right]$

  ○ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{9}{2}$

  ○ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = -2$

- Multiply by denominators

$$2x^2(2+x)\,y'' + x(7x-4)\,y' + (3x-5)\,y = 0$$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

$$(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (7u^2 - 32u + 36)\left(\frac{d}{du}y(u)\right) + (3u-11)\,y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(7+2r)\,u^{-1+r} + \left(4a_1(1+r)(9+2r) - a_0(8r^2+24r+11)\right)u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+r+1)(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(7+2r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{7}{2}\right\}$

- Each term must be 0

  $4a_1(1+r)(9+2r) - a_0(8r^2+24r+11) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(-4a_k + a_{k-1} + 4a_{k+1})k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})r - 24a_k + a_{k-1} + 44a_{k+1})k + 2(-4a_k +$

- Shift index using $k- >k+1$

  $2(-4a_{k+1} + a_k + 4a_{k+2})(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})r - 24a_{k+1} + a_k + 44a_{k+2})(k+1)$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+4kr+2r^2+15k+15r+22)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$

- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$

- Recursion relation for $r = -\frac{7}{2}$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}$

- Solution for $r = -\frac{7}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$

- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2a_k - 8k^2a_{1+k} + 5ka_k - 40ka_{1+k} + 3a_k - 43a_{1+k}}{4(2k^2+15k+22)}, 36a\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
Order:=6;
dsolve(2*x^2*(2+x)*diff(y(x),x$2)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \frac{c_1 x^3 \left(1 - \frac{7}{4}x + \frac{63}{32}x^2 - \frac{231}{128}x^3 + \frac{3003}{2048}x^4 - \frac{9009}{8192}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-\frac{45}{32}x^3 + \frac{315}{128}x^4 - \frac{2835}{1024}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 98

```
AsymptoticDSolveValue[2*x^2*(2+x)*y''[x]-x*(4-7*x)*y'[x]-(5-3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{3003x^{13/2}}{2048} - \frac{231x^{11/2}}{128} + \frac{63x^{9/2}}{32} - \frac{7x^{7/2}}{4} + x^{5/2}\right) + c_1 \left(\frac{15}{512}(7x-4)x^{5/2}\log(x) + \frac{809x^4 - 548x^3 + 96x^2 + 128x + 1024}{1024\sqrt{x}}\right)$$

16.2 problem Example 7.7.2 page 383

Internal problem ID [1414]
Internal file name [`OUTPUT/1415_Sunday_June_05_2022_02_15_52_AM_30298835/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: Example 7.7.2 page 383.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{9x-8}{x\,(2x-1)}$$
$$q(x)=\frac{3x-6}{x^2\,(2x-1)}$$

Table 774: Table $p(x), q(x)$ singularites.

$p(x) = \frac{9x-8}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = \frac{3x-6}{x^2(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x-1) + \left(-9x^2+8x\right)y' + (6-3x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(2x-1) + \left(-9x^2+8x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (6-3x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-9x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r) (n+r-1)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-9x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-9a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-2a_{n-1} (n+r-1) (n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-9a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + 8x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 8x^r a_0 r + 6a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 8x^r r + 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 7r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 7r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -6$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 7r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^6}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-6}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ - 9a_{n-1}(n+r-1) + 8a_n(n+r) + 6a_n - 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n^2 + 4nr + 2r^2 + 3n + 3r - 2)}{n^2 + 2nr + r^2 + 7n + 7r + 6} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = \frac{a_{n-1}(2n^2 - n - 3)}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r^2 + 7r + 3}{r^2 + 9r + 14}$$

Which for the root $r = -1$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^3 + 24r^2 + 35r + 12}{(r+8)(r+7)(r+2)}$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{1}{3}$
a_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^4 + 76r^3 + 226r^2 + 245r + 75}{(r+9)(r+2)(r+7)(r+8)}$$

Which for the root $r = -1$ becomes

$$a_3 = -\frac{1}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{1}{3}$
a_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$-\frac{1}{14}$
a_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{1}{28}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^5 + 224r^4 + 1112r^3 + 2416r^2 + 2217r + 630}{(r+10)(r+9)(r+2)(r+7)(r+8)}$$

Which for the root $r = -1$ becomes

$$a_4 = -\frac{25}{1008}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{1}{3}$
a_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$-\frac{1}{14}$
a_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{1}{28}$
a_4	$\frac{16r^5+224r^4+1112r^3+2416r^2+2217r+630}{(r+10)(r+9)(r+2)(r+7)(r+8)}$	$-\frac{25}{1008}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{(r+11)(r+8)(r+2)(r+9)(r+10)}$$

Which for the root $r = -1$ becomes

$$a_5 = -\frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{1}{3}$
a_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$-\frac{1}{14}$
a_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{1}{28}$
a_4	$\frac{16r^5+224r^4+1112r^3+2416r^2+2217r+630}{(r+10)(r+9)(r+2)(r+7)(r+8)}$	$-\frac{25}{1008}$
a_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{(r+11)(r+8)(r+2)(r+9)(r+10)}$	$-\frac{1}{48}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \frac{1 - \frac{x}{3} - \frac{x^2}{14} - \frac{x^3}{28} - \frac{25x^4}{1008} - \frac{x^5}{48} + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{(r+11)(r+8)(r+2)(r+9)(r+10)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{(r+11)(r+8)(r+2)(r+9)(r+10)} &= \lim_{r\to -6} \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{(r+11)(r+8)(r+2)(r+9)(r+10)} \\ &= \frac{693}{32} \end{aligned}$$

The limit is $\frac{693}{32}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-6} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ &- 9b_{n-1}(n+r-1) + 8b_n(n+r) + 6b_n - 3b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -6$ becomes

$$-2b_{n-1}(n-7)(n-8) + b_n(n-6)(n-7) - 9b_{n-1}(n-7) + 8b_n(n-6) + 6b_n - 3b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(2n^2 + 4nr + 2r^2 + 3n + 3r - 2)}{n^2 + 2nr + r^2 + 7n + 7r + 6} \tag{5}$$

Which for the root $r = -6$ becomes

$$b_n = \frac{b_{n-1}(2n^2 - 21n + 52)}{n^2 - 5n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -6$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2r^2 + 7r + 3}{r^2 + 9r + 14}$$

Which for the root $r = -6$ becomes

$$b_1 = -\frac{33}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{33}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^3 + 24r^2 + 35r + 12}{(r + 8)(r^2 + 9r + 14)}$$

Which for the root $r = -6$ becomes

$$b_2 = \frac{99}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{33}{4}$
b_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$\frac{99}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^4 + 76r^3 + 226r^2 + 245r + 75}{(r + 9)(r + 8)(r^2 + 9r + 14)}$$

Which for the root $r = -6$ becomes

$$b_3 = -\frac{231}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{33}{4}$
b_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$\frac{99}{4}$
b_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{231}{8}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^5 + 224r^4 + 1112r^3 + 2416r^2 + 2217r + 630}{(r + 10)(r + 9)(r + 8)(r^2 + 9r + 14)}$$

Which for the root $r = -6$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{33}{4}$
b_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$\frac{99}{4}$
b_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{231}{8}$
b_4	$\frac{16r^5+224r^4+1112r^3+2416r^2+2217r+630}{(r+10)(r+9)(r+2)(r+7)(r+8)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{(r + 11)(r + 8)(r + 2)(r + 9)(r + 10)}$$

Which for the root $r = -6$ becomes

$$b_5 = \frac{693}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2r^2+7r+3}{r^2+9r+14}$	$-\frac{33}{4}$
b_2	$\frac{4r^3+24r^2+35r+12}{(r+8)(r+7)(r+2)}$	$\frac{99}{4}$
b_3	$\frac{8r^4+76r^3+226r^2+245r+75}{(r+9)(r+2)(r+7)(r+8)}$	$-\frac{231}{8}$
b_4	$\frac{16r^5+224r^4+1112r^3+2416r^2+2217r+630}{(r+10)(r+9)(r+2)(r+7)(r+8)}$	0
b_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{(r+11)(r+8)(r+2)(r+9)(r+10)}$	$\frac{693}{32}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \frac{1}{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{33x}{4} + \frac{99x^2}{4} - \frac{231x^3}{8} + \frac{693x^5}{32} + O(x^6)}{x^6}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - \frac{x}{3} - \frac{x^2}{14} - \frac{x^3}{28} - \frac{25x^4}{1008} - \frac{x^5}{48} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{33x}{4} + \frac{99x^2}{4} - \frac{231x^3}{8} + \frac{693x^5}{32} + O(x^6)\right)}{x^6}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - \frac{x}{3} - \frac{x^2}{14} - \frac{x^3}{28} - \frac{25x^4}{1008} - \frac{x^5}{48} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{33x}{4} + \frac{99x^2}{4} - \frac{231x^3}{8} + \frac{693x^5}{32} + O(x^6)\right)}{x^6}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\left(1 - \frac{x}{3} - \frac{x^2}{14} - \frac{x^3}{28} - \frac{25x^4}{1008} - \frac{x^5}{48} + O(x^6)\right)}{x} \\
&\quad + \frac{c_2\left(1 - \frac{33x}{4} + \frac{99x^2}{4} - \frac{231x^3}{8} + \frac{693x^5}{32} + O(x^6)\right)}{x^6}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \left(1-\frac{x}{3}-\frac{x^2}{14}-\frac{x^3}{28}-\frac{25x^4}{1008}-\frac{x^5}{48}+O(x^6)\right)}{x} + \frac{c_2 \left(1-\frac{33x}{4}+\frac{99x^2}{4}-\frac{231x^3}{8}+\frac{693x^5}{32}+O(x^6)\right)}{x^6}$$

Verified OK.

16.2.1 Maple step by step solution

Let's solve

$-y''x^2(2x-1)+(-9x^2+8x)\,y'+(6-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3(-2+x)y}{x^2(2x-1)}-\frac{(9x-8)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(9x-8)y'}{x(2x-1)}+\frac{3(-2+x)y}{x^2(2x-1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{9x-8}{x(2x-1)}, P_3(x)=\frac{3(-2+x)}{x^2(2x-1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=8$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(9x-8)\,y'+(3x-6)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ◦ Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(6+r)\,(1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+6)\,(k+r+1) + a_{k-1}(k+2+r)\,(2k-1+2r)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(6+r)\,(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, -1\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+2+r)\left(k+r-\tfrac{1}{2}\right) a_{k-1} - a_k(k+r+6)\,(k+r+1) = 0$$

- Shift index using $k- >k + 1$

$$2(k+r+3)\left(k+\frac{1}{2}+r\right)a_k - a_{k+1}(k+7+r)(k+2+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$$

- Recursion relation for $r = -6$; series terminates at $k = 3$

$$a_{k+1} = \frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{33a_0}{4}$$

- Apply recursion relation for $k = 1$

$$a_2 = -3a_1$$

- Express in terms of a_0

$$a_2 = \frac{99a_0}{4}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{7a_2}{6}$$

- Express in terms of a_0

$$a_3 = -\frac{231a_0}{8}$$

- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second

$$y = a_0 \cdot \left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3\right)$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{1+k} = \frac{(k+2)(2k-1)b_k}{(k+6)(1+k)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = \frac{c_1\left(1 - \frac{1}{3}x - \frac{1}{14}x^2 - \frac{1}{28}x^3 - \frac{25}{1008}x^4 - \frac{1}{48}x^5 + \mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2(2880 - 23760x + 71280x^2 - 83160x^3 + 62370x^5 + \mathrm{O}\left(x^6\right))}{x^6}$$

✓ Solution by Mathematica
Time used: 0.055 (sec). Leaf size: 61

```
AsymptoticDSolveValue[x^2*(1-2*x)*y''[x]+x*(8-9*x)*y'[x]+(6-3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{25x^3}{1008} - \frac{x^2}{28} - \frac{x}{14} + \frac{1}{x} - \frac{1}{3}\right) + c_1\left(\frac{1}{x^6} - \frac{33}{4x^5} + \frac{99}{4x^4} - \frac{231}{8x^3}\right)$$

16.3 problem Example 7.7.3 page 385

Internal problem ID [1415]
Internal file name [OUTPUT/1416_Sunday_June_05_2022_02_15_55_AM_70064381/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: Example 7.7.3 page 385.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+x(10x^2+3)\,y'-(-14x^2+15)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(10x^3+3x)\,y'+(14x^2-15)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{10x^2+3}{x\,(x^2+1)}$$
$$q(x)=\frac{14x^2-15}{x^2\,(x^2+1)}$$

Table 776: Table $p(x), q(x)$ singularites.

$p(x) = \frac{10x^2+3}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{14x^2-15}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\, y'' + (10x^3+3x)\, y' + (14x^2-15)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (10x^3+3x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (14x^2-15)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 10 x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 14 x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-15 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r} \\
\sum_{n=0}^{\infty} 10 x^{n+r+2} a_n (n+r) &= \sum_{n=2}^{\infty} 10 a_{n-2} (n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 14 x^{n+r+2} a_n &= \sum_{n=2}^{\infty} 14 a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=2}^{\infty} 10 a_{n-2} (n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3 x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=2}^{\infty} 14 a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-15 a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + 3 x^{n+r} a_n (n+r) - 15 a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r - 15a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r - 15x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(5+r)\,(r-3)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(5+r)\,(r-3) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -5 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(5+r)\,(r-3)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 8$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^5} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-5} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r)+a_n(n+r)(n+r-1) \\ +10a_{n-2}(n+r-2)+3a_n(n+r)+14a_{n-2}-15a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\,a_{n-2}}{n-3+r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{(n+3)\,a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2-r}{-1+r}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{5}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2 + 6r + 8}{r^2 - 1}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{35}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{r^2-1}$	$\frac{35}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{r^2-1}$	$\frac{35}{8}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-r^3 - 12r^2 - 44r - 48}{(r^2 - 1)(3 + r)}$$

Which for the root $r = 3$ becomes

$$a_6 = -\frac{105}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{r^2-1}$	$\frac{35}{8}$
a_5	0	0
a_6	$\frac{-r^3-12r^2-44r-48}{(r^2-1)(3+r)}$	$-\frac{105}{16}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{r^2-1}$	$\frac{35}{8}$
a_5	0	0
a_6	$\frac{-r^3-12r^2-44r-48}{(r^2-1)(3+r)}$	$-\frac{105}{16}$
a_7	0	0

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{r^4 + 20r^3 + 140r^2 + 400r + 384}{(r^2 - 1)(3 + r)(5 + r)}$$

Which for the root $r = 3$ becomes

$$a_8 = \frac{1155}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-2-r}{-1+r}$	$-\frac{5}{2}$
a_3	0	0
a_4	$\frac{r^2+6r+8}{r^2-1}$	$\frac{35}{8}$
a_5	0	0
a_6	$\frac{-r^3-12r^2-44r-48}{(r^2-1)(3+r)}$	$-\frac{105}{16}$
a_7	0	0
a_8	$\frac{r^4+20r^3+140r^2+400r+384}{(r^2-1)(3+r)(5+r)}$	$\frac{1155}{128}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 \dots\right) \\
&= x^3\left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O\left(x^9\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 8$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_8(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_8 \\
&= \frac{r^4 + 20r^3 + 140r^2 + 400r + 384}{(r^2 - 1)(3 + r)(5 + r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{r^4 + 20r^3 + 140r^2 + 400r + 384}{(r^2 - 1)(3 + r)(5 + r)} &= \lim_{r\to -5} \frac{r^4 + 20r^3 + 140r^2 + 400r + 384}{(r^2 - 1)(3 + r)(5 + r)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n + r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n + r_2)(-1 + n + r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2(x^2+1)\,y''+(10x^3+3x)\,y'+(14x^2-15)\,y=0$ gives

$$x^2(x^2+1)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(10x^3+3x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(14x^2-15)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2(x^2+1)\,y_1''(x)+(10x^3+3x)\,y_1'(x)+(14x^2-15)\,y_1(x)\right)\ln(x)+x^2(x^2+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(10x^3+3x)\,y_1(x)}{x}\right)C+x^2(x^2+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(10x^3+3x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(14x^2-15)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2(x^2+1)\,y_1''(x)+(10x^3+3x)\,y_1'(x)+(14x^2-15)\,y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2(x^2+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(10x^3+3x)\,y_1(x)}{x}\right)C+x^2(x^2+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(10x^3+3x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(14x^2-15)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(2x\left(x^2+1\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)+\left(9x^2+2\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+\left(x^4+x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)\left(-1+n+r_2\right)\right) \\
&+\left(10x^3+3x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)+\left(14x^2-15\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned} \tag{9}
$$

Since $r_1 = 3$ and $r_2 = -5$ then the above becomes

$$
\begin{aligned}
&\left(2x\left(x^2+1\right)\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right)+\left(9x^2+2\right)\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C \\
&+\left(x^4+x^2\right)\left(\sum_{n=0}^{\infty} x^{-7+n} b_n (n-5)\left(-6+n\right)\right) \\
&+\left(10x^3+3x\right)\left(\sum_{n=0}^{\infty} x^{-6+n} b_n (n-5)\right)+\left(14x^2-15\right)\left(\sum_{n=0}^{\infty} b_n x^{n-5}\right)=0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+3)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right)+\left(\sum_{n=0}^{\infty} 9C\, x^{n+5} a_n\right) \\
&+\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n\right)+\left(\sum_{n=0}^{\infty} x^{n-3} b_n (-6+n)(n-5)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-5} b_n (-6+n)(n-5)\right)+\left(\sum_{n=0}^{\infty} 10x^{n-3} b_n (n-5)\right) \\
&+\left(\sum_{n=0}^{\infty} 3x^{n-5} b_n (n-5)\right)+\left(\sum_{n=0}^{\infty} 14x^{n-3} b_n\right)+\sum_{n=0}^{\infty}\left(-15 b_n x^{n-5}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-5$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-5} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+3) = \sum_{n=10}^{\infty} 2C a_{n-10} (-7+n)\, x^{n-5}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=8}^{\infty} 2C a_{n-8} (n-5)\, x^{n-5}$$

$$\sum_{n=0}^{\infty} 9C\, x^{n+5} a_n = \sum_{n=10}^{\infty} 9C a_{n-10} x^{n-5}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n = \sum_{n=8}^{\infty} 2C a_{n-8} x^{n-5}$$

$$\sum_{n=0}^{\infty} x^{n-3} b_n (-6+n)\,(n-5) = \sum_{n=2}^{\infty} b_{n-2} (n-8)\,(-7+n)\, x^{n-5}$$

$$\sum_{n=0}^{\infty} 10 x^{n-3} b_n (n-5) = \sum_{n=2}^{\infty} 10 b_{n-2} (-7+n)\, x^{n-5}$$

$$\sum_{n=0}^{\infty} 14 x^{n-3} b_n = \sum_{n=2}^{\infty} 14 b_{n-2} x^{n-5}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-5$.

$$\begin{aligned}
&\left(\sum_{n=10}^{\infty} 2C a_{n-10} (-7+n)\, x^{n-5}\right) + \left(\sum_{n=8}^{\infty} 2C a_{n-8} (n-5)\, x^{n-5}\right) \\
&+ \left(\sum_{n=10}^{\infty} 9C a_{n-10} x^{n-5}\right) + \left(\sum_{n=8}^{\infty} 2C a_{n-8} x^{n-5}\right) \\
&+ \left(\sum_{n=2}^{\infty} b_{n-2} (n-8)\,(-7+n)\, x^{n-5}\right) + \left(\sum_{n=0}^{\infty} x^{n-5} b_n (-6+n)\,(n-5)\right) \\
&+ \left(\sum_{n=2}^{\infty} 10 b_{n-2} (-7+n)\, x^{n-5}\right) + \left(\sum_{n=0}^{\infty} 3 x^{n-5} b_n (n-5)\right) \\
&+ \left(\sum_{n=2}^{\infty} 14 b_{n-2} x^{n-5}\right) + \sum_{n=0}^{\infty} \left(-15 b_n x^{n-5}\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-7b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-7b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$-6b_0 - 12b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6 - 12b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$-6b_1 - 15b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-15b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$-4b_2 - 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 - 16b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{8}$$

For $n = 5$, Eq (2B) gives

$$-15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = 6$, Eq (2B) gives

$$6b_4 - 12b_6 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{3}{4} - 12b_6 = 0$$

Solving the above for b_6 gives

$$b_6 = \frac{1}{16}$$

For $n = 7$, Eq (2B) gives

$$-7b_7 + 14b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-7b_7 = 0$$

Solving the above for b_7 gives

$$b_7 = 0$$

For $n = N$, where $N = 8$ which is the difference between the two roots, we are free to choose $b_8 = 0$. Hence for $n = 8$, Eq (2B) gives

$$8C + \frac{3}{2} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{3}{16}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{3}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{3}{16}\left(x^3\left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right)\right)\ln(x) + \frac{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + O(x^9)}{x^5}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \\
&\quad + c_2 \left(-\frac{3}{16}\left(x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right)\right) \ln(x) \right. \\
&\qquad \left. + \frac{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + O(x^9)}{x^5}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \\
&\quad + c_2 \left(-\frac{3x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \ln(x)}{16} \right. \\
&\qquad \left. + \frac{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + O(x^9)}{x^5}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \\
&\quad + c_2 \left(-\frac{3x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \ln(x)}{16} \right. \\
&\qquad \left. + \frac{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + O(x^9)}{x^5}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) + c_2 \left(-\frac{3x^3 \left(1 - \frac{5x^2}{2} + \frac{35x^4}{8} - \frac{105x^6}{16} + \frac{1155x^8}{128} + O(x^9)\right) \ln(x)}{16} + \frac{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{16} + O(x^9)}{x^5}\right)$$

Verified OK.

16.3.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (10x^3 + 3x)\, y' + (14x^2 - 15)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(14x^2-15)y}{x^2(x^2+1)} - \frac{(10x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(10x^2+3)y'}{x(x^2+1)} + \frac{(14x^2-15)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{10x^2+3}{x(x^2+1)}, P_3(x) = \frac{14x^2-15}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -15$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(10x^2+3)\, y' + (14x^2-15)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+r)\,(-3+r)\, x^r + a_1(6+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+5)\,(k+r-3) + a_{k-2}(k+r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-5, 3\}$

- Each term must be 0

$$a_1(6+r)(-2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+5)\left(a_k(k+r-3) + a_{k-2}(k+r)\right) = 0$$

- Shift index using $k- >k+2$

$$(k+r+7)\left(a_{k+2}(k+r-1) + a_k(k+r+2)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+2)}{k+r-1}$$

- Recursion relation for $r = -5$

$$a_{k+2} = -\frac{a_k(k-3)}{k-6}$$

- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 6$

$$a_{k+2} = -\frac{a_k(k-3)}{k-6}$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k(k+5)}{k+2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+5)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(3+10*x^2)*diff(y(x),x)-(15-14*x^2)*y(x)=0,y(x),type='se
```

$$y(x) = c_1 x^3 \left(1 - \frac{5}{2}x^2 + \frac{35}{8}x^4 + O\left(x^6\right)\right) + \frac{c_2(-203212800 + 101606400x^2 - 25401600x^4 + O\left(x^6\right))}{x^5}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 46

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*(3+10*x^2)*y'[x]-(15-14*x^2)*y[x]==0,y[x],{x,0,5}
```

$$y(x) \to c_1 \left(\frac{1}{x^5} - \frac{1}{2x^3} + \frac{1}{8x}\right) + c_2 \left(\frac{35x^7}{8} - \frac{5x^5}{2} + x^3\right)$$

16.4 problem Example 7.7.4 page 387

Internal problem ID [1416]
Internal file name [`OUTPUT/1417_Sunday_June_05_2022_02_16_00_AM_37075915/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: Example 7.7.4 page 387.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-2x^4+x^2)\,y''+(-13x^3+7x)\,y'-14x^2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{13x^2-7}{x\,(2x^2-1)}$$
$$q(x)=\frac{14}{2x^2-1}$$

Table 778: Table $p(x), q(x)$ singularites.

$p(x) = \frac{13x^2-7}{x(2x^2-1)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{\sqrt{2}}{2}$	"regular"
$x = \frac{\sqrt{2}}{2}$	"regular"

$q(x) = \frac{14}{2x^2-1}$	
singularity	type
$x = -\frac{\sqrt{2}}{2}$	"regular"
$x = \frac{\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(2x^2-1) + (-13x^3+7x)\,y' - 14x^2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(2x^2-1) + (-13x^3+7x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 14x^2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(-2x^{n+r+2}a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-13x^{n+r+2}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}7x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-14x^{n+r+2}a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-2x^{n+r+2}a_n(n+r)(n+r-1)\right)&=\sum_{n=2}^{\infty}\left(-2a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-13x^{n+r+2}a_n(n+r)\right)&=\sum_{n=2}^{\infty}\left(-13a_{n-2}(n+r-2)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-14x^{n+r+2}a_n\right)&=\sum_{n=2}^{\infty}\left(-14a_{n-2}x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left(-2a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-13a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty}7x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-14a_{n-2}x^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r}a_n(n+r)(n+r-1)+7x^{n+r}a_n(n+r)=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1 + r) + 7x^r a_0 r = 0$$

Or

$$(x^r r(-1 + r) + 7x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(6 + r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(6 + r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -6 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(6 + r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^6} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-6} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \le n$ the recursive equation is

$$\begin{aligned} -2a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ -13a_{n-2}(n+r-2) + 7a_n(n+r) - 14a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(2n+2r+3)\, a_{n-2}}{n+6+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{(2n+3)\, a_{n-2}}{n+6} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{7+2r}{8+r}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{7}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{7+2r}{8+r}$	$\frac{7}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{7+2r}{8+r}$	$\frac{7}{8}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 36r + 77}{(8+r)(10+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{77}{80}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{7+2r}{8+r}$	$\frac{7}{8}$
a_3	0	0
a_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{77}{80}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{7+2r}{8+r}$	$\frac{7}{8}$
a_3	0	0
a_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{77}{80}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{8r^3 + 132r^2 + 694r + 1155}{(8+r)(10+r)(12+r)}$$

Which for the root $r = 0$ becomes

$$a_6 = \frac{77}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{7+2r}{8+r}$	$\frac{7}{8}$
a_3	0	0
a_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{77}{80}$
a_5	0	0
a_6	$\frac{8r^3+132r^2+694r+1155}{(8+r)(10+r)(12+r)}$	$\frac{77}{64}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \ldots \\ &= 1 + \frac{7x^2}{8} + \frac{77x^4}{80} + \frac{77x^6}{64} + O(x^7) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= \frac{8r^3 + 132r^2 + 694r + 1155}{(8 + r)(10 + r)(12 + r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{8r^3 + 132r^2 + 694r + 1155}{(8 + r)(10 + r)(12 + r)} &= \lim_{r\to -6} \frac{8r^3 + 132r^2 + 694r + 1155}{(8 + r)(10 + r)(12 + r)} \\ &= \frac{5}{16} \end{aligned}$$

The limit is $\frac{5}{16}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-6} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &-2b_{n-2}(n + r - 2)(n - 3 + r) + b_n(n + r)(n + r - 1) \\ &- 13b_{n-2}(n + r - 2) + 7b_n(n + r) - 14b_{n-2} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -6$ becomes

$$-2b_{n-2}(n-8)(n-9) + b_n(n-6)(n-7) - 13b_{n-2}(n-8) + 7b_n(n-6) - 14b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(2n + 2r + 3)\, b_{n-2}}{n + 6 + r} \tag{5}$$

Which for the root $r = -6$ becomes

$$b_n = \frac{(2n-9)\, b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -6$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{7+2r}{8+r}$$

Which for the root $r = -6$ becomes

$$b_2 = -\frac{5}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{7+2r}{8+r}$	$-\frac{5}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{7+2r}{8+r}$	$-\frac{5}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 36r + 77}{(8 + r)(10 + r)}$$

Which for the root $r = -6$ becomes

$$b_4 = \frac{5}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{7+2r}{8+r}$	$-\frac{5}{2}$
b_3	0	0
b_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{5}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{7+2r}{8+r}$	$-\frac{5}{2}$
b_3	0	0
b_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{5}{8}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{8r^3 + 132r^2 + 694r + 1155}{(8 + r)(10 + r)(12 + r)}$$

Which for the root $r = -6$ becomes

$$b_6 = \frac{5}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{7+2r}{8+r}$	$-\frac{5}{2}$
b_3	0	0
b_4	$\frac{4r^2+36r+77}{(8+r)(10+r)}$	$\frac{5}{8}$
b_5	0	0
b_6	$\frac{8r^3+132r^2+694r+1155}{(8+r)(10+r)(12+r)}$	$\frac{5}{16}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 \dots\right) \\ &= \frac{1 - \frac{5x^2}{2} + \frac{5x^4}{8} + \frac{5x^6}{16} + O(x^7)}{x^6} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \left(1 + \frac{7x^2}{8} + \frac{77x^4}{80} + \frac{77x^6}{64} + O(x^7)\right) + \frac{c_2 \left(1 - \frac{5x^2}{2} + \frac{5x^4}{8} + \frac{5x^6}{16} + O(x^7)\right)}{x^6} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \left(1 + \frac{7x^2}{8} + \frac{77x^4}{80} + \frac{77x^6}{64} + O(x^7)\right) + \frac{c_2 \left(1 - \frac{5x^2}{2} + \frac{5x^4}{8} + \frac{5x^6}{16} + O(x^7)\right)}{x^6} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(1 + \frac{7x^2}{8} + \frac{77x^4}{80} + \frac{77x^6}{64} + O(x^7)\right) + \frac{c_2 \left(1 - \frac{5x^2}{2} + \frac{5x^4}{8} + \frac{5x^6}{16} + O(x^7)\right)}{x^6} \tag{1}$$

Verification of solutions

$$y = c_1\left(1+\frac{7x^2}{8}+\frac{77x^4}{80}+\frac{77x^6}{64}+O\left(x^7\right)\right)+\frac{c_2\left(1-\frac{5x^2}{2}+\frac{5x^4}{8}+\frac{5x^6}{16}+O(x^7)\right)}{x^6}$$

Verified OK.

16.4.1 Maple step by step solution

Let's solve

$-y''x^2(2x^2-1)+\left(-13x^3+7x\right)y'-14x^2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{14y}{2x^2-1}-\frac{(13x^2-7)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(13x^2-7)y'}{x(2x^2-1)}+\frac{14y}{2x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{13x^2-7}{x(2x^2-1)}, P_3(x)=\frac{14}{2x^2-1}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=7$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $14yx+\left(13x^2-7\right)y'+xy''(2x^2-1)=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(6+r)\, x^{-1+r} - a_1(1+r)\,(7+r)\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+r+1)\,(k+7+r) + a_{k-1}(2k+5+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, 0\}$$

- Each term must be 0

$$-a_1(1+r)\,(7+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r+1)\left(\left(k+r+\frac{5}{2}\right) a_{k-1} - \frac{a_{k+1}(k+7+r)}{2}\right) = 0$$

- Shift index using $k->k+1$

  $$2(k+r+2)\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$$

- Recursion relation for $r=-6$

  $$a_{k+2}=\frac{(2k-5)a_k}{k+2}$$

- Solution for $r=-6$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$$

- Recursion relation for $r=0$

  $$a_{k+2}=\frac{(2k+7)a_k}{k+8}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k}, a_{k+2}=\frac{(2k+7)a_k}{k+8}, -7a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-6}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k}\right), a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0, b_{k+2}=\frac{(2k+7)b_k}{k+8}, -7b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x^2*(1-2*x^2)*diff(y(x),x$2)+x*(7-13*x^2)*diff(y(x),x)-14*x^2*y(x)=0,y(x),type='serie
```

$$y(x) = c_1 \left(1 + \frac{7}{8}x^2 + \frac{77}{80}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-86400 + 216000x^2 - 54000x^4 + \mathrm{O}\left(x^6\right))}{x^6}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x^2*(1-2*x^2)*y''[x]+x*(7-13*x^2)*y'[x]-14*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{77x^4}{80} + \frac{7x^2}{8} + 1\right) + c_1 \left(\frac{1}{x^6} - \frac{5}{2x^4} + \frac{5}{8x^2}\right)$$

16.5 problem 1

Internal problem ID [1417]
Internal file name [OUTPUT/1418_Sunday_June_05_2022_02_16_03_AM_57993504/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + (4x + 3)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3y'x + (4x + 3)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4x + 3}{x^2}$$

Table 780: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x+3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 3y'x + (4x + 3)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\,a_n x^{n+r-2}\right) - 3\left(\sum_{n=0}^{\infty} (n + r)\,a_n x^{n+r-1}\right) x + (4x + 3)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4 x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 4 a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3 x^{n+r} a_n (n+r) + 3 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3 x^r a_0 r + 3 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3 x^r r + 3 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_n(n+r) + 4a_{n-1} + 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r + 3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{4a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{r(r-2)}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{4}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{r(r-2)(r^2-1)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{r^6 - 5r^4 + 4r^2}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{8}{45}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{r^2 (r^4 - 5r^2 + 4) (r^2 + 4r + 3)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{4}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{r^2 (r-2) (r+2)^2 (-1+r) (r+1)^2 (r+3) (r+4)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{16}{4725}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{r(r-2)}$	$-\frac{4}{3}$
a_2	$\frac{16}{r(r-2)(r^2-1)}$	$\frac{2}{3}$
a_3	$-\frac{64}{r^6-5r^4+4r^2}$	$-\frac{8}{45}$
a_4	$\frac{256}{r^2(r^4-5r^2+4)(r^2+4r+3)}$	$\frac{4}{135}$
a_5	$-\frac{1024}{r^2(r-2)(r+2)^2(-1+r)(r+1)^2(r+3)(r+4)}$	$-\frac{16}{4725}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^3\left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{16}{r\left(r-2\right)\left(r^2-1\right)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{16}{r\left(r-2\right)\left(r^2-1\right)} &= \lim_{r\to 1} \frac{16}{r\left(r-2\right)\left(r^2-1\right)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2 y'' - 3y'x + (4x+3)\, y = 0$ gives

$$\begin{aligned}
&x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- 3\left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x \\
&+ (4x+3)\left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^{2} y_1''(x)-3 y_1'(x) x+(4 x+3) y_1(x)\right) \ln (x)+x^{2}\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)\right.\right.$$
$$\left.\left.-3 y_1(x)\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}\left(n+r_2\right)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}\left(n+r_2\right)}{x^{2}}\right)\right)\right) \tag{7}$$
$$-3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}\left(n+r_2\right)}{x}\right) x+(4 x+3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_1''(x)-3 y_1'(x) x+(4 x+3) y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^{2}\left(\frac{2 y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)-3 y_1(x)\right) C$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}\left(n+r_2\right)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}\left(n+r_2\right)}{x^{2}}\right)\right) \tag{8}$$
$$-3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}\left(n+r_2\right)}{x}\right) x+(4 x+3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

Substituting $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n\left(n+r_1\right)\right) x-4\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n\left(n+r_2\right)\left(-1+n+r_2\right)\right) x^{2} \tag{9}$$
$$-3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n\left(n+r_2\right)\right) x+(4 x+3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0$$

Since $r_1=3$ and $r_2=1$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{n+2} a_n(n+3)\right) x-4\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C+\left(\sum_{n=0}^{\infty} x^{n-1} b_n(1+n) n\right) x^{2} \tag{10}$$
$$-3\left(\sum_{n=0}^{\infty} x^{n} b_n(1+n)\right) x+(4 x+3)\left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)=0$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \sum_{n=0}^{\infty} \left(-4C a_n x^{n+3}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+2} b_n\right) + \left(\sum_{n=0}^{\infty} 3b_n x^{1+n}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=2}^{\infty} 2C a_{-2+n} (1+n)\, x^{1+n}$$

$$\sum_{n=0}^{\infty} \left(-4C a_n x^{n+3}\right) = \sum_{n=2}^{\infty} \left(-4C a_{-2+n} x^{1+n}\right)$$

$$\sum_{n=0}^{\infty} 4x^{n+2} b_n = \sum_{n=1}^{\infty} 4b_{n-1} x^{1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $1+n$.

$$\left(\sum_{n=2}^{\infty} 2C a_{-2+n} (1+n)\, x^{1+n}\right) + \sum_{n=2}^{\infty} \left(-4C a_{-2+n} x^{1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) \\ + \sum_{n=0}^{\infty} \left(-3x^{1+n} b_n (1+n)\right) + \left(\sum_{n=1}^{\infty} 4b_{n-1} x^{1+n}\right) + \left(\sum_{n=0}^{\infty} 3b_n x^{1+n}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 + 4b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 + 4 = 0$$

Solving the above for b_1 gives

$$b_1 = 4$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 16 = 0$$

Which is solved for C. Solving for C gives

$$C = -8$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + 4b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 + \frac{128}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{128}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + 4b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{800}{9} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{100}{9}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + 4b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{2512}{45} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{2512}{675}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -8$ and all b_n, then the second solution becomes

$$
\begin{aligned}
y_2(x) = (-8) \left(x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right)\right) \ln(x) \\
+ x \left(1 + 4x - \frac{128x^3}{9} + \frac{100x^4}{9} - \frac{2512x^5}{675} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \\
&\quad + c_2 \left((-8) \left(x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right)\right) \ln(x) \right. \\
&\quad \left. + x \left(1 + 4x - \frac{128x^3}{9} + \frac{100x^4}{9} - \frac{2512x^5}{675} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \\
&\quad + c_2 \left(-8x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \ln(x) \right. \\
&\quad \left. + x \left(1 + 4x - \frac{128x^3}{9} + \frac{100x^4}{9} - \frac{2512x^5}{675} + O(x^6)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \\
+ c_2 \left(-8x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \ln(x) \right. \\
\left. + x \left(1 + 4x - \frac{128x^3}{9} + \frac{100x^4}{9} - \frac{2512x^5}{675} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) + c_2 \left(-8x^3 \left(1 - \frac{4x}{3} + \frac{2x^2}{3} - \frac{8x^3}{45} + \frac{4x^4}{135} - \frac{16x^5}{4725} + O(x^6)\right) \ln(x) + x \left(1 + 4x - \frac{128x^3}{9} + \frac{100x^4}{9} - \frac{2512x^5}{675} + O(x^6)\right)\right)$$

Verified OK.

16.5.1 Maple step by step solution

Let's solve

$x^2 y'' - 3y'x + (4x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{(4x+3)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{(4x+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{3}{x}, P_3(x) = \frac{4x+3}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 3$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$x^2 y'' - 3y'x + (4x+3)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)\,(k+r-3) + 4a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+r)\,(k-2+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k}{(k+r)(k-2+r)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{4a_k}{(k+1)(k-1)}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{4a_k}{(k+1)(k-1)}$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{4a_k}{(k+3)(k+1)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{4a_k}{(k+3)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+(3+4*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(c_1 x^2 \left(1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{8}{45}x^3 + \frac{4}{135}x^4 - \frac{16}{4725}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(16x^2 - \frac{64}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{45}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-2 - 8x + \frac{256}{9}x^3 - \frac{200}{9}x^4 + \frac{5024}{675}x^5 + \mathrm{O}\left(x^6\right)\right)\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+(3+4*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{9}x\left(124x^4-176x^3+36x^2+36x+9\right)-\frac{8}{3}x^3\left(2x^2-4x+3\right)\log(x)\right) + c_2\left(\frac{4x^7}{135}-\frac{8x^6}{45}+\frac{2x^5}{3}-\frac{4x^4}{3}+x^3\right)$$

16.6 problem 2

Internal problem ID [1418]
Internal file name [OUTPUT/1419_Sunday_June_05_2022_02_16_07_AM_56052247/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x}$$

Table 782: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{(n+1)\, n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{(1+r)\, r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2\, r\, (2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(1+r)^2\, r\, (2+r)^2\, (3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{(1+r)r}$	$-\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{1}{2880}$
a_5	$-\frac{1}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{(1+r)\,r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(1+r)\,r} &= \lim_{r \to 0} -\frac{1}{(1+r)\,r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' + y = 0$ gives

$$
\left(C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x
$$
$$
+ C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x + y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)xC + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \sum_{n=0}^{\infty} (-C\, x^n a_n) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} (-C\, x^n a_n) = \sum_{n=1}^{\infty} (-C a_{n-1} x^{n-1})$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - 1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n\,=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 + b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 - \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{35}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1)&\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ &+ 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{x^5}{86400} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} \right. \\ &\quad \left. - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O\left(x^6\right)\right) \\
& + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{3x^2}{4} \right. \\
& \left. +\frac{7x^3}{36}-\frac{35x^4}{1728}+\frac{101x^5}{86400}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O\left(x^6\right)\right) \\
& + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{x^5}{86400}+O\left(x^6\right)\right)\ln\left(x\right)+1-\frac{3x^2}{4}+\frac{7x^3}{36} \right. \\
& \left. -\frac{35x^4}{1728}+\frac{101x^5}{86400}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

16.6.1 Maple step by step solution

Let's solve

$y''x+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{1}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1) k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = -\frac{a_k}{(1+k) k}, b_{1+k} = -\frac{b_k}{(k+2)(1+k)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{1}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{144}x\left(x^3 - 12x^2 + 72x - 144\right)\log(x) + \frac{-47x^4 + 480x^3 - 2160x^2 + 1728x + 1728}{1728}\right) + c_2\left(\frac{x^5}{2880} - \frac{x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

16.7 problem 3

Internal problem ID [1419]
Internal file name [OUTPUT/1420_Sunday_June_05_2022_02_16_11_AM_75225682/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x+1)\,y''+4x(1+2x)\,y'-(3x+1)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1+2x}{x\,(x+1)}$$
$$q(x)=-\frac{3x+1}{4x^2\,(x+1)}$$

Table 784: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+2x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{3x+1}{4x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\, y'' + \left(8x^2 + 4x\right) y' + (-3x - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(8x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-3x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)\right)+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) \\
&+\sum_{n=0}^{\infty}\left(-3x^{1+n+r} a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty}\left(-3x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty}\left(-3a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right)+\sum_{n=1}^{\infty}\left(-3a_{n-1} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1)+4x^{n+r} a_n (n+r)-a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 4x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}4a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1)\\+8a_{n-1}(n+r-1)+4a_n(n+r)-3a_{n-1}-a_n=0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-3)\,a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(n-1)\,a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2r+1}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+2r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r-1}{3+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+2r}$	0
a_2	$\frac{2r-1}{3+2r}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-2r+1}{5+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+2r}$	0
a_2	$\frac{2r-1}{3+2r}$	0
a_3	$\frac{-2r+1}{5+2r}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r-1}{7+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+2r}$	0
a_2	$\frac{2r-1}{3+2r}$	0
a_3	$\frac{-2r+1}{5+2r}$	0
a_4	$\frac{2r-1}{7+2r}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-2r+1}{9+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{1+2r}$	0
a_2	$\frac{2r-1}{3+2r}$	0
a_3	$\frac{-2r+1}{5+2r}$	0
a_4	$\frac{2r-1}{7+2r}$	0
a_5	$\frac{-2r+1}{9+2r}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{-2r + 1}{1 + 2r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \rightarrow r_2} \frac{-2r + 1}{1 + 2r} &= \lim_{r \rightarrow -\frac{1}{2}} \frac{-2r + 1}{1 + 2r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2) (-1 + n + r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $4x^2(x + 1) y'' + (8x^2 + 4x) y' + (-3x - 1) y = 0$

gives

$$
\begin{aligned}
&4x^2(x+1)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(8x^2+4x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(-3x-1)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(4x^2(x+1)\,y_1''(x)+\left(8x^2+4x\right)y_1'(x)+(-3x-1)\,y_1(x)\right)\ln(x)\right.\\
&\left.+4x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(8x^2+4x\right)y_1(x)}{x}\right)C\\
&+4x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(8x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-3x-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
4x^2(x+1)\,y_1''(x)+\left(8x^2+4x\right)y_1'(x)+(-3x-1)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(4x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(8x^2+4x\right)y_1(x)}{x}\right)C\\
&+4x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(8x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(-3x-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(8x(x+1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)+4\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)x\right)C \\
&+\left(4x^3+4x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \\
&+\left(8x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)+(-3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{9}
$$

Since $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$
\begin{aligned}
&\left(8x(x+1)\left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} a_n \left(n+\frac{1}{2}\right)\right)+4\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\right)x\right)C \\
&+\left(4x^3+4x^2\right)\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right) \\
&+\left(8x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n \left(n-\frac{1}{2}\right)\right)+(-3x-1)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)=0
\end{aligned}
\tag{10}
$$

Expanding $4C\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
4C\sqrt{x} &= 4C\sqrt{x} + \dots \\
&= 4C\sqrt{x}
\end{aligned}
$$

Expanding $\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\sqrt{x} &= \sqrt{x} + \dots \\
&= \sqrt{x}
\end{aligned}
$$

Expanding $\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}} + \dots \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

Expanding $4\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
4\sqrt{x} &= 4\sqrt{x} + \dots \\
&= 4\sqrt{x}
\end{aligned}
$$

Expanding $\frac{2}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{2}{\sqrt{x}} &= \frac{2}{\sqrt{x}} + \dots \\ &= \frac{2}{\sqrt{x}}\end{aligned}$$

Expanding $-3\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-3\sqrt{x} &= -3\sqrt{x} + \dots \\ &= -3\sqrt{x}\end{aligned}$$

Expanding $-\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{1}{\sqrt{x}} &= -\frac{1}{\sqrt{x}} + \dots \\ &= -\frac{1}{\sqrt{x}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} (8n+4)\, C a_n x^{\frac{3}{2}+n}\right) + \left(\sum_{n=0}^{\infty} (8n+4)\, C a_n x^{n+\frac{1}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} 4C\, x^{\frac{3}{2}+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+\frac{1}{2}} b_n \left(4n^2 - 8n + 3\right)\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2 - 8n + 3\right)\right) + \left(\sum_{n=0}^{\infty} (8n-4)\, b_n x^{n+\frac{1}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} (4n-2)\, b_n x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+\frac{1}{2}} b_n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-\frac{1}{2}}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n - \frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty} (8n+4)\, C a_n x^{\frac{3}{2}+n} &= \sum_{n=2}^{\infty} C a_{n-2} (8n-12)\, x^{n-\frac{1}{2}} \\ \sum_{n=0}^{\infty} (8n+4)\, C a_n x^{n+\frac{1}{2}} &= \sum_{n=1}^{\infty} C a_{n-1} (8n-4)\, x^{n-\frac{1}{2}}\end{aligned}$$

$$\sum_{n=0}^{\infty} 4C\, x^{\frac{3}{2}+n} a_n = \sum_{n=2}^{\infty} 4C a_{n-2} x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} x^{n+\frac{1}{2}} b_n \left(4n^2-8n+3\right) = \sum_{n=1}^{\infty} b_{n-1}\left(4(n-1)^2-8n+11\right) x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} (8n-4)\, b_n x^{n+\frac{1}{2}} = \sum_{n=1}^{\infty} b_{n-1}(8n-12)\, x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+\frac{1}{2}} b_n\right) = \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-\frac{1}{2}}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} C a_{n-2}(8n-12)\, x^{n-\frac{1}{2}}\right) + \left(\sum_{n=1}^{\infty} C a_{n-1}(8n-4)\, x^{n-\frac{1}{2}}\right) \\ &+ \left(\sum_{n=2}^{\infty} 4C a_{n-2} x^{n-\frac{1}{2}}\right) + \left(\sum_{n=1}^{\infty} b_{n-1}\left(4(n-1)^2-8n+11\right) x^{n-\frac{1}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n \left(4n^2-8n+3\right)\right) + \left(\sum_{n=1}^{\infty} b_{n-1}(8n-12)\, x^{n-\frac{1}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} (4n-2)\, b_n x^{n-\frac{1}{2}}\right) + \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-\frac{1}{2}}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$4C-4=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$(8a_0+12a_1)\, C+8b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8+8b_2=0$$

Solving the above for b_2 gives

$$b_2 = -1$$

For $n = 3$, Eq (2B) gives

$$(16a_1 + 20a_2) C + 12b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-12 + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{1}{2}$$

For $n = 4$, Eq (2B) gives

$$(24a_2 + 28a_3) C + 32b_3 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$16 + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{3}$$

For $n = 5$, Eq (2B) gives

$$(32a_3 + 36a_4) C + 60b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20 + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{4}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\left(\sqrt{x}\,(1 + O(x^6))\right) \ln(x) + \frac{1 - x^2 + \frac{x^3}{2} - \frac{x^4}{3} + \frac{x^5}{4} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\,\left(1+O\left(x^6\right)\right)+c_2\left(1\left(\sqrt{x}\,\left(1+O\left(x^6\right)\right)\right)\ln\left(x\right)+\frac{1-x^2+\frac{x^3}{2}-\frac{x^4}{3}+\frac{x^5}{4}+O\left(x^6\right)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\,\left(1+O\left(x^6\right)\right)+c_2\left(\sqrt{x}\,\left(1+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-x^2+\frac{x^3}{2}-\frac{x^4}{3}+\frac{x^5}{4}+O\left(x^6\right)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\,\left(1+O\left(x^6\right)\right) \\
& + c_2\left(\sqrt{x}\,\left(1+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-x^2+\frac{x^3}{2}-\frac{x^4}{3}+\frac{x^5}{4}+O\left(x^6\right)}{\sqrt{x}}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\left(1+O\left(x^6\right)\right)+c_2\left(\sqrt{x}\,\left(1+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-x^2+\frac{x^3}{2}-\frac{x^4}{3}+\frac{x^5}{4}+O\left(x^6\right)}{\sqrt{x}}\right)
$$

Verified OK.

16.7.1 Maple step by step solution

Let's solve

$4x^2(x+1)\,y''+\left(8x^2+4x\right)y'+\left(-3x-1\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(3x+1)y}{4x^2(x+1)}-\frac{(1+2x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+2x)y'}{x(x+1)}-\frac{(3x+1)y}{4x^2(x+1)}=0$

- Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1+2x}{x(x+1)}, P_3(x) = -\frac{3x+1}{4x^2(x+1)}\right]$
  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 1$
  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- Multiply by denominators

  $4x^2(x+1)\, y'' + 4x(1+2x)\, y' + (-3x-1)\, y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2} y(u)\right) + (8u^2 - 12u + 4)\left(\frac{d}{du} y(u)\right) + (-3u+2)\, y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..3$

  $$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

  $$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$4 a_{0} r^{2} u^{-1+r}+\left(4 a_{1} \left(1+r \right)^{2}-2 a_{0} \left(4 r^{2}+2 r -1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty} \left(4 a_{k +1} \left(k +1+r \right)^{2}-2 a_{k} \left(4 k^{2}+8 k r -\right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$4 r^{2}=0$$

- Values of r that satisfy the indicial equation

  $$r =0$$

- Each term must be 0

  $$4 a_{1} \left(1+r \right)^{2}-2 a_{0} \left(4 r^{2}+2 r -1\right)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(4 k^{2}-4 k -3\right) a_{k -1}+\left(-8 k^{2}-4 k +2\right) a_{k}+4 a_{k +1} \left(k +1\right)^{2}=0$$

- Shift index using $k->k +1$

  $$\left(4 \left(k +1\right)^{2}-4 k -7\right) a_{k}+\left(-8 \left(k +1\right)^{2}-4 k -2\right) a_{k +1}+4 a_{k +2} \left(k +2\right)^{2}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k a_{k}-20 k a_{k +1}-3 a_{k}-10 a_{k +1}}{4 \left(k +2\right)^{2}}$$

- Recursion relation for $r =0$

  $$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k a_{k}-20 k a_{k +1}-3 a_{k}-10 a_{k +1}}{4 \left(k +2\right)^{2}}$$

- Solution for $r =0$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty} a_{k} u^{k}, a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k a_{k}-20 k a_{k +1}-3 a_{k}-10 a_{k +1}}{4 \left(k +2\right)^{2}}, 4 a_{1}+2 a_{0}=0\right]$$

- Revert the change of variables $u =x +1$

  $$\left[y =\sum_{k =0}^{\infty} a_{k} \left(x +1\right)^{k}, a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k a_{k}-20 k a_{k +1}-3 a_{k}-10 a_{k +1}}{4 \left(k +2\right)^{2}}, 4 a_{1}+2 a_{0}=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x),type='series'
```

$$y(x) = \frac{c_1(1+\mathrm{O}(x^6))\,x+\ln(x)\,(x+\mathrm{O}(x^6))\,c_2+\left(1-x-x^2+\frac{1}{2}x^3-\frac{1}{3}x^4+\frac{1}{4}x^5+\mathrm{O}(x^6)\right)c_2}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 53

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]+4*x*(1+2*x)*y'[x]-(1+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\sqrt{x}\log(x)-\frac{2x^4-3x^3+6x^2+6x-6}{6\sqrt{x}}\right)+c_2\sqrt{x}$$

16.8 problem 4

Internal problem ID [1420]
Internal file name [`OUTPUT/1421_Sunday_June_05_2022_02_16_16_AM_38138722/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+1)\,y''+y'x+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2+x\right)y''+y'x+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{x+1}$$
$$q(x)=\frac{1}{x\,(x+1)}$$

Table 786: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x+1}$	
singularity	type
$x = -1$	"regular"

$q(x) = \frac{1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+1) y'' + y'x + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+1) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 2)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n^2 + 1)}{(n+1)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 - 1}{(1+r)r}$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-1}{(1+r)r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r^2+1)(r^2+2r+2)}{(1+r)^2 r(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{5}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-1}{(1+r)r}$	-1
a_2	$\frac{(r^2+1)(r^2+2r+2)}{(1+r)^2 r(2+r)}$	$\frac{5}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{(1+r)^2 r (2+r)^2 (3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{25}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-1}{(1+r)r}$	-1
a_2	$\frac{(r^2+1)(r^2+2r+2)}{(1+r)^2 r(2+r)}$	$\frac{5}{6}$
a_3	$-\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{(1+r)^2 r(2+r)^2(3+r)}$	$-\frac{25}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{85}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-1}{(1+r)r}$	-1
a_2	$\frac{(r^2+1)(r^2+2r+2)}{(1+r)^2 r(2+r)}$	$\frac{5}{6}$
a_3	$-\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{25}{36}$
a_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{85}{144}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{221}{432}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2-1}{(1+r)r}$	-1
a_2	$\frac{(r^2+1)(r^2+2r+2)}{(1+r)^2 r(2+r)}$	$\frac{5}{6}$
a_3	$-\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{(1+r)^2 r(2+r)^2 (3+r)}$	$-\frac{25}{36}$
a_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)}$	$\frac{85}{144}$
a_5	$-\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(1+r)^2 r(2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$	$-\frac{221}{432}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-r^2-1}{(1+r)\,r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-r^2-1}{(1+r)\,r} &= \lim_{r\to 0} \frac{-r^2-1}{(1+r)\,r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x(x+1)\,y''+y'x+y=0$ gives

$$x(x+1)\left(Cy_1''(x)\ln\left(x\right)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+\left(Cy_1'(x)\ln\left(x\right)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x+Cy_1(x)\ln\left(x\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

Which can be written as

$$\left(\left(x(x+1)\,y_1''(x)+y_1'(x)\,x+y_1(x)\right)\ln\left(x\right)+x(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C+x(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x(x+1)\,y_1''(x)+y_1'(x)\,x+y_1(x)=0$$

Eq (7) simplifes to

$$\left(x(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+y_1(x)\right)C+x(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n} a_n (n+1)\right) \\ &+ \sum_{n=0}^{\infty} \left(-C\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} x^n b_n n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) = \sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n} a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n} a_{n}\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{n} b_{n} n(n-1) = \sum_{n=1}^{\infty} (n-1)\, b_{n-1}(-2+n)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} x^{n} b_{n} n = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}$$

$$\sum_{n=0}^{\infty} b_{n} x^{n} = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\, x^{n-1}\right) + \left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_{n}(n-1)\right) \\
&+ \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$(2a_0 + 3a_1)\, C + 2b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(4a_1 + 5a_2) C + 5b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{8}{3} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{4}{9}$$

For $n = 4$, Eq (2B) gives

$$(6a_2 + 7a_3) C + 10b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{155}{36} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{155}{432}$$

For $n = 5$, Eq (2B) gives

$$(8a_3 + 9a_4) C + 17b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1265}{216} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{253}{864}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1) \left(x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O(x^6)\right)\right) \ln(x) + 1 - \frac{x^2}{2} + \frac{4x^3}{9} - \frac{155x^4}{432} + \frac{253x^5}{864} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right) \\
&\quad + c_2\left((-1)\left(x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right)\right)\ln\left(x\right) + 1 \right. \\
&\quad \left. - \frac{x^2}{2} + \frac{4x^3}{9} - \frac{155x^4}{432} + \frac{253x^5}{864} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right)\ln\left(x\right) + 1 - \frac{x^2}{2} + \frac{4x^3}{9} \right. \\
&\quad \left. - \frac{155x^4}{432} + \frac{253x^5}{864} + O\left(x^6\right)\right)
\end{aligned}
$$

<ins>Summary</ins>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right)\ln\left(x\right) + 1 - \frac{x^2}{2} \right. \\
&\quad \left. + \frac{4x^3}{9} - \frac{155x^4}{432} + \frac{253x^5}{864} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

<ins>Verification of solutions</ins>

$$
\begin{aligned}
y &= c_1 x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right) \\
&\quad + c_2\left(-x\left(1 - x + \frac{5x^2}{6} - \frac{25x^3}{36} + \frac{85x^4}{144} - \frac{221x^5}{432} + O\left(x^6\right)\right)\ln\left(x\right) + 1 - \frac{x^2}{2} + \frac{4x^3}{9} \right. \\
&\quad \left. - \frac{155x^4}{432} + \frac{253x^5}{864} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

16.8.1 Maple step by step solution

Let's solve

$x(x+1)\,y''+y'x+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x+1)}-\frac{y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x+1}+\frac{y}{x(x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{1}{x+1}, P_3(x)=\frac{1}{x(x+1)}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(x+1)\,y''+y'x+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum\limits_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)^2 + a_k(k^2+2kr+r^2+1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+1)^2 + a_k(k^2+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k^2+1)}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k^2+1)}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+1)}{(k+1)^2}\right]$$

- Revert the change of variables $u = x+1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k^2+1)}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
      -> solution has integrals; searching for one without integrals...
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric solution without integrals succesful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 60

```
Order:=6;
dsolve(x*(1+x)*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x \left(1 - x + \frac{5}{6}x^2 - \frac{25}{36}x^3 + \frac{85}{144}x^4 - \frac{221}{432}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(-x + x^2 - \frac{5}{6}x^3 + \frac{25}{36}x^4 - \frac{85}{144}x^5 + \mathrm{O}\left(x^6\right)\right) \ln(x) c_2 \\
& + \left(1 - x + \frac{1}{2}x^2 - \frac{7}{18}x^3 + \frac{145}{432}x^4 - \frac{257}{864}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*(1+x)*y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{36} x \left(25x^3 - 30x^2 + 36x - 36 \right) \log(x) + \frac{1}{432} \left(-455x^4 + 552x^3 - 648x^2 + 432x + 432 \right) \right) + c_2 \left(\frac{85x^5}{144} - \frac{25x^4}{36} + \frac{5x^3}{6} - x^2 + x \right)$$

16.9 problem 5

Internal problem ID [1421]
Internal file name [OUTPUT/1422_Sunday_June_05_2022_02_16_21_AM_75049073/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(3x+2)\,y''+x(4+21x)\,y'-(1-9x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4+21x}{2x\,(3x+2)}$$
$$q(x)=\frac{9x-1}{2x^2\,(3x+2)}$$

Table 788: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4+21x}{2x(3x+2)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

$q(x) = \frac{9x-1}{2x^2(3x+2)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{2}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(3x+2)\, y'' + \left(21x^2+4x\right) y' + (9x-1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2(3x+2)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(21x^2+4x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (9x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 6x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 21x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 21a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 9x^{1+n+r} a_n = \sum_{n=1}^{\infty} 9a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 21a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 9a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$6a_{n-1}(n+r-1)(n+r-2)+4a_n(n+r)(n+r-1) \\ +21a_{n-1}(n+r-1)+4a_n(n+r)+9a_{n-1}-a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3(n+r)\,a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{3(2n+1)\,a_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3-3r}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{9}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{1+2r}$	$-\frac{9}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r^2+27r+18}{4r^2+8r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{135}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{1+2r}$	$-\frac{9}{4}$
a_2	$\frac{9r^2+27r+18}{4r^2+8r+3}$	$\frac{135}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^3 - 162r^2 - 297r - 162}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{945}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{1+2r}$	$-\frac{9}{4}$
a_2	$\frac{9r^2+27r+18}{4r^2+8r+3}$	$\frac{135}{32}$
a_3	$\frac{-27r^3-162r^2-297r-162}{8r^3+36r^2+46r+15}$	$-\frac{945}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^4 + 810r^3 + 2835r^2 + 4050r + 1944}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{25515}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{1+2r}$	$-\frac{9}{4}$
a_2	$\frac{9r^2+27r+18}{4r^2+8r+3}$	$\frac{135}{32}$
a_3	$\frac{-27r^3-162r^2-297r-162}{8r^3+36r^2+46r+15}$	$-\frac{945}{128}$
a_4	$\frac{81r^4+810r^3+2835r^2+4050r+1944}{16r^4+128r^3+344r^2+352r+105}$	$\frac{25515}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243(5+r)(4+r)(3+r)(2+r)(1+r)}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{168399}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3-3r}{1+2r}$	$-\frac{9}{4}$
a_2	$\frac{9r^2+27r+18}{4r^2+8r+3}$	$\frac{135}{32}$
a_3	$\frac{-27r^3-162r^2-297r-162}{8r^3+36r^2+46r+15}$	$-\frac{945}{128}$
a_4	$\frac{81r^4+810r^3+2835r^2+4050r+1944}{16r^4+128r^3+344r^2+352r+105}$	$\frac{25515}{2048}$
a_5	$-\frac{243(5+r)(4+r)(3+r)(2+r)(1+r)}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{168399}{8192}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-3-3r}{1+2r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{-3-3r}{1+2r} &= \lim_{r\rightarrow -\frac{1}{2}} \frac{-3-3r}{1+2r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $2x^2(3x+2)\, y'' + (21x^2+4x)\, y' + (9x-1)\, y =$

0 gives

$$
\begin{aligned}
&2x^2(3x+2)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(21x^2+4x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(9x-1)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(2x^2(3x+2)\,y_1''(x)+\left(21x^2+4x\right)y_1'(x)+(9x-1)\,y_1(x)\right)\ln(x)\right.\\
&\left.+2x^2(3x+2)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(21x^2+4x\right)y_1(x)}{x}\right)C\\
&+2x^2(3x+2)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(21x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(9x-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
2x^2(3x+2)\,y_1''(x)+\left(21x^2+4x\right)y_1'(x)+(9x-1)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(2x^2(3x+2)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(21x^2+4x\right)y_1(x)}{x}\right)C\\
&+2x^2(3x+2)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(21x^2+4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(9x-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(12\left(x+\frac{2}{3}\right)x\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)+15\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)x\right)C \\ +\left(6x^3+4x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \\ +\left(21x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)+(9x-1)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{9}$$

Since $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$\left(12\left(x+\frac{2}{3}\right)x\left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} a_n \left(n+\frac{1}{2}\right)\right)+15\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\right)x\right)C \\ +\left(6x^3+4x^2\right)\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right) \\ +\left(21x^2+4x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n \left(n-\frac{1}{2}\right)\right)+(9x-1)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)=0 \tag{10}$$

Expanding $4C\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 4C\sqrt{x} &= 4C\sqrt{x}+\dots \\ &= 4C\sqrt{x} \end{aligned}$$

Expanding $\frac{3\sqrt{x}}{2}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \frac{3\sqrt{x}}{2} &= \frac{3\sqrt{x}}{2}+\dots \\ &= \frac{3\sqrt{x}}{2} \end{aligned}$$

Expanding $\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}}+\dots \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Expanding $\frac{21\sqrt{x}}{2}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{21\sqrt{x}}{2} &= \frac{21\sqrt{x}}{2} + \dots \\ &= \frac{21\sqrt{x}}{2}\end{aligned}$$

Expanding $\frac{2}{\sqrt{x}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{2}{\sqrt{x}} &= \frac{2}{\sqrt{x}} + \dots \\ &= \frac{2}{\sqrt{x}}\end{aligned}$$

Expanding $9\sqrt{x}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}9\sqrt{x} &= 9\sqrt{x} + \dots \\ &= 9\sqrt{x}\end{aligned}$$

Expanding $-\frac{1}{\sqrt{x}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{1}{\sqrt{x}} &= -\frac{1}{\sqrt{x}} + \dots \\ &= -\frac{1}{\sqrt{x}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} (12n+6)\, C a_n x^{\frac{3}{2}+n}\right) + \left(\sum_{n=0}^{\infty} (8n+4)\, C a_n x^{n+\frac{1}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} 15C\, x^{\frac{3}{2}+n} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{3x^{n+\frac{1}{2}} b_n (4n^2-8n+3)}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n (4n^2-8n+3)\right) + \left(\sum_{n=0}^{\infty} \frac{21x^{n+\frac{1}{2}} b_n (2n-1)}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} (4n-2)\, b_n x^{n-\frac{1}{2}}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+\frac{1}{2}} b_n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-\frac{1}{2}}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-\frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(12n+6\right) C a_{n} x^{\frac{3}{2}+n} = \sum_{n=2}^{\infty} C a_{n-2} \left(12n-18\right) x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \left(8n+4\right) C a_{n} x^{n+\frac{1}{2}} = \sum_{n=1}^{\infty} C a_{n-1} \left(8n-4\right) x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} 15 C\, x^{\frac{3}{2}+n} a_{n} = \sum_{n=2}^{\infty} 15 C a_{n-2} x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \frac{3 x^{n+\frac{1}{2}} b_{n} \left(4n^{2}-8n+3\right)}{2} = \sum_{n=1}^{\infty} \frac{3 b_{n-1} \left(4 \left(n-1\right)^{2}-8n+11\right) x^{n-\frac{1}{2}}}{2}$$

$$\sum_{n=0}^{\infty} \frac{21 x^{n+\frac{1}{2}} b_{n} \left(2n-1\right)}{2} = \sum_{n=1}^{\infty} \frac{21 b_{n-1} \left(-3+2n\right) x^{n-\frac{1}{2}}}{2}$$

$$\sum_{n=0}^{\infty} 9 x^{n+\frac{1}{2}} b_{n} = \sum_{n=1}^{\infty} 9 b_{n-1} x^{n-\frac{1}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} C a_{n-2} \left(12n-18\right) x^{n-\frac{1}{2}}\right) + \left(\sum_{n=1}^{\infty} C a_{n-1} \left(8n-4\right) x^{n-\frac{1}{2}}\right) \\ &+ \left(\sum_{n=2}^{\infty} 15 C a_{n-2} x^{n-\frac{1}{2}}\right) + \left(\sum_{n=1}^{\infty} \frac{3 b_{n-1} \left(4 \left(n-1\right)^{2}-8n+11\right) x^{n-\frac{1}{2}}}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_{n} \left(4n^{2}-8n+3\right)\right) + \left(\sum_{n=1}^{\infty} \frac{21 b_{n-1} \left(-3+2n\right) x^{n-\frac{1}{2}}}{2}\right) \\ &+ \left(\sum_{n=0}^{\infty} \left(4n-2\right) b_{n} x^{n-\frac{1}{2}}\right) + \left(\sum_{n=1}^{\infty} 9 b_{n-1} x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-b_{n} x^{n-\frac{1}{2}}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$4C+3=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{3}{4}$$

For $n = 2$, Eq (2B) gives

$$(21a_0 + 12a_1) C + 18b_1 + 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{9}{2} + 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{9}{16}$$

For $n = 3$, Eq (2B) gives

$$(33a_1 + 20a_2) C + 45b_2 + 24b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1053}{32} + 24b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{351}{256}$$

For $n = 4$, Eq (2B) gives

$$(45a_2 + 28a_3) C + 84b_3 + 48b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{8181}{64} + 48b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{2727}{1024}$$

For $n = 5$, Eq (2B) gives

$$(57a_3 + 36a_4) C + 135b_4 + 80b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{778815}{2048} + 80b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{155763}{32768}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{3}{4}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{3}{4}\left(\sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{9x^2}{16} + \frac{351x^3}{256} - \frac{2727x^4}{1024} + \frac{155763x^5}{32768} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{3}{4}\left(\sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right)\right)\ln(x) \right. \\ &\qquad \left. + \frac{1 - \frac{9x^2}{16} + \frac{351x^3}{256} - \frac{2727x^4}{1024} + \frac{155763x^5}{32768} + O(x^6)}{\sqrt{x}}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{3\sqrt{x}\left(1 - \frac{9x}{4} + \frac{135x^2}{32} - \frac{945x^3}{128} + \frac{25515x^4}{2048} - \frac{168399x^5}{8192} + O(x^6)\right)\ln(x)}{4} \right. \\ &\qquad \left. + \frac{1 - \frac{9x^2}{16} + \frac{351x^3}{256} - \frac{2727x^4}{1024} + \frac{155763x^5}{32768} + O(x^6)}{\sqrt{x}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{9x}{4}+\frac{135x^2}{32}-\frac{945x^3}{128}+\frac{25515x^4}{2048}-\frac{168399x^5}{8192}+O(x^6)\right) + c_2\left(-\frac{3\sqrt{x}\left(1-\frac{9x}{4}+\frac{135x^2}{32}-\frac{945x^3}{128}+\frac{25515x^4}{2048}-\frac{168399x^5}{8192}+O(x^6)\right)\ln(x)}{4} + \frac{1-\frac{9x^2}{16}+\frac{351x^3}{256}-\frac{2727x^4}{1024}+\frac{155763x^5}{32768}+O(x^6)}{\sqrt{x}}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{9x}{4}+\frac{135x^2}{32}-\frac{945x^3}{128}+\frac{25515x^4}{2048}-\frac{168399x^5}{8192}+O(x^6)\right) + c_2\left(-\frac{3\sqrt{x}\left(1-\frac{9x}{4}+\frac{135x^2}{32}-\frac{945x^3}{128}+\frac{25515x^4}{2048}-\frac{168399x^5}{8192}+O(x^6)\right)\ln(x)}{4} + \frac{1-\frac{9x^2}{16}+\frac{351x^3}{256}-\frac{2727x^4}{1024}+\frac{155763x^5}{32768}+O(x^6)}{\sqrt{x}}\right)$$

Verified OK.

16.9.1 Maple step by step solution

Let's solve

$2x^2(3x+2)\,y''+(21x^2+4x)\,y'+(9x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x-1)y}{2x^2(3x+2)}-\frac{(4+21x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4+21x)y'}{2x(3x+2)}+\frac{(9x-1)y}{2x^2(3x+2)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{4+21x}{2x(3x+2)}, P_3(x)=\frac{9x-1}{2x^2(3x+2)}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0}=1$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{4}$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(3x+2)\,y''+x(4+21x)\,y'+(9x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..1$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..2$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m=2..3$

  $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 3a_{k-1}(2k+2r+1)(k+r)\right) x^{k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{3a_{k-1}(k+r)}{2}\right) = 0$$

- Shift index using $k->k+1$

$$4\left(k+\frac{3}{2}+r\right)\left(\left(k+r+\frac{1}{2}\right)a_{k+1} + \frac{3a_k(k+r+1)}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r+1)}{2k+2r+1}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{1+k} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 69

```
Order:=6;
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x)=0,y(x),type='series
```

$$y(x) = \frac{c_1 x\left(1-\frac{9}{4}x+\frac{135}{32}x^2-\frac{945}{128}x^3+\frac{25515}{2048}x^4-\frac{168399}{8192}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(-\frac{3}{4}x+\frac{27}{16}x^2-\frac{405}{128}x^3+\frac{2835}{512}x^4-\right.\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 108

```
AsymptoticDSolveValue[2*x^2*(2+3*x)*y''[x]+x*(4+21*x)*y'[x]-(1-9*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{25515x^{9/2}}{2048}-\frac{945x^{7/2}}{128}+\frac{135x^{5/2}}{32}-\frac{9x^{3/2}}{4}+\sqrt{x}\right)+c_1\left(\frac{3}{512}\sqrt{x}\left(945x^3-540x^2+288x-128\right)\log(x)+\frac{8613x^4-5076x^3+2880x^2-1536x+1024}{1024\sqrt{x}}\right)$$

16.10 problem 6

Internal problem ID [1422]
Internal file name [`OUTPUT/1423_Sunday_June_05_2022_02_16_25_AM_99354858/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(2+x)\,y' - (-3x+2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 + 2x\right)y' + (3x-2)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2+x}{x}$$
$$q(x) = \frac{3x-2}{x^2}$$

Table 790: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3x-2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^2 + 2x)\, y' + (3x - 2)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (x^2 + 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (3x - 2) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 2x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 2x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 2x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_{n-1}(n + r - 1) + 2a_n(n + r) + 3a_{n-1} - 2a_n = 0 \qquad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(1+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r(1+r)(r+2)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(r+2)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(1+r)(r+2)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(r+2)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(r+2)(3+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{r(1+r)(r+2)(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$-\frac{1}{r(1+r)(r+2)}$	$-\frac{1}{6}$
a_4	$\frac{1}{r(1+r)(r+2)(3+r)}$	$\frac{1}{24}$
a_5	$-\frac{1}{r(1+r)(r+2)(3+r)(4+r)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\big(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\big) \\ &= x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{1}{r(1+r)(r+2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r(1+r)(r+2)} &= \lim_{r\to -2} -\frac{1}{r(1+r)(r+2)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + (x^2+2x)\,y' + (3x-2)\,y = 0$ gives

$$
\begin{aligned}
& x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (x^2+2x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ (3x-2)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2y_1''(x) + (x^2+2x)\,y_1'(x) + (3x-2)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left. + \frac{(x^2+2x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (x^2+2x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + (x^2 + 2x)\, y_1'(x) + (3x - 2)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2+2x)\, y_1(x)}{x}\right) C \\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (x^2+2x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ (x^2+2x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = -2$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n (1+n)\right) x + (x+1)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 \\
&+ (x^2+2x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + (3x-2)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\,x^{1+n} a_n (1+n)\right)+\left(\sum_{n=0}^{\infty} C\,x^{n+2} a_n\right)+\left(\sum_{n=0}^{\infty} C\,x^{1+n} a_n\right)$$
$$+\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-2)\right)$$
$$+\left(\sum_{n=0}^{\infty} 2x^{n-2} b_n (n-2)\right)+\left(\sum_{n=0}^{\infty} 3x^{n-1} b_n\right)+\sum_{n=0}^{\infty}\left(-2b_n x^{n-2}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\,x^{1+n} a_n (1+n)=\sum_{n=3}^{\infty} 2C a_{-3+n} (n-2)\,x^{n-2}$$

$$\sum_{n=0}^{\infty} C\,x^{n+2} a_n=\sum_{n=4}^{\infty} C a_{-4+n} x^{n-2}$$

$$\sum_{n=0}^{\infty} C\,x^{1+n} a_n=\sum_{n=3}^{\infty} C a_{-3+n} x^{n-2}$$

$$\sum_{n=0}^{\infty} x^{n-1} b_n (n-2)=\sum_{n=1}^{\infty} b_{n-1}(-3+n)\,x^{n-2}$$

$$\sum_{n=0}^{\infty} 3x^{n-1} b_n=\sum_{n=1}^{\infty} 3b_{n-1} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\left(\sum_{n=3}^{\infty} 2C a_{-3+n} (n-2)\,x^{n-2}\right)+\left(\sum_{n=4}^{\infty} C a_{-4+n} x^{n-2}\right)+\left(\sum_{n=3}^{\infty} C a_{-3+n} x^{n-2}\right)$$
$$+\left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}(-3+n)\,x^{n-2}\right)$$
$$+\left(\sum_{n=0}^{\infty} 2x^{n-2} b_n (n-2)\right)+\left(\sum_{n=1}^{\infty} 3b_{n-1} x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-2b_n x^{n-2}\right)=0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$b_0 - 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 2b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$2b_1 - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{2}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C + \frac{3}{2} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 4$, Eq (2B) gives

$$(a_0 + 5a_1)\, C + 4b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 4b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{2}$$

For $n = 5$, Eq (2B) gives

$$(a_1 + 7a_2)\, C + 5b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{15}{4}+10b_5=0$$

Solving the above for b_5 gives

$$b_5=\frac{3}{8}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=-\frac{1}{2}$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)=&-\frac{1}{2}\left(x\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O\left(x^6\right)\right)\right)\ln(x)\\&+\frac{1+\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}+\frac{3x^5}{8}+O(x^6)}{x^2}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O\left(x^6\right)\right)\\&\quad+c_2\left(-\frac{1}{2}\left(x\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O\left(x^6\right)\right)\right)\ln(x)\right.\\&\quad\left.+\frac{1+\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}+\frac{3x^5}{8}+O(x^6)}{x^2}\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O\left(x^6\right)\right)\\&\quad+c_2\left(-\frac{x\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)\ln(x)}{2}\right.\\&\quad\left.+\frac{1+\frac{x}{2}+\frac{x^2}{2}-\frac{x^4}{2}+\frac{3x^5}{8}+O(x^6)}{x^2}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x)}{2} + \frac{1 + \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{2} + \frac{3x^5}{8} + O(x^6)}{x^2}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)\ln(x)}{2} + \frac{1 + \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{2} + \frac{3x^5}{8} + O(x^6)}{x^2}\right)
$$

Verified OK.

16.10.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 + 2x) y' + (3x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{x^2} - \frac{(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2+x)y'}{x} + \frac{(3x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2+x}{x}, P_3(x) = \frac{3x-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(2+x)\, y' + (3x-2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(a_k(k+r-1)+a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r+3)(a_{k+1}(k+r)+a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{k-2}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(2+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1x^3\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2(\ln\left(x\right)\left(\left(-6\right)x^3 + 6x^4 - 3x^5 + \mathrm{O}\left(x^6\right)\right) + (12 + 6}{x^2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 74

```
AsymptoticDSolveValue[x^2*y''[x]+x*(2+x)*y'[x]-(2-3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4 - 3x^3 + 2x^2 + 2x + 4}{4x^2} + \frac{1}{2}(x-1)x\log(x)\right) + c_2\left(\frac{x^5}{24} - \frac{x^4}{6} + \frac{x^3}{2} - x^2 + x\right)$$

16.11 problem 7

Internal problem ID [1423]
Internal file name [`OUTPUT/1424_Sunday_June_05_2022_02_16_29_AM_38437327/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4y'x - (9 - x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 4y'x + (-9 + x)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{-9 + x}{4x^2} \end{aligned}$$

Table 792: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{-9+x}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 4y'x + (-9 + x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + 4\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x + (-9 + x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)\,(n+r-1) + 4x^{n+r} a_n(n+r) - 9a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - 9a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 4x^r r - 9x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 9\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{3}{2} \\
r_2 &= -\frac{3}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(4r^2 - 9)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{3}{2}}}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) + 4a_n(n+r) - 9a_n + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{4n^2 + 8nr + 4r^2 - 9} \tag{4}
$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{4n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4r^2 + 8r - 5}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+8r-5}$	$-\frac{1}{16}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16r^4 + 96r^3 + 136r^2 - 24r - 35}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+8r-5}$	$-\frac{1}{16}$
a_2	$\frac{1}{16r^4+96r^3+136r^2-24r-35}$	$\frac{1}{640}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+8r-5}$	$-\frac{1}{16}$
a_2	$\frac{1}{16r^4+96r^3+136r^2-24r-35}$	$\frac{1}{640}$
a_3	$-\frac{1}{64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945}$	$-\frac{1}{46080}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945)(4r^2 + 32r + 55)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{5160960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+8r-5}$	$-\frac{1}{16}$
a_2	$\frac{1}{16r^4+96r^3+136r^2-24r-35}$	$\frac{1}{640}$
a_3	$-\frac{1}{64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945}$	$-\frac{1}{46080}$
a_4	$\frac{1}{(64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945)(4r^2+32r+55)}$	$\frac{1}{5160960}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945)(4r^2 + 32r + 55)(4r^2 + 40r + 91)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{1}{825753600}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+8r-5}$	$-\frac{1}{16}$
a_2	$\frac{1}{16r^4+96r^3+136r^2-24r-35}$	$\frac{1}{640}$
a_3	$-\frac{1}{64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945}$	$-\frac{1}{46080}$
a_4	$\frac{1}{(64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945)(4r^2+32r+55)}$	$\frac{1}{5160960}$
a_5	$-\frac{1}{(64r^6+768r^5+3280r^4+5760r^3+2956r^2-1488r-945)(4r^2+32r+55)(4r^2+40r+91)}$	$-\frac{1}{825753600}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{1}{64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945} &= \lim_{r\to -\frac{3}{2}} -\frac{1}{64r^6 + 768r^5 + 3280r^4 + 5760r^3 + 2956r^2 - 1488r - 945} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $4x^2y'' + 4y'x + (-9+x)\,y = 0$ gives

$$\begin{aligned}
&4x^2\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&\quad + 4\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x \\
&\quad + (-9+x)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(4x^2 y_1''(x) + 4y_1'(x)\, x + (-9+x)\, y_1(x)\right) \ln(x) + 4x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + 4y_1(x)\right) C + 4x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + 4\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + (-9+x) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$4x^2 y_1''(x) + 4y_1'(x)\, x + (-9+x)\, y_1(x) = 0$$

Eq (7) simplifes to

$$\left(4x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + 4y_1(x)\right) C + 4x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) + 4\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + (-9+x) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2)\right) x^2 + 8x \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) C + 4\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x + (-9+x) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = \frac{3}{2}$ and $r_2 = -\frac{3}{2}$ then the above becomes

$$4\left(\sum_{n=0}^{\infty} x^{-\frac{7}{2}+n} b_n \left(n-\frac{3}{2}\right) \left(-\frac{5}{2}+n\right)\right) x^2 + 8x \left(\sum_{n=0}^{\infty} x^{\frac{1}{2}+n} a_n \left(n+\frac{3}{2}\right)\right) C + 4\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{3}{2}\right)\right) x + (-9+x) \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}\right) = 0 \tag{10}$$

Expanding $-\frac{9}{x^{\frac{3}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{9}{x^{\frac{3}{2}}} &= -\frac{9}{x^{\frac{3}{2}}} + \dots \\ &= -\frac{9}{x^{\frac{3}{2}}}\end{aligned}$$

Expanding $\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}} + \dots \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} x^{n-\frac{3}{2}} b_n \left(4n^2 - 16n + 15\right)\right) + \left(\sum_{n=0}^{\infty} (8n + 12)\, C a_n x^{n+\frac{3}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} (-6 + 4n)\, b_n x^{n-\frac{3}{2}}\right) + \sum_{n=0}^{\infty} \left(-9 b_n x^{n-\frac{3}{2}}\right) + \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n - \frac{3}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{3}{2}}$ and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty} (8n + 12)\, C a_n x^{n+\frac{3}{2}} &= \sum_{n=3}^{\infty} C a_{n-3} (8n - 12)\, x^{n-\frac{3}{2}} \\ \sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-\frac{3}{2}}\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - \frac{3}{2}$.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} x^{n-\frac{3}{2}} b_n \left(4n^2 - 16n + 15\right)\right) + \left(\sum_{n=3}^{\infty} C a_{n-3} (8n - 12)\, x^{n-\frac{3}{2}}\right) \\ &+ \left(\sum_{n=0}^{\infty} (-6 + 4n)\, b_n x^{n-\frac{3}{2}}\right) + \sum_{n=0}^{\infty} \left(-9 b_n x^{n-\frac{3}{2}}\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-\frac{3}{2}}\right) = 0\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-8b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{8}$$

For $n = 2$, Eq (2B) gives

$$-8b_2 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_2 + \frac{1}{8} = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{64}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$12C + \frac{1}{64} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{768}$$

For $n = 4$, Eq (2B) gives

$$20Ca_1 + b_3 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$16b_4 + \frac{5}{3072} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{49152}$$

For $n = 5$, Eq (2B) gives

$$28Ca_2 + b_4 + 40b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$40b_5 - \frac{13}{81920} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{13}{3276800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{768}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{768}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{x}{8} + \frac{x^2}{64} - \frac{5x^4}{49152} + \frac{13x^5}{3276800} + O(x^6)}{x^{\frac{3}{2}}}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{1}{768}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right)\right)\ln(x) \right. \\ &\quad \left. + \frac{1 + \frac{x}{8} + \frac{x^2}{64} - \frac{5x^4}{49152} + \frac{13x^5}{3276800} + O(x^6)}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right) \\ &\quad + c_2\left(-\frac{x^{\frac{3}{2}}\left(1 - \frac{x}{16} + \frac{x^2}{640} - \frac{x^3}{46080} + \frac{x^4}{5160960} - \frac{x^5}{825753600} + O(x^6)\right)\ln(x)}{768} \right. \\ &\quad \left. + \frac{1 + \frac{x}{8} + \frac{x^2}{64} - \frac{5x^4}{49152} + \frac{13x^5}{3276800} + O(x^6)}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{3}{2}} \left(1-\frac{x}{16}+\frac{x^2}{640}-\frac{x^3}{46080}+\frac{x^4}{5160960}-\frac{x^5}{825753600}+O\left(x^6\right)\right) \\
& + c_2 \left(-\frac{x^{\frac{3}{2}} \left(1-\frac{x}{16}+\frac{x^2}{640}-\frac{x^3}{46080}+\frac{x^4}{5160960}-\frac{x^5}{825753600}+O\left(x^6\right)\right) \ln\left(x\right)}{768} \right. \\
& \left. + \frac{1+\frac{x}{8}+\frac{x^2}{64}-\frac{5x^4}{49152}+\frac{13x^5}{3276800}+O\left(x^6\right)}{x^{\frac{3}{2}}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^{\frac{3}{2}} \left(1-\frac{x}{16}+\frac{x^2}{640}-\frac{x^3}{46080}+\frac{x^4}{5160960}-\frac{x^5}{825753600}+O\left(x^6\right)\right) \\
& + c_2 \left(-\frac{x^{\frac{3}{2}} \left(1-\frac{x}{16}+\frac{x^2}{640}-\frac{x^3}{46080}+\frac{x^4}{5160960}-\frac{x^5}{825753600}+O\left(x^6\right)\right) \ln\left(x\right)}{768} \right. \\
& \left. + \frac{1+\frac{x}{8}+\frac{x^2}{64}-\frac{5x^4}{49152}+\frac{13x^5}{3276800}+O\left(x^6\right)}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Verified OK.

16.11.1 Maple step by step solution

Let's solve

$4x^2y'' + 4y'x + \left(-9+x\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{\left(-9+x\right)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{\left(-9+x\right)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{-9+x}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{9}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' + 4y'x + (-9 + x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3 + 2r)\,(-3 + 2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k + 2r + 3)\,(2k + 2r - 3) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 9) + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $a_{k+1}\left(4(k+1)^2 + 8(k+1)\,r + 4r^2 - 9\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{4k^2+8kr+4r^2+8k+8r-5}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = -\frac{a_k}{4k^2-4k-8}$

- Series not valid for $r = -\frac{3}{2}$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k}{4k^2-4k-8}$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{a_k}{4k^2+20k+16}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{a_k}{4k^2+20k+16}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 63

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)-(9-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^3 \left(1 - \frac{1}{16}x + \frac{1}{640}x^2 - \frac{1}{46080}x^3 + \frac{1}{5160960}x^4 - \frac{1}{825753600}x^5 + O\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-\frac{1}{64}x^3 + \frac{1}{1024}x^4 - \frac{1}{40960}x\right.\right.}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 96

```
AsymptoticDSolveValue[4*x^2*y''[x]+4*x*y'[x]-(9-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^{11/2}}{5160960} - \frac{x^{9/2}}{46080} + \frac{x^{7/2}}{640} - \frac{x^{5/2}}{16} + x^{3/2}\right) + c_1 \left(\frac{(x-16)x^{3/2}\log(x)}{12288} - \frac{19x^4 - 64x^3 - 2304x^2 - 18432x - 147456}{147456x^{3/2}}\right)$$

16.12 problem 8

Internal problem ID [1424]
Internal file name [`OUTPUT/1425_Sunday_June_05_2022_02_16_34_AM_38405556/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 10y'x + (x + 14)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 10y'x + (x + 14)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{10}{x}$$
$$q(x) = \frac{x + 14}{x^2}$$

Table 794: Table $p(x), q(x)$ singularites.

$p(x) = \frac{10}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+14}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 10y'x + (x + 14)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + 10\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) x + (x + 14) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 10 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 14 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 10 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 14 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 10 x^{n+r} a_n (n+r) + 14 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 10 x^r a_0 r + 14 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 10 x^r r + 14 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+7)(r+2)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+7)(r+2) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -2$$
$$r_2 = -7$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+7)(r+2)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x^2}$$
$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^7}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-7}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 10a_n(n+r) + a_{n-1} + 14a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 + 9n + 9r + 14} \tag{4}$$

Which for the root $r = -2$ becomes

$$a_n = -\frac{a_{n-1}}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 + 11r + 24}$$

Which for the root $r = -2$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+11r+24}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+8)(r+3)(r+9)(r+4)}$$

Which for the root $r = -2$ becomes

$$a_2 = \frac{1}{84}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+11r+24}$	$-\frac{1}{6}$
a_2	$\frac{1}{(r+8)(r+3)(r+9)(r+4)}$	$\frac{1}{84}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)}$$

Which for the root $r = -2$ becomes

$$a_3 = -\frac{1}{2016}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+11r+24}$	$-\frac{1}{6}$
a_2	$\frac{1}{(r+8)(r+3)(r+9)(r+4)}$	$\frac{1}{84}$
a_3	$-\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)}$	$-\frac{1}{2016}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)}$$

Which for the root $r = -2$ becomes

$$a_4 = \frac{1}{72576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+11r+24}$	$-\frac{1}{6}$
a_2	$\frac{1}{(r+8)(r+3)(r+9)(r+4)}$	$\frac{1}{84}$
a_3	$-\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)}$	$-\frac{1}{2016}$
a_4	$\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)}$	$\frac{1}{72576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)(r+12)(r+7)}$$

Which for the root $r = -2$ becomes

$$a_5 = -\frac{1}{3628800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+11r+24}$	$-\frac{1}{6}$
a_2	$\frac{1}{(r+8)(r+3)(r+9)(r+4)}$	$\frac{1}{84}$
a_3	$-\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)}$	$-\frac{1}{2016}$
a_4	$\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)}$	$\frac{1}{72576}$
a_5	$-\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)(r+12)(r+7)}$	$-\frac{1}{3628800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x}{6} + \frac{x^2}{84} - \frac{x^3}{2016} + \frac{x^4}{72576} - \frac{x^5}{3628800} + O(x^6)}{x^2} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= -\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)(r+12)(r+7)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{1}{(r+8)(r+3)(r+9)(r+4)(r+10)(5+r)(r+11)(r+6)(r+12)(r+7)} &= \lim_{r \to -7} -\frac{}{(r+8)(r+} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + 10y'x + (x+14)\,y = 0$ gives

$$\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ 10\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x \\
&+ (x+14)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(x^{2} y_{1}''(x)+10 y_{1}'(x) x+(x+14) y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right. \\&\left.+10 y_{1}(x)\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \\&+10\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right) x+(x+14)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+10 y_{1}'(x) x+(x+14) y_{1}(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+10 y_{1}(x)\right) C \\&+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \\&+10\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right) x+(x+14)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x+9\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C \\&+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2} \\&+10\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right) x+(x+14)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0\end{aligned} \tag{9}$$

Since $r_1 = -2$ and $r_2 = -7$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-3+n} a_n (n-2)\right) x + 9\left(\sum_{n=0}^{\infty} a_n x^{n-2}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-9+n} b_n (n-7)(-8+n)\right) x^2 \\
&+ 10\left(\sum_{n=0}^{\infty} x^{-8+n} b_n (n-7)\right) x + (x+14)\left(\sum_{n=0}^{\infty} b_n x^{n-7}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n-2} a_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 9C a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-7} b_n (-8+n)(n-7)\right) \\
&+ \left(\sum_{n=0}^{\infty} 10 x^{n-7} b_n (n-7)\right) + \left(\sum_{n=0}^{\infty} x^{n-6} b_n\right) + \left(\sum_{n=0}^{\infty} 14 b_n x^{n-7}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-7$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-7} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n-2} a_n (n-2) &= \sum_{n=5}^{\infty} 2C a_{n-5} (n-7)\, x^{n-7} \\
\sum_{n=0}^{\infty} 9C a_n x^{n-2} &= \sum_{n=5}^{\infty} 9C a_{n-5} x^{n-7} \\
\sum_{n=0}^{\infty} x^{n-6} b_n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-7}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-7$.

$$\left(\sum_{n=5}^{\infty} 2C a_{n-5}(n-7)\, x^{n-7}\right)+\left(\sum_{n=5}^{\infty} 9C a_{n-5} x^{n-7}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n-7} b_{n}(-8+n)\,(n-7)\right)+\left(\sum_{n=0}^{\infty} 10 x^{n-7} b_{n}(n-7)\right) \\ +\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-7}\right)+\left(\sum_{n=0}^{\infty} 14 b_{n} x^{n-7}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-4b_1+b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_1+1=0$$

Solving the above for b_1 gives

$$b_1=\frac{1}{4}$$

For $n=2$, Eq (2B) gives

$$-6b_2+b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6b_2+\frac{1}{4}=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{24}$$

For $n=3$, Eq (2B) gives

$$-6b_3+b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6b_3+\frac{1}{24}=0$$

Solving the above for b_3 gives

$$b_3=\frac{1}{144}$$

For $n=4$, Eq (2B) gives

$$-4b_4+b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_4 + \frac{1}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{576}$$

For $n = N$, where $N = 5$ which is the difference between the two roots, we are free to choose $b_5 = 0$. Hence for $n = 5$, Eq (2B) gives

$$5C + \frac{1}{576} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2880}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2880}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2880}\left(\frac{1 - \frac{x}{6} + \frac{x^2}{84} - \frac{x^3}{2016} + \frac{x^4}{72576} - \frac{x^5}{3628800} + O(x^6)}{x^2}\right)\ln(x) + \frac{1 + \frac{x}{4} + \frac{x^2}{24} + \frac{x^3}{144} + \frac{x^4}{576} + O(x^6)}{x^7}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x}{6} + \frac{x^2}{84} - \frac{x^3}{2016} + \frac{x^4}{72576} - \frac{x^5}{3628800} + O(x^6)\right)}{x^2} \\ &\quad + c_2\left(-\frac{1}{2880}\left(\frac{1 - \frac{x}{6} + \frac{x^2}{84} - \frac{x^3}{2016} + \frac{x^4}{72576} - \frac{x^5}{3628800} + O(x^6)}{x^2}\right)\ln(x) + \frac{1 + \frac{x}{4} + \frac{x^2}{24} + \frac{x^3}{144} + \frac{x^4}{576} + O(x^6)}{x^7}\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1 \left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)}{x^2} \\
&\quad + c_2 \left(-\frac{\left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)\ln(x)}{2880x^2} \right. \\
&\quad \left. + \frac{1+\frac{x}{4}+\frac{x^2}{24}+\frac{x^3}{144}+\frac{x^4}{576}+O(x^6)}{x^7}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)}{x^2} \\
&\quad + c_2 \left(-\frac{\left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)\ln(x)}{2880x^2} \right. \\
&\quad \left. + \frac{1+\frac{x}{4}+\frac{x^2}{24}+\frac{x^3}{144}+\frac{x^4}{576}+O(x^6)}{x^7}\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= \frac{c_1 \left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)}{x^2} \\
&\quad + c_2 \left(-\frac{\left(1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^3}{2016}+\frac{x^4}{72576}-\frac{x^5}{3628800}+O(x^6)\right)\ln(x)}{2880x^2} \right. \\
&\quad \left. + \frac{1+\frac{x}{4}+\frac{x^2}{24}+\frac{x^3}{144}+\frac{x^4}{576}+O(x^6)}{x^7}\right)
\end{aligned}
$$

Verified OK.

16.12.1 Maple step by step solution

Let's solve

$x^2y'' + 10y'x + (x + 14)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{10y'}{x} - \frac{(x+14)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{10y'}{x} + \frac{(x+14)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{10}{x}, P_3(x) = \frac{x+14}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 10$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 14$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 10y'x + (x + 14)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (7+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+7)\,(k+r+2) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(7+r)\,(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-7, -2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r+7)\,(k+r+2) + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1} (k+8+r)\,(k+3+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{(k+8+r)(k+3+r)}$$

- Recursion relation for $r = -7$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-4)}$$

- Series not valid for $r = -7$, division by 0 in the recursion relation at $k = 4$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-4)}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k}{(k+6)(k+1)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{(k+6)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+10*x*diff(y(x),x)+(14+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{1}{6}x+\frac{1}{84}x^2-\frac{1}{2016}x^3+\frac{1}{72576}x^4-\frac{1}{3628800}x^5+\mathrm{O}\left(x^6\right)\right)x^5+c_2(\ln\left(x\right)\left(-x^5+\mathrm{O}\left(x^6\right)\right)+(2880+720}{x^7}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 68

```
AsymptoticDSolveValue[x^2*y''[x]+10*x*y'[x]+(14+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^2}{72576}+\frac{1}{x^2}-\frac{x}{2016}-\frac{1}{6x}+\frac{1}{84}\right)+c_1\left(\frac{1}{x^7}+\frac{1}{4x^6}+\frac{1}{24x^5}+\frac{1}{144x^4}+\frac{1}{576x^3}\right)$$

16.13 problem 9

Internal problem ID [1425]
Internal file name [`OUTPUT/1426_Sunday_June_05_2022_02_16_38_AM_26967055/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x+1)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{3+8x}{x\,(x+1)}\\q(x)&=\frac{49x-5}{4x^2\,(x+1)}\end{aligned}$$

Table 796: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3+8x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{49x-5}{4x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\,y'' + \left(32x^2+12x\right)y' + (49x-5)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(32x^2+12x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (49x-5)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 32x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 49x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 32x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 32a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 49x^{1+n+r} a_n = \sum_{n=1}^{\infty} 49a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 32a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 12x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 49a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 12x^{n+r} a_n (n+r) - 5a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 12x^r a_0 r - 5a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 12x^r r - 5x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 8r - 5\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 8r - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{5}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 8r - 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{5}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}
&4a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\
&+ 32a_{n-1}(n+r-1) + 12a_n(n+r) + 49a_{n-1} - 5a_n = 0
\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(5+2n+2r)\, a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(3+n)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-7-2r}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-7-2r}{1+2r}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+32r+63}{4r^2+8r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = 10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-7-2r}{1+2r}$	-4
a_2	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 - 108r^2 - 478r - 693}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -20$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-7-2r}{1+2r}$	-4
a_2	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
a_3	$\frac{-8r^3-108r^2-478r-693}{8r^3+36r^2+46r+15}$	-20

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{8r^3 + 132r^2 + 718r + 1287}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = 35$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-7-2r}{1+2r}$	-4
a_2	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
a_3	$\frac{-8r^3-108r^2-478r-693}{8r^3+36r^2+46r+15}$	-20
a_4	$\frac{8r^3+132r^2+718r+1287}{8r^3+36r^2+46r+15}$	35

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-8r^3 - 156r^2 - 1006r - 2145}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -56$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-7-2r}{1+2r}$	-4
a_2	$\frac{4r^2+32r+63}{4r^2+8r+3}$	10
a_3	$\frac{-8r^3-108r^2-478r-693}{8r^3+36r^2+46r+15}$	-20
a_4	$\frac{8r^3+132r^2+718r+1287}{8r^3+36r^2+46r+15}$	35
a_5	$\frac{-8r^3-156r^2-1006r-2145}{8r^3+36r^2+46r+15}$	-56

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{-8r^3 - 108r^2 - 478r - 693}{8r^3 + 36r^2 + 46r + 15}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\rightarrow r_2} \frac{-8r^3 - 108r^2 - 478r - 693}{8r^3 + 36r^2 + 46r + 15} &= \lim_{r\rightarrow -\frac{5}{2}} \frac{-8r^3 - 108r^2 - 478r - 693}{8r^3 + 36r^2 + 46r + 15} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $4x^2(x+1)\,y'' + (32x^2+12x)\,y' + (49x-5)\,y =$

0 gives

$$
\begin{aligned}
&4x^2(x+1)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(32x^2+12x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(49x-5)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(4x^2(x+1)\,y_1''(x)+\left(32x^2+12x\right)y_1'(x)+(49x-5)\,y_1(x)\right)\ln(x)\right.\\
&\left.+4x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(32x^2+12x)\,y_1(x)}{x}\right)C\\
&+4x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(32x^2+12x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(49x-5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
4x^2(x+1)\,y_1''(x)+\left(32x^2+12x\right)y_1'(x)+(49x-5)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(4x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(32x^2+12x)\,y_1(x)}{x}\right)C\\
&+4x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+\left(32x^2+12x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(49x-5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(8x(x+1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)+4(7x+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C \\ +4\left(x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right) \\ +4\left(8x^2+3x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)+(49x-5)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)=0 \tag{9}$$

Since $r_1 = \frac{1}{2}$ and $r_2 = -\frac{5}{2}$ then the above becomes

$$\left(8x(x+1)\left(\sum_{n=0}^{\infty} x^{-\frac{1}{2}+n} a_n\left(n+\frac{1}{2}\right)\right)+4(7x+2)\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\right)\right)C \\ +4\left(x^3+x^2\right)\left(\sum_{n=0}^{\infty} x^{-\frac{9}{2}+n} b_n\left(n-\frac{5}{2}\right)\left(-\frac{7}{2}+n\right)\right) \\ +4\left(8x^2+3x\right)\left(\sum_{n=0}^{\infty} x^{-\frac{7}{2}+n} b_n\left(n-\frac{5}{2}\right)\right)+(49x-5)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right)=0 \tag{10}$$

Expanding $4C\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 4C\sqrt{x} &= 4C\sqrt{x}+\dots \\ &= 4C\sqrt{x} \end{aligned}$$

Expanding $28C\,x^{\frac{3}{2}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 28C\,x^{\frac{3}{2}} &= 28C\,x^{\frac{3}{2}}+\dots \\ &= 28C\,x^{\frac{3}{2}} \end{aligned}$$

Expanding $8C\sqrt{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 8C\sqrt{x} &= 8C\sqrt{x}+\dots \\ &= 8C\sqrt{x} \end{aligned}$$

Expanding $\frac{1}{x^{\frac{3}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{1}{x^{\frac{3}{2}}} &= \frac{1}{x^{\frac{3}{2}}} + \dots \\ &= \frac{1}{x^{\frac{3}{2}}}\end{aligned}$$

Expanding $\frac{1}{x^{\frac{5}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{1}{x^{\frac{5}{2}}} &= \frac{1}{x^{\frac{5}{2}}} + \dots \\ &= \frac{1}{x^{\frac{5}{2}}}\end{aligned}$$

Expanding $\frac{16}{x^{\frac{3}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{16}{x^{\frac{3}{2}}} &= \frac{16}{x^{\frac{3}{2}}} + \dots \\ &= \frac{16}{x^{\frac{3}{2}}}\end{aligned}$$

Expanding $\frac{6}{x^{\frac{5}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{6}{x^{\frac{5}{2}}} &= \frac{6}{x^{\frac{5}{2}}} + \dots \\ &= \frac{6}{x^{\frac{5}{2}}}\end{aligned}$$

Expanding $\frac{49}{x^{\frac{3}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{49}{x^{\frac{3}{2}}} &= \frac{49}{x^{\frac{3}{2}}} + \dots \\ &= \frac{49}{x^{\frac{3}{2}}}\end{aligned}$$

Expanding $-\frac{5}{x^{\frac{5}{2}}}$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned}-\frac{5}{x^{\frac{5}{2}}} &= -\frac{5}{x^{\frac{5}{2}}} + \dots \\ &= -\frac{5}{x^{\frac{5}{2}}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\left(8n+4\right)C a_{n} x^{\frac{3}{2}+n}\right)+\left(\sum_{n=0}^{\infty}\left(8n+4\right)C a_{n} x^{n+\frac{1}{2}}\right)\\&+\left(\sum_{n=0}^{\infty}28C\,x^{\frac{3}{2}+n} a_{n}\right)+\left(\sum_{n=0}^{\infty}8C\,x^{n+\frac{1}{2}} a_{n}\right)\\&+\left(\sum_{n=0}^{\infty}x^{n-\frac{3}{2}} b_{n}\left(4n^{2}-24n+35\right)\right)+\left(\sum_{n=0}^{\infty}x^{n-\frac{5}{2}} b_{n}\left(4n^{2}-24n+35\right)\right)\\&+\left(\sum_{n=0}^{\infty}\left(32n-80\right) b_{n} x^{n-\frac{3}{2}}\right)+\left(\sum_{n=0}^{\infty}\left(12n-30\right) b_{n} x^{n-\frac{5}{2}}\right)\\&+\left(\sum_{n=0}^{\infty}49x^{n-\frac{3}{2}} b_{n}\right)+\sum_{n=0}^{\infty}\left(-5b_{n} x^{n-\frac{5}{2}}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-\frac{5}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{5}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(8n+4\right)C a_{n} x^{\frac{3}{2}+n}=\sum_{n=4}^{\infty}C a_{n-4}\left(8n-28\right) x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}\left(8n+4\right)C a_{n} x^{n+\frac{1}{2}}=\sum_{n=3}^{\infty}C a_{n-3}\left(8n-20\right) x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}28C\,x^{\frac{3}{2}+n} a_{n}=\sum_{n=4}^{\infty}28C a_{n-4} x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}8C\,x^{n+\frac{1}{2}} a_{n}=\sum_{n=3}^{\infty}8C a_{n-3} x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}x^{n-\frac{3}{2}} b_{n}\left(4n^{2}-24n+35\right)=\sum_{n=1}^{\infty}b_{n-1}\left(4(n-1)^{2}-24n+59\right) x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}\left(32n-80\right) b_{n} x^{n-\frac{3}{2}}=\sum_{n=1}^{\infty}b_{n-1}\left(32n-112\right) x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty}49x^{n-\frac{3}{2}} b_{n}=\sum_{n=1}^{\infty}49b_{n-1} x^{n-\frac{5}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - \frac{5}{2}$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} C a_{n-4}(8n-28)\, x^{n-\frac{5}{2}}\right) + \left(\sum_{n=3}^{\infty} C a_{n-3}(8n-20)\, x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=4}^{\infty} 28 C a_{n-4} x^{n-\frac{5}{2}}\right) + \left(\sum_{n=3}^{\infty} 8 C a_{n-3} x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=1}^{\infty} b_{n-1}\left(4(n-1)^2 - 24n + 59\right) x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{5}{2}} b_n \left(4n^2 - 24n + 35\right)\right) + \left(\sum_{n=1}^{\infty} b_{n-1}(32n-112)\, x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} (12n-30)\, b_n x^{n-\frac{5}{2}}\right) + \left(\sum_{n=1}^{\infty} 49 b_{n-1} x^{n-\frac{5}{2}}\right) + \sum_{n=0}^{\infty} \left(-5 b_n x^{n-\frac{5}{2}}\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$4b_0 - 8b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$4 - 8b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$16b_1 - 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8 - 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = 1$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$12C + 36 = 0$$

Which is solved for C. Solving for C gives

$$C = -3$$

For $n = 4$, Eq (2B) gives

$$(32a_0 + 20a_1) C + 64b_3 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$144 + 16b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -9$$

For $n = 5$, Eq (2B) gives

$$(40a_1 + 28a_2) C + 100b_4 + 40b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-1260 + 40b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{63}{2}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -3$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) &= (-3) \left(\sqrt{x}\left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + O(x^6)\right)\right) \ln(x) \\ &\quad + \frac{1 + \frac{x}{2} + x^2 - 9x^4 + \frac{63x^5}{2} + O(x^6)}{x^{\frac{5}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + O(x^6)\right) \\ &\quad + c_2 \left((-3) \left(\sqrt{x}\left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + O(x^6)\right)\right) \ln(x) \right. \\ &\quad \left. + \frac{1 + \frac{x}{2} + x^2 - 9x^4 + \frac{63x^5}{2} + O(x^6)}{x^{\frac{5}{2}}} \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right) \\ &\quad + c_2\left(-3\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\ln\left(x\right) + \frac{1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O(x^6)}{x^{\frac{5}{2}}}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right) \\ &\quad + c_2\left(-3\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\ln\left(x\right) + \frac{1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O(x^6)}{x^{\frac{5}{2}}}\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right) \\ &\quad + c_2\left(-3\sqrt{x}\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\ln\left(x\right) + \frac{1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O(x^6)}{x^{\frac{5}{2}}}\right)\end{aligned}$$

Verified OK.

16.13.1 Maple step by step solution

Let's solve

$4x^2(x+1)\,y''+(32x^2+12x)\,y'+(49x-5)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(49x-5)y}{4x^2(x+1)}-\frac{(3+8x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+8x)y'}{x(x+1)} + \frac{(49x-5)y}{4x^2(x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3+8x}{x(x+1)}, P_3(x) = \frac{49x-5}{4x^2(x+1)}\right]$

  - $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = 5$

  - $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4x^2(x+1)\, y'' + 4x(3+8x)\, y' + (49x-5)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (32u^2 - 52u + 20)\left(\frac{d}{du}y(u)\right) + (49u - 54)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

  $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $4a_{0}r(4+r)\,u^{-1+r}+\left(4a_{1}(1+r)\,(5+r)-2a_{0}(4r^{2}+22r+27)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)\,(k+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-4,0\}$

- Each term must be 0

  $4a_{1}(1+r)\,(5+r)-2a_{0}(4r^{2}+22r+27)=0$

- Each term in the series must be 0, giving the recursion relation

  $4a_{k+1}(k+1+r)\,(k+5+r)-2a_{k}(4k^{2}+8kr+4r^{2}+22k+22r+27)+a_{k-1}(2k+5+2r)^{2}=($

- Shift index using $k->k+1$

  $4a_{k+2}(k+2+r)\,(k+6+r)-2a_{k+1}\left(4(k+1)^{2}+8(k+1)\,r+4r^{2}+22k+49+22r\right)+a_{k}(2k+$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+8kra_{k}-16kra_{k+1}+4r^{2}a_{k}-8r^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+28ra_{k}-60ra_{k+1}+49a_{k}-106a_{k+1}}{4(k+2+r)(k+6+r)}$

- Recursion relation for $r=-4$

  $a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}+4ka_{k+1}+a_{k}+6a_{k+1}}{4(k-2)(k+2)}$

- Series not valid for $r=-4$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}+4ka_{k+1}+a_{k}+6a_{k+1}}{4(k-2)(k+2)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+49a_{k}-106a_{k+1}}{4(k+2)(k+6)}$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x)=0,y(x),type='series
```

$$y(x) = \frac{c_1x^3(1-4x+10x^2-20x^3+35x^4-56x^5+\mathrm{O}(x^6))+c_2(\ln(x)((-36)x^3+144x^4-360x^5+\mathrm{O}(x^6))+}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]+4*x*(3+8*x)*y'[x]-(5-49*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(35x^{9/2} - 20x^{7/2} + 10x^{5/2} - 4x^{3/2} + \sqrt{x}\right) + c_1\left(\frac{62x^4 - 20x^3 + 2x^2 + x + 2}{2x^{5/2}} + 3\sqrt{x}(4x - 1)\log(x)\right)$$

16.14 problem 10

Internal problem ID [1426]
Internal file name [`OUTPUT/1427_Sunday_June_05_2022_02_16_43_AM_54492134/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''-x(3+10x)\,y'+30yx=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$x^2(x+1)\,y''+\left(-10x^2-3x\right)y'+30yx=0$$

Or

$$x\left(x^2y''-10y'x+y''x+30y-3y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(x^2+x\right)y''+(-3-10x)\,y'+30y=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(-10x^2-3x\right)y'+30yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = -\frac{3+10x}{x(x+1)}$$
$$q(x) = \frac{30}{x(x+1)}$$

Table 798: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3+10x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{30}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + \left(-10x^2 - 3x\right) y' + 30yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^2(x+1)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\
&+\left(-10x^2-3x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+30\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)x=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\,(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-10x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}30x^{1+n+r}a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)\,(n+r-1)&=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-10x^{1+n+r}a_n(n+r)\right)&=\sum_{n=1}^{\infty}\left(-10a_{n-1}(n+r-1)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty}30x^{1+n+r}a_n&=\sum_{n=1}^{\infty}30a_{n-1}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-10a_{n-1}(n+r-1)\,x^{n+r}\right) \\
&+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=1}^{\infty}30a_{n-1}x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) - 3x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ -10a_{n-1}(n+r-1) - 3a_n(n+r) + 30a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 13n - 13r + 42)}{n^2 + 2nr + r^2 - 4n - 4r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - 5n + 6)}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 11r - 30}{r^2 - 2r - 3}$$

Which for the root $r = 4$ becomes

$$a_1 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-30}{r^2-2r-3}$	$-\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-4+r)(r-5)^2(r-6)}{r^4 - 2r^3 - 7r^2 + 8r + 12}$$

Which for the root $r = 4$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-30}{r^2-2r-3}$	$-\frac{2}{5}$
a_2	$\frac{(-4+r)(r-5)^2(r-6)}{r^4-2r^3-7r^2+8r+12}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r-6)(r-5)^2(-4+r)^2}{r^5 + 3r^4 - 5r^3 - 15r^2 + 4r + 12}$$

Which for the root $r = 4$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-30}{r^2-2r-3}$	$-\frac{2}{5}$
a_2	$\frac{(-4+r)(r-5)^2(r-6)}{r^4-2r^3-7r^2+8r+12}$	0
a_3	$-\frac{(r-6)(r-5)^2(-4+r)^2}{r^5+3r^4-5r^3-15r^2+4r+12}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)\,r\,(r^4 + 5r^3 + 5r^2 - 5r - 6)}$$

Which for the root $r = 4$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-30}{r^2-2r-3}$	$-\frac{2}{5}$
a_2	$\frac{(-4+r)(r-5)^2(r-6)}{r^4-2r^3-7r^2+8r+12}$	0
a_3	$-\frac{(r-6)(r-5)^2(-4+r)^2}{r^5+3r^4-5r^3-15r^2+4r+12}$	0
a_4	$\frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)r(r^4+5r^3+5r^2-5r-6)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r-6)(r-5)^2(-4+r)^2(r-3)(r-2)}{(5+r)(r+1)^2(r+3)(r+2)r(r+4)}$$

Which for the root $r = 4$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-30}{r^2-2r-3}$	$-\frac{2}{5}$
a_2	$\frac{(-4+r)(r-5)^2(r-6)}{r^4-2r^3-7r^2+8r+12}$	0
a_3	$-\frac{(r-6)(r-5)^2(-4+r)^2}{r^5+3r^4-5r^3-15r^2+4r+12}$	0
a_4	$\frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)r(r^4+5r^3+5r^2-5r-6)}$	0
a_5	$-\frac{(r-6)(r-5)^2(-4+r)^2(r-3)(r-2)}{(5+r)(r+1)^2(r+3)(r+2)r(r+4)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^4\left(1 - \frac{2x}{5} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)\,r\,(r^4+5r^3+5r^2-5r-6)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)\,r\,(r^4+5r^3+5r^2-5r-6)} &= \lim_{r\to 0} \frac{(r-6)(r-5)^2(-4+r)^2(r-3)}{(r+4)\,r\,(r^4+5r^3+5r^2-5r-6)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x)\ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x)\ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2(x+1)\,y'' + (-10x^2-3x)\,y' + 30yx = 0$

gives

$$
\begin{aligned}
&x^{2}(x+1)\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^{2}}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&+\left(-10x^{2}-3x\right)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+30\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)x=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^{2}(x+1)\,y_1''(x)+\left(-10x^{2}-3x\right)y_1'(x)+30y_1(x)\,x\right)\ln(x)\right.\\
&\left.+x^{2}(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)+\frac{\left(-10x^{2}-3x\right)y_1(x)}{x}\right)C\\
&+x^{2}(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&+\left(-10x^{2}-3x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+30\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^{2}(x+1)\,y_1''(x)+\left(-10x^{2}-3x\right)y_1'(x)+30y_1(x)\,x=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^{2}(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)+\frac{\left(-10x^{2}-3x\right)y_1(x)}{x}\right)C\\
&+x^{2}(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^{2}}{x^{2}}-\frac{b_n x^{n+r_2}(n+r_2)}{x^{2}}\right)\right)\\
&+\left(-10x^{2}-3x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+30\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{gathered}
\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)+(-11x-4)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
+x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \\
+\left(-10x^2-3x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)+30\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x=0
\end{gathered} \tag{9}$$

Since $r_1 = 4$ and $r_2 = 0$ then the above becomes

$$\begin{gathered}
\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{3+n} a_n(n+4)\right)+(-11x-4)\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\
+x^2(x+1)\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) \\
+\left(-10x^2-3x\right)\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right)+30\left(\sum_{n=0}^{\infty} b_n x^n\right) x=0
\end{gathered} \tag{10}$$

Which simplifies to

$$\begin{gathered}
\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n(n+4)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n(n+4)\right)+\sum_{n=0}^{\infty}\left(-11C\, x^{n+5} a_n\right) \\
+\sum_{n=0}^{\infty}\left(-4C\, x^{n+4} a_n\right)+\left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n(n-1)\right)+\left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) \\
+\sum_{n=0}^{\infty}\left(-10n\, x^{1+n} b_n\right)+\sum_{n=0}^{\infty}\left(-3x^n b_n n\right)+\left(\sum_{n=0}^{\infty} 30x^{1+n} b_n\right)=0
\end{gathered} \tag{2A}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+4) = \sum_{n=5}^{\infty} 2C a_{n-5} (n-1)\, x^n$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) = \sum_{n=4}^{\infty} 2C a_{-4+n} n\, x^n$$

$$\sum_{n=0}^{\infty} \left(-11C\, x^{n+5} a_n\right) = \sum_{n=5}^{\infty} \left(-11C a_{n-5} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-4C\, x^{n+4} a_n\right) = \sum_{n=4}^{\infty} \left(-4C a_{-4+n} x^n\right)$$

$$\sum_{n=0}^{\infty} n\, x^{1+n} b_n (n-1) = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} (-2+n)\, x^n$$

$$\sum_{n=0}^{\infty} \left(-10n\, x^{1+n} b_n\right) = \sum_{n=1}^{\infty} \left(-10(n-1)\, b_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} 30 x^{1+n} b_n = \sum_{n=1}^{\infty} 30 b_{n-1} x^n$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=5}^{\infty} 2C a_{n-5} (n-1)\, x^n\right) + \left(\sum_{n=4}^{\infty} 2C a_{-4+n} n\, x^n\right) + \sum_{n=5}^{\infty} \left(-11C a_{n-5} x^n\right) \\ &+ \sum_{n=4}^{\infty} \left(-4C a_{-4+n} x^n\right) + \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} (-2+n)\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) + \sum_{n=1}^{\infty} \left(-10(n-1)\, b_{n-1} x^n\right) \\ &+ \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \left(\sum_{n=1}^{\infty} 30 b_{n-1} x^n\right) = 0 \end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-3b_1 + 30b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 + 30 = 0$$

Solving the above for b_1 gives

$$b_1 = 10$$

For $n = 2$, Eq (2B) gives

$$-4b_2 + 20b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2 + 200 = 0$$

Solving the above for b_2 gives

$$b_2 = 50$$

For $n = 3$, Eq (2B) gives

$$12b_2 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$600 - 3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 200$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 1200 = 0$$

Which is solved for C. Solving for C gives

$$C = -300$$

For $n = 5$, Eq (2B) gives

$$(-3a_0 + 6a_1)\, C + 2b_4 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1620 + 5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -324$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -300$ and all b_n, then the second solution becomes

$$y_2(x) = (-300)\left(x^4\left(1-\frac{2x}{5}+O(x^6)\right)\right)\ln(x)+1+10x+50x^2+200x^3-324x^5+O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x)\\ &= c_1x^4\left(1-\frac{2x}{5}+O(x^6)\right) + c_2\left((-300)\left(x^4\left(1-\frac{2x}{5}+O(x^6)\right)\right)\ln(x)+1+10x\right.\\ &\qquad \left. + 50x^2+200x^3-324x^5+O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1x^4\left(1-\frac{2x}{5}+O(x^6)\right)\\ &\quad + c_2\left(-300x^4\left(1-\frac{2x}{5}+O(x^6)\right)\ln(x)+1+10x+50x^2+200x^3-324x^5+O(x^6)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1x^4\left(1-\frac{2x}{5}+O(x^6)\right)\\ &\quad + c_2\left(-300x^4\left(1-\frac{2x}{5}+O(x^6)\right)\ln(x)+1+10x+50x^2+200x^3-324x^5+O(x^6)\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1x^4\left(1-\frac{2x}{5}+O(x^6)\right)\\ &\quad + c_2\left(-300x^4\left(1-\frac{2x}{5}+O(x^6)\right)\ln(x)+1+10x+50x^2+200x^3-324x^5+O(x^6)\right)\end{aligned}$$

Verified OK.

16.14.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y''+(-10x^2-3x)\,y'+30yx=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{30y}{x(x+1)}+\frac{(3+10x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(3+10x)y'}{x(x+1)}+\frac{30y}{x(x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{3+10x}{x(x+1)},P_3(x)=\frac{30}{x(x+1)}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Bigg|_{x=-1}=-7$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Bigg|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(x+1)\,y''+(-3-10x)\,y'+30y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(7-10u)\left(\frac{d}{du}y(u)\right)+30y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-8+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 8\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{30a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{3}$

- Express in terms of a_0

  $a_2 = \frac{50a_0}{7}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{5}$
- Express in terms of a_0

  $a_3 = -\frac{40a_0}{7}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{3a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{15a_0}{7}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{2a_4}{15}$
- Express in terms of a_0

  $a_5 = -\frac{2a_0}{7}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{30}{7}u + \frac{50}{7}u^2 - \frac{40}{7}u^3 + \frac{15}{7}u^4 - \frac{2}{7}u^5\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$
- Recursion relation for $r = 8$

  $a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$
- Solution for $r = 8$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+8}\right), b_{1+k} = \frac{b_k(k+3)(k+2)}{(k+9)(1+k)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 48

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3+10*x)*diff(y(x),x)+30*x*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^4 \left(1 - \frac{2}{5}x + \mathrm{O}\left(x^6\right)\right) + \ln\left(x\right)\left(43200x^4 - 17280x^5 + \mathrm{O}\left(x^6\right)\right) c_2 \\
& + \left(-144 - 1440x - 7200x^2 - 28800x^3 - 90720x^4 + 82944x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 48

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-x*(3+10*x)*y'[x]+30*x*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_2 \left(x^4 - \frac{2x^5}{5}\right) + c_1 \left(745x^4 - 300x^4 \log(x) + 200x^3 + 50x^2 + 10x + 1\right)
$$

16.15 problem 11

Internal problem ID [1427]
Internal file name [`OUTPUT/1428_Sunday_June_05_2022_02_16_47_AM_97370590/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x+1)\,y' - 3y(x+3) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^2 + x)\,y' + (-3x - 9)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{x}$$
$$q(x) = -\frac{3(x+3)}{x^2}$$

Table 800: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{3(x+3)}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(x^2 + x\right) y' + (-3x - 9)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(x^2 + x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-3x - 9) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + x^{n+r} a_n (n+r) - 9a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 9a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 9x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 9\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 9\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) - 3a_{n-1} - 9a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-4)}{n^2+2nr+r^2-9} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}(n-1)}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3-r}{r^2+2r-8}$$

Which for the root $r = 3$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-3+r}{r^3+8r^2+11r-20}$$

Which for the root $r = 3$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0
a_2	$\frac{-3+r}{r^3+8r^2+11r-20}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3 - r}{(r + 6) r (5 + r) (r + 4)}$$

Which for the root $r = 3$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0
a_2	$\frac{-3+r}{r^3+8r^2+11r-20}$	0
a_3	$\frac{3-r}{(r+6)r(5+r)(r+4)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-3 + r}{(r + 6) (5 + r) (r + 4) (r + 7) (r + 1)}$$

Which for the root $r = 3$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0
a_2	$\frac{-3+r}{r^3+8r^2+11r-20}$	0
a_3	$\frac{3-r}{(r+6)r(5+r)(r+4)}$	0
a_4	$\frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+1)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{3 - r}{(r + 6) (5 + r) (r + 4) (r + 7) (r + 8) (r + 2)}$$

Which for the root $r = 3$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0
a_2	$\frac{-3+r}{r^3+8r^2+11r-20}$	0
a_3	$\frac{3-r}{(r+6)r(5+r)(r+4)}$	0
a_4	$\frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+1)}$	0
a_5	$\frac{3-r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r+2)}$	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-3 + r}{(r + 6)(5 + r)(r + 4)(r + 7)(r + 8)(r + 9)(r + 3)}$$

Which for the root $r = 3$ becomes

$$a_6 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-r}{r^2+2r-8}$	0
a_2	$\frac{-3+r}{r^3+8r^2+11r-20}$	0
a_3	$\frac{3-r}{(r+6)r(5+r)(r+4)}$	0
a_4	$\frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+1)}$	0
a_5	$\frac{3-r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r+2)}$	0
a_6	$\frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r+9)(r+3)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\ &= x^3 \left(1 + O(x^7)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_6 \\
&= \frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r+9)(r+3)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r+9)(r+3)} &= \lim_{r\to -3} \frac{-3+r}{(r+6)(5+r)(r+4)(r+7)(r+8)(r-} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2 y'' + (x^2+x)\,y' + (-3x-9)\,y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^2+x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + (-3x-9)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2 y_1''(x) + (x^2+x)\,y_1'(x) + (-3x-9)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2+x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^2+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-3x-9)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) + (x^2+x)\,y_1'(x) + (-3x-9)\,y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^2+x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^2+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-3x-9)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + (x^2+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (-3x-9)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 3$ and $r_2 = -3$ then the above becomes

$$\left(\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right) x + 2\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-5+n} b_n (n-3)(-4+n)\right) x^2 + (x^2+x)\left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-3)\right) + (-3x-9)\left(\sum_{n=0}^{\infty} b_n x^{n-3}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} C\, x^{4+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \left(\sum_{n=0}^{\infty} x^{n-3} b_n (-4+n)(n-3)\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-3)\right) + \left(\sum_{n=0}^{\infty} x^{n-3} b_n (n-3)\right) + \sum_{n=0}^{\infty} \left(-3x^{n-2} b_n\right) + \sum_{n=0}^{\infty} \left(-9 b_n x^{n-3}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-3$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-3} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} C\, x^{4+n} a_n = \sum_{n=7}^{\infty} C a_{n-7} x^{n-3}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=6}^{\infty} 2C a_{n-6} (n-3)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} x^{n-2} b_n (n-3) = \sum_{n=1}^{\infty} b_{n-1} (-4+n)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} \left(-3x^{n-2} b_n\right) = \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-3}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-3$.

$$\begin{aligned} &\left(\sum_{n=7}^{\infty} C a_{n-7} x^{n-3}\right) + \left(\sum_{n=6}^{\infty} 2C a_{n-6} (n-3)\, x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-3} b_n (-4+n)\,(n-3)\right) + \left(\sum_{n=1}^{\infty} b_{n-1}(-4+n)\, x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-3} b_n (n-3)\right) + \sum_{n=1}^{\infty} \left(-3b_{n-1} x^{n-3}\right) + \sum_{n=0}^{\infty} \left(-9b_n x^{n-3}\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-5b_1 - 6b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_1 - 6 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{6}{5}$$

For $n=2$, Eq (2B) gives

$$-8b_2 - 5b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_2 + 6 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$-4b_2 - 9b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3 - 9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{3}$$

For $n = 4$, Eq (2B) gives

$$-8b_4 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-8b_4 + 1 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{8}$$

For $n = 5$, Eq (2B) gives

$$-5b_5 - 2b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 - \frac{1}{4} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{1}{20}$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + \frac{1}{20} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{120}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{120}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{120}\left(x^3\left(1+O\left(x^7\right)\right)\right)\ln(x) + \frac{1 - \frac{6x}{5} + \frac{3x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^7)}{x^3}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3\left(1+O\left(x^7\right)\right) \\ &\quad + c_2\left(-\frac{1}{120}\left(x^3\left(1+O\left(x^7\right)\right)\right)\ln(x) + \frac{1 - \frac{6x}{5} + \frac{3x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^7)}{x^3}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3\left(1+O\left(x^7\right)\right) + c_2\left(-\frac{x^3(1+O(x^7))\ln(x)}{120} + \frac{1 - \frac{6x}{5} + \frac{3x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^7)}{x^3}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^3\left(1+O\left(x^7\right)\right) \\ &\quad + c_2\left(-\frac{x^3(1+O(x^7))\ln(x)}{120} + \frac{1 - \frac{6x}{5} + \frac{3x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^7)}{x^3}\right) \end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^3\left(1+O\left(x^7\right)\right) + c_2\left(-\frac{x^3(1+O(x^7))\ln(x)}{120} + \frac{1 - \frac{6x}{5} + \frac{3x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^5}{20} + O(x^7)}{x^3}\right)$$

Verified OK.

16.15.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\, y' + (-3x - 9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x+3)y}{x^2} - \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{x} - \frac{3(x+3)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x+1}{x}, P_3(x) = -\frac{3(x+3)}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -9$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x + 1)\, y' + (-3x - 9)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(3+r)\,(-3+r)\, x^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k}(k+r+3)\,(k+r-3)+a_{k-1}(k-4+r)\right) x^{k+r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$(3+r)\,(-3+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{-3,3\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k}(k+r+3)\,(k+r-3)+a_{k-1}(k-4+r)=0$$

- Shift index using $k->k+1$

  $$a_{k+1}(k+4+r)\,(k-2+r)+a_{k}(k+r-3)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=-\frac{a_{k}(k+r-3)}{(k+4+r)(k-2+r)}$$

- Recursion relation for $r=-3$; series terminates at $k=6$

  $$a_{k+1}=-\frac{a_{k}(k-6)}{(k+1)(k-5)}$$

- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=5$

  $$a_{k+1}=-\frac{a_{k}(k-6)}{(k+1)(k-5)}$$

- Recursion relation for $r=3$

  $$a_{k+1}=-\frac{a_{k}k}{(k+7)(k+1)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-3*(3+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 (1 + \mathrm{O}(x^6)) + \frac{c_2(-86400 + 103680x - 64800x^2 + 28800x^3 - 10800x^4 + 4320x^5 + \mathrm{O}(x^6))}{x^3}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1+x)*y'[x]-3*(3+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^3 + c_1 \left(\frac{1}{x^3} - \frac{6}{5x^2} + \frac{x}{8} + \frac{3}{4x} - \frac{1}{3}\right)$$

16.16 problem 12

Internal problem ID [1428]
Internal file name [`OUTPUT/1429_Sunday_June_05_2022_02_16_52_AM_48502818/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1-2x)\,y' - (x+4)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-2x^2 + x\right)y' + (-x-4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x-1}{x}$$
$$q(x) = -\frac{x+4}{x^2}$$

Table 802: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-2x^2 + x)\, y' + (-x - 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (-2x^2 + x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x - 4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} - 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(2n+2r-1)}{n^2+2nr+r^2-4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-1}(2n+3)}{n\,(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+2r}{r^2+2r-3}$$

Which for the root $r = 2$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r^2+2r-3}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2+8r+3}{(r+3)\,(-1+r)\,r\,(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{7}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r^2+2r-3}$	1
a_2	$\frac{4r^2+8r+3}{(r+3)(-1+r)r(r+4)}$	$\frac{7}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 36r^2 + 46r + 15}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r^2+2r-3}$	1
a_2	$\frac{4r^2+8r+3}{(r+3)(-1+r)r(r+4)}$	$\frac{7}{12}$
a_3	$\frac{8r^3+36r^2+46r+15}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{4}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{11}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r^2+2r-3}$	1
a_2	$\frac{4r^2+8r+3}{(r+3)(-1+r)r(r+4)}$	$\frac{7}{12}$
a_3	$\frac{8r^3+36r^2+46r+15}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{4}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$	$\frac{11}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}{r(-1+r)(5+r)(r+4)(r+3)^2(r+1)(r+6)(r+2)(r+7)}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{143}{5760}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+2r}{r^2+2r-3}$	1
a_2	$\frac{4r^2+8r+3}{(r+3)(-1+r)r(r+4)}$	$\frac{7}{12}$
a_3	$\frac{8r^3+36r^2+46r+15}{(r+3)(-1+r)r(r+4)(5+r)(r+1)}$	$\frac{1}{4}$
a_4	$\frac{16r^4+128r^3+344r^2+352r+105}{(r+3)r(r+4)(5+r)(r^2+8r+12)(r^2-1)}$	$\frac{11}{128}$
a_5	$\frac{32r^5+400r^4+1840r^3+3800r^2+3378r+945}{r(-1+r)(5+r)(r+4)(r+3)^2(r+1)(r+6)(r+2)(r+7)}$	$\frac{143}{5760}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{(r+3)\, r\,(r+4)\,(5+r)\,(r^2+8r+12)\,(r^2-1)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{(r+3)\, r\,(r+4)\,(5+r)\,(r^2+8r+12)\,(r^2-1)} &= \lim_{r\to -2} \frac{16r^4 + 128r^3 + 344r^2 + 352r + 105}{(r+3)\, r\,(r+4)\,(5+r)\,(r^2+8r+12)\,(r^2-1)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x) \ln (x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= Cy_1'(x) \ln (x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x) \ln (x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= Cy_1''(x) \ln (x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x) \ln (x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2 y'' + (-2x^2 + x) y' + (-x-4) y = 0$ gives

$$\begin{aligned}
&x^2 \left(Cy_1''(x) \ln (x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-2x^2 + x) \left(Cy_1'(x) \ln (x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ (-x-4) \left(Cy_1(x) \ln (x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^{2} y_{1}''(x)+\left(-2x^{2}+x\right) y_{1}'(x)+\left(-x-4\right) y_{1}(x)\right) \ln \left(x\right)+x^{2} \left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right.\right.$$
$$\left.\left.+\frac{\left(-2x^{2}+x\right) y_{1}(x)}{x}\right) C+x^{2} \left(\sum_{n=0}^{\infty} \left(\frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)}{x^{2}}\right)\right)\right) \tag{7}$$
$$+\left(-2x^{2}+x\right) \left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)}{x}\right)+\left(-x-4\right) \left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+\left(-2x^{2}+x\right) y_{1}'(x)+\left(-x-4\right) y_{1}(x)=0$$

Eq (7) simplifes to

$$\left(x^{2} \left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+\frac{\left(-2x^{2}+x\right) y_{1}(x)}{x}\right) C$$
$$+x^{2} \left(\sum_{n=0}^{\infty} \left(\frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)}{x^{2}}\right)\right)\Bigg) \tag{8}$$
$$+\left(-2x^{2}+x\right) \left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}} \left(n+r_{2}\right)}{x}\right)+\left(-x-4\right) \left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n} \left(n+r_{1}\right)\right) x-2 \left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right) x\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n} \left(n+r_{2}\right) \left(-1+n+r_{2}\right)\right) x^{2} \tag{9}$$
$$+\left(-2x^{2}+x\right) \left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n} \left(n+r_{2}\right)\right)+\left(-x-4\right) \left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) x\right) C \\ + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 \\ + (-2x^2 + x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + (-x-4)\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{3+n} a_n\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2 - 5n + 6)\right) + \sum_{n=0}^{\infty} \left(-2x^{n-1} b_n (n-2)\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-x^{n-1} b_n\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n} (n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{3+n} a_n\right) = \sum_{n=5}^{\infty} \left(-2C a_{n-5} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{n-1} b_n (n-2)\right) = \sum_{n=1}^{\infty} \left(-2 b_{n-1} (-3+n)\, x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{n-1} b_n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{n-2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\left(\sum_{n=4}^{\infty} 2Ca_{-4+n}(n-2)\, x^{n-2}\right) + \sum_{n\, =5}^{\infty} \left(-2Ca_{n-5}x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2-5n+6)\right) + \sum_{n\, =1}^{\infty} \left(-2b_{n-1}(-3+n)\, x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \sum_{n\, =1}^{\infty} \left(-b_{n-1}x^{n-2}\right) + \sum_{n\, =0}^{\infty} \left(-4b_n x^{n-2}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1+3b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1+3=0$$

Solving the above for b_1 gives

$$b_1=1$$

For $n=2$, Eq (2B) gives

$$b_1-4b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1-4b_2=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$-3b_3-b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3-\frac{1}{4}=0$$

Solving the above for b_3 gives

$$b_3=-\frac{1}{12}$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$(-2a_0 + 6a_1) C - 5b_4 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{4} + 5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{1}{20}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(x^2\left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right)\right) \ln(x) + \frac{1 + x + \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^5}{20} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{16}\left(x^2\left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right)\right) \ln(x) \right.\\
&\quad \left. + \frac{1 + x + \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^5}{20} + O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right) \ln(x)}{16} \right. \\
&\qquad \left. + \frac{1 + x + \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^5}{20} + O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right) \ln(x)}{16} \right. \\
&\qquad \left. + \frac{1 + x + \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^5}{20} + O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x^2 \left(1 + x + \frac{7x^2}{12} + \frac{x^3}{4} + \frac{11x^4}{128} + \frac{143x^5}{5760} + O(x^6)\right) \ln(x)}{16} \right. \\
&\qquad \left. + \frac{1 + x + \frac{x^2}{4} - \frac{x^3}{12} + \frac{x^5}{20} + O(x^6)}{x^2}\right)
\end{aligned}
$$

Verified OK.

16.16.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 + x)\, y' + (-x - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+4)y}{x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} - \frac{(x+4)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = -\frac{x+4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(2x - 1)\, y' + (-x - 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-2) - a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+2)\,(k+r-2) - 2\left(k-\tfrac{1}{2}+r\right) a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+3+r)\,(k+r-1) - 2\left(k+\tfrac{1}{2}+r\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(2k+2r+1)a_k}{(k+3+r)(k+r-1)}$$

- Recursion relation for $r = -2$

$$a_{k+1} = \frac{(2k-3)a_k}{(k+1)(k-3)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 3$

$$a_{k+1} = \frac{(2k-3)a_k}{(k+1)(k-3)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{(2k+5)a_k}{(k+5)(k+1)}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+1}=\frac{(2k+5)a_k}{(k+5)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x)*diff(y(x),x)-(4+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4\left(1+x+\frac{7}{12}x^2+\frac{1}{4}x^3+\frac{11}{128}x^4+\frac{143}{5760}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(9x^4+9x^5+\mathrm{O}\left(x^6\right)\right)+\left(-144-144x\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 75

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-2*x)*y'[x]-(4+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{3x^4-16x^3+48x^2+192x+192}{192x^2}-\frac{1}{16}x^2\log(x)\right)+c_2\left(\frac{11x^6}{128}+\frac{x^5}{4}+\frac{7x^4}{12}+x^3+x^2\right)$$

16.17 problem 13

Internal problem ID [1429]
Internal file name [`OUTPUT/1430_Sunday_June_05_2022_02_16_56_AM_12855953/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x+1)\, y'' - 4y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2 + x\right) y'' - 2y - 4y' = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{4}{x\,(x+1)}$$
$$q(x) = -\frac{2}{x\,(x+1)}$$

Table 804: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{4}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{2}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x+1)\,y'' - 4y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) - 4\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2 a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 4(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 4 r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-5+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-5+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^5\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) - 4a_n(n+r) - 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-3)\, a_{n-1}}{n-5+r} \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = -\frac{(n+2)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2-r}{-4+r}$$

Which for the root $r = 5$ becomes

$$a_1 = -3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{-4+r}$	-3

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2-3r+2}{(-4+r)\,(r-3)}$$

Which for the root $r = 5$ becomes

$$a_2 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{-4+r}$	-3
a_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	6

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(-1+r)}{(-4+r)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_3 = -10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{-4+r}$	-3
a_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	6
a_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	-10

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)}{(-4+r)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_4 = 15$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{-4+r}$	-3
a_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	6
a_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	-10
a_4	$\frac{r(1+r)}{(-4+r)(r-3)}$	15

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(1+r)(2+r)}{(-4+r)(r-3)}$$

Which for the root $r = 5$ becomes

$$a_5 = -21$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{-4+r}$	-3
a_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	6
a_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	-10
a_4	$\frac{r(1+r)}{(-4+r)(r-3)}$	15
a_5	$-\frac{(1+r)(2+r)}{(-4+r)(r-3)}$	-21

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^5\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^5\left(1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= -\frac{(1+r)(2+r)}{(-4+r)(r-3)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{(1+r)(2+r)}{(-4+r)(r-3)} &= \lim_{r\to 0} -\frac{(1+r)(2+r)}{(-4+r)(r-3)} \\ &= -\frac{1}{6} \end{aligned}$$

The limit is $-\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) - 4(n+r) b_n - 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)(n-2) + b_n n(n-1) - 4n b_n - 2b_{n-1} = 0 \quad \text{(4A)}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r-3) b_{n-1}}{n-5+r} \quad (5)$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{(n-3) b_{n-1}}{n-5} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{-2+r}{-4+r}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{-4+r}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{(-4+r)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{-4+r}$	$-\frac{1}{2}$
b_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r(-1+r)}{(-4+r)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{-4+r}$	$-\frac{1}{2}$
b_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	$\frac{1}{6}$
b_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(1+r)}{(-4+r)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{-4+r}$	$-\frac{1}{2}$
b_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	$\frac{1}{6}$
b_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	0
b_4	$\frac{r(1+r)}{(-4+r)(r-3)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(1+r)(2+r)}{(-4+r)(r-3)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{-4+r}$	$-\frac{1}{2}$
b_2	$\frac{r^2-3r+2}{(-4+r)(r-3)}$	$\frac{1}{6}$
b_3	$-\frac{r(-1+r)}{(-4+r)(r-3)}$	0
b_4	$\frac{r(1+r)}{(-4+r)(r-3)}$	0
b_5	$-\frac{(1+r)(2+r)}{(-4+r)(r-3)}$	$-\frac{1}{6}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^5}{6} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^5 \left(1-3x+6x^2-10x^3+15x^4-21x^5+O(x^6)\right)+c_2\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^5}{6}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^5 \left(1-3x+6x^2-10x^3+15x^4-21x^5+O(x^6)\right)+c_2\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^5}{6}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^5 \left(1-3x+6x^2-10x^3+15x^4-21x^5+O(x^6)\right)+c_2\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^5}{6}+O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^5 \left(1-3x+6x^2-10x^3+15x^4-21x^5+O(x^6)\right)+c_2\left(1-\frac{x}{2}+\frac{x^2}{6}-\frac{x^5}{6}+O(x^6)\right)
$$

Verified OK.

16.17.1 Maple step by step solution

Let's solve

$x(x+1)\, y'' - 4y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x+1)} + \frac{4y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x(x+1)} - \frac{2y}{x(x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{4}{x(x+1)}, P_3(x) = -\frac{2}{x(x+1)}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right) \Big|_{x=-1} = 4$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right) \Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)\, y'' - 4y' - 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - u)\left(\frac{d^2}{du^2}y(u)\right) - 4\frac{d}{du}y(u) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k+4+r) + a_k(k+1+r)\,(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $((-k-r-4)\, a_{k+1} + a_k(k+r-2))\,(k+1+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{k+4+r}$
- Recursion relation for $r = -3$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)}{k+1}$
- Apply recursion relation for $k = 0$

  $a_1 = -5a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -2a_1$
- Express in terms of a_0

  $a_2 = 10a_0$
- Apply recursion relation for $k = 2$

  $a_3 = -a_2$
- Express in terms of a_0

  $a_3 = -10a_0$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{2}$
- Express in terms of a_0

  $a_4 = 5a_0$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -a_0$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y(u) = a_0 \cdot (-u^5 + 5u^4 - 10u^3 + 10u^2 - 5u + 1)$
- Revert the change of variables $u = x + 1$

  $[y = -a_0 x^5]$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{k+4}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{5}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{10}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{1}{2}u + \frac{1}{10}u^2\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = a_0\left(\frac{3}{5} - \frac{3}{10}x + \frac{1}{10}x^2\right)\right]$
- Combine solutions and rename parameters

  $\left[y = -a_0 x^5 + b_0\left(\frac{3}{5} - \frac{3}{10}x + \frac{1}{10}x^2\right)\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
Order:=6;
dsolve(x*(1+x)*diff(y(x),x$2)-4*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^5 (1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + O(x^6)) + c_2 (2880 - 1440x + 480x^2 - 480x^5 + O(x^6))$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 48

```
AsymptoticDSolveValue[x*(1+x)*y''[x]-4*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^2}{6} - \frac{x}{2} + 1\right) + c_2 (15x^9 - 10x^8 + 6x^7 - 3x^6 + x^5)$$

16.18 problem 14

Internal problem ID [1430]
Internal file name [`OUTPUT/1431_Sunday_June_05_2022_02_16_59_AM_71911011/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+2x)\,y''+x(9+13x)\,y'+(7+5x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(7+5x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{9+13x}{x\,(1+2x)}$$
$$q(x)=\frac{7+5x}{x^2\,(1+2x)}$$

Table 806: Table $p(x), q(x)$ singularites.

$p(x) = \frac{9+13x}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = \frac{7+5x}{x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\, y'' + \left(13x^2 + 9x\right) y' + (7+5x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(1+2x) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(13x^2+9x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (7+5x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 13x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 13x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 13a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n = \sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 13a_{n-1}(n+r-1)\, x^{n+r}\right) \\
&+ \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 7a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 9x^{n+r} a_n (n+r) + 7a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 9x^r a_0 r + 7a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 9x^r r + 7x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 8r + 7\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 8r + 7 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -7$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 8r + 7\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^7}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-7}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + 13a_{n-1}(n+r-1) + 9a_n(n+r) + 7a_n + 5a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(2n^2 + 4nr + 2r^2 + 7n + 7r - 4)}{n^2 + 2nr + r^2 + 8n + 8r + 7} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-1}(2n^2 + 3n - 9)}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2r^2 - 11r - 5}{r^2 + 10r + 16}$$

Which for the root $r = -1$ becomes

$$a_1 = \frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^4 + 52r^3 + 211r^2 + 273r + 90}{(r+8)(r+2)(r+9)(r+3)}$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{5}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$
a_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$-\frac{5}{28}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^6 - 180r^5 - 1550r^4 - 6375r^3 - 12752r^2 - 11265r - 3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$$

Which for the root $r = -1$ becomes

$$a_3 = \frac{5}{42}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$
a_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$-\frac{5}{28}$
a_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	$\frac{5}{42}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^6 + 336r^5 + 2680r^4 + 10200r^3 + 19129r^2 + 16149r + 4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$$

Which for the root $r = -1$ becomes

$$a_4 = -\frac{5}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$
a_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$-\frac{5}{28}$
a_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	$\frac{5}{42}$
a_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	$-\frac{5}{48}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-32r^6 - 624r^5 - 4640r^4 - 16680r^3 - 29978r^2 - 24591r - 6615}{(r + 12)(r + 2)(r + 3)(r + 10)(r + 4)(r + 11)}$$

Which for the root $r = -1$ becomes

$$a_5 = \frac{7}{66}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$
a_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$-\frac{5}{28}$
a_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	$\frac{5}{42}$
a_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	$-\frac{5}{48}$
a_5	$\frac{-32r^6-624r^5-4640r^4-16680r^3-29978r^2-24591r-6615}{(r+12)(r+2)(r+3)(r+10)(r+4)(r+11)}$	$\frac{7}{66}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{(r + 13)(r + 11)(r + 4)(r + 3)(r + 2)(r + 12)}$$

Which for the root $r = -1$ becomes

$$a_6 = -\frac{21}{176}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{4}{7}$
a_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$-\frac{5}{28}$
a_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	$\frac{5}{42}$
a_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	$-\frac{5}{48}$
a_5	$\frac{-32r^6-624r^5-4640r^4-16680r^3-29978r^2-24591r-6615}{(r+12)(r+2)(r+3)(r+10)(r+4)(r+11)}$	$\frac{7}{66}$
a_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{(r+13)(r+11)(r+4)(r+3)(r+2)(r+12)}$	$-\frac{21}{176}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\ &= \frac{1 + \frac{4x}{7} - \frac{5x^2}{28} + \frac{5x^3}{42} - \frac{5x^4}{48} + \frac{7x^5}{66} - \frac{21x^6}{176} + O(x^7)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{(r+13)(r+11)(r+4)(r+3)(r+2)(r+12)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{(r+13)(r+11)(r+4)(r+3)(r+2)(r+12)} &= \lim_{r\to -7} \frac{64r^6 + 1152r^5 + 8080r^4 + 2}{(r+13)(r+11)(r} \\ &= -\frac{3003}{160} \end{aligned}$$

The limit is $-\frac{3003}{160}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-7}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\
&+ 13b_{n-1}(n+r-1) + 9b_n(n+r) + 7b_n + 5b_{n-1} = 0
\end{aligned} \tag{4}
$$

Which for for the root $r = -7$ becomes

$$
2b_{n-1}(n-8)(n-9) + b_n(n-7)(n-8) + 13b_{n-1}(n-8) + 9b_n(n-7) + 7b_n + 5b_{n-1} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(2n^2 + 4nr + 2r^2 + 7n + 7r - 4)}{n^2 + 2nr + r^2 + 8n + 8r + 7} \tag{5}
$$

Which for the root $r = -7$ becomes

$$
b_n = -\frac{b_{n-1}(2n^2 - 21n + 45)}{n^2 - 6n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -7$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2r^2 + 11r + 5}{r^2 + 10r + 16}
$$

Which for the root $r = -7$ becomes

$$
b_1 = \frac{26}{5}
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^4 + 52r^3 + 211r^2 + 273r + 90}{(r^2 + 10r + 16)(r^2 + 12r + 27)}$$

Which for the root $r = -7$ becomes

$$b_2 = \frac{143}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$
b_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$\frac{143}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8r^6 + 180r^5 + 1550r^4 + 6375r^3 + 12752r^2 + 11265r + 3150}{(r^2 + 10r + 16)(r^2 + 12r + 27)(r^2 + 14r + 40)}$$

Which for the root $r = -7$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$
b_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$\frac{143}{20}$
b_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^6 + 336r^5 + 2680r^4 + 10200r^3 + 19129r^2 + 16149r + 4410}{(r + 11)(r^2 + 14r + 40)(r^2 + 12r + 27)(r + 2)}$$

Which for the root $r = -7$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$
b_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$\frac{143}{20}$
b_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	0
b_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32r^6 + 624r^5 + 4640r^4 + 16680r^3 + 29978r^2 + 24591r + 6615}{(r + 12)(r + 2)(r + 3)(r^2 + 14r + 40)(r + 11)}$$

Which for the root $r = -7$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$
b_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$\frac{143}{20}$
b_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	0
b_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	0
b_5	$\frac{-32r^6-624r^5-4640r^4-16680r^3-29978r^2-24591r-6615}{(r+12)(r+2)(r+3)(r+10)(r+4)(r+11)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64r^6 + 1152r^5 + 8080r^4 + 27840r^3 + 48556r^2 + 39048r + 10395}{(r+13)(r+11)(r+4)(r+3)(r+2)(r+12)}$$

Which for the root $r = -7$ becomes

$$b_6 = -\frac{3003}{160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r^2-11r-5}{r^2+10r+16}$	$\frac{26}{5}$
b_2	$\frac{4r^4+52r^3+211r^2+273r+90}{(r+8)(r+2)(r+9)(r+3)}$	$\frac{143}{20}$
b_3	$\frac{-8r^6-180r^5-1550r^4-6375r^3-12752r^2-11265r-3150}{(r+8)(r+2)(r+9)(r+3)(r+10)(r+4)}$	0
b_4	$\frac{16r^6+336r^5+2680r^4+10200r^3+19129r^2+16149r+4410}{(r+11)(r+4)(r+10)(r+3)(r+9)(r+2)}$	0
b_5	$\frac{-32r^6-624r^5-4640r^4-16680r^3-29978r^2-24591r-6615}{(r+12)(r+2)(r+3)(r+10)(r+4)(r+11)}$	0
b_6	$\frac{64r^6+1152r^5+8080r^4+27840r^3+48556r^2+39048r+10395}{(r+13)(r+11)(r+4)(r+3)(r+2)(r+12)}$	$-\frac{3003}{160}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 \dots\right) \\ &= \frac{1 + \frac{26x}{5} + \frac{143x^2}{20} - \frac{3003x^6}{160} + O(x^7)}{x^7} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= \frac{c_1\left(1 + \frac{4x}{7} - \frac{5x^2}{28} + \frac{5x^3}{42} - \frac{5x^4}{48} + \frac{7x^5}{66} - \frac{21x^6}{176} + O(x^7)\right)}{x} \\ &+ \frac{c_2\left(1 + \frac{26x}{5} + \frac{143x^2}{20} - \frac{3003x^6}{160} + O(x^7)\right)}{x^7} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1+\frac{4x}{7}-\frac{5x^2}{28}+\frac{5x^3}{42}-\frac{5x^4}{48}+\frac{7x^5}{66}-\frac{21x^6}{176}+O(x^7)\right)}{x} \\
&+ \frac{c_2\left(1+\frac{26x}{5}+\frac{143x^2}{20}-\frac{3003x^6}{160}+O(x^7)\right)}{x^7}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = &\frac{c_1\left(1+\frac{4x}{7}-\frac{5x^2}{28}+\frac{5x^3}{42}-\frac{5x^4}{48}+\frac{7x^5}{66}-\frac{21x^6}{176}+O(x^7)\right)}{x} \\
&+ \frac{c_2\left(1+\frac{26x}{5}+\frac{143x^2}{20}-\frac{3003x^6}{160}+O(x^7)\right)}{x^7}
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = &\frac{c_1\left(1+\frac{4x}{7}-\frac{5x^2}{28}+\frac{5x^3}{42}-\frac{5x^4}{48}+\frac{7x^5}{66}-\frac{21x^6}{176}+O(x^7)\right)}{x} \\
&+ \frac{c_2\left(1+\frac{26x}{5}+\frac{143x^2}{20}-\frac{3003x^6}{160}+O(x^7)\right)}{x^7}
\end{aligned}
$$

Verified OK.

16.18.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y''+(13x^2+9x)\,y'+(7+5x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(7+5x)y}{x^2(1+2x)}-\frac{(9+13x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(9+13x)y'}{x(1+2x)}+\frac{(7+5x)y}{x^2(1+2x)}=0$

- □ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{9+13x}{x(1+2x)}, P_3(x) = \frac{7+5x}{x^2(1+2x)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 9$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 7$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(1+2x)\, y'' + x(9+13x)\, y' + (7+5x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(7+r)\,(1+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r+7)\,(k+r+1)+a_{k-1}(k+4+r)\,(2k-1+2r)\right)x^{k+r}\right)=$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-7,-1\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+4+r)\,a_{k-1}+a_{k}(k+r+7)\,(k+r+1)=0$

- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+r+5)\,a_{k}+a_{k+1}(k+8+r)\,(k+2+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(2k+2r+1)(k+r+5)a_{k}}{(k+8+r)(k+2+r)}$

- Recursion relation for $r=-7$; series terminates at $k=2$

  $a_{k+1}=-\frac{(2k-13)(k-2)a_{k}}{(k+1)(k-5)}$

- Apply recursion relation for $k=0$

  $a_{1}=\frac{26a_{0}}{5}$

- Apply recursion relation for $k=1$

  $a_{2}=\frac{11a_{1}}{8}$

- Express in terms of a_{0}

  $a_{2}=\frac{143a_{0}}{20}$

- Terminating series solution of the ODE for $r=-7$. Use reduction of order to find the second

  $y=a_{0}\cdot\left(1+\frac{26}{5}x+\frac{143}{20}x^{2}\right)$

- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(2k-1)(k+4)a_{k}}{(k+7)(k+1)}$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{(2k-1)(k+4)a_k}{(k+7)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\cdot\left(1+\frac{26}{5}x+\frac{143}{20}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{1+k}=-\frac{(2k-1)(4+k)b_k}{(k+7)(1+k)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x)=0,y(x),type='series',
```

$$y(x)=\frac{c_1\left(1+\frac{4}{7}x-\frac{5}{28}x^2+\frac{5}{42}x^3-\frac{5}{48}x^4+\frac{7}{66}x^5+\mathrm{O}\left(x^6\right)\right)}{x}+\frac{c_2\left(-86400-449280x-617760x^2+\mathrm{O}\left(x^6\right)\right)}{x^7}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 54

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]+x*(9+13*x)*y'[x]+(7+5*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(-\frac{5x^3}{48}+\frac{5x^2}{42}-\frac{5x}{28}+\frac{1}{x}+\frac{4}{7}\right)+c_1\left(\frac{1}{x^7}+\frac{26}{5x^6}+\frac{143}{20x^5}\right)$$

16.19 problem 15

Internal problem ID [1431]
Internal file name [`OUTPUT/1432_Sunday_June_05_2022_02_17_03_AM_1594100/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1+2x)\,y''-2x(-x+4)\,y'-(7+5x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{x-4}{2x\,(1+2x)}\\q(x)&=-\frac{7+5x}{4x^2\,(1+2x)}\end{aligned}$$

Table 808: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-4}{2x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = -\frac{7+5x}{4x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(1+2x)\, y'' + \left(2x^2 - 8x\right) y' + (-5x - 7)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(1+2x) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(2x^2 - 8x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-5x-7) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-8x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-7a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-5x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 8a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-8x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-5a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - 8x^{n+r} a_n (n+r) - 7a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) - 8x^r a_0 r - 7a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 8x^r r - 7x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 12r - 7\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 12r - 7 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 12r - 7\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{7}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 8a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + 2a_{n-1}(n+r-1) - 8a_n(n+r) - 5a_{n-1} - 7a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(8n^2 + 16nr + 8r^2 - 22n - 22r + 9)}{4n^2 + 8nr + 4r^2 - 12n - 12r - 7} \tag{4}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_n = -\frac{a_{n-1}(4n^2 + 17n + 15)}{2n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-8r^2 + 6r + 5}{4r^2 - 4r - 15}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_1 = -\frac{18}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	$-\frac{18}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{32r^3 - 32r^2 - 14r + 5}{8r^3 - 12r^2 - 50r + 75}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_2 = \frac{39}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	$-\frac{18}{5}$
a_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	$\frac{39}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-128r^4 + 32r^3 + 152r^2 + 22r - 15}{(4r^2 + 12r - 7)(4r^2 - 16r + 15)}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_3 = -\frac{663}{28}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	$-\frac{18}{5}$
a_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	$\frac{39}{4}$
a_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{(4r^2+12r-7)(4r^2-16r+15)}$	$-\frac{663}{28}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^4 + 256r^3 - 544r^2 - 304r + 105}{16r^4 - 232r^2 + 384r - 135}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_4 = \frac{13923}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	$-\frac{18}{5}$
a_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	$\frac{39}{4}$
a_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{(4r^2+12r-7)(4r^2-16r+15)}$	$-\frac{663}{28}$
a_4	$\frac{256r^4+256r^3-544r^2-304r+105}{16r^4-232r^2+384r-135}$	$\frac{13923}{256}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1024r^5 - 3840r^4 - 640r^3 + 7200r^2 + 2924r - 1155}{32r^5 + 80r^4 - 560r^3 + 40r^2 + 1098r - 495}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_5 = -\frac{7735}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	$-\frac{18}{5}$
a_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	$\frac{39}{4}$
a_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{(4r^2+12r-7)(4r^2-16r+15)}$	$-\frac{663}{28}$
a_4	$\frac{256r^4+256r^3-544r^2-304r+105}{16r^4-232r^2+384r-135}$	$\frac{13923}{256}$
a_5	$\frac{-1024r^5-3840r^4-640r^3+7200r^2+2924r-1155}{32r^5+80r^4-560r^3+40r^2+1098r-495}$	$-\frac{7735}{64}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{7}{2}}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^{\frac{7}{2}}\left(1 - \frac{18x}{5} + \frac{39x^2}{4} - \frac{663x^3}{28} + \frac{13923x^4}{256} - \frac{7735x^5}{64} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{256r^4 + 256r^3 - 544r^2 - 304r + 105}{16r^4 - 232r^2 + 384r - 135} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{256r^4 + 256r^3 - 544r^2 - 304r + 105}{16r^4 - 232r^2 + 384r - 135} &= \lim_{r\rightarrow -\frac{1}{2}} \frac{256r^4 + 256r^3 - 544r^2 - 304r + 105}{16r^4 - 232r^2 + 384r - 135} \\ &= -\frac{35}{128} \end{aligned}$$

The limit is $-\frac{35}{128}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 8b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\ + 2b_{n-1}(n+r-1) - 8b_n(n+r) - 5b_{n-1} - 7b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$\begin{aligned} 8b_{n-1}\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) + 4b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \\ + 2b_{n-1}\left(n-\frac{3}{2}\right) - 8b_n\left(n-\frac{1}{2}\right) - 5b_{n-1} - 7b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(8n^2 + 16nr + 8r^2 - 22n - 22r + 9)}{4n^2 + 8nr + 4r^2 - 12n - 12r - 7} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-1}(8n^2 - 30n + 22)}{4n^2 - 16n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{8r^2 - 6r - 5}{4r^2 - 4r - 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{32r^3 - 32r^2 - 14r + 5}{(4r^2 + 4r - 15)(2r - 5)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	0
b_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{128r^4 - 32r^3 - 152r^2 - 22r + 15}{(2r-5)(2r-3)(4r^2+12r-7)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	0
b_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	0
b_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{16r^4-16r^3-160r^2+292r-105}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256r^4 + 256r^3 - 544r^2 - 304r + 105}{(2r+9)(2r-1)(2r-5)(2r-3)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = -\frac{35}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	0
b_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	0
b_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{16r^4-16r^3-160r^2+292r-105}$	0
b_4	$\frac{256r^4+256r^3-544r^2-304r+105}{16r^4-232r^2+384r-135}$	$-\frac{35}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024r^5 + 3840r^4 + 640r^3 - 7200r^2 - 2924r + 1155}{(2r-1)(2r-5)(2r-3)(4r^2+28r+33)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{63}{64}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-8r^2+6r+5}{4r^2-4r-15}$	0
b_2	$\frac{32r^3-32r^2-14r+5}{8r^3-12r^2-50r+75}$	0
b_3	$\frac{-128r^4+32r^3+152r^2+22r-15}{16r^4-16r^3-160r^2+292r-105}$	0
b_4	$\frac{256r^4+256r^3-544r^2-304r+105}{16r^4-232r^2+384r-135}$	$-\frac{35}{128}$
b_5	$\frac{-1024r^5-3840r^4-640r^3+7200r^2+2924r-1155}{32r^5+80r^4-560r^3+40r^2+1098r-495}$	$\frac{63}{64}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{7}{2}}(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots) \\
&= \frac{1 - \frac{35x^4}{128} + \frac{63x^5}{64} + O(x^6)}{\sqrt{x}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{7}{2}} \left(1 - \frac{18x}{5} + \frac{39x^2}{4} - \frac{663x^3}{28} + \frac{13923x^4}{256} - \frac{7735x^5}{64} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - \frac{35x^4}{128} + \frac{63x^5}{64} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{7}{2}} \left(1 - \frac{18x}{5} + \frac{39x^2}{4} - \frac{663x^3}{28} + \frac{13923x^4}{256} - \frac{7735x^5}{64} + O(x^6)\right) \\
&+ \frac{c_2 \left(1 - \frac{35x^4}{128} + \frac{63x^5}{64} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{7}{2}} \left(1 - \frac{18x}{5} + \frac{39x^2}{4} - \frac{663x^3}{28} + \frac{13923x^4}{256} - \frac{7735x^5}{64} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{35x^4}{128} + \frac{63x^5}{64} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{7}{2}} \left(1 - \frac{18x}{5} + \frac{39x^2}{4} - \frac{663x^3}{28} + \frac{13923x^4}{256} - \frac{7735x^5}{64} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{35x^4}{128} + \frac{63x^5}{64} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

16.19.1 Maple step by step solution

Let's solve

$4x^2(1+2x)\,y'' + (2x^2-8x)\,y' + (-5x-7)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(7+5x)y}{4x^2(1+2x)} - \frac{(x-4)y'}{2x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-4)y'}{2x(1+2x)} - \frac{(7+5x)y}{4x^2(1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-4}{2x(1+2x)}, P_3(x) = -\frac{7+5x}{4x^2(1+2x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{7}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(1+2x)\, y'' + 2x(x-4)\, y' + (-5x-7)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-7+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-7) + a_{k-1}(2k-1+2r)\,(4k-9+4r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-7+2r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(k-\frac{1}{2}+r\right)\left(k-\frac{9}{4}+r\right)a_{k-1}+4\left(k+r-\frac{7}{2}\right)\left(k+r+\frac{1}{2}\right)a_k=0$$

- Shift index using $k->k+1$

$$8\left(k+r+\frac{1}{2}\right)\left(k-\frac{5}{4}+r\right)a_k+4\left(k-\frac{5}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{(2k+2r+1)(4k+4r-5)a_k}{(2k-5+2r)(2k+3+2r)}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{2k(4k-7)a_k}{(2k-6)(2k+2)}$$

- Series not valid for $r=-\frac{1}{2}$, division by 0 in the recursion relation at $k=3$

$$a_{k+1}=-\frac{2k(4k-7)a_k}{(2k-6)(2k+2)}$$

- Recursion relation for $r=\frac{7}{2}$

$$a_{k+1}=-\frac{(2k+8)(4k+9)a_k}{(2k+2)(2k+10)}$$

- Solution for $r=\frac{7}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1}=-\frac{(2k+8)(4k+9)a_k}{(2k+2)(2k+10)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
Order:=6;
dsolve(4*x^2*(1+2*x)*diff(y(x),x$2)-2*x*(4-x)*diff(y(x),x)-(7+5*x)*y(x)=0,y(x),type='series'
```

$$y(x) = \frac{c_1x^4\left(1-\frac{18}{5}x+\frac{39}{4}x^2-\frac{663}{28}x^3+\frac{13923}{256}x^4-\frac{7735}{64}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(-144-\frac{405}{8}x^4+\frac{729}{4}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 67

```
AsymptoticDSolveValue[4*x^2*(1+2*x)*y''[x]-2*x*(4-x)*y'[x]-(7+5*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{1}{\sqrt{x}}-\frac{35x^{7/2}}{128}\right)+c_2\left(\frac{13923x^{15/2}}{256}-\frac{663x^{13/2}}{28}+\frac{39x^{11/2}}{4}-\frac{18x^{9/2}}{5}+x^{7/2}\right)$$

16.20 problem 16

Internal problem ID [1432]
Internal file name [OUTPUT/1433_Sunday_June_05_2022_02_17_06_AM_65559690/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2(x+3)\,y''-x(15+x)\,y'-20y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{15+x}{3x\,(x+3)}$$
$$q(x)=-\frac{20}{3x^2\,(x+3)}$$

Table 810: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{15+x}{3x(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{20}{3x^2(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2(x+3)\, y'' + \left(-x^2 - 15x\right) y' - 20y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2(x+3)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 - 15x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 20\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-15x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-20 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-15x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-20 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) - 15x^{n+r} a_n (n+r) - 20 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) - 15x^r a_0 r - 20 a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) - 15x^r r - 20x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 24r - 20\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 24r - 20 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{10}{3} \\ r_2 &= -\frac{2}{3} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 24r - 20\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^{\frac{10}{3}} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{2}{3}}} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{10}{3}} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \le n$ the recursive equation is

$$\begin{aligned} 3a_{n-1}(n+r-1)(n+r-2) + 9a_n(n+r)(n+r-1) \\ - a_{n-1}(n+r-1) - 15a_n(n+r) - 20a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(3n^2 + 6nr + 3r^2 - 10n - 10r + 7)}{9n^2 + 18nr + 9r^2 - 24n - 24r - 20} \tag{4}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_n = -\frac{a_{n-1}(3n^2 + 10n + 7)}{9n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{10}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-3r^2 + 4r}{9r^2 - 6r - 35}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_1 = -\frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$-\frac{4}{9}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(3r^2 + 2r - 1)}{(3r+8)(9r^2 - 6r - 35)}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_2 = \frac{13}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$-\frac{4}{9}$
a_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$\frac{13}{81}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-3r^4 - 11r^3 - 12r^2 - 4r}{81r^4 + 459r^3 + 135r^2 - 2523r - 3080}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_3 = -\frac{832}{15309}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$-\frac{4}{9}$
a_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$\frac{13}{81}$
a_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	$-\frac{832}{15309}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+1)(r+2)r(r+3)}{81r^4 + 702r^3 + 1107r^2 - 3738r - 8624}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_4 = \frac{2470}{137781}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$-\frac{4}{9}$
a_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$\frac{13}{81}$
a_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	$-\frac{832}{15309}$
a_4	$\frac{(r+1)(r+2)r(r+3)}{81r^4+702r^3+1107r^2-3738r-8624}$	$\frac{2470}{137781}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+4)(r+1)(r+2)r(r+3)}{243r^5 + 3240r^4 + 12420r^3 - 90r^2 - 76503r - 91630}$$

Which for the root $r = \frac{10}{3}$ becomes

$$a_5 = -\frac{21736}{3720087}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$-\frac{4}{9}$
a_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$\frac{13}{81}$
a_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	$-\frac{832}{15309}$
a_4	$\frac{(r+1)(r+2)r(r+3)}{81r^4+702r^3+1107r^2-3738r-8624}$	$\frac{2470}{137781}$
a_5	$-\frac{(r+4)(r+1)(r+2)r(r+3)}{243r^5+3240r^4+12420r^3-90r^2-76503r-91630}$	$-\frac{21736}{3720087}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{10}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{10}{3}}\left(1 - \frac{4x}{9} + \frac{13x^2}{81} - \frac{832x^3}{15309} + \frac{2470x^4}{137781} - \frac{21736x^5}{3720087} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{(r+1)(r+2)r(r+3)}{81r^4 + 702r^3 + 1107r^2 - 3738r - 8624} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{(r+1)(r+2)r(r+3)}{81r^4 + 702r^3 + 1107r^2 - 3738r - 8624} &= \lim_{r \to -\frac{2}{3}} \frac{(r+1)(r+2)r(r+3)}{81r^4 + 702r^3 + 1107r^2 - 3738r - 8624} \\ &= \frac{7}{59049} \end{aligned}$$

The limit is $\frac{7}{59049}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &3b_{n-1}(n+r-1)(n+r-2) + 9b_n(n+r)(n+r-1) \\ &- b_{n-1}(n+r-1) - 15b_n(n+r) - 20b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{2}{3}$ becomes

$$\begin{aligned} &3b_{n-1}\left(n-\frac{5}{3}\right)\left(n-\frac{8}{3}\right) + 9b_n\left(n-\frac{2}{3}\right)\left(n-\frac{5}{3}\right) \\ &- b_{n-1}\left(n-\frac{5}{3}\right) - 15b_n\left(n-\frac{2}{3}\right) - 20b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(3n^2 + 6nr + 3r^2 - 10n - 10r + 7)}{9n^2 + 18nr + 9r^2 - 24n - 24r - 20} \tag{5}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_n = -\frac{b_{n-1}(3n^2 - 14n + 15)}{9n^2 - 36n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r(3r - 4)}{9r^2 - 6r - 35}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_1 = \frac{4}{27}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$\frac{4}{27}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(3r^2 + 2r - 1)}{(3r + 8)(9r^2 - 6r - 35)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_2 = -\frac{1}{243}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$\frac{4}{27}$
b_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$-\frac{1}{243}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(3r^2+8r+4)\, r(r+1)}{(3r+11)\,(3r+8)\,(9r^2-6r-35)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$\frac{4}{27}$
b_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$-\frac{1}{243}$
b_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r+1)\,(r+2)\, r(r+3)}{(3r+14)\,(3r-7)\,(3r+8)\,(3r+11)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_4 = \frac{7}{59049}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$\frac{4}{27}$
b_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$-\frac{1}{243}$
b_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	0
b_4	$\frac{(r+1)(r+2)r(r+3)}{81r^4+702r^3+1107r^2-3738r-8624}$	$\frac{7}{59049}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r+4)\,(r+1)\,(r+2)\, r(r+3)}{(3r+11)\,(3r-7)\,(3r+14)\,(9r^2+66r+85)}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_5 = -\frac{28}{531441}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{9r^2-6r-35}$	$\frac{4}{27}$
b_2	$\frac{r(3r^2+2r-1)}{(3r+8)(9r^2-6r-35)}$	$-\frac{1}{243}$
b_3	$\frac{-3r^4-11r^3-12r^2-4r}{81r^4+459r^3+135r^2-2523r-3080}$	0
b_4	$\frac{(r+1)(r+2)r(r+3)}{81r^4+702r^3+1107r^2-3738r-8624}$	$\frac{7}{59049}$
b_5	$-\frac{(r+4)(r+1)(r+2)r(r+3)}{(3r+11)(3r-7)(3r+14)(9r^2+66r+85)}$	$-\frac{28}{531441}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{10}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{4x}{27} - \frac{x^2}{243} + \frac{7x^4}{59049} - \frac{28x^5}{531441} + O(x^6)}{x^{\frac{2}{3}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{10}{3}}\left(1 - \frac{4x}{9} + \frac{13x^2}{81} - \frac{832x^3}{15309} + \frac{2470x^4}{137781} - \frac{21736x^5}{3720087} + O(x^6)\right) \\ &+ \frac{c_2\left(1 + \frac{4x}{27} - \frac{x^2}{243} + \frac{7x^4}{59049} - \frac{28x^5}{531441} + O(x^6)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{10}{3}}\left(1 - \frac{4x}{9} + \frac{13x^2}{81} - \frac{832x^3}{15309} + \frac{2470x^4}{137781} - \frac{21736x^5}{3720087} + O(x^6)\right) \\ &+ \frac{c_2\left(1 + \frac{4x}{27} - \frac{x^2}{243} + \frac{7x^4}{59049} - \frac{28x^5}{531441} + O(x^6)\right)}{x^{\frac{2}{3}}} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{10}{3}} \left(1 - \frac{4x}{9} + \frac{13x^2}{81} - \frac{832x^3}{15309} + \frac{2470x^4}{137781} - \frac{21736x^5}{3720087} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{4x}{27} - \frac{x^2}{243} + \frac{7x^4}{59049} - \frac{28x^5}{531441} + O(x^6)\right)}{x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{10}{3}} \left(1 - \frac{4x}{9} + \frac{13x^2}{81} - \frac{832x^3}{15309} + \frac{2470x^4}{137781} - \frac{21736x^5}{3720087} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{4x}{27} - \frac{x^2}{243} + \frac{7x^4}{59049} - \frac{28x^5}{531441} + O(x^6)\right)}{x^{\frac{2}{3}}}$$

Verified OK.

16.20.1 Maple step by step solution

Let's solve

$3x^2(x+3)\, y'' + (-x^2 - 15x)\, y' - 20y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(15+x)y'}{3x(x+3)} + \frac{20y}{3x^2(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(15+x)y'}{3x(x+3)} - \frac{20y}{3x^2(x+3)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{15+x}{3x(x+3)}, P_3(x) = -\frac{20}{3x^2(x+3)}\right]$

- ◦ $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

  $\left((x+3) \cdot P_2(x)\right)\Big|_{x=-3} = \frac{4}{3}$

- ◦ $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

$$\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = 0$$

- $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $3x^2(x+3)\,y'' - x(15+x)\,y' - 20y = 0$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(3u^3 - 18u^2 + 27u)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2 - 9u + 36)\left(\frac{d}{du}y(u)\right) - 20y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $9a_0 r(1+3r)\,u^{-1+r} + (9a_1(1+r)(4+3r) - a_0(18r^2 - 9r + 20))\,u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)(3k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9r(1+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, -\frac{1}{3}\right\}$

- Each term must be 0

$$9a_1(1+r)(4+3r) - a_0(18r^2 - 9r + 20) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3(-6a_k + a_{k-1} + 9a_{k+1})k^2 + (6(-6a_k + a_{k-1} + 9a_{k+1})r + 9a_k - 10a_{k-1} + 63a_{k+1})k + 3(-6a_k +$$

- Shift index using $k- > k+1$

$$3(-6a_{k+1} + a_k + 9a_{k+2})(k+1)^2 + (6(-6a_{k+1} + a_k + 9a_{k+2})r + 9a_{k+1} - 10a_k + 63a_{k+2})(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2a_k - 18r^2a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2 + 6kr + 3r^2 + 13k + 13r + 14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2 + 13k + 14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2 + 13k + 14)}, 36a_1 - 20a_0 = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2 + 13k + 14)}, 36a_1 - 20a_0 = 0\right]$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2 + 11k + 10)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2 + 11k + 10)}, 18a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{1}{3}}, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2 + 11k + 10)}, 18a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{1}{3}}\right), a_{k+2} = -\frac{3k^2a_k - 18k^2a_{1+k} - 4ka_k - 27ka_{1+k} - 29a_{1+k}}{9(3k^2 + 13k + 14)}, 36a_1 -\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(3*x^2*(3+x)*diff(y(x),x$2)-x*(15+x)*diff(y(x),x)-20*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{4}{9}x + \frac{13}{81}x^2 - \frac{832}{15309}x^3 + \frac{2470}{137781}x^4 - \frac{21736}{3720087}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-144 - \frac{64}{3}x + \frac{16}{27}x^2 - \frac{112}{6561}x^4 + \frac{448}{5904\ldots}\right.}{x^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 85

```
AsymptoticDSolveValue[3*x^2*(3+x)*y''[x]-x*(15+x)*y'[x]-20*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{7x^{10/3}}{59049} - \frac{x^{4/3}}{243} + \frac{1}{x^{2/3}} + \frac{4\sqrt[3]{x}}{27}\right) + c_2\left(\frac{2470x^{22/3}}{137781} - \frac{832x^{19/3}}{15309} + \frac{13x^{16/3}}{81} - \frac{4x^{13/3}}{9} + x^{10/3}\right)$$

16.21 problem 17

Internal problem ID [1433]
Internal file name [OUTPUT/1434_Sunday_June_05_2022_02_17_10_AM_51484014/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''+x(1-10x)\,y'-(9-10x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{10x-1}{x\,(x+1)}$$
$$q(x)=\frac{10x-9}{x^2\,(x+1)}$$

Table 812: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{10x-1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{10x-9}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + \left(-10x^2 + x\right) y' + (10x - 9)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(-10x^2 + x\right)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (10x - 9)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)$$
$$+ \sum_{n=0}^{\infty} \left(-10x^{1+n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} 10x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-10x^{1+n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 10x^{1+n+r} a_n = \sum_{n=1}^{\infty} 10a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)$$
$$+ \sum_{n=1}^{\infty} \left(-10a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \tag{2B}$$
$$+ \left(\sum_{n=1}^{\infty} 10a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-9a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) - 9a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 9a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r r - 9x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 - 9)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 9)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ -10a_{n-1}(n+r-1) + a_n(n+r) + 10a_{n-1} - 9a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 13n - 13r + 22)}{n^2 + 2nr + r^2 - 9} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - 7n - 8)}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 11r - 10}{r^2 + 2r - 8}$$

Which for the root $r = 3$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r-9)\,r(r-10)}{r^3 + 7r^2 + 2r - 40}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{9}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2
a_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{9}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r-9)(r-10)(r+1)(r-8)}{r^4+13r^3+44r^2-28r-240}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{5}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2
a_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{9}{4}$
a_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{r^4+13r^3+44r^2-28r-240}$	$\frac{5}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{r^5+20r^4+135r^3+280r^2-436r-1680}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{5}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2
a_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{9}{4}$
a_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{r^4+13r^3+44r^2-28r-240}$	$\frac{5}{3}$
a_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{r^5+20r^4+135r^3+280r^2-436r-1680}$	$\frac{5}{6}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+3)(r-6)(r-9)(r-10)(r-8)(r-7)}{(r+8)(r+7)(r+5)(r+4)(r-2)(r+6)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{3}{11}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2
a_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{9}{4}$
a_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{r^4+13r^3+44r^2-28r-240}$	$\frac{5}{3}$
a_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{r^5+20r^4+135r^3+280r^2-436r-1680}$	$\frac{5}{6}$
a_5	$-\frac{(r+3)(r-6)(r-9)(r-10)(r-8)(r-7)}{(r+8)(r+7)(r+5)(r+4)(r-2)(r+6)}$	$\frac{3}{11}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{7}{132}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	2
a_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{9}{4}$
a_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{r^4+13r^3+44r^2-28r-240}$	$\frac{5}{3}$
a_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{r^5+20r^4+135r^3+280r^2-436r-1680}$	$\frac{5}{6}$
a_5	$-\frac{(r+3)(r-6)(r-9)(r-10)(r-8)(r-7)}{(r+8)(r+7)(r+5)(r+4)(r-2)(r+6)}$	$\frac{3}{11}$
a_6	$\frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)}$	$\frac{7}{132}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \dots\right) \\ &= x^3\left(1 + 2x + \frac{9x^2}{4} + \frac{5x^3}{3} + \frac{5x^4}{6} + \frac{3x^5}{11} + \frac{7x^6}{132} + O(x^7)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= \frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)} &= \lim_{r\to -3} \frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)} \\ &= -\frac{1716}{5} \end{aligned}$$

The limit is $-\frac{1716}{5}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-3} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ - 10b_{n-1}(n+r-1) + b_n(n+r) + 10b_{n-1} - 9b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -3$ becomes

$$b_{n-1}(n-4)(n-5)+b_n(n-3)(n-4)-10b_{n-1}(n-4)+b_n(n-3)+10b_{n-1}-9b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 13n - 13r + 22)}{n^2 + 2nr + r^2 - 9} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = -\frac{b_{n-1}(n^2 - 19n + 70)}{n^2 - 6n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r^2 - 11r + 10}{r^2 + 2r - 8}$$

Which for the root $r = -3$ becomes

$$b_1 = \frac{52}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r-9)\, r(r-10)}{(r+5)\,(r^2+2r-8)}$$

Which for the root $r = -3$ becomes

$$b_2 = \frac{234}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$
b_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{234}{5}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r-9)\,(r-10)\,(r^2-7r-8)}{(r+5)\,(r^2+2r-8)\,(r+6)}$$

Which for the root $r = -3$ becomes

$$b_3 = \frac{572}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$
b_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{234}{5}$
b_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{(r+5)(r+4)(r-2)(r+6)}$	$\frac{572}{5}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 5r - 14)(r - 8)(r - 10)(r - 9)}{(r + 7)(r + 5)(r^2 + 2r - 8)(r + 6)}$$

Which for the root $r = -3$ becomes

$$b_4 = 143$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$
b_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{234}{5}$
b_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{(r+5)(r+4)(r-2)(r+6)}$	$\frac{572}{5}$
b_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{(r+7)(r+5)(r+4)(r-2)(r+6)}$	143

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2 - 3r - 18)(r - 9)(r - 10)(r - 8)(r - 7)}{(r + 8)(r + 7)(r + 5)(r^2 + 2r - 8)(r + 6)}$$

Which for the root $r = -3$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$
b_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{234}{5}$
b_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{(r+5)(r+4)(r-2)(r+6)}$	$\frac{572}{5}$
b_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{(r+7)(r+5)(r+4)(r-2)(r+6)}$	143
b_5	$-\frac{(r+3)(r-6)(r-9)(r-10)(r-8)(r-7)}{(r+8)(r+7)(r+5)(r+4)(r-2)(r+6)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)}$$

Which for the root $r = -3$ becomes

$$b_6 = -\frac{1716}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+11r-10}{r^2+2r-8}$	$\frac{52}{5}$
b_2	$\frac{(r-9)r(r-10)}{r^3+7r^2+2r-40}$	$\frac{234}{5}$
b_3	$-\frac{(r-9)(r-10)(r+1)(r-8)}{(r+5)(r+4)(r-2)(r+6)}$	$\frac{572}{5}$
b_4	$\frac{(r+2)(r-7)(r-8)(r-10)(r-9)}{(r+7)(r+5)(r+4)(r-2)(r+6)}$	143
b_5	$-\frac{(r+3)(r-6)(r-9)(r-10)(r-8)(r-7)}{(r+8)(r+7)(r+5)(r+4)(r-2)(r+6)}$	0
b_6	$\frac{(r-7)(r-8)(r-10)(r-9)(r-6)(r-5)}{(r+9)(r+6)(r-2)(r+5)(r+7)(r+8)}$	$-\frac{1716}{5}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^3 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 \dots\right) \\ &= \frac{1 + \frac{52x}{5} + \frac{234x^2}{5} + \frac{572x^3}{5} + 143x^4 - \frac{1716x^6}{5} + O(x^7)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 + 2x + \frac{9x^2}{4} + \frac{5x^3}{3} + \frac{5x^4}{6} + \frac{3x^5}{11} + \frac{7x^6}{132} + O(x^7)\right) \\ &\quad + \frac{c_2 \left(1 + \frac{52x}{5} + \frac{234x^2}{5} + \frac{572x^3}{5} + 143x^4 - \frac{1716x^6}{5} + O(x^7)\right)}{x^3} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1+2x+\frac{9x^2}{4}+\frac{5x^3}{3}+\frac{5x^4}{6}+\frac{3x^5}{11}+\frac{7x^6}{132}+O(x^7)\right) \\
&\quad + \frac{c_2 \left(1+\frac{52x}{5}+\frac{234x^2}{5}+\frac{572x^3}{5}+143x^4-\frac{1716x^6}{5}+O(x^7)\right)}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1+2x+\frac{9x^2}{4}+\frac{5x^3}{3}+\frac{5x^4}{6}+\frac{3x^5}{11}+\frac{7x^6}{132}+O(x^7)\right) \\
&\quad + \frac{c_2 \left(1+\frac{52x}{5}+\frac{234x^2}{5}+\frac{572x^3}{5}+143x^4-\frac{1716x^6}{5}+O(x^7)\right)}{x^3}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^3 \left(1+2x+\frac{9x^2}{4}+\frac{5x^3}{3}+\frac{5x^4}{6}+\frac{3x^5}{11}+\frac{7x^6}{132}+O(x^7)\right) \\
&\quad + \frac{c_2 \left(1+\frac{52x}{5}+\frac{234x^2}{5}+\frac{572x^3}{5}+143x^4-\frac{1716x^6}{5}+O(x^7)\right)}{x^3}
\end{aligned}
$$

Verified OK.

16.21.1 Maple step by step solution

Let's solve

$x^2(x+1)\, y'' + (-10x^2+x)\, y' + (10x-9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x-9)y}{x^2(x+1)} + \frac{(10x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(10x-1)y'}{x(x+1)} + \frac{(10x-9)y}{x^2(x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{10x-1}{x(x+1)}, P_3(x) = \frac{10x-9}{x^2(x+1)}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = -11$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\, y'' - x(10x-1)\, y' + (10x-9)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (-10u^2 + 21u - 11)\left(\frac{d}{du}y(u)\right) + (10u - 19)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-12+r)\, u^{-1+r} + (a_1(1+r)(-11+r) - a_0(2r^2-23r+19))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-12+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 12\}$$

- Each term must be 0

$$a_1(1+r)(-11+r) - a_0(2r^2-23r+19) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a$$

- Shift index using $k->k+1$

$$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})\,(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 11ka_k + 19ka_{k+1} - 11ra_k + 19ra_{k+1} + 10a_k + 2a_{k+1}}{k^2 + 2kr + r^2 - 8k - 8r - 20}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Recursion relation for $r = 12$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}$$

- Solution for $r = 12$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k+12}, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}+13k a_k-29k a_{k+1}+22a_k-58a_{k+1}}{k^2+16k+28}, 13a_1-31a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x)=0,y(x),type='series',x
```

$$y(x)=c_1 x^3\left(1+2x+\frac{9}{4}x^2+\frac{5}{3}x^3+\frac{5}{6}x^4+\frac{3}{11}x^5+\mathrm{O}\left(x^6\right)\right)+\frac{c_2(-86400-898560x-4043520x^2-9884160x^3-12355200x^4+\mathrm{O}(x^6))}{x^3}$$

✓ Solution by Mathematica
Time used: 0.046 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]+x*(1-10*x)*y'[x]-(9-10*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{1}{x^3}+\frac{52}{5x^2}+143x+\frac{234}{5x}+\frac{572}{5}\right)+c_2\left(\frac{5x^7}{6}+\frac{5x^6}{3}+\frac{9x^5}{4}+2x^4+x^3\right)$$

16.22 problem 18

Internal problem ID [1434]
Internal file name [`OUTPUT/1435_Sunday_June_05_2022_02_17_13_AM_47093617/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''+3y'x^2-(6-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+3y'x^2+(x-6)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x+1}$$
$$q(x)=\frac{x-6}{x^2\,(x+1)}$$

Table 814: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x+1}$	
singularity	type
$x = -1$	"regular"

$q(x) = \frac{x-6}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + 3y'x^2 + (x-6)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + (x-6) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-6a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 6a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 6a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 6x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1)+3a_{n-1}(n+r-1)+a_{n-1}-6a_n=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2)}{n^2 + 2nr + r^2 - n - r - 6} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}(n+3)^2}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{(r+1)^2}{r^2 + r - 6}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{8}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{r^2+r-6}$	$-\frac{8}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+1)^2 (r+2)^2}{r^4 + 4r^3 - 7r^2 - 22r + 24}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{100}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{r^2+r-6}$	$-\frac{8}{3}$
a_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	$\frac{100}{21}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r+3)(r+1)^2(r+2)^2}{r^5+6r^4-5r^3-42r^2+40r}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{50}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{r^2+r-6}$	$-\frac{8}{3}$
a_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	$\frac{100}{21}$
a_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{r^5+6r^4-5r^3-42r^2+40r}$	$-\frac{50}{7}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+2)^2(r+1)(r+3)(r+4)}{r^5+8r^4-r^3-68r^2+60r}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{175}{18}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{r^2+r-6}$	$-\frac{8}{3}$
a_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	$\frac{100}{21}$
a_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{r^5+6r^4-5r^3-42r^2+40r}$	$-\frac{50}{7}$
a_4	$\frac{(r+2)^2(r+1)(r+3)(r+4)}{r^5+8r^4-r^3-68r^2+60r}$	$\frac{175}{18}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r+4)(r+3)(r+1)(r+2)(r+5)}{r^5+10r^4+5r^3-100r^2+84r}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{112}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{(r+1)^2}{r^2+r-6}$	$-\frac{8}{3}$
a_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	$\frac{100}{21}$
a_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{r^5+6r^4-5r^3-42r^2+40r}$	$-\frac{50}{7}$
a_4	$\frac{(r+2)^2(r+1)(r+3)(r+4)}{r^5+8r^4-r^3-68r^2+60r}$	$\frac{175}{18}$
a_5	$-\frac{(r+4)(r+3)(r+1)(r+2)(r+5)}{r^5+10r^4+5r^3-100r^2+84r}$	$-\frac{112}{9}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^3\left(1 - \frac{8x}{3} + \frac{100x^2}{21} - \frac{50x^3}{7} + \frac{175x^4}{18} - \frac{112x^5}{9} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= -\frac{(r+4)(r+3)(r+1)(r+2)(r+5)}{r^5+10r^4+5r^3-100r^2+84r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} -\frac{(r+4)(r+3)(r+1)(r+2)(r+5)}{r^5+10r^4+5r^3-100r^2+84r} &= \lim_{r\rightarrow -2} -\frac{(r+4)(r+3)(r+1)(r+2)(r+5)}{r^5+10r^4+5r^3-100r^2+84r} \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2)+b_n(n+r)(n+r-1)+3b_{n-1}(n+r-1)+b_{n-1}-6b_n = 0 \quad (4)$$

Which for for the root $r = -2$ becomes

$$b_{n-1}(n-3)(n-4)+b_n(n-2)(n-3)+3b_{n-1}(n-3)+b_{n-1}-6b_n = 0 \quad \text{(4A)}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2+2nr+r^2)}{n^2+2nr+r^2-n-r-6} \quad (5)$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-1}(n^2-4n+4)}{n^2-5n} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r^2 + 2r + 1}{r^2 + r - 6}$$

Which for the root $r = -2$ becomes

$$b_1 = \frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{r^2+r-6}$	$\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^4 + 6r^3 + 13r^2 + 12r + 4}{(r^2 + r - 6)(r^2 + 3r - 4)}$$

Which for the root $r = -2$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{r^2+r-6}$	$\frac{1}{4}$
b_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r^5 + 9r^4 + 31r^3 + 51r^2 + 40r + 12}{(r^2 + 3r - 4)(r - 2)\, r\,(r + 5)}$$

Which for the root $r = -2$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{r^2+r-6}$	$\frac{1}{4}$
b_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	0
b_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{(r+5)r(-1+r)(r-2)(r+4)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^5 + 12r^4 + 55r^3 + 120r^2 + 124r + 48}{(r+6)(r+5)r(r-2)(-1+r)}$$

Which for the root $r = -2$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{r^2+r-6}$	$\frac{1}{4}$
b_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	0
b_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{(r+5)r(-1+r)(r-2)(r+4)}$	0
b_4	$\frac{(r+2)^2(r+1)(r+3)(r+4)}{(r+6)(r+5)(-1+r)(r-2)r}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r^5 + 15r^4 + 85r^3 + 225r^2 + 274r + 120}{(r+7)(-1+r)(r-2)r(r+6)}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{(r+1)^2}{r^2+r-6}$	$\frac{1}{4}$
b_2	$\frac{(r+1)^2(r+2)^2}{r^4+4r^3-7r^2-22r+24}$	0
b_3	$-\frac{(r+3)(r+1)^2(r+2)^2}{(r+5)r(-1+r)(r-2)(r+4)}$	0
b_4	$\frac{(r+2)^2(r+1)(r+3)(r+4)}{(r+6)(r+5)(-1+r)(r-2)r}$	0
b_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{(r+7)(-1+r)(r-2)r(r+6)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^3\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{4} + O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{8x}{3} + \frac{100x^2}{21} - \frac{50x^3}{7} + \frac{175x^4}{18} - \frac{112x^5}{9} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{4} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 - \frac{8x}{3} + \frac{100x^2}{21} - \frac{50x^3}{7} + \frac{175x^4}{18} - \frac{112x^5}{9} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{4} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 \left(1 - \frac{8x}{3} + \frac{100x^2}{21} - \frac{50x^3}{7} + \frac{175x^4}{18} - \frac{112x^5}{9} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{4} + O(x^6)\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 \left(1 - \frac{8x}{3} + \frac{100x^2}{21} - \frac{50x^3}{7} + \frac{175x^4}{18} - \frac{112x^5}{9} + O(x^6)\right) + \frac{c_2\left(1 + \frac{x}{4} + O(x^6)\right)}{x^2}
$$

Verified OK.

16.22.1 Maple step by step solution

Let's solve

$x^2(x+1)\, y'' + 3y'x^2 + (x-6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x+1} - \frac{(x-6)y}{x^2(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x+1} + \frac{(x-6)y}{x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3}{x+1}, P_3(x) = \frac{x-6}{x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 3$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\, y'' + 3y'x^2 + (x-6)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 6u + 3)\left(\frac{d}{du}y(u)\right) + (u-7)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 0\}$$

- Each term must be 0

  $$a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2 + (2r+2)\,k + r^2 + 2r + \frac{7}{2}\right) a_k = 0$$

- Shift index using $k- >k + 1$

  $$a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2 + (2r+2)\,(k+1) + r^2 + 2r + \frac{7}{2}\right) a_{k+1}$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} + 2ka_k - 8ka_{k+1} + 2ra_k - 8ra_{k+1} + a_k - 13a_{k+1}}{(k+r+2)(k+4+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)+3*x^2*diff(y(x),x)-(6-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 - \frac{8}{3}x + \frac{100}{21}x^2 - \frac{50}{7}x^3 + \frac{175}{18}x^4 - \frac{112}{9}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(2880 + 720x + \mathrm{O}\left(x^6\right))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]+3*x^2*y'[x]-(6-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{x^2} + \frac{1}{4x}\right) + c_2 \left(\frac{175x^7}{18} - \frac{50x^6}{7} + \frac{100x^5}{21} - \frac{8x^4}{3} + x^3\right)$$

16.23 problem 19

Internal problem ID [1435]
Internal file name [OUTPUT/1436_Sunday_June_05_2022_02_17_17_AM_47446477/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{2(3+14x)}{x\,(1+2x)}$$
$$q(x)=\frac{6+100x}{x^2\,(1+2x)}$$

Table 816: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(3+14x)}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = \frac{6+100x}{x^2(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\, y'' + \left(-28x^2 - 6x\right) y' + (6 + 100x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(1+2x) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-28x^2 - 6x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (6+100x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-28x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 100x^{1+n+r} a_n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}
$$

$$
\sum_{n=0}^{\infty} \left(-28x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-28a_{n-1}(n+r-1)\, x^{n+r}\right)
$$

$$
\sum_{n=0}^{\infty} 100x^{1+n+r} a_n = \sum_{n=1}^{\infty} 100a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \sum_{n=1}^{\infty} \left(-28a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r} a_n (n+r)\right) \\
&+ \left(\sum_{n=0}^{\infty} 6a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 100a_{n-1} x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) - 6x^{n+r} a_n (n+r) + 6a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 6x^r a_0 r + 6a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 6x^r r + 6x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 7r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 7r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 7r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^6\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+6}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&2a_{n-1}(n+r-1)(n+r-2)+a_n(n+r)(n+r-1) \\
&\quad -28a_{n-1}(n+r-1)-6a_n(n+r)+6a_n+100a_{n-1}=0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2(n+r-11)\,a_{n-1}}{n+r-1} \tag{4}
$$

Which for the root $r = 6$ becomes

$$
a_n = -\frac{2(n-5)\,a_{n-1}}{n+5} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{20-2r}{r}
$$

Which for the root $r = 6$ becomes

$$
a_1 = \frac{4}{3}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{20-2r}{r}$	$\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{4r^2-76r+360}{r(1+r)}
$$

Which for the root $r = 6$ becomes

$$a_2 = \frac{8}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{20-2r}{r}$	$\frac{4}{3}$
a_2	$\frac{4r^2-76r+360}{r(1+r)}$	$\frac{8}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 + 216r^2 - 1936r + 5760}{r(1+r)(2+r)}$$

Which for the root $r = 6$ becomes

$$a_3 = \frac{4}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{20-2r}{r}$	$\frac{4}{3}$
a_2	$\frac{4r^2-76r+360}{r(1+r)}$	$\frac{8}{7}$
a_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	$\frac{4}{7}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 - 544r^3 + 6896r^2 - 38624r + 80640}{r(1+r)(2+r)(3+r)}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{8}{63}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{20-2r}{r}$	$\frac{4}{3}$
a_2	$\frac{4r^2-76r+360}{r(1+r)}$	$\frac{8}{7}$
a_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	$\frac{4}{7}$
a_4	$\frac{16r^4-544r^3+6896r^2-38624r+80640}{r(1+r)(2+r)(3+r)}$	$\frac{8}{63}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32(-10+r)(-6+r)(-7+r)(-8+r)(-9+r)}{(4+r)(1+r)(2+r)r(3+r)}$$

Which for the root $r = 6$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{20-2r}{r}$	$\frac{4}{3}$
a_2	$\frac{4r^2-76r+360}{r(1+r)}$	$\frac{8}{7}$
a_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	$\frac{4}{7}$
a_4	$\frac{16r^4-544r^3+6896r^2-38624r+80640}{r(1+r)(2+r)(3+r)}$	$\frac{8}{63}$
a_5	$-\frac{32(-10+r)(-6+r)(-7+r)(-8+r)(-9+r)}{(4+r)(1+r)(2+r)r(3+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^6\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \ldots\right) \\
&= x^6\left(1 + \frac{4x}{3} + \frac{8x^2}{7} + \frac{4x^3}{7} + \frac{8x^4}{63} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_5 \\ &= -\frac{32(-10+r)(-6+r)(-7+r)(-8+r)(-9+r)}{(4+r)(1+r)(2+r)r(3+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{32(-10+r)(-6+r)(-7+r)(-8+r)(-9+r)}{(4+r)(1+r)(2+r)r(3+r)} &= \lim_{r \to 1} -\frac{32(-10+r)(-6+r)(-7+r)(-8+r)}{(4+r)(1+r)(2+r)r(3+r)} \\ &= 4032 \end{aligned}$$

The limit is 4032. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{1+n} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ &- 28b_{n-1}(n+r-1) - 6b_n(n+r) + 6b_n + 100b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 1$ becomes

$$2b_{n-1}n(n-1) + b_n(1+n)n - 28b_{n-1}n - 6b_n(1+n) + 6b_n + 100b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2(n+r-11)b_{n-1}}{n+r-1} \tag{5}$$

Which for the root $r = 1$ becomes

$$b_n = -\frac{2(n-10)b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2(-10 + r)}{r}$$

Which for the root $r = 1$ becomes

$$b_1 = 18$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{20-2r}{r}$	18

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 76r + 360}{r(1 + r)}$$

Which for the root $r = 1$ becomes

$$b_2 = 144$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{20-2r}{r}$	18
b_2	$\frac{4r^2-76r+360}{r(1+r)}$	144

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8(r^3 - 27r^2 + 242r - 720)}{r(1 + r)(2 + r)}$$

Which for the root $r = 1$ becomes

$$b_3 = 672$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{20-2r}{r}$	18
b_2	$\frac{4r^2-76r+360}{r(1+r)}$	144
b_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	672

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^4 - 544r^3 + 6896r^2 - 38624r + 80640}{r(1+r)(2+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$b_4 = 2016$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{20-2r}{r}$	18
b_2	$\frac{4r^2-76r+360}{r(1+r)}$	144
b_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	672
b_4	$\frac{16r^4-544r^3+6896r^2-38624r+80640}{r(1+r)(2+r)(3+r)}$	2016

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32(r^5 - 40r^4 + 635r^3 - 5000r^2 + 19524r - 30240)}{r(1+r)(2+r)(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$b_5 = 4032$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{20-2r}{r}$	18
b_2	$\frac{4r^2-76r+360}{r(1+r)}$	144
b_3	$\frac{-8r^3+216r^2-1936r+5760}{r(1+r)(2+r)}$	672
b_4	$\frac{16r^4-544r^3+6896r^2-38624r+80640}{r(1+r)(2+r)(3+r)}$	2016
b_5	$-\frac{32(-10+r)(-6+r)(-7+r)(-8+r)(-9+r)}{(4+r)(1+r)(2+r)r(3+r)}$	4032

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^6\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x\left(1 + 18x + 144x^2 + 672x^3 + 2016x^4 + 4032x^5 + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^6 \left(1 + \frac{4x}{3} + \frac{8x^2}{7} + \frac{4x^3}{7} + \frac{8x^4}{63} + O(x^6)\right) \\
&\quad + c_2 x\left(1 + 18x + 144x^2 + 672x^3 + 2016x^4 + 4032x^5 + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^6 \left(1 + \frac{4x}{3} + \frac{8x^2}{7} + \frac{4x^3}{7} + \frac{8x^4}{63} + O(x^6)\right) \\
&\quad + c_2 x\left(1 + 18x + 144x^2 + 672x^3 + 2016x^4 + 4032x^5 + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = {}& c_1 x^6 \left(1 + \frac{4x}{3} + \frac{8x^2}{7} + \frac{4x^3}{7} + \frac{8x^4}{63} + O(x^6)\right) \\
& + c_2 x\left(1 + 18x + 144x^2 + 672x^3 + 2016x^4 + 4032x^5 + O(x^6)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^6 \left(1+\frac{4x}{3}+\frac{8x^2}{7}+\frac{4x^3}{7}+\frac{8x^4}{63}+O(x^6)\right) + c_2 x\left(1+18x+144x^2+672x^3+2016x^4+4032x^5+O(x^6)\right)$$

Verified OK.

16.23.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y''+(-28x^2-6x)\,y'+(6+100x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3+50x)y}{x^2(1+2x)}+\frac{2(3+14x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(3+14x)y'}{x(1+2x)}+\frac{2(3+50x)y}{x^2(1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2(3+14x)}{x(1+2x)}, P_3(x)=\frac{2(3+50x)}{x^2(1+2x)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-6$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=6$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)(-6+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-6) + 2a_{k-1}(k+r-6)(k-11+r)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)(-6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-6)\left((2k+2r-22)\, a_{k-1} + a_k(k+r-1)\right) = 0$$

- Shift index using $k- >k + 1$

$$(k+r-5)\left((2k+2r-20)\, a_k + a_{k+1}(k+r)\right) = 0$$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{2(k+r-10)a_k}{k+r}$

- Recursion relation for $r = 1$; series terminates at $k = 9$

  $a_{k+1} = -\frac{2(k-9)a_k}{k+1}$

- Recursion relation that defines the terminating series solution of the ODE for $r = 1$

  $\left[y = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$

- Recursion relation for $r = 6$; series terminates at $k = 4$

  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{4a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{6a_1}{7}$

- Express in terms of a_0

  $a_2 = \frac{8a_0}{7}$

- Apply recursion relation for $k = 2$

  $a_3 = \frac{a_2}{2}$

- Express in terms of a_0

  $a_3 = \frac{4a_0}{7}$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{2a_3}{9}$

- Express in terms of a_0

  $a_4 = \frac{8a_0}{63}$

- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{8} a_k x^{1+k}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{1+k} = -\frac{2(k-9)a_k}{1+k}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
Order:=6;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x)=0,y(x),type='seri
```

$$y(x) = c_1 x^6 \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x \left(2880 + 51840x + 414720x^2 + 1935360x^3 + 5806080x^4 + 11612160x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]-2*x*(3+14*x)*y'[x]+(6+100*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(2016x^5 + 672x^4 + 144x^3 + 18x^2 + x\right) + c_2 \left(\frac{8x^{10}}{63} + \frac{4x^9}{7} + \frac{8x^8}{7} + \frac{4x^7}{3} + x^6\right)$$

16.24 problem 20

Internal problem ID [1436]
Internal file name [`OUTPUT/1437_Sunday_June_05_2022_02_17_20_AM_96646602/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''-x(6+11x)\,y'+(6+32x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(-11x^2-6x\right)y'+(6+32x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{6+11x}{x\,(x+1)}$$
$$q(x)=\frac{6+32x}{x^2\,(x+1)}$$

Table 818: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{6+11x}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{6+32x}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + \left(-11x^2 - 6x\right) y' + (6 + 32x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-11x^2 - 6x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (6 + 32x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-11 x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-6 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6 a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 32 x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-11 x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-11 a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 32 x^{1+n+r} a_n = \sum_{n=1}^{\infty} 32 a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-11 a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-6 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6 a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 32 a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 6 x^{n+r} a_n (n+r) + 6 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 6x^r a_0 r + 6a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 6x^r r + 6x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 7r + 6\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 7r + 6 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 7r + 6\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^6 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+6}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ &- 11a_{n-1}(n+r-1) - 6a_n(n+r) + 6a_n + 32a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 14n - 14r + 45)}{n^2 + 2nr + r^2 - 7n - 7r + 6} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - 2n - 3)}{n(n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + 12r - 32}{r(r-5)}$$

Which for the root $r = 6$ becomes

$$a_1 = \frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^3 - 18r^2 + 101r - 168}{(r+1)r(r-5)}$$

Which for the root $r = 6$ becomes

$$a_2 = \frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{2}{3}$
a_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{1}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^4 + 23r^3 - 188r^2 + 628r - 672}{(r+2)(r+1)r(r-5)}$$

Which for the root $r = 6$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{2}{3}$
a_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{1}{7}$
a_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 - 22r^3 + 167r^2 - 482r + 336}{(r+3)r(r+1)(r+2)}$$

Which for the root $r = 6$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{2}{3}$
a_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{1}{7}$
a_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	0
a_4	$\frac{r^4-22r^3+167r^2-482r+336}{(r+3)r(r+1)(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^4 + 25r^3 - 230r^2 + 920r - 1344}{(r+4)(r+3)(r+1)(r+2)}$$

Which for the root $r = 6$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{2}{3}$
a_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{1}{7}$
a_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	0
a_4	$\frac{r^4-22r^3+167r^2-482r+336}{(r+3)r(r+1)(r+2)}$	0
a_5	$\frac{-r^4+25r^3-230r^2+920r-1344}{(r+4)(r+3)(r+1)(r+2)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^6 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^6 \left(1 + \frac{2x}{3} + \frac{x^2}{7} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_5(r)$. If this limit exists, then $C=0$, else we need to keep the log term and $C\neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_5 \\
&= \frac{-r^4+25r^3-230r^2+920r-1344}{(r+4)(r+3)(r+1)(r+2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{-r^4+25r^3-230r^2+920r-1344}{(r+4)(r+3)(r+1)(r+2)} &= \lim_{r\rightarrow 1} \frac{-r^4+25r^3-230r^2+920r-1344}{(r+4)(r+3)(r+1)(r+2)} \\
&= -\frac{21}{4}
\end{aligned}
$$

The limit is $-\frac{21}{4}$. Since the limit exists then the log term is not needed and we can set $C=0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{1+n}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0=1$. For $1\leq n$ the recursive equation is

$$
\begin{aligned}
&b_{n-1}(n+r-1)(n+r-2)+b_n(n+r)(n+r-1) \\
&-11b_{n-1}(n+r-1)-6b_n(n+r)+6b_n+32b_{n-1}=0
\end{aligned}
\tag{4}
$$

Which for for the root $r=1$ becomes

$$
b_{n-1}n(n-1)+b_n(1+n)n-11b_{n-1}n-6b_n(1+n)+6b_n+32b_{n-1}=0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n^2+2nr+r^2-14n-14r+45)}{n^2+2nr+r^2-7n-7r+6} \tag{5}
$$

Which for the root $r=1$ becomes

$$
b_n = -\frac{b_{n-1}(n^2-12n+32)}{n^2-5n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r^2 - 12r + 32}{r (r - 5)}$$

Which for the root $r = 1$ becomes

$$b_1 = \frac{21}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{21}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^3 - 18r^2 + 101r - 168}{(r + 1) r (r - 5)}$$

Which for the root $r = 1$ becomes

$$b_2 = \frac{21}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{21}{4}$
b_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{21}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r^4 - 23r^3 + 188r^2 - 628r + 672}{(r + 2) (r + 1) r (r - 5)}$$

Which for the root $r = 1$ becomes

$$b_3 = \frac{35}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{21}{4}$
b_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{21}{2}$
b_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	$\frac{35}{4}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 - 22r^3 + 167r^2 - 482r + 336}{(r+3)\, r\, (r+1)\, (r+2)}$$

Which for the root $r = 1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{21}{4}$
b_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{21}{2}$
b_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	$\frac{35}{4}$
b_4	$\frac{r^4-22r^3+167r^2-482r+336}{(r+3)r(r+1)(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r^4 - 25r^3 + 230r^2 - 920r + 1344}{(r+4)\,(r+3)\,(r+1)\,(r+2)}$$

Which for the root $r = 1$ becomes

$$b_5 = -\frac{21}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+12r-32}{r(r-5)}$	$\frac{21}{4}$
b_2	$\frac{r^3-18r^2+101r-168}{(r+1)r(r-5)}$	$\frac{21}{2}$
b_3	$\frac{-r^4+23r^3-188r^2+628r-672}{(r+2)(r+1)r(r-5)}$	$\frac{35}{4}$
b_4	$\frac{r^4-22r^3+167r^2-482r+336}{(r+3)r(r+1)(r+2)}$	0
b_5	$\frac{-r^4+25r^3-230r^2+920r-1344}{(r+4)(r+3)(r+1)(r+2)}$	$-\frac{21}{4}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^6(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= x\left(1 + \frac{21x}{4} + \frac{21x^2}{2} + \frac{35x^3}{4} - \frac{21x^5}{4} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^6\left(1 + \frac{2x}{3} + \frac{x^2}{7} + O(x^6)\right) + c_2x\left(1 + \frac{21x}{4} + \frac{21x^2}{2} + \frac{35x^3}{4} - \frac{21x^5}{4} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x^6\left(1 + \frac{2x}{3} + \frac{x^2}{7} + O(x^6)\right) + c_2x\left(1 + \frac{21x}{4} + \frac{21x^2}{2} + \frac{35x^3}{4} - \frac{21x^5}{4} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^6\left(1 + \frac{2x}{3} + \frac{x^2}{7} + O(x^6)\right) + c_2x\left(1 + \frac{21x}{4} + \frac{21x^2}{2} + \frac{35x^3}{4} - \frac{21x^5}{4} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1x^6\left(1 + \frac{2x}{3} + \frac{x^2}{7} + O(x^6)\right) + c_2x\left(1 + \frac{21x}{4} + \frac{21x^2}{2} + \frac{35x^3}{4} - \frac{21x^5}{4} + O(x^6)\right)$$

Verified OK.

16.24.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y''+(-11x^2-6x)\,y'+(6+32x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3+16x)y}{x^2(x+1)}+\frac{(6+11x)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(6+11x)y'}{x(x+1)}+\frac{2(3+16x)y}{x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6+11x}{x(x+1)}, P_3(x)=\frac{2(3+16x)}{x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-5$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(x+1)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-6+r)\, u^{-1+r} + \left(a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k-5\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-6+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 6\}$$

- Each term must be 0

  $$a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1} -$$

- Shift index using $k- >k + 1$

  $$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 12ka_k + 14ka_{k+1} - 12ra_k + 14ra_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Recursion relation for $r = 6$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x)=0,y(x),type='series',x
```

$$y(x) = c_1 x^6 \left(1 + \frac{2}{3}x + \frac{1}{7}x^2 + \mathrm{O}\left(x^6\right)\right) + c_2 x\left(2880 + 15120x + 30240x^2 + 25200x^3 - 15120x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 51

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-x*(6+11*x)*y'[x]+(6+32*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^8}{7} + \frac{2x^7}{3} + x^6\right) + c_1\left(\frac{35x^4}{4} + \frac{21x^3}{2} + \frac{21x^2}{4} + x\right)$$

16.25 problem 21

Internal problem ID [1437]
Internal file name [OUTPUT/1438_Sunday_June_05_2022_02_17_23_AM_43414470/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x+1)\,y''+4x(4x+1)\,y'-(49+27x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x+1}{x\,(x+1)}$$
$$q(x)=-\frac{49+27x}{4x^2\,(x+1)}$$

Table 820: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4x+1}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{49+27x}{4x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\, y'' + \left(16x^2 + 4x\right) y' + (-27x - 49)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(16x^2 + 4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-27x - 49) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 16x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-27x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-49a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} 16x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 16a_{n-1}(n+r-1)\, x^{n+r} \\
\sum_{n=0}^{\infty} \left(-27x^{1+n+r} a_n\right) &= \sum_{n=1}^{\infty} \left(-27a_{n-1} x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=1}^{\infty} 16a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=1}^{\infty} \left(-27a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-49a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) - 49a_n x^{n+r} = 0
$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 4x^r a_0 r - 49a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 4x^r r - 49x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 49\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 49 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{2}$$
$$r_2 = -\frac{7}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 49\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 7$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{7}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{7}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + 16a_{n-1}(n+r-1) + 4a_n(n+r) - 27a_{n-1} - 49a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-5)\, a_{n-1}}{2n+2r-7} \tag{4}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_n = -\frac{(1+n)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3-2r}{2r-5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2r-1}{2r-5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1-2r}{2r-5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_3 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3
a_3	$\frac{-1-2r}{2r-5}$	-4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{3+2r}{2r-5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3
a_3	$\frac{-1-2r}{2r-5}$	-4
a_4	$\frac{3+2r}{2r-5}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-5 - 2r}{2r - 5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_5 = -6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3
a_3	$\frac{-1-2r}{2r-5}$	-4
a_4	$\frac{3+2r}{2r-5}$	5
a_5	$\frac{-5-2r}{2r-5}$	-6

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{7 + 2r}{2r - 5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_6 = 7$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3
a_3	$\frac{-1-2r}{2r-5}$	-4
a_4	$\frac{3+2r}{2r-5}$	5
a_5	$\frac{-5-2r}{2r-5}$	-6
a_6	$\frac{7+2r}{2r-5}$	7

For $n = 7$, using the above recursive equation gives

$$a_7 = \frac{-9 - 2r}{2r - 5}$$

Which for the root $r = \frac{7}{2}$ becomes

$$a_7 = -8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3-2r}{2r-5}$	-2
a_2	$\frac{2r-1}{2r-5}$	3
a_3	$\frac{-1-2r}{2r-5}$	-4
a_4	$\frac{3+2r}{2r-5}$	5
a_5	$\frac{-5-2r}{2r-5}$	-6
a_6	$\frac{7+2r}{2r-5}$	7
a_7	$\frac{-9-2r}{2r-5}$	-8

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{7}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \dots\right) \\ &= x^{\frac{7}{2}}\left(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + O(x^8)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 7$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_7(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_7 \\ &= \frac{-9 - 2r}{2r - 5} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-9 - 2r}{2r - 5} &= \lim_{r \to -\frac{7}{2}} \frac{-9 - 2r}{2r - 5} \\ &= \frac{1}{6} \end{aligned}$$

The limit is $\frac{1}{6}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &4b_{n-1}(n + r - 1)(n + r - 2) + 4b_n(n + r)(n + r - 1) \\ &+ 16b_{n-1}(n + r - 1) + 4b_n(n + r) - 27b_{n-1} - 49b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -\frac{7}{2}$ becomes

$$\begin{aligned} &4b_{n-1}\left(n - \frac{9}{2}\right)\left(n - \frac{11}{2}\right) + 4b_n\left(n - \frac{7}{2}\right)\left(n - \frac{9}{2}\right) \\ &+ 16b_{n-1}\left(n - \frac{9}{2}\right) + 4b_n\left(n - \frac{7}{2}\right) - 27b_{n-1} - 49b_n = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(2n+2r-5)\, b_{n-1}}{2n+2r-7} \tag{5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_n = -\frac{(2n-12)\, b_{n-1}}{2n-14} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{7}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{-3+2r}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_1 = -\frac{5}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2r-1}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1+2r}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_3 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$
b_3	$\frac{-1-2r}{2r-5}$	$-\frac{1}{2}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{3+2r}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_4 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$
b_3	$\frac{-1-2r}{2r-5}$	$-\frac{1}{2}$
b_4	$\frac{3+2r}{2r-5}$	$\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{5+2r}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_5 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$
b_3	$\frac{-1-2r}{2r-5}$	$-\frac{1}{2}$
b_4	$\frac{3+2r}{2r-5}$	$\frac{1}{3}$
b_5	$\frac{-5-2r}{2r-5}$	$-\frac{1}{6}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{7+2r}{2r-5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$
b_3	$\frac{-1-2r}{2r-5}$	$-\frac{1}{2}$
b_4	$\frac{3+2r}{2r-5}$	$\frac{1}{3}$
b_5	$\frac{-5-2r}{2r-5}$	$-\frac{1}{6}$
b_6	$\frac{7+2r}{2r-5}$	0

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{9 + 2r}{2r - 5}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_7 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3-2r}{2r-5}$	$-\frac{5}{6}$
b_2	$\frac{2r-1}{2r-5}$	$\frac{2}{3}$
b_3	$\frac{-1-2r}{2r-5}$	$-\frac{1}{2}$
b_4	$\frac{3+2r}{2r-5}$	$\frac{1}{3}$
b_5	$\frac{-5-2r}{2r-5}$	$-\frac{1}{6}$
b_6	$\frac{7+2r}{2r-5}$	0
b_7	$\frac{-9-2r}{2r-5}$	$\frac{1}{6}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{7}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \dots\right) \\ &= \frac{1 - \frac{5x}{6} + \frac{2x^2}{3} - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{6} + \frac{x^7}{6} + O(x^8)}{x^{\frac{7}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{7}{2}}\left(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + O(x^8)\right) \\ &+ \frac{c_2\left(1 - \frac{5x}{6} + \frac{2x^2}{3} - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{6} + \frac{x^7}{6} + O(x^8)\right)}{x^{\frac{7}{2}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{7}{2}}\left(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + O(x^8)\right) \\ &+ \frac{c_2\left(1 - \frac{5x}{6} + \frac{2x^2}{3} - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{6} + \frac{x^7}{6} + O(x^8)\right)}{x^{\frac{7}{2}}} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{7}{2}} \left(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+O(x^8)\right) + \frac{c_2 \left(1-\frac{5x}{6}+\frac{2x^2}{3}-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{6}+\frac{x^7}{6}+O(x^8)\right)}{x^{\frac{7}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{7}{2}} \left(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+O(x^8)\right) + \frac{c_2 \left(1-\frac{5x}{6}+\frac{2x^2}{3}-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{6}+\frac{x^7}{6}+O(x^8)\right)}{x^{\frac{7}{2}}}$$

Verified OK.

16.25.1 Maple step by step solution

Let's solve

$4x^2(x+1)\, y'' + (16x^2+4x)\, y' + (-27x-49)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(49+27x)y}{4x^2(x+1)} - \frac{(4x+1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x+1)y'}{x(x+1)} - \frac{(49+27x)y}{4x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4x+1}{x(x+1)}, P_3(x) = -\frac{49+27x}{4x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 3$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$4x^2(x+1)\,y'' + 4x(4x+1)\,y' + (-27x-49)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

$(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (16u^2 - 28u + 12)\left(\frac{d}{du}y(u)\right) + (-27u - 22)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$4a_0 r(2+r)\,u^{-1+r} + (4a_1(1+r)\,(3+r) - 2a_0(4r^2+10r+11))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$4r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term must be 0

  $4a_1(1+r)(3+r) - 2a_0(4r^2 + 10r + 11) = 0$

- Each term in the series must be 0, giving the recursion relation

  $4(-2a_k + a_{k-1} + a_{k+1})k^2 + 4(2(-2a_k + a_{k-1} + a_{k+1})r - 5a_k + a_{k-1} + 4a_{k+1})k + 4(-2a_k + a_{k-}$

- Shift index using $k- >k+1$

  $4(-2a_{k+1} + a_k + a_{k+2})(k+1)^2 + 4(2(-2a_{k+1} + a_k + a_{k+2})r - 5a_{k+1} + a_k + 4a_{k+2})(k+1) + 4($

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 12ka_k - 36ka_{k+1} + 12ra_k - 36ra_{k+1} - 27a_k - 50a_{k+1}}{4(k^2 + 2kr + r^2 + 6k + 6r + 8)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2 + 2k)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2 + 2k)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2 + 6k + 8)}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2 + 6k + 8)}, 12a_1 - 22a_0 = 0\right]$$

- Revert the change of variables $u = x + 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2 + 6k + 8)}, 12a_1 - 22a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+4*x)*diff(y(x),x)-(49+27*x)*y(x)=0,y(x),type='serie
```

$$y(x) = \frac{c_1 x^7 (1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \mathrm{O}(x^6)) + c_2 (3628800 - 3024000x + 2419200x^2 - 1814400x^3 +}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]+4*x*(1+4*x)*y'[x]-(49+27*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{2}{3x^{3/2}} - \frac{5}{6x^{5/2}} + \frac{1}{x^{7/2}} + \frac{\sqrt{x}}{3} - \frac{1}{2\sqrt{x}} \right) + c_2 \left(5x^{15/2} - 4x^{13/2} + 3x^{11/2} - 2x^{9/2} + x^{7/2}\right)$$

16.26 problem 22

Internal problem ID [1438]
Internal file name [OUTPUT/1439_Sunday_June_05_2022_02_17_28_AM_15986656/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2(1+2x)\,y''-x(9+8x)\,y'-12yx=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$\left(2x^3+x^2\right)y''+\left(-8x^2-9x\right)y'-12yx=0$$

Or

$$x\left(2x^2y''+y''x-8y'x-9y'-12y\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(2x^2+x\right)y''+(-8x-9)\,y'-12y=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+x^2\right)y''+\left(-8x^2-9x\right)y'-12yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = -\frac{9+8x}{(1+2x)\,x}$$
$$q(x) = -\frac{12}{x\,(1+2x)}$$

Table 822: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{9+8x}{(1+2x)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

$q(x) = -\frac{12}{x(1+2x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(1+2x)\,y'' + \left(-8x^2 - 9x\right) y' - 12yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
& x^2(1+2x)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\
& +\left(-8x^2-9x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)-12\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)x=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
& \left(\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n\right. \\
& \left.+r-1)\right)+\sum_{n=0}^{\infty}\left(-8x^{1+n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-9x^{n+r}a_n(n\right. \\
& \left.+r)\right)+\sum_{n=0}^{\infty}\left(-12x^{1+n+r}a_n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r)(n+r-1) &= \sum_{n=1}^{\infty}2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-8x^{1+n+r}a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-8a_{n-1}(n+r-1)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-12x^{1+n+r}a_n\right) &= \sum_{n=1}^{\infty}\left(-12a_{n-1}x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-8a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-9x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-12a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 9x^{n+r} a_n (n+r) = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 9x^r a_0 r = 0$$

Or

$$\left(x^r r(-1+r) - 9x^r r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-10+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-10+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 10$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-10+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 10$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{10} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+10}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ - 8a_{n-1}(n+r-1) - 9a_n(n+r) - 12a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2(n+r-7)\,a_{n-1}}{n-10+r} \tag{4}$$

Which for the root $r = 10$ becomes

$$a_n = -\frac{2(n+3)\,a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 10$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{12 - 2r}{-9 + r}$$

Which for the root $r = 10$ becomes

$$a_1 = -8$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4r^2 - 44r + 120}{(-9 + r)(-8 + r)}$$

Which for the root $r = 10$ becomes

$$a_2 = 40$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^3 + 120r^2 - 592r + 960}{(-9 + r)(-8 + r)(r - 7)}$$

Which for the root $r = 10$ becomes

$$a_3 = -160$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^3 - 192r^2 + 752r - 960}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 10$ becomes

$$a_4 = 560$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 10$ becomes

$$a_5 = -1792$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 10$ becomes

$$a_6 = 5376$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792
a_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	5376

For $n = 7$, using the above recursive equation gives

$$a_7 = -\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 10$ becomes

$$a_7 = -15360$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792
a_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	5376
a_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	-15360

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{256r^3 - 256r}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 10$ becomes

$$a_8 = 42240$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792
a_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	5376
a_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	-15360
a_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	42240

For $n = 9$, using the above recursive equation gives

$$a_9 = -\frac{512(2+r)\,r(1+r)}{(-9+r)\,(-8+r)\,(r-7)}$$

Which for the root $r = 10$ becomes

$$a_9 = -112640$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792
a_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	5376
a_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	-15360
a_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	42240
a_9	$-\frac{512(2+r)r(1+r)}{(-9+r)(-8+r)(r-7)}$	-112640

For $n = 10$, using the above recursive equation gives

$$a_{10} = \frac{1024(2+r)\,(1+r)\,(3+r)}{(-9+r)\,(-8+r)\,(r-7)}$$

Which for the root $r = 10$ becomes

$$a_{10} = 292864$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{12-2r}{-9+r}$	-8
a_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	40
a_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	-160
a_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	560
a_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	-1792
a_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	5376
a_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	-15360
a_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	42240
a_9	$-\frac{512(2+r)r(1+r)}{(-9+r)(-8+r)(r-7)}$	-112640
a_{10}	$\frac{1024(2+r)(1+r)(3+r)}{(-9+r)(-8+r)(r-7)}$	292864

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{10}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11}\dots\right) \\ &= x^{10}\left(1 - 8x + 40x^2 - 160x^3 + 560x^4 - 1792x^5 + 5376x^6 - 15360x^7 + 42240x^8 - 112640x^9 + 2928\right.\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 10$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_{10}(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_{10} \\ &= \frac{1024(2+r)(1+r)(3+r)}{(-9+r)(-8+r)(r-7)}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{1024(2+r)(1+r)(3+r)}{(-9+r)(-8+r)(r-7)} &= \lim_{r\to 0} \frac{1024(2+r)(1+r)(3+r)}{(-9+r)(-8+r)(r-7)} \\ &= -\frac{256}{21}\end{aligned}$$

The limit is $-\frac{256}{21}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) \\ &- 8b_{n-1}(n+r-1) - 9b_n(n+r) - 12b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$2b_{n-1}(n-1)(n-2) + b_n n(n-1) - 8b_{n-1}(n-1) - 9b_n n - 12b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2(n+r-7)\, b_{n-1}}{n-10+r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{2(n-7)\, b_{n-1}}{n-10} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{2(-6+r)}{-9+r}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4r^2 - 44r + 120}{(-9+r)(-8+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{5}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8(r^3 - 15r^2 + 74r - 120)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{40}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^3 - 192r^2 + 752r - 960}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{40}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32(r^2 - 7r + 12)(-2+r)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{32}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_6 = \frac{16}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$
b_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	$\frac{16}{21}$

For $n = 7$, using the above recursive equation gives

$$b_7 = -\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$
b_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	$\frac{16}{21}$
b_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	0

For $n = 8$, using the above recursive equation gives

$$b_8 = \frac{256r(-1+r)(1+r)}{(-9+r)(-8+r)(r-7)}$$

Which for the root $r = 0$ becomes

$$b_8 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$
b_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	$\frac{16}{21}$
b_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	0
b_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	0

For $n = 9$, using the above recursive equation gives

$$b_9 = -\frac{512(2+r)\, r(1+r)}{(-9+r)\,(-8+r)\,(r-7)}$$

Which for the root $r = 0$ becomes

$$b_9 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$
b_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	$\frac{16}{21}$
b_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	0
b_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	0
b_9	$-\frac{512(2+r)r(1+r)}{(-9+r)(-8+r)(r-7)}$	0

For $n = 10$, using the above recursive equation gives

$$b_{10} = \frac{1024(2+r)\,(1+r)\,(3+r)}{(-9+r)\,(-8+r)\,(r-7)}$$

Which for the root $r = 0$ becomes

$$b_{10} = -\frac{256}{21}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{12-2r}{-9+r}$	$-\frac{4}{3}$
b_2	$\frac{4r^2-44r+120}{(-9+r)(-8+r)}$	$\frac{5}{3}$
b_3	$\frac{-8r^3+120r^2-592r+960}{(-9+r)(-8+r)(r-7)}$	$-\frac{40}{21}$
b_4	$\frac{16r^3-192r^2+752r-960}{(-9+r)(-8+r)(r-7)}$	$\frac{40}{21}$
b_5	$-\frac{32(-3+r)(-4+r)(-2+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{32}{21}$
b_6	$\frac{64(-3+r)(-2+r)(-1+r)}{(-9+r)(-8+r)(r-7)}$	$\frac{16}{21}$
b_7	$-\frac{128(-2+r)(-1+r)r}{(-9+r)(-8+r)(r-7)}$	0
b_8	$\frac{256r^3-256r}{(-9+r)(-8+r)(r-7)}$	0
b_9	$-\frac{512(2+r)r(1+r)}{(-9+r)(-8+r)(r-7)}$	0
b_{10}	$\frac{1024(2+r)(1+r)(3+r)}{(-9+r)(-8+r)(r-7)}$	$-\frac{256}{21}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 + b_9x^9 + b_{10}x^{10} + b_{11}x^{11}\dots \\
&= 1 - \frac{4x}{3} + \frac{5x^2}{3} - \frac{40x^3}{21} + \frac{40x^4}{21} - \frac{32x^5}{21} + \frac{16x^6}{21} - \frac{256x^{10}}{21} + O(x^{11})
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^{10}\big(1 - 8x + 40x^2 - 160x^3 + 560x^4 - 1792x^5 + 5376x^6 - 15360x^7 + 42240x^8 \\
&\quad - 112640x^9 + 292864x^{10} + O(x^{11})\big) \\
&\quad + c_2\left(1 - \frac{4x}{3} + \frac{5x^2}{3} - \frac{40x^3}{21} + \frac{40x^4}{21} - \frac{32x^5}{21} + \frac{16x^6}{21} - \frac{256x^{10}}{21} + O(x^{11})\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^{10}\big(1 - 8x + 40x^2 - 160x^3 + 560x^4 - 1792x^5 + 5376x^6 - 15360x^7 + 42240x^8 \\
&\quad - 112640x^9 + 292864x^{10} + O(x^{11})\big) \\
&\quad + c_2\left(1 - \frac{4x}{3} + \frac{5x^2}{3} - \frac{40x^3}{21} + \frac{40x^4}{21} - \frac{32x^5}{21} + \frac{16x^6}{21} - \frac{256x^{10}}{21} + O(x^{11})\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x^{10}\left(1-8x+40x^2-160x^3+560x^4-1792x^5+5376x^6-15360x^7+42240x^8-112640x^9+292864x^{10}+O\left(x^{11}\right)\right) + c_2\left(1-\frac{4x}{3}+\frac{5x^2}{3}-\frac{40x^3}{21}+\frac{40x^4}{21}-\frac{32x^5}{21}+\frac{16x^6}{21}-\frac{256x^{10}}{21}+O\left(x^{11}\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{10}\left(1-8x+40x^2-160x^3+560x^4-1792x^5+5376x^6-15360x^7+42240x^8-112640x^9+292864x^{10}+O\left(x^{11}\right)\right) + c_2\left(1-\frac{4x}{3}+\frac{5x^2}{3}-\frac{40x^3}{21}+\frac{40x^4}{21}-\frac{32x^5}{21}+\frac{16x^6}{21}-\frac{256x^{10}}{21}+O\left(x^{11}\right)\right)$$

Verified OK.

16.26.1 Maple step by step solution

Let's solve

$x^2(1+2x)\,y''+\left(-8x^2-9x\right)y'-12yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{12y}{x(1+2x)}+\frac{(9+8x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(9+8x)y'}{x(1+2x)}-\frac{12y}{x(1+2x)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{9+8x}{(1+2x)x}, P_3(x)=-\frac{12}{x(1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=-9$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(1+2x)\,y''+(-8x-9)\,y'-12y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=1..2$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-10+r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+r+1)\,(k-9+r)+2a_k(k+r+1)\,(k+r-6)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-10+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,10\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(\frac{a_{k+1}(k-9+r)}{2}+a_k(k+r-6)\right)=0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k(k+r-6)}{k-9+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = -\frac{2a_k(k-6)}{k-9}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y = \sum_{k=0}^{5} a_k x^k, a_{k+1} = -\frac{2a_k(k-6)}{k-9}\right]$$

- Recursion relation for $r = 10$

$$a_{k+1} = -\frac{2a_k(k+4)}{k+1}$$

- Solution for $r = 10$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+10}, a_{k+1} = -\frac{2a_k(k+4)}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{5} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+10}\right), a_{1+k} = -\frac{2a_k(k-6)}{k-9}, b_{1+k} = -\frac{2b_k(4+k)}{1+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(x^2*(1+2*x)*diff(y(x),x$2)-x*(9+8*x)*diff(y(x),x)-12*x*y(x)=0,y(x),type='series',x=0)
```

$$\begin{aligned} y(x) = {} & c_1 x^{10}\left(1 - 8x + 40x^2 - 160x^3 + 560x^4 - 1792x^5 + \mathrm{O}\left(x^6\right)\right) \\ & + c_2\left(-1316818944000 + 1755758592000x - 2194698240000x^2\right. \\ & \left. + 2508226560000x^3 - 2508226560000x^4 + 2006581248000x^5 + \mathrm{O}\left(x^6\right)\right) \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 62

```
AsymptoticDSolveValue[x^2*(1+2*x)*y''[x]-x*(9+8*x)*y'[x]-12*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{40x^4}{21} - \frac{40x^3}{21} + \frac{5x^2}{3} - \frac{4x}{3} + 1\right) + c_2 (560x^{14} - 160x^{13} + 40x^{12} - 8x^{11} + x^{10})$$

16.27 problem 23

Internal problem ID [1439]
Internal file name [`OUTPUT/1440_Sunday_June_05_2022_02_17_33_AM_88030696/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-x(-2x^2+7)\,y'+12y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(2x^3-7x)\,y'+12y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2-7}{x\,(x^2+1)}$$
$$q(x)=\frac{12}{x^2\,(x^2+1)}$$

Table 824: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2-7}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{12}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (2x^3-7x)\,y' + 12y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3-7x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 12\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 12a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n(n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 12a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - 7x^{n+r} a_n(n+r) + 12a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 7x^r a_0 r + 12a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 7x^r r + 12x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)(r-6)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)(r-6) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)(r-6)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^6\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^2\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+6}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) - 7a_n(n+r) + 12a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-1)\, a_{n-2}}{n+r-6} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{(n+5)\, a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1-r}{-4+r}$$

Which for the root $r = 6$ becomes

$$a_2 = -\frac{7}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-4+r}$	$-\frac{7}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-4+r}$	$-\frac{7}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3 + r)(1 + r)}{(-4 + r)(r - 2)}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{63}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-4+r}$	$-\frac{7}{2}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(-4+r)(r-2)}$	$\frac{63}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-1-r}{-4+r}$	$-\frac{7}{2}$
a_3	0	0
a_4	$\frac{(3+r)(1+r)}{(-4+r)(r-2)}$	$\frac{63}{8}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^6 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^6 \left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{(3+r)(1+r)}{(-4+r)(r-2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \rightarrow r_2} \frac{(3+r)(1+r)}{(-4+r)(r-2)} &= \lim_{r \rightarrow 2} \frac{(3+r)(1+r)}{(-4+r)(r-2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$

$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$

$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2(x^2+1)\,y'' + (2x^3-7x)\,y' + 12y = 0$ gives

$$x^2(x^2+1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (2x^3-7x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + 12Cy_1(x)\ln(x) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(x^2(x^2+1)\,y_1''(x) + (2x^3-7x)\,y_1'(x) + 12y_1(x)\right)\ln(x) + x^2(x^2+1)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(2x^3-7x)\,y_1(x)}{x}\right)C + x^2(x^2+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (2x^3-7x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2(x^2+1)\,y_1''(x)+(2x^3-7x)\,y_1'(x)+12y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^2(x^2+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3-7x)\,y_1(x)}{x}\right)C\\&+x^2(x^2+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+(2x^3-7x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+12\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2x(x^2+1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(x^2-8)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C\\&+(x^4+x^2)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)\\&+(2x^3-7x)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)+12\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{9}$$

Since $r_1=6$ and $r_2=2$ then the above becomes

$$\begin{aligned}&\left(2x(x^2+1)\left(\sum_{n=0}^{\infty}x^{n+5}a_n(n+6)\right)+(x^2-8)\left(\sum_{n=0}^{\infty}a_n x^{n+6}\right)\right)C\\&+(x^4+x^2)\left(\sum_{n=0}^{\infty}x^{n}b_n(n+2)(1+n)\right)\\&+(2x^3-7x)\left(\sum_{n=0}^{\infty}x^{1+n}b_n(n+2)\right)+12\left(\sum_{n=0}^{\infty}b_n x^{n+2}\right)=0\end{aligned}\tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+8} a_n (n+6)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{n+6} a_n (n+6)\right) \\
&+\left(\sum_{n=0}^{\infty} C\, x^{n+8} a_n\right)+\sum_{n\,=0}^{\infty}\left(-8C\, x^{n+6} a_n\right)+\left(\sum_{n=0}^{\infty} x^{n+4} b_n \left(n^2+3n+2\right)\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+4} b_n (n+2)\right) \\
&+\sum_{n\,=0}^{\infty}\left(-7x^{n+2} b_n (n+2)\right)+\left(\sum_{n=0}^{\infty} 12 b_n x^{n+2}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n\,=0}^{\infty} 2C\, x^{n+8} a_n (n+6) &= \sum_{n=6}^{\infty} 2C a_{n-6} n\, x^{n+2} \\
\sum_{n\,=0}^{\infty} 2C\, x^{n+6} a_n (n+6) &= \sum_{n=4}^{\infty} 2C a_{n-4} (n+2)\, x^{n+2} \\
\sum_{n\,=0}^{\infty} C\, x^{n+8} a_n &= \sum_{n=6}^{\infty} C a_{n-6} x^{n+2} \\
\sum_{n\,=0}^{\infty} \left(-8C\, x^{n+6} a_n\right) &= \sum_{n=4}^{\infty} \left(-8C a_{n-4} x^{n+2}\right) \\
\sum_{n\,=0}^{\infty} x^{n+4} b_n \left(n^2+3n+2\right) &= \sum_{n=2}^{\infty} b_{n-2} \left((n-2)^2+3n-4\right) x^{n+2} \\
\sum_{n\,=0}^{\infty} 2x^{n+4} b_n (n+2) &= \sum_{n=2}^{\infty} 2 b_{n-2} n\, x^{n+2}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n+2$.

$$\left(\sum_{n=6}^{\infty} 2C a_{n-6} n\, x^{n+2}\right) + \left(\sum_{n=4}^{\infty} 2C a_{n-4}(n+2)\, x^{n+2}\right) + \left(\sum_{n=6}^{\infty} C a_{n-6} x^{n+2}\right) + \sum_{n=4}^{\infty} \left(-8C a_{n-4} x^{n+2}\right) + \left(\sum_{n=2}^{\infty} b_{n-2}\left((n-2)^2+3n-4\right) x^{n+2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right) + \left(\sum_{n=2}^{\infty} 2b_{n-2} n\, x^{n+2}\right) + \sum_{n=0}^{\infty} \left(-7x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 12 b_n x^{n+2}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$6b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{2}$$

For $n=3$, Eq (2B) gives

$$12b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 30 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{15}{2}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 30b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{15}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{15}{2}\left(x^6\left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right)\right)\ln(x) + x^2\left(1 + \frac{3x^2}{2} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^6\left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{15}{2}\left(x^6\left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right)\right)\ln(x) + x^2\left(1 + \frac{3x^2}{2} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^6\left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{15x^6\left(1 - \frac{7x^2}{2} + \frac{63x^4}{8} + O(x^6)\right)\ln(x)}{2} + x^2\left(1 + \frac{3x^2}{2} + O(x^6)\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^6 \left(1-\frac{7x^2}{2}+\frac{63x^4}{8}+O(x^6)\right) \\
& + c_2\left(-\frac{15x^6\left(1-\frac{7x^2}{2}+\frac{63x^4}{8}+O(x^6)\right)\ln(x)}{2}+x^2\left(1+\frac{3x^2}{2}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^6 \left(1-\frac{7x^2}{2}+\frac{63x^4}{8}+O(x^6)\right) \\
& + c_2\left(-\frac{15x^6\left(1-\frac{7x^2}{2}+\frac{63x^4}{8}+O(x^6)\right)\ln(x)}{2}+x^2\left(1+\frac{3x^2}{2}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

16.27.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\,y''+(2x^3-7x)\,y'+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{12y}{x^2(x^2+1)}-\frac{(2x^2-7)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x^2-7)y'}{x(x^2+1)}+\frac{12y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x^2-7}{x(x^2+1)}, P_3(x)=\frac{12}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-7$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 12$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(2x^2-7)\, y' + 12y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-6+r)\, x^r + a_1(-1+r)\,(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)\,(k+r-6) + a_{k-2}(k +\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0

  $a_1(-1+r)\,(-5+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r-6)+a_{k-2}(k+r-1)\right)=0$

- Shift index using $k->k+2$

  $(k+r)\left(a_{k+2}(k-4+r)+a_k(k+r+1)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+1)}{k-4+r}$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k(k+3)}{k-2}$

- Series not valid for $r=2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=-\frac{a_k(k+3)}{k-2}$

- Recursion relation for $r=6$

  $a_{k+2}=-\frac{a_k(k+7)}{k+2}$

- Solution for $r=6$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2}=-\frac{a_k(k+7)}{k+2}, a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(7-2*x^2)*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = \left(c_1 x^4 \left(1 - \frac{7}{2}x^2 + \frac{63}{8}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(1080x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 216x^2 + 2106x^4 + \mathrm{O}\left(x^6\right)\right)\right)\right) x^2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 57

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-x*(7-2*x^2)*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{63x^{10}}{8} - \frac{7x^8}{2} + x^6\right) + c_1\left(-\frac{15}{2}x^6\log(x) - \frac{1}{4}\left(31x^4 - 6x^2 - 4\right)x^2\right)$$

16.28 problem 24

Internal problem ID [1440]
Internal file name [`OUTPUT/1441_Sunday_June_05_2022_02_17_37_AM_11746741/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x\left(-x^2+7\right)y' + 12y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^3-7x\right)y' + 12y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{x^2-7}{x} \\ q(x) &= \frac{12}{x^2} \end{aligned}$$

Table 826: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-7}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{12}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(x^3 - 7x\right) y' + 12y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(x^3 - 7x\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 12 \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-7 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7 x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 7 x^{n+r} a_n (n+r) + 12 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 7 x^r a_0 r + 12 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 7 x^r r + 12 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)(r-6)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)(r-6) = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 6 \\
r_2 &= 2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r-2)(r-6)x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^6\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^2\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+6} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - 7a_n(n+r) + 12a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n + r - 6} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{-4 + r}$$

Which for the root $r = 6$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{-4+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{-4+r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(-4+r)(r-2)}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{-4+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(-4+r)(r-2)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{-4+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(-4+r)(r-2)}$	$\frac{1}{8}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^6 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{(-4+r)(r-2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{(-4+r)(r-2)} &= \lim_{r\to 2} \frac{1}{(-4+r)(r-2)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + (x^3 - 7x)\,y' + 12y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^3 - 7x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + 12Cy_1(x)\ln(x) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x) + (x^3 - 7x)\,y_1'(x) + 12y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^3 - 7x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^3 - 7x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) + (x^3 - 7x)\,y_1'(x) + 12y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(x^3 - 7x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (x^3 - 7x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x^2-8)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ (x^3-7x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 6$ and $r_2 = 2$ then the above becomes

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{5+n} a_n (n+6)\right) x + (x^2-8)\left(\sum_{n=0}^{\infty} a_n x^{n+6}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{n} b_n (n+2)(1+n)\right) x^2 \\
&+ (x^3-7x)\left(\sum_{n=0}^{\infty} x^{1+n} b_n (n+2)\right) + 12\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+6} a_n (n+6)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+8} a_n\right) + \sum_{n=0}^{\infty} \left(-8C\, x^{n+6} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+2} b_n (n^2+3n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n+4} b_n (n+2)\right) \\
&+ \sum_{n=0}^{\infty} \left(-7x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 12 b_n x^{n+2}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\,x^{n+6} a_n (n+6) = \sum_{n=4}^{\infty} 2C a_{n-4} (n+2)\,x^{n+2}$$

$$\sum_{n=0}^{\infty} C\,x^{n+8} a_n = \sum_{n=6}^{\infty} C a_{n-6} x^{n+2}$$

$$\sum_{n=0}^{\infty} \left(-8C\,x^{n+6} a_n\right) = \sum_{n=4}^{\infty} \left(-8C a_{n-4} x^{n+2}\right)$$

$$\sum_{n=0}^{\infty} x^{n+4} b_n (n+2) = \sum_{n=2}^{\infty} b_{n-2} n\,x^{n+2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$\begin{aligned}\left(\sum_{n=4}^{\infty} 2C a_{n-4} (n+2)\,x^{n+2}\right) + \left(\sum_{n=6}^{\infty} C a_{n-6} x^{n+2}\right) + \sum_{n=4}^{\infty} \left(-8C a_{n-4} x^{n+2}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n+2} b_n \left(n^2+3n+2\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} n\,x^{n+2}\right) \\ + \sum_{n=0}^{\infty} \left(-7x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} 12 b_n x^{n+2}\right) = 0\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$-4b_2 + 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2 + 2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$-3b_3 + 3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 5b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x^6\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\right)\ln(x) + x^2\left(1 + \frac{x^2}{2} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{1}{2}\left(x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\right) \ln(x) + x^2 \left(1 + \frac{x^2}{2} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x)}{2} + x^2 \left(1 + \frac{x^2}{2} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x)}{2} + x^2 \left(1 + \frac{x^2}{2} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^6 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \ln(x)}{2} + x^2 \left(1 + \frac{x^2}{2} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

16.28.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 - 7x)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y}{x^2} - \frac{(x^2-7)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-7)y'}{x} + \frac{12y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -7$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 12$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2 - 7)\,y' + 12y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)\,(-6+r)\, x^r + a_1(-1+r)\,(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)\,(-6+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{2, 6\}$$

- Each term must be 0

  $$a_1(-1+r)\,(-5+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\,(a_k(k+r-6) + a_{k-2}) = 0$$

- Shift index using $k->k+2$

  $$(k+r)\,(a_{k+2}(k-4+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{k-4+r}$$

- Recursion relation for $r=2$

  $$a_{k+2} = -\frac{a_k}{k-2}$$

- Series not valid for $r=2$, division by 0 in the recursion relation at $k=2$

  $$a_{k+2} = -\frac{a_k}{k-2}$$

- Recursion relation for $r=6$

  $$a_{k+2} = -\frac{a_k}{k+2}$$

- Solution for $r=6$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2}=-\frac{a_k}{k+2}, a_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(7-x^2)*diff(y(x),x)+12*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(c_1 x^4\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(72x^4+\mathrm{O}\left(x^6\right)\right)+\left(-144-72x^2+54x^4+\mathrm{O}\left(x^6\right)\right)\right)\right)x^2$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 55

```
AsymptoticDSolveValue[x^2*y''[x]-x*(7-x^2)*y'[x]+12*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^{10}}{8}-\frac{x^8}{2}+x^6\right)+c_1\left(-\frac{1}{2}x^6\log(x)-\frac{1}{4}\left(x^4-2x^2-4\right)x^2\right)$$

16.29 problem 25

Internal problem ID [1441]
Internal file name [`OUTPUT/1442_Sunday_June_05_2022_02_17_41_AM_46253651/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' - 5y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 5y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = 1$$

Table 828: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 5y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1) - 5(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 5r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-6+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 6$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-6+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^6\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+6}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 5a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 6n - 6r} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 - 2r - 8}$$

Which for the root $r = 6$ becomes

$$a_2 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 - 20r^2 + 64}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}$$

Which for the root $r = 6$ becomes

$$a_6 = -\frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2-2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{r^4-20r^2+64}$	$\frac{1}{640}$
a_5	0	0
a_6	$-\frac{1}{(r^4-20r^2+64)r(r+6)}$	$-\frac{1}{46080}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^6\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\
&= x^6\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_6 \\
&= -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} &= \lim_{r\rightarrow 0} -\frac{1}{(r^4 - 20r^2 + 64)\, r\, (r + 6)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $xy'' - 5y' + yx = 0$ gives

$$\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x - 5Cy_1'(x)\ln(x) - \frac{5Cy_1(x)}{x} \\
&- 5\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) x = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(y_1''(x)\,x + y_1(x)\,x - 5y_1'(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - \frac{5y_1(x)}{x}\right)C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x - 5\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x + y_1(x)\,x - 5y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{5y_1(x)}{x}\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ +\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x-5\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-6\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x^2+\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2-5\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x} \\ =0 \tag{9}$$

Since $r_1=6$ and $r_2=0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{5+n}a_n(n+6)\right)x-6\left(\sum_{n=0}^{\infty}a_n x^{n+6}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x^2+\left(\sum_{n=0}^{\infty}x^{n-2}b_n n(-1+n)\right)x^2-5\left(\sum_{n=0}^{\infty}x^{-1+n}b_n n\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^{5+n}a_n(n+6)\right)+\sum_{n=0}^{\infty}\left(-6C\,x^{5+n}a_n\right)+\left(\sum_{n=0}^{\infty}x^{1+n}b_n\right) \\ +\left(\sum_{n=0}^{\infty}n\,x^{-1+n}b_n(-1+n)\right)+\sum_{n=0}^{\infty}\left(-5x^{-1+n}b_n n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{5+n} a_n (n+6) = \sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n}$$

$$\sum_{n=0}^{\infty} \left(-6C\, x^{5+n} a_n\right) = \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{-1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-1+n$.

$$\left(\sum_{n=6}^{\infty} 2C a_{-6+n} n\, x^{-1+n}\right) + \sum_{n=6}^{\infty} \left(-6C a_{-6+n} x^{-1+n}\right) + \left(\sum_{n=2}^{\infty} b_{n-2} x^{-1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{-1+n} b_n (-1+n)\right) + \sum_{n=0}^{\infty} \left(-5x^{-1+n} b_n n\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-5b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$b_0 - 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{8}$$

For $n = 3$, Eq (2B) gives

$$b_1 - 9b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$b_2 - 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{8} - 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{64}$$

For $n = 5$, Eq (2B) gives

$$b_3 - 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + \frac{1}{64} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{384}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{384}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{1}{384}\left(x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\right)\ln(x)+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right) \\
&\quad + c_2\left(-\frac{x^6\left(1-\frac{x^2}{16}+\frac{x^4}{640}-\frac{x^6}{46080}+O(x^7)\right)\ln(x)}{384}+1+\frac{x^2}{8}+\frac{x^4}{64}+O(x^7)\right)
\end{aligned}$$

Verified OK.

16.29.1 Maple step by step solution

Let's solve

$y''x - 5y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + y = 0$

- Simplify ODE

  $x^2y'' - 5y'x + x^2y = 0$

- Make a change of variables

  $y = x^3u(x)$

- Compute y'

  $y' = 3x^2u(x) + x^3u'(x)$

- Compute y''

  $y'' = 6xu(x) + 6x^2u'(x) + x^3u''(x)$

- Apply change of variables to the ODE

  $x^2u(x) + u''(x)\,x^2 + u'(x)\,x - 9u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x)$

- Make the change from y back to y

  $y = (c_1 \mathit{BesselJ}(3, x) + c_2 \mathit{BesselY}(3, x))\,x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)-5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^6 \left(1 - \frac{1}{16}x^2 + \frac{1}{640}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-86400 - 10800x^2 - 1350x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x*y''[x]-5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{64} + \frac{x^2}{8} + 1\right) + c_2\left(\frac{x^{10}}{640} - \frac{x^8}{16} + x^6\right)$$

16.30 problem 26

Internal problem ID [1442]
Internal file name [`OUTPUT/1443_Sunday_June_05_2022_02_17_45_AM_21889668/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(2x^2 + 1)\, y' - (-10x^2 + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (2x^3 + x)\, y' + (10x^2 - 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2x^2 + 1}{x}$$
$$q(x) = \frac{10x^2 - 1}{x^2}$$

Table 830: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{10x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (2x^3 + x)\, y' + (10x^2 - 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (2x^3 + x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (10x^2 - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 10x^{n+r+2} a_n = \sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-2}(n+r-2) + a_n(n+r) + 10a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}(n+r+3)}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{2a_{n-2}(n+4)}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-10 - 2r}{r^2 + 4r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10-2r}{r^2+4r+3}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10-2r}{r^2+4r+3}$	$-\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{28 + 4r}{(r + 3)^2 (1 + r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10-2r}{r^2+4r+3}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{28+4r}{(r+3)^2(1+r)}$	1

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10-2r}{r^2+4r+3}$	$-\frac{3}{2}$
a_3	0	0
a_4	$\frac{28+4r}{(r+3)^2(1+r)}$	1
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 - \frac{3x^2}{2} + x^4 + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{-10 - 2r}{r^2 + 4r + 3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{-10 - 2r}{r^2 + 4r + 3} &= \lim_{r\to -1} \frac{-10 - 2r}{r^2 + 4r + 3} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n + r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n + r_2)(-1 + n + r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y''+(2x^3+x)\,y'+(10x^2-1)\,y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(2x^3+x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(10x^2-1)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+(2x^3+x)\,y_1'(x)+(10x^2-1)\,y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(2x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(10x^2-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x)+(2x^3+x)\,y_1'(x)+(10x^2-1)\,y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(2x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(10x^2-1)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\ &+ (2x^3+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ &+ 10\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x\right) C \\ &+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 + (2x^3+x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) \\ &+ 10\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} 2C\, x^{3+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2-3n+2)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+1} b_n (n-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+1} b_n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{3+n} a_n = \sum_{n=4}^{\infty} 2C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) = \sum_{n=2}^{\infty} 2C a_{n-2} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 2x^{n+1} b_n (n-1) = \sum_{n=2}^{\infty} 2b_{n-2} (-3+n)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} 10x^{n+1} b_n = \sum_{n=2}^{\infty} 10 b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}&\left(\sum_{n=4}^{\infty} 2C a_{n-4} x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2}(n-1)\, x^{n-1}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right) + \left(\sum_{n=2}^{\infty} 2b_{n-2}(-3+n)\, x^{n-1}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 10 b_{n-2} x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 8 = 0$$

Which is solved for C. Solving for C gives

$$C = -4$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + 10b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(2a_0 + 6a_2) C + 12b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$28 + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{7}{2}$$

For $n = 5$, Eq (2B) gives

$$(2a_1 + 8a_3) C + 14b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -4$ and all b_n, then the second solution becomes

$$y_2(x) = (-4) \left(x\left(1 - \frac{3x^2}{2} + x^4 + O(x^6)\right)\right) \ln(x) + \frac{1 - \frac{7x^4}{2} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\\ &+ c_2\left((-4)\left(x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\right)\ln\left(x\right)+\frac{1-\frac{7x^4}{2}+O(x^6)}{x}\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\\ &+ c_2\left(-4x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-\frac{7x^4}{2}+O(x^6)}{x}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned}y &= c_1 x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\\ &+ c_2\left(-4x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-\frac{7x^4}{2}+O(x^6)}{x}\right)\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1 x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\\ &+ c_2\left(-4x\left(1-\frac{3x^2}{2}+x^4+O\left(x^6\right)\right)\ln\left(x\right)+\frac{1-\frac{7x^4}{2}+O(x^6)}{x}\right)\end{aligned}$$

Verified OK.

16.30.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^3 + x)\,y' + (10x^2 - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x^2-1)y}{x^2} - \frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+1)y'}{x} + \frac{(10x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+1}{x}, P_3(x) = \frac{10x^2-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x^2 + 1)\,y' + (10x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(1+r)\,(-1+r)\, x^{r}+a_{1}(2+r)\, r\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+1)\,(k+r-1)+2a_{k-2}(k+3+r)\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{-1,1\}$$

- Each term must be 0

  $$a_{1}(2+r)\, r=0$$

- Solve for the dependent coefficient(s)

  $$a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k}(k+r+1)\,(k+r-1)+2a_{k-2}(k+3+r)=0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+3+r)\,(k+r+1)+2a_{k}(k+r+5)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{2a_{k}(k+r+5)}{(k+3+r)(k+r+1)}$$

- Recursion relation for $r=-1$

  $$a_{k+2}=-\frac{2a_{k}(k+4)}{(k+2)k}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{k+2} = -\frac{2a_k(4+k)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(4+k)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1+2*x^2)*diff(y(x),x)-(1-10*x^2)*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = \frac{c_1x^2\left(1-\frac{3}{2}x^2+x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(8x^2-12x^4+\mathrm{O}\left(x^6\right)\right)+\left(-2+2x^2+4x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1+2*x^2)*y'[x]-(1-10*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x^5 - \frac{3x^3}{2} + x\right) + c_1\left(2x(3x^2 - 2)\log(x) - \frac{5x^4 - x^2 - 1}{x}\right)$$

16.31 problem 27

Internal problem ID [1443]
Internal file name [`OUTPUT/1444_Sunday_June_05_2022_02_17_49_AM_56478400/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x - (-x^2 + 3)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + (x^2 - 3)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^2 - 3}{x^2}$$

Table 832: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x + (x^2 - 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + (x^2 - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) - 3 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r - 3 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r - 3 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 2r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 3 \\
r_2 &= -1
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(r^2 - 2r - 3\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} - 3a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r - 3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 2r - 3}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r^4+8r^3+14r^2-8r-15}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r^4+8r^3+14r^2-8r-15}$	$\frac{1}{384}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} &= \lim_{r\to -1} \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' - y'x + (x^2-3)\,y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x + (x^2-3)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x) - y_1'(x)\,x + (x^2-3)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x + (x^2-3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) - y_1'(x)\,x + (x^2-3)\,y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - y_1(x)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x + (x^2-3)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 3$ and $r_2 = -1$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 - \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) x - 3\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \sum_{n=0}^{\infty} \left(-2C a_n x^{n+3}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) + \sum_{n=0}^{\infty} \left(-x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-1}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=4}^{\infty} 2C a_{n-4} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C a_n x^{n+3}\right) = \sum_{n=4}^{\infty} \left(-2C a_{n-4} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
&\left(\sum_{n=4}^{\infty} 2C a_{n-4}(n-1)\, x^{n-1}\right)+\sum_{n\,=4}^{\infty}\left(-2C a_{n-4} x^{n-1}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n-1} b_{n}\left(n^{2}-3n+2\right)\right)+\left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right) \\
&+\sum_{n\,=0}^{\infty}\left(-x^{n-1} b_{n}(n-1)\right)+\sum_{n\,=0}^{\infty}\left(-3 b_{n} x^{n-1}\right)=0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{16}\left(x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\ln(x)}{16} + \frac{1 + \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= c_1 x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \\
&+ c_2 \left(-\frac{x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \ln\left(x\right)}{16}+\frac{1+\frac{x^2}{4}+O\left(x^6\right)}{x}\right) \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1 x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \\
&+ c_2 \left(-\frac{x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O\left(x^6\right)\right) \ln\left(x\right)}{16}+\frac{1+\frac{x^2}{4}+O\left(x^6\right)}{x}\right)
\end{align*}
$$

Verified OK.

16.31.1 Maple step by step solution

Let's solve

$x^2 y'' - y' x + \left(x^2-3\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{\left(x^2-3\right) y}{x^2}+\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x}+\frac{\left(x^2-3\right) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{x^2-3}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -3$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - y'x + (x^2 - 3)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1 + r)\, (-3 + r)\, x^r + a_1(2 + r)\, (-2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 1)\, (k + r - 3) + a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r)\, (-3 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 3\}$

- Each term must be 0

  $a_1(2 + r)\, (-2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-3) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)(k+r-1) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r-1)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k}{(k+6)(k+2)}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-(3-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{1}{12}x^2 + \frac{1}{384}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 (\ln(x) \left(9x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 36x^2 + \mathrm{O}\left(x^6\right)\right))}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]-(3-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{(x^2+8)^2}{64x} - \frac{1}{16}x^3 \log(x)\right) + c_2 \left(\frac{x^7}{384} - \frac{x^5}{12} + x^3\right)$$

16.32 problem 28

Internal problem ID [1444]
Internal file name [OUTPUT/1445_Sunday_June_05_2022_02_17_53_AM_50493699/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x(x^2+8)\,y' + (3x^2+5)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (2x^3+16x)\,y' + (3x^2+5)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2+8}{2x}$$
$$q(x) = \frac{3x^2+5}{4x^2}$$

Table 834: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+8}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{3x^2+5}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (2x^3 + 16x)\, y' + (3x^2 + 5)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (2x^3 + 16x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (3x^2 + 5) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 5a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 16x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 16x^{n+r} a_n (n+r) + 5a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 16x^r a_0 r + 5a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + 16x^r r + 5x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(4r^2 + 12r + 5)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 12r + 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -\frac{5}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 12r + 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sqrt{x}}$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{5}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1)+2a_{n-2}(n+r-2)+16a_n(n+r)+3a_{n-2}+5a_n=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{a_{n-2}(2n+2r-1)}{4n^2+8nr+4r^2+12n+12r+5} \quad (4)$$

Which for the root $r=-\frac{1}{2}$ becomes

$$a_n=-\frac{a_{n-2}(n-1)}{2n(n+2)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=-\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=\frac{-3-2r}{4r^2+28r+45}$$

Which for the root $r=-\frac{1}{2}$ becomes

$$a_2=-\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-2r}{4r^2+28r+45}$	$-\frac{1}{16}$

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-2r}{4r^2+28r+45}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 20r + 21}{(4r^2 + 28r + 45)(4r^2 + 44r + 117)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = \frac{1}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-2r}{4r^2+28r+45}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{4r^2+20r+21}{(4r^2+28r+45)(4r^2+44r+117)}$	$\frac{1}{256}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-3-2r}{4r^2+28r+45}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{4r^2+20r+21}{(4r^2+28r+45)(4r^2+44r+117)}$	$\frac{1}{256}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= \frac{1 - \frac{x^2}{16} + \frac{x^4}{256} + O(x^6)}{\sqrt{x}}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$a_N = a_2$$
$$= \frac{-3 - 2r}{4r^2 + 28r + 45}$$

Therefore

$$\lim_{r\to r_2} \frac{-3 - 2r}{4r^2 + 28r + 45} = \lim_{r\to -\frac{5}{2}} \frac{-3 - 2r}{4r^2 + 28r + 45}$$
$$= \textit{undefined}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\frac{d}{dx} y_2(x) = C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right)$$
$$= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right)$$
$$\frac{d^2}{dx^2} y_2(x) = C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right)$$
$$= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2)(-1 + n + r_2)\right)$$

Substituting these back into the given ode $4x^2y''+(2x^3+16x)\,y'+(3x^2+5)\,y=0$ gives

$$
\begin{aligned}
&4x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(2x^3+16x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(3x^2+5)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(4x^2y_1''(x)+(2x^3+16x)\,y_1'(x)+(3x^2+5)\,y_1(x)\right)\ln(x)\right.\\
&\left.+4x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3+16x)\,y_1(x)}{x}\right)C\\
&+4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(2x^3+16x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(3x^2+5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
4x^2y_1''(x)+(2x^3+16x)\,y_1'(x)+(3x^2+5)\,y_1(x)=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(4x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(2x^3+16x)\,y_1(x)}{x}\right)C\\
&+4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(2x^3+16x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(3x^2+5)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(8\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + 2(x^2+6)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C$$
$$+ 4\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \tag{9}$$
$$+ 2(x^3+8x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (3x^2+5)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Since $r_1 = -\frac{1}{2}$ and $r_2 = -\frac{5}{2}$ then the above becomes

$$\left(8\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} a_n \left(n-\frac{1}{2}\right)\right) x + 2(x^2+6)\left(\sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}\right)\right) C$$
$$+ 4\left(\sum_{n=0}^{\infty} x^{-\frac{9}{2}+n} b_n \left(n-\frac{5}{2}\right)\left(-\frac{7}{2}+n\right)\right) x^2 \tag{10}$$
$$+ 2(x^3+8x)\left(\sum_{n=0}^{\infty} x^{-\frac{7}{2}+n} b_n \left(n-\frac{5}{2}\right)\right) + (3x^2+5)\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right) = 0$$

Expanding $2C\,x^{\frac{3}{2}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 2C\,x^{\frac{3}{2}} &= 2C\,x^{\frac{3}{2}} + \dots \\ &= 2C\,x^{\frac{3}{2}} \end{aligned}$$

Expanding $\frac{12C}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \frac{12C}{\sqrt{x}} &= \frac{12C}{\sqrt{x}} + \dots \\ &= \frac{12C}{\sqrt{x}} \end{aligned}$$

Expanding $\frac{1}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}} + \dots \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Expanding $\frac{8}{x^{\frac{5}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{8}{x^{\frac{5}{2}}} &= \frac{8}{x^{\frac{5}{2}}} + \dots \\ &= \frac{8}{x^{\frac{5}{2}}}\end{aligned}$$

Expanding $\frac{3}{\sqrt{x}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{3}{\sqrt{x}} &= \frac{3}{\sqrt{x}} + \dots \\ &= \frac{3}{\sqrt{x}}\end{aligned}$$

Expanding $\frac{5}{x^{\frac{5}{2}}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\frac{5}{x^{\frac{5}{2}}} &= \frac{5}{x^{\frac{5}{2}}} + \dots \\ &= \frac{5}{x^{\frac{5}{2}}}\end{aligned}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} \left(8n-4\right) C a_n x^{n-\frac{1}{2}}\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+\frac{3}{2}} a_n\right) + \left(\sum_{n=0}^{\infty} 12C\, x^{n-\frac{1}{2}} a_n\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-\frac{5}{2}} b_n \left(4n^2-24n+35\right)\right) + \left(\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n (2n-5)\right) \\ &+ \left(\sum_{n=0}^{\infty} \left(16n-40\right) b_n x^{n-\frac{5}{2}}\right) + \left(\sum_{n=0}^{\infty} 3x^{n-\frac{1}{2}} b_n\right) + \left(\sum_{n=0}^{\infty} 5b_n x^{n-\frac{5}{2}}\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-\frac{5}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{5}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(8n-4\right) C a_n x^{n-\frac{1}{2}} = \sum_{n=2}^{\infty} C a_{n-2} (8n-20)\, x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+\frac{3}{2}} a_n = \sum_{n=4}^{\infty} 2C a_{n-4} x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty} 12C\, x^{n-\frac{1}{2}} a_n = \sum_{n=2}^{\infty} 12C a_{n-2} x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty} x^{n-\frac{1}{2}} b_n (2n-5) = \sum_{n=2}^{\infty} b_{n-2}(-9+2n)\, x^{n-\frac{5}{2}}$$

$$\sum_{n=0}^{\infty} 3x^{n-\frac{1}{2}} b_n = \sum_{n=2}^{\infty} 3b_{n-2} x^{n-\frac{5}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{5}{2}$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} C a_{n-2}(8n-20)\, x^{n-\frac{5}{2}}\right) + \left(\sum_{n=4}^{\infty} 2C a_{n-4} x^{n-\frac{5}{2}}\right) + \left(\sum_{n=2}^{\infty} 12C a_{n-2} x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-\frac{5}{2}} b_n \left(4n^2-24n+35\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2}(-9+2n)\, x^{n-\frac{5}{2}}\right) \\
&+ \left(\sum_{n=0}^{\infty} (16n-40)\, b_n x^{n-\frac{5}{2}}\right) + \left(\sum_{n=2}^{\infty} 3b_{n-2} x^{n-\frac{5}{2}}\right) + \left(\sum_{n=0}^{\infty} 5b_n x^{n-\frac{5}{2}}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-4b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$8C - 2 = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{4}$$

For $n=3$, Eq (2B) gives

$$16C a_1 + 12b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(2a_0 + 24a_2) C + 2b_2 + 32b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{8} + 32b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{256}$$

For $n = 5$, Eq (2B) gives

$$(2a_1 + 32a_3) C + 4b_3 + 60b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$60b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{4}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{1}{4}\left(\frac{1 - \frac{x^2}{16} + \frac{x^4}{256} + O(x^6)}{\sqrt{x}}\right) \ln(x) + \frac{1 - \frac{x^4}{256} + O(x^6)}{x^{\frac{5}{2}}}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= \frac{c_1\left(1 - \frac{x^2}{16} + \frac{x^4}{256} + O(x^6)\right)}{\sqrt{x}} + c_2\left(\frac{1}{4}\left(\frac{1 - \frac{x^2}{16} + \frac{x^4}{256} + O(x^6)}{\sqrt{x}}\right) \ln(x) + \frac{1 - \frac{x^4}{256} + O(x^6)}{x^{\frac{5}{2}}}\right)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)}{\sqrt{x}} + c_2\left(\frac{\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)\ln(x)}{4\sqrt{x}} + \frac{1-\frac{x^4}{256}+O(x^6)}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)}{\sqrt{x}} \\
& + c_2\left(\frac{\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)\ln(x)}{4\sqrt{x}} + \frac{1-\frac{x^4}{256}+O(x^6)}{x^{\frac{5}{2}}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)}{\sqrt{x}} + c_2\left(\frac{\left(1-\frac{x^2}{16}+\frac{x^4}{256}+O(x^6)\right)\ln(x)}{4\sqrt{x}} + \frac{1-\frac{x^4}{256}+O(x^6)}{x^{\frac{5}{2}}}\right)
$$

Verified OK.

16.32.1 Maple step by step solution

Let's solve

$4x^2y'' + (2x^3+16x)\, y' + (3x^2+5)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2+5)y}{4x^2} - \frac{(x^2+8)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+8)y'}{2x} + \frac{(3x^2+5)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x^2+8}{2x}, P_3(x) = \frac{3x^2+5}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 4$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{5}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2 y'' + 2x(x^2 + 8)\, y' + (3x^2 + 5)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (5 + 2r)\,(1 + 2r)\, x^r + a_1 (7 + 2r)\,(3 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (2k + 2r + 5)\,(2k + 2r + 1) + a_{k-2}(2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, -\frac{1}{2}\right\}$
- Each term must be 0

  $a_1(7+2r)(3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(k+r+\frac{5}{2}\right)a_k + 2\left(k-\frac{1}{2}+r\right)a_{k-2} = 0$
- Shift index using $k- >k+2$

  $4\left(k+r+\frac{5}{2}\right)\left(k+\frac{9}{2}+r\right)a_{k+2} + 2\left(k+\frac{3}{2}+r\right)a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r+3)a_k}{(2k+2r+5)(2k+9+2r)}$
- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{(2k-2)a_k}{2k(2k+4)}$
- Solution for $r = -\frac{5}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-2)a_k}{2k(2k+4)}, a_1 = 0\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{(2k+2)a_k}{(2k+4)(2k+8)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{(2k+4)(2k+8)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+2} = -\frac{(2k-2)a_k}{2k(2k+4)}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{(2k+4)(2k+8)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+2*x*(8+x^2)*diff(y(x),x)+(5+3*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_1\left(1-\frac{1}{16}x^2+\frac{1}{256}x^4+\mathrm{O}\left(x^6\right)\right)x^2+c_2\left(\ln\left(x\right)\left(-\frac{1}{2}x^2+\frac{1}{32}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-2+\frac{1}{2}x^2-\frac{3}{128}x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 72

```
AsymptoticDSolveValue[4*x^2*y''[x]+2*x*(8+x^2)*y'[x]+(5+3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^{7/2}}{256}-\frac{x^{3/2}}{16}+\frac{1}{\sqrt{x}}\right)+c_1\left(\frac{5x^4-96x^2+256}{256x^{5/2}}-\frac{(x^2-16)\log(x)}{64\sqrt{x}}\right)$$

16.33 problem 29

Internal problem ID [1445]
Internal file name [`OUTPUT/1446_Sunday_June_05_2022_02_17_57_AM_16696302/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + x(x^2+1)\,y' - (-3x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3+x)\,y' + (3x^2-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2+1}{x}$$
$$q(x) = \frac{3x^2-1}{x^2}$$

Table 836: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{3x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 + x) y' + (3x^2 - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + (x^3 + x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + (3x^2 - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{n+r+2} a_n = \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) + a_n(n+r) + 3a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{1+r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(1+r)(3+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{1+r}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{1}{(1+r)(3+r)}$	$\frac{1}{8}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)\\ &= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_2\\ &= -\frac{1}{1+r}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} -\frac{1}{1+r} &= \lim_{r\to -1} -\frac{1}{1+r}\\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $x^2 y'' + (x^3 + x) y' + (3x^2 - 1) y = 0$ gives

$$
\begin{aligned}
&x^2 \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \right) \\
&+ (x^3 + x) \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) \right) \\
&+ (3x^2 - 1) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) \right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2 y_1''(x) + (x^3 + x) y_1'(x) + (3x^2 - 1) y_1(x) \right) \ln(x) + x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) \right. \\
&\left. + \frac{(x^3 + x) y_1(x)}{x} \right) C + x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \right) \\
&+ (x^3 + x) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) + (3x^2 - 1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2 y_1''(x) + (x^3 + x) y_1'(x) + (3x^2 - 1) y_1(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^2 \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) + \frac{(x^3 + x) y_1(x)}{x} \right) C \\
&+ x^2 \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n + r_2)}{x^2} \right) \right) \\
&+ (x^3 + x) \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n + r_2)}{x} \right) + (3x^2 - 1) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C$$
$$+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2$$
$$+ (x^3+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right)$$
$$+ 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = -1$ then the above becomes

$$\left(\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x\right) C$$
$$+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 + (x^3+x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right)$$
$$+ 3\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 - \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} C\, x^{3+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right)$$
$$+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2-3n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n+1} b_n (n-1)\right)$$
$$+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+1} b_n\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} C\,x^{3+n} a_n = \sum_{n=4}^{\infty} C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} 2C\,x^{n+1} a_n (n+1) = \sum_{n=2}^{\infty} 2C a_{n-2}(n-1)\,x^{n-1}$$

$$\sum_{n=0}^{\infty} x^{n+1} b_n (n-1) = \sum_{n=2}^{\infty} b_{n-2}(-3+n)\,x^{n-1}$$

$$\sum_{n=0}^{\infty} 3x^{n+1} b_n = \sum_{n=2}^{\infty} 3b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} C a_{n-4} x^{n-1}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2}(n-1)\,x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2}(-3+n)\,x^{n-1}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right) + \left(\sum_{n=2}^{\infty} 3b_{n-2} x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = N$, where $N = 2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n = 2$, Eq (2B) gives

$$2C + 2 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + 3b_1 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$(a_0 + 6a_2)\, C + 4b_2 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{4}$$

For $n = 5$, Eq (2B) gives

$$(a_1 + 8a_3)\, C + 5b_3 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1) \left(x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\right) \ln(x) + \frac{1 - \frac{x^4}{4} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
&\quad + c_2 \left((-1)\left(x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{x^4}{4} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$

$$= c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) + c_2 \left(-x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + \frac{1 - \frac{x^4}{4} + O(x^6)}{x}\right)$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \\
+ c_2 \left(-x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + \frac{1 - \frac{x^4}{4} + O(x^6)}{x}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) + c_2 \left(-x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)\ln(x) + \frac{1 - \frac{x^4}{4} + O(x^6)}{x}\right)$$

Verified OK.

16.33.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 + x) y' + (3x^2 - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2-1)y}{x^2} - \frac{(x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+1)y'}{x} + \frac{(3x^2-1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x^2+1}{x}, P_3(x) = \frac{3x^2-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2y'' + x(x^2+1)\, y' + (3x^2-1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)(k+r-1) + a_{k-2}(k+r+1))\, x^{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2+r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)(a_k(k+r-1) + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(k+r+3)(a_{k+2}(k+r+1) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{k}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{k}, a_1 = 0\right]$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{k+2} = -\frac{a_k}{k}, a_1 = 0, b_{k+2} = -\frac{b_k}{k+2}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1+x^2)*diff(y(x),x)-(1-3*x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + O\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(2x^2 - x^4 + O\left(x^6\right)\right) + \left(-2 + x^2 + O\left(x^6\right)\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 51

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1+x^2)*y'[x]-(1-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8} - \frac{x^3}{2} + x\right) + c_1\left(\frac{1}{2}x\left(x^2 - 2\right)\log(x) - \frac{x^4 - 4}{4x}\right)$$

16.34 problem 30

Internal problem ID [1446]
Internal file name [OUTPUT/1447_Sunday_June_05_2022_02_18_01_AM_20020391/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x\left(-2x^2+1\right)y' - 4y\left(2x^2+1\right) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-2x^3+x\right)y' + \left(-8x^2-4\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2-1}{x}$$
$$q(x) = -\frac{4(2x^2+1)}{x^2}$$

Table 838: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{4(2x^2+1)}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-2x^3 + x\right) y' + \left(-8x^2 - 4\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-2x^3 + x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(-8x^2 - 4\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-8x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-8a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 4)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-2}(n+r-2) + a_n(n+r) - 8a_{n-2} - 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{2a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{r}$$

Which for the root $r = 2$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r}$	1
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{r(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r}$	1
a_3	0	0
a_4	$\frac{4}{r(r+2)}$	$\frac{1}{2}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{r}$	1
a_3	0	0
a_4	$\frac{4}{r(r+2)}$	$\frac{1}{2}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 + x^2 + \frac{x^4}{2} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{4}{r\,(r+2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{4}{r\,(r+2)} &= \lim_{r\to -2} \frac{4}{r\,(r+2)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + (-2x^3 + x)\,y' + (-8x^2 - 4)\,y = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-2x^3 + x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) + (-8x^2 - 4)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(x^2y_1''(x) + (-2x^3 + x)\,y_1'(x) + (-8x^2 - 4)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-2x^3 + x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-2x^3 + x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-8x^2 - 4)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x) + (-2x^3 + x)\,y_1'(x) + (-8x^2 - 4)\,y_1(x) = 0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-2x^3 + x)\,y_1(x)}{x}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) + (-2x^3 + x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (-8x^2 - 4)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned} & \left(-2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C \\ & + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\ & + (-2x^3+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ & - 8\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \end{aligned} \tag{9}$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$\begin{aligned} & \left(-2\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) x\right) C \\ & + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + (-2x^3+x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) \\ & - 8\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0 \end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(-2C\, x^{4+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) \\ & + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2-5n+6)\right) + \sum_{n=0}^{\infty} \left(-2x^n b_n (n-2)\right) \\ & + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-8 b_n x^n\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2C\, x^{4+n} a_n\right) = \sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=4}^{\infty} 2C a_{-4+n} (n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-2x^n b_n (n-2)\right) = \sum_{n=2}^{\infty} \left(-2b_{n-2}(-4+n)\, x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} \left(-8b_n x^n\right) = \sum_{n=2}^{\infty} \left(-8b_{n-2} x^{n-2}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned}\sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right) + \left(\sum_{n=4}^{\infty} 2C a_{-4+n}(n-2)\, x^{n-2}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \sum_{n=2}^{\infty} \left(-2b_{n-2}(-4+n)\, x^{n-2}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \sum_{n=2}^{\infty} \left(-8b_{n-2} x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-4b_n x^{n-2}\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$-4b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -1$$

For $n = 3$, Eq (2B) gives

$$-6b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 8 = 0$$

Which is solved for C. Solving for C gives

$$C = -2$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 - 10b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -2$ and all b_n, then the second solution becomes

$$y_2(x) = (-2)\left(x^2\left(1 + x^2 + \frac{x^4}{2} + O(x^6)\right)\right)\ln(x) + \frac{1 - x^2 + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 + x^2 + \frac{x^4}{2} + O(x^6)\right) \\
&\quad + c_2\left((-2)\left(x^2\left(1 + x^2 + \frac{x^4}{2} + O(x^6)\right)\right)\ln(x) + \frac{1 - x^2 + O(x^6)}{x^2}\right)
\end{aligned}$$

Hence the final solution is

$$y = y_h$$
$$= c_1 x^2 \left(1+x^2+\frac{x^4}{2}+O(x^6)\right)+c_2\left(-2x^2\left(1+x^2+\frac{x^4}{2}+O(x^6)\right)\ln(x)+\frac{1-x^2+O(x^6)}{x^2}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \left(1+x^2+\frac{x^4}{2}+O(x^6)\right) + c_2\left(-2x^2\left(1+x^2+\frac{x^4}{2}+O(x^6)\right)\ln(x)+\frac{1-x^2+O(x^6)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1+x^2+\frac{x^4}{2}+O(x^6)\right) + c_2\left(-2x^2\left(1+x^2+\frac{x^4}{2}+O(x^6)\right)\ln(x)+\frac{1-x^2+O(x^6)}{x^2}\right)$$

Verified OK.

16.34.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^3 + x) y' + (-8x^2 - 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4(2x^2+1)y}{x^2} + \frac{(2x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-1)y'}{x} - \frac{4(2x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(2x^2 - 1)\, y' + (-8x^2 - 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2 + r)\,(-2 + r)\, x^r + a_1(3 + r)\,(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r + 2)\,(k + r - 2) - 2a_{k-2}(k + r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1(3+r)(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(a_k(k+r-2)-2a_{k-2})=0$

- Shift index using $k->k+2$

  $(k+r+4)(a_{k+2}(k+r)-2a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{k+r}$

- Recursion relation for $r=-2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Recursion relation for $r=2$

  $a_{k+2} = \frac{2a_k}{k+2}$

- Solution for $r=2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x^2)*diff(y(x),x)-4*(1+2*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = \frac{c_1x^4\left(1+x^2+\frac{1}{2}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(288x^4+\mathrm{O}\left(x^6\right)\right)+\left(-144+144x^2+216x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 45

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-2*x^2)*y'[x]-4*(1+2*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(-2x^2\log(x)-\frac{x^4+x^2-1}{x^2}\right)+c_2\left(\frac{x^6}{2}+x^4+x^2\right)$$

16.35 problem 31

Internal problem ID [1447]
Internal file name [OUTPUT/1448_Sunday_June_05_2022_02_18_05_AM_59615361/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 8y'x - (-x^2 + 35)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + 8y'x + (x^2 - 35)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{x^2 - 35}{4x^2}$$

Table 840: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-35}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + 8y'x + \left(x^2 - 35\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 8\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + \left(x^2 - 35\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)\right) \\
+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-35 a_n x^{n+r}\right)=0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)\right) \\
+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-35 a_n x^{n+r}\right)=0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n(n+r)(n+r-1)+8x^{n+r} a_n(n+r)-35 a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r)+8x^r a_0 r-35 a_0 x^r=0
$$

Or

$$
\left(4x^r r(-1+r)+8x^r r-35x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(4r^2+4r-35\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$
4r^2+4r-35=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{5}{2} \\
r_2 &= -\frac{7}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(4r^2 + 4r - 35)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{\frac{5}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{7}{2}}}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) + 8a_n(n+r) + a_{n-2} - 35a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{4n^2 + 8nr + 4r^2 + 4n + 4r - 35} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{4n\,(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{4r^2 + 20r - 11}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = -\frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+20r-11}$	$-\frac{1}{64}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+20r-11}$	$-\frac{1}{64}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 224r^3 + 856r^2 + 504r - 495}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{1}{10240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+20r-11}$	$-\frac{1}{64}$
a_3	0	0
a_4	$\frac{1}{16r^4+224r^3+856r^2+504r-495}$	$\frac{1}{10240}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+20r-11}$	$-\frac{1}{64}$
a_3	0	0
a_4	$\frac{1}{16r^4+224r^3+856r^2+504r-495}$	$\frac{1}{10240}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{64r^6 + 1728r^5 + 17200r^4 + 76320r^3 + 138076r^2 + 41292r - 65835}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_6 = -\frac{1}{2949120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{4r^2+20r-11}$	$-\frac{1}{64}$
a_3	0	0
a_4	$\frac{1}{16r^4+224r^3+856r^2+504r-495}$	$\frac{1}{10240}$
a_5	0	0
a_6	$-\frac{1}{64r^6+1728r^5+17200r^4+76320r^3+138076r^2+41292r-65835}$	$-\frac{1}{2949120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\ &= x^{\frac{5}{2}}\left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= -\frac{1}{64r^6 + 1728r^5 + 17200r^4 + 76320r^3 + 138076r^2 + 41292r - 65835} \end{aligned}$$

Therefore

$$\lim_{r\to r_2} -\frac{1}{64r^6+1728r^5+17200r^4+76320r^3+138076r^2+41292r-65835} = \lim_{r\to -\frac{7}{2}} -\frac{}{64r^6+1728r^5+1720}$$
$$= undefined$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $4x^2y''+8y'x+(x^2-35)\,y=0$ gives

$$\begin{aligned}
&4x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+ 8\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x\\
&+ (x^2-35)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{gathered}\left(\left(\left(4x^2y_1''(x)+8y_1'(x)\,x+\left(x^2-35\right)y_1(x)\right)\ln\left(x\right)+4x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)\right.\right.\\ \left.+8y_1(x)\right)C+4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\ +8\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(x^2-35\right)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{gathered} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$4x^2y_1''(x)+8y_1'(x)\,x+\left(x^2-35\right)y_1(x)=0$$

Eq (7) simplifes to

$$\begin{gathered}\left(4x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+8y_1(x)\right)C\\ +4x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\ +8\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(x^2-35\right)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{gathered} \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{gathered}\left(8\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+4\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C\\ +4\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\left(-1+n+r_2\right)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x^2\\ +8\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x-35\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{gathered} \tag{9}$$

Since $r_1 = \frac{5}{2}$ and $r_2 = -\frac{7}{2}$ then the above becomes

$$\left(8\left(\sum_{n=0}^{\infty} x^{\frac{3}{2}+n} a_n \left(n+\frac{5}{2}\right)\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}\right)\right) C$$
$$+ 4\left(\sum_{n=0}^{\infty} x^{-\frac{11}{2}+n} b_n \left(n-\frac{7}{2}\right)\left(-\frac{9}{2}+n\right)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right) x^2 \tag{10}$$
$$+ 8\left(\sum_{n=0}^{\infty} x^{-\frac{9}{2}+n} b_n \left(n-\frac{7}{2}\right)\right) x - 35\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right) = 0$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (8n+20)\, C a_n x^{n+\frac{5}{2}}\right) + \left(\sum_{n=0}^{\infty} 4C a_n x^{n+\frac{5}{2}}\right)$$
$$+ \left(\sum_{n=0}^{\infty} x^{n-\frac{7}{2}} b_n (4n^2 - 32n + 63)\right) + \left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} (8n-28)\, b_n x^{n-\frac{7}{2}}\right) + \sum_{n=0}^{\infty} \left(-35 b_n x^{n-\frac{7}{2}}\right) = 0$$

The next step is to make all powers of x be $n-\frac{7}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{7}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} (8n+20)\, C a_n x^{n+\frac{5}{2}} = \sum_{n=6}^{\infty} C a_{n-6} (8n-28)\, x^{n-\frac{7}{2}}$$

$$\sum_{n=0}^{\infty} 4C a_n x^{n+\frac{5}{2}} = \sum_{n=6}^{\infty} 4C a_{n-6} x^{n-\frac{7}{2}}$$

$$\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-\frac{7}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-\frac{7}{2}$.

$$\left(\sum_{n=6}^{\infty} C a_{n-6}(8n-28)\, x^{n-\frac{7}{2}}\right)+\left(\sum_{n=6}^{\infty} 4C a_{n-6} x^{n-\frac{7}{2}}\right)$$
$$+\left(\sum_{n=0}^{\infty} x^{n-\frac{7}{2}} b_n \left(4n^2-32n+63\right)\right)+\left(\sum_{n=2}^{\infty} b_{n-2} x^{n-\frac{7}{2}}\right) \tag{2B}$$
$$+\left(\sum_{n=0}^{\infty} (8n-28)\, b_n x^{n-\frac{7}{2}}\right)+\sum_{n=0}^{\infty}\left(-35 b_n x^{n-\frac{7}{2}}\right)=0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-20b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$-32b_2+b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-32b_2+1=0$$

Solving the above for b_2 gives

$$b_2=\frac{1}{32}$$

For $n=3$, Eq (2B) gives

$$-36b_3+b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-36b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n=4$, Eq (2B) gives

$$-32b_4+b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-32b_4 + \frac{1}{32} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{1024}$$

For $n = 5$, Eq (2B) gives

$$-20b_5 + b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$24C + \frac{1}{1024} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{24576}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{24576}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{24576}\left(x^{\frac{5}{2}}\left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{32} + \frac{x^4}{1024} + O(x^7)}{x^{\frac{7}{2}}}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \\
&\quad + c_2 \left(-\frac{1}{24576} \left(x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right)\right) \ln(x) \right. \\
&\quad \left. + \frac{1 + \frac{x^2}{32} + \frac{x^4}{1024} + O(x^7)}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \\
&\quad + c_2 \left(-\frac{x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \ln(x)}{24576} + \frac{1 + \frac{x^2}{32} + \frac{x^4}{1024} + O(x^7)}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \\
&\quad + c_2 \left(-\frac{x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \ln(x)}{24576} + \frac{1 + \frac{x^2}{32} + \frac{x^4}{1024} + O(x^7)}{x^{\frac{7}{2}}}\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= c_1 x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \\
&\quad + c_2 \left(-\frac{x^{\frac{5}{2}} \left(1 - \frac{x^2}{64} + \frac{x^4}{10240} - \frac{x^6}{2949120} + O(x^7)\right) \ln(x)}{24576} + \frac{1 + \frac{x^2}{32} + \frac{x^4}{1024} + O(x^7)}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Verified OK.

16.35.1 Maple step by step solution

Let's solve

$4x^2y'' + 8y'x + (x^2 - 35)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-35)y}{4x^2} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{(x^2-35)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^2-35}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{35}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 8y'x + (x^2 - 35)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+2r)\,(-5+2r)\,x^{r}+a_1(9+2r)\,(-3+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+7)\,(2k+2r-5)+a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+2r)\,(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{7}{2},\frac{5}{2}\right\}$

- Each term must be 0

  $a_1(9+2r)\,(-3+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{7}{2}\right)\left(k+r-\frac{5}{2}\right)a_k+a_{k-2}=0$

- Shift index using $k->k+2$

  $4\left(k+\frac{11}{2}+r\right)\left(k-\frac{1}{2}+r\right)a_{k+2}+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(2k+11+2r)(2k-1+2r)}$

- Recursion relation for $r=-\frac{7}{2}$

  $a_{k+2}=-\frac{a_k}{(2k+4)(2k-8)}$

- Series not valid for $r=-\frac{7}{2}$, division by 0 in the recursion relation at $k=4$

  $a_{k+2}=-\frac{a_k}{(2k+4)(2k-8)}$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{a_k}{(2k+16)(2k+4)}$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{a_k}{(2k+16)(2k+4)}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+8*x*diff(y(x),x)-(35-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^6 \left(1 - \frac{1}{64}x^2 + \frac{1}{10240}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-86400 - 2700x^2 - \frac{675}{8}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 58

```
AsymptoticDSolveValue[4*x^2*y''[x]+8*x*y'[x]-(35-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{32x^{3/2}} + \frac{1}{x^{7/2}} + \frac{\sqrt{x}}{1024} \right) + c_2 \left(\frac{x^{13/2}}{10240} - \frac{x^{9/2}}{64} + x^{5/2} \right)$$

16.36 problem 32

Internal problem ID [1448]
Internal file name [`OUTPUT/1449_Sunday_June_05_2022_02_18_09_AM_38044428/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' - 3x\left(2x^2+11\right)y' + \left(10x^2+13\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + \left(-6x^3 - 33x\right)y' + \left(10x^2+13\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= -\frac{2x^2+11}{3x}\\ q(x) &= \frac{10x^2+13}{9x^2}\end{aligned}$$

Table 842: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2+11}{3x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{10x^2+13}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + \left(-6x^3 - 33x\right) y' + \left(10x^2 + 13\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-6x^3 - 33x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(10x^2 + 13\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-33x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 13a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-6x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 10x^{n+r+2} a_n = \sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-6a_{n-2}(n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-33x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 10a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 13a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) - 33x^{n+r} a_n (n+r) + 13a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) - 33x^r a_0 r + 13a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) - 33x^r r + 13x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 42r + 13\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 42r + 13 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{13}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 42r + 13\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{13}{3}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{\frac{1}{3}} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{13}{3}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) - 6a_{n-2}(n+r-2) - 33a_n(n+r) + 10a_{n-2} + 13a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}(3n+3r-11)}{9n^2+18nr+9r^2-42n-42r+13} \quad (4)$$

Which for the root $r = \frac{13}{3}$ becomes

$$a_n = \frac{2a_{n-2}(3n+2)}{9n(n+4)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{13}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-10+6r}{9r^2-6r-35}$$

Which for the root $r = \frac{13}{3}$ becomes

$$a_2 = \frac{4}{27}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10+6r}{9r^2-6r-35}$	$\frac{4}{27}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10+6r}{9r^2-6r-35}$	$\frac{4}{27}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{36r^2 - 48r - 20}{(9r^2 - 6r - 35)(9r^2 + 30r - 11)}$$

Which for the root $r = \frac{13}{3}$ becomes

$$a_4 = \frac{7}{486}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10+6r}{9r^2-6r-35}$	$\frac{4}{27}$
a_3	0	0
a_4	$\frac{36r^2-48r-20}{(9r^2-6r-35)(9r^2+30r-11)}$	$\frac{7}{486}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-10+6r}{9r^2-6r-35}$	$\frac{4}{27}$
a_3	0	0
a_4	$\frac{36r^2-48r-20}{(9r^2-6r-35)(9r^2+30r-11)}$	$\frac{7}{486}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{13}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{13}{3}}\left(1 + \frac{4x^2}{27} + \frac{7x^4}{486} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{36r^2 - 48r - 20}{(9r^2 - 6r - 35)(9r^2 + 30r - 11)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{36r^2 - 48r - 20}{(9r^2 - 6r - 35)(9r^2 + 30r - 11)} &= \lim_{r\to \frac{1}{3}} \frac{36r^2 - 48r - 20}{(9r^2 - 6r - 35)(9r^2 + 30r - 11)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2)(-1 + n + r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $9x^2y'' + (-6x^3 - 33x)\,y' + (10x^2 + 13)\,y = 0$ gives

$$\begin{aligned}
&9x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\
&\left.+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(-6x^3-33x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\
&+(10x^2+13)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left(9x^2y_1''(x) + (-6x^3-33x)\,y_1'(x) + (10x^2+13)\,y_1(x)\right)\ln(x)\right.\\
&\left.+9x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-6x^3-33x)\,y_1(x)}{x}\right)C\\
&+9x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(-6x^3-33x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (10x^2+13)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$9x^2y_1''(x) + (-6x^3-33x)\,y_1'(x) + (10x^2+13)\,y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(9x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) + \frac{(-6x^3-33x)\,y_1(x)}{x}\right)C\\
&+9x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+(-6x^3-33x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (10x^2+13)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(18\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + 6(-x^2-7)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ 9\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ 3(-2x^3-11x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + (10x^2+13)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = \frac{13}{3}$ and $r_2 = \frac{1}{3}$ then the above becomes

$$
\begin{aligned}
&\left(18\left(\sum_{n=0}^{\infty} x^{\frac{10}{3}+n} a_n \left(n+\frac{13}{3}\right)\right) x + 6(-x^2-7)\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{13}{3}}\right)\right) C \\
&+ 9\left(\sum_{n=0}^{\infty} x^{-\frac{5}{3}+n} b_n \left(n+\frac{1}{3}\right)\left(-\frac{2}{3}+n\right)\right) x^2 \\
&+ 3(-2x^3-11x)\left(\sum_{n=0}^{\infty} x^{-\frac{2}{3}+n} b_n \left(n+\frac{1}{3}\right)\right) + (10x^2+13)\left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}\right) = 0
\end{aligned} \tag{10}
$$

Expanding $-6C\,x^{\frac{19}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-6C\,x^{\frac{19}{3}} &= -6C\,x^{\frac{19}{3}} + \dots \\
&= -6C\,x^{\frac{19}{3}}
\end{aligned}
$$

Expanding $-42C\,x^{\frac{13}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-42C\,x^{\frac{13}{3}} &= -42C\,x^{\frac{13}{3}} + \dots \\
&= -42C\,x^{\frac{13}{3}}
\end{aligned}
$$

Expanding $-2x^{\frac{7}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-2x^{\frac{7}{3}} &= -2x^{\frac{7}{3}} + \dots \\
&= -2x^{\frac{7}{3}}
\end{aligned}
$$

Expanding $-11x^{\frac{1}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-11x^{\frac{1}{3}} &= -11x^{\frac{1}{3}} + \dots \\
&= -11x^{\frac{1}{3}}
\end{aligned}
$$

Expanding $10x^{\frac{7}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
10x^{\frac{7}{3}} &= 10x^{\frac{7}{3}} + \dots \\
&= 10x^{\frac{7}{3}}
\end{aligned}
$$

Expanding $13x^{\frac{1}{3}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
13x^{\frac{1}{3}} &= 13x^{\frac{1}{3}} + \dots \\
&= 13x^{\frac{1}{3}}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} (18n+78)\, C a_n x^{n+\frac{13}{3}}\right) + \sum_{n=0}^{\infty} \left(-6C\, x^{n+\frac{19}{3}} a_n\right) + \sum_{n=0}^{\infty} \left(-42C\, x^{n+\frac{13}{3}} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n+\frac{1}{3}} b_n \left(9n^2-3n-2\right)\right) + \left(\sum_{n=0}^{\infty} (-6n-2)\, b_n x^{n+\frac{7}{3}}\right) \\
&+ \left(\sum_{n=0}^{\infty} (-33n-11)\, b_n x^{n+\frac{1}{3}}\right) + \left(\sum_{n=0}^{\infty} 10x^{n+\frac{7}{3}} b_n\right) + \left(\sum_{n=0}^{\infty} 13 b_n x^{n+\frac{1}{3}}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n + \frac{1}{3}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n+\frac{1}{3}}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} (18n+78)\, C a_n x^{n+\frac{13}{3}} &= \sum_{n=4}^{\infty} C a_{n-4} (18n+6)\, x^{n+\frac{1}{3}} \\
\sum_{n=0}^{\infty} \left(-6C\, x^{n+\frac{19}{3}} a_n\right) &= \sum_{n=6}^{\infty} \left(-6C a_{n-6} x^{n+\frac{1}{3}}\right) \\
\sum_{n=0}^{\infty} \left(-42C\, x^{n+\frac{13}{3}} a_n\right) &= \sum_{n=4}^{\infty} \left(-42C a_{n-4} x^{n+\frac{1}{3}}\right)
\end{aligned}
$$

$$\sum_{n=0}^{\infty} \left(-6n -2\right) b_{n} x^{n +\frac{7}{3}} = \sum_{n=2}^{\infty} b_{n -2} \left(-6n +10\right) x^{n +\frac{1}{3}}$$

$$\sum_{n=0}^{\infty} 10 x^{n +\frac{7}{3}} b_{n} = \sum_{n=2}^{\infty} 10 b_{n -2} x^{n +\frac{1}{3}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n +\frac{1}{3}$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} C a_{n -4} \left(18 n +6\right) x^{n +\frac{1}{3}}\right)+\sum_{n=6}^{\infty} \left(-6 C a_{n -6} x^{n +\frac{1}{3}}\right) \\
&+\sum_{n=4}^{\infty} \left(-42 C a_{n -4} x^{n +\frac{1}{3}}\right)+\left(\sum_{n=0}^{\infty} x^{n +\frac{1}{3}} b_{n} \left(9 n^{2}-3 n -2\right)\right) \\
&+\left(\sum_{n=2}^{\infty} b_{n -2} \left(-6 n +10\right) x^{n +\frac{1}{3}}\right)+\left(\sum_{n=0}^{\infty} \left(-33 n -11\right) b_{n} x^{n +\frac{1}{3}}\right) \\
&+\left(\sum_{n=2}^{\infty} 10 b_{n -2} x^{n +\frac{1}{3}}\right)+\left(\sum_{n=0}^{\infty} 13 b_{n} x^{n +\frac{1}{3}}\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-27 b_{1} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-27 b_{1} = 0$$

Solving the above for b_1 gives

$$b_{1} = 0$$

For $n = 2$, Eq (2B) gives

$$-36 b_{2}+8 b_{0} = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-36 b_{2}+8 = 0$$

Solving the above for b_2 gives

$$b_{2} = \frac{2}{9}$$

For $n = 3$, Eq (2B) gives

$$-27b_3 + 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-27b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$36C - \frac{8}{9} = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{2}{81}$$

For $n = 5$, Eq (2B) gives

$$54Ca_1 - 10b_3 + 45b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$45b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{2}{81}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{2}{81}\left(x^{\frac{13}{3}}\left(1 + \frac{4x^2}{27} + \frac{7x^4}{486} + O(x^6)\right)\right)\ln(x) + x^{\frac{1}{3}}\left(1 + \frac{2x^2}{9} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^{\frac{13}{3}}\left(1 + \frac{4x^2}{27} + \frac{7x^4}{486} + O(x^6)\right) \\
&\quad + c_2\left(\frac{2}{81}\left(x^{\frac{13}{3}}\left(1 + \frac{4x^2}{27} + \frac{7x^4}{486} + O(x^6)\right)\right)\ln(x) + x^{\frac{1}{3}}\left(1 + \frac{2x^2}{9} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \\
&+ c_2 \left(\frac{2x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \ln(x)}{81} + x^{\frac{1}{3}} \left(1+\frac{2x^2}{9}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \\
+ c_2 \left(\frac{2x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \ln(x)}{81} + x^{\frac{1}{3}} \left(1+\frac{2x^2}{9}+O(x^6)\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \\
+ c_2 \left(\frac{2x^{\frac{13}{3}} \left(1+\frac{4x^2}{27}+\frac{7x^4}{486}+O(x^6)\right) \ln(x)}{81} + x^{\frac{1}{3}} \left(1+\frac{2x^2}{9}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

16.36.1 Maple step by step solution

Let's solve

$9x^2 y'' + (-6x^3 - 33x) y' + (10x^2 + 13) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x^2+13)y}{9x^2} + \frac{(2x^2+11)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2+11)y'}{3x} + \frac{(10x^2+13)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2x^2+11}{3x}, P_3(x) = \frac{10x^2+13}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{11}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{13}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2 y'' - 3x(2x^2 + 11)\, y' + (10x^2 + 13)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)(-13+3r)\, x^r + a_1(2+3r)(-10+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)(3k+3r-13)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)(-13+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{3}, \frac{13}{3}\right\}$$

- Each term must be 0

  $$a_1(2+3r)(-10+3r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$9\left(k+r-\frac{1}{3}\right)\left(k+r-\frac{13}{3}\right) a_k - 6\left(k-\frac{11}{3}+r\right) a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$9\left(k+\frac{5}{3}+r\right)\left(k-\frac{7}{3}+r\right) a_{k+2} - 6\left(k-\frac{5}{3}+r\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2(3k+3r-5)a_k}{(3k+5+3r)(3k-7+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+2} = \frac{2(3k-4)a_k}{(3k+6)(3k-6)}$$

- Series not valid for $r = \frac{1}{3}$, division by 0 in the recursion relation at $k = 2$

  $$a_{k+2} = \frac{2(3k-4)a_k}{(3k+6)(3k-6)}$$

- Recursion relation for $r = \frac{13}{3}$

  $$a_{k+2} = \frac{2(3k+8)a_k}{(3k+18)(3k+6)}$$

- Solution for $r = \frac{13}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{13}{3}}, a_{k+2} = \frac{2(3k+8)a_k}{(3k+18)(3k+6)}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)-3*x*(11+2*x^2)*diff(y(x),x)+(13+10*x^2)*y(x)=0,y(x),type='series
```

$$y(x) = x^{\frac{1}{3}}\left(\left(1+\frac{4}{27}x^2+\frac{7}{486}x^4+\mathrm{O}\left(x^6\right)\right)x^4c_1 + c_2\left(\ln\left(x\right)\left(-\frac{32}{9}x^4+\mathrm{O}\left(x^6\right)\right)+\left(-144-32x^2-\frac{8}{3}x^4+\mathrm{O}\left(x^6\right)\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 62

```
AsymptoticDSolveValue[9*x^2*y''[x]-3*x*(11+2*x^2)*y'[x]+(13+10*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{7x^{25/3}}{486}+\frac{4x^{19/3}}{27}+x^{13/3}\right)+c_1\left(\frac{2}{81}x^{13/3}\log(x)+\frac{1}{81}\left(x^2+9\right)^2\sqrt[3]{x}\right)$$

16.37 problem 33

Internal problem ID [1449]
Internal file name [`OUTPUT/1450_Sunday_June_05_2022_02_18_14_AM_4833379/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2y'' + x(-2x^2 + 1)\,y' - 4(1 - x^2)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (-2x^3 + x)\,y' + (4x^2 - 4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x^2 - 1}{x}$$
$$q(x) = \frac{4x^2 - 4}{x^2}$$

Table 844: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{4x^2-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-2x^3 + x\right) y' + \left(4x^2 - 4\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-2x^3 + x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(4x^2 - 4\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 4) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-2}(n+r-2) + a_n(n+r) + 4a_{n-2} - 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}(n+r-4)}{n^2+2nr+r^2-4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{2a_{n-2}(n-2)}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4+2r}{r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4+2r}{r(r+4)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4+2r}{r(r+4)}$	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-8+4r}{(r+4)(r+6)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4+2r}{r(r+4)}$	0
a_3	0	0
a_4	$\frac{-8+4r}{(r+4)(r+6)(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4+2r}{r(r+4)}$	0
a_3	0	0
a_4	$\frac{-8+4r}{(r+4)(r+6)(r+2)}$	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_4 \\
&= \frac{-8 + 4r}{(r + 4)(r + 6)(r + 2)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} \frac{-8 + 4r}{(r + 4)(r + 6)(r + 2)} &= \lim_{r \to -2} \frac{-8 + 4r}{(r + 4)(r + 6)(r + 2)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n + r_2)(-1 + n + r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y''+(-2x^3+x)\,y'+(4x^2-4)\,y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-2x^3+x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(4x^2-4)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+(-2x^3+x)\,y_1'(x)+(4x^2-4)\,y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-2x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-2x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(4x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x)+(-2x^3+x)\,y_1'(x)+(4x^2-4)\,y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-2x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-2x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(4x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(-2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + (-2x^3+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$\left(-2\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) x\right) C + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + (-2x^3+x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0 \tag{10}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-2C\, x^{n+4} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \sum_{n=0}^{\infty} \left(-2x^n b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 4 b_n x^n\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-2C\,x^{n+4}a_n\right)=\sum_{n=6}^{\infty}\left(-2Ca_{n-6}x^{n-2}\right)$$

$$\sum_{n=0}^{\infty}2C\,x^{n+2}a_n(n+2)=\sum_{n=4}^{\infty}2Ca_{-4+n}(n-2)\,x^{n-2}$$

$$\sum_{n=0}^{\infty}\left(-2x^n b_n(n-2)\right)=\sum_{n=2}^{\infty}\left(-2b_{n-2}(-4+n)\,x^{n-2}\right)$$

$$\sum_{n=0}^{\infty}4b_n x^n=\sum_{n=2}^{\infty}4b_{n-2}x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned}&\sum_{n=6}^{\infty}\left(-2Ca_{n-6}x^{n-2}\right)+\left(\sum_{n=4}^{\infty}2Ca_{-4+n}(n-2)\,x^{n-2}\right)\\&+\left(\sum_{n=0}^{\infty}x^{n-2}b_n\left(n^2-5n+6\right)\right)+\sum_{n=2}^{\infty}\left(-2b_{n-2}(-4+n)\,x^{n-2}\right)\\&+\left(\sum_{n=0}^{\infty}x^{n-2}b_n(n-2)\right)+\left(\sum_{n=2}^{\infty}4b_{n-2}x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-4b_n x^{n-2}\right)=0\end{aligned}\tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1=0$$

Solving the above for b_1 gives

$$b_1=0$$

For $n=2$, Eq (2B) gives

$$8b_0-4b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8-4b_2=0$$

Solving the above for b_2 gives

$$b_2 = 2$$

For $n = 3$, Eq (2B) gives

$$6b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 8 = 0$$

Which is solved for C. Solving for C gives

$$C = -2$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 2b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -2$ and all b_n, then the second solution becomes

$$y_2(x) = (-2)\left(x^2\left(1 + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 + 2x^2 + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2\left(1 + O\left(x^6\right)\right) + c_2\left((-2)\left(x^2\left(1 + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 + 2x^2 + O(x^6)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^2 \left(1+O\left(x^6\right)\right)+c_2 \left(-2x^2 \left(1+O\left(x^6\right)\right) \ln \left(x \right)+\frac{1+2x^2+O\left(x^6\right)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1+O\left(x^6\right)\right)+c_2 \left(-2x^2 \left(1+O\left(x^6\right)\right) \ln \left(x \right)+\frac{1+2x^2+O\left(x^6\right)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1+O\left(x^6\right)\right)+c_2 \left(-2x^2 \left(1+O\left(x^6\right)\right) \ln \left(x \right)+\frac{1+2x^2+O\left(x^6\right)}{x^2}\right)$$

Verified OK.

16.37.1 Maple step by step solution

Let's solve

$x^2 y''+\left(-2x^3+x\right) y'+\left(4x^2-4\right) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4\left(x^2-1\right) y}{x^2}+\frac{\left(2x^2-1\right) y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{\left(2x^2-1\right) y'}{x}+\frac{4\left(x^2-1\right) y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{2x^2-1}{x}, P_3(x)=\frac{4\left(x^2-1\right)}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0}=1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(2x^2 - 1) y' + (4x^2 - 4) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-2+r) x^r + a_1(3+r)(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k-4\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1(3+r)(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2)-2a_{k-2}(k-4+r)=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+r)-2a_k(k+r-2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2a_k(k+r-2)}{(k+4+r)(k+r)}$

- Recursion relation for $r=-2$; series terminates at $k=4$

  $a_{k+2}=\frac{2a_k(k-4)}{(k+2)(k-2)}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=\frac{2a_k(k-4)}{(k+2)(k-2)}$

- Recursion relation for $r=2$

  $a_{k+2}=\frac{2a_k k}{(k+6)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{2a_k k}{(k+6)(k+2)}, a_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x^2)*diff(y(x),x)-4*(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(\ln\left(x\right) \left(288x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 288x^2 - 216x^4 + \mathrm{O}\left(x^6\right)\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 37

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-2*x^2)*y'[x]-4*(1-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^2 + c_1 \left(\frac{2x^4 + 2x^2 + 1}{x^2} - 2x^2 \log(x)\right)$$

16.38 problem 34

Internal problem ID [1450]
Internal file name [`OUTPUT/1451_Sunday_June_05_2022_02_18_18_AM_91871725/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-3x^2+1)\,y' - 4(-3x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (-3x^3+x)\,y' + (12x^2-4)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3x^2-1}{x}$$
$$q(x) = \frac{12x^2-4}{x^2}$$

Table 846: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{12x^2-4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-3x^3 + x)\,y' + (12x^2 - 4)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (-3x^3 + x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (12x^2 - 4)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 12 x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 12 x^{n+r+2} a_n = \sum_{n=2}^{\infty} 12 a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-3a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 12 a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 4 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 4 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 4)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_{n-2}(n+r-2) + a_n(n+r) + 12a_{n-2} - 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{3a_{n-2}(n+r-6)}{n^2+2nr+r^2-4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{3a_{n-2}(n-4)}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-12+3r}{r(r+4)}$$

Which for the root $r = 2$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-12+3r}{r(r+4)}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-12+3r}{r(r+4)}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{9(-4+r)(-2+r)}{r(r+4)(r+6)(r+2)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-12+3r}{r(r+4)}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{9(-4+r)(-2+r)}{r(r+4)(r+6)(r+2)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-12+3r}{r(r+4)}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{9(-4+r)(-2+r)}{r(r+4)(r+6)(r+2)}$	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(1 - \frac{x^2}{2} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_4 \\
&= \frac{9(-4 + r)(-2 + r)}{r(r + 4)(r + 6)(r + 2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{9(-4 + r)(-2 + r)}{r(r + 4)(r + 6)(r + 2)} &= \lim_{r \to -2} \frac{9(-4 + r)(-2 + r)}{r(r + 4)(r + 6)(r + 2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n + r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n + r_2}(n + r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1 + n + r_2} b_n (n + r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n + r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n + r_2}(n + r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2 C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2 + n + r_2} b_n (n + r_2)(-1 + n + r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y''+(-3x^3+x)\,y'+(12x^2-4)\,y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-3x^3+x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)+(12x^2-4)\left(Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)\right)=0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+(-3x^3+x)\,y_1'(x)+(12x^2-4)\,y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-3x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-3x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(12x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x)+(-3x^3+x)\,y_1'(x)+(12x^2-4)\,y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-3x^3+x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-3x^3+x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+(12x^2-4)\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(-3\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ (-3x^3+x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
&+ 12\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{9}
$$

Since $r_1 = 2$ and $r_2 = -2$ then the above becomes

$$
\begin{aligned}
&\left(-3\left(\sum_{n=0}^{\infty} a_n x^{n+2}\right) x^2 + 2\left(\sum_{n=0}^{\infty} x^{1+n} a_n (n+2)\right) x\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{n-4} b_n (n-2)(-3+n)\right) x^2 + (-3x^3+x)\left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) \\
&+ 12\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 - 4\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(-3C\, x^{n+4} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n^2-5n+6)\right) + \sum_{n=0}^{\infty} \left(-3x^n b_n (n-2)\right) \\
&\quad + \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \left(\sum_{n=0}^{\infty} 12 b_n x^n\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and

adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3C\, x^{n+4} a_n\right) = \sum_{n=6}^{\infty} \left(-3C a_{n-6} x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+2} a_n (n+2) = \sum_{n=4}^{\infty} 2C a_{n-4} (n-2)\, x^{n-2}$$

$$\sum_{n=0}^{\infty} \left(-3x^n b_n (n-2)\right) = \sum_{n=2}^{\infty} \left(-3b_{n-2}(n-4)\, x^{n-2}\right)$$

$$\sum_{n=0}^{\infty} 12 b_n x^n = \sum_{n=2}^{\infty} 12 b_{n-2} x^{n-2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$\begin{aligned}
&\sum_{n=6}^{\infty} \left(-3C a_{n-6} x^{n-2}\right) + \left(\sum_{n=4}^{\infty} 2C a_{n-4}(n-2)\, x^{n-2}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2 - 5n + 6\right)\right) + \sum_{n=2}^{\infty} \left(-3b_{n-2}(n-4)\, x^{n-2}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-2)\right) + \left(\sum_{n=2}^{\infty} 12 b_{n-2} x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-4 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$18b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$18 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{9}{2}$$

For $n = 3$, Eq (2B) gives

$$15b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 54 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{27}{2}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 9b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{27}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{27}{2}\left(x^2\left(1 - \frac{x^2}{2} + O(x^6)\right)\right) \ln(x) + \frac{1 + \frac{9x^2}{2} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1-\frac{x^2}{2}+O\left(x^6\right)\right)+c_2\left(-\frac{27}{2}\left(x^2\left(1-\frac{x^2}{2}+O\left(x^6\right)\right)\right)\ln\left(x\right)+\frac{1+\frac{9x^2}{2}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1-\frac{x^2}{2}+O\left(x^6\right)\right)+c_2\left(-\frac{27x^2\left(1-\frac{x^2}{2}+O(x^6)\right)\ln\left(x\right)}{2}+\frac{1+\frac{9x^2}{2}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2 \left(1-\frac{x^2}{2}+O\left(x^6\right)\right) \\
& + c_2\left(-\frac{27x^2\left(1-\frac{x^2}{2}+O(x^6)\right)\ln\left(x\right)}{2}+\frac{1+\frac{9x^2}{2}+O(x^6)}{x^2}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \left(1-\frac{x^2}{2}+O\left(x^6\right)\right)+c_2\left(-\frac{27x^2\left(1-\frac{x^2}{2}+O(x^6)\right)\ln\left(x\right)}{2}+\frac{1+\frac{9x^2}{2}+O(x^6)}{x^2}\right)
$$

Verified OK.

16.38.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-3x^3+x\right) y' + \left(12x^2-4\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4(3x^2-1)y}{x^2} + \frac{(3x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(3x^2-1)y'}{x} + \frac{4(3x^2-1)y}{x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$$\left[P_2(x) = -\frac{3x^2-1}{x}, P_3(x) = \frac{4(3x^2-1)}{x^2}\right]$$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = -4$$

  ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$x^2y'' - x(3x^2 - 1)\, y' + (12x^2 - 4)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-2+r)\, x^r + a_1(3+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term must be 0

  $$a_1(3+r)\,(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6+r) = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}(k+4+r)\,(k+r) - 3a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$$

- Recursion relation for $r = -2$; series terminates at $k = 6$

  $$a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $$a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$$

- Recursion relation for $r = 2$; series terminates at $k = 2$

  $$a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(1-3*x^2)*diff(y(x),x)-4*(1-3*x^2)*y(x)=0,y(x),type='series',x=0
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{2}x^2 + O\left(x^6\right)\right) + \frac{c_2 (\ln(x) \left(1944x^4 + O\left(x^6\right)\right) + \left(-144 - 648x^2 - 810x^4 + O\left(x^6\right)\right))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 50

```
AsymptoticDSolveValue[x^2*y''[x]+x*(1-3*x^2)*y'[x]-4*(1-3*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x^2 - \frac{x^4}{2}\right) + c_1\left(\frac{18x^4 + 9x^2 + 2}{2x^2} - \frac{27}{2}x^2\log(x)\right)$$

16.39 problem 35

Internal problem ID [1451]
Internal file name [OUTPUT/1452_Sunday_June_05_2022_02_18_22_AM_85121220/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0$$

Or

$$x(y''x^3+11y'x^2+24yx+y''x+5y')=0$$

For $x\neq 0$ the above simplifies to

$$(x^3+x)\,y''+11y'x^2+24yx+5y'=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = \frac{11x^2+5}{x(x^2+1)}$$
$$q(x) = \frac{24}{x^2+1}$$

Table 848: Table $p(x), q(x)$ singularites.

$p(x) = \frac{11x^2+5}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{24}{x^2+1}$	
singularity	type
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\, y'' + (11x^3+5x)\, y' + 24x^2 y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{gathered}
x^{2}\left(x^{2}+1\right)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1) a_{n} x^{n+r-2}\right) \\
+\left(11 x^{3}+5 x\right)\left(\sum_{n=0}^{\infty}(n+r) a_{n} x^{n+r-1}\right)+24 x^{2}\left(\sum_{n=0}^{\infty} a_{n} x^{n+r}\right)=0
\end{gathered} \tag{1}
$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} x^{n+r+2} a_{n}(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_{n}(n+r)(n+r-1)\right) \\
+\left(\sum_{n=0}^{\infty} 11 x^{n+r+2} a_{n}(n+r)\right)+\left(\sum_{n=0}^{\infty} 5 x^{n+r} a_{n}(n+r)\right)+\left(\sum_{n=0}^{\infty} 24 x^{n+r+2} a_{n}\right)=0
\end{gathered} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r+2} a_{n}(n+r)(n+r-1) &= \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r} \\
\sum_{n=0}^{\infty} 11 x^{n+r+2} a_{n}(n+r) &= \sum_{n=2}^{\infty} 11 a_{n-2}(n+r-2) x^{n+r} \\
\sum_{n=0}^{\infty} 24 x^{n+r+2} a_{n} &= \sum_{n=2}^{\infty} 24 a_{n-2} x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{gathered}
\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r) x^{n+r}\right) \\
+\left(\sum_{n=0}^{\infty} x^{n+r} a_{n}(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 11 a_{n-2}(n+r-2) x^{n+r}\right) \\
+\left(\sum_{n=0}^{\infty} 5 x^{n+r} a_{n}(n+r)\right)+\left(\sum_{n=2}^{\infty} 24 a_{n-2} x^{n+r}\right)=0
\end{gathered} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 5x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) + 5x^r r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -4 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^4} \end{aligned}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-4}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ + 11a_{n-2}(n+r-2) + 5a_n(n+r) + 24a_{n-2} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r+2)\, a_{n-2}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{(n+2)\, a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-4-r}{r+2}$$

Which for the root $r = 0$ becomes

$$a_2 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{r+2}$	-2

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{r+2}$	-2
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6+r}{r+2}$$

Which for the root $r = 0$ becomes

$$a_4 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{r+2}$	-2
a_3	0	0
a_4	$\frac{6+r}{r+2}$	3

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-4-r}{r+2}$	-2
a_3	0	0
a_4	$\frac{6+r}{r+2}$	3
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - 2x^2 + 3x^4 + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{6+r}{r+2} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{6+r}{r+2} &= \lim_{r \to -4} \frac{6+r}{r+2} \\ &= -1 \end{aligned}$$

The limit is -1. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-4} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_{n-2}(n+r-2)(n-3+r) + b_n(n+r)(n+r-1) \\ + 11b_{n-2}(n+r-2) + 5b_n(n+r) + 24b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -4$ becomes

$$b_{n-2}(n-6)(n-7) + b_n(n-4)(n-5) + 11b_{n-2}(n-6) + 5b_n(n-4) + 24b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r+2)\, b_{n-2}}{n+r} \tag{5}$$

Which for the root $r = -4$ becomes

$$b_n = -\frac{(n-2)\, b_{n-2}}{n-4} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -4$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4+r}{r+2}$$

Which for the root $r = -4$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{r+2}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{r+2}$	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6+r}{r+2}$$

Which for the root $r = -4$ becomes

$$b_4 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{r+2}$	0
b_3	0	0
b_4	$\frac{6+r}{r+2}$	-1

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-4-r}{r+2}$	0
b_3	0	0
b_4	$\frac{6+r}{r+2}$	-1
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - x^4 + O(x^6)}{x^4}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - 2x^2 + 3x^4 + O\left(x^6\right)\right) + \frac{c_2\left(1 - x^4 + O(x^6)\right)}{x^4}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - 2x^2 + 3x^4 + O\left(x^6\right)\right) + \frac{c_2\left(1 - x^4 + O(x^6)\right)}{x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1 - 2x^2 + 3x^4 + O\left(x^6\right)\right) + \frac{c_2\left(1 - x^4 + O(x^6)\right)}{x^4} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1 - 2x^2 + 3x^4 + O\left(x^6\right)\right) + \frac{c_2\left(1 - x^4 + O(x^6)\right)}{x^4}
$$

Verified OK.

16.39.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\,y''+(11x^3+5x)\,y'+24x^2y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{24y}{x^2+1}-\frac{(11x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(11x^2+5)y'}{x(x^2+1)}+\frac{24y}{x^2+1}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{11x^2+5}{x(x^2+1)}, P_3(x)=\frac{24}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$24yx+(11x^2+5)\,y'+x(x^2+1)\,y''=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

$x\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+1}$

- Shift index using $k->k-1$

  $x\cdot y=\sum_{k=1}^{\infty} a_{k-1}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=0..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m\cdot y''$ to series expansion for $m=1..3$

  $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(4+r)\, x^{-1+r}+a_1(1+r)\,(5+r)\, x^r+\left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+5+r)+a_{k-1}(k+5+r)\,(k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-4,0\}$

- Each term must be 0

  $a_1(1+r)\,(5+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+5+r)\,(a_{k+1}(k+r+1)+a_{k-1}(k+3+r))=0$

- Shift index using $k->k+1$

  $(k+r+6)\,(a_{k+2}(k+2+r)+a_k(k+r+4))=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+4)}{k+2+r}$

- Recursion relation for $r=-4$

  $a_{k+2}=-\frac{a_k k}{k-2}$

- Series not valid for $r=-4$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=-\frac{a_k k}{k-2}$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k(k+4)}{k+2}$
- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+4)}{k+2}, 5a_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+11*x^2)*diff(y(x),x)+24*x^2*y(x)=0,y(x),type='series'
```

$$y(x)=c_1\left(1-2x^2+3x^4+\mathrm{O}\left(x^6\right)\right)+\frac{c_2\left(-144+432x^4+\mathrm{O}\left(x^6\right)\right)}{x^4}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 27

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*(5+11*x^2)*y'[x]+24*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{x^4}-1\right)+c_2\left(3x^4-2x^2+1\right)$$

16.40 problem 36

Internal problem ID [1452]
Internal file name [`OUTPUT/1453_Sunday_June_05_2022_02_18_25_AM_23966063/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+1)\, y''+8y'x-(-x^2+35)\, y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+4x^2)\, y''+8y'x+(x^2-35)\, y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x\,(x^2+1)} \\ q(x) &= \frac{x^2-35}{4x^2\,(x^2+1)} \end{aligned}$$

Table 850: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = \frac{x^2-35}{4x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x^2+1)\,y'' + 8y'x + (x^2-35)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 8\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) x + (x^2-35)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-35 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-35 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 8x^{n+r} a_n (n+r) - 35 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 8x^r a_0 r - 35 a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 8x^r r - 35x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 4r - 35\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 4r - 35 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{2}$$
$$r_2 = -\frac{7}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 4r - 35\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{5}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{7}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4a_{n-2}(n+r-2)(n-3+r)+4a_n(n+r)(n+r-1)+8a_n(n+r)+a_{n-2}-35a_n=0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(-5+2n+2r)\, a_{n-2}}{2n+2r+7} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = -\frac{n a_{n-2}}{n+6} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1-2r}{11+2r}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{11+2r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{11+2r}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 4r - 3}{4r^2 + 52r + 165}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{11+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	$\frac{1}{10}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{11+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	$\frac{1}{10}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-8r^3 - 36r^2 - 22r + 21}{8r^3 + 180r^2 + 1318r + 3135}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_6 = -\frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{11+2r}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	$\frac{1}{10}$
a_5	0	0
a_6	$\frac{-8r^3-36r^2-22r+21}{8r^3+180r^2+1318r+3135}$	$-\frac{1}{20}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\ &= x^{\frac{5}{2}}\left(1 - \frac{x^2}{4} + \frac{x^4}{10} - \frac{x^6}{20} + O(x^7)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_6 \\ &= \frac{-8r^3 - 36r^2 - 22r + 21}{8r^3 + 180r^2 + 1318r + 3135} \end{aligned}$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{-8r^3-36r^2-22r+21}{8r^3+180r^2+1318r+3135} &= \lim_{r\rightarrow -\frac{7}{2}} \frac{-8r^3-36r^2-22r+21}{8r^3+180r^2+1318r+3135} \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{7}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
4b_{n-2}(n+r-2)(n-3+r)+4b_n(n+r)(n+r-1)+8b_n(n+r)+b_{n-2}-35b_n = 0 \tag{4}
$$

Which for for the root $r = -\frac{7}{2}$ becomes

$$
4b_{n-2}\left(n-\frac{11}{2}\right)\left(n-\frac{13}{2}\right)+4b_n\left(n-\frac{7}{2}\right)\left(n-\frac{9}{2}\right)+8b_n\left(n-\frac{7}{2}\right)+b_{n-2}-35b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{(-5+2n+2r)\, b_{n-2}}{2n+2r+7} \tag{5}
$$

Which for the root $r = -\frac{7}{2}$ becomes

$$
b_n = -\frac{(-12+2n)\, b_{n-2}}{2n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{7}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{2r - 1}{11 + 2r}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_2 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{11+2r}$	2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{11+2r}$	2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 4r - 3}{(11 + 2r)(15 + 2r)}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{11+2r}$	2
b_3	0	0
b_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	1

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{11+2r}$	2
b_3	0	0
b_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	1
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{8r^3 + 36r^2 + 22r - 21}{(11 + 2r)(15 + 2r)(19 + 2r)}$$

Which for the root $r = -\frac{7}{2}$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{11+2r}$	2
b_3	0	0
b_4	$\frac{4r^2+4r-3}{4r^2+52r+165}$	1
b_5	0	0
b_6	$\frac{-8r^3-36r^2-22r+21}{8r^3+180r^2+1318r+3135}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^{\frac{5}{2}}(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 \dots) \\ &= \frac{1 + 2x^2 + x^4 + O(x^7)}{x^{\frac{7}{2}}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^{\frac{5}{2}}\left(1 - \frac{x^2}{4} + \frac{x^4}{10} - \frac{x^6}{20} + O(x^7)\right) + \frac{c_2(1 + 2x^2 + x^4 + O(x^7))}{x^{\frac{7}{2}}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1x^{\frac{5}{2}}\left(1 - \frac{x^2}{4} + \frac{x^4}{10} - \frac{x^6}{20} + O(x^7)\right) + \frac{c_2(1 + 2x^2 + x^4 + O(x^7))}{x^{\frac{7}{2}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^{\frac{5}{2}}\left(1 - \frac{x^2}{4} + \frac{x^4}{10} - \frac{x^6}{20} + O(x^7)\right) + \frac{c_2(1 + 2x^2 + x^4 + O(x^7))}{x^{\frac{7}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1x^{\frac{5}{2}}\left(1 - \frac{x^2}{4} + \frac{x^4}{10} - \frac{x^6}{20} + O(x^7)\right) + \frac{c_2(1 + 2x^2 + x^4 + O(x^7))}{x^{\frac{7}{2}}}$$

Verified OK.

16.40.1 Maple step by step solution

Let's solve

$4x^2(x^2+1)\,y''+8y'x+(x^2-35)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2-35)y}{4x^2(x^2+1)}-\frac{2y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x(x^2+1)}+\frac{(x^2-35)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2}{x(x^2+1)}, P_3(x)=\frac{x^2-35}{4x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=2$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{35}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\,y''+8y'x+(x^2-35)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+2r)\,(-5+2r)\, x^r + a_1(9+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)\,(2k+2r-5) + a_k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$
- Each term must be 0

  $a_1(9+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right)\left(k+r-\frac{5}{2}\right) = 0$
- Shift index using $k->k+2$

  $4\left(\left(k-\frac{1}{2}+r\right) a_k + a_{k+2}\left(k+\frac{11}{2}+r\right)\right)\left(k-\frac{1}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+11+2r}$
- Recursion relation for $r = -\frac{7}{2}$; series terminates at $k = 4$

  $a_{k+2} = -\frac{(2k-8)a_k}{2k+4}$

- Solution for $r = -\frac{7}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}, a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $$a_{k+2} = -\frac{(2k+4)a_k}{2k+16}$$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+16}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+4)b_k}{2k+16}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+8*x*diff(y(x),x)-(35-x^2)*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = \frac{c_1 x^6 \left(1 - \frac{1}{4}x^2 + \frac{1}{10}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-86400 - 172800x^2 - 86400x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 52

```
AsymptoticDSolveValue[4*x^2*(1+x^2)*y''[x]+8*x*y'[x]-(35-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{2}{x^{3/2}} + \frac{1}{x^{7/2}} + \sqrt{x} \right) + c_2 \left(\frac{x^{13/2}}{10} - \frac{x^{9/2}}{4} + x^{5/2} \right)$$

16.41 problem 37

Internal problem ID [1453]
Internal file name [`OUTPUT/1454_Sunday_June_05_2022_02_18_28_AM_33762512/index.tex`]

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Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-x(-x^2+5)\,y'-(25x^2+7)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\,y''+(x^3-5x)\,y'+(-25x^2-7)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2-5}{x\,(x^2+1)}$$
$$q(x)=-\frac{25x^2+7}{x^2\,(x^2+1)}$$

Table 852: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-5}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = -\frac{25x^2+7}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (x^3-5x)\,y' + (-25x^2-7)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (x^3-5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (-25x^2-7)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-25x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-7a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-25x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-25a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-25a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 5x^{n+r} a_n (n+r) - 7a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 5x^r a_0 r - 7a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 5x^r r - 7x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 - 6r - 7)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 6r - 7 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 7 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 6r - 7)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 8$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^7\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+7} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ + a_{n-2}(n+r-2) - 5a_n(n+r) - 25a_{n-2} - 7a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r+3)\, a_{n-2}}{n+1+r} \tag{4}$$

Which for the root $r = 7$ becomes

$$a_n = -\frac{(n+10)\, a_{n-2}}{n+8} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 7$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-5-r}{r+3}$$

Which for the root $r = 7$ becomes

$$a_2 = -\frac{6}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{7+r}{r+3}$$

Which for the root $r = 7$ becomes

$$a_4 = \frac{7}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0
a_4	$\frac{7+r}{r+3}$	$\frac{7}{5}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0
a_4	$\frac{7+r}{r+3}$	$\frac{7}{5}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{-9-r}{r+3}$$

Which for the root $r = 7$ becomes

$$a_6 = -\frac{8}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0
a_4	$\frac{7+r}{r+3}$	$\frac{7}{5}$
a_5	0	0
a_6	$\frac{-9-r}{r+3}$	$-\frac{8}{5}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0
a_4	$\frac{7+r}{r+3}$	$\frac{7}{5}$
a_5	0	0
a_6	$\frac{-9-r}{r+3}$	$-\frac{8}{5}$
a_7	0	0

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{11 + r}{r + 3}$$

Which for the root $r = 7$ becomes

$$a_8 = \frac{9}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{-5-r}{r+3}$	$-\frac{6}{5}$
a_3	0	0
a_4	$\frac{7+r}{r+3}$	$\frac{7}{5}$
a_5	0	0
a_6	$\frac{-9-r}{r+3}$	$-\frac{8}{5}$
a_7	0	0
a_8	$\frac{11+r}{r+3}$	$\frac{9}{5}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^7 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 \dots\right) \\ &= x^7 \left(1 - \frac{6x^2}{5} + \frac{7x^4}{5} - \frac{8x^6}{5} + \frac{9x^8}{5} + O(x^9)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 8$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_8(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_8 \\ &= \frac{11+r}{r+3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{11+r}{r+3} &= \lim_{r\to -1} \frac{11+r}{r+3} \\ &= 5 \end{aligned}$$

The limit is 5. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &b_{n-2}(n+r-2)(n-3+r) + b_n(n+r)(n+r-1) \\ &+ b_{n-2}(n+r-2) - 5b_n(n+r) - 25b_{n-2} - 7b_n = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_{n-2}(n-3)(n-4) + b_n(n-1)(n-2) + b_{n-2}(n-3) - 5b_n(n-1) - 25b_{n-2} - 7b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r+3)\, b_{n-2}}{n+1+r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{(n+2)\, b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{5+r}{r+3}$$

Which for the root $r = -1$ becomes

$$b_2 = -2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{7+r}{r+3}$$

Which for the root $r = -1$ becomes

$$b_4 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0
b_4	$\frac{7+r}{r+3}$	3

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0
b_4	$\frac{7+r}{r+3}$	3
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{9+r}{r+3}$$

Which for the root $r = -1$ becomes

$$b_6 = -4$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0
b_4	$\frac{7+r}{r+3}$	3
b_5	0	0
b_6	$\frac{-9-r}{r+3}$	-4

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0
b_4	$\frac{7+r}{r+3}$	3
b_5	0	0
b_6	$\frac{-9-r}{r+3}$	-4
b_7	0	0

For $n = 8$, using the above recursive equation gives

$$b_8 = \frac{11 + r}{r + 3}$$

Which for the root $r = -1$ becomes

$$b_8 = 5$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{-5-r}{r+3}$	-2
b_3	0	0
b_4	$\frac{7+r}{r+3}$	3
b_5	0	0
b_6	$\frac{-9-r}{r+3}$	-4
b_7	0	0
b_8	$\frac{11+r}{r+3}$	5

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^7\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + b_9 x^9 \dots\right) \\
&= \frac{1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 + O(x^9)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^7 \left(1 - \frac{6x^2}{5} + \frac{7x^4}{5} - \frac{8x^6}{5} + \frac{9x^8}{5} + O(x^9)\right) + \frac{c_2(1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 + O(x^9))}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^7 \left(1 - \frac{6x^2}{5} + \frac{7x^4}{5} - \frac{8x^6}{5} + \frac{9x^8}{5} + O(x^9)\right) + \frac{c_2(1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 + O(x^9))}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 x^7 \left(1 - \frac{6x^2}{5} + \frac{7x^4}{5} - \frac{8x^6}{5} + \frac{9x^8}{5} + O(x^9)\right) \\
+ \frac{c_2(1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 + O(x^9))}{x}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^7 \left(1 - \frac{6x^2}{5} + \frac{7x^4}{5} - \frac{8x^6}{5} + \frac{9x^8}{5} + O(x^9)\right) + \frac{c_2(1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 + O(x^9))}{x}$$

Verified OK.

16.41.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (x^3-5x)\, y' + (-25x^2-7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(25x^2+7)y}{x^2(x^2+1)} - \frac{(x^2-5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-5)y'}{x(x^2+1)} - \frac{(25x^2+7)y}{x^2(x^2+1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x^2-5}{x(x^2+1)}, P_3(x) = -\frac{25x^2+7}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -7$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(x^2-5)\, y' + (-25x^2-7)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\, x^r + a_1(2+r)\,(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 7\}$$

- Each term must be 0

$$a_1(2+r)\,(-6+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-7)\left(a_k(k+r+1)+a_{k-2}(k+3+r)\right)=0$$

- Shift index using $k->k+2$

$$(k+r-5)\left(a_{k+2}(k+3+r)+a_k(k+r+5)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+5)}{k+3+r}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{a_k(k+4)}{k+2}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0\right]$$

- Recursion relation for $r=7$

$$a_{k+2}=-\frac{a_k(k+12)}{k+10}$$

- Solution for $r=7$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2}=-\frac{a_k(k+12)}{k+10}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2}=-\frac{a_k(4+k)}{k+2}, a_1=0, b_{k+2}=-\frac{b_k(k+12)}{k+10}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(5-x^2)*diff(y(x),x)-(7+25*x^2)*y(x)=0,y(x),type='series
```

$$y(x) = c_1 x^7 \left(1 - \frac{6}{5}x^2 + \frac{7}{5}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-203212800 + 406425600x^2 - 609638400x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]-x*(5-x^2)*y'[x]-(7+25*x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(3x^3 - 2x + \frac{1}{x}\right) + c_2\left(\frac{7x^{11}}{5} - \frac{6x^9}{5} + x^7\right)$$

16.42 problem 38

Internal problem ID [1454]
Internal file name [`OUTPUT/1455_Sunday_June_05_2022_02_18_31_AM_33591993/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\, y''+x(2x^2+5)\, y'-21y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^4+x^2)\, y''+(2x^3+5x)\, y'-21y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x^2+5}{x\,(x^2+1)}$$
$$q(x)=-\frac{21}{x^2\,(x^2+1)}$$

Table 854: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x^2+5}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = -\frac{21}{x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2+1)\,y'' + (2x^3+5x)\,y' - 21y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + (2x^3+5x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 21\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-21 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} (n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-21 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - 21 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 5x^r a_0 r - 21 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 5x^r r - 21 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+7)(r-3)x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+7)(r-3) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -7$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+7)(r-3)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 10$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^7}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-7}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots

of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) \\ + 2a_{n-2}(n+r-2) + 5a_n(n+r) - 21a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n^2+2nr+r^2-3n-3r+2)}{n^2+2nr+r^2+4n+4r-21} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-2}(n^2+3n+2)}{n(n+10)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r(1+r)}{r^2+8r-9}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{15}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$$

Which for the root $r = 3$ becomes

$$a_6 = -\frac{5}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0
a_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$-\frac{5}{32}$

For $n = 7$, using the above recursive equation gives

$$a_7 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0
a_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$-\frac{5}{32}$
a_7	0	0

For $n = 8$, using the above recursive equation gives

$$a_8 = \frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$$

Which for the root $r = 3$ becomes

$$a_8 = \frac{25}{256}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0
a_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$-\frac{5}{32}$
a_7	0	0
a_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	$\frac{25}{256}$

For $n = 9$, using the above recursive equation gives

$$a_9 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0
a_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$-\frac{5}{32}$
a_7	0	0
a_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	$\frac{25}{256}$
a_9	0	0

For $n = 10$, using the above recursive equation gives

$$a_{10} = -\frac{(r+4)\,r(r+2)\,(r+6)\,(r+8)}{(r+17)\,(r+13)\,(r+11)\,(-1+r)\,(r+15)}$$

Which for the root $r = 3$ becomes

$$a_{10} = -\frac{33}{512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r(1+r)}{r^2+8r-9}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{15}{56}$
a_5	0	0
a_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$-\frac{5}{32}$
a_7	0	0
a_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	$\frac{25}{256}$
a_9	0	0
a_{10}	$-\frac{(r+4)r(r+2)(r+6)(r+8)}{(r+17)(r+13)(r+11)(-1+r)(r+15)}$	$-\frac{33}{512}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + a_{10} x^{10} + a_{11} x^{11} \dots\right) \\
&= x^3\left(1 - \frac{x^2}{2} + \frac{15x^4}{56} - \frac{5x^6}{32} + \frac{25x^8}{256} - \frac{33x^{10}}{512} + O(x^{11})\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 10$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_{10}(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_{10} \\
&= -\frac{(r+4)\, r(r+2)\,(r+6)\,(r+8)}{(r+17)\,(r+13)\,(r+11)\,(-1+r)\,(r+15)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} -\frac{(r+4)\, r(r+2)\,(r+6)\,(r+8)}{(r+17)\,(r+13)\,(r+11)\,(-1+r)\,(r+15)} &= \lim_{r\to -7} -\frac{(r+4)\, r(r+2)\,(r+6)\,(r+8)}{(r+17)\,(r+13)\,(r+11)\,(-1+r)\,(r+15)} \\
&= \frac{7}{1024}
\end{aligned}$$

The limit is $\frac{7}{1024}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-7}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
&b_{n-2}(n+r-2)(n-3+r) + b_n(n+r)(n+r-1) \\
&+ 2b_{n-2}(n+r-2) + 5b_n(n+r) - 21b_n = 0
\end{aligned} \tag{4}
$$

Which for for the root $r = -7$ becomes

$$
b_{n-2}(n-9)(n-10) + b_n(n-7)(n-8) + 2b_{n-2}(n-9) + 5b_n(n-7) - 21b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}(n^2 + 2nr + r^2 - 3n - 3r + 2)}{n^2 + 2nr + r^2 + 4n + 4r - 21} \tag{5}
$$

Which for the root $r = -7$ becomes

$$
b_n = -\frac{b_{n-2}(n^2 - 17n + 72)}{n^2 - 10n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -7$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$
b_2 = -\frac{r(1+r)}{r^2 + 8r - 9}
$$

Which for the root $r = -7$ becomes

$$b_2 = \frac{21}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r^2 + 5r + 6)}{(r + 11)(r^2 + 8r - 9)}$$

Which for the root $r = -7$ becomes

$$b_4 = \frac{35}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0

For $n = 6$, using the above recursive equation gives

$$b_6 = -\frac{(r^2 + 9r + 20)\, r(r + 2)}{(r + 13)\,(r + 11)\,(r^2 + 8r - 9)}$$

Which for the root $r = -7$ becomes

$$b_6 = \frac{35}{64}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0
b_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$\frac{35}{64}$

For $n = 7$, using the above recursive equation gives

$$b_7 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0
b_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$\frac{35}{64}$
b_7	0	0

For $n = 8$, using the above recursive equation gives

$$b_8 = \frac{(r^2 + 13r + 42)(r + 2)r(r + 4)}{(r + 15)(r + 13)(r + 11)(r^2 + 8r - 9)}$$

Which for the root $r = -7$ becomes

$$b_8 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0
b_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$\frac{35}{64}$
b_7	0	0
b_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	0

For $n = 9$, using the above recursive equation gives

$$b_9 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0
b_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$\frac{35}{64}$
b_7	0	0
b_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	0
b_9	0	0

For $n = 10$, using the above recursive equation gives

$$b_{10} = -\frac{(r+4)\, r(r+2)\, (r+6)\, (r+8)}{(r+17)\, (r+13)\, (r+11)\, (-1+r)\, (r+15)}$$

Which for the root $r = -7$ becomes

$$b_{10} = \frac{7}{1024}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r(1+r)}{r^2+8r-9}$	$\frac{21}{8}$
b_3	0	0
b_4	$\frac{r(r^2+5r+6)}{(r+11)(r^2+8r-9)}$	$\frac{35}{16}$
b_5	0	0
b_6	$-\frac{(r+5)(r+4)r(r+2)}{(r+13)(r+11)(r+9)(-1+r)}$	$\frac{35}{64}$
b_7	0	0
b_8	$\frac{(r+7)(r+6)(r+2)r(r+4)}{(r+15)(-1+r)(r+9)(r+11)(r+13)}$	0
b_9	0	0
b_{10}	$-\frac{(r+4)r(r+2)(r+6)(r+8)}{(r+17)(r+13)(r+11)(-1+r)(r+15)}$	$\frac{7}{1024}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^3\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + b_9 x^9 + b_{10} x^{10} + b_{11} x^{11} \dots\right) \\
&= \frac{1 + \frac{21x^2}{8} + \frac{35x^4}{16} + \frac{35x^6}{64} + \frac{7x^{10}}{1024} + O(x^{11})}{x^7}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{x^2}{2} + \frac{15x^4}{56} - \frac{5x^6}{32} + \frac{25x^8}{256} - \frac{33x^{10}}{512} + O(x^{11})\right) \\
&\quad + \frac{c_2 \left(1 + \frac{21x^2}{8} + \frac{35x^4}{16} + \frac{35x^6}{64} + \frac{7x^{10}}{1024} + O(x^{11})\right)}{x^7}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 - \frac{x^2}{2} + \frac{15x^4}{56} - \frac{5x^6}{32} + \frac{25x^8}{256} - \frac{33x^{10}}{512} + O(x^{11})\right) \\
&\quad + \frac{c_2 \left(1 + \frac{21x^2}{8} + \frac{35x^4}{16} + \frac{35x^6}{64} + \frac{7x^{10}}{1024} + O(x^{11})\right)}{x^7}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 \left(1 - \frac{x^2}{2} + \frac{15x^4}{56} - \frac{5x^6}{32} + \frac{25x^8}{256} - \frac{33x^{10}}{512} + O(x^{11})\right) + \frac{c_2 \left(1 + \frac{21x^2}{8} + \frac{35x^4}{16} + \frac{35x^6}{64} + \frac{7x^{10}}{1024} + O(x^{11})\right)}{x^7} \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \left(1 - \frac{x^2}{2} + \frac{15x^4}{56} - \frac{5x^6}{32} + \frac{25x^8}{256} - \frac{33x^{10}}{512} + O(x^{11})\right) + \frac{c_2 \left(1 + \frac{21x^2}{8} + \frac{35x^4}{16} + \frac{35x^6}{64} + \frac{7x^{10}}{1024} + O(x^{11})\right)}{x^7}$$

Verified OK.

16.42.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + (2x^3+5x)\, y' - 21y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{21y}{x^2(x^2+1)} - \frac{(2x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+5)y'}{x(x^2+1)} - \frac{21y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+5}{x(x^2+1)}, P_3(x) = -\frac{21}{x^2(x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 5$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -21$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\,y'' + x(2x^2+5)\,y' - 21y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(-3+r)\, x^r + a_1(8+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-7, 3\}$

- Each term must be 0

  $a_1(8+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+7)(k+r-3) + a_{k-2}(k-2+r)(k+r-1) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+9+r)(k+r-1) + a_k(k+r)(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$

- Recursion relation for $r = -7$; series terminates at $k = 6$

  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$

- Solution for $r = -7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-7}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0, b_{k+2} = -\frac{b_k(k+3)(4+k)}{(k+12)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+2*x^2)*diff(y(x),x)-21*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = c_1 x^3 \left(1 - \frac{1}{2}x^2 + \frac{15}{56}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-1316818944000 - 3456649728000x^2 - 2880541440000x^4 + \mathrm{O}\left(x^6\right))}{x^7}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 46

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*(5+2*x^2)*y'[x]-21*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{x^7} + \frac{21}{8x^5} + \frac{35}{16x^3}\right) + c_2 \left(\frac{15x^7}{56} - \frac{x^5}{2} + x^3\right)$$

16.43 problem 39

Internal problem ID [1455]
Internal file name [`OUTPUT/1456_Sunday_June_05_2022_02_18_36_AM_19784511/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+2x^2)\,y''-x(x^2+3)\,y'-2yx=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$(2x^4+x^2)\,y''+(-x^3-3x)\,y'-2yx=0$$

Or

$$x(2y''x^3-y'x^2+y''x-3y'-2y)=0$$

For $x\neq 0$ the above simplifies to

$$(2x^3+x)\,y''-y'x^2-2y-3y'=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^4+x^2)\,y''+(-x^3-3x)\,y'-2yx=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$
\begin{aligned}
p(x) &= -\frac{x^2+3}{x\left(1+2x^2\right)} \\
q(x) &= -\frac{2}{\left(1+2x^2\right)x}
\end{aligned}
$$

Table 856: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2+3}{x(1+2x^2)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

$q(x) = -\frac{2}{(1+2x^2)x}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{i\sqrt{2}}{2}$	"regular"
$x = \frac{i\sqrt{2}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{i\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
x^2\left(1+2x^2\right)y'' + \left(-x^3-3x\right)y' - 2yx = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
x^2(1+2x^2)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}\right) \\
+(-x^3-3x)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right)-2\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)x=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}2x^{n+r+2}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) \\
+\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}2x^{n+r+2}a_n(n+r)(n+r-1)&=\sum_{n=2}^{\infty}2a_{n-2}(n+r-2)(n-3+r)x^{n+r} \\
\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n(n+r)\right)&=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)x^{n+r}\right) \\
\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n\right)&=\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
\left(\sum_{n=2}^{\infty}2a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right) \\
+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)x^{n+r}\right) \\
+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) - 3x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^4 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2}{r^2 - 2r - 3}$$

For $2 \leq n$ the recursive equation is

$$2a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) - a_{n-2}(n+r-2) - 3a_n(n+r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2n^2 a_{n-2} + 4nr a_{n-2} + 2r^2 a_{n-2} - 11n a_{n-2} - 11r a_{n-2} + 14a_{n-2} - 2a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{-2n^2 a_{n-2} - 5n a_{n-2} - 2a_{n-2} + 2a_{n-1}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2-2r-3}$	$\frac{2}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2r^4 + 7r^3 - 9r + 4}{(r^2 - 2r - 3)(r^2 - 4)}$$

Which for the root $r = 4$ becomes

$$a_2 = -\frac{8}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-2r^4+7r^3-9r+4}{(r^2-2r-3)(r^2-4)}$	$-\frac{8}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-8r^4 + 12r^3 + 18r^2 - 10r}{r^6 - 14r^4 + 49r^2 - 36}$$

Which for the root $r = 4$ becomes

$$a_3 = -\frac{86}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-2r^4+7r^3-9r+4}{(r^2-2r-3)(r^2-4)}$	$-\frac{8}{5}$
a_3	$\frac{-8r^4+12r^3+18r^2-10r}{r^6-14r^4+49r^2-36}$	$-\frac{86}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^7 - 51r^5 + 5r^4 + 117r^3 - 33r^2 - 54r + 24}{r^7 + 3r^6 - 17r^5 - 39r^4 + 88r^3 + 108r^2 - 144r}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{445}{168}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-2r^4+7r^3-9r+4}{(r^2-2r-3)(r^2-4)}$	$-\frac{8}{5}$
a_3	$\frac{-8r^4+12r^3+18r^2-10r}{r^6-14r^4+49r^2-36}$	$-\frac{86}{105}$
a_4	$\frac{4r^7-51r^5+5r^4+117r^3-33r^2-54r+24}{r^7+3r^6-17r^5-39r^4+88r^3+108r^2-144r}$	$\frac{445}{168}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{24r^7 + 72r^6 - 126r^5 - 378r^4 - 72r^3 + 114r^2 - 42r + 24}{(r+5)(1+r)^2(-1+r)(r-2)(r-3)(r+4)(r+3)r}$$

Which for the root $r = 4$ becomes

$$a_5 = \frac{9571}{6300}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2}{r^2-2r-3}$	$\frac{2}{5}$
a_2	$\frac{-2r^4+7r^3-9r+4}{(r^2-2r-3)(r^2-4)}$	$-\frac{8}{5}$
a_3	$\frac{-8r^4+12r^3+18r^2-10r}{r^6-14r^4+49r^2-36}$	$-\frac{86}{105}$
a_4	$\frac{4r^7-51r^5+5r^4+117r^3-33r^2-54r+24}{r^7+3r^6-17r^5-39r^4+88r^3+108r^2-144r}$	$\frac{445}{168}$
a_5	$\frac{24r^7+72r^6-126r^5-378r^4-72r^3+114r^2-42r+24}{(r+5)(1+r)^2(-1+r)(r-2)(r-3)(r+4)(r+3)r}$	$\frac{9571}{6300}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^4\left(1 + \frac{2x}{5} - \frac{8x^2}{5} - \frac{86x^3}{105} + \frac{445x^4}{168} + \frac{9571x^5}{6300} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_4 \\
&= \frac{4r^7 - 51r^5 + 5r^4 + 117r^3 - 33r^2 - 54r + 24}{r^7 + 3r^6 - 17r^5 - 39r^4 + 88r^3 + 108r^2 - 144r}
\end{aligned}$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} \frac{4r^7-51r^5+5r^4+117r^3-33r^2-54r+24}{r^7+3r^6-17r^5-39r^4+88r^3+108r^2-144r} &= \lim_{r\rightarrow 0} \frac{4r^7-51r^5+5r^4+117r^3-33r^2-54r+24}{r^7+3r^6-17r^5-39r^4+88r^3+108r^2-144r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2(1+2x^2)\, y'' + (-x^3-3x)\, y' - 2yx = 0$ gives

$$
\begin{aligned}
&x^2(1+2x^2)\left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ (-x^3-3x)\left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&- 2\left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) x = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\Bigg(\left(x^{2}\left(1+2x^{2}\right) y_1''(x)+\left(-x^{3}-3x\right) y_1'(x)-2y_1(x)\, x\right) \ln\left(x\right) \\
&+x^{2}\left(1+2x^{2}\right) \left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)+\frac{\left(-x^{3}-3x\right) y_1(x)}{x}\Bigg) C \\
&+x^{2}\left(1+2x^{2}\right) \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2} \left(n+r_2\right)^{2}}{x^{2}}-\frac{b_n x^{n+r_2} \left(n+r_2\right)}{x^{2}}\right)\right) \\
&+\left(-x^{3}-3x\right) \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2} \left(n+r_2\right)}{x}\right)-2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right) x=0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^{2}\left(1+2x^{2}\right) y_1''(x)+\left(-x^{3}-3x\right) y_1'(x)-2y_1(x)\, x=0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(x^{2}\left(1+2x^{2}\right) \left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^{2}}\right)+\frac{\left(-x^{3}-3x\right) y_1(x)}{x}\right) C \\
&+x^{2}\left(1+2x^{2}\right) \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2} \left(n+r_2\right)^{2}}{x^{2}}-\frac{b_n x^{n+r_2} \left(n+r_2\right)}{x^{2}}\right)\right) \\
&+\left(-x^{3}-3x\right) \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2} \left(n+r_2\right)}{x}\right)-2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right) x=0
\end{aligned}
\tag{8}
$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(2\left(2x^{3}+x\right) \left(\sum_{n=0}^{\infty}x^{-1+n+r_1} a_n \left(n+r_1\right)\right)+\left(-3x^{2}-4\right) \left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right) C \\
&+\left(2x^{4}+x^{2}\right) \left(\sum_{n=0}^{\infty}x^{-2+n+r_2} b_n \left(n+r_2\right) \left(-1+n+r_2\right)\right) \\
&+\left(-x^{3}-3x\right) \left(\sum_{n=0}^{\infty}x^{-1+n+r_2} b_n \left(n+r_2\right)\right)-2\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right) x=0
\end{aligned}
\tag{9}
$$

Since $r_1 = 4$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}
&\left(2(2x^3+x)\left(\sum_{n=0}^{\infty} x^{3+n} a_n (n+4)\right) + (-3x^2-4)\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\
&+ (2x^4+x^2)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) \\
&+ (-x^3-3x)\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) - 2\left(\sum_{n=0}^{\infty} b_n x^n\right) x = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4C\, x^{n+6} a_n (n+4)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-3C\, x^{n+6} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-4C\, x^{n+4} a_n\right) + \left(\sum_{n=0}^{\infty} 2n\, x^{n+2} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) + \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n} b_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 4C\, x^{n+6} a_n (n+4) &= \sum_{n=6}^{\infty} 4C a_{n-6} (n-2)\, x^n \\
\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) &= \sum_{n=4}^{\infty} 2C a_{n-4} n\, x^n \\
\sum_{n=0}^{\infty} \left(-3C\, x^{n+6} a_n\right) &= \sum_{n=6}^{\infty} \left(-3C a_{n-6} x^n\right) \\
\sum_{n=0}^{\infty} \left(-4C\, x^{n+4} a_n\right) &= \sum_{n=4}^{\infty} \left(-4C a_{n-4} x^n\right) \\
\sum_{n=0}^{\infty} 2n\, x^{n+2} b_n (n-1) &= \sum_{n=2}^{\infty} 2(n-2)\, b_{n-2} (n-3)\, x^n
\end{aligned}$$

$$\sum_{n=0}^{\infty} \left(-n\, x^{n+2} b_n\right) = \sum_{n=2}^{\infty} \left(-(n-2)\, b_{n-2} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n} b_n\right) = \sum_{n=1}^{\infty} \left(-2b_{n-1} x^n\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=6}^{\infty} 4C a_{n-6}(n-2)\, x^n\right) + \left(\sum_{n=4}^{\infty} 2C a_{n-4} n\, x^n\right) + \sum_{n=6}^{\infty} \left(-3C a_{n-6} x^n\right) \\
&\quad + \sum_{n=4}^{\infty} \left(-4C a_{n-4} x^n\right) + \left(\sum_{n=2}^{\infty} 2(n-2)\, b_{n-2}(n-3)\, x^n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-(n-2)\, b_{n-2} x^n\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-3x^n b_n n\right) + \sum_{n=1}^{\infty} \left(-2b_{n-1} x^n\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-3b_1 - 2b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 - 2 = 0$$

Solving the above for b_1 gives

$$b_1 = -\frac{2}{3}$$

For $n = 2$, Eq (2B) gives

$$-4b_2 - 2b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2 + \frac{4}{3} = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{3}$$

For $n = 3$, Eq (2B) gives

$$-3b_3 - b_1 - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{2}{3} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{6}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 9b_3 - 2b_4 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 - \frac{2}{5} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{2}{25}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{6}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{6}\left(x^4\left(1 + \frac{2x}{5} - \frac{8x^2}{5} - \frac{86x^3}{105} + \frac{445x^4}{168} + \frac{9571x^5}{6300} + O(x^6)\right)\right) \ln(x) + 1 - \frac{2x}{3} + \frac{x^2}{3} + \frac{2x^5}{25} + O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^4 \left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{1}{6}\left(x^4\left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right)\right)\ln(x)+1 \right. \\
&\quad \left. -\frac{2x}{3}+\frac{x^2}{3}+\frac{2x^5}{25}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^4 \left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^4\left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right)\ln(x)}{6}+1-\frac{2x}{3}+\frac{x^2}{3} \right. \\
&\quad \left. +\frac{2x^5}{25}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^4 \left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^4\left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right)\ln(x)}{6}+1-\frac{2x}{3} \right. \\
&\quad \left. +\frac{x^2}{3}+\frac{2x^5}{25}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1x^4\left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right) + c_2\left(-\frac{x^4\left(1+\frac{2x}{5}-\frac{8x^2}{5}-\frac{86x^3}{105}+\frac{445x^4}{168}+\frac{9571x^5}{6300}+O(x^6)\right)\ln(x)}{6}+1-\frac{2x}{3}+\frac{x^2}{3} + \frac{2x^5}{25}+O(x^6)\right)$$

Verified OK.

16.43.1 Maple step by step solution

Let's solve

$x^2(1+2x^2)\,y''+(-x^3-3x)\,y'-2yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(1+2x^2)}+\frac{(x^2+3)y'}{x(1+2x^2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2+3)y'}{x(1+2x^2)}-\frac{2y}{x(1+2x^2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2+3}{x(1+2x^2)}, P_3(x)=-\frac{2}{(1+2x^2)x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x(1+2x^2)\, y'' + (-x^2-3)\, y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-4+r)\, x^{-1+r} + (a_1(1+r)\,(-3+r) - 2a_0)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-3+r) - 2a_k +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term must be 0

  $a_1(1+r)\,(-3+r) - 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{5}{2}\right)(k+r-1)\, a_{k-1} + a_{k+1}(k+1+r)\,(k-3+r) - 2a_k = 0$

- Shift index using $k- >k+1$

  $2\left(k-\frac{3}{2}+r\right)(k+r)\, a_k + a_{k+2}(k+2+r)\,(k-2+r) - 2a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k + 4kr a_k + 2r^2 a_k - 3k a_k - 3r a_k - 2a_{k+1}}{(k+2+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 3k a_k - 2a_{k+1}}{(k+2)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = -\frac{2k^2 a_k - 3k a_k - 2a_{k+1}}{(k+2)(k-2)}$$

- Recursion relation for $r = 4$

$$a_{k+2} = -\frac{2k^2 a_k + 13k a_k + 20a_k - 2a_{k+1}}{(k+6)(k+2)}$$

- Solution for $r = 4$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{2k^2 a_k + 13k a_k + 20a_k - 2a_{k+1}}{(k+6)(k+2)}, 5a_1 - 2a_0 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0, g <
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve(x^2*(1+2*x^2)*diff(y(x),x$2)-x*(3+x^2)*diff(y(x),x)-2*x*y(x)=0,y(x),type='series',x=0
```

$$y(x) = c_1 x^4 \left(1 + \frac{2}{5}x - \frac{8}{5}x^2 - \frac{86}{105}x^3 + \frac{445}{168}x^4 + \frac{9571}{6300}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(24x^4 + \frac{48}{5}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-144 + 96x - 48x^2 + 210x^4 + \frac{1812}{25}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1+2*x^2)*y''[x]-x*(3+x^2)*y'[x]-2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{12} \left(3x^4 + 4x^2 - 8x + 12 \right) - \frac{1}{6} x^4 \log(x) \right) + c_2 \left(\frac{445x^8}{168} - \frac{86x^7}{105} - \frac{8x^6}{5} + \frac{2x^5}{5} + x^4 \right)$$

16.44 problem 40

Internal problem ID [1456]
Internal file name [OUTPUT/1457_Sunday_June_05_2022_02_18_40_AM_84123780/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 7 Series Solutions of Linear Second Equations. 7.6 THE METHOD OF FROBENIUS III. Exercises 7.7. Page 389
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+1)\,y''+4x(x^2+2)\,y'-(x^2+15)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(4x^4+4x^2)\,y''+(4x^3+8x)\,y'+(-x^2-15)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+2}{x(x^2+1)}$$
$$q(x)=-\frac{x^2+15}{4x^2(x^2+1)}$$

Table 858: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+2}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

$q(x) = -\frac{x^2+15}{4x^2(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x^2+1)\, y'' + (4x^3+8x)\, y' + (-x^2-15)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x^2+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (4x^3+8x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x^2-15) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-15 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r)(n+r-1) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{n+r+2} a_n(n+r) = \sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} 4a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n(n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-15 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n(n+r)(n+r-1) + 8x^{n+r} a_n(n+r) - 15 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) + 8x^r a_0 r - 15a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 8x^r r - 15x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 4r - 15\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 4r - 15 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{5}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 4r - 15\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^{\frac{5}{2}}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 4a_{n-2}(n+r-2)(n-3+r) + 4a_n(n+r)(n+r-1) \\ + 4a_{n-2}(n+r-2) + 8a_n(n+r) - a_{n-2} - 15a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(2n+2r-5)\, a_{n-2}}{2n+2r+5} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{(n-1)\, a_{n-2}}{n+4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1-2r}{9+2r}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{9+2r}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{9+2r}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^2 + 4r - 3}{4r^2 + 44r + 117}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{9+2r}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{4r^2+44r+117}$	$\frac{1}{16}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-2r}{9+2r}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{4r^2+4r-3}{4r^2+44r+117}$	$\frac{1}{16}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x^2}{6} + \frac{x^4}{16} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_4 \\ &= \frac{4r^2 + 4r - 3}{4r^2 + 44r + 117}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} \frac{4r^2 + 4r - 3}{4r^2 + 44r + 117} &= \lim_{r\to -\frac{5}{2}} \frac{4r^2 + 4r - 3}{4r^2 + 44r + 117} \\ &= \frac{3}{8}\end{aligned}$$

The limit is $\frac{3}{8}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{5}{2}}\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \le n$ the recursive equation is

$$4b_{n-2}(n+r-2)(n-3+r) + 4b_n(n+r)(n+r-1) \\ + 4b_{n-2}(n+r-2) + 8b_n(n+r) - b_{n-2} - 15b_n = 0 \tag{4}$$

Which for for the root $r = -\frac{5}{2}$ becomes

$$4b_{n-2}\left(n-\frac{9}{2}\right)\left(n-\frac{11}{2}\right) + 4b_n\left(n-\frac{5}{2}\right)\left(n-\frac{7}{2}\right) \\ + 4b_{n-2}\left(n-\frac{9}{2}\right) + 8b_n\left(n-\frac{5}{2}\right) - b_{n-2} - 15b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(2n+2r-5)\, b_{n-2}}{2n+2r+5} \tag{5}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_n = -\frac{(2n-10)\, b_{n-2}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{2r-1}{9+2r}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_2 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{9+2r}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{9+2r}$	$\frac{3}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^2 + 4r - 3}{(9 + 2r)(13 + 2r)}$$

Which for the root $r = -\frac{5}{2}$ becomes

$$b_4 = \frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{9+2r}$	$\frac{3}{2}$
b_3	0	0
b_4	$\frac{4r^2+4r-3}{4r^2+44r+117}$	$\frac{3}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1-2r}{9+2r}$	$\frac{3}{2}$
b_3	0	0
b_4	$\frac{4r^2+4r-3}{4r^2+44r+117}$	$\frac{3}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{3x^2}{2} + \frac{3x^4}{8} + O(x^6)}{x^{\frac{5}{2}}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{6} + \frac{x^4}{16} + O(x^6)\right) + \frac{c_2\left(1 + \frac{3x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^{\frac{5}{2}}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{6} + \frac{x^4}{16} + O(x^6)\right) + \frac{c_2\left(1 + \frac{3x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^{\frac{5}{2}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{6} + \frac{x^4}{16} + O(x^6)\right) + \frac{c_2\left(1 + \frac{3x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x^2}{6} + \frac{x^4}{16} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{3x^2}{2} + \frac{3x^4}{8} + O(x^6)\right)}{x^{\frac{5}{2}}}$$

Verified OK.

16.44.1 Maple step by step solution

Let's solve

$4x^2(x^2+1)\,y'' + (4x^3+8x)\,y' + (-x^2-15)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y'}{x(x^2+1)} + \frac{(x^2+15)y}{4x^2(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+2)y'}{x(x^2+1)} - \frac{(x^2+15)y}{4x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x^2+2}{x(x^2+1)}, P_3(x) = -\frac{x^2+15}{4x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{15}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\,y'' + 4x(x^2+2)\,y' + (-x^2-15)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5+2r)\,(-3+2r)\, x^r + a_1(7+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-3) + a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{5}{2}\right)a_{k-2}+a_k\left(k+r+\frac{5}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k+a_{k+2}\left(k+\frac{9}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r-1)a_k}{2k+9+2r}$$

- Recursion relation for $r=-\frac{5}{2}$

$$a_{k+2}=-\frac{(2k-6)a_k}{2k+4}$$

- Solution for $r=-\frac{5}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{(2k-6)a_k}{2k+4}, a_1=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+2}=-\frac{(2k+2)a_k}{2k+12}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{(2k+2)a_k}{2k+12}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{(2k-6)a_k}{2k+4}, a_1=0, b_{k+2}=-\frac{(2k+2)b_k}{2k+12}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(2+x^2)*diff(y(x),x)-(15+x^2)*y(x)=0,y(x),type='seri
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{1}{6}x^2 + \frac{1}{16}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(-144 - 216x^2 - 54x^4 + \mathrm{O}\left(x^6\right)\right)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 58

```
AsymptoticDSolveValue[4*x^2*(1+x^2)*y''[x]+4*x*(2+x^2)*y'[x]-(15+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3x^{3/2}}{8} + \frac{1}{x^{5/2}} + \frac{3}{2\sqrt{x}}\right) + c_2\left(\frac{x^{11/2}}{16} - \frac{x^{7/2}}{6} + x^{3/2}\right)$$

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17.1 problem section 9.1, problem 2

Internal problem ID [1457]
Internal file name [`OUTPUT/1458_Sunday_June_05_2022_02_18_44_AM_72604641/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^3y''' - x^2y'' - 2y'x + 6y = 0}$$

With initial conditions

$$[y(-1) = -4, y'(-1) = -14, y''(-1) = -20]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 2y'x + 6y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 6x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 6x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = -1$ in the above gives

$$-4 = -c_1 + c_2 - c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 2c_2 x + 3c_3 x^2$$

substituting $y' = -14$ and $x = -1$ in the above gives

$$-14 = -c_1 - 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_2 + 6c_3 x$$

substituting $y'' = -20$ and $x = -1$ in the above gives

$$-20 = -2c_1 + 2c_2 - 6c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{25}{3} \\ c_2 &= \frac{22}{3} \\ c_3 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{9x^4 + 22x^3 + 25}{3x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{9x^4 + 22x^3 + 25}{3x} \tag{1}$$

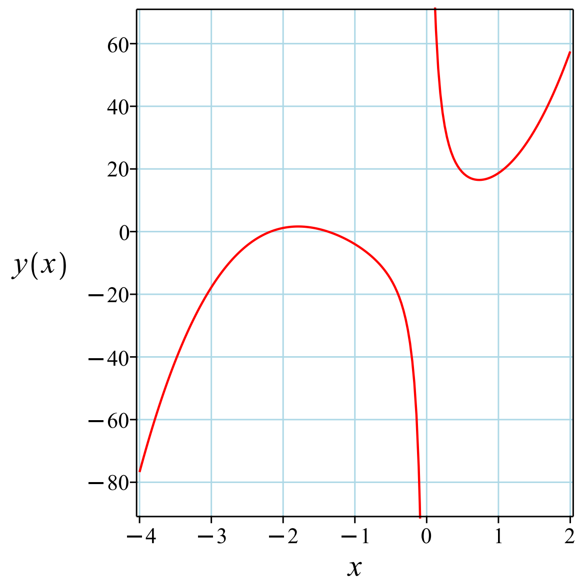


Figure 537: Solution plot

Verification of solutions

$$y = \frac{9x^4 + 22x^3 + 25}{3x}$$

Verified OK.

17.1.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - x^2 y'' - 2y'x + 6y = 0, y(-1) = -4, y'\Big|_{\{x=-1\}} = -14, y''\Big|_{\{x=-1\}} = -20\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{6y}{x^3} + \frac{y''x + 2y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} - \frac{2y'}{x^2} + \frac{6y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' - x^2 y'' - 2y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

    Substitute the change of variables back into the ODE

    $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y(t)$

  - Define new variable $y_2(t)$

    $y_2(t) = \frac{d}{dt}y(t)$

  - Define new variable $y_3(t)$

    $y_3(t) = \frac{d^2}{dt^2}y(t)$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

$\frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)$

Convert linear ODE into a system of first order ODEs

$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 e^{-t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$$

- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x}+\frac{c_2x^2}{4}+\frac{c_3x^3}{9}$

- Use the initial condition $y(-1)=-4$

  $-4=-c_1+\frac{c_2}{4}-\frac{c_3}{9}$

- Calculate the 1st derivative of the solution

  $y'=-\frac{c_1}{x^2}+\frac{c_2x}{2}+\frac{c_3x^2}{3}$

- Use the initial condition $y'\Big|_{\{x=-1\}}=-14$

  $-14=-c_1-\frac{c_2}{2}+\frac{c_3}{3}$

- Calculate the 2nd derivative of the solution

  $y''=\frac{2c_1}{x^3}+\frac{c_2}{2}+\frac{2c_3x}{3}$

- Use the initial condition $y''\Big|_{\{x=-1\}}=-20$

  $-20=-2c_1+\frac{c_2}{2}-\frac{2c_3}{3}$

- Solve for the unknown coefficients

  $\left\{c_1=\frac{25}{3},c_2=\frac{88}{3},c_3=27,x=x\right\}$

- Solution to the IVP

  $y=\frac{25}{3x}+\frac{22x^2}{3}+3x^3$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(-1) = -4, D(y)(-1)
```

$$y(x) = 3x^3 + \frac{22x^2}{3} + \frac{25}{3x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 24

```
DSolve[{x^3*y'''[x]-x^2*y''[x]-2*x*y'[x]+6*y[x]==0,{y[-1]==-4,y'[-1]==-14,y''[-1]==-20}},y[x
```

$$y(x) \to \frac{9x^4 + 22x^3 + 25}{3x}$$

17.2 problem section 9.1, problem 3

Internal problem ID [1458]
Internal file name [`OUTPUT/1459_Sunday_June_05_2022_02_18_45_AM_77226135/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 3.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' + y''' - 7y'' - y' + 6y = 0}$$

With initial conditions

$$[y(0) = 5, y'(0) = -6, y''(0) = 10, y'''(0) = -36]$$

The characteristic equation is

$$\lambda^4 + \lambda^3 - 7\lambda^2 - \lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -3$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{-3x} \\
y_4 &= \mathrm{e}^{2x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_3 + \mathrm{e}^{2x} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$
5 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - 3\,\mathrm{e}^{-3x} c_3 + 2\,\mathrm{e}^{2x} c_4
$$

substituting $y' = -6$ and $x = 0$ in the above gives

$$
-6 = -c_1 + c_2 - 3c_3 + 2c_4 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 9\,\mathrm{e}^{-3x} c_3 + 4\,\mathrm{e}^{2x} c_4
$$

substituting $y'' = 10$ and $x = 0$ in the above gives

$$
10 = c_1 + c_2 + 9c_3 + 4c_4 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - 27\,\mathrm{e}^{-3x} c_3 + 8\,\mathrm{e}^{2x} c_4
$$

substituting $y''' = -36$ and $x = 0$ in the above gives

$$
-36 = -c_1 + c_2 - 27c_3 + 8c_4 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 2$$
$$c_3 = 1$$
$$c_4 = -1$$

Substituting these values back in above solution results in

$$y = 3\,e^{-x} + 2\,e^{x} + e^{-3x} - e^{2x}$$

Summary

The solution(s) found are the following

$$y = 3\,e^{-x} + 2\,e^{x} + e^{-3x} - e^{2x} \tag{1}$$

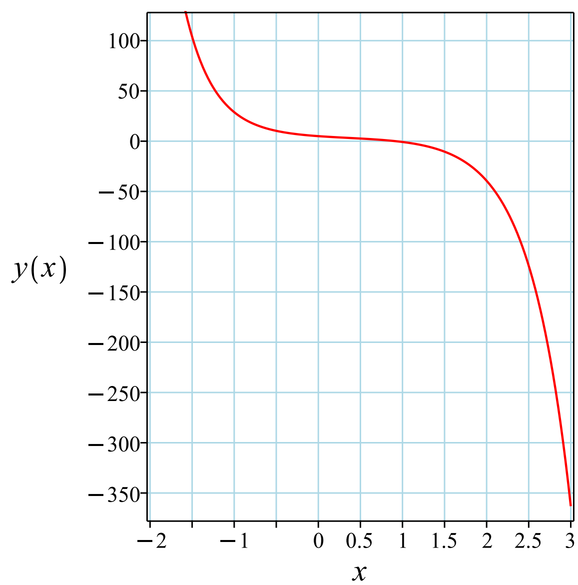


Figure 538: Solution plot

Verification of solutions

$$y = 3\,e^{-x} + 2\,e^{x} + e^{-3x} - e^{2x}$$

Verified OK.

17.2.1 Maple step by step solution

Let's solve

$$\left[y'''' + y''' - 7y'' - y' + 6y = 0, y(0) = 5, y'\Big|_{\{x=0\}} = -6, y''\Big|_{\{x=0\}} = 10, y'''\Big|_{\{x=0\}} = -36\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_4(x) + 7y_3(x) + y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_4(x) + 7y_3(x) + y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + c_4 \overrightarrow{y}_4$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \mathrm{e}^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(27 c_2 \mathrm{e}^{2x} - 27\,\mathrm{e}^{4x} c_3 - \frac{27\,\mathrm{e}^{5x} c_4}{8} + c_1\right) \mathrm{e}^{-3x}}{27}$$

- Use the initial condition $y(0) = 5$

$$5 = -c_2 + c_3 + \frac{c_4}{8} - \frac{c_1}{27}$$

- Calculate the 1st derivative of the solution

$$y' = -\frac{\left(54 c_2 \mathrm{e}^{2x} - 108\,\mathrm{e}^{4x} c_3 - \frac{135\,\mathrm{e}^{5x} c_4}{8}\right) \mathrm{e}^{-3x}}{27} + \frac{\left(27 c_2 \mathrm{e}^{2x} - 27\,\mathrm{e}^{4x} c_3 - \frac{27\,\mathrm{e}^{5x} c_4}{8} + c_1\right) \mathrm{e}^{-3x}}{9}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = -6$

$$-6 = c_2 + c_3 + \frac{c_4}{4} + \frac{c_1}{9}$$

- Calculate the 2nd derivative of the solution

$$y'' = -\frac{\left(108 c_2 \mathrm{e}^{2x} - 432\,\mathrm{e}^{4x} c_3 - \frac{675\,\mathrm{e}^{5x} c_4}{8}\right) \mathrm{e}^{-3x}}{27} + \frac{2\left(54 c_2 \mathrm{e}^{2x} - 108\,\mathrm{e}^{4x} c_3 - \frac{135\,\mathrm{e}^{5x} c_4}{8}\right) \mathrm{e}^{-3x}}{9} - \frac{\left(27 c_2 \mathrm{e}^{2x} - 27\,\mathrm{e}^{4x} c_3 - \frac{27\,\mathrm{e}^{5x} c_4}{8} + c_1\right.}{3}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 10$

$$10 = -c_2 + c_3 + \frac{c_4}{2} - \frac{c_1}{3}$$

- Calculate the 3rd derivative of the solution

$$y''' = -\frac{\left(216 c_2 \mathrm{e}^{2x} - 1728\,\mathrm{e}^{4x} c_3 - \frac{3375\,\mathrm{e}^{5x} c_4}{8}\right) \mathrm{e}^{-3x}}{27} + \frac{\left(108 c_2 \mathrm{e}^{2x} - 432\,\mathrm{e}^{4x} c_3 - \frac{675\,\mathrm{e}^{5x} c_4}{8}\right) \mathrm{e}^{-3x}}{3} - \Big(54 c_2 \mathrm{e}^{2x} - 108\,\mathrm{e}^{4x} c_3 -$$

- Use the initial condition $y'''\Big|_{\{x=0\}} = -36$

$$-36 = c_1 + c_2 + c_3 + c_4$$

- Solve for the unknown coefficients

$$\{c_1 = -27, c_2 = -3, c_3 = 2, c_4 = -8\}$$

- Solution to the IVP

$$y = \left(-\mathrm{e}^{5x} + 2\,\mathrm{e}^{4x} + 3\,\mathrm{e}^{2x} + 1\right) \mathrm{e}^{-3x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$4)+diff(y(x),x$3)-7*diff(y(x),x$2)-diff(y(x),x)+6*y(x)=0,y(0) = 5, D(y)(
```

$$y(x) = \left(-e^{5x} + 2\,e^{4x} + 3\,e^{2x} + 1\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 30

```
DSolve[{y''''[x]+y'''[x]-7*y''[x]-y'[x]+6*y[x]==0,{y[0]==5,y'[0]==-6,y''[0]==10,y'''[0]==-36
```

$$y(x) \to e^{-3x} + 3e^{-x} + 2e^{x} - e^{2x}$$

17.3 problem section 9.1, problem 5(b) 1

Internal problem ID [1459]
Internal file name [`OUTPUT/1460_Sunday_June_05_2022_02_18_47_AM_32992798/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 5(b) 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

`[[_3rd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x^3y''' - x^2y'' - 2y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0, y''(1) = 0]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 2y'x + 6y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 6x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 6x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-2\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2x^2 + c_3x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 2c_2 x + 3c_3 x^2$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = -c_1 + 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_2 + 6c_3 x$$

substituting $y'' = 0$ and $x = 1$ in the above gives

$$0 = 2c_1 + 2c_2 + 6c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 1$$
$$c_3 = -\frac{1}{2}$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - 2x^3 - 1}{2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4 - 2x^3 - 1}{2x} \tag{1}$$

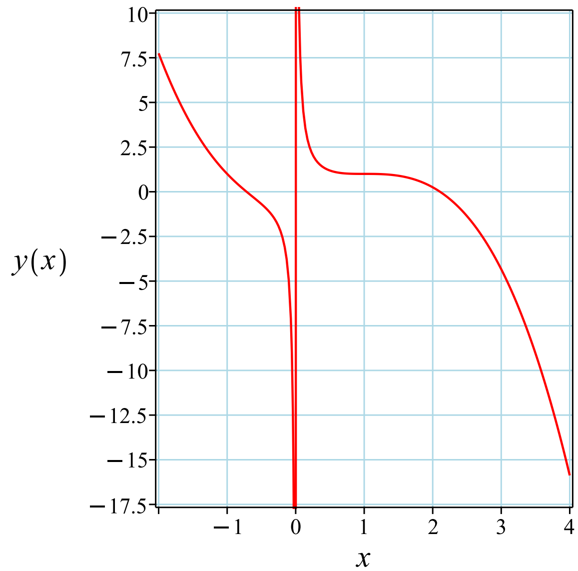


Figure 539: Solution plot

Verification of solutions

$$y = -\frac{x^4 - 2x^3 - 1}{2x}$$

Verified OK.

17.3.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - x^2 y'' - 2y'x + 6y = 0, y(1) = 1, y'\Big|_{\{x=1\}} = 0, y''\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{6y}{x^3} + \frac{y''x + 2y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} - \frac{2y'}{x^2} + \frac{6y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' - x^2 y'' - 2y'x + 6y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 6y(t) = 0$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)\right]$$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{2t}}{4} + \frac{c_3 \mathrm{e}^{3t}}{9}$$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \frac{c_2 x^2}{4} + \frac{c_3 x^3}{9}$
- Use the initial condition $y(1) = 1$

  $1 = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$
- Calculate the 1st derivative of the solution

  $y' = -\frac{c_1}{x^2} + \frac{c_2 x}{2} + \frac{c_3 x^2}{3}$
- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

  $0 = -c_1 + \frac{c_2}{2} + \frac{c_3}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{2c_1}{x^3} + \frac{c_2}{2} + \frac{2c_3 x}{3}$
- Use the initial condition $y''\Big|_{\{x=1\}} = 0$

  $0 = 2c_1 + \frac{c_2}{2} + \frac{2c_3}{3}$
- Solve for the unknown coefficients

  $\left\{c_1 = \frac{1}{2}, c_2 = 4, c_3 = -\frac{9}{2}, x = x\right\}$
- Solution to the IVP

  $y = \frac{1}{2x} + x^2 - \frac{x^3}{2}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(1) = 1, D(y)(1) =
```

$$y(x) = -\frac{x^3}{2} + x^2 + \frac{1}{2x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 23

```
DSolve[{x^3*y'''[x]-x^2*y''[x]-2*x*y'[x]+6*y[x]==0,{y[1]==1,y'[1]==0,y''[1]==0}},y[x],x,Incl
```

$$y(x) \to -\frac{x^3}{2} + x^2 + \frac{1}{2x}$$

17.4 problem section 9.1, problem 5(b) 2

Internal problem ID [1460]
Internal file name [`OUTPUT/1461_Sunday_June_05_2022_02_18_49_AM_34964901/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 5(b) 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

`[[_3rd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x^3y''' - x^2y'' - 2y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 1, y''(1) = 0]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 2y'x + 6y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 6x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 6x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-2\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 3\\ \lambda_3 &= -1\end{aligned}$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \frac{1}{x}\\ y_2 &= x^2\\ y_3 &= x^3\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 2c_2 x + 3c_3 x^2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -c_1 + 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_2 + 6c_3 x$$

substituting $y'' = 0$ and $x = 1$ in the above gives

$$0 = 2c_1 + 2c_2 + 6c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{3} \\ c_2 &= \frac{1}{3} \\ c_3 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 1}{3x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 1}{3x} \tag{1}$$

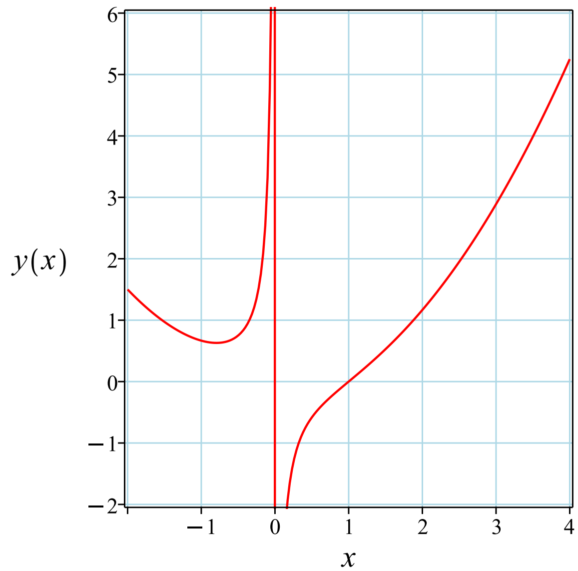


Figure 540: Solution plot

Verification of solutions

$$y = \frac{x^3 - 1}{3x}$$

Verified OK.

17.4.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - x^2 y'' - 2y'x + 6y = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 1, y''\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''
- Isolate 3rd derivative

  $y''' = -\frac{6y}{x^3} + \frac{y''x + 2y'}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} - \frac{2y'}{x^2} + \frac{6y}{x^3} = 0$
- Multiply by denominators of the ODE

  $x^3 y''' - x^2 y'' - 2y'x + 6y = 0$
- Make a change of variables

$t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\,t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)\right]$$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y(t) = c_1 \mathrm{e}^{-t} + \frac{c_2 \mathrm{e}^{2t}}{4} + \frac{c_3 \mathrm{e}^{3t}}{9}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \frac{c_2 x^2}{4} + \frac{c_3 x^3}{9}$
- Use the initial condition $y(1) = 0$

  $0 = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$
- Calculate the 1st derivative of the solution

  $y' = -\frac{c_1}{x^2} + \frac{c_2 x}{2} + \frac{c_3 x^2}{3}$
- Use the initial condition $y'\Big|_{\{x=1\}} = 1$

  $1 = -c_1 + \frac{c_2}{2} + \frac{c_3}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{2c_1}{x^3} + \frac{c_2}{2} + \frac{2c_3 x}{3}$
- Use the initial condition $y''\Big|_{\{x=1\}} = 0$

  $0 = 2c_1 + \frac{c_2}{2} + \frac{2c_3}{3}$
- Solve for the unknown coefficients

  $\left\{c_1 = -\frac{1}{3}, c_2 = \frac{4}{3}, c_3 = 0, x = x\right\}$
- Solution to the IVP

  $y = \frac{x^3 - 1}{3x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(1) = 0, D(y)(1) =
```

$$y(x) = \frac{x^3 - 1}{3x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 17

```
DSolve[{x^3*y'''[x]-x^2*y''[x]-2*x*y'[x]+6*y[x]==0,{y[1]==0,y'[1]==1,y''[1]==0}},y[x],x,Incl
```

$$y(x) \to \frac{x^3 - 1}{3x}$$

17.5 problem section 9.1, problem 5(b) 3

Internal problem ID [1461]
Internal file name [`OUTPUT/1462_Sunday_June_05_2022_02_18_50_AM_77436272/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 5(b) 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

`[[_3rd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x^3y''' - x^2y'' - 2y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 0, y''(1) = 1]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 2y'x + 6y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 6x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 6x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-2\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 2c_2 x + 3c_3 x^2$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = -c_1 + 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_2 + 6c_3 x$$

substituting $y'' = 1$ and $x = 1$ in the above gives

$$1 = 2c_1 + 2c_2 + 6c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{12} \\ c_2 &= -\frac{1}{3} \\ c_3 &= \frac{1}{4} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{3x^4 - 4x^3 + 1}{12x}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^4 - 4x^3 + 1}{12x} \tag{1}$$

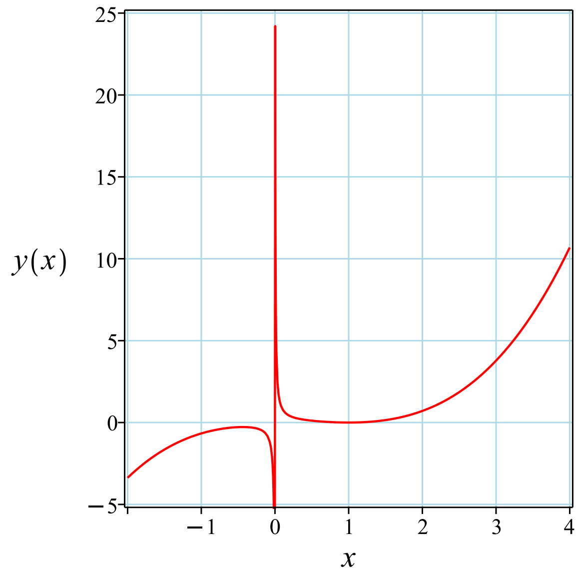


Figure 541: Solution plot

Verification of solutions

$$y = \frac{3x^4 - 4x^3 + 1}{12x}$$

Verified OK.

17.5.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - x^2 y'' - 2y'x + 6y = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 0, y''\Big|_{\{x=1\}} = 1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''
- Isolate 3rd derivative

  $y''' = -\frac{6y}{x^3} + \frac{y''x+2y'}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} - \frac{2y'}{x^2} + \frac{6y}{x^3} = 0$
- Multiply by denominators of the ODE

  $x^3 y''' - x^2 y'' - 2y'x + 6y = 0$
- Make a change of variables

$t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)$

Convert linear ODE into a system of first order ODEs

$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y(t) = c_1 e^{-t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \frac{c_2 x^2}{4} + \frac{c_3 x^3}{9}$
- Use the initial condition $y(1) = 0$

  $0 = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$
- Calculate the 1st derivative of the solution

  $y' = -\frac{c_1}{x^2} + \frac{c_2 x}{2} + \frac{c_3 x^2}{3}$
- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

  $0 = -c_1 + \frac{c_2}{2} + \frac{c_3}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{2c_1}{x^3} + \frac{c_2}{2} + \frac{2c_3 x}{3}$
- Use the initial condition $y''\Big|_{\{x=1\}} = 1$

  $1 = 2c_1 + \frac{c_2}{2} + \frac{2c_3}{3}$
- Solve for the unknown coefficients

  $\left\{c_1 = \frac{1}{12}, c_2 = -\frac{4}{3}, c_3 = \frac{9}{4}, x = x\right\}$
- Solution to the IVP

  $y = \frac{3x^4 - 4x^3 + 1}{12x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve([x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(1) = 0, D(y)(1) =
```

$$y(x) = \frac{3x^4 - 4x^3 + 1}{12x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 24

```
DSolve[{x^3*y'''[x]-x^2*y''[x]-2*x*y'[x]+6*y[x]==0,{y[1]==0,y'[1]==0,y''[1]==1}},y[x],x,Incl
```

$$y(x) \to \frac{3x^4 - 4x^3 + 1}{12x}$$

17.6 problem section 9.1, problem 5(b)

Internal problem ID [1462]
Internal file name [`OUTPUT/1463_Sunday_June_05_2022_02_18_52_AM_32337383/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 5(b).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

`[[_3rd_order, _exact, _linear, _homogeneous]]`

$$\boxed{x^3y''' - x^2y'' - 2y'x + 6y = 0}$$

With initial conditions

$$[y(1) = k_0, y'(1) = k_1, y''(1) = k_2]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 2y'x + 6y = 0$$

gives

$$-2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 6x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 6x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda - \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x^2 + c_3 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = k_0$ and $x = 1$ in the above gives

$$k_0 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + 2c_2 x + 3c_3 x^2$$

substituting $y' = k_1$ and $x = 1$ in the above gives

$$k_1 = -c_1 + 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_2 + 6c_3 x$$

substituting $y'' = k_2$ and $x = 1$ in the above gives

$$k_2 = 2c_1 + 2c_2 + 6c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{k_0}{2} - \frac{k_1}{3} + \frac{k_2}{12} \\ c_2 &= \frac{k_1}{3} + k_0 - \frac{k_2}{3} \\ c_3 &= \frac{k_2}{4} - \frac{k_0}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{6x^4 k_0 - 3x^4 k_2 - 12x^3 k_0 - 4x^3 k_1 + 4x^3 k_2 - 6k_0 + 4k_1 - k_2}{12x}$$

Which simplifies to

$$y = \frac{3(k_2 - 2k_0)\, x^4 + 4(k_1 + 3k_0 - k_2)\, x^3 + 6k_0 - 4k_1 + k_2}{12x}$$

Summary
The solution(s) found are the following

$$y = \frac{3(k_2 - 2k_0)\, x^4 + 4(k_1 + 3k_0 - k_2)\, x^3 + 6k_0 - 4k_1 + k_2}{12x} \tag{1}$$

Verification of solutions

$$y = \frac{3(k_2 - 2k_0)\, x^4 + 4(k_1 + 3k_0 - k_2)\, x^3 + 6k_0 - 4k_1 + k_2}{12x}$$

Verified OK.

17.6.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - x^2 y'' - 2y'x + 6y = 0, y(1) = k_0, y'\Big|_{\{x=1\}} = k_1, y''\Big|_{\{x=1\}} = k_2\right]$$

- Highest derivative means the order of the ODE is 3
  y'''
- Isolate 3rd derivative
  $y''' = -\frac{6y}{x^3} + \frac{y''x + 2y'}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y''' - \frac{y''}{x} - \frac{2y'}{x^2} + \frac{6y}{x^3} = 0$
- Multiply by denominators of the ODE
  $x^3 y''' - x^2 y'' - 2y'x + 6y = 0$
- Make a change of variables
  $t = \ln(x)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule
    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$
  - Compute derivative
    $y' = \frac{\frac{d}{dt} y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule
    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - y_2(t) - 6y_1(t)\right]$

- Define vector

  $\vec{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right]$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 e^{-t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{4} + \frac{c_3 x^3}{9}$$

- Use the initial condition $y(1) = k_0$

$$k_0 = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$$

- Calculate the 1st derivative of the solution

$$y' = -\frac{c_1}{x^2} + \frac{c_2 x}{2} + \frac{c_3 x^2}{3}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = k_1$

$$k_1 = -c_1 + \frac{c_2}{2} + \frac{c_3}{3}$$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{2c_1}{x^3} + \frac{c_2}{2} + \frac{2c_3 x}{3}$

- Use the initial condition $y''\Big|_{\{x=1\}} = k_2$

  $k_2 = 2c_1 + \frac{c_2}{2} + \frac{2c_3}{3}$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{k_0}{2} - \frac{k_1}{3} + \frac{k_2}{12}, c_2 = -\frac{4k_2}{3} + 4k_0 + \frac{4k_1}{3}, c_3 = -\frac{9k_0}{2} + \frac{9k_2}{4}, x = x\right\}$

- Solution to the IVP

  $y = \frac{3(k_2-2k_0)x^4+4(k_1+3k_0-k_2)x^3+6k_0-4k_1+k_2}{12x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
dsolve([x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+6*y(x)=0,y(1) = k__0, D(y)(1)
```

$$y(x) = \frac{3(-2k_0 + k_2)\,x^4 + 4(k_1 + 3k_0 - k_2)\,x^3 + 6k_0 - 4k_1 + k_2}{12x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 47

```
DSolve[{x^3*y'''[x]-x^2*y''[x]-2*x*y'[x]+6*y[x]==0,{y[1]==k0,y'[1]==k1,y''[1]==k2}},y[x],x,I
```

$$y(x) \to \frac{-6\text{k0}(x^4 - 2x^3 - 1) + 4\text{k1}(x^3 - 1) + 3\text{k2}x^4 - 4\text{k2}x^3 + \text{k2}}{12x}$$

17.7 problem section 9.1, problem 6(a)

Internal problem ID [1463]
Internal file name [`OUTPUT/1464_Sunday_June_05_2022_02_18_54_AM_93893289/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 6(a).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - y' - y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\\ \lambda_3 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-x}\\ y_2 &= x\,\mathrm{e}^{-x}\\ y_3 &= \mathrm{e}^{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x\, e^{-x} c_2 + c_3 e^{x}$$

Verified OK.

17.7.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + y_2(x) + y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\, \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I)\cdot \overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A-\lambda \cdot I)\cdot \overrightarrow{p}=\overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{array}\right]-(-1)\cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot \overrightarrow{p}=\left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{y}_2(x)=\mathrm{e}^{-x}\cdot \left(x\cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]+\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3=\mathrm{e}^{x}\cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((x+1) c_2 + c_1) \mathrm{e}^{-x} + c_3 \mathrm{e}^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) \mathrm{e}^{-x} + \mathrm{e}^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 x + c_3 e^{2x} + c_1 \right)$$

17.8 problem section 9.1, problem 6(b)

Internal problem ID [1464]
Internal file name [`OUTPUT/1465_Sunday_June_05_2022_02_18_55_AM_2191687/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.1. Page 471
Problem number: section 9.1, problem 6(b).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 7y' - 5y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1 - 2i$$
$$\lambda_3 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{(1+2i)x}$$
$$y_3 = e^{(1-2i)x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3$$

Verified OK.

17.8.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 7y' - 5y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) - 7y_2(x) + 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) - 7y_2(x) + 5y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1-2\,\mathrm{I}, \begin{bmatrix} -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} -\frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-2I)x} \cdot \begin{bmatrix} -\frac{3}{25}+\frac{4I}{25} \\ \frac{1}{5}+\frac{2I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{3}{25}+\frac{4I}{25} \\ \frac{1}{5}+\frac{2I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \left(-\frac{3}{25}+\frac{4I}{25}\right)(\cos(2x) - I\sin(2x)) \\ \left(\frac{1}{5}+\frac{2I}{5}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} \frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} \frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{x}(3c_2\cos(2x) - 4c_3\cos(2x) - 4c_2\sin(2x) - 3c_3\sin(2x) - 25c_1)}{25}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+7*diff(y(x),x)-5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x}(c_1 + \sin(2x)\, c_2 + \cos(2x)\, c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]-3*y''[x]+7*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(2x) + c_1 \sin(2x) + c_3)$$

18 Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483

18.1 problem section 9.2, problem 1

Internal problem ID [1465]
Internal file name [`OUTPUT/1466_Sunday_June_05_2022_02_18_56_AM_89074572/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 3y' - y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + x^2 e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = x^2 e^x$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 x\, e^x + x^2 e^x c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2 x\, e^x + x^2 e^x c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x (c_3 x^2 + c_2 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 21

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (x(c_3 x + c_2) + c_1)$$

18.2 problem section 9.2, problem 2

Internal problem ID [1466]
Internal file name [`OUTPUT/1467_Sunday_June_05_2022_02_18_57_AM_34749542/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 8y'' - 9y = 0}$$

The characteristic equation is

$$\lambda^4 + 8\lambda^2 - 9 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 3i$$
$$\lambda_4 = -3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-3ix} c_3 + e^{3ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-3ix}$$
$$y_4 = e^{3ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3ix} c_3 + e^{3ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3ix} c_3 + e^{3ix} c_4$$

Verified OK.

18.2.1 Maple step by step solution

Let's solve

$y'''' + 8y'' - 9y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -8y_3(x) + 9y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -8y_3(x) + 9y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & -8 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & -8 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ -\frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\begin{bmatrix} -3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution from eigenpair

$$e^{-3\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3x) - \mathrm{I}\sin(3x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{27}(\cos(3x) - \mathrm{I}\sin(3x)) \\ -\frac{\cos(3x)}{9} + \frac{\mathrm{I}\sin(3x)}{9} \\ \frac{\mathrm{I}}{3}(\cos(3x) - \mathrm{I}\sin(3x)) \\ \cos(3x) - \mathrm{I}\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(3x)}{27} \\ -\frac{\cos(3x)}{9} \\ \frac{\sin(3x)}{3} \\ \cos(3x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(3x)}{27} \\ \frac{\sin(3x)}{9} \\ \frac{\cos(3x)}{3} \\ -\sin(3x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2\mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3\sin(3x)}{27} - \frac{c_4\cos(3x)}{27} \\ -\frac{c_3\cos(3x)}{9} + \frac{c_4\sin(3x)}{9} \\ \frac{c_3\sin(3x)}{3} + \frac{c_4\cos(3x)}{3} \\ c_3\cos(3x) - c_4\sin(3x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} - \frac{c_4\cos(3x)}{27} - \frac{c_3\sin(3x)}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)+8*diff(y(x),x$2)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}c_1 + c_2\mathrm{e}^{x} + c_3 \sin(3x) + c_4 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 34

```
DSolve[y''''[x]+8*y''[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-x} + c_4 e^{x} + c_1 \cos(3x) + c_2 \sin(3x)$$

18.3 problem section 9.2, problem 3

Internal problem ID [1467]
Internal file name [`OUTPUT/1468_Sunday_June_05_2022_02_18_58_AM_77789978/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' + 16y' - 16y = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 16\lambda - 16 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 4i$$
$$\lambda_3 = -4i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-4ix} c_2 + e^{4ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{-4ix}$$
$$y_3 = e^{4ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{-4ix} c_2 + e^{4ix} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{-4ix} c_2 + e^{4ix} c_3$$

Verified OK.

18.3.1 Maple step by step solution

Let's solve

$y''' - y'' + 16y' - 16y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) - 16y_2(x) + 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - 16y_2(x) + 16y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 16 & -16 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 16 & -16 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-4\,\mathrm{I}, \begin{bmatrix} -\frac{1}{16} \\ \frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[4\,\mathrm{I}, \begin{bmatrix} -\frac{1}{16} \\ -\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-4\,\mathrm{I}, \begin{bmatrix} -\frac{1}{16} \\ \frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-4 I x} \cdot \begin{bmatrix} -\frac{1}{16} \\ \frac{I}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(4x) - I\sin(4x)) \cdot \begin{bmatrix} -\frac{1}{16} \\ \frac{I}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(4x)}{16} + \frac{I\sin(4x)}{16} \\ \frac{I}{4}(\cos(4x) - I\sin(4x)) \\ \cos(4x) - I\sin(4x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(4x)}{16} \\ \frac{\sin(4x)}{4} \\ \cos(4x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin(4x)}{16} \\ \frac{\cos(4x)}{4} \\ -\sin(4x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_2\cos(4x)}{16} + \frac{c_3\sin(4x)}{16} \\ \frac{c_2\sin(4x)}{4} + \frac{c_3\cos(4x)}{4} \\ c_2\cos(4x) - c_3\sin(4x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^x + \frac{c_3\sin(4x)}{16} - \frac{c_2\cos(4x)}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+16*diff(y(x),x)-16*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x} c_1 + \sin(4x) c_2 + c_3 \cos(4x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]-y''[x]+16*y'[x]-16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^x + c_1 \cos(4x) + c_2 \sin(4x)$$

18.4 problem section 9.2, problem 4

Internal problem ID [1468]
Internal file name [OUTPUT/1469_Sunday_June_05_2022_02_19_00_AM_92486463/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' + 3y'' - 2y' - 3y = 0}$$

The characteristic equation is

$$2\lambda^3 + 3\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -\frac{3}{2}\\ \lambda_3 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-\frac{3x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-x}\\ y_2 &= \mathrm{e}^{x}\\ y_3 &= \mathrm{e}^{-\frac{3x}{2}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-\frac{3x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-\frac{3x}{2}} c_3$$

Verified OK.

18.4.1 Maple step by step solution

Let's solve

$2y''' + 3y'' - 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{3y''}{2} + y' + \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{3y''}{2} - y' - \frac{3y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{3y_3(x)}{2} + y_2(x) + \frac{3y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{3y_3(x)}{2} + y_2(x) + \frac{3y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & -\frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & -\frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{3x}{2}} \cdot \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{3x}{2}} \cdot \left[\begin{array}{c} \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{array}\right] + c_2 e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + c_3 e^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(9c_3 e^{\frac{5x}{2}} + 9c_2 e^{\frac{x}{2}} + 4c_1\right) e^{-\frac{3x}{2}}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(2*diff(y(x),x$3)+3*diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 e^{\frac{5x}{2}} + c_2 e^{\frac{x}{2}} + c_1\right) e^{-\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[2*y'''[x]+3*y''[x]-2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x/2} + c_2 e^{-x} + c_3 e^x$$

18.5 problem section 9.2, problem 5

Internal problem ID [1469]
Internal file name [`OUTPUT/1470_Sunday_June_05_2022_02_19_01_AM_68935447/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 5y'' + 9y' + 5y = 0}$$

The characteristic equation is

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -2 - i$$
$$\lambda_3 = -2 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{(-2+i)x} c_2 + e^{(-2-i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{(-2+i)x}$$
$$y_3 = e^{(-2-i)x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{(-2+i)x} c_2 + e^{(-2-i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{(-2+i)x} c_2 + e^{(-2-i)x} c_3$$

Verified OK.

18.5.1 Maple step by step solution

Let's solve

$y''' + 5y'' + 9y' + 5y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -5y_3(x) - 9y_2(x) - 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -5y_3(x) - 9y_2(x) - 5y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-2-\mathrm{I}, \begin{bmatrix} \frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[-2+\mathrm{I}, \begin{bmatrix} \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-\mathrm{I}, \begin{bmatrix} \frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-2-I)x} \cdot \begin{bmatrix} \frac{3}{25} - \frac{4I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-2x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{3}{25} - \frac{4I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2x} \cdot \begin{bmatrix} \left(\frac{3}{25} - \frac{4I}{25}\right)(\cos(x) - I\sin(x)) \\ \left(-\frac{2}{5} + \frac{I}{5}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-2x} \cdot \begin{bmatrix} \frac{3\cos(x)}{25} - \frac{4\sin(x)}{25} \\ -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{-2x} \cdot \begin{bmatrix} -\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25} \\ \frac{2\sin(x)}{5} + \frac{\cos(x)}{5} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{3\cos(x)}{25} - \frac{4\sin(x)}{25} \\ -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} \\ \cos(x) \end{bmatrix} + c_3 e^{-2x} \cdot \begin{bmatrix} -\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25} \\ \frac{2\sin(x)}{5} + \frac{\cos(x)}{5} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\cos(x)(3c_2 - 4c_3) - 4\left(c_2 + \frac{3c_3}{4}\right)\sin(x)\right)e^{-2x}}{25} + c_1 e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)+5*diff(y(x),x$2)+9*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{-2x}\sin(x) + c_3e^{-2x}\cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]+5*y''[x]+9*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(c_3e^x + c_2\cos(x) + c_1\sin(x))$$

18.6 problem section 9.2, problem 6

Internal problem ID [1470]
Internal file name [`OUTPUT/1471_Sunday_June_05_2022_02_19_03_AM_20746883/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' - 8y'' + 5y' - y = 0}$$

The characteristic equation is

$$4\lambda^3 - 8\lambda^2 + 5\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{\frac{x}{2}}$$
$$y_3 = e^{\frac{x}{2}} x$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3$$

Verified OK.

18.6.1 Maple step by step solution

Let's solve

$4y''' - 8y'' + 5y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 2y'' - \frac{5y'}{4} + \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 2y'' + \frac{5y'}{4} - \frac{y}{4} = 0$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) - \frac{5y_2(x)}{4} + \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - \frac{5y_2(x)}{4} + \frac{y_1(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_1(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{array}\right] - \frac{1}{2} \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \left(x \cdot \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right] + \left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}\right) + c_3 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((4x - 8) c_2 + 4c_1) e^{\frac{x}{2}} + c_3 e^x$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*diff(y(x),x$3)-8*diff(y(x),x$2)+5*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) e^{\frac{x}{2}} + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[4*y'''[x]-8*y''[x]+5*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2}\left(c_2x + c_3e^{x/2} + c_1\right)$$

18.7 problem section 9.2, problem 7

Internal problem ID [1471]
Internal file name [`OUTPUT/1472_Sunday_June_05_2022_02_19_04_AM_89291697/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 7.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{27y''' + 27y'' + 9y' + y = 0}$$

The characteristic equation is

$$27\lambda^3 + 27\lambda^2 + 9\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{3}$$
$$\lambda_2 = -\frac{1}{3}$$
$$\lambda_3 = -\frac{1}{3}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-\frac{x}{3}} + c_2 x\,\mathrm{e}^{-\frac{x}{3}} + x^2 \mathrm{e}^{-\frac{x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-\frac{x}{3}}$$
$$y_2 = x\,\mathrm{e}^{-\frac{x}{3}}$$
$$y_3 = x^2 \mathrm{e}^{-\frac{x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x}{3}} + c_2 x \,\mathrm{e}^{-\frac{x}{3}} + x^2 \mathrm{e}^{-\frac{x}{3}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x}{3}} + c_2 x \,\mathrm{e}^{-\frac{x}{3}} + x^2 \mathrm{e}^{-\frac{x}{3}} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(27*diff(y(x),x$3)+27*diff(y(x),x$2)+9*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x}{3}} \left(c_3 x^2 + c_2 x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[27*y'''[x]+27*y''[x]+9*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/3}(x(c_3 x + c_2) + c_1)$$

18.8 problem section 9.2, problem 8

Internal problem ID [1472]
Internal file name [`OUTPUT/1473_Sunday_June_05_2022_02_19_05_AM_84385381/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y'' = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = \mathrm{e}^{-ix}$$
$$y_4 = \mathrm{e}^{ix}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

Verified OK.

18.8.1 Maple step by step solution

Let's solve

$y'''' + y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -c_4\cos(x) - c_3\sin(x) + c_1 \\ c_4\sin(x) - c_3\cos(x) \\ c_4\cos(x) + c_3\sin(x) \\ -c_4\sin(x) + c_3\cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_4\cos(x) - c_3\sin(x) + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$4)+diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2x + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 24

```
DSolve[y''''[x]+y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_4x - c_1 \cos(x) - c_2 \sin(x) + c_3$$

18.9 problem section 9.2, problem 9

Internal problem ID [1473]
Internal file name [`OUTPUT/1474_Sunday_June_05_2022_02_19_06_AM_78568510/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 9.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 16 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\\ \lambda_3 &= 2i\\ \lambda_4 &= -2i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= e^{-2x}\\ y_2 &= e^{2x}\\ y_3 &= e^{2ix}\\ y_4 &= e^{-2ix}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

Verified OK.

18.9.1 Maple step by step solution

Let's solve

$y'''' - 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3 \sin(2x)}{8} - \frac{c_4 \cos(2x)}{8} \\ -\frac{c_3 \cos(2x)}{4} + \frac{c_4 \sin(2x)}{4} \\ \frac{c_3 \sin(2x)}{2} + \frac{c_4 \cos(2x)}{2} \\ c_3 \cos(2x) - c_4 \sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_1 \mathrm{e}^{-2x}}{8} + \frac{c_2 \mathrm{e}^{2x}}{8} - \frac{c_4 \cos(2x)}{8} - \frac{c_3 \sin(2x)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)-16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-2x} c_2 + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 36

```
DSolve[y''''[x]-16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2x} + c_3 e^{-2x} + c_2 \cos(2x) + c_4 \sin(2x)$$

18.10 problem section 9.2, problem 10

Internal problem ID [1474]
Internal file name [`OUTPUT/1475_Sunday_June_05_2022_02_19_07_AM_84899194/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 10.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 12y'' + 36y = 0}$$

The characteristic equation is

$$\lambda^4 + 12\lambda^2 + 36 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{6}$$
$$\lambda_2 = -i\sqrt{6}$$
$$\lambda_3 = i\sqrt{6}$$
$$\lambda_4 = -i\sqrt{6}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-i\sqrt{6}\,x} c_1 + x\,\mathrm{e}^{-i\sqrt{6}\,x} c_2 + \mathrm{e}^{i\sqrt{6}\,x} c_3 + x\,\mathrm{e}^{i\sqrt{6}\,x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-i\sqrt{6}\,x}$$
$$y_2 = x\,\mathrm{e}^{-i\sqrt{6}\,x}$$
$$y_3 = \mathrm{e}^{i\sqrt{6}\,x}$$
$$y_4 = x\,\mathrm{e}^{i\sqrt{6}\,x}$$

Summary

The solution(s) found are the following

$$y = e^{-i\sqrt{6}\, x} c_1 + x\, e^{-i\sqrt{6}\, x} c_2 + e^{i\sqrt{6}\, x} c_3 + x\, e^{i\sqrt{6}\, x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{-i\sqrt{6}\, x} c_1 + x\, e^{-i\sqrt{6}\, x} c_2 + e^{i\sqrt{6}\, x} c_3 + x\, e^{i\sqrt{6}\, x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)+12*diff(y(x),x$2)+36*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos\left(\sqrt{6}\, x\right) + \sin\left(\sqrt{6}\, x\right) (c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 38

```
DSolve[y''''[x]+12*y''[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos\left(\sqrt{6} x\right) + (c_4 x + c_3) \sin\left(\sqrt{6} x\right)$$

18.11 problem section 9.2, problem 11

Internal problem ID [1475]
Internal file name [`OUTPUT/1476_Sunday_June_05_2022_02_19_09_AM_66588318/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 11.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{16y'''' - 72y'' + 81y = 0}$$

The characteristic equation is

$$16\lambda^4 - 72\lambda^2 + 81 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{3}{2}$$
$$\lambda_2 = \frac{3}{2}$$
$$\lambda_3 = -\frac{3}{2}$$
$$\lambda_4 = -\frac{3}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-\frac{3x}{2}} + c_2 x\, e^{-\frac{3x}{2}} + e^{\frac{3x}{2}} c_3 + x\, e^{\frac{3x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-\frac{3x}{2}} \\
y_2 &= x\,\mathrm{e}^{-\frac{3x}{2}} \\
y_3 &= \mathrm{e}^{\frac{3x}{2}} \\
y_4 &= x\,\mathrm{e}^{\frac{3x}{2}}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}} + \mathrm{e}^{\frac{3x}{2}} c_3 + x\,\mathrm{e}^{\frac{3x}{2}} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 x\,\mathrm{e}^{-\frac{3x}{2}} + \mathrm{e}^{\frac{3x}{2}} c_3 + x\,\mathrm{e}^{\frac{3x}{2}} c_4
$$

Verified OK.

18.11.1 Maple step by step solution

Let's solve

$16y'''' - 72y'' + 81y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{9y''}{2} - \frac{81y}{16}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{9y''}{2} + \frac{81y}{16} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{9y_3(x)}{2} - \frac{81y_1(x)}{16}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{9y_3(x)}{2} - \frac{81y_1(x)}{16}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{81}{16} & 0 & \frac{9}{2} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{81}{16} & 0 & \frac{9}{2} & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-\frac{3}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{3}{2}, \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right], \left[\frac{3}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{3}{2}, \left[\begin{array}{c} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue $-\frac{3}{2}$

$$\overrightarrow{y}_{1}(x) = \mathrm{e}^{-\frac{3x}{2}} \cdot \left[\begin{array}{c} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -\frac{3}{2}$ is the eigenvalue, a

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{2}(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{2}(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{3}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{81}{16} & 0 & \frac{9}{2} & 0 \end{bmatrix} - -\frac{3}{2} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{16}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $-\frac{3}{2}$

$$\overrightarrow{y}_2(x) = \mathrm{e}^{-\frac{3x}{2}} \cdot \left(x \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{16}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{3}{2}, \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{3}{2}$

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\frac{3x}{2}} \cdot \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = \frac{3}{2}$ is the eigenvalue, an

$$\overrightarrow{y}_4(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt

- Substitute $\overrightarrow{y}_4(x)$ into the homogeneous system

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v}+\overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_4(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{3}{2}$

  $$\left(\left[\begin{array}{cccc}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{81}{16} & 0 & \frac{9}{2} & 0\end{array}\right]-\frac{3}{2}\cdot\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}\frac{8}{27}\\ \frac{4}{9}\\ \frac{2}{3}\\ 1\end{array}\right]$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c}-\frac{16}{81}\\ 0\\ 0\\ 0\end{array}\right]$$
- Second solution from eigenvalue $\frac{3}{2}$

  $$\overrightarrow{y}_4(x)=\mathrm{e}^{\frac{3x}{2}}\cdot\left(x\cdot\left[\begin{array}{c}\frac{8}{27}\\ \frac{4}{9}\\ \frac{2}{3}\\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{16}{81}\\ 0\\ 0\\ 0\end{array}\right]\right)$$
- General solution to the system of ODEs

  $\overrightarrow{y}=c_1\overrightarrow{y}_1(x)+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+c_4\overrightarrow{y}_4(x)$
- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-\frac{3x}{2}} \cdot \left(x \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{16}{81} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \mathrm{e}^{\frac{3x}{2}} c_3 \cdot \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \mathrm{e}^{\frac{3x}{2}} c_4 \cdot \left(x \cdot \right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{8((-3x-2)c_2-3c_1)\mathrm{e}^{-\frac{3x}{2}}}{81} + \frac{8\left(\left(x-\frac{2}{3}\right)c_4+c_3\right)\mathrm{e}^{\frac{3x}{2}}}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(16*diff(y(x),x$4)-72*diff(y(x),x$2)+81*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{3x}{2}}(c_2x + c_1) + \mathrm{e}^{\frac{3x}{2}}(c_4x + c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 37

```
DSolve[16*y''''[x]-72*y''[x]+81*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/2}\left(c_3e^{3x} + x\left(c_4e^{3x} + c_2\right) + c_1\right)$$

18.12 problem section 9.2, problem 12

Internal problem ID [1476]
Internal file name [`OUTPUT/1477_Sunday_June_05_2022_02_19_10_AM_42483123/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 12.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{6y'''' + 5y''' + 7y'' + 5y' + y = 0}$$

The characteristic equation is

$$6\lambda^4 + 5\lambda^3 + 7\lambda^2 + 5\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -\frac{1}{3}$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-ix}c_1 + \mathrm{e}^{ix}c_2 + \mathrm{e}^{-\frac{x}{2}}c_3 + \mathrm{e}^{-\frac{x}{3}}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-ix}$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-\frac{x}{2}}$$
$$y_4 = e^{-\frac{x}{3}}$$

Summary
The solution(s) found are the following

$$y = e^{-ix}c_1 + e^{ix}c_2 + e^{-\frac{x}{2}}c_3 + e^{-\frac{x}{3}}c_4 \tag{1}$$

Verification of solutions

$$y = e^{-ix}c_1 + e^{ix}c_2 + e^{-\frac{x}{2}}c_3 + e^{-\frac{x}{3}}c_4$$

Verified OK.

18.12.1 Maple step by step solution

Let's solve

$6y'''' + 5y''' + 7y'' + 5y' + y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -\frac{5y'''}{6} - \frac{7y''}{6} - \frac{5y'}{6} - \frac{y}{6}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{5y'''}{6} + \frac{7y''}{6} + \frac{5y'}{6} + \frac{y}{6} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -\frac{5y_4(x)}{6} - \frac{7y_3(x)}{6} - \frac{5y_2(x)}{6} - \frac{y_1(x)}{6}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -\frac{5y_4(x)}{6} - \frac{7y_3(x)}{6} - \frac{5y_2(x)}{6} - \frac{y_1(x)}{6}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} & -\frac{7}{6} & -\frac{5}{6} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} & -\frac{7}{6} & -\frac{5}{6} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{3}, \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}\right], \left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right], \left[I, \begin{bmatrix} I \\ -1 \\ -I \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{3}, \left[\begin{array}{c} -27 \\ 9 \\ -3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{3}} \cdot \left[\begin{array}{c} -27 \\ 9 \\ -3 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(x\right) - I\sin\left(x\right)\right) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos\left(x\right) - I\sin\left(x\right)) \\ -\cos\left(x\right) + I\sin\left(x\right) \\ I(\cos\left(x\right) - I\sin\left(x\right)) \\ \cos\left(x\right) - I\sin\left(x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin\left(x\right) \\ -\cos\left(x\right) \\ \sin\left(x\right) \\ \cos\left(x\right) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos\left(x\right) \\ \sin\left(x\right) \\ \cos\left(x\right) \\ -\sin\left(x\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-\frac{x}{3}} \cdot \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_3\sin\left(x\right) - c_4\cos\left(x\right) \\ c_4\sin\left(x\right) - c_3\cos\left(x\right) \\ c_4\cos\left(x\right) + c_3\sin\left(x\right) \\ -c_4\sin\left(x\right) + c_3\cos\left(x\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -8c_1 e^{-\frac{x}{2}} - 27c_2 e^{-\frac{x}{3}} - c_4\cos\left(x\right) - c_3\sin\left(x\right)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(6*diff(y(x),x$4)+5*diff(y(x),x$3)+7*diff(y(x),x$2)+5*diff(y(x),x)+y(x)=0,y(x), singsol
```

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{x}{3}} + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 37

```
DSolve[6*y''''[x]+5*y'''[x]+7*y''[x]+5*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-x/2}\left(c_3 e^{x/6} + c_4\right) + c_1 \cos(x) + c_2 \sin(x)$$

18.13 problem section 9.2, problem 13

Internal problem ID [1477]
Internal file name [OUTPUT/1478_Sunday_June_05_2022_02_19_11_AM_51871714/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 13.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{4y'''' + 12y''' + 3y'' - 13y' - 6y = 0}$$

The characteristic equation is

$$4\lambda^4 + 12\lambda^3 + 3\lambda^2 - 13\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{3}{2}$$
$$\lambda_3 = -\frac{1}{2}$$
$$\lambda_4 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x + e^{-\frac{x}{2}} c_3 + e^{-\frac{3x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-\frac{x}{2}}$$
$$y_4 = e^{-\frac{3x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{-\frac{x}{2}} c_3 + e^{-\frac{3x}{2}} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{-\frac{x}{2}} c_3 + e^{-\frac{3x}{2}} c_4$$

Verified OK.

18.13.1 Maple step by step solution

Let's solve

$4y'''' + 12y''' + 3y'' - 13y' - 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -3y''' - \frac{3y''}{4} + \frac{13y'}{4} + \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 3y''' + \frac{3y''}{4} - \frac{13y'}{4} - \frac{3y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

- ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

- ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

- ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

- ◦ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -3y_4(x) - \frac{3y_3(x)}{4} + \frac{13y_2(x)}{4} + \frac{3y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3y_4(x) - \frac{3y_3(x)}{4} + \frac{13y_2(x)}{4} + \frac{3y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{2} & \frac{13}{4} & -\frac{3}{4} & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{2} & \frac{13}{4} & -\frac{3}{4} & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + e^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + e^{x} c_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(\frac{64 c_2 e^{\frac{x}{2}}}{27} + 64 e^{\frac{3x}{2}} c_3 - 8 e^{3x} c_4 + c_1\right) e^{-2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(4*diff(y(x),x$4)+12*diff(y(x),x$3)+3*diff(y(x),x$2)-13*diff(y(x),x)-6*y(x)=0,y(x), si
```

$$y(x) = \left(c_4 e^{3x} + c_3 e^{\frac{3x}{2}} + c_1 e^{\frac{x}{2}} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[4*y''''[x]+12*y'''[x]+3*y''[x]-13*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-2x}\left(c_1 e^{x/2} + c_2 e^{3x/2} + c_4 e^{3x} + c_3\right)$$

18.14 problem section 9.2, problem 14

Internal problem ID [1478]
Internal file name [`OUTPUT/1479_Sunday_June_05_2022_02_19_12_AM_14232300/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y''' + 7y'' - 6y' + 2y = 0}$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 7\lambda^2 - 6\lambda + 2 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 1 - i \\ \lambda_2 &= 1 + i \\ \lambda_3 &= 1 \\ \lambda_4 &= 1 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= x\,\mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{(1-i)x} \\
y_4 &= \mathrm{e}^{(1+i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4
$$

Verified OK.

18.14.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 7y'' - 6y' + 2y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 4y_4(x) - 7y_3(x) + 6y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 4y_4(x) - 7y_3(x) + 6y_2(x) - 2y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} -\frac{1}{4} + \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} -\frac{1}{4} - \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 1, & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \left(x \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I, \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^x \cdot (\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x}\cdot\left[\begin{array}{c}\left(-\frac{1}{4}+\frac{I}{4}\right)\left(\cos\left(x\right)-I\sin\left(x\right)\right)\\ \frac{I}{2}\left(\cos\left(x\right)-I\sin\left(x\right)\right)\\ \left(\frac{1}{2}+\frac{I}{2}\right)\left(\cos\left(x\right)-I\sin\left(x\right)\right)\\ \cos\left(x\right)-I\sin\left(x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=e^{x}\cdot\left[\begin{array}{c}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\sin(x)}{2}\\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ \cos\left(x\right)\end{array}\right],\vec{y}_{4}(x)=e^{x}\cdot\left[\begin{array}{c}\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\\ \frac{\cos(x)}{2}\\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2}\\ -\sin\left(x\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}(x)+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}e^{x}\cdot\left[\begin{array}{c}1\\ 1\\ 1\\ 1\end{array}\right]+c_{2}e^{x}\cdot\left(x\cdot\left[\begin{array}{c}1\\ 1\\ 1\\ 1\end{array}\right]+\left[\begin{array}{c}-1\\ 0\\ 0\\ 0\end{array}\right]\right)+c_{3}e^{x}\cdot\left[\begin{array}{c}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\sin(x)}{2}\\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ \cos\left(x\right)\end{array}\right]+e^{x}c_{4}\cdot\left[\begin{array}{c}\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\\ \frac{\cos(x)}{2}\\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2}\\ -\sin\left(x\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\left(\frac{(-c_{3}+c_{4})\cos(x)}{4}+\frac{(c_{3}+c_{4})\sin(x)}{4}+\left(x-1\right)c_{2}+c_{1}\right)e^{x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+7*diff(y(x),x$2)-6*diff(y(x),x)+2*y(x)=0,y(x), singso
```

$$y(x) = e^{x}(c_1 + c_2 x + c_3 \sin(x) + c_4 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[y''''[x]-4*y'''[x]+7*y''[x]-6*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^x(c_4 x + c_2 \cos(x) + c_1 \sin(x) + c_3)$$

18.15 problem section 9.2, problem 15

Internal problem ID [1479]
Internal file name [OUTPUT/1480_Sunday_June_05_2022_02_19_14_AM_82956636/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' + 4y' - 8y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2, y''(0) = 2]$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{2x} \\
y_2 &= e^{2ix} \\
y_3 &= e^{-2ix}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + e^{2ix} c_2 + e^{-2ix} c_3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + 2i e^{2ix} c_2 - 2i e^{-2ix} c_3$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = 2c_2 i - 2c_3 i + 2c_1 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 4c_1 e^{2x} - 4\,e^{2ix} c_2 - 4\,e^{-2ix} c_3$$

substituting $y'' = 2$ and $x = 0$ in the above gives

$$2 = 4c_1 - 4c_2 - 4c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{5}{4} \\
c_2 &= \frac{3}{8} + \frac{9i}{8} \\
c_3 &= \frac{3}{8} - \frac{9i}{8}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{5\,e^{2x}}{4} + \frac{3\cos(2x)}{4} - \frac{9\sin(2x)}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{5\,e^{2x}}{4} + \frac{3\cos(2x)}{4} - \frac{9\sin(2x)}{4} \tag{1}$$

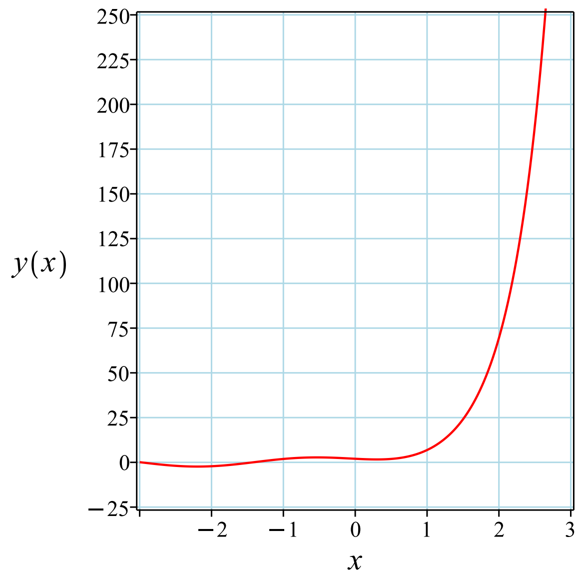


Figure 542: Solution plot

Verification of solutions

$$y = \frac{5\,e^{2x}}{4} + \frac{3\cos(2x)}{4} - \frac{9\sin(2x)}{4}$$

Verified OK.

18.15.1 Maple step by step solution

Let's solve

$$\left[y''' - 2y'' + 4y' - 8y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -2, y''\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 2y_3(x) - 4y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - 4y_2(x) + 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos \left(2x\right)-\mathrm{I} \sin \left(2x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(2x)}{4}+\frac{\mathrm{I} \sin(2x)}{4} \\ \frac{\mathrm{I}}{2}\left(\cos \left(2x\right)-\mathrm{I} \sin \left(2x\right)\right) \\ \cos \left(2x\right)-\mathrm{I} \sin \left(2x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_{2}\left(x\right) = \left[\begin{array}{c} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos \left(2x\right) \end{array}\right], \overrightarrow{y}_{3}\left(x\right) = \left[\begin{array}{c} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin \left(2x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x)$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_2 \cos(2x)}{4} + \frac{c_3 \sin(2x)}{4} \\ \frac{c_2 \sin(2x)}{2} + \frac{c_3 \cos(2x)}{2} \\ c_2 \cos(2x) - c_3 \sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = \frac{c_1 e^{2x}}{4} + \frac{c_3 \sin(2x)}{4} - \frac{c_2 \cos(2x)}{4}$

- Use the initial condition $y(0) = 2$

$2 = \frac{c_1}{4} - \frac{c_2}{4}$

- Calculate the 1st derivative of the solution

$y' = \frac{c_1 e^{2x}}{2} + \frac{c_3 \cos(2x)}{2} + \frac{c_2 \sin(2x)}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -2$

$-2 = \frac{c_1}{2} + \frac{c_3}{2}$

- Calculate the 2nd derivative of the solution

$y'' = c_1 e^{2x} - c_3 \sin(2x) + c_2 \cos(2x)$

- Use the initial condition $y''\Big|_{\{x=0\}} = 2$

$2 = c_1 + c_2$

- Solve for the unknown coefficients

$\{c_1 = 5, c_2 = -3, c_3 = -9\}$

- Solution to the IVP

$y = \frac{5 e^{2x}}{4} + \frac{3 \cos(2x)}{4} - \frac{9 \sin(2x)}{4}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$3)-2*diff(y(x),x$2)+4*diff(y(x),x)-8*y(x)=0,y(0) = 2, D(y)(0) = -2, (D@@
```

$$y(x) = \frac{5\,e^{2x}}{4} - \frac{9\sin(2x)}{4} + \frac{3\cos(2x)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 21

```
DSolve[{y'''[x]-2*y''[x]+4*y'[x]-8*y[x]==0,{y[0]==2,y'[0]==-2,y''[0]==0}},y[x],x,IncludeSing
```

$$y(x) \to e^{2x} - 2\sin(2x) + \cos(2x)$$

18.16 problem section 9.2, problem 16

Internal problem ID [1480]
Internal file name [`OUTPUT/1481_Sunday_June_05_2022_02_19_16_AM_55814989/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - y' - 3y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 14, y''(0) = -40]$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{-3x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - 3\,\mathrm{e}^{-3x} c_3
$$

substituting $y' = 14$ and $x = 0$ in the above gives

$$
14 = -c_1 + c_2 - 3c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 9\,\mathrm{e}^{-3x} c_3
$$

substituting $y'' = -40$ and $x = 0$ in the above gives

$$
-40 = c_1 + c_2 + 9c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 2 \\
c_3 &= -5
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x} - 5\,\mathrm{e}^{-3x}
$$

Summary

The solution(s) found are the following

$$y = 3\,e^{-x} + 2\,e^{x} - 5\,e^{-3x} \tag{1}$$

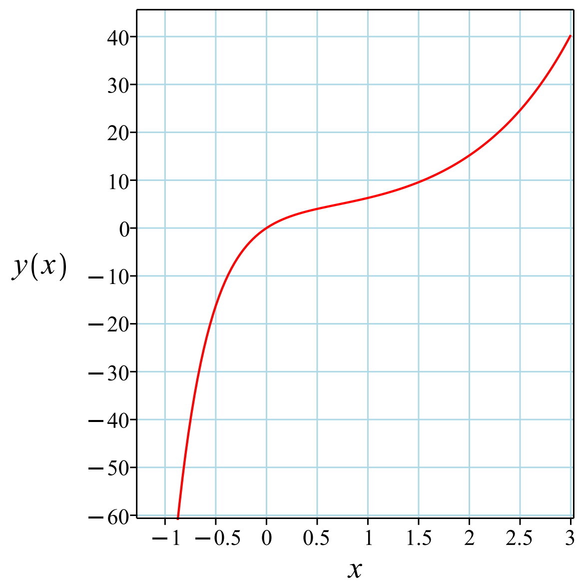


Figure 543: Solution plot

Verification of solutions

$$y = 3\,e^{-x} + 2\,e^{x} - 5\,e^{-3x}$$

Verified OK.

18.16.1 Maple step by step solution

Let's solve

$$\left[y''' + 3y'' - y' - 3y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 14, y''\Big|_{\{x=0\}} = -40\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -3y_3(x) + y_2(x) + 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + y_2(x) + 3y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{3} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_{1}\overrightarrow{y}_{1} + c_{2}\overrightarrow{y}_{2} + c_{3}\overrightarrow{y}_{3}$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(9\,e^{4x}c_3 + 9c_2 e^{2x} + c_1)\,e^{-3x}}{9}$
- Use the initial condition $y(0) = 0$

  $0 = c_3 + c_2 + \frac{c_1}{9}$
- Calculate the 1st derivative of the solution

  $y' = \frac{(36\,e^{4x}c_3 + 18c_2 e^{2x})\,e^{-3x}}{9} - \frac{(9\,e^{4x}c_3 + 9c_2 e^{2x} + c_1)\,e^{-3x}}{3}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 14$

  $14 = c_3 - c_2 - \frac{c_1}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{(144\,e^{4x}c_3 + 36c_2 e^{2x})\,e^{-3x}}{9} - \frac{2(36\,e^{4x}c_3 + 18c_2 e^{2x})\,e^{-3x}}{3} + (9\,e^{4x}c_3 + 9c_2 e^{2x} + c_1)\,e^{-3x}$
- Use the initial condition $y''\Big|_{\{x=0\}} = -40$

  $-40 = c_1 + c_2 + c_3$
- Solve for the unknown coefficients

  $\{c_1 = -45, c_2 = 3, c_3 = 2\}$
- Solution to the IVP

  $y = (2\,e^{4x} + 3\,e^{2x} - 5)\,e^{-3x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$3)+3*diff(y(x),x$2)-diff(y(x),x)-3*y(x)=0,y(0) = 0, D(y)(0) = 14, (D@@2)
```

$$y(x) = \left(2\,e^{4x} + 3\,e^{2x} - 5\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[{y'''[x]+3*y''[x]-y'[x]-3*y[x]==0,{y[0]==0,y'[0]==14,y''[0]==-40}},y[x],x,IncludeSing
```

$$y(x) \to -5e^{-3x} + 3e^{-x} + 2e^{x}$$

18.17 problem section 9.2, problem 17

Internal problem ID [1481]
Internal file name [OUTPUT/1482_Sunday_June_05_2022_02_19_17_AM_40863832/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' - y' + y = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = 9, y''(0) = 4]$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= x\,\mathrm{e}^{x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$
-2 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3
$$

substituting $y' = 9$ and $x = 0$ in the above gives

$$
9 = -c_1 + c_2 + c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 2c_3 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3
$$

substituting $y'' = 4$ and $x = 0$ in the above gives

$$
4 = c_1 + c_2 + 2c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -4 \\
c_2 &= 2 \\
c_3 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}
$$

Which simplifies to

$$y = -4\,e^{-x} + e^{x}(3x+2)$$

Summary

The solution(s) found are the following

$$y = -4\,e^{-x} + e^{x}(3x+2) \tag{1}$$

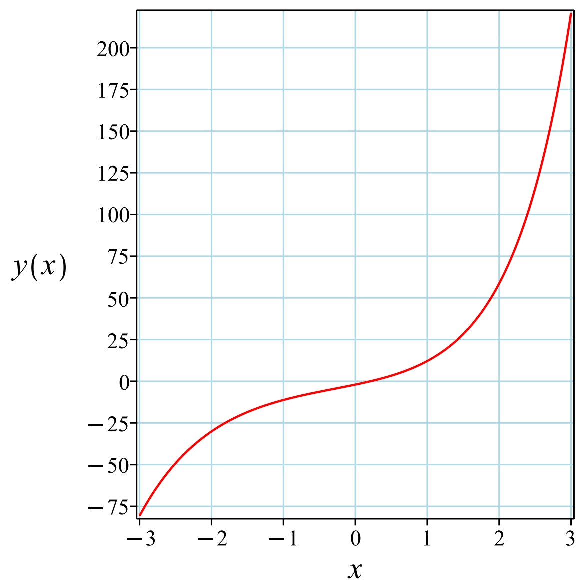


Figure 544: Solution plot

Verification of solutions

$$y = -4\,e^{-x} + e^{x}(3x+2)$$

Verified OK.

18.17.1 Maple step by step solution

Let's solve

$$\left[y''' - y'' - y' + y = 0, y(0) = -2, y'\Big|_{\{x=0\}} = 9, y''\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = y_3(x) + y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) + y_2(x) - y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_{3}(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{3}(x)$ into the homogeneous system

$$\lambda\, \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\, \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \mathrm{e}^{\lambda x} \left(\lambda x \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{3}(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1\end{array}\right]-1\cdot\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c}-1 \\ 0 \\ 0\end{array}\right]$$

- Second solution from eigenvalue 1

$$\vec{y}_3(x)=\mathrm{e}^x\cdot\left(x\cdot\left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-x}\cdot\left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]+c_2\mathrm{e}^x\cdot\left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]+c_3\mathrm{e}^x\cdot\left(x\cdot\left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0 \\ 0\end{array}\right]\right)$$

- First component of the vector is the solution to the ODE

$$y=c_1\mathrm{e}^{-x}+\left((x-1)\,c_3+c_2\right)\mathrm{e}^x$$

- Use the initial condition $y(0)=-2$

$$-2=c_1+c_2-c_3$$

- Calculate the 1st derivative of the solution

$$y'=-c_1\mathrm{e}^{-x}+c_3\mathrm{e}^x+\left((x-1)\,c_3+c_2\right)\mathrm{e}^x$$

- Use the initial condition $y'\Big|_{\{x=0\}}=9$

$$9=-c_1+c_2$$

- Calculate the 2nd derivative of the solution

$$y'' = c_1 \mathrm{e}^{-x} + 2c_3 \mathrm{e}^{x} + ((x-1)\, c_3 + c_2)\, \mathrm{e}^{x}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 4$

$$4 = c_1 + c_2 + c_3$$

- Solve for the unknown coefficients

$$\{c_1 = -4, c_2 = 5, c_3 = 3\}$$

- Solution to the IVP

$$y = -4\,\mathrm{e}^{-x} + \mathrm{e}^{x}(3x+2)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 20

```
dsolve([diff(y(x),x$3)-diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(0) = -2, D(y)(0) = 9, (D@@2)(y)(
```

$$y(x) = -4\,\mathrm{e}^{-x} + (3x+2)\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
DSolve[{y'''[x]-y''[x]-y'[x]+y[x]==0,{y[0]==-2,y'[0]==9,y''[0]==4}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^{x}(3x+2) - 4e^{-x}$$

18.18 problem section 9.2, problem 18

Internal problem ID [1482]
Internal file name [`OUTPUT/1483_Sunday_June_05_2022_02_19_19_AM_45254044/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 18.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y' - 4y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = 3, y''(0) = 22]$$

The characteristic equation is

$$\lambda^3 - 2\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1 - i$$
$$\lambda_3 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(-1-i)x}c_1 + c_2e^{2x} + e^{(-1+i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{(-1-i)x} \\
y_2 &= e^{2x} \\
y_3 &= e^{(-1+i)x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{(-1-i)x}c_1 + c_2 e^{2x} + e^{(-1+i)x}c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$
6 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = (-1-i)\,e^{(-1-i)x}c_1 + 2c_2 e^{2x} + (-1+i)\,e^{(-1+i)x}c_3
$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$
3 = (-1-i)\,c_1 + 2c_2 + (-1+i)\,c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 2ie^{(-1-i)x}c_1 + 4c_2 e^{2x} - 2ie^{(-1+i)x}c_3
$$

substituting $y'' = 22$ and $x = 0$ in the above gives

$$
22 = 2c_1 i - 2c_3 i + 4c_2 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 - \frac{3i}{2} \\
c_2 &= 4 \\
c_3 &= 1 + \frac{3i}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = e^{(-1-i)x} - \frac{3ie^{(-1-i)x}}{2} + 4\,e^{2x} + e^{(-1+i)x} + \frac{3ie^{(-1+i)x}}{2}
$$

Summary

The solution(s) found are the following

$$y = \left(1-\frac{3i}{2}\right) \mathrm{e}^{(-1-i)x}+\left(1+\frac{3i}{2}\right) \mathrm{e}^{(-1+i)x}+4\,\mathrm{e}^{2x} \tag{1}$$

Verification of solutions

$$y = \left(1-\frac{3i}{2}\right) \mathrm{e}^{(-1-i)x}+\left(1+\frac{3i}{2}\right) \mathrm{e}^{(-1+i)x}+4\,\mathrm{e}^{2x}$$

Verified OK.

18.18.1 Maple step by step solution

Let's solve

$\left[y'''-2y'-4y=0, y(0)=6, y'\Big|_{\{x=0\}}=3, y''\Big|_{\{x=0\}}=22\right]$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x)=y$
  - Define new variable $y_2(x)$

    $y_2(x)=y'$
  - Define new variable $y_3(x)$

    $y_3(x)=y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x)=2y_2(x)+4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x)=y_1'(x), y_3(x)=y_2'(x), y_3'(x)=2y_2(x)+4y_1(x)]$

- Define vector

  $\vec{y}(x)=\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1-\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-1-I)x} \cdot \left[\begin{array}{c} -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-x} \cdot \left[\begin{array}{c} -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-x} \cdot \left[\begin{array}{c} -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_3(x) = e^{-x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + c_2 e^{-x} \cdot \left[\begin{array}{c} -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right] + c_3 e^{-x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-2c_2\sin(x) - 2c_3\cos(x))e^{-x}}{4} + \frac{c_1 e^{2x}}{4}$$

- Use the initial condition $y(0) = 6$

$$6 = -\frac{c_3}{2} + \frac{c_1}{4}$$

- Calculate the 1st derivative of the solution

$$y' = \frac{(-2c_2\cos(x)+2c_3\sin(x))\,\mathrm{e}^{-x}}{4} - \frac{(-2c_2\sin(x)-2c_3\cos(x))\,\mathrm{e}^{-x}}{4} + \frac{c_1\mathrm{e}^{2x}}{2}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

$$3 = -\frac{c_2}{2} + \frac{c_3}{2} + \frac{c_1}{2}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{(2c_2\sin(x)+2c_3\cos(x))\,\mathrm{e}^{-x}}{4} - \frac{(-2c_2\cos(x)+2c_3\sin(x))\,\mathrm{e}^{-x}}{2} + \frac{(-2c_2\sin(x)-2c_3\cos(x))\,\mathrm{e}^{-x}}{4} + c_1\mathrm{e}^{2x}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 22$

$$22 = c_1 + c_2$$

- Solve for the unknown coefficients

$$\{c_1 = 16, c_2 = 6, c_3 = -4\}$$

- Solution to the IVP

$$y = (-3\sin(x) + 2\cos(x))\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{2x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve([diff(y(x),x$3)-2*diff(y(x),x)-4*y(x)=0,y(0) = 6, D(y)(0) = 3, (D@@2)(y)(0) = 22],y(x
```

$$y(x) = 4\,\mathrm{e}^{2x} + (-3\sin(x) + 2\cos(x))\,\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[{y'''[x]-2*y'[x]-4*y[x]==0,{y[0]==6,y'[0]==3,y''[0]==22}},y[x],x,IncludeSingularSolut
```

$$y(x) \to e^{-x}\left(4e^{3x} - 3\sin(x) + 2\cos(x)\right)$$

18.19 problem section 9.2, problem 19

Internal problem ID [1483]
Internal file name [OUTPUT/1484_Sunday_June_05_2022_02_19_21_AM_92724626/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{3y''' - y'' - 7y' + 5y = 0}$$

With initial conditions

$$\left[y(0) = \frac{14}{5}, y'(0) = 0, y''(0) = 10\right]$$

The characteristic equation is

$$3\lambda^3 - \lambda^2 - 7\lambda + 5 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{5}{3}$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x e^x + e^{-\frac{5x}{3}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= x\,\mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{-\frac{5x}{3}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{-\frac{5x}{3}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{14}{5}$ and $x = 0$ in the above gives

$$
\frac{14}{5} = c_1 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} - \frac{5\,\mathrm{e}^{-\frac{5x}{3}} c_3}{3}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 - \frac{5c_3}{3} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{x} + 2c_2 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \frac{25\,\mathrm{e}^{-\frac{5x}{3}} c_3}{9}
$$

substituting $y'' = 10$ and $x = 0$ in the above gives

$$
10 = c_1 + 2c_2 + \frac{25c_3}{9} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 2 \\
c_3 &= \frac{9}{5}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = 2x\,e^{x} + \frac{9\,e^{-\frac{5x}{3}}}{5} + e^{x}$$

Summary
The solution(s) found are the following

$$y = 2x\,e^{x} + \frac{9\,e^{-\frac{5x}{3}}}{5} + e^{x} \tag{1}$$

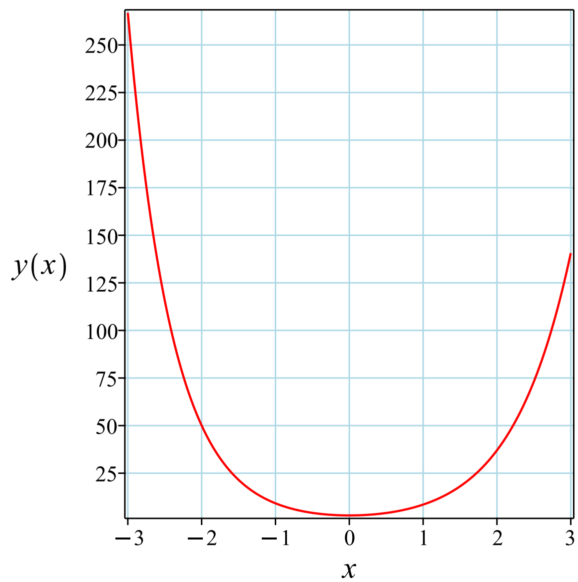


Figure 545: Solution plot

Verification of solutions

$$y = 2x\,e^{x} + \frac{9\,e^{-\frac{5x}{3}}}{5} + e^{x}$$

Verified OK.

18.19.1 Maple step by step solution

Let's solve

$$\left[3y''' - y'' - 7y' + 5y = 0, y(0) = \frac{14}{5}, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 10\right]$$

- Highest derivative means the order of the ODE is 3

y'''

- Isolate 3rd derivative

$y''' = \frac{y''}{3} + \frac{7y'}{3} - \frac{5y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' - \frac{y''}{3} - \frac{7y'}{3} + \frac{5y}{3} = 0$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{y_3(x)}{3} + \frac{7y_2(x)}{3} - \frac{5y_1(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{y_3(x)}{3} + \frac{7y_2(x)}{3} - \frac{5y_1(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{3} & \frac{7}{3} & \frac{1}{3} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{3} & \frac{7}{3} & \frac{1}{3} \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-\frac{5}{3}, \left[\begin{array}{c} \frac{9}{25} \\ -\frac{3}{5} \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$
- Consider eigenpair

  $$\left[-\frac{5}{3}, \left[\begin{array}{c} \frac{9}{25} \\ -\frac{3}{5} \\ 1 \end{array}\right]\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \mathrm{e}^{-\frac{5x}{3}} \cdot \left[\begin{array}{c} \frac{9}{25} \\ -\frac{3}{5} \\ 1 \end{array}\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$
- First solution from eigenvalue 1

  $$\overrightarrow{y}_2(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$
- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

  $\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$
- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{5}{3} & \frac{7}{3} & \frac{1}{3}\end{bmatrix}-1\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_3(x)=\mathrm{e}^{x}\cdot\left(x\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-\frac{5x}{3}}\cdot\begin{bmatrix}\frac{9}{25}\\ -\frac{3}{5}\\ 1\end{bmatrix}+c_2\mathrm{e}^{x}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+c_3\mathrm{e}^{x}\cdot\left(x\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}\right)$$

- First component of the vector is the solution to the ODE

$$y = \left(\left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}+\frac{9 c_1}{25}\right) {\mathrm e}^{-\frac{5 x}{3}}$$

- Use the initial condition $y \left(0\right)=\frac{14}{5}$

$$\frac{14}{5}=c_2-c_3+\frac{9 c_1}{25}$$

- Calculate the 1st derivative of the solution

$$y^{\prime}=\left(c_3 {\mathrm e}^{\frac{8 x}{3}}+\frac{8 \left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}}{3}\right) {\mathrm e}^{-\frac{5 x}{3}}-\frac{5 \left(\left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}+\frac{9 c_1}{25}\right) {\mathrm e}^{-\frac{5 x}{3}}}{3}$$

- Use the initial condition $y^{\prime}{\Big|}_{\left\{x =0\right\}}=0$

$$0=c_2-\frac{3 c_1}{5}$$

- Calculate the 2nd derivative of the solution

$$y^{\prime \prime}=\left(\frac{16 c_3 {\mathrm e}^{\frac{8 x}{3}}}{3}+\frac{64 \left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}}{9}\right) {\mathrm e}^{-\frac{5 x}{3}}-\frac{10 \left(c_3 {\mathrm e}^{\frac{8 x}{3}}+\frac{8 \left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}}{3}\right) {\mathrm e}^{-\frac{5 x}{3}}}{3}+\frac{25 \left(\left(\left(x-1\right) c_3+c_2\right) {\mathrm e}^{\frac{8 x}{3}}+\frac{9 c_1}{25}\right) {\mathrm e}^{-\frac{5 x}{3}}}{9}$$

- Use the initial condition $y^{\prime \prime}{\Big|}_{\left\{x =0\right\}}=10$

$$10=c_1+c_2+c_3$$

- Solve for the unknown coefficients

$$\left\{c_1 =5, c_2 =3, c_3 =2\right\}$$

- Solution to the IVP

$$y = {\mathrm e}^{-\frac{5 x}{3}} \left(\frac{9}{5}+\left(1+2 x \right) {\mathrm e}^{\frac{8 x}{3}}\right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([3*diff(y(x),x$3)-diff(y(x),x$2)-7*diff(y(x),x)+5*y(x)=0,y(0) = 14/5, D(y)(0) = 0, (D
```

$$y(x) = e^{-\frac{5x}{3}} \left(\frac{9}{5} + (2x+1)\, e^{\frac{8x}{3}}\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[{3*y'''[x]-y''[x]-7*y'[x]+5*y[x]==0,{y[0]==14/5,y'[0]==0,y''[0]==10}},y[x],x,IncludeS
```

$$y(x) \to e^x(2x+1) + \frac{9}{5}e^{-5x/3}$$

18.20 problem section 9.2, problem 20

Internal problem ID [1484]
Internal file name [`OUTPUT/1485_Sunday_June_05_2022_02_19_22_AM_88452520/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1, y''(0) = -4]$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 x\, e^{2x} + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= x\,\mathrm{e}^{2x} \\
y_3 &= x^2\mathrm{e}^{2x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1\mathrm{e}^{2x} + c_2 x\,\mathrm{e}^{2x} + x^2\mathrm{e}^{2x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{2x} + 2c_2 x\,\mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} c_3 + 2x^2\mathrm{e}^{2x} c_3
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = 2c_1 + c_2 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 4c_1\mathrm{e}^{2x} + 4c_2\mathrm{e}^{2x} + 4c_2 x\,\mathrm{e}^{2x} + 2c_3\mathrm{e}^{2x} + 8x\,\mathrm{e}^{2x} c_3 + 4x^2\mathrm{e}^{2x} c_3
$$

substituting $y'' = -4$ and $x = 0$ in the above gives

$$
-4 = 4c_1 + 4c_2 + 2c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -3 \\
c_3 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2x^2\mathrm{e}^{2x} - 3x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}
$$

Which simplifies to

$$y = e^{2x}(2x^2 - 3x + 1)$$

Summary

The solution(s) found are the following

$$y = e^{2x}(2x^2 - 3x + 1) \tag{1}$$

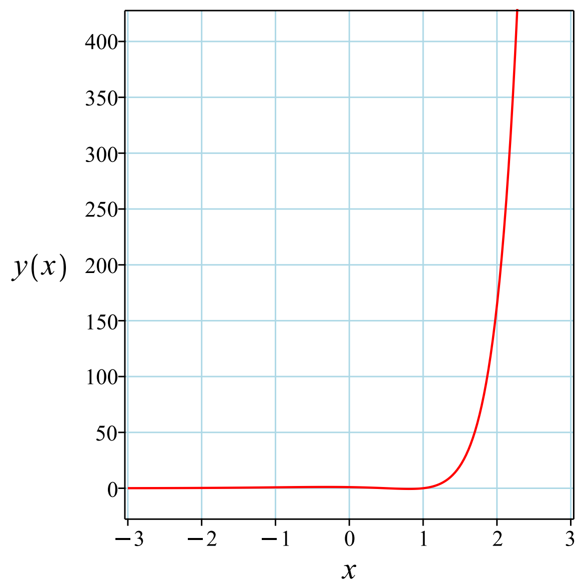


Figure 546: Solution plot

Verification of solutions

$$y = e^{2x}(2x^2 - 3x + 1)$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=0,y(0) = 1, D(y)(0) = -1, (D@
```

$$y(x) = e^{2x}(2x^2 - 3x + 1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 21

```
DSolve[{y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==0,{y[0]==1,y'[0]==-1,y''[0]==-4}},y[x],x,IncludeSi
```

$$y(x) \to e^{2x}(2x^2 - 3x + 1)$$

18.21 problem section 9.2, problem 21

Internal problem ID [1485]
Internal file name [`OUTPUT/1486_Sunday_June_05_2022_02_19_24_AM_93199483/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 21.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - 11y'' + 12y' + 9y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = 3, y''(0) = 13]$$

The characteristic equation is

$$2\lambda^3 - 11\lambda^2 + 12\lambda + 9 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = 3$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{3x} + c_2 x\, e^{3x} + e^{-\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{3x} \\
y_2 &= x\,\mathrm{e}^{3x} \\
y_3 &= \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} + \mathrm{e}^{-\frac{x}{2}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$
6 = c_1 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 3c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} + 3c_2 x\,\mathrm{e}^{3x} - \frac{\mathrm{e}^{-\frac{x}{2}} c_3}{2}
$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$
3 = 3c_1 + c_2 - \frac{c_3}{2} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 9c_1 \mathrm{e}^{3x} + 6c_2 \mathrm{e}^{3x} + 9c_2 x\,\mathrm{e}^{3x} + \frac{\mathrm{e}^{-\frac{x}{2}} c_3}{4}
$$

substituting $y'' = 13$ and $x = 0$ in the above gives

$$
13 = 9c_1 + 6c_2 + \frac{c_3}{4} \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -1 \\
c_3 &= 4
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -x\,\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x} + 4\,\mathrm{e}^{-\frac{x}{2}}
$$

Which simplifies to

$$y = 4\,e^{-\frac{x}{2}} + (2-x)\,e^{3x}$$

Summary

The solution(s) found are the following

$$y = 4\,e^{-\frac{x}{2}} + (2-x)\,e^{3x} \tag{1}$$

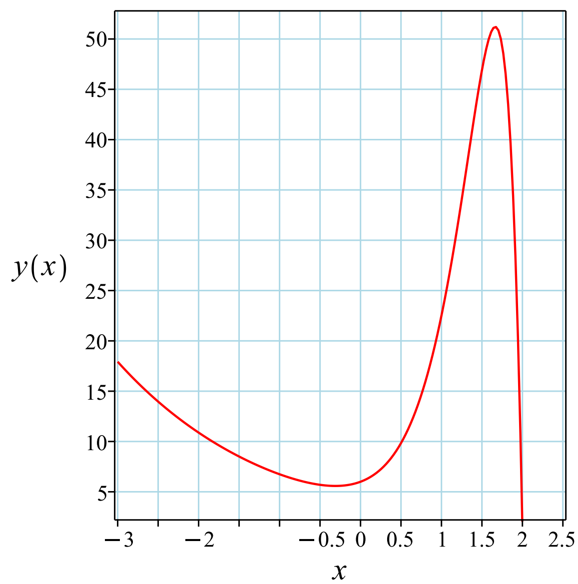


Figure 547: Solution plot

Verification of solutions

$$y = 4\,e^{-\frac{x}{2}} + (2-x)\,e^{3x}$$

Verified OK.

18.21.1 Maple step by step solution

Let's solve

$$\left[2y''' - 11y'' + 12y' + 9y = 0, y(0) = 6, y'\Big|_{\{x=0\}} = 3, y''\Big|_{\{x=0\}} = 13\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

$y''' = \frac{11y''}{2} - 6y' - \frac{9y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{11y''}{2} + 6y' + \frac{9y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{11y_3(x)}{2} - 6y_2(x) - \frac{9y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{11y_3(x)}{2} - 6y_2(x) - \frac{9y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{2} & -6 & \frac{11}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{2} & -6 & \frac{11}{2} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 3

$$\overrightarrow{y}_2(x) = \mathrm{e}^{3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{9}{2} & -6 & \frac{11}{2} \end{array}\right]-3\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c} -\frac{1}{27} \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3

$$\vec{y}_3(x)=\mathrm{e}^{3x}\cdot\left(x\cdot\left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{27} \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-\frac{x}{2}}\cdot\left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]+c_2\mathrm{e}^{3x}\cdot\left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]+c_3\mathrm{e}^{3x}\cdot\left(x\cdot\left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{27} \\ 0 \\ 0 \end{array}\right]\right)$$

- First component of the vector is the solution to the ODE

$$y=4c_1\mathrm{e}^{-\frac{x}{2}}+\frac{\left(\left(x-\frac{1}{3}\right)c_3+c_2\right)\mathrm{e}^{3x}}{9}$$

- Use the initial condition $y(0)=6$

$6 = 4c_1 - \frac{c_3}{27} + \frac{c_2}{9}$

- Calculate the 1st derivative of the solution

  $y' = -2c_1 e^{-\frac{x}{2}} + \frac{c_3 e^{3x}}{9} + \frac{\left(\left(x-\frac{1}{3}\right)c_3+c_2\right)e^{3x}}{3}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = -2c_1 + \frac{c_2}{3}$

- Calculate the 2nd derivative of the solution

  $y'' = c_1 e^{-\frac{x}{2}} + \frac{2c_3 e^{3x}}{3} + \left(\left(x-\frac{1}{3}\right)c_3 + c_2\right)e^{3x}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 13$

  $13 = c_1 + \frac{c_3}{3} + c_2$

- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = 15, c_3 = -9\}$

- Solution to the IVP

  $y = 4\,e^{-\frac{x}{2}} + (2-x)\,e^{3x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve([2*diff(y(x),x$3)-11*diff(y(x),x$2)+12*diff(y(x),x)+9*y(x)=0,y(0) = 6, D(y)(0) = 3, (
```

$$y(x) = 4\,e^{-\frac{x}{2}} + (2-x)\,e^{3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[{2*y'''[x]-11*y''[x]+12*y'[x]+9*y[x]==0,{y[0]==6,y'[0]==3,y''[0]==13}},y[x],x,Include
```

$$y(x) \to 4e^{-x/2} - e^{3x}(x - 2)$$

18.22 problem section 9.2, problem 22

Internal problem ID [1486]
Internal file name [`OUTPUT/1487_Sunday_June_05_2022_02_19_25_AM_18782849/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{8y''' - 4y'' - 2y' + y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = -3, y''(0) = -1]$$

The characteristic equation is

$$8\lambda^3 - 4\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}} + x\,\mathrm{e}^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-\frac{x}{2}} \\
y_2 &= \mathrm{e}^{\frac{x}{2}} \\
y_3 &= \mathrm{e}^{\frac{x}{2}} x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}} + x\,\mathrm{e}^{\frac{x}{2}} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$
4 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{2} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{2} + c_3 \mathrm{e}^{\frac{x}{2}} + \frac{x\,\mathrm{e}^{\frac{x}{2}} c_3}{2}
$$

substituting $y' = -3$ and $x = 0$ in the above gives

$$
-3 = -\frac{c_1}{2} + \frac{c_2}{2} + c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = \frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{4} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{4} + c_3 \mathrm{e}^{\frac{x}{2}} + \frac{x\,\mathrm{e}^{\frac{x}{2}} c_3}{4}
$$

substituting $y'' = -1$ and $x = 0$ in the above gives

$$
-1 = \frac{c_1}{4} + \frac{c_2}{4} + c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 1 \\
c_3 &= -2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -2\,\mathrm{e}^{\frac{x}{2}} x + 3\,\mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{\frac{x}{2}}
$$

Summary

The solution(s) found are the following

$$y = -2\,e^{\frac{x}{2}}x + 3\,e^{-\frac{x}{2}} + e^{\frac{x}{2}} \tag{1}$$

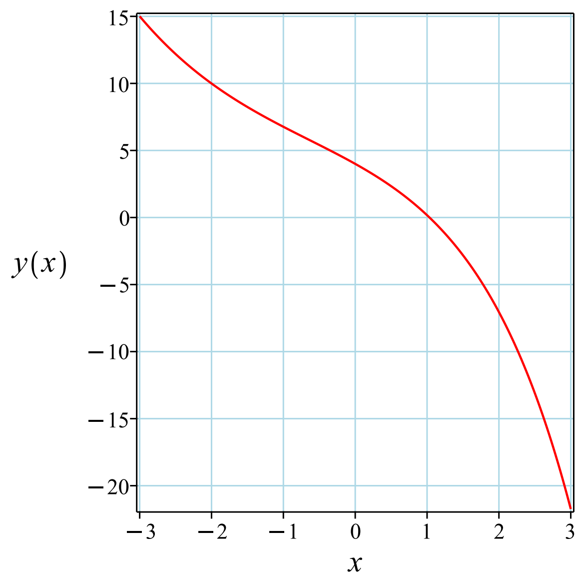


Figure 548: Solution plot

Verification of solutions

$$y = -2\,e^{\frac{x}{2}}x + 3\,e^{-\frac{x}{2}} + e^{\frac{x}{2}}$$

Verified OK.

18.22.1 Maple step by step solution

Let's solve

$\left[8y''' - 4y'' - 2y' + y = 0, y(0) = 4, y'\Big|_{\{x=0\}} = -3, y''\Big|_{\{x=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''}{2} + \frac{y'}{4} - \frac{y}{8}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{2} - \frac{y'}{4} + \frac{y}{8} = 0$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{y_3(x)}{2} + \frac{y_2(x)}{4} - \frac{y_1(x)}{8}$

    Convert linear ODE into a system of first order ODEs

    $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{y_3(x)}{2} + \frac{y_2(x)}{4} - \frac{y_1(x)}{8}\right]$
- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right], \left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right], \left[\frac{1}{2}, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} \end{array}\right]-\frac{1}{2}\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_3(x)=\mathrm{e}^{\frac{x}{2}}\cdot\left(x\cdot\left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-\frac{x}{2}}\cdot\left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]+c_2\mathrm{e}^{\frac{x}{2}}\cdot\left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]+c_3\mathrm{e}^{\frac{x}{2}}\cdot\left(x\cdot\left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]\right)$$

- First component of the vector is the solution to the ODE

$$y=\left(\left(4x-8\right)c_3+4c_2\right)\mathrm{e}^{\frac{x}{2}}+4c_1\mathrm{e}^{-\frac{x}{2}}$$

- Use the initial condition $y(0)=4$

$$4=-8c_3+4c_2+4c_1$$

- Calculate the 1st derivative of the solution

$$y' = 4c_3 e^{\frac{x}{2}} + \frac{((4x-8)c_3 + 4c_2) e^{\frac{x}{2}}}{2} - 2c_1 e^{-\frac{x}{2}}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = -3$

$$-3 = 2c_2 - 2c_1$$

- Calculate the 2nd derivative of the solution

$$y'' = 4c_3 e^{\frac{x}{2}} + \frac{((4x-8)c_3 + 4c_2) e^{\frac{x}{2}}}{4} + c_1 e^{-\frac{x}{2}}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = -1$

$$-1 = c_1 + c_2 + 2c_3$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{3}{4}, c_2 = -\frac{3}{4}, c_3 = -\frac{1}{2}\right\}$$

- Solution to the IVP

$$y = -2\,e^{\frac{x}{2}} x + 3\,e^{-\frac{x}{2}} + e^{\frac{x}{2}}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([8*diff(y(x),x$3)-4*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(0) = 4, D(y)(0) = -3, (D@@
```

$$y(x) = 3\,e^{-\frac{x}{2}} + e^{\frac{x}{2}} - 2\,e^{\frac{x}{2}} x$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 57

```
DSolve[{8*y'''[x]-4*y''[x]-2*y'[x]-2*y[x]==0,{y[0]==4,y'[0]==-3,y''[0]==-1}},y[x],x,IncludeS
```

$$y(x) \to -\frac{2}{21}e^{-x/4}\left(9e^{5x/4} + 13\sqrt{3}\sin\left(\frac{\sqrt{3}x}{4}\right) - 51\cos\left(\frac{\sqrt{3}x}{4}\right)\right)$$

18.23 problem section 9.2, problem 23

Internal problem ID [1487]
Internal file name [OUTPUT/1488_Sunday_June_05_2022_02_19_27_AM_35409308/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 23.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 16y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 2, y''(0) = -2, y'''(0) = 0]$$

The characteristic equation is

$$\lambda^4 - 16 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{2ix} \\
y_4 &= \mathrm{e}^{-2ix}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + 2c_2 \mathrm{e}^{2x} + 2i \mathrm{e}^{2ix} c_3 - 2i \mathrm{e}^{-2ix} c_4
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = 2c_3 i - 2c_4 i - 2c_1 + 2c_2 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 4c_1 \mathrm{e}^{-2x} + 4c_2 \mathrm{e}^{2x} - 4\,\mathrm{e}^{2ix} c_3 - 4\,\mathrm{e}^{-2ix} c_4
$$

substituting $y'' = -2$ and $x = 0$ in the above gives

$$
-2 = 4c_1 + 4c_2 - 4c_3 - 4c_4 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -8c_1 \mathrm{e}^{-2x} + 8c_2 \mathrm{e}^{2x} - 8i \mathrm{e}^{2ix} c_3 + 8i \mathrm{e}^{-2ix} c_4
$$

substituting $y''' = 0$ and $x = 0$ in the above gives

$$
0 = -8c_3 i + 8c_4 i - 8c_1 + 8c_2 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = \frac{1}{8}$$
$$c_2 = \frac{5}{8}$$
$$c_3 = \frac{5}{8} - \frac{i}{4}$$
$$c_4 = \frac{5}{8} + \frac{i}{4}$$

Substituting these values back in above solution results in

$$y = \frac{e^{-2x}}{8} + \frac{5\,e^{2x}}{8} + \frac{5\cos(2x)}{4} + \frac{\sin(2x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}}{8} + \frac{5\,e^{2x}}{8} + \frac{5\cos(2x)}{4} + \frac{\sin(2x)}{2} \tag{1}$$

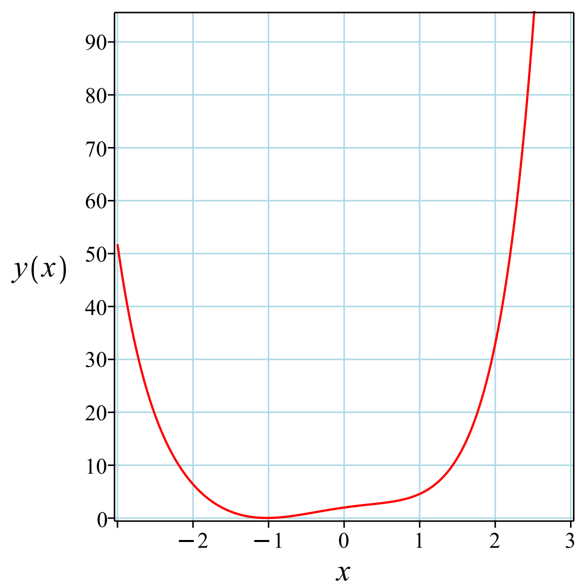


Figure 549: Solution plot

Verification of solutions

$$y = \frac{e^{-2x}}{8} + \frac{5\,e^{2x}}{8} + \frac{5\cos(2x)}{4} + \frac{\sin(2x)}{2}$$

Verified OK.

18.23.1 Maple step by step solution

Let's solve

$$\left[y'''' - 16y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 2, y''\Big|_{\{x=0\}} = -2, y'''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3\sin(2x)}{8} - \frac{c_4\cos(2x)}{8} \\ -\frac{c_3\cos(2x)}{4} + \frac{c_4\sin(2x)}{4} \\ \frac{c_3\sin(2x)}{2} + \frac{c_4\cos(2x)}{2} \\ c_3\cos(2x) - c_4\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = -\frac{c_1 e^{-2x}}{8} + \frac{c_2 e^{2x}}{8} - \frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8}$

- Use the initial condition $y(0) = 2$

$2 = -\frac{c_1}{8} + \frac{c_2}{8} - \frac{c_4}{8}$

- Calculate the 1st derivative of the solution

$y' = \frac{c_1 e^{-2x}}{4} + \frac{c_2 e^{2x}}{4} + \frac{c_4\sin(2x)}{4} - \frac{c_3\cos(2x)}{4}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 2$

$2 = \frac{c_1}{4} + \frac{c_2}{4} - \frac{c_3}{4}$

- Calculate the 2nd derivative of the solution

$y'' = -\frac{c_1 e^{-2x}}{2} + \frac{c_2 e^{2x}}{2} + \frac{c_4\cos(2x)}{2} + \frac{c_3\sin(2x)}{2}$

- Use the initial condition $y''\Big|_{\{x=0\}} = -2$

$-2 = -\frac{c_1}{2} + \frac{c_2}{2} + \frac{c_4}{2}$

- Calculate the 3rd derivative of the solution

$y''' = c_1 e^{-2x} + c_2 e^{2x} - c_4\sin(2x) + c_3\cos(2x)$

- Use the initial condition $y'''\Big|_{\{x=0\}} = 0$

$0 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $$\{c_1 = -1, c_2 = 5, c_3 = -4, c_4 = -10\}$$

- Solution to the IVP

  $$y = \frac{e^{-2x}}{8} + \frac{5 e^{2x}}{8} + \frac{5\cos(2x)}{4} + \frac{\sin(2x)}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve([diff(y(x),x$4)-16*y(x)=0,y(0) = 2, D(y)(0) = 2, (D@@2)(y)(0) = -2, (D@@3)(y)(0) = 0]
```

$$y(x) = \frac{5\,e^{2x}}{8} + \frac{e^{-2x}}{8} + \frac{\sin(2x)}{2} + \frac{5\cos(2x)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 34

```
DSolve[{y''''[x]-16*y[x]==0,{y[0]==2,y'[0]==2,y''[0]==-2,y'''[0]==0}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{8}\left(e^{-2x} + 5e^{2x} + 4\sin(2x) + 10\cos(2x)\right)$$

18.24 problem section 9.2, problem 24

Internal problem ID [1488]
Internal file name [`OUTPUT/1489_Sunday_June_05_2022_02_19_29_AM_18528297/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 24.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 6y''' + 7y'' + 6y' - 8y = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = -8, y''(0) = -14, y'''(0) = -62]$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 7\lambda^2 + 6\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 4$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 e^{4x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{2x} \\
y_4 &= \mathrm{e}^{4x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{2x} + c_4 \mathrm{e}^{4x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$
-2 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 2c_3 \mathrm{e}^{2x} + 4c_4 \mathrm{e}^{4x}
$$

substituting $y' = -8$ and $x = 0$ in the above gives

$$
-8 = -c_1 + c_2 + 2c_3 + 4c_4 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 4c_3 \mathrm{e}^{2x} + 16c_4 \mathrm{e}^{4x}
$$

substituting $y'' = -14$ and $x = 0$ in the above gives

$$
-14 = c_1 + c_2 + 4c_3 + 16c_4 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + 8c_3 \mathrm{e}^{2x} + 64c_4 \mathrm{e}^{4x}
$$

substituting $y''' = -62$ and $x = 0$ in the above gives

$$
-62 = -c_1 + c_2 + 8c_3 + 64c_4 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -4 \\
c_3 &= 1 \\
c_4 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} - \mathrm{e}^{4x}
$$

Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} - \mathrm{e}^{4x} \tag{1}
$$

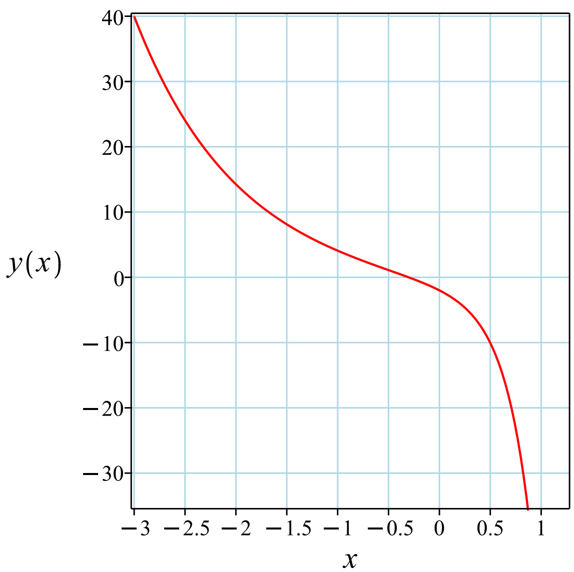


Figure 550: Solution plot

Verification of solutions

$$
y = 2\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} - \mathrm{e}^{4x}
$$

Verified OK.

18.24.1 Maple step by step solution

Let's solve

$$\left[y'''' - 6y''' + 7y'' + 6y' - 8y = 0, y(0) = -2, y'\Big|_{\{x=0\}} = -8, y''\Big|_{\{x=0\}} = -14, y'''\Big|_{\{x=0\}} = -62\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- ☐ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 6y_4(x) - 7y_3(x) - 6y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 6y_4(x) - 7y_3(x) - 6y_2(x) + 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & -6 & -7 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & -6 & -7 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 4, \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{4x} \cdot \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_4 \mathrm{e}^{4x} \cdot \begin{bmatrix} \frac{1}{64} \\ \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{c_3 \mathrm{e}^{2x}}{8} + \frac{c_4 \mathrm{e}^{4x}}{64}$

- Use the initial condition $y(0) = -2$

  $-2 = -c_1 + c_2 + \frac{c_3}{8} + \frac{c_4}{64}$

- Calculate the 1st derivative of the solution

  $y' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{c_3 \mathrm{e}^{2x}}{4} + \frac{c_4 \mathrm{e}^{4x}}{16}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -8$

  $-8 = c_1 + c_2 + \frac{c_3}{4} + \frac{c_4}{16}$

- Calculate the 2nd derivative of the solution

  $y'' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{c_3 \mathrm{e}^{2x}}{2} + \frac{c_4 \mathrm{e}^{4x}}{4}$

- Use the initial condition $y''\Big|_{\{x=0\}} = -14$

  $-14 = -c_1 + c_2 + \frac{c_3}{2} + \frac{c_4}{4}$

- Calculate the 3rd derivative of the solution

  $y''' = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{2x} + c_4 \mathrm{e}^{4x}$

- Use the initial condition $y'''\Big|_{\{x=0\}} = -62$

  $-62 = c_1 + c_2 + c_3 + c_4$

- Solve for the unknown coefficients

  $\{c_1 = -2, c_2 = -4, c_3 = 8, c_4 = -64\}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} - \mathrm{e}^{4x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$4)-6*diff(y(x),x$3)+7*diff(y(x),x$2)+6*diff(y(x),x)-8*y(x)=0,y(0) = -2,
```

$$y(x) = e^{2x} - e^{4x} + 2\,e^{-x} - 4\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{y''''[x]-6*y'''[x]+7*y''[x]+6*y'[x]-8*y[x]==0,{y[0]==-2,y'[0]==-8,y''[0]==-14,y'''[0
```

$$y(x) \to 2e^{-x} - 4e^{x} + e^{2x} - e^{4x}$$

18.25 problem section 9.2, problem 25

Internal problem ID [1489]
Internal file name [`OUTPUT/1490_Sunday_June_05_2022_02_19_30_AM_21780042/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 25.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{4y'''' - 13y'' + 9y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3, y''(0) = 1, y'''(0) = 3]$$

The characteristic equation is

$$4\lambda^4 - 13\lambda^2 + 9 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -\frac{3}{2}$$
$$\lambda_4 = \frac{3}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-\frac{3x}{2}} c_3 + \mathrm{e}^{\frac{3x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= e^{x} \\
y_3 &= e^{-\frac{3x}{2}} \\
y_4 &= e^{\frac{3x}{2}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-x} + c_2 e^{x} + e^{-\frac{3x}{2}} c_3 + e^{\frac{3x}{2}} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 e^{-x} + c_2 e^{x} - \frac{3\,e^{-\frac{3x}{2}} c_3}{2} + \frac{3\,e^{\frac{3x}{2}} c_4}{2}
$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$
3 = -c_1 + c_2 - \frac{3c_3}{2} + \frac{3c_4}{2} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 e^{-x} + c_2 e^{x} + \frac{9\,e^{-\frac{3x}{2}} c_3}{4} + \frac{9\,e^{\frac{3x}{2}} c_4}{4}
$$

substituting $y'' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 + \frac{9c_3}{4} + \frac{9c_4}{4} \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -c_1 e^{-x} + c_2 e^{x} - \frac{27\,e^{-\frac{3x}{2}} c_3}{8} + \frac{27\,e^{\frac{3x}{2}} c_4}{8}
$$

substituting $y''' = 3$ and $x = 0$ in the above gives

$$
3 = -c_1 + c_2 - \frac{27c_3}{8} + \frac{27c_4}{8} \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 2 \\
c_3 &= 0 \\
c_4 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{-x} + 2\,\mathrm{e}^{x} \tag{1}$$

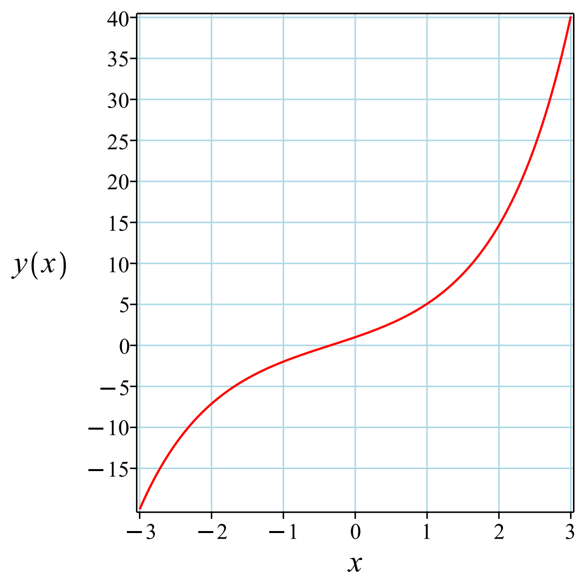


Figure 551: Solution plot

Verification of solutions

$$y = -\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$$

Verified OK.

18.25.1 Maple step by step solution

Let's solve

$$\left[4y'''' - 13y'' + 9y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 3, y''\Big|_{\{x=0\}} = 1, y'''\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{13y''}{4} - \frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{13y''}{4} + \frac{9y}{4} = 0$

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = \frac{13y_3(x)}{4} - \frac{9y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{13y_3(x)}{4} - \frac{9y_1(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{9}{4} & 0 & \frac{13}{4} & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{9}{4} & 0 & \frac{13}{4} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[\frac{3}{2}, \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{3} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{3}{2}, \left[\begin{array}{c} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{\frac{3x}{2}} \cdot \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{\frac{3x}{2}} c_4 \cdot \begin{bmatrix} \frac{8}{27} \\ \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(-8\,e^{3x} c_4 - 27 c_3 e^{\frac{5x}{2}} + 27 c_2 e^{\frac{x}{2}} + 8 c_1\right) e^{-\frac{3x}{2}}}{27}$$

- Use the initial condition $y(0) = 1$

$$1 = \frac{8c_4}{27} + c_3 - c_2 - \frac{8c_1}{27}$$

- Calculate the 1st derivative of the solution

$$y' = -\frac{\left(-24\,e^{3x} c_4 - \frac{135 c_3 e^{\frac{5x}{2}}}{2} + \frac{27 c_2 e^{\frac{x}{2}}}{2}\right) e^{-\frac{3x}{2}}}{27} + \frac{\left(-8\,e^{3x} c_4 - 27 c_3 e^{\frac{5x}{2}} + 27 c_2 e^{\frac{x}{2}} + 8 c_1\right) e^{-\frac{3x}{2}}}{18}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

$$3 = \frac{4c_4}{9} + c_3 + c_2 + \frac{4c_1}{9}$$

- Calculate the 2nd derivative of the solution

$$y'' = -\frac{\left(-72\,e^{3x} c_4 - \frac{675 c_3 e^{\frac{5x}{2}}}{4} + \frac{27 c_2 e^{\frac{x}{2}}}{4}\right) e^{-\frac{3x}{2}}}{27} + \frac{\left(-24\,e^{3x} c_4 - \frac{135 c_3 e^{\frac{5x}{2}}}{2} + \frac{27 c_2 e^{\frac{x}{2}}}{2}\right) e^{-\frac{3x}{2}}}{9} - \frac{\left(-8\,e^{3x} c_4 - 27 c_3 e^{\frac{5x}{2}} + 27 c_2 e^{\frac{x}{2}} + 8\right.}{12}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 1$

$$1 = \frac{2c_4}{3} + c_3 - c_2 - \frac{2c_1}{3}$$

- Calculate the 3rd derivative of the solution

$$y''' = -\frac{\left(-216\,e^{3x} c_4 - \frac{3375 c_3 e^{\frac{5x}{2}}}{8} + \frac{27 c_2 e^{\frac{x}{2}}}{8}\right) e^{-\frac{3x}{2}}}{27} + \frac{\left(-72\,e^{3x} c_4 - \frac{675 c_3 e^{\frac{5x}{2}}}{4} + \frac{27 c_2 e^{\frac{x}{2}}}{4}\right) e^{-\frac{3x}{2}}}{6} - \frac{\left(-24\,e^{3x} c_4 - \frac{135 c_3 e^{\frac{5x}{2}}}{2} + \frac{27 c_2}{2}\right.}{4}$$

- Use the initial condition $y'''\Big|_{\{x=0\}} = 3$

  $3 = c_1 + c_2 + c_3 + c_4$

- Solve for the unknown coefficients

  $\{c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 0\}$

- Solution to the IVP

  $y = -\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([4*diff(y(x),x$4)-13*diff(y(x),x$2)+9*y(x)=0,y(0) = 1, D(y)(0) = 3, (D@@2)(y)(0) = 1,
```

$$y(x) = -\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 18

```
DSolve[{4*y''''[x]-13*y''[x]+9*y[x]==0,{y[0]==1,y'[0]==3,y''[0]==1,y'''[0]==3}},y[x],x,Inclu
```

$$y(x) \to 2e^{x} - e^{-x}$$

18.26 problem section 9.2, problem 26

Internal problem ID [1490]
Internal file name [`OUTPUT/1491_Sunday_June_05_2022_02_19_32_AM_68701485/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 26.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' - 2y'' - 8y' - 8y = 0}$$

With initial conditions

$$[y(0) = 5, y'(0) = -2, y''(0) = 6, y'''(0) = 8]$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 2\lambda^2 - 8\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{(-1-i)x} c_2 + c_3 \mathrm{e}^{2x} + \mathrm{e}^{(-1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{(-1-i)x} \\
y_3 &= \mathrm{e}^{2x} \\
y_4 &= \mathrm{e}^{(-1+i)x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{(-1-i)x} c_2 + c_3 \mathrm{e}^{2x} + \mathrm{e}^{(-1+i)x} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$
5 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + (-1-i)\,\mathrm{e}^{(-1-i)x} c_2 + 2c_3 \mathrm{e}^{2x} + (-1+i)\,\mathrm{e}^{(-1+i)x} c_4
$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$
-2 = -2c_1 + (-1-i)\,c_2 + 2c_3 + (-1+i)\,c_4 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 4c_1 \mathrm{e}^{-2x} + 2i\mathrm{e}^{(-1-i)x} c_2 + 4c_3 \mathrm{e}^{2x} - 2i\mathrm{e}^{(-1+i)x} c_4
$$

substituting $y'' = 6$ and $x = 0$ in the above gives

$$
6 = 2c_2 i - 2c_4 i + 4c_1 + 4c_3 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -8c_1 \mathrm{e}^{-2x} + (2-2i)\,\mathrm{e}^{(-1-i)x} c_2 + 8c_3 \mathrm{e}^{2x} + (2+2i)\,\mathrm{e}^{(-1+i)x} c_4
$$

substituting $y''' = 8$ and $x = 0$ in the above gives

$$
8 = -8c_1 + (2-2i)\,c_2 + 8c_3 + (2+2i)\,c_4 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= \frac{3}{2} + \frac{i}{2} \\
c_3 &= 1 \\
c_4 &= \frac{3}{2} - \frac{i}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-2x} + \frac{3\,\mathrm{e}^{(-1-i)x}}{2} + \frac{i\mathrm{e}^{(-1-i)x}}{2} + \mathrm{e}^{2x} + \frac{3\,\mathrm{e}^{(-1+i)x}}{2} - \frac{i\mathrm{e}^{(-1+i)x}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{3}{2} + \frac{i}{2}\right)\mathrm{e}^{(-1-i)x} + \left(\frac{3}{2} - \frac{i}{2}\right)\mathrm{e}^{(-1+i)x} + \mathrm{e}^{-2x} + \mathrm{e}^{2x} \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{3}{2} + \frac{i}{2}\right)\mathrm{e}^{(-1-i)x} + \left(\frac{3}{2} - \frac{i}{2}\right)\mathrm{e}^{(-1+i)x} + \mathrm{e}^{-2x} + \mathrm{e}^{2x}
$$

Verified OK.

18.26.1 Maple step by step solution

Let's solve

$$
\left[y'''' + 2y''' - 2y'' - 8y' - 8y = 0, y(0) = 5, y'\Big|_{\{x=0\}} = -2, y''\Big|_{\{x=0\}} = 6, y'''\Big|_{\{x=0\}} = 8\right]
$$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -2y_4(x) + 2y_3(x) + 8y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) + 2y_3(x) + 8y_2(x) + 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 8 & 2 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 8 & 2 & -2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[-1+\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ -\frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4}+\frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_4(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{-x} \cdot \begin{bmatrix} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix} + c_4 e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(c_2 e^{4x} + ((2c_3+2c_4)\cos(x) + 2(c_3-c_4)\sin(x))e^x - c_1)e^{-2x}}{8}$

- Use the initial condition $y(0) = 5$

  $5 = \frac{c_2}{8} + \frac{c_3}{4} + \frac{c_4}{4} - \frac{c_1}{8}$

- Calculate the 1st derivative of the solution

  $y' = \frac{(4c_2 e^{4x} + (-(2c_3+2c_4)\sin(x) + 2(c_3-c_4)\cos(x))e^x + ((2c_3+2c_4)\cos(x) + 2(c_3-c_4)\sin(x))e^x)e^{-2x}}{8} - \frac{(c_2 e^{4x} + ((2c_3+2c_4)\cos(x}{}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -2$

  $-2 = \frac{c_2}{4} - \frac{c_4}{2} + \frac{c_1}{4}$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{(16c_2 e^{4x} + (-(2c_3+2c_4)\cos(x) - 2(c_3-c_4)\sin(x))e^x + 2(-(2c_3+2c_4)\sin(x) + 2(c_3-c_4)\cos(x))e^x + ((2c_3+2c_4)\cos(x) + 2(c_3-c_4)}{8}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 6$

  $6 = \frac{c_2}{2} - \frac{c_3}{2} + \frac{c_4}{2} - \frac{c_1}{2}$

- Calculate the 3rd derivative of the solution

  $y''' = \frac{(64c_2 e^{4x} + ((2c_3+2c_4)\sin(x) - 2(c_3-c_4)\cos(x))e^x + 3(-(2c_3+2c_4)\cos(x) - 2(c_3-c_4)\sin(x))e^x + 3(-(2c_3+2c_4)\sin(x) + 2(c_3-c_4}{8}$

- Use the initial condition $y'''\Big|_{\{x=0\}} = 8$

  $8 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $\{c_1 = -8, c_2 = 8, c_3 = 8, c_4 = 4\}$

- Solution to the IVP

  $y = (e^{4x} + (3\cos(x) + \sin(x))e^x + 1)e^{-2x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 28

```
dsolve([diff(y(x),x$4)+2*diff(y(x),x$3)-2*diff(y(x),x$2)-8*diff(y(x),x)-8*y(x)=0,y(0) = 5, D
```

$$y(x) = \left(1 + (\sin(x) + 3\cos(x))\,e^{x} + e^{4x}\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 31

```
DSolve[{y''''[x]+2*y'''[x]-2*y''[x]-8*y'[x]-8*y[x]==0,{y[0]==5,y'[0]==-2,y''[0]==6,y'''[0]==
```

$$y(x) \to e^{-2x}\left(e^{4x} + e^{x}\sin(x) + 3e^{x}\cos(x) + 1\right)$$

18.27 problem section 9.2, problem 27

Internal problem ID [1491]
Internal file name [`OUTPUT/1492_Sunday_June_05_2022_02_19_34_AM_95161317/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 27.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{4y'''' + 8y''' + 19y'' + 32y' + 12y = 0}$$

With initial conditions

$$\left[y(0) = 3, y'(0) = -3, y''(0) = -\frac{7}{2}, y'''(0) = \frac{31}{4}\right]$$

The characteristic equation is

$$4\lambda^4 + 8\lambda^3 + 19\lambda^2 + 32\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{3}{2}$$
$$\lambda_2 = -\frac{1}{2}$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3 + \mathrm{e}^{-\frac{3x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-\frac{x}{2}} \\
y_2 &= e^{2ix} \\
y_3 &= e^{-2ix} \\
y_4 &= e^{-\frac{3x}{2}}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-\frac{x}{2}} + e^{2ix} c_2 + e^{-2ix} c_3 + e^{-\frac{3x}{2}} c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{c_1 e^{-\frac{x}{2}}}{2} + 2i e^{2ix} c_2 - 2i e^{-2ix} c_3 - \frac{3\,e^{-\frac{3x}{2}} c_4}{2}
$$

substituting $y' = -3$ and $x = 0$ in the above gives

$$
-3 = -\frac{1}{2} c_1 + 2ic_2 - 2ic_3 - \frac{3}{2} c_4 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = \frac{c_1 e^{-\frac{x}{2}}}{4} - 4\,e^{2ix} c_2 - 4\,e^{-2ix} c_3 + \frac{9\,e^{-\frac{3x}{2}} c_4}{4}
$$

substituting $y'' = -\frac{7}{2}$ and $x = 0$ in the above gives

$$
-\frac{7}{2} = \frac{c_1}{4} - 4c_2 - 4c_3 + \frac{9c_4}{4} \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -\frac{c_1 e^{-\frac{x}{2}}}{8} - 8i e^{2ix} c_2 + 8i e^{-2ix} c_3 - \frac{27\,e^{-\frac{3x}{2}} c_4}{8}
$$

substituting $y''' = \frac{31}{4}$ and $x = 0$ in the above gives

$$
\frac{31}{4} = -\frac{1}{8} c_1 - 8ic_2 + 8ic_3 - \frac{27}{8} c_4 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= \frac{1}{2} + \frac{i}{2} \\
c_3 &= \frac{1}{2} - \frac{i}{2} \\
c_4 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-\frac{x}{2}} + \cos(2x) - \sin(2x)$$

Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{-\frac{x}{2}} + \cos(2x) - \sin(2x) \tag{1}$$

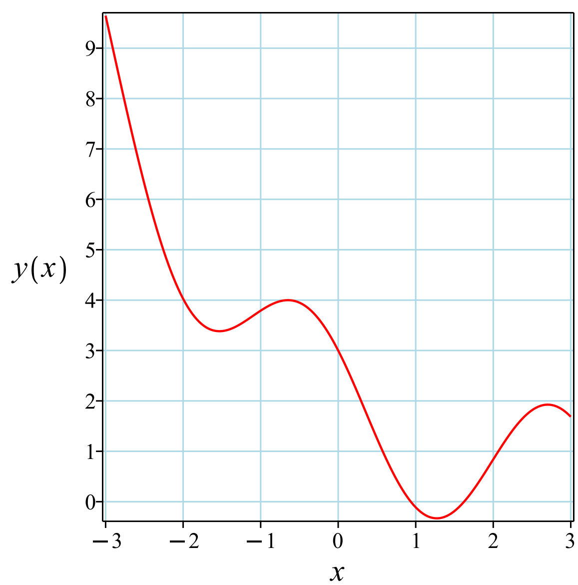


Figure 552: Solution plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-\frac{x}{2}} + \cos(2x) - \sin(2x)$$

Verified OK.

18.27.1 Maple step by step solution

Let's solve

$$\left[4y'''' + 8y''' + 19y'' + 32y' + 12y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = -3, y''\Big|_{\{x=0\}} = -\frac{7}{2}, y'''\Big|_{\{x=0\}} = \frac{31}{4}\right]$$

- Highest derivative means the order of the ODE is 4

y''''

- Isolate 4th derivative

$y'''' = -2y''' - \frac{19y''}{4} - 8y' - 3y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'''' + 2y''' + \frac{19y''}{4} + 8y' + 3y = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

  $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -2y_4(x) - \frac{19y_3(x)}{4} - 8y_2(x) - 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) - \frac{19y_3(x)}{4} - 8y_2(x) - 3y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -8 & -\frac{19}{4} & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -8 & -\frac{19}{4} & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{3}{2}, \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{3x}{2}} \cdot \begin{bmatrix} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c}-8 \\ 4 \\ -2 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c}-8 \\ 4 \\ -2 \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos (2x) - \mathrm{I} \sin (2x)) \cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{I}{8}(\cos(2x)-I\sin(2x)) \\ -\frac{\cos(2x)}{4}+\frac{I\sin(2x)}{4} \\ \frac{I}{2}(\cos(2x)-I\sin(2x)) \\ \cos(2x)-I\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\left[\begin{array}{c} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{array}\right], \vec{y}_4(x)=\left[\begin{array}{c} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y}=c_1 e^{-\frac{3x}{2}}\cdot\left[\begin{array}{c} -\frac{8}{27} \\ \frac{4}{9} \\ -\frac{2}{3} \\ 1 \end{array}\right]+c_2 e^{-\frac{x}{2}}\cdot\left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{c_3\sin(2x)}{8}-\frac{c_4\cos(2x)}{8} \\ -\frac{c_3\cos(2x)}{4}+\frac{c_4\sin(2x)}{4} \\ \frac{c_3\sin(2x)}{2}+\frac{c_4\cos(2x)}{2} \\ c_3\cos(2x)-c_4\sin(2x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$y=-\frac{8c_1 e^{-\frac{3x}{2}}}{27}-8c_2 e^{-\frac{x}{2}}-\frac{c_4\cos(2x)}{8}-\frac{c_3\sin(2x)}{8}$

- Use the initial condition $y(0)=3$

$3=-\frac{8c_1}{27}-8c_2-\frac{c_4}{8}$

- Calculate the 1st derivative of the solution

$y'=\frac{4c_1 e^{-\frac{3x}{2}}}{9}+4c_2 e^{-\frac{x}{2}}+\frac{c_4\sin(2x)}{4}-\frac{c_3\cos(2x)}{4}$

- Use the initial condition $y'\Big|_{\{x=0\}}=-3$

$-3=\frac{4c_1}{9}+4c_2-\frac{c_3}{4}$

- Calculate the 2nd derivative of the solution

$y''=-\frac{2c_1 e^{-\frac{3x}{2}}}{3}-2c_2 e^{-\frac{x}{2}}+\frac{c_4\cos(2x)}{2}+\frac{c_3\sin(2x)}{2}$

- Use the initial condition $y''\Big|_{\{x=0\}} = -\frac{7}{2}$

  $-\frac{7}{2} = -\frac{2c_1}{3} - 2c_2 + \frac{c_4}{2}$
- Calculate the 3rd derivative of the solution

  $y''' = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} - c_4 \sin(2x) + c_3 \cos(2x)$
- Use the initial condition $y'''\Big|_{\{x=0\}} = \frac{31}{4}$

  $\frac{31}{4} = c_1 + c_2 + c_3$
- Solve for the unknown coefficients

  $\left\{c_1 = 0, c_2 = -\frac{1}{4}, c_3 = 8, c_4 = -8\right\}$
- Solution to the IVP

  $y = 2\,\mathrm{e}^{-\frac{x}{2}} + \cos(2x) - \sin(2x)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve([4*diff(y(x),x$4)+8*diff(y(x),x$3)+19*diff(y(x),x$2)+32*diff(y(x),x)+12*y(x)=0,y(0) =
```

$$y(x) = 2\,\mathrm{e}^{-\frac{x}{2}} - \sin(2x) + \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 31

```
DSolve[{y''''[x]+2*y'''[x]-2*y''[x]-8*y'[x]-8*y[x]==0,{y[0]==5,y'[0]==-2,y''[0]==6,y'''[0]==
```

$$y(x) \to e^{-2x}\left(e^{4x} + e^x \sin(x) + 3e^x \cos(x) + 1\right)$$

18.28 problem section 9.2, problem 43(a)

Internal problem ID [1492]
Internal file name [`OUTPUT/1493_Sunday_June_05_2022_02_19_36_AM_96881931/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(a).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y = 0}$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = \mathrm{e}^{-ix}$$
$$y_4 = \mathrm{e}^{ix}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

Verified OK.

18.28.1 Maple step by step solution

Let's solve

$y'''' - y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_3\sin(x) - c_4\cos(x) \\ c_4\sin(x) - c_3\cos(x) \\ c_4\cos(x) + c_3\sin(x) \\ -c_4\sin(x) + c_3\cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} - c_4\cos(x) - c_3\sin(x)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$4)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_1 + c_2 \mathrm{e}^{x} + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 30

```
DSolve[y''''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_3 e^{-x} + c_2 \cos(x) + c_4 \sin(x)$$

18.29 problem section 9.2, problem 43(b)

Internal problem ID [1493]
Internal file name [`OUTPUT/1494_Sunday_June_05_2022_02_19_37_AM_76649214/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(b).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y = 0}$$

The characteristic equation is

$$\lambda^4 + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_2 &= -\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_3 &= -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2} \\
\lambda_4 &= \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)x} c_1 + e^{\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x} c_2 + e^{\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)x} c_3 + e^{\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} \\
y_2 &= \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} \\
y_3 &= \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} \\
y_4 &= \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_1 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_1 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_4
$$

Verified OK.

18.29.1 Maple step by step solution

Let's solve

$y'''' + y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_1(x)$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}}\\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}}\\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{\sqrt{2}\,x}{2}}\cdot\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}}\\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{\sqrt{2}\,x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}}\\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}}\\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}}\\ \cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \mathrm{e}^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right], \vec{y}_2(x) = \mathrm{e}^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} - \frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} + \frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{\frac{\sqrt{2}\,x}{2}} \cdot \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x) = e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right], \vec{y}_{4}(x) = e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}-\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right] + c_2 e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}-\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right] + c_3 e^{\frac{\sqrt{2}\,x}{2}}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left((c_1+c_2)e^{-\frac{\sqrt{2}\,x}{2}} - e^{\frac{\sqrt{2}\,x}{2}}(c_3-c_4)\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right) + \sin\left(\frac{\sqrt{2}\,x}{2}\right)\left((c_1-c_2)e^{-\frac{\sqrt{2}\,x}{2}} + e^{\frac{\sqrt{2}\,x}{2}}(c_3+c_4)\right)\right)\sqrt{2}}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$4)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(-\mathrm{e}^{-\frac{\sqrt{2}\,x}{2}} c_1 - \mathrm{e}^{\frac{\sqrt{2}\,x}{2}} c_2\right) \sin\left(\frac{\sqrt{2}\,x}{2}\right) + \left(c_3 \mathrm{e}^{-\frac{\sqrt{2}\,x}{2}} + c_4 \mathrm{e}^{\frac{\sqrt{2}\,x}{2}}\right) \cos\left(\frac{\sqrt{2}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
DSolve[y''''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x}{\sqrt{2}}} \left(\left(c_1 e^{\sqrt{2}x} + c_2\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \left(c_4 e^{\sqrt{2}x} + c_3\right) \sin\left(\frac{x}{\sqrt{2}}\right)\right)$$

18.30 problem section 9.2, problem 43(c)

Internal problem ID [1494]
Internal file name [`OUTPUT/1495_Sunday_June_05_2022_02_19_38_AM_63443292/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(c).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 64y = 0}$$

The characteristic equation is

$$\lambda^4 + 64 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 - 2i \\
\lambda_2 &= 2 + 2i \\
\lambda_3 &= -2 - 2i \\
\lambda_4 &= -2 + 2i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(2-2i)x}c_1 + \mathrm{e}^{(-2+2i)x}c_2 + \mathrm{e}^{(-2-2i)x}c_3 + \mathrm{e}^{(2+2i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{(2-2i)x} \\
y_2 &= \mathrm{e}^{(-2+2i)x} \\
y_3 &= \mathrm{e}^{(-2-2i)x} \\
y_4 &= \mathrm{e}^{(2+2i)x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{(2-2i)x} c_1 + e^{(-2+2i)x} c_2 + e^{(-2-2i)x} c_3 + e^{(2+2i)x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{(2-2i)x} c_1 + e^{(-2+2i)x} c_2 + e^{(-2-2i)x} c_3 + e^{(2+2i)x} c_4$$

Verified OK.

18.30.1 Maple step by step solution

Let's solve

$y'''' + 64y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -64y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -64y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -64 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -64 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2-2\mathrm{I}, \begin{bmatrix} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[-2+2\mathrm{I}, \begin{bmatrix} \frac{1}{32}-\frac{\mathrm{I}}{32} \\ \frac{\mathrm{I}}{8} \\ -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[2-2\mathrm{I}, \begin{bmatrix} -\frac{1}{32}+\frac{\mathrm{I}}{32} \\ \frac{\mathrm{I}}{8} \\ \frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[2+2\mathrm{I}, \begin{bmatrix} -\frac{1}{3\ldots} \\ \\ \frac{1}{4} \\ \end{bmatrix}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-2\mathrm{I}, \begin{bmatrix} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-2-2\,I)x} \cdot \begin{bmatrix} \frac{1}{32}+\frac{I}{32} \\ -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-2x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} \frac{1}{32}+\frac{I}{32} \\ -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2x} \cdot \begin{bmatrix} \left(\frac{1}{32}+\frac{I}{32}\right)(\cos(2x) - I\sin(2x)) \\ -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ \left(-\frac{1}{4}+\frac{I}{4}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32}+\frac{\sin(2x)}{32} \\ -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4}+\frac{\sin(2x)}{4} \\ \cos(2x) \end{bmatrix}, \vec{y}_2(x) = e^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32}-\frac{\sin(2x)}{32} \\ -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ -\sin(2x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-2\,I, \begin{bmatrix} -\frac{1}{32}+\frac{I}{32} \\ \frac{I}{8} \\ \frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(2-2I)x} \cdot \begin{bmatrix} -\frac{1}{32}+\frac{I}{32} \\ \frac{I}{8} \\ \frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{2x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{1}{32}+\frac{I}{32} \\ \frac{I}{8} \\ \frac{1}{4}+\frac{I}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2x} \cdot \begin{bmatrix} \left(-\frac{1}{32}+\frac{I}{32}\right)(\cos(2x) - I\sin(2x)) \\ \frac{I}{8}(\cos(2x) - I\sin(2x)) \\ \left(\frac{1}{4}+\frac{I}{4}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{2x} \cdot \begin{bmatrix} -\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32} \\ \frac{\sin(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = e^{2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32}+\frac{\sin(2x)}{32} \\ \frac{\cos(2x)}{8} \\ \frac{\cos(2x)}{4}-\frac{\sin(2x)}{4} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32}+\frac{\sin(2x)}{32} \\ -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4}+\frac{\sin(2x)}{4} \\ \cos(2x) \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32}-\frac{\sin(2x)}{32} \\ -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ -\sin(2x) \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} -\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32} \\ \frac{\sin(2x)}{8} \\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4} \\ \cos(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((c_1+c_2)e^{-2x}-(c_3-c_4)e^{2x})\cos(2x)}{32} + \frac{((c_1-c_2)e^{-2x}+(c_3+c_4)e^{2x})\sin(2x)}{32}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$4)+64*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_2e^{2x} + c_4e^{-2x})\cos(2x) + \sin(2x)(c_1e^{2x} + c_3e^{-2x})$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 44

```
DSolve[y''''[x]+64*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left((c_4e^{4x} + c_1)\cos(2x) + (c_3e^{4x} + c_2)\sin(2x)\right)$$

18.31 problem section 9.2, problem 43(d)

Internal problem ID [1495]
Internal file name [`OUTPUT/1496_Sunday_June_05_2022_02_19_40_AM_23184824/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(d).
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)} - y = 0}$$

The characteristic equation is

$$\lambda^6 - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1 \\
\lambda_3 &= \frac{\sqrt{-2i\sqrt{3}-2}}{2} \\
\lambda_4 &= -\frac{\sqrt{-2i\sqrt{3}-2}}{2} \\
\lambda_5 &= \frac{\sqrt{2i\sqrt{3}-2}}{2} \\
\lambda_6 &= -\frac{\sqrt{2i\sqrt{3}-2}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_3 + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_4 + \mathrm{e}^{-\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_5 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} \\
y_4 &= \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} \\
y_5 &= \mathrm{e}^{-\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} \\
y_6 &= \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_3 + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_4 + \mathrm{e}^{-\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_5 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_6 \quad (1)$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_3 + \mathrm{e}^{\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_4 + \mathrm{e}^{-\frac{\sqrt{2i\sqrt{3}-2}\,x}{2}} c_5 + \mathrm{e}^{-\frac{\sqrt{-2i\sqrt{3}-2}\,x}{2}} c_6$$

Verified OK.

18.31.1 Maple step by step solution

Let's solve

$y^{(6)} - y = 0$

- Highest derivative means the order of the ODE is 6

  $y^{(6)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

$y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Define new variable $y_5(x)$

  $y_5(x) = y''''$

- Define new variable $y_6(x)$

  $y_6(x) = y^{(5)}$

- Isolate for $y_6'(x)$ using original ODE

  $y_6'(x) = y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_6(x) = y_5'(x), y_6'(x) = y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^5} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}\right.\right.$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{I\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}}\right) \\ \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_{4}(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}}\right) \\ \Im\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{I\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{5}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{5}(x) = e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}}\right) \\ \Re\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ -\cos\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right], \vec{y}_{6}(x) = e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{5}}\right) \\ \Im\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)+c_{5}\vec{y}_{5}(x)+c_{6}\vec{y}_{6}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^5}\right) \\ \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^4}\right) \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + c_4 e^{-\frac{x}{2}} \cdot \begin{bmatrix} \Im\left(\frac{\cos\left(\frac{\sqrt{3}}{2}\right.}{(}\right. \\ \Im\left(\frac{\cos\left(\frac{\sqrt{3}}{2}\right.}{(}\right. \\ -\mathrm{s} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right.}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\mathrm{s} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(c_2 e^{2x} + 32 c_5 e^{\frac{3x}{2}} \Re\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) + I\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\sqrt{3}+I\right)^5}\right) + 32 c_6 e^{\frac{3x}{2}} \Im\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) + I\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\sqrt{3}+I\right)^5}\right) + 32 c_3 \Re\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right.}{(}\right.\right.$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(diff(y(x),x$6)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(e^{\frac{x}{2}} c_6 + e^{\frac{3x}{2}} c_4\right) \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \left(e^{\frac{x}{2}} c_5 + c_3 e^{\frac{3x}{2}}\right) \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 e^{2x} + c_1\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 78

```
DSolve[y''''''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1e^{2x} + e^{x/2}(c_2e^x + c_3)\cos\left(\frac{\sqrt{3}x}{2}\right) + e^{x/2}(c_6e^x + c_5)\sin\left(\frac{\sqrt{3}x}{2}\right) + c_4\right)$$

18.32 problem section 9.2, problem 43(e)

Internal problem ID [1496]
Internal file name [`OUTPUT/1497_Sunday_June_05_2022_02_19_41_AM_26235213/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(e).
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 64y = 0}$$

The characteristic equation is

$$\lambda^4 + 64 = 0$$

The roots of the above equation are

$$\lambda_1 = 2 - 2i$$
$$\lambda_2 = 2 + 2i$$
$$\lambda_3 = -2 - 2i$$
$$\lambda_4 = -2 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(2-2i)x}c_1 + e^{(-2+2i)x}c_2 + e^{(-2-2i)x}c_3 + e^{(2+2i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{(2-2i)x}$$
$$y_2 = e^{(-2+2i)x}$$
$$y_3 = e^{(-2-2i)x}$$
$$y_4 = e^{(2+2i)x}$$

Summary
The solution(s) found are the following

$$y = e^{(2-2i)x}c_1 + e^{(-2+2i)x}c_2 + e^{(-2-2i)x}c_3 + e^{(2+2i)x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{(2-2i)x}c_1 + e^{(-2+2i)x}c_2 + e^{(-2-2i)x}c_3 + e^{(2+2i)x}c_4$$

Verified OK.

18.32.1 Maple step by step solution

Let's solve

$y'''' + 64y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -64y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -64y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -64 & 0 & 0 & 0 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -64 & 0 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2-2\mathrm{I}, \left[\begin{array}{c} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right], \left[-2+2\mathrm{I}, \left[\begin{array}{c} \frac{1}{32}-\frac{\mathrm{I}}{32} \\ \frac{\mathrm{I}}{8} \\ -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right], \left[2-2\mathrm{I}, \left[\begin{array}{c} -\frac{1}{32}+\frac{\mathrm{I}}{32} \\ \frac{\mathrm{I}}{8} \\ \frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right], \left[2+2\mathrm{I}, \left[\begin{array}{c} -\frac{1}{3} \\ \\ \frac{1}{4} \\ \end{array}\right.\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-2\mathrm{I}, \left[\begin{array}{c} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-2-2 I)x} \cdot \left[\begin{array}{c} \frac{1}{32}+\frac{I}{32} \\ -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-2x} \cdot \left(\cos \left(2x\right)-I \sin \left(2x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{32}+\frac{I}{32} \\ -\frac{I}{8} \\ -\frac{1}{4}+\frac{I}{4} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-2x} \cdot \left[\begin{array}{c} \left(\frac{1}{32}+\frac{I}{32}\right) \left(\cos \left(2x\right)-I \sin \left(2x\right)\right) \\ -\frac{I}{8}\left(\cos \left(2x\right)-I \sin \left(2x\right)\right) \\ \left(-\frac{1}{4}+\frac{I}{4}\right) \left(\cos \left(2x\right)-I \sin \left(2x\right)\right) \\ \cos \left(2x\right)-I \sin \left(2x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{1}\left(x\right)=e^{-2x} \cdot \left[\begin{array}{c} \frac{\cos \left(2x\right)}{32}+\frac{\sin \left(2x\right)}{32} \\ -\frac{\sin \left(2x\right)}{8} \\ -\frac{\cos \left(2x\right)}{4}+\frac{\sin \left(2x\right)}{4} \\ \cos \left(2x\right) \end{array}\right], \vec{y}_{2}\left(x\right)=e^{-2x} \cdot \left[\begin{array}{c} \frac{\cos \left(2x\right)}{32}-\frac{\sin \left(2x\right)}{32} \\ -\frac{\cos \left(2x\right)}{8} \\ \frac{\sin \left(2x\right)}{4}+\frac{\cos \left(2x\right)}{4} \\ -\sin \left(2x\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-2 I, \left[\begin{array}{c} -\frac{1}{32}+\frac{I}{32} \\ \frac{I}{8} \\ \frac{1}{4}+\frac{I}{4} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(2-2\,\mathrm{I})x}\cdot\begin{bmatrix}-\frac{1}{32}+\frac{\mathrm{I}}{32}\\ \frac{\mathrm{I}}{8}\\ \frac{1}{4}+\frac{\mathrm{I}}{4}\\ 1\end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{2x}\cdot(\cos(2x)-\mathrm{I}\sin(2x))\cdot\begin{bmatrix}-\frac{1}{32}+\frac{\mathrm{I}}{32}\\ \frac{\mathrm{I}}{8}\\ \frac{1}{4}+\frac{\mathrm{I}}{4}\\ 1\end{bmatrix}$$

- Simplify expression

$$e^{2x}\cdot\begin{bmatrix}\left(-\frac{1}{32}+\frac{\mathrm{I}}{32}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\ \frac{\mathrm{I}}{8}(\cos(2x)-\mathrm{I}\sin(2x))\\ \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\ \cos(2x)-\mathrm{I}\sin(2x)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=e^{2x}\cdot\begin{bmatrix}-\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32}\\ \frac{\sin(2x)}{8}\\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\\ \cos(2x)\end{bmatrix},\vec{y}_4(x)=e^{2x}\cdot\begin{bmatrix}\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32}\\ \frac{\cos(2x)}{8}\\ \frac{\cos(2x)}{4}-\frac{\sin(2x)}{4}\\ -\sin(2x)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1e^{-2x}\cdot\begin{bmatrix}\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32}\\ -\frac{\sin(2x)}{8}\\ -\frac{\cos(2x)}{4}+\frac{\sin(2x)}{4}\\ \cos(2x)\end{bmatrix}+c_2e^{-2x}\cdot\begin{bmatrix}\frac{\cos(2x)}{32}-\frac{\sin(2x)}{32}\\ -\frac{\cos(2x)}{8}\\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\\ -\sin(2x)\end{bmatrix}+c_3e^{2x}\cdot\begin{bmatrix}-\frac{\cos(2x)}{32}+\frac{\sin(2x)}{32}\\ \frac{\sin(2x)}{8}\\ \frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\\ \cos(2x)\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((c_1+c_2)\mathrm{e}^{-2x} - (c_3-c_4)\mathrm{e}^{2x})\cos(2x)}{32} + \frac{((c_1-c_2)\mathrm{e}^{-2x} + (c_3+c_4)\mathrm{e}^{2x})\sin(2x)}{32}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$4)+64*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2\mathrm{e}^{2x} + c_4\mathrm{e}^{-2x}\right)\cos\left(2x\right) + \sin\left(2x\right)\left(c_1\mathrm{e}^{2x} + c_3\mathrm{e}^{-2x}\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 44

```
DSolve[y''''[x]+64*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(\left(c_4e^{4x} + c_1\right)\cos(2x) + \left(c_3e^{4x} + c_2\right)\sin(2x)\right)$$

18.33 problem section 9.2, problem 43(g)

Internal problem ID [1497]
Internal file name [`OUTPUT/1498_Sunday_June_05_2022_02_19_42_AM_93887519/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.2. constant coefficient. Page 483
Problem number: section 9.2, problem 43(g).
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)}+y''''+y'''+y''+y'+y=0}$$

The characteristic equation is

$$\lambda^5+\lambda^4+\lambda^3+\lambda^2+\lambda+1=0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1&=-1\\ \lambda_2&=-\frac{1}{2}-\frac{i\sqrt{3}}{2}\\ \lambda_3&=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\\ \lambda_4&=\frac{1}{2}-\frac{i\sqrt{3}}{2}\\ \lambda_5&=\frac{1}{2}+\frac{i\sqrt{3}}{2}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x)=c_1\mathrm{e}^{-x}+\mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_2+\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_3+\mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_4+\mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \\
y_3 &= \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} \\
y_4 &= \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \\
y_5 &= \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_3 + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_4 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_5 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_3 + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_4 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_5
$$

Verified OK.

18.33.1 Maple step by step solution

Let's solve

$y^{(5)} + y'''' + y''' + y'' + y' + y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Define new variable $y_5(x)$

$y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = -y_5(x) - y_4(x) - y_3(x) - y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = -y_5(x) - y_4(x) - y_3(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1\\ -1\\ 1\\ -1\\ 1\end{array}\right]\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right], \left[\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(\frac{1}{2}-\right.}\\ \frac{1}{\left(\frac{1}{2}-\right.}\\ \frac{1}{\left(\frac{1}{2}-\right.}\\ \frac{1}{\frac{1}{2}-}\end{array}\right.\right.\right.$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1\\ -1\\ 1\\ -1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1\\ -1\\ 1\\ -1\\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}}\\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{4}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-\frac{x}{2}} \cdot \begin{bmatrix} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^4}\right) \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}, \vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \begin{bmatrix} \Im\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^4}\right) \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^4} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}} \\ \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ -\cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_5(x) = e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \Im\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{4}}\right) \\ \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{x}{2}} \cdot \begin{bmatrix} \Re\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^4}\right) \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + e^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} \Im\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^4}\right) \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = 16\,e^{-x}\left(c_4 e^{\frac{3x}{2}} \Im\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\sqrt{3}+\mathrm{I}\right)^4}\right) - c_5 e^{\frac{3x}{2}} \Re\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\sqrt{3}+\mathrm{I}\right)^4}\right) + c_2 \Im\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\mathrm{I}+\sqrt{3}\right)^4}\right)\right.$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$5)+diff(y(x),x$4)+diff(y(x),x$3)+diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),
```

$$y(x) = e^{-x}\left(\left(e^{\frac{x}{2}} c_5 + c_3 e^{\frac{3x}{2}}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \left(c_2 e^{\frac{3x}{2}} + c_4 e^{\frac{x}{2}}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 70

```
DSolve[y'''''[x]+y''''[x]+y'''[x]+y''[x]+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-x}\left(e^{x/2}\left(c_3 e^x + c_2\right)\cos\left(\frac{\sqrt{3}x}{2}\right) + e^{x/2}\left(c_4 e^x + c_1\right)\sin\left(\frac{\sqrt{3}x}{2}\right) + c_5\right)$$

19 Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495

19.1 problem section 9.3, problem 1

Internal problem ID [1498]
Internal file name [`OUTPUT/1499_Sunday_June_05_2022_02_19_44_AM_49068004/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 6y'' + 11y' - 6y = -e^{x}\left(-24x^{2} + 76x + 4\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 11y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 11y' - 6y = -e^x(-24x^2 + 76x + 4)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^x(-24x^2 + 76x + 4)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}, e^{3x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x, x^2 e^x, e^x x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x + A_3 e^x x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_3 e^x x^2 - 18A_3 e^x x + 4A_2 x e^x + 6A_3 e^x - 6A_2 e^x + 2A_1 e^x = -e^x(-24x^2 + 76x + 4)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -17, A_2 = -1, A_3 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -17x\,e^x - x^2e^x + 4\,e^x x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^x + c_2e^{2x} + c_3e^{3x}\right) + \left(-17x\,e^x - x^2e^x + 4\,e^x x^3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - 17x\,e^x - x^2e^x + 4\,e^x x^3 \tag{1}$$

Verification of solutions

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - 17x\,e^x - x^2e^x + 4\,e^x x^3$$

Verified OK.

19.1.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 11y' - 6y = -e^x(-24x^2 + 76x + 4)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 6y + 24x^2e^x - 76x\,e^x - 4\,e^x + 6y'' - 11y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 6y'' + 11y' - 6y = 4\,e^x(6x^2 - 19x - 1)$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

◦ Define new variable $y_2(x)$

$y_2(x) = y'$

◦ Define new variable $y_3(x)$

$y_3(x) = y''$

◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 24x^2 e^x - 76x\, e^x + 6y_3(x) - 11y_2(x) + 6y_1(x) - 4\, e^x$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 24x^2 e^x - 76x\, e^x + 6y_3(x) - 11y_2(x) + 6y_1(x) - 4\, e^x]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 24x^2 e^x - 76x\, e^x - 4\, e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 24x^2 e^x - 76x\, e^x - 4\, e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right], \left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 3e^{x} - 3e^{2x} + e^{3x} & -\frac{5e^{x}}{2} + 4e^{2x} - \frac{3e^{3x}}{2} & \frac{e^{x}}{2} - e^{2x} + \frac{e^{3x}}{2} \\ 3e^{x} - 6e^{2x} + 3e^{3x} & -\frac{5e^{x}}{2} + 8e^{2x} - \frac{9e^{3x}}{2} & \frac{e^{x}}{2} - 2e^{2x} + \frac{3e^{3x}}{2} \\ 3e^{x} - 12e^{2x} + 9e^{3x} & -\frac{5e^{x}}{2} + 16e^{2x} - \frac{27e^{3x}}{2} & \frac{e^{x}}{2} - 4e^{2x} + \frac{9e^{3x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x)=\left[\begin{array}{c} 32\,\mathrm{e}^{2x}-\frac{15\,\mathrm{e}^{3x}}{2}+\frac{(8x^3-2x^2-34x-49)\mathrm{e}^{x}}{2} \\ 64\,\mathrm{e}^{2x}-\frac{45\,\mathrm{e}^{3x}}{2}+\frac{(8x^3+22x^2-38x-83)\mathrm{e}^{x}}{2} \\ 128\,\mathrm{e}^{2x}-\frac{135\,\mathrm{e}^{3x}}{2}+\frac{(8x^3+46x^2+6x-121)\mathrm{e}^{x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3+\left[\begin{array}{c} 32\,\mathrm{e}^{2x}-\frac{15\,\mathrm{e}^{3x}}{2}+\frac{(8x^3-2x^2-34x-49)\mathrm{e}^{x}}{2} \\ 64\,\mathrm{e}^{2x}-\frac{45\,\mathrm{e}^{3x}}{2}+\frac{(8x^3+22x^2-38x-83)\mathrm{e}^{x}}{2} \\ 128\,\mathrm{e}^{2x}-\frac{135\,\mathrm{e}^{3x}}{2}+\frac{(8x^3+46x^2+6x-121)\mathrm{e}^{x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=\frac{(128+c_2)\mathrm{e}^{2x}}{4}+\frac{(-135+2c_3)\mathrm{e}^{3x}}{18}+4\left(x^3-\frac{1}{4}x^2-\frac{17}{4}x+\frac{1}{4}c_1-\frac{49}{8}\right)\mathrm{e}^{x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=-exp(x)*(4+76*x-24*x^2),y(x),
```

$$y(x) = e^{x}\left(c_3 e^{2x} + 4x^3 + c_2 e^{x} - x^2 + c_1 - 17x\right)$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 47

```
DSolve[y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==-Exp[x]*(4+76*x-24*x^2),y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{2} e^{x}\left(8x^3 - 2x^2 - 34x + 2c_2 e^{x} + 2c_3 e^{2x} - 49 + 2c_1\right)$$

19.2 problem section 9.3, problem 2

Internal problem ID [1499]
Internal file name [`OUTPUT/1500_Sunday_June_05_2022_02_19_46_AM_21485494/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' - 5y' + 6y = \mathrm{e}^{-3x}\left(6x^2 - 23x + 32\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' - 5y' + 6y = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^{x} \\ y_3 &= e^{3x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' - 5y' + 6y = e^{-3x}(6x^2 - 23x + 32)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-3x}(6x^2 - 23x + 32)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-3x}, x^2 e^{-3x}, e^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-3x} + A_2 x^2 e^{-3x} + A_3 e^{-3x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$34A_1 e^{-3x} - 24A_1 x\,e^{-3x} - 22A_2 e^{-3x} + 68A_2 x\,e^{-3x} - 24A_2 x^2 e^{-3x} - 24A_3 e^{-3x} \\ = e^{-3x}(6x^2 - 23x + 32)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = -\frac{1}{4}, A_3 = -\frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-3x}}{4} - \frac{x^2\mathrm{e}^{-3x}}{4} - \frac{3\,\mathrm{e}^{-3x}}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{3x}\right) + \left(\frac{x\,\mathrm{e}^{-3x}}{4} - \frac{x^2\mathrm{e}^{-3x}}{4} - \frac{3\,\mathrm{e}^{-3x}}{4}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{3x} + \frac{x\,\mathrm{e}^{-3x}}{4} - \frac{x^2\mathrm{e}^{-3x}}{4} - \frac{3\,\mathrm{e}^{-3x}}{4} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{3x} + \frac{x\,\mathrm{e}^{-3x}}{4} - \frac{x^2\mathrm{e}^{-3x}}{4} - \frac{3\,\mathrm{e}^{-3x}}{4}$$

Verified OK.

19.2.1 Maple step by step solution

Let's solve

$y''' - 2y'' - 5y' + 6y = \mathrm{e}^{-3x}(6x^2 - 23x + 32)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 6x^2 e^{-3x} - 23x\, e^{-3x} + 2y_3(x) + 5y_2(x) - 6y_1(x) + 32\, e^{-3x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6x^2 e^{-3x} - 23x\, e^{-3x} + 2y_3(x) + 5y_2(x) - 6y_1(x) + 32\, e^{-3x}]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 6x^2 e^{-3x} - 23x\, e^{-3x} + 32\, e^{-3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 6x^2 e^{-3x} - 23x\, e^{-3x} + 32\, e^{-3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{3x}}{9} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{3x}}{3} \\ e^{-2x} & e^{x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{3x}}{9} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{3x}}{3} \\ e^{-2x} & e^{x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{9} \\ -\frac{1}{2} & 1 & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} -\frac{(e^{5x}-5e^{3x}-1)e^{-2x}}{5} & \frac{(3e^{5x}+5e^{3x}-8)e^{-2x}}{30} & \frac{(3e^{5x}-5e^{3x}+2)e^{-2x}}{30} \\ -\frac{(3e^{5x}-5e^{3x}+2)e^{-2x}}{5} & \frac{(9e^{5x}+5e^{3x}+16)e^{-2x}}{30} & \frac{(9e^{5x}-5e^{3x}-4)e^{-2x}}{30} \\ -\frac{(9e^{5x}-5e^{3x}-4)e^{-2x}}{5} & \frac{(27e^{5x}+5e^{3x}-32)e^{-2x}}{30} & \frac{(27e^{5x}-5e^{3x}+8)e^{-2x}}{30} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}\frac{\left(19\,\mathrm{e}^{6x}-45\,\mathrm{e}^{4x}-10x^2+56\,\mathrm{e}^{x}+10x-30\right)\mathrm{e}^{-3x}}{40}\\ -\frac{\left(-57\,\mathrm{e}^{6x}+45\,\mathrm{e}^{4x}-30x^2+112\,\mathrm{e}^{x}+50x-100\right)\mathrm{e}^{-3x}}{40}\\ \frac{\left(171\,\mathrm{e}^{6x}-45\,\mathrm{e}^{4x}-90x^2+224\,\mathrm{e}^{x}+210x-350\right)\mathrm{e}^{-3x}}{40}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c}\frac{\left(19\,\mathrm{e}^{6x}-45\,\mathrm{e}^{4x}-10x^2+56\,\mathrm{e}^{x}+10x-30\right)\mathrm{e}^{-3x}}{40}\\ -\frac{\left(-57\,\mathrm{e}^{6x}+45\,\mathrm{e}^{4x}-30x^2+112\,\mathrm{e}^{x}+50x-100\right)\mathrm{e}^{-3x}}{40}\\ \frac{\left(171\,\mathrm{e}^{6x}-45\,\mathrm{e}^{4x}-90x^2+224\,\mathrm{e}^{x}+210x-350\right)\mathrm{e}^{-3x}}{40}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\left(c_2-\frac{9}{8}\right)\mathrm{e}^{-3x}\mathrm{e}^{4x}-\frac{\left(90x^2-90c_1\mathrm{e}^{x}-40c_3\mathrm{e}^{6x}-90x-504\,\mathrm{e}^{x}-171\,\mathrm{e}^{6x}+270\right)\mathrm{e}^{-3x}}{360}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=exp(-3*x)*(32-23*x+6*x^2),y(x),
```

$$y(x) = \frac{(4c_3 e^{6x} + 4\,e^{4x} c_1 + 4c_2 e^{x} - x^2 + x - 3)\,e^{-3x}}{4}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 45

```
DSolve[y'''[x]-2*y''[x]-5*y'[x]+6*y[x]==Exp[-3*x]*(32-23*x+6*x^2),y[x],x,IncludeSingularSolu
```

$$y(x) \to -\frac{1}{4}e^{-3x}\left(x^2 - x + 3\right) + c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x}$$

19.3 problem section 9.3, problem 3

Internal problem ID [1500]
Internal file name [`OUTPUT/1501_Sunday_June_05_2022_02_19_48_AM_67624774/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y''' + 8y'' - y' - 2y = -e^{x}(6x^2 + 45x + 4)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$4y''' + 8y'' - y' - 2y = 0$$

The characteristic equation is

$$4\lambda^3 + 8\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-\frac{x}{2}} + c_3 e^{\frac{x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{-\frac{x}{2}}$$
$$y_3 = e^{\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$4y''' + 8y'' - y' - 2y = -e^x (6x^2 + 45x + 4)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^x (6x^2 + 45x + 4)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2x}, e^{-\frac{x}{2}}, e^{\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x + A_3 e^x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$27A_1 e^x + 9A_1 x e^x + 40A_2 e^x + 54A_2 x e^x + 9A_2 x^2 e^x + 9A_3 e^x = -e^x (6x^2 + 45x + 4)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = -\frac{2}{3}, A_3 = \frac{149}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^x - \frac{2x^2\mathrm{e}^x}{3} + \frac{149\,\mathrm{e}^x}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-\frac{x}{2}} + c_3\mathrm{e}^{\frac{x}{2}}\right) + \left(-x\,\mathrm{e}^x - \frac{2x^2\mathrm{e}^x}{3} + \frac{149\,\mathrm{e}^x}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-\frac{x}{2}} + c_3\mathrm{e}^{\frac{x}{2}} - x\,\mathrm{e}^x - \frac{2x^2\mathrm{e}^x}{3} + \frac{149\,\mathrm{e}^x}{27} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-\frac{x}{2}} + c_3\mathrm{e}^{\frac{x}{2}} - x\,\mathrm{e}^x - \frac{2x^2\mathrm{e}^x}{3} + \frac{149\,\mathrm{e}^x}{27}$$

Verified OK.

19.3.1 Maple step by step solution

Let's solve

$4y''' + 8y'' - y' - 2y = -\mathrm{e}^x(6x^2 + 45x + 4)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y}{2} - 2y'' + \frac{y'}{4} - \frac{3x^2\mathrm{e}^x}{2} - \frac{45x\,\mathrm{e}^x}{4} - \mathrm{e}^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + 2y'' - \frac{y'}{4} - \frac{y}{2} = -\frac{\mathrm{e}^x(6x^2+45x+4)}{4}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -\frac{3x^2 e^x}{2} - \frac{45x\,e^x}{4} - e^x - 2y_3(x) + \frac{y_2(x)}{4} + \frac{y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{3x^2 e^x}{2} - \frac{45x\,e^x}{4} - e^x - 2y_3(x) + \frac{y_2(x)}{4} + \frac{y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ -\frac{3x^2 e^x}{2} - \frac{45x\,e^x}{4} - e^x \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ -\frac{3x^2 e^x}{2} - \frac{45x\,e^x}{4} - e^x \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 4\,e^{-\frac{x}{2}} & 4\,e^{\frac{x}{2}} \\ -\frac{e^{-2x}}{2} & -2\,e^{-\frac{x}{2}} & 2\,e^{\frac{x}{2}} \\ e^{-2x} & e^{-\frac{x}{2}} & e^{\frac{x}{2}} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 4\,e^{-\frac{x}{2}} & 4\,e^{\frac{x}{2}} \\ -\frac{e^{-2x}}{2} & -2\,e^{-\frac{x}{2}} & 2\,e^{\frac{x}{2}} \\ e^{-2x} & e^{-\frac{x}{2}} & e^{\frac{x}{2}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 4 & 4 \\ -\frac{1}{2} & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\left(6\,e^{\frac{5x}{2}}+10\,e^{\frac{3x}{2}}-1\right)e^{-2x}}{15} & -e^{-\frac{x}{2}} + e^{\frac{x}{2}} & \frac{2\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+2\right)e^{-2x}}{15} \\ \frac{\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+2\right)e^{-2x}}{15} & \frac{e^{-\frac{x}{2}}}{2} + \frac{e^{\frac{x}{2}}}{2} & \frac{\left(3\,e^{\frac{5x}{2}}+5\,e^{\frac{3x}{2}}-8\right)e^{-2x}}{15} \\ \frac{\left(3\,e^{\frac{5x}{2}}+5\,e^{\frac{3x}{2}}-8\right)e^{-2x}}{30} & -\frac{e^{-\frac{x}{2}}}{4} + \frac{e^{\frac{x}{2}}}{4} & \frac{\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+32\right)e^{-2x}}{30} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{\left(90x^2 e^{3x}+135x\, e^{3x}-745\, e^{3x}+1026\, e^{\frac{5x}{2}}-310\, e^{\frac{3x}{2}}+29\right) e^{-2x}}{135} \\ -\frac{\left(90x^2 e^{3x}+315x\, e^{3x}-610\, e^{3x}+513\, e^{\frac{5x}{2}}+155\, e^{\frac{3x}{2}}-58\right) e^{-2x}}{135} \\ -\frac{\left(180x^2 e^{3x}+990x\, e^{3x}-590\, e^{3x}+513\, e^{\frac{5x}{2}}-155\, e^{\frac{3x}{2}}+232\right) e^{-2x}}{270} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \left[\begin{array}{c} -\frac{\left(90x^2 e^{3x}+135x\, e^{3x}-745\, e^{3x}+1026\, e^{\frac{5x}{2}}-310\, e^{\frac{3x}{2}}+29\right) e^{-2x}}{135} \\ -\frac{\left(90x^2 e^{3x}+315x\, e^{3x}-610\, e^{3x}+513\, e^{\frac{5x}{2}}+155\, e^{\frac{3x}{2}}-58\right) e^{-2x}}{135} \\ -\frac{\left(180x^2 e^{3x}+990x\, e^{3x}-590\, e^{3x}+513\, e^{\frac{5x}{2}}-155\, e^{\frac{3x}{2}}+232\right) e^{-2x}}{270} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{2\left(\left(-6c_2-\frac{31}{9}\right) e^{\frac{3x}{2}}+\left(-6c_3+\frac{57}{5}\right) e^{\frac{5x}{2}}+\left(x^2+\frac{3}{2}x-\frac{149}{18}\right) e^{3x}-\frac{3c_1}{8}+\frac{29}{90}\right) e^{-2x}}{3}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve(4*diff(y(x),x$3)+8*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=-exp(x)*(4+45*x+6*x^2),y(x), si
```

$$y(x) = \frac{\left(-18x^2 e^{3x} - 27x\, e^{3x} + 149\, e^{3x} + 27c_3 e^{\frac{5x}{2}} + 27c_2 e^{\frac{3x}{2}} + 27c_1\right) e^{-2x}}{27}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 52

```
DSolve[4*y'''[x]+8*y''[x]-y'[x]-2*y[x]==-Exp[x]*(4+45*x+6*x^2),y[x],x,IncludeSingularSolutio
```

$$y(x) \to e^x \left(-\frac{2x^2}{3} - x + \frac{149}{27}\right) + c_1 e^{-x/2} + c_2 e^{x/2} + c_3 e^{-2x}$$

19.4 problem section 9.3, problem 4

Internal problem ID [1501]
Internal file name [`OUTPUT/1502_Sunday_June_05_2022_02_19_50_AM_99008672/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 3y'' - y' - 3y = \mathrm{e}^{-2x}(3x^2 - 17x + 2)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' - y' - 3y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-3x}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' - y' - 3y = e^{-2x}(3x^2 - 17x + 2)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}(3x^2 - 17x + 2)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{-2x}, e^{-2x} x, e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-3x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{-2x} + A_2 e^{-2x} x + A_3 e^{-2x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{-2x} - 2A_1 x e^{-2x} + 3A_1 x^2 e^{-2x} + 3A_2 e^{-2x} x - A_2 e^{-2x} + 3A_3 e^{-2x} = e^{-2x}(3x^2 - 17x + 2)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -5, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{-2x} - 5 e^{-2x} x + e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3\right) + \left(x^2 e^{-2x} - 5 e^{-2x} x + e^{-2x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + x^2 e^{-2x} - 5 e^{-2x} x + e^{-2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + x^2 e^{-2x} - 5 e^{-2x} x + e^{-2x}$$

Verified OK.

19.4.1 Maple step by step solution

Let's solve

$y''' + 3y'' - y' - 3y = e^{-2x}(3x^2 - 17x + 2)$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 3x^2 e^{-2x} - 17\,e^{-2x}x + 2\,e^{-2x} - 3y_3(x) + y_2(x) + 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3x^2 e^{-2x} - 17\,e^{-2x}x + 2\,e^{-2x} - 3y_3(x) + y_2(x) + 3y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3x^2 e^{-2x} - 17\,e^{-2x}x + 2\,e^{-2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3x^2 e^{-2x} - 17\,e^{-2x}x + 2\,e^{-2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

□ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & e^{x} \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{x} \\ e^{-3x} & e^{-x} & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & e^{x} \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{x} \\ e^{-3x} & e^{-x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & 1 & 1 \\ -\frac{1}{3} & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(3\,e^{4x}+6\,e^{2x}-1)e^{-3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{(e^{4x}-2\,e^{2x}+1)e^{-3x}}{8} \\ \frac{3(e^{4x}-2\,e^{2x}+1)e^{-3x}}{8} & \frac{e^{x}}{2} + \frac{e^{-x}}{2} & \frac{(e^{4x}+2\,e^{2x}-3)e^{-3x}}{8} \\ \frac{3(e^{4x}+2\,e^{2x}-3)e^{-3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{(e^{4x}-2\,e^{2x}+9)e^{-3x}}{8} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(-25+18\,e^{2x}-e^{4x}+8\left(x^{2}-5x+1\right)e^{x}\right)e^{-3x}}{8} \\ \frac{\left(-18\,e^{2x}-e^{4x}+75+\left(-16x^{2}+96x-56\right)e^{x}\right)e^{-3x}}{8} \\ \frac{\left(18\,e^{2x}-e^{4x}-225+\left(32x^{2}-224x+208\right)e^{x}\right)e^{-3x}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} \frac{\left(-25+18\,e^{2x}-e^{4x}+8\left(x^{2}-5x+1\right)e^{x}\right)e^{-3x}}{8} \\ \frac{\left(-18\,e^{2x}-e^{4x}+75+\left(-16x^{2}+96x-56\right)e^{x}\right)e^{-3x}}{8} \\ \frac{\left(18\,e^{2x}-e^{4x}-225+\left(32x^{2}-224x+208\right)e^{x}\right)e^{-3x}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(72\,e^{4x}c_3-9\,e^{4x}+72c_2e^{2x}+72x^{2}e^{x}+162\,e^{2x}-360x\,e^{x}+72\,e^{x}+8c_1-225\right)e^{-3x}}{72}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-diff(y(x),x)-3*y(x)=exp(-2*x)*(2-17*x+3*x^2),y(x), si
```

$$y(x) = \left(c_3 e^{2x} + e^{4x} c_1 + \left(x^2 - 5x + 1\right) e^x + c_2\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 42

```
DSolve[y'''[x]+3*y''[x]-y'[x]-3*y[x]==Exp[-2*x]*(2-17*x+3*x^2),y[x],x,IncludeSingularSolutio
```

$$y(x) \to e^{-3x}\left(e^x\left(x^2 - 5x + 1\right) + c_2 e^{2x} + c_3 e^{4x} + c_1\right)$$

19.5 problem section 9.3, problem 5

Internal problem ID [1502]
Internal file name [`OUTPUT/1503_Sunday_June_05_2022_02_19_52_AM_4034753/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 3y'' - y' - 3y = e^{x}(16x^{3} + 24x^{2} + 2x - 1)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' - y' - 3y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^x + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^x \\ y_3 &= e^{-3x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' - y' - 3y = e^x(16x^3 + 24x^2 + 2x - 1)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x(16x^3 + 24x^2 + 2x - 1)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^x, x^2 e^x, e^x x^3, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-3x}, e^{-x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x, x^2 e^x, e^x x^3, e^x x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^x + A_2 x^2 e^x + A_3 e^x x^3 + A_4 e^x x^4$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &24A_3 e^x x^2 + 32A_4 e^x x^3 + 36A_3 e^x x + 72A_4 e^x x^2 + 24A_4 e^x x + 16A_2 x\, e^x \\ &+ 12A_2 e^x + 8A_1 e^x + 6A_3 e^x = e^x(16x^3 + 24x^2 + 2x - 1) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{2}, A_3 = -\frac{1}{2}, A_4 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^x}{2} + \frac{x^2 e^x}{2} - \frac{e^x x^3}{2} + \frac{e^x x^4}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + e^{-3x} c_3\right) + \left(-\frac{x\,e^x}{2} + \frac{x^2 e^x}{2} - \frac{e^x x^3}{2} + \frac{e^x x^4}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^x + e^{-3x} c_3 - \frac{x\,e^x}{2} + \frac{x^2 e^x}{2} - \frac{e^x x^3}{2} + \frac{e^x x^4}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^x + e^{-3x} c_3 - \frac{x\,e^x}{2} + \frac{x^2 e^x}{2} - \frac{e^x x^3}{2} + \frac{e^x x^4}{2}$$

Verified OK.

19.5.1 Maple step by step solution

Let's solve

$y''' + 3y'' - y' - 3y = e^x(16x^3 + 24x^2 + 2x - 1)$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 16\,\mathrm{e}^x x^3 + 24x^2\mathrm{e}^x + 2x\,\mathrm{e}^x - \mathrm{e}^x - 3y_3(x) + y_2(x) + 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 16\,\mathrm{e}^x x^3 + 24x^2\mathrm{e}^x + 2x\,\mathrm{e}^x - \mathrm{e}^x - 3y_3(x) + y_2(x) + 3y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 16\,\mathrm{e}^x x^3 + 24x^2\mathrm{e}^x + 2x\,\mathrm{e}^x - \mathrm{e}^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 16\,\mathrm{e}^x x^3 + 24x^2\mathrm{e}^x + 2x\,\mathrm{e}^x - \mathrm{e}^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \left[\begin{array}{c}\frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]\right], \left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right], \left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c}\frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c}\frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & e^{x} \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{x} \\ e^{-3x} & e^{-x} & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & e^{-x} & e^{x} \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{x} \\ e^{-3x} & e^{-x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & 1 & 1 \\ -\frac{1}{3} & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(3e^{4x}+6e^{2x}-1)e^{-3x}}{8} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{(e^{4x}-2e^{2x}+1)e^{-3x}}{8} \\ \frac{3(e^{4x}-2e^{2x}+1)e^{-3x}}{8} & \frac{e^{x}}{2}+\frac{e^{-x}}{2} & \frac{(e^{4x}+2e^{2x}-3)e^{-3x}}{8} \\ \frac{3(e^{4x}+2e^{2x}-3)e^{-3x}}{8} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{(e^{4x}-2e^{2x}+9)e^{-3x}}{8} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{e^{-x}}{4} + \frac{(2x^4-2x^3+2x^2-2x+1)e^x}{4} \\ \frac{e^{-x}}{4} + \frac{(2x^4+6x^3-4x^2+2x-1)e^x}{4} \\ -\frac{e^{-x}}{4} + \frac{(2x^4+14x^3+14x^2-6x+1)e^x}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \left[\begin{array}{c} -\frac{e^{-x}}{4} + \frac{(2x^4-2x^3+2x^2-2x+1)e^x}{4} \\ \frac{e^{-x}}{4} + \frac{(2x^4+6x^3-4x^2+2x-1)e^x}{4} \\ -\frac{e^{-x}}{4} + \frac{(2x^4+14x^3+14x^2-6x+1)e^x}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(x^4-x^3+x^2-x+2c_3+\frac{1}{2}\right)e^{4x}+\left(2c_2-\frac{1}{2}\right)e^{2x}+\frac{2c_1}{9}\right)e^{-3x}}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-diff(y(x),x)-3*y(x)=exp(x)*(-1+2*x+24*x^2+16*x^3),y(x
```

$$y(x) = \frac{\left(\left(x^4 - x^3 + x^2 + 2c_1 - x\right) e^{4x} + 2c_3 e^{2x} + 2c_2\right) e^{-3x}}{2}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 53

```
DSolve[y'''[x]+3*y''[x]-y'[x]-3*y[x]==Exp[x]*(-1+2*x+24*x^2+16*x^3),y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{4} e^x \left(2x^4 - 2x^3 + 2x^2 - 2x + 1 + 4c_3\right) + c_1 e^{-3x} + c_2 e^{-x}$$

19.6 problem section 9.3, problem 6

Internal problem ID [1503]
Internal file name [`OUTPUT/1504_Sunday_June_05_2022_02_19_54_AM_73131463/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' - 2y = e^{x}(15x^{2} + 34x + 14)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - 2y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1 - i$$
$$\lambda_3 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{(-1-i)x}$$
$$y_3 = e^{(-1+i)x}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - 2y = e^x (15x^2 + 34x + 14)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x (15x^2 + 34x + 14)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{(-1-i)x}, e^{(-1+i)x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x, x^2 e^x, e^x x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x + A_3 e^x x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^x + 8A_2 e^x + 10A_2 x e^x + 15A_3 e^x x^2 + 24A_3 e^x x + 6A_3 e^x = e^x (15x^2 + 34x + 14)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^x + e^x x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3\right) + \left(x^2 e^x + e^x x^3\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3 + x^2 e^x + e^x x^3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3 + x^2 e^x + e^x x^3$$

Verified OK.

19.6.1 Maple step by step solution

Let's solve

$y''' + y'' - 2y = e^x(15x^2 + 34x + 14)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 15x^2 e^x + 34x\, e^x - y_3(x) + 2y_1(x) + 14\, e^x$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 15x^2 e^x + 34x\, e^x - y_3(x) + 2y_1(x) + 14\, e^x]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 15x^2 e^x + 34x\, e^x + 14\, e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 15x^2 e^x + 34x\, e^x + 14\, e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ -\frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot \left(\cos \left(x\right)-\mathrm{I} \sin \left(x\right)\right) \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2}\left(\cos \left(x\right)-\mathrm{I} \sin \left(x\right)\right) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right) \left(\cos \left(x\right)-\mathrm{I} \sin \left(x\right)\right) \\ \cos \left(x\right)-\mathrm{I} \sin \left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2\left(x\right) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -\frac{\sin \left(x\right)}{2} \\ -\frac{\cos \left(x\right)}{2}+\frac{\sin \left(x\right)}{2} \\ \cos \left(x\right) \end{array}\right], \overrightarrow{y}_3\left(x\right) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -\frac{\cos \left(x\right)}{2} \\ \frac{\cos \left(x\right)}{2}+\frac{\sin \left(x\right)}{2} \\ -\sin \left(x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} \\ e^x & e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ e^x & \cos(x)e^{-x} & -\sin(x)e^{-x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^x & -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} \\ e^x & e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ e^x & \cos(x)e^{-x} & -\sin(x)e^{-x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(3\cos(x)+\sin(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(-2\cos(x)+\sin(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(-2\sin(x)-\cos(x))e^{-x}}{5} + \frac{e^x}{5} \\ \frac{(-4\sin(x)-2\cos(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(3\cos(x)+\sin(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(-\cos(x)+3\sin(x))e^{-x}}{5} + \frac{e^x}{5} \\ \frac{(-2\cos(x)+6\sin(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(-4\sin(x)-2\cos(x))e^{-x}}{5} + \frac{2e^x}{5} & \frac{(4\cos(x)-2\sin(x))e^{-x}}{5} + \frac{e^x}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(4\sin(x)+2\cos(x))\mathrm{e}^{-x}}{5} + \frac{\mathrm{e}^x(5x^3+5x^2-2)}{5} \\ \frac{(2\cos(x)-6\sin(x))\mathrm{e}^{-x}}{5} + \left(x^3+4x^2+2x-\frac{2}{5}\right)\mathrm{e}^x \\ \frac{(-8\cos(x)+4\sin(x))\mathrm{e}^{-x}}{5} + \left(10x+7x^2+\frac{8}{5}+x^3\right)\mathrm{e}^x \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{(4\sin(x)+2\cos(x))\mathrm{e}^{-x}}{5} + \frac{\mathrm{e}^x(5x^3+5x^2-2)}{5} \\ \frac{(2\cos(x)-6\sin(x))\mathrm{e}^{-x}}{5} + \left(x^3+4x^2+2x-\frac{2}{5}\right)\mathrm{e}^x \\ \frac{(-8\cos(x)+4\sin(x))\mathrm{e}^{-x}}{5} + \left(10x+7x^2+\frac{8}{5}+x^3\right)\mathrm{e}^x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((-5c_3+4)\cos(x)+(-5c_2+8)\sin(x))\mathrm{e}^{-x}}{10} + \left(c_1+x^3+x^2-\frac{2}{5}\right)\mathrm{e}^x$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-2*y(x)=exp(x)*(14+34*x+15*x^2),y(x), singsol=all)
```

$$y(x) = (c_2 \cos(x) + c_3 \sin(x)) e^{-x} + e^{x}(x^3 + x^2 + c_1)$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 49

```
DSolve[y'''[x]+y''[x]-2*y[x]==Exp[x]*(14+34*x+15*x^2),y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{5}e^{-x}\left(e^{2x}\left(5x^3 + 5x^2 - 2 + 5c_3\right) + 5c_2\cos(x) + 5c_1\sin(x)\right)$$

19.7 problem section 9.3, problem 7

Internal problem ID [1504]
Internal file name [`OUTPUT/1505_Sunday_June_05_2022_02_19_56_AM_24707965/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 7.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y''' + 8y'' - y' - 2y = -\mathrm{e}^{-2x}(1 - 15x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$4y''' + 8y'' - y' - 2y = 0$$

The characteristic equation is

$$4\lambda^3 + 8\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-\frac{x}{2}} + c_3 \mathrm{e}^{\frac{x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-\frac{x}{2}}$$
$$y_3 = \mathrm{e}^{\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$4y''' + 8y'' - y' - 2y = -\mathrm{e}^{-2x}(1 - 15x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\mathrm{e}^{-2x}(1 - 15x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x} x, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-2x}, \mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{\frac{x}{2}}\right\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-2x}, \mathrm{e}^{-2x} x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-2x} + A_2 \mathrm{e}^{-2x} x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1 \mathrm{e}^{-2x} + 30A_1 x \,\mathrm{e}^{-2x} + 15A_2 \mathrm{e}^{-2x} = -\mathrm{e}^{-2x}(1 - 15x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{-2x}}{2} + e^{-2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-\frac{x}{2}} + c_3 e^{\frac{x}{2}}\right) + \left(\frac{x^2 e^{-2x}}{2} + e^{-2x} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-\frac{x}{2}} + c_3 e^{\frac{x}{2}} + \frac{x^2 e^{-2x}}{2} + e^{-2x} x \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-\frac{x}{2}} + c_3 e^{\frac{x}{2}} + \frac{x^2 e^{-2x}}{2} + e^{-2x} x$$

Verified OK.

19.7.1 Maple step by step solution

Let's solve

$4y''' + 8y'' - y' - 2y = -e^{-2x}(1 - 15x)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y}{2} - 2y'' + \frac{y'}{4} + \frac{15 e^{-2x} x}{4} - \frac{e^{-2x}}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + 2y'' - \frac{y'}{4} - \frac{y}{2} = \frac{e^{-2x}(15x-1)}{4}$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \frac{15\,\mathrm{e}^{-2x}x}{4} - \frac{\mathrm{e}^{-2x}}{4} - 2y_3(x) + \frac{y_2(x)}{4} + \frac{y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{15\,\mathrm{e}^{-2x}x}{4} - \frac{\mathrm{e}^{-2x}}{4} - 2y_3(x) + \frac{y_2(x)}{4} + \frac{y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \frac{15\,\mathrm{e}^{-2x}x}{4} - \frac{\mathrm{e}^{-2x}}{4} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{15\,\mathrm{e}^{-2x}x}{4} - \frac{\mathrm{e}^{-2x}}{4} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right], \left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 4\,e^{-\frac{x}{2}} & 4\,e^{\frac{x}{2}} \\ -\frac{e^{-2x}}{2} & -2\,e^{-\frac{x}{2}} & 2\,e^{\frac{x}{2}} \\ e^{-2x} & e^{-\frac{x}{2}} & e^{\frac{x}{2}} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 4\,e^{-\frac{x}{2}} & 4\,e^{\frac{x}{2}} \\ -\frac{e^{-2x}}{2} & -2\,e^{-\frac{x}{2}} & 2\,e^{\frac{x}{2}} \\ e^{-2x} & e^{-\frac{x}{2}} & e^{\frac{x}{2}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 4 & 4 \\ -\frac{1}{2} & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\left(6\,e^{\frac{5x}{2}}+10\,e^{\frac{3x}{2}}-1\right)e^{-2x}}{15} & -e^{-\frac{x}{2}}+e^{\frac{x}{2}} & \frac{2\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+2\right)e^{-2x}}{15} \\ \frac{\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+2\right)e^{-2x}}{15} & \frac{e^{-\frac{x}{2}}}{2}+\frac{e^{\frac{x}{2}}}{2} & \frac{\left(3\,e^{\frac{5x}{2}}+5\,e^{\frac{3x}{2}}-8\right)e^{-2x}}{15} \\ \frac{\left(3\,e^{\frac{5x}{2}}+5\,e^{\frac{3x}{2}}-8\right)e^{-2x}}{30} & -\frac{e^{-\frac{x}{2}}}{4}+\frac{e^{\frac{x}{2}}}{4} & \frac{\left(3\,e^{\frac{5x}{2}}-5\,e^{\frac{3x}{2}}+32\right)e^{-2x}}{30} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}-10\,\mathrm{e}^{\frac{3x}{2}}+5x^2+10x+8\right)\mathrm{e}^{-2x}}{10}\\ \frac{\left(\mathrm{e}^{\frac{5x}{2}}+5\,\mathrm{e}^{\frac{3x}{2}}-10x^2-10x-6\right)\mathrm{e}^{-2x}}{10}\\ \frac{\left(\mathrm{e}^{\frac{5x}{2}}-5\,\mathrm{e}^{\frac{3x}{2}}+40x^2+4\right)\mathrm{e}^{-2x}}{20}\end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3+\left[\begin{array}{c}\frac{\left(2\,\mathrm{e}^{\frac{5x}{2}}-10\,\mathrm{e}^{\frac{3x}{2}}+5x^2+10x+8\right)\mathrm{e}^{-2x}}{10}\\ \frac{\left(\mathrm{e}^{\frac{5x}{2}}+5\,\mathrm{e}^{\frac{3x}{2}}-10x^2-10x-6\right)\mathrm{e}^{-2x}}{10}\\ \frac{\left(\mathrm{e}^{\frac{5x}{2}}-5\,\mathrm{e}^{\frac{3x}{2}}+40x^2+4\right)\mathrm{e}^{-2x}}{20}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{\left(80c_3\mathrm{e}^{\frac{5x}{2}}+4\,\mathrm{e}^{\frac{5x}{2}}+80c_2\mathrm{e}^{\frac{3x}{2}}-20\,\mathrm{e}^{\frac{3x}{2}}+10x^2+5c_1+20x+16\right)\mathrm{e}^{-2x}}{20}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(4*diff(y(x),x$3)+8*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=-exp(-2*x)*(1-15*x),y(x), sings
```

$$y(x) = \frac{\left(2c_3 e^{\frac{5x}{2}} + 2c_2 e^{\frac{3x}{2}} + x^2 + 2c_1 + 2x\right) e^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 53

```
DSolve[4*y'''[x]+8*y''[x]-y'[x]-2*y[x]==-Exp[-2*x]*(1-15*x),y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{10} e^{-2x} \left(5x^2 + 10x + 2\left(5c_1 e^{3x/2} + 5c_2 e^{5x/2} + 4 + 5c_3\right)\right)$$

19.8 problem section 9.3, problem 8

Internal problem ID [1505]
Internal file name [`OUTPUT/1506_Sunday_June_05_2022_02_19_58_AM_19111488/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - y' + y = -e^{x}(7 + 6x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - y' + y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = x e^{x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - y' + y = -e^{x}(7 + 6x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{x}(7 + 6x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{x}, e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{x}, x^2 e^{x}\}]$$

Since $x e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{x}, e^{x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{x} + A_2 e^{x} x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{x} + 12A_2 e^{x} x + 6A_2 e^{x} = -e^{x}(7 + 6x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 \mathrm{e}^x - \frac{\mathrm{e}^x x^3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^x + x\,\mathrm{e}^x c_3\right) + \left(-x^2 \mathrm{e}^x - \frac{\mathrm{e}^x x^3}{2}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^x (c_3 x + c_2) - x^2 \mathrm{e}^x - \frac{\mathrm{e}^x x^3}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^x (c_3 x + c_2) - x^2 \mathrm{e}^x - \frac{\mathrm{e}^x x^3}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^x (c_3 x + c_2) - x^2 \mathrm{e}^x - \frac{\mathrm{e}^x x^3}{2}$$

Verified OK.

19.8.1 Maple step by step solution

Let's solve

$y''' - y'' - y' + y = -\mathrm{e}^x (7 + 6x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -6x\,\mathrm{e}^{x} + y_3(x) + y_2(x) - y_1(x) - 7\,\mathrm{e}^{x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -6x\,\mathrm{e}^{x} + y_3(x) + y_2(x) - y_1(x) - 7\,\mathrm{e}^{x}]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ -6x\,\mathrm{e}^{x} - 7\,\mathrm{e}^{x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ -6x\,\mathrm{e}^{x} - 7\,\mathrm{e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_2(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 1 & 1\end{array}\right]-1\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-1\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue 1

$$\overrightarrow{y}_3(x)=\mathrm{e}^{x}\cdot\left(x\cdot\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]+\left[\begin{array}{c}-1\\ 0\\ 0\end{array}\right]\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+\overrightarrow{y}_p(x)$$

□ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{ccc}\mathrm{e}^{-x} & \mathrm{e}^{x} & (x-1)\,\mathrm{e}^{x}\\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & x\,\mathrm{e}^{x}\\ \mathrm{e}^{-x} & \mathrm{e}^{x} & x\,\mathrm{e}^{x}\end{array}\right]$$

- ○ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & e^{x} & (x-1)\,e^{x} \\ -e^{-x} & e^{x} & x\,e^{x} \\ e^{-x} & e^{x} & x\,e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -(x-1)\,e^{x} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}-\frac{e^{x}}{2}+x\,e^{x} \\ -x\,e^{x} & \frac{e^{x}}{2}+\frac{e^{-x}}{2} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2}+x\,e^{x} \\ -x\,e^{x} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}+\frac{e^{x}}{2}+x\,e^{x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$
\vec{y}_p(x) = \begin{bmatrix} e^{-x} + (-x^3 - 2x^2 + 2x - 1)\,e^{x} \\ -e^{-x} + (-x^3 - 5x^2 - 2x + 1)\,e^{x} \\ e^{-x} + (-x^3 - 5x^2 - 5x - 1)\,e^{x} \end{bmatrix}
$$

- Plug particular solution back into general solution

$$
\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} e^{-x} + (-x^3 - 2x^2 + 2x - 1)\,e^{x} \\ -e^{-x} + (-x^3 - 5x^2 - 2x + 1)\,e^{x} \\ e^{-x} + (-x^3 - 5x^2 - 5x - 1)\,e^{x} \end{bmatrix}
$$

- First component of the vector is the solution to the ODE

$$
y = (1 + c_1)\,e^{-x} - e^{x}(x^3 + 2x^2 + (-c_3 - 2)\,x - c_2 + c_3 + 1)
$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-diff(y(x),x)+y(x)=-exp(x)*(7+6*x),y(x), singsol=all)
```

$$
y(x) = e^{-x}c_2 - \frac{e^{x}(x^3 - 2c_3x + 2x^2 - 2c_1)}{2}
$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 41

```
DSolve[y'''[x]-y''[x]-y'[x]+y[x]==-Exp[x]*(7+6*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(-\frac{x^3}{2} - x^2 + x + c_3 x - \frac{1}{2} + c_2\right) + c_1 e^{-x}$$

19.9 problem section 9.3, problem 9

Internal problem ID [1506]
Internal file name [`OUTPUT/1507_Sunday_June_05_2022_02_20_00_AM_31149122/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$2y''' - 7y'' + 4y' + 4y = 0$$

The characteristic equation is

$$2\lambda^3 - 7\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 x\, e^{2x} + e^{-\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = x\, e^{2x}$$
$$y_3 = e^{-\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(17 + 30x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\, e^{2x}, e^{2x}, e^{-\frac{x}{2}}\right\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{2x}, x^2 e^{2x}\}]$$

Since $x\, e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x} + A_2 e^{2x} x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1\mathrm{e}^{2x} + 30A_2\mathrm{e}^{2x}x + 12A_2\mathrm{e}^{2x} = \mathrm{e}^{2x}(17 + 30x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x}x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2 x\,\mathrm{e}^{2x} + \mathrm{e}^{-\frac{x}{2}}c_3\right) + \left(\frac{x^2\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x}x^3\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-\frac{x}{2}}c_3 + \mathrm{e}^{2x}(c_2x + c_1) + \frac{x^2\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x}x^3$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}}c_3 + \mathrm{e}^{2x}(c_2x + c_1) + \frac{x^2\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x}x^3 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{x}{2}}c_3 + \mathrm{e}^{2x}(c_2x + c_1) + \frac{x^2\mathrm{e}^{2x}}{2} + \mathrm{e}^{2x}x^3$$

Verified OK.

19.9.1 Maple step by step solution

Let's solve

$2y''' - 7y'' + 4y' + 4y = e^{2x}(17 + 30x)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -2y + \frac{7y''}{2} - 2y' + 15x\,e^{2x} + \frac{17\,e^{2x}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{7y''}{2} + 2y' + 2y = \frac{e^{2x}(17+30x)}{2}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 15x\,e^{2x} + \frac{17\,e^{2x}}{2} + \frac{7y_3(x)}{2} - 2y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 15x\,e^{2x} + \frac{17\,e^{2x}}{2} + \frac{7y_3(x)}{2} - 2y_2(x) - 2y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & \frac{7}{2} \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 15x\,e^{2x} + \frac{17\,e^{2x}}{2} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 15x\,\mathrm{e}^{2x} + \frac{17\,\mathrm{e}^{2x}}{2} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & \frac{7}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

  $\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

  $\lambda \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x \overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \mathrm{e}^{\lambda x} \left(\lambda x \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & \frac{7}{2} \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

- Choice of $\overrightarrow{p}$

  $\overrightarrow{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$

- Second solution from eigenvalue 2

$$\vec{y}_3(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 4\,e^{-\frac{x}{2}} & \frac{e^{2x}}{4} & e^{2x}\left(\frac{x}{4} - \frac{1}{8}\right) \\ -2\,e^{-\frac{x}{2}} & \frac{e^{2x}}{2} & \frac{x\,e^{2x}}{2} \\ e^{-\frac{x}{2}} & e^{2x} & x\,e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 4\,e^{-\frac{x}{2}} & \frac{e^{2x}}{4} & e^{2x}\left(\frac{x}{4} - \frac{1}{8}\right) \\ -2\,e^{-\frac{x}{2}} & \frac{e^{2x}}{2} & \frac{x\,e^{2x}}{2} \\ e^{-\frac{x}{2}} & e^{2x} & x\,e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 4 & \frac{1}{4} & -\frac{1}{8} \\ -2 & \frac{1}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (1-2x)\,e^{2x} & -\frac{8\,e^{-\frac{x}{2}}}{5} + \frac{8\,e^{2x}}{5} - 3x\,e^{2x} & \frac{4\,e^{-\frac{x}{2}}}{5} - \frac{4\,e^{2x}}{5} + 2x\,e^{2x} \\ -4x\,e^{2x} & \frac{4\,e^{-\frac{x}{2}}}{5} + \frac{e^{2x}}{5} - 6x\,e^{2x} & -\frac{2\,e^{-\frac{x}{2}}}{5} + \frac{2\,e^{2x}}{5} + 4x\,e^{2x} \\ -8x\,e^{2x} & -\frac{2\,e^{-\frac{x}{2}}}{5} + \frac{2\,e^{2x}}{5} - 12x\,e^{2x} & \frac{e^{-\frac{x}{2}}}{5} + \frac{4\,e^{2x}}{5} + 8x\,e^{2x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot \Phi(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot \left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x)=\left[\begin{array}{c} -\frac{4\,\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(50x^3+25x^2-20x+8\right)\mathrm{e}^{2x}}{10} \\ \frac{2\,\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(50x^3+100x^2+5x-2\right)\mathrm{e}^{2x}}{5} \\ -\frac{\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(100x^3+200x^2+40x+1\right)\mathrm{e}^{2x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\left[\begin{array}{c} -\frac{4\,\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(50x^3+25x^2-20x+8\right)\mathrm{e}^{2x}}{10} \\ \frac{2\,\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(50x^3+100x^2+5x-2\right)\mathrm{e}^{2x}}{5} \\ -\frac{\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{\left(100x^3+200x^2+40x+1\right)\mathrm{e}^{2x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y=\frac{\left(160c_1-32\right)\mathrm{e}^{-\frac{x}{2}}}{40}+5\left(x^3+\frac{x^2}{2}+\left(\frac{c_3}{20}-\frac{2}{5}\right)x+\frac{c_2}{20}-\frac{c_3}{40}+\frac{4}{25}\right)\mathrm{e}^{2x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(2*diff(y(x),x$3)-7*diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=exp(2*x)*(17+30*x),y(x), sing
```

$$y(x) = \mathrm{e}^{-\frac{x}{2}} c_2 + \mathrm{e}^{2x}\left(x^3 + \frac{1}{2}x^2 + c_1 + c_3 x\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 46

```
DSolve[2*y'''[x]-7*y''[x]+4*y'[x]+4*y[x]==Exp[2*x]*(17+30*x),y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{2x}\left(x^3 + \frac{x^2}{2} + \left(-\frac{2}{5} + c_3\right)x + \frac{4}{25} + c_2\right) + c_1 e^{-x/2}$$

19.10 problem section 9.3, problem 10

Internal problem ID [1507]
Internal file name [`OUTPUT/1508_Sunday_June_05_2022_02_20_02_AM_9230329/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 5y'' + 3y' + 9y = 2\,\mathrm{e}^{3x}(11 - 24x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 5y'' + 3y' + 9y = 0$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 3$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{3x} x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{3x}$$
$$y_3 = x e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 5y'' + 3y' + 9y = 2 e^{3x}(11 - 24x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 e^{3x}(11 - 24x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{3x}, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{3x}, e^{-x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{3x}, x^2 e^{3x}\}]$$

Since $x e^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{3x}, x^3 e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{3x} + A_2 x^3 e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 e^{3x} + 6A_2 e^{3x} + 24A_2 x\, e^{3x} = 2\, e^{3x}(11 - 24x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{17}{4}, A_2 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{17x^2 e^{3x}}{4} - 2x^3 e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{3x} + c_3 e^{3x} x\right) + \left(\frac{17x^2 e^{3x}}{4} - 2x^3 e^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\, e^{3x} + c_1 e^{-x} + \frac{17x^2 e^{3x}}{4} - 2x^3 e^{3x}$$

Summary

The solution(s) found are the following

$$y = (c_3 x + c_2)\, e^{3x} + c_1 e^{-x} + \frac{17x^2 e^{3x}}{4} - 2x^3 e^{3x} \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\, e^{3x} + c_1 e^{-x} + \frac{17x^2 e^{3x}}{4} - 2x^3 e^{3x}$$

Verified OK.

19.10.1 Maple step by step solution

Let's solve

$y''' - 5y'' + 3y' + 9y = 2\,e^{3x}(11 - 24x)$

- Highest derivative means the order of the ODE is 3

y'''

- Isolate 3rd derivative

$y''' = -9y - 48x\,e^{3x} + 5y'' - 3y' + 22\,e^{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' - 5y'' + 3y' + 9y = -2\,e^{3x}(-11 + 24x)$

☐ Convert linear ODE into a system of first order ODEs

- ○ Define new variable $y_1(x)$

$y_1(x) = y$

- ○ Define new variable $y_2(x)$

$y_2(x) = y'$

- ○ Define new variable $y_3(x)$

$y_3(x) = y''$

- ○ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = -48x\,e^{3x} + 22\,e^{3x} + 5y_3(x) - 3y_2(x) - 9y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -48x\,e^{3x} + 22\,e^{3x} + 5y_3(x) - 3y_2(x) - 9y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -3 & 5 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -48x\,e^{3x} + 22\,e^{3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -48x\,\mathrm{e}^{3x} + 22\,\mathrm{e}^{3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -3 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\vec{y}_2(x) = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\vec{y}_3(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -3 & 5 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 3

$$\vec{y}_3(x) = e^{3x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & \frac{e^{3x}}{9} & e^{3x}\left(\frac{x}{9} - \frac{1}{27}\right) \\ -e^{-x} & \frac{e^{3x}}{3} & \frac{x\,e^{3x}}{3} \\ e^{-x} & e^{3x} & x\,e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & \frac{e^{3x}}{9} & e^{3x}\left(\frac{x}{9} - \frac{1}{27}\right) \\ -e^{-x} & \frac{e^{3x}}{3} & \frac{x\,e^{3x}}{3} \\ e^{-x} & e^{3x} & x\,e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{9} & -\frac{1}{27} \\ -1 & \frac{1}{3} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (-3x+1)\,e^{3x} & -\frac{3\,e^{-x}}{4} + \frac{3\,e^{3x}}{4} - 2x\,e^{3x} & \frac{e^{-x}}{4} - \frac{e^{3x}}{4} + x\,e^{3x} \\ -9x\,e^{3x} & \frac{3\,e^{-x}}{4} + \frac{e^{3x}}{4} - 6x\,e^{3x} & -\frac{e^{-x}}{4} + \frac{e^{3x}}{4} + 3x\,e^{3x} \\ -27x\,e^{3x} & -\frac{3\,e^{-x}}{4} + \frac{3\,e^{3x}}{4} - 18x\,e^{3x} & \frac{e^{-x}}{4} + \frac{3\,e^{3x}}{4} + 9x\,e^{3x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(-64x^3+136x^2-68x+17)\mathrm{e}^{3x}}{8} - \frac{17\,\mathrm{e}^{-x}}{8} \\ \frac{(-192x^3+216x^2+68x-17)\mathrm{e}^{3x}}{8} + \frac{17\,\mathrm{e}^{-x}}{8} \\ \frac{(-576x^3+648x^2+108x+17)\mathrm{e}^{3x}}{8} - \frac{17\,\mathrm{e}^{-x}}{8} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{(-64x^3+136x^2-68x+17)\mathrm{e}^{3x}}{8} - \frac{17\,\mathrm{e}^{-x}}{8} \\ \frac{(-192x^3+216x^2+68x-17)\mathrm{e}^{3x}}{8} + \frac{17\,\mathrm{e}^{-x}}{8} \\ \frac{(-576x^3+648x^2+108x+17)\mathrm{e}^{3x}}{8} - \frac{17\,\mathrm{e}^{-x}}{8} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-1728x^3+3672x^2+(24c_3-1836)x+24c_2-8c_3+459)\mathrm{e}^{3x}}{216} + \frac{(216c_1-459)\mathrm{e}^{-x}}{216}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)-5*diff(y(x),x$2)+3*diff(y(x),x)+9*y(x)=2*exp(3*x)*(11-24*x),y(x), sing
```

$$y(x) = \frac{(-8x^3 + 4c_3x + 17x^2 + 4c_2)\,\mathrm{e}^{3x}}{4} + \mathrm{e}^{-x}c_1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 46

```
DSolve[y'''[x]-5*y''[x]+3*y'[x]+9*y[x]==2*Exp[3*x]*(11-24*x),y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{3x}\left(-2x^3 + \frac{17x^2}{4} + \left(-\frac{17}{8} + c_3\right)x + \frac{17}{32} + c_2\right) + c_1e^{-x}$$

19.11 problem section 9.3, problem 11

Internal problem ID [1508]
Internal file name [`OUTPUT/1509_Sunday_June_05_2022_02_20_04_AM_97221995/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 7y'' + 8y' + 16y = 2\,\mathrm{e}^{4x}(13 + 15x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 7y'' + 8y' + 16y = 0$$

The characteristic equation is

$$\lambda^3 - 7\lambda^2 + 8\lambda + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 4$$
$$\lambda_3 = 4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{4x} + x\, e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{4x}$$
$$y_3 = x\, e^{4x}$$

Now the particular solution to the given ODE is found

$$y''' - 7y'' + 8y' + 16y = 2\, e^{4x}(13 + 15x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{4x}(13 + 15x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{4x}, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{4x}, e^{-x}, e^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{4x}, x^2 e^{4x}\}]$$

Since $x\, e^{4x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{4x}, e^{4x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{4x} + A_2 e^{4x} x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{4x} + 30A_2 e^{4x} x + 6A_2 e^{4x} = 2\,e^{4x}(13 + 15x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 e^{4x} + e^{4x} x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{4x} + x\,e^{4x} c_3\right) + \left(2x^2 e^{4x} + e^{4x} x^3\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\,e^{4x} + c_1 e^{-x} + 2x^2 e^{4x} + e^{4x} x^3$$

Summary
The solution(s) found are the following

$$y = (c_3 x + c_2)\,e^{4x} + c_1 e^{-x} + 2x^2 e^{4x} + e^{4x} x^3 \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\,e^{4x} + c_1 e^{-x} + 2x^2 e^{4x} + e^{4x} x^3$$

Verified OK.

19.11.1 Maple step by step solution

Let's solve

$y''' - 7y'' + 8y' + 16y = 2\,e^{4x}(13 + 15x)$

- Highest derivative means the order of the ODE is 3

y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 30x\,e^{4x} + 26\,e^{4x} + 7y_3(x) - 8y_2(x) - 16y_1(x)$

    Convert linear ODE into a system of first order ODEs

    $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 30x\,e^{4x} + 26\,e^{4x} + 7y_3(x) - 8y_2(x) - 16y_1(x)]$
- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -8 & 7 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 30x\,e^{4x} + 26\,e^{4x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 30x\,e^{4x} + 26\,e^{4x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -8 & 7 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A\cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[4, \left[\begin{array}{c} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{array}\right]\right], \left[4, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x}\cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[4, \left[\begin{array}{c} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 4

$$\overrightarrow{y}_2(x) = \mathrm{e}^{4x}\cdot \left[\begin{array}{c} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

  $$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ -16 & -8 & 7\end{array}\right]-4\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}\frac{1}{16}\\ \frac{1}{4}\\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c}-\frac{1}{64}\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue 4

  $$\overrightarrow{y}_3(x)=\mathrm{e}^{4x}\cdot\left(x\cdot\left[\begin{array}{c}\frac{1}{16}\\ \frac{1}{4}\\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{64}\\ 0\\ 0\end{array}\right]\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+\overrightarrow{y}_p(x)$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x)=\left[\begin{array}{ccc}\mathrm{e}^{-x} & \frac{\mathrm{e}^{4x}}{16} & \mathrm{e}^{4x}\left(\frac{x}{16}-\frac{1}{64}\right)\\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{4x}}{4} & \frac{x\,\mathrm{e}^{4x}}{4}\\ \mathrm{e}^{-x} & \mathrm{e}^{4x} & x\,\mathrm{e}^{4x}\end{array}\right]$$

  ◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{4x}}{16} & \mathrm{e}^{4x}\left(\frac{x}{16}-\frac{1}{64}\right) \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{4x}}{4} & \frac{x\,\mathrm{e}^{4x}}{4} \\ \mathrm{e}^{-x} & \mathrm{e}^{4x} & x\,\mathrm{e}^{4x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & \frac{1}{16} & -\frac{1}{64} \\ -1 & \frac{1}{4} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} (-4x+1)\,\mathrm{e}^{4x} & -\frac{4\,\mathrm{e}^{-x}}{5}+\frac{4\,\mathrm{e}^{4x}}{5}-3x\,\mathrm{e}^{4x} & \frac{\mathrm{e}^{-x}}{5}-\frac{\mathrm{e}^{4x}}{5}+x\,\mathrm{e}^{4x} \\ -16x\,\mathrm{e}^{4x} & \frac{4\,\mathrm{e}^{-x}}{5}+\frac{\mathrm{e}^{4x}}{5}-12x\,\mathrm{e}^{4x} & -\frac{\mathrm{e}^{-x}}{5}+\frac{\mathrm{e}^{4x}}{5}+4x\,\mathrm{e}^{4x} \\ -64x\,\mathrm{e}^{4x} & -\frac{4\,\mathrm{e}^{-x}}{5}+\frac{4\,\mathrm{e}^{4x}}{5}-48x\,\mathrm{e}^{4x} & \frac{\mathrm{e}^{-x}}{5}+\frac{4\,\mathrm{e}^{4x}}{5}+16x\,\mathrm{e}^{4x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(25x^3+50x^2-20x+4)\,e^{4x}}{5} - \frac{4\,e^{-x}}{5} \\ \frac{(100x^3+275x^2+20x-4)\,e^{4x}}{5} + \frac{4\,e^{-x}}{5} \\ \frac{2(200x^3+550x^2+55x+2)\,e^{4x}}{5} - \frac{4\,e^{-x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(25x^3+50x^2-20x+4)\,e^{4x}}{5} - \frac{4\,e^{-x}}{5} \\ \frac{(100x^3+275x^2+20x-4)\,e^{4x}}{5} + \frac{4\,e^{-x}}{5} \\ \frac{2(200x^3+550x^2+55x+2)\,e^{4x}}{5} - \frac{4\,e^{-x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(256+1600x^3+3200x^2+20(-64+c_3)x+20c_2-5c_3)\,e^{4x}}{320} + \frac{e^{-x}(5c_1-4)}{5}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$3)-7*diff(y(x),x$2)+8*diff(y(x),x)+16*y(x)=2*exp(4*x)*(13+15*x),y(x), sin
```

$$y(x) = \left(x^3 + c_3x + 2x^2 + c_2\right) e^{4x} + e^{-x}c_1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 42

```
DSolve[y'''[x]-7*y''[x]+8*y'[x]+16*y[x]==2*Exp[4*x]*(13+15*x),y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{4x}\left(x^3 + 2x^2 + \left(-\frac{4}{5} + c_3\right)x + \frac{4}{25} + c_2\right) + c_1 e^{-x}$$

19.12 problem section 9.3, problem 12

Internal problem ID [1509]
Internal file name [`OUTPUT/1510_Sunday_June_05_2022_02_20_06_AM_92045200/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 12.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{8y''' - 12y'' + 6y' - y = e^{\frac{x}{2}}(4x + 1)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$8y''' - 12y'' + 6y' - y = 0$$

The characteristic equation is

$$8\lambda^3 - 12\lambda^2 + 6\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x + x^2 e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\frac{x}{2}}$$
$$y_2 = e^{\frac{x}{2}} x$$
$$y_3 = x^2 e^{\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$8y''' - 12y'' + 6y' - y = e^{\frac{x}{2}} (4x + 1)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{\frac{x}{2}} (4x + 1)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{e^{\frac{x}{2}} x, e^{\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x^2 e^{\frac{x}{2}}, e^{\frac{x}{2}} x, e^{\frac{x}{2}}\right\}$$

Since $e^{\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x^2 e^{\frac{x}{2}}, e^{\frac{x}{2}} x\right\}\right]$$

Since $e^{\frac{x}{2}} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x^2 e^{\frac{x}{2}}, x^3 e^{\frac{x}{2}}\right\}\right]$$

Since $x^2 e^{\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x^3 e^{\frac{x}{2}}, x^4 e^{\frac{x}{2}}\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 e^{\frac{x}{2}} + A_2 x^4 e^{\frac{x}{2}}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$48A_1 e^{\frac{x}{2}} + 192A_2 x\, e^{\frac{x}{2}} = e^{\frac{x}{2}}(4x+1)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{48}, A_2 = \frac{1}{48}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3 e^{\frac{x}{2}}}{48} + \frac{x^4 e^{\frac{x}{2}}}{48}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x + x^2 e^{\frac{x}{2}} c_3\right) + \left(\frac{x^3 e^{\frac{x}{2}}}{48} + \frac{x^4 e^{\frac{x}{2}}}{48}\right) \end{aligned}$$

Which simplifies to

$$y = e^{\frac{x}{2}}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{\frac{x}{2}}}{48} + \frac{x^4 e^{\frac{x}{2}}}{48}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x}{2}}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{\frac{x}{2}}}{48} + \frac{x^4 e^{\frac{x}{2}}}{48} \tag{1}$$

Verification of solutions

$$y = e^{\frac{x}{2}}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{x^3 e^{\frac{x}{2}}}{48} + \frac{x^4 e^{\frac{x}{2}}}{48}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(8*diff(y(x),x$3)-12*diff(y(x),x$2)+6*diff(y(x),x)-y(x)=exp(x/2)*(1+4*x),y(x), singsol
```

$$y(x) = \frac{\left(x^4 + x^3 + \left(48c_2 + \frac{3}{16}\right)x^2 + 48c_3x + 48c_1\right)e^{\frac{x}{2}}}{48}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 39

```
DSolve[8*y'''[x]-12*y''[x]+6*y'[x]-y[x]==Exp[x/2]*(1+4*x),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{48}e^{x/2}\left(x^4 + x^3 + 48c_3x^2 + 48c_2x + 48c_1\right)$$

19.13 problem section 9.3, problem 13

Internal problem ID [1510]
Internal file name [OUTPUT/1511_Sunday_June_05_2022_02_20_08_AM_9104730/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 13.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 3y''' - 3y'' - 7y' + 6y = -3\,\mathrm{e}^{-x}\left(-8x^2 + 8x + 12\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y''' - 3y'' - 7y' + 6y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 - 3\lambda^2 - 7\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3 + \mathrm{e}^{-3x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-2x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= x\,\mathrm{e}^{x} \\ y_4 &= \mathrm{e}^{-3x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y''' - 3y'' - 7y' + 6y = -3\,\mathrm{e}^{-x}\left(-8x^2 + 8x + 12\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\,\mathrm{e}^{-x}\left(-8x^2 + 8x + 12\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}, \mathrm{e}^{-3x}, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 x^2 \mathrm{e}^{-x} + A_3 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &8A_2 x\,\mathrm{e}^{-x} + 8A_1 x\,\mathrm{e}^{-x} + 8A_2 x^2 \mathrm{e}^{-x} + 8A_3 \mathrm{e}^{-x} - 12A_2 \mathrm{e}^{-x} + 4A_1 \mathrm{e}^{-x} \\ &= -3\,\mathrm{e}^{-x}\left(-8x^2 + 8x + 12\right) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -6, A_2 = 3, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -6x\,e^{-x} + 3x^2e^{-x} + 3\,e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-2x} + c_2e^{x} + x\,e^{x}c_3 + e^{-3x}c_4\right) + \left(-6x\,e^{-x} + 3x^2e^{-x} + 3\,e^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = \left((c_3x + c_2)\,e^{4x} + c_1e^{x} + c_4\right)e^{-3x} - 6x\,e^{-x} + 3x^2e^{-x} + 3\,e^{-x}$$

Summary

The solution(s) found are the following

$$y = \left((c_3x + c_2)\,e^{4x} + c_1e^{x} + c_4\right)e^{-3x} - 6x\,e^{-x} + 3x^2e^{-x} + 3\,e^{-x} \tag{1}$$

Verification of solutions

$$y = \left((c_3x + c_2)\,e^{4x} + c_1e^{x} + c_4\right)e^{-3x} - 6x\,e^{-x} + 3x^2e^{-x} + 3\,e^{-x}$$

Verified OK.

19.13.1 Maple step by step solution

Let's solve

$y'''' + 3y''' - 3y'' - 7y' + 6y = -3\,e^{-x}(-8x^2 + 8x + 12)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -6y + 24x^2e^{-x} - 24x\,e^{-x} - 36\,e^{-x} - 3y''' + 3y'' + 7y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 3y''' - 3y'' - 7y' + 6y = 12\,e^{-x}(2x^2 - 2x - 3)$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 24x^2 e^{-x} - 24x\, e^{-x} - 36\, e^{-x} - 3y_4(x) + 3y_3(x) + 7y_2(x) - 6y_1(x)$

    Convert linear ODE into a system of first order ODEs

    $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 24x^2 e^{-x} - 24x\, e^{-x} - 36\, e^{-x} - 3y_4(x) + 3y$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 7 & 3 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 24x^2 e^{-x} - 24x\, e^{-x} - 36\, e^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 24x^2 e^{-x} - 24x\, e^{-x} - 36\, e^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 7 & 3 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_3(x) = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_4(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_4(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_4(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 7 & 3 & -3 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\overrightarrow{y}_4(x) = \mathrm{e}^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3(x) + c_4 \overrightarrow{y}_4(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{\mathrm{e}^{-3x}}{27} & -\frac{\mathrm{e}^{-2x}}{8} & \mathrm{e}^x & (x-1)\,\mathrm{e}^x \\ \frac{\mathrm{e}^{-3x}}{9} & \frac{\mathrm{e}^{-2x}}{4} & \mathrm{e}^x & x\,\mathrm{e}^x \\ -\frac{\mathrm{e}^{-3x}}{3} & -\frac{\mathrm{e}^{-2x}}{2} & \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^{-3x} & \mathrm{e}^{-2x} & \mathrm{e}^x & x\,\mathrm{e}^x \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{cccc} -\frac{e^{-3x}}{27} & -\frac{e^{-2x}}{8} & e^{x} & (x-1)\,e^{x} \\ \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{x} & x\,e^{x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & e^{x} & x\,e^{x} \\ e^{-3x} & e^{-2x} & e^{x} & x\,e^{x} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cccc} -\frac{1}{27} & -\frac{1}{8} & 1 & -1 \\ \frac{1}{9} & \frac{1}{4} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{cccc} -(x-1)\,e^{x} & \frac{((2+x)e^{4x}-3\,e^{x}+1)e^{-3x}}{6} & \frac{(8x\,e^{4x}-3\,e^{4x}+4\,e^{x}-1)e^{-3x}}{12} & \frac{(2x\,e^{4x}-e^{4x}+2\,e^{x}-1)e^{-3x}}{12} \\ -x\,e^{x} & \frac{((x+3)e^{4x}+6\,e^{x}-3)e^{-3x}}{6} & -\frac{(-8x\,e^{4x}-5\,e^{4x}+8\,e^{x}-3)e^{-3x}}{12} & \frac{(2x\,e^{4x}+e^{4x}-4\,e^{x}+3)e^{-3x}}{12} \\ -x\,e^{x} & \frac{((x+3)e^{4x}-12\,e^{x}+9)e^{-3x}}{6} & \frac{(8x\,e^{4x}+5\,e^{4x}+16\,e^{x}-9)e^{-3x}}{12} & \frac{(2x\,e^{4x}+e^{4x}+8\,e^{x}-9)e^{-3x}}{12} \\ -x\,e^{x} & \frac{((x+3)e^{4x}+24\,e^{x}-27)e^{-3x}}{6} & -\frac{(-8x\,e^{4x}-5\,e^{4x}+32\,e^{x}-27)e^{-3x}}{12} & \frac{(2x\,e^{4x}+e^{4x}-16\,e^{x}+27)e^{-3x}}{12} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} e^{-3x}\left((5x^2-9x+4)\,e^{2x} - 3x\,e^{4x} - 6\,e^{x} + \frac{5\,e^{4x}}{2} - \frac{1}{2}\right) \\ e^{-3x}\left((-5x^2+19x-13)\,e^{2x} - 3x\,e^{4x} + 12\,e^{x} - \frac{e^{4x}}{2} + \frac{3}{2}\right) \\ e^{-3x}\left((7x^2-29x+29)\,e^{2x} - 3x\,e^{4x} - 24\,e^{x} - \frac{e^{4x}}{2} - \frac{9}{2}\right) \\ e^{-3x}\left((-5x^2+43x-61)\,e^{2x} - 3x\,e^{4x} + 48\,e^{x} - \frac{e^{4x}}{2} + \frac{27}{2}\right) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} e^{-3x}\left((5x^2-9x+4)\,e^{2x} - 3x\,e^{4x} - 6\,e^{x} + \frac{5\,e^{4x}}{2} \right. \\ e^{-3x}\left((-5x^2+19x-13)\,e^{2x} - 3x\,e^{4x} + 12\,e^{x} - \frac{e^{4x}}{2} \right. \\ e^{-3x}\left((7x^2-29x+29)\,e^{2x} - 3x\,e^{4x} - 24\,e^{x} - \frac{e^{4x}}{2} \right. \\ e^{-3x}\left((-5x^2+43x-61)\,e^{2x} - 3x\,e^{4x} + 48\,e^{x} - \frac{e^{4x}}{2} \right. \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = 5\left(-\frac{1}{10} + \left(\frac{1}{2} + \frac{(c_4-3)x}{5} + \frac{c_3}{5} - \frac{c_4}{5}\right)e^{4x} + \left(x^2 - \frac{9}{5}x + \frac{4}{5}\right)e^{2x} + \frac{\left(-6-\frac{c_2}{8}\right)e^{x}}{5} - \frac{c_1}{135}\right)e^{-3x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$3)-3*diff(y(x),x$2)-7*diff(y(x),x)+6*y(x)=-3*exp(-x)*(12
```

$$y(x) = 3\left((x-1)^2 e^{2x} + \frac{(c_4 x + c_1) e^{4x}}{3} + \frac{c_3 e^x}{3} + \frac{c_2}{3}\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 45

```
DSolve[y''''[x]+3*y'''[x]-3*y''[x]-7*y'[x]+6*y[x]==-3*Exp[-x]*(12+8*x-8*x^2),y[x],x,IncludeS
```

$$y(x) \to e^{-3x}\left(3e^{2x}(x-1)^2 + c_2 e^x + e^{4x}(c_4 x + c_3) + c_1\right)$$

19.14 problem section 9.3, problem 14

Internal problem ID [1511]
Internal file name [OUTPUT/1512_Sunday_June_05_2022_02_20_10_AM_58351539/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 3y''' + y'' - 3y' - 2y = -3\,\mathrm{e}^{2x}(11 + 12x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y''' + y'' - 3y' - 2y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 + \lambda^2 - 3\lambda - 2 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= 1\\ \lambda_3 &= -1\\ \lambda_4 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\,e^{-x} c_2 + e^{-2x} c_3 + e^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= x\,e^{-x} \\ y_3 &= e^{-2x} \\ y_4 &= e^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y''' + y'' - 3y' - 2y = -3\,e^{2x}(11 + 12x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-3\,e^{2x}(11 + 12x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{-x}, e^{x}, e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$69A_1 e^{2x} + 36A_1 x\,e^{2x} + 36A_2 e^{2x} = -3\,e^{2x}(11 + 12x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{-2x} c_3 + \mathrm{e}^{x} c_4\right) + \left(-x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \left(\mathrm{e}^{3x} c_4 + \mathrm{e}^{x}(c_2 x + c_1) + c_3\right) \mathrm{e}^{-2x} - x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = \left(\mathrm{e}^{3x} c_4 + \mathrm{e}^{x}(c_2 x + c_1) + c_3\right) \mathrm{e}^{-2x} - x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} \tag{1}$$

Verification of solutions

$$y = \left(\mathrm{e}^{3x} c_4 + \mathrm{e}^{x}(c_2 x + c_1) + c_3\right) \mathrm{e}^{-2x} - x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}$$

Verified OK.

19.14.1 Maple step by step solution

Let's solve

$y'''' + 3y''' + y'' - 3y' - 2y = -3\,\mathrm{e}^{2x}(11 + 12x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -36x\,\mathrm{e}^{2x} - 33\,\mathrm{e}^{2x} - 3y_4(x) - y_3(x) + 3y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = -36x\,\mathrm{e}^{2x} - 33\,\mathrm{e}^{2x} - 3y_4(x) - y_3(x) + 3y_2($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & -1 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -36x\,\mathrm{e}^{2x} - 33\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -36x\,\mathrm{e}^{2x} - 33\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & -1 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$

- Simplify equation

$\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$

- Make use of the identity matrix I

$(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(x)$ to be a solution to the homogeneous system

$(A - \lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & -1 & -3 \end{array}\right] - (-1)\cdot\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p} = \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{y}_3(x) = \mathrm{e}^{-x}\cdot\left(x\cdot\left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4 + \vec{y}_p(x)$

☐ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{-x}(-x-1) & e^x \\ \frac{e^{-2x}}{4} & e^{-x} & x\,e^{-x} & e^x \\ -\frac{e^{-2x}}{2} & -e^{-x} & -x\,e^{-x} & e^x \\ e^{-2x} & e^{-x} & x\,e^{-x} & e^x \end{bmatrix}$$

  ○ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  ○ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{-x}(-x-1) & e^x \\ \frac{e^{-2x}}{4} & e^{-x} & x\,e^{-x} & e^x \\ -\frac{e^{-2x}}{2} & -e^{-x} & -x\,e^{-x} & e^x \\ e^{-2x} & e^{-x} & x\,e^{-x} & e^x \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & -1 & 1 \\ \frac{1}{4} & 1 & 0 & 1 \\ -\frac{1}{2} & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}}$$

  ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (x+1)\,e^{-x} & \frac{(2\,e^{3x}+1+(3x-3)e^{x})e^{-2x}}{6} & -\frac{e^{-x}}{2} - x\,e^{-x} + \frac{e^{x}}{2} & \frac{(e^{3x}-3x\,e^{x}-1)e^{-2x}}{6} \\ -x\,e^{-x} & \frac{(2\,e^{3x}-2+(6-3x)e^{x})e^{-2x}}{6} & -\frac{e^{-x}}{2} + x\,e^{-x} + \frac{e^{x}}{2} & \frac{(e^{3x}+2+(3x-3)e^{x})e^{-2x}}{6} \\ x\,e^{-x} & \frac{(2\,e^{3x}+4+(3x-6)e^{x})e^{-2x}}{6} & \frac{e^{-x}}{2} - x\,e^{-x} + \frac{e^{x}}{2} & \frac{(e^{3x}-4+(-3x+3)e^{x})e^{-2x}}{6} \\ -x\,e^{-x} & \frac{(2\,e^{3x}-8+(6-3x)e^{x})e^{-2x}}{6} & -\frac{e^{-x}}{2} + x\,e^{-x} + \frac{e^{x}}{2} & \frac{(e^{3x}+8+(3x-3)e^{x})e^{-2x}}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{(5x\,e^{4x}-4\,e^{4x}+e^{3x}+7x\,e^{x}+e^{x}+2)e^{-2x}}{2} \\ -\frac{(10x\,e^{4x}-3\,e^{4x}+e^{3x}-7x\,e^{x}+6\,e^{x}-4)e^{-2x}}{2} \\ -\frac{(8x\,e^{4x}-3\,e^{4x}+e^{3x}+7x\,e^{x}-6\,e^{x}+8)e^{-2x}}{2} \\ -\frac{(28x\,e^{4x}+9\,e^{4x}+e^{3x}-7x\,e^{x}+6\,e^{x}-16)e^{-2x}}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4+\left[\begin{array}{c} -\frac{\left(5x\,\mathrm{e}^{4x}-4\,\mathrm{e}^{4x}+\mathrm{e}^{3x}+7x\,\mathrm{e}^{x}+\mathrm{e}^{x}+2\right)\mathrm{e}^{-2x}}{2} \\ -\frac{\left(10x\,\mathrm{e}^{4x}-3\,\mathrm{e}^{4x}+\mathrm{e}^{3x}-7x\,\mathrm{e}^{x}+6\,\mathrm{e}^{x}-4\right)\mathrm{e}^{-2x}}{2} \\ -\frac{\left(8x\,\mathrm{e}^{4x}-3\,\mathrm{e}^{4x}+\mathrm{e}^{3x}+7x\,\mathrm{e}^{x}-6\,\mathrm{e}^{x}+8\right)\mathrm{e}^{-2x}}{2} \\ -\frac{\left(28x\,\mathrm{e}^{4x}+9\,\mathrm{e}^{4x}+\mathrm{e}^{3x}-7x\,\mathrm{e}^{x}+6\,\mathrm{e}^{x}-16\right)\mathrm{e}^{-2x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\left(\left(-c_4+\frac{1}{2}\right)\mathrm{e}^{3x}+\left(\frac{5x}{2}-2\right)\mathrm{e}^{4x}+\left(\left(c_3+\frac{7}{2}\right)x+c_2+c_3+\frac{1}{2}\right)\mathrm{e}^{x}+\frac{c_1}{8}+1\right)\mathrm{e}^{-2x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$3)+diff(y(x),x$2)-3*diff(y(x),x)-2*y(x)=-3*exp(2*x)*(11+
```

$$y(x)=\left(-\mathrm{e}^{4x}x+\mathrm{e}^{4x}+c_1\mathrm{e}^{3x}+c_4x\,\mathrm{e}^{x}+c_3\mathrm{e}^{x}+c_2\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 43

```
DSolve[y''''[x]+3*y'''[x]+y''[x]-3*y'[x]-2*y[x]==-3*Exp[2*x]*(11+12*x),y[x],x,IncludeSingula
```

$$y(x)\to e^{-2x}\left(-e^{4x}(x-1)+e^{x}(c_3x+c_2)+c_4e^{3x}+c_1\right)$$

19.15 problem section 9.3, problem 15

Internal problem ID [1512]
Internal file name [OUTPUT/1513_Sunday_June_05_2022_02_20_12_AM_13644452/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 15.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 8y''' + 24y'' + 32y' = -16\,\mathrm{e}^{-2x}\left(-x^3 + x^2 + x + 1\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 8y''' + 24y'' + 32y' = 0$$

The characteristic equation is

$$\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -4$$
$$\lambda_3 = -2 - 2i$$
$$\lambda_4 = -2 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-4x} + e^{(-2+2i)x} c_3 + e^{(-2-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= e^{-4x} \\ y_3 &= e^{(-2+2i)x} \\ y_4 &= e^{(-2-2i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 8y''' + 24y'' + 32y' = -16\, e^{-2x} \left(-x^3 + x^2 + x + 1\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-16\, e^{-2x} \left(-x^3 + x^2 + x + 1\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{-2x}, x^3 e^{-2x}, e^{-2x} x, e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{(-2-2i)x}, e^{(-2+2i)x}, e^{-4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{-2x} + A_2 x^3 e^{-2x} + A_3 e^{-2x} x + A_4 e^{-2x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-16A_4 e^{-2x} - 16A_1 x^2 e^{-2x} - 16A_2 x^3 e^{-2x} - 16A_3 e^{-2x} x = -16\, e^{-2x} \left(-x^3 + x^2 + x + 1\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1, A_3 = 1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \mathrm{e}^{-2x} - x^3 \mathrm{e}^{-2x} + \mathrm{e}^{-2x} x + \mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 \mathrm{e}^{-4x} + \mathrm{e}^{(-2+2i)x} c_3 + \mathrm{e}^{(-2-2i)x} c_4\right) + \left(x^2 \mathrm{e}^{-2x} - x^3 \mathrm{e}^{-2x} + \mathrm{e}^{-2x} x + \mathrm{e}^{-2x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-4x} + \mathrm{e}^{(-2+2i)x} c_3 + \mathrm{e}^{(-2-2i)x} c_4 + x^2 \mathrm{e}^{-2x} - x^3 \mathrm{e}^{-2x} + \mathrm{e}^{-2x} x + \mathrm{e}^{-2x} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \mathrm{e}^{-4x} + \mathrm{e}^{(-2+2i)x} c_3 + \mathrm{e}^{(-2-2i)x} c_4 + x^2 \mathrm{e}^{-2x} - x^3 \mathrm{e}^{-2x} + \mathrm{e}^{-2x} x + \mathrm{e}^{-2x}$$

Verified OK.

19.15.1 Maple step by step solution

Let's solve

$y'''' + 8y''' + 24y'' + 32y' = -16\,\mathrm{e}^{-2x}(-x^3 + x^2 + x + 1)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = 16x^3 \mathrm{e}^{-2x} - 16x^2 \mathrm{e}^{-2x} - 16\,\mathrm{e}^{-2x} x - 16\,\mathrm{e}^{-2x} - 8y''' - 24y'' - 32y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 8y''' + 24y'' + 32y' = 16\,\mathrm{e}^{-2x}(x^3 - x^2 - x - 1)$

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 16x^3 e^{-2x} - 16x^2 e^{-2x} - 16\, e^{-2x} x - 16\, e^{-2x} - 8y_4(x) - 24y_3(x) - 32y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16x^3 e^{-2x} - 16x^2 e^{-2x} - 16\, e^{-2x} x - 16\, e^{-2x}$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -32 & -24 & -8 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16x^3 e^{-2x} - 16x^2 e^{-2x} - 16\, e^{-2x} x - 16\, e^{-2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16x^3 e^{-2x} - 16x^2 e^{-2x} - 16\, e^{-2x} x - 16\, e^{-2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -32 & -24 & -8 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \left[\begin{array}{c} -\frac{1}{64} \\ \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[-2-2\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right], \left[-2+2\,\mathrm{I}, \left[\begin{array}{c} \frac{1}{32}-\frac{\mathrm{I}}{32} \\ \frac{\mathrm{I}}{8} \\ -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c} -\frac{1}{64} \\ \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-4x} \cdot \left[\begin{array}{c} -\frac{1}{64} \\ \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-2\,\mathrm{I})x} \cdot \begin{bmatrix} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-2x} \cdot (\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} \frac{1}{32}+\frac{\mathrm{I}}{32} \\ -\frac{\mathrm{I}}{8} \\ -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-2x} \cdot \begin{bmatrix} \left(\frac{1}{32}+\frac{\mathrm{I}}{32}\right)(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32} + \frac{\sin(2x)}{32} \\ -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} + \frac{\sin(2x)}{4} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{\cos(2x)}{32} - \frac{\sin(2x)}{32} \\ -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} + \frac{\cos(2x)}{4} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{\mathrm{e}^{-4x}}{64} & 1 & \mathrm{e}^{-2x}\left(\frac{\cos(2x)}{32} + \frac{\sin(2x)}{32}\right) & \mathrm{e}^{-2x}\left(\frac{\cos(2x)}{32} - \frac{\sin(2x)}{32}\right) \\ \frac{\mathrm{e}^{-4x}}{16} & 0 & -\frac{\mathrm{e}^{-2x}\sin(2x)}{8} & -\frac{\mathrm{e}^{-2x}\cos(2x)}{8} \\ -\frac{\mathrm{e}^{-4x}}{4} & 0 & \mathrm{e}^{-2x}\left(-\frac{\cos(2x)}{4} + \frac{\sin(2x)}{4}\right) & \mathrm{e}^{-2x}\left(\frac{\sin(2x)}{4} + \frac{\cos(2x)}{4}\right) \\ \mathrm{e}^{-4x} & 0 & \mathrm{e}^{-2x}\cos(2x) & -\mathrm{e}^{-2x}\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{\mathrm{e}^{-4x}}{64} & 1 & \mathrm{e}^{-2x}\left(\frac{\cos(2x)}{32} + \frac{\sin(2x)}{32}\right) & \mathrm{e}^{-2x}\left(\frac{\cos(2x)}{32} - \frac{\sin(2x)}{32}\right) \\ \frac{\mathrm{e}^{-4x}}{16} & 0 & -\frac{\mathrm{e}^{-2x}\sin(2x)}{8} & -\frac{\mathrm{e}^{-2x}\cos(2x)}{8} \\ -\frac{\mathrm{e}^{-4x}}{4} & 0 & \mathrm{e}^{-2x}\left(-\frac{\cos(2x)}{4} + \frac{\sin(2x)}{4}\right) & \mathrm{e}^{-2x}\left(\frac{\sin(2x)}{4} + \frac{\cos(2x)}{4}\right) \\ \mathrm{e}^{-4x} & 0 & \mathrm{e}^{-2x}\cos(2x) & -\mathrm{e}^{-2x}\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{64} & 1 & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{16} & 0 & 0 & -\frac{1}{8} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{(-2\cos(2x)-2\sin(2x))\mathrm{e}^{-2x}}{4} - \frac{\mathrm{e}^{-4x}}{4} + \frac{3}{4} & \frac{(-\cos(2x)-3\sin(2x))\mathrm{e}^{-2x}}{8} - \frac{\mathrm{e}^{-4x}}{8} + \frac{1}{4} \\ 0 & \mathrm{e}^{-4x} + 2\,\mathrm{e}^{-2x}\sin(2x) & \frac{(-\cos(2x)+2\sin(2x))\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-4x}}{2} \\ 0 & -4\,\mathrm{e}^{-4x} + (4\cos(2x) - 4\sin(2x))\,\mathrm{e}^{-2x} & (3\cos(2x) - \sin(2x))\,\mathrm{e}^{-2x} - 2\,\mathrm{e}^{-4x} \\ 0 & 16\,\mathrm{e}^{-4x} - 16\,\mathrm{e}^{-2x}\cos(2x) & (-8\cos(2x) - 4\sin(2x))\,\mathrm{e}^{-2x} + 8\,\mathrm{e}^{-4x} \quad (\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms
  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$
  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution
  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{5}{16} + \frac{\left(-8x^3+8x^2+8x-2\cos(2x)-5\sin(2x)+8\right)\mathrm{e}^{-2x}}{8} - \frac{7\,\mathrm{e}^{-4x}}{16} \\ \frac{\left(8x^3-20x^2-3\cos(2x)+7\sin(2x)-4\right)\mathrm{e}^{-2x}}{4} + \frac{7\,\mathrm{e}^{-4x}}{4} \\ \left(-4x^3+16x^2-10x+5\cos\left(2x\right)-2\sin\left(2x\right)+2\right)\mathrm{e}^{-2x} - 7\,\mathrm{e}^{-4x} \\ \left(8x^3-44x^2+52x-14\cos\left(2x\right)-6\sin\left(2x\right)-14\right)\mathrm{e}^{-2x} + 28\,\mathrm{e}^{-4x} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} -\frac{5}{16} + \frac{\left(-8x^3+8x^2+8x-2\cos(2x)-5\sin(2x)+8\right)\mathrm{e}^{-2x}}{8} \\ \frac{\left(8x^3-20x^2-3\cos(2x)+7\sin(2x)-4\right)\mathrm{e}^{-2x}}{4} + \frac{7\,\mathrm{e}^{-}}{4} \\ \left(-4x^3+16x^2-10x+5\cos\left(2x\right)-2\sin\left(2x\right)+2 \right. \\ \left(8x^3-44x^2+52x-14\cos\left(2x\right)-6\sin\left(2x\right)-14 \right. \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -\frac{5}{16} + \frac{(32+(-8+c_3+c_4)\cos(2x)+(-20+c_3-c_4)\sin(2x)-32x^3+32x^2+32x)\,\mathrm{e}^{-2x}}{32} + \frac{(-c_1-28)\,\mathrm{e}^{-4x}}{64} + c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = 16*exp(-2*_a)*_a^3-
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$4)+8*diff(y(x),x$3)+24*diff(y(x),x$2)+32*diff(y(x),x)=-16*exp(-2*x)*(1+x+
```

$$y(x) = \frac{((-c_2 - c_3)\cos(2x) + (c_2 - c_3)\sin(2x) - 4x^3 + 4x^2 + 4x + 4)\,\mathrm{e}^{-2x}}{4} - \frac{\mathrm{e}^{-4x} c_1}{4} + c_4$$

✓ Solution by Mathematica

Time used: 0.712 (sec). Leaf size: 64

```
DSolve[y''''[x]+8*y'''[x]+24*y''[x]+32*y'[x]==-16*Exp[-2*x]*(1+x+x^2-x^3),y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{4}e^{-2x}\left(-4x^3 + 4x^2 + 4x - c_3 e^{-2x} - (c_1 + c_2)\cos(2x) + (c_2 - c_1)\sin(2x) + 4\right) + c_4$$

19.16 problem section 9.3, problem 16

Internal problem ID [1513]
Internal file name [OUTPUT/1514_Sunday_June_05_2022_02_20_14_AM_64334905/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 16.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'''' - 11y'' - 9y' - 2y = -e^{x}(1 - 6x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$4y'''' - 11y'' - 9y' - 2y = 0$$

The characteristic equation is

$$4\lambda^4 - 11\lambda^2 - 9\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -\frac{1}{2}$$
$$\lambda_4 = -\frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + e^{-\frac{x}{2}} c_3 + x\, e^{-\frac{x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{-\frac{x}{2}}$$
$$y_4 = x\, e^{-\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$4y'''' - 11y'' - 9y' - 2y = -e^{x}(1 - 6x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{x}(1 - 6x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\, e^{-\frac{x}{2}}, e^{-x}, e^{2x}, e^{-\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, e^{x} + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-15A_1 e^{x} - 18A_1 x\, e^{x} - 18A_2 e^{x} = -e^{x}(1 - 6x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^x}{3} + \frac{e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{2x} + e^{-\frac{x}{2}} c_3 + x\,e^{-\frac{x}{2}} c_4\right) + \left(-\frac{x\,e^x}{3} + \frac{e^x}{3}\right) \end{aligned}$$

Which simplifies to

$$y = \left(\left(c_4 x + c_3\right) e^{\frac{x}{2}} + c_2 e^{3x} + c_1\right) e^{-x} - \frac{x\,e^x}{3} + \frac{e^x}{3}$$

Summary
The solution(s) found are the following

$$y = \left(\left(c_4 x + c_3\right) e^{\frac{x}{2}} + c_2 e^{3x} + c_1\right) e^{-x} - \frac{x\,e^x}{3} + \frac{e^x}{3} \tag{1}$$

Verification of solutions

$$y = \left(\left(c_4 x + c_3\right) e^{\frac{x}{2}} + c_2 e^{3x} + c_1\right) e^{-x} - \frac{x\,e^x}{3} + \frac{e^x}{3}$$

Verified OK.

19.16.1 Maple step by step solution

Let's solve

$4y'''' - 11y'' - 9y' - 2y = -e^x(1 - 6x)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{y}{2} + \frac{11y''}{4} + \frac{9y'}{4} + \frac{3x\,e^x}{2} - \frac{e^x}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{11y''}{4} - \frac{9y'}{4} - \frac{y}{2} = \frac{e^x(-1+6x)}{4}$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{3x\,e^x}{2} - \frac{e^x}{4} + \frac{11y_3(x)}{4} + \frac{9y_2(x)}{4} + \frac{y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{3x\,e^x}{2} - \frac{e^x}{4} + \frac{11y_3(x)}{4} + \frac{9y_2(x)}{4} + \frac{y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{9}{4} & \frac{11}{4} & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{3x\,e^x}{2} - \frac{e^x}{4} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{3x\,e^x}{2} - \frac{e^x}{4} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{9}{4} & \frac{11}{4} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $-\frac{1}{2}$

$$\vec{y}_2(x) = e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -\frac{1}{2}$ is the eigenvalue, a

$$\vec{y}_3(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{9}{4} & \frac{11}{4} & 0 \end{bmatrix} - -\frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -16 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $-\frac{1}{2}$

$$\vec{y}_3(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -16 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4 + \vec{y}_p(x)$

☐ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & -8\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{-\frac{x}{2}}(-8x-16) & \frac{\mathrm{e}^{2x}}{8} \\ \mathrm{e}^{-x} & 4\,\mathrm{e}^{-\frac{x}{2}} & 4x\,\mathrm{e}^{-\frac{x}{2}} & \frac{\mathrm{e}^{2x}}{4} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-\frac{x}{2}} & -2x\,\mathrm{e}^{-\frac{x}{2}} & \frac{\mathrm{e}^{2x}}{2} \\ \mathrm{e}^{-x} & \mathrm{e}^{-\frac{x}{2}} & x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{2x} \end{bmatrix}$$

- ○ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- ○ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & -8\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{-\frac{x}{2}}(-8x-16) & \frac{\mathrm{e}^{2x}}{8} \\ \mathrm{e}^{-x} & 4\,\mathrm{e}^{-\frac{x}{2}} & 4x\,\mathrm{e}^{-\frac{x}{2}} & \frac{\mathrm{e}^{2x}}{4} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-\frac{x}{2}} & -2x\,\mathrm{e}^{-\frac{x}{2}} & \frac{\mathrm{e}^{2x}}{2} \\ \mathrm{e}^{-x} & \mathrm{e}^{-\frac{x}{2}} & x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -8 & -16 & \frac{1}{8} \\ 1 & 4 & 0 & \frac{1}{4} \\ -1 & -2 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{-\frac{x}{2}}(2+x)}{2} & \frac{\left(2\,\mathrm{e}^{3x}+75\,\mathrm{e}^{\frac{x}{2}}x-42\,\mathrm{e}^{\frac{x}{2}}+40\right)\mathrm{e}^{-x}}{60} & \frac{\left(2\,\mathrm{e}^{3x}+5\,\mathrm{e}^{\frac{x}{2}}x-22\,\mathrm{e}^{\frac{x}{2}}+20\right)\mathrm{e}^{-x}}{20} & \frac{\left(2\,\mathrm{e}^{3x}-15\,\mathrm{e}^{\frac{x}{2}}x+18\,\mathrm{e}^{\frac{x}{2}}-\cdots\right.}{30} \\ -\frac{x\,\mathrm{e}^{-\frac{x}{2}}}{4} & \frac{\left(8\,\mathrm{e}^{3x}-75\,\mathrm{e}^{\frac{x}{2}}x+192\,\mathrm{e}^{\frac{x}{2}}-80\right)\mathrm{e}^{-x}}{120} & \frac{\left(8\,\mathrm{e}^{3x}-5\,\mathrm{e}^{\frac{x}{2}}x+32\,\mathrm{e}^{\frac{x}{2}}-40\right)\mathrm{e}^{-x}}{40} & \frac{\left(8\,\mathrm{e}^{3x}+15\,\mathrm{e}^{\frac{x}{2}}x-48\,\mathrm{e}^{\frac{x}{2}}+\cdots\right.}{60} \\ \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8} & \frac{\left(32\,\mathrm{e}^{3x}+75\,\mathrm{e}^{\frac{x}{2}}x-192\,\mathrm{e}^{\frac{x}{2}}+160\right)\mathrm{e}^{-x}}{240} & \frac{\left(32\,\mathrm{e}^{3x}+5\,\mathrm{e}^{\frac{x}{2}}x-32\,\mathrm{e}^{\frac{x}{2}}+80\right)\mathrm{e}^{-x}}{80} & \frac{\left(32\,\mathrm{e}^{3x}-15\,\mathrm{e}^{\frac{x}{2}}x+48\,\mathrm{e}^{\frac{x}{2}}\cdots\right.}{120} \\ -\frac{x\,\mathrm{e}^{-\frac{x}{2}}}{16} & \frac{\left(128\,\mathrm{e}^{3x}-75\,\mathrm{e}^{\frac{x}{2}}x+192\,\mathrm{e}^{\frac{x}{2}}-320\right)\mathrm{e}^{-x}}{480} & \frac{\left(128\,\mathrm{e}^{3x}-5\,\mathrm{e}^{\frac{x}{2}}x+32\,\mathrm{e}^{\frac{x}{2}}-160\right)\mathrm{e}^{-x}}{160} & \frac{\left(128\,\mathrm{e}^{3x}+15\,\mathrm{e}^{\frac{x}{2}}x-48\,\mathrm{e}^{\frac{x}{2}}\cdots\right.}{240} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}_p{}'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}\,'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms
  $\Phi(x) \cdot \vec{v}\,'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}\,'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$
  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{\left(e^{3x}-4x\,e^{2x}+3\,e^{2x}-5\,e^{\frac{x}{2}}x-4\right)e^{-x}}{12} \\ \frac{\left(4\,e^{3x}-8x\,e^{2x}-2\,e^{2x}+5\,e^{\frac{x}{2}}x-10\,e^{\frac{x}{2}}+8\right)e^{-x}}{24} \\ \frac{e^{-x}\left(-\frac{5(-2+x)e^{\frac{x}{2}}}{2}-14x\,e^{2x}-5\,e^{2x}+8\,e^{3x}-8\right)}{24} \\ \frac{e^{-x}\left(\frac{5(-2+x)e^{\frac{x}{2}}}{2}-22x\,e^{2x}-43\,e^{2x}+32\,e^{3x}+16\right)}{48} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + c_4\overrightarrow{y}_4 + \begin{bmatrix} \frac{\left(e^{3x}-4x\,e^{2x}+3\,e^{2x}-5\,e^{\frac{x}{2}}x-4\right)e^{-x}}{12} \\ \frac{\left(4\,e^{3x}-8x\,e^{2x}-2\,e^{2x}+5\,e^{\frac{x}{2}}x-10\,e^{\frac{x}{2}}+8\right)e^{-x}}{24} \\ \frac{e^{-x}\left(-\frac{5(-2+x)e^{\frac{x}{2}}}{2}-14x\,e^{2x}-5\,e^{2x}+8\,e^{3x}-8\right)}{24} \\ \frac{e^{-x}\left(\frac{5(-2+x)e^{\frac{x}{2}}}{2}-22x\,e^{2x}-43\,e^{2x}+32\,e^{3x}+16\right)}{48} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -8\left(\left(\left(c_3 + \frac{5}{96}\right)x + c_2 + 2c_3\right)e^{\frac{x}{2}} + \left(\frac{x}{24} - \frac{1}{32}\right)e^{2x} + \left(-\frac{c_4}{64} - \frac{1}{96}\right)e^{3x} + \frac{c_1}{8} + \frac{1}{24}\right)e^{-x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(4*diff(y(x),x$4)-11*diff(y(x),x$2)-9*diff(y(x),x)-2*y(x)=-exp(x)*(1-6*x),y(x), singso
```

$$y(x) = e^{-x}\left((c_4x + c_3)\,e^{\frac{x}{2}} - \frac{e^{2x}x}{3} + c_2e^{3x} + c_1 + \frac{e^{2x}}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 47

```
DSolve[4*y''''[x]-11*y''[x]-9*y'[x]-2*y[x]==-Exp[x]*(1-6*x),y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{1}{3}e^x(x-1) + e^{-x/2}(c_2x + c_1) + c_3e^{-x} + c_4e^{2x}$$

19.17 problem section 9.3, problem 17

Internal problem ID [1514]
Internal file name [`OUTPUT/1515_Sunday_June_05_2022_02_20_17_AM_28180072/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 17.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2y''' + 3y' - y = e^{x}\left(x^{2} + 4x + 3\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^{4} - 2\lambda^{3} + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= \mathrm{RootOf}\left(_Z^{4} - 2_Z^{3} + 3_Z - 1, \mathrm{index} = 1\right) \\
\lambda_2 &= \mathrm{RootOf}\left(_Z^{4} - 2_Z^{3} + 3_Z - 1, \mathrm{index} = 2\right) \\
\lambda_3 &= \mathrm{RootOf}\left(_Z^{4} - 2_Z^{3} + 3_Z - 1, \mathrm{index} = 3\right) \\
\lambda_4 &= \mathrm{RootOf}\left(_Z^{4} - 2_Z^{3} + 3_Z - 1, \mathrm{index} = 4\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=3\right)x} c_1+\mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=2\right)x} c_2+\mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=3\right)x}$$
$$y_2 = \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=2\right)x}$$
$$y_3 = \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=1\right)x}$$
$$y_4 = \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=4\right)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + 3y' - y = \mathrm{e}^x \left(x^2 + 4x + 3\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x \left(x^2 + 4x + 3\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^x, x^2\mathrm{e}^x, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=1\right)x}, \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=2\right)x}, \mathrm{e}^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1, \mathrm{index}=3\right)x}, \mathrm{e}^{\mathrm{RootOf}\left(\right.}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^x + A_2 x^2 \mathrm{e}^x + A_3 \mathrm{e}^x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^x + A_1 x\,\mathrm{e}^x + 2A_2 x\,\mathrm{e}^x + A_2 x^2\mathrm{e}^x + A_3\mathrm{e}^x = \mathrm{e}^x \left(x^2 + 4x + 3\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,e^x + x^2 e^x + e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=3)x} c_1 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=2)x} c_2 \right. \\ &\quad \left. + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=1)x} c_3 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=4)x} c_4\right) \\ &\quad + \left(2x\,e^x + x^2 e^x + e^x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=3)x} c_1 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=2)x} c_2 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=1)x} c_3 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=4)x} c_4 + 2x\,e^x + x^2 e^x + e^x \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=3)x} c_1 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=2)x} c_2 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=1)x} c_3 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=4)x} c_4 + 2x\,e^x + x^2 e^x + e^x \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 101

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+0*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=exp(x)*(3+4*x+x^
```

$$y(x) = c_1 e^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=1\right)x} + c_2 e^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=2\right)x} + c_3 e^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=3\right)x} + c_4 e^{\mathrm{RootOf}\left(\_Z^4-2\_Z^3+3\_Z-1,\mathrm{index}=4\right)x} + (x+1)^2 e^x$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 123

```
DSolve[y''''[x]-2*y'''[x]+0*y''[x]+3*y'[x]-y[x]==Exp[x]*(3+4*x+x^2),y[x],x,IncludeSingularSo
```

$$y(x) \to c_2 \exp\left(x\mathrm{Root}\left[\#1^4-2\#1^3+3\#1-1\&,2\right]\right) + c_3 \exp\left(x\mathrm{Root}\left[\#1^4-2\#1^3+3\#1-1\&,3\right]\right) + c_4 \exp\left(x\mathrm{Root}\left[\#1^4-2\#1^3+3\#1-1\&,4\right]\right) + c_1 \exp\left(x\mathrm{Root}\left[\#1^4-2\#1^3+3\#1-1\&,1\right]\right) + e^x(x+1)^2$$

19.18 problem section 9.3, problem 18

Internal problem ID [1515]
Internal file name [`OUTPUT/1516_Sunday_June_05_2022_02_20_19_AM_9905438/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 4y''' + 6y'' - 4y' + 2y = \mathrm{e}^{2x}\left(x^4 + x + 24\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 6y'' - 4y' + 2y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 2 = 0$$

The roots of the above equation are

$$
\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1 \\
\lambda_2 &= -\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1 \\
\lambda_3 &= -\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1 \\
\lambda_4 &= \frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1
\end{aligned}
$$

Therefore the homogeneous solution is

$$
y_h(x) = \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} c_1 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} c_2 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x} c_3 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x} c_4
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} \\
y_2 &= \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} \\
y_3 &= \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x} \\
y_4 &= \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y'''' - 4y''' + 6y'' - 4y' + 2y = \mathrm{e}^{2x}\left(x^4 + x + 24\right)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{2x}\left(x^4 + x + 24\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{2x}, x^2\mathrm{e}^{2x}, \mathrm{e}^{2x}x^3, \mathrm{e}^{2x}x^4, \mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x}, \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x}, \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x}, \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{2x} + A_2 x^2 \mathrm{e}^{2x} + A_3 \mathrm{e}^{2x} x^3 + A_4 \mathrm{e}^{2x} x^4 + A_5 \mathrm{e}^{2x}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_2 x\,\mathrm{e}^{2x} + 12A_3 \mathrm{e}^{2x} x^2 + 16A_4 \mathrm{e}^{2x} x^3 + 36A_3 \mathrm{e}^{2x} x + 72A_4 \mathrm{e}^{2x} x^2 \\ &+ 96A_4 \mathrm{e}^{2x} x + 2A_4 \mathrm{e}^{2x} x^4 + 12A_2 \mathrm{e}^{2x} + 2A_5 \mathrm{e}^{2x} + 2A_2 x^2 \mathrm{e}^{2x} + 2A_3 \mathrm{e}^{2x} x^3 \\ &+ 4A_1 \mathrm{e}^{2x} + 2A_1 x\,\mathrm{e}^{2x} + 24A_3 \mathrm{e}^{2x} + 24A_4 \mathrm{e}^{2x} = \mathrm{e}^{2x}\left(x^4 + x + 24\right)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{49}{2}, A_2 = 6, A_3 = -4, A_4 = \frac{1}{2}, A_5 = -31\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{49x\,\mathrm{e}^{2x}}{2} + 6x^2 \mathrm{e}^{2x} - 4\,\mathrm{e}^{2x} x^3 + \frac{\mathrm{e}^{2x} x^4}{2} - 31\,\mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(\mathrm{e}^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} c_1 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}+1\right)x} c_2 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x} c_3 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}+1\right)x} c_4\right) \\ &\quad + \left(\frac{49x\,\mathrm{e}^{2x}}{2} + 6x^2 \mathrm{e}^{2x} - 4\,\mathrm{e}^{2x} x^3 + \frac{\mathrm{e}^{2x} x^4}{2} - 31\,\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$\begin{aligned}y &= \mathrm{e}^{\frac{\left(2+(-1+i)\sqrt{2}\right)x}{2}} c_1 + \mathrm{e}^{\frac{x\left(2+(1+i)\sqrt{2}\right)}{2}} c_2 + \mathrm{e}^{-\frac{\left(-2+(-1+i)\sqrt{2}\right)x}{2}} c_3 \\ &\quad + \mathrm{e}^{-\frac{\left(-2+(1+i)\sqrt{2}\right)x}{2}} c_4 + \frac{49x\,\mathrm{e}^{2x}}{2} + 6x^2 \mathrm{e}^{2x} - 4\,\mathrm{e}^{2x} x^3 + \frac{\mathrm{e}^{2x} x^4}{2} - 31\,\mathrm{e}^{2x}\end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & e^{\frac{\left(2+(-1+i)\sqrt{2}\right)x}{2}} c_1 + e^{\frac{x\left(2+(1+i)\sqrt{2}\right)}{2}} c_2 + e^{-\frac{\left(-2+(-1+i)\sqrt{2}\right)x}{2}} c_3 \\
& + e^{-\frac{\left(-2+(1+i)\sqrt{2}\right)x}{2}} c_4 + \frac{49x\, e^{2x}}{2} + 6x^2 e^{2x} - 4\, e^{2x} x^3 + \frac{e^{2x} x^4}{2} - 31\, e^{2x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & e^{\frac{\left(2+(-1+i)\sqrt{2}\right)x}{2}} c_1 + e^{\frac{x\left(2+(1+i)\sqrt{2}\right)}{2}} c_2 + e^{-\frac{\left(-2+(-1+i)\sqrt{2}\right)x}{2}} c_3 \\
& + e^{-\frac{\left(-2+(1+i)\sqrt{2}\right)x}{2}} c_4 + \frac{49x\, e^{2x}}{2} + 6x^2 e^{2x} - 4\, e^{2x} x^3 + \frac{e^{2x} x^4}{2} - 31\, e^{2x}
\end{aligned}
$$

Verified OK.

19.18.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 6y'' - 4y' + 2y = e^{2x}(x^4 + x + 24)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = e^{2x} x^4 + x\, e^{2x} + 24\, e^{2x} + 4y_4(x) - 6y_3(x) + 4y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^{2x} x^4 + x\, e^{2x} + 24\, e^{2x} + 4y_4(x) - 6y_3(x) +$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 4 & -6 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^{2x} x^4 + x\,\mathrm{e}^{2x} + 24\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^{2x} x^4 + x\,\mathrm{e}^{2x} + 24\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 4 & -6 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2} + 1, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ 1 \end{bmatrix}\right], \left[-\frac{\sqrt{2}}{2} + \frac{\mathrm{I}\sqrt{2}}{2} + 1, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}+\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{2}}{2} - \frac{\mathrm{I}\sqrt{2}}{2} + 1, \right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1, \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\left(-\frac{\sqrt{2}}{2}+1\right)x} \cdot \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\left(-\frac{\sqrt{2}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}+1} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{\left(-\frac{\sqrt{2}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+3\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-2+\sqrt{2}\right)^3} \\ -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-2+\sqrt{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\cos\left(\frac{\sqrt{2}\,x}{2}\right)}{2\left(-2+\sqrt{2}\right)} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right], \vec{y}_2(x) = e^{\left(-\frac{\sqrt{2}}{2}+1\right)x} \cdot \left[\begin{array}{c} - \end{array}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2} + 1, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\left(\frac{\sqrt{2}}{2}+1\right)x} \cdot \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\left(\frac{\sqrt{2}}{2}+1\right)x} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^3} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1\right)^2} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}+1} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{\left(\frac{\sqrt{2}}{2}+1\right)x} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+2\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\cos\left(\frac{\sqrt{2}\,x}{2}\right)+3\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(2+\sqrt{2}\right)^3} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(2+\sqrt{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+2\cos\left(\frac{\sqrt{2}\,x}{2}\right)}{2\left(2+\sqrt{2}\right)} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{bmatrix}, \vec{y}_4(x) = e^{\left(\frac{\sqrt{2}}{2}+1\right)x} \cdot \cdots\right.$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(\cdots$

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst$\cdots$

$$\phi(x) = \begin{bmatrix} \frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+3\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{\left(-2+\sqrt{2}\right)^3} & \cdots \\ -\frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{\left(-2+\sqrt{2}\right)^2} & \cdots \\ \frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\cos\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{2\left(-2+\sqrt{2}\right)} & \cdots \\ e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\cos\left(\frac{\sqrt{2}\,x}{2}\right) & -e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\sin\cdots \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t$\cdots$

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{cc} \frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+3\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{\left(-2+\sqrt{2}\right)^{3}} & -\frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(2\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\sin\left(\frac{\sqrt{2}}{2}\cdots\right.\right.}{\left(-2+\sqrt{2}\cdots\right.} \\ -\frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{\left(-2+\sqrt{2}\right)^{2}} & \frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\sin\left(\frac{\sqrt{2}\,x}{2}\cdots\right.\right.}{\left(-2+\sqrt{2}\right)\cdots} \\ \frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}-2\cos\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{2\left(-2+\sqrt{2}\right)} & -\frac{e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}+\sin\cdots\right.}{2\left(-2+\cdots\right.} \\ e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\cos\left(\frac{\sqrt{2}\,x}{2}\right) & -e^{\left(-\frac{\sqrt{2}}{2}+1\right)x}\sin\cdots \end{array}\right.$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{c} \frac{\left(\left(-8\sqrt{2}+12\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)-4\sin\left(\frac{\sqrt{2}\,x}{2}\right)\left(\sqrt{2}-1\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}-8\left(\left(\sqrt{2}-\frac{3}{2}\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\frac{5\left(\sqrt{2}-\frac{7}{5}\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{2}\right)e^{\frac{(2+\cdots}{}}\cdots}{\left(-2+\sqrt{2}\right)^{3}\left(2+\sqrt{2}\right)^{3}\left(-3+2\sqrt{2}\right)} \\ \frac{6\left(\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}-\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{\frac{\left(2+\sqrt{2}\right)x}{2}}\right)\left(\sqrt{2}-\frac{4}{3}\right)}{\left(-2+\sqrt{2}\right)^{2}\left(2+\sqrt{2}\right)^{2}\left(-3+2\sqrt{2}\right)} \\ \frac{\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)\left(3\sqrt{2}-4\right)+\sin\left(\frac{\sqrt{2}\,x}{2}\right)\left(-10+7\sqrt{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}-3\left(\left(\sqrt{2}-\frac{4}{3}\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\frac{\left(-2+\sqrt{2}\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{3}\right)e^{\frac{(2+\cdots}{}}\cdots}{-6+4\sqrt{2}} \\ \frac{2\sin\left(\frac{\sqrt{2}\,x}{2}\right)\left(e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}\left(-3+2\sqrt{2}\right)+e^{\frac{\left(2+\sqrt{2}\right)x}{2}}\right)}{-2+\sqrt{2}} \end{array}\right.\cdots$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x)\cdot\vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}'_p(x) = \Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A\cdot\Phi(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}\frac{\left(\left(-2267904\sqrt{2}+3207296\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(-1956096\sqrt{2}+2766336\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}+\left(\left(371200\sqrt{2}-524928\right)\right.}{\left(-2+\sqrt{2}\right)^{8}\left(-3+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(4+\right.} \\ \frac{\left(\left(1244192\sqrt{2}-1759552\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(2471456\sqrt{2}-3495168\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}+\left(\left(-96992\sqrt{2}+13715\right.\right.}{\left(-2+\sqrt{2}\right)^{7}\left(-3+2\sqrt{2}\right)\left(\sqrt{2}-1\right)\left(4+\right.} \\ \frac{\left(\left(-548432\sqrt{2}+775600\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(-7429520\sqrt{2}+10506928\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}+\left(\left(-75632\sqrt{2}+10696\right.\right.}{\left(70\sqrt{2}-99\right)\left(2+\sqrt{2}\right)^{5}\left(-3\right.} \\ \frac{\left(\left(-740720\sqrt{2}+1047536\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(2242592\sqrt{2}-3171504\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)e^{-\frac{\left(-2+\sqrt{2}\right)x}{2}}+\left(\left(21808\sqrt{2}-30832\right.\right.}{\left(-2+\sqrt{2}\right)^{6}\left(70\sqrt{2}-9\right.}\end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\left[\begin{array}{c}\frac{\left(\left(-2267904\sqrt{2}+3207296\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(-1956096\sqrt{2}+27\right.\right.}{} \\ \frac{\left(\left(1244192\sqrt{2}-1759552\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(2471456\sqrt{2}-3\right.\right.}{} \\ \frac{\left(\left(-548432\sqrt{2}+775600\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(-7429520\sqrt{2}+105\right.\right.}{} \\ \frac{\left(\left(-740720\sqrt{2}+1047536\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left(2242592\sqrt{2}-\right.\right.}{}\end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left((-3712c_1+8960c_2-2267904)\sqrt{2}+5248c_1-12672c_2+3207296\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+8960\left(\left(c_1+\frac{29c_2}{70}-\frac{7641}{35}\right)\sqrt{2}-\frac{99c_1}{70}-\frac{41c_2}{70}+\frac{10806}{35}\right)\right.}{}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 87

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+6*diff(y(x),x$2)-4*diff(y(x),x)+2*y(x)=exp(2*x)*(24+x
```

$$y(x) = e^{\frac{(2+\sqrt{2})x}{2}}\left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)c_1+\sin\left(\frac{\sqrt{2}\,x}{2}\right)c_2\right) + \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)c_3+\sin\left(\frac{\sqrt{2}\,x}{2}\right)c_4\right)e^{-\frac{(\sqrt{2}-2)x}{2}} + \frac{e^{2x}(x^4-8x^3+12x^2+49x-62)}{2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 102

```
DSolve[y''''[x]-4*y'''[x]+6*y''[x]-4*y'[x]+2*y[x]==Exp[2*x]*(24+x+x^4),y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{2}e^{x-\frac{x}{\sqrt{2}}}\left(e^{\frac{x}{\sqrt{2}}+x}(x^4-8x^3+12x^2+49x-62)+2\left(c_4e^{\sqrt{2}x}+c_2\right)\cos\left(\frac{x}{\sqrt{2}}\right) + 2\left(c_1e^{\sqrt{2}x}+c_3\right)\sin\left(\frac{x}{\sqrt{2}}\right)\right)$$

19.19 problem section 9.3, problem 19

Internal problem ID [1516]
Internal file name [`OUTPUT/1517_Sunday_June_05_2022_02_20_21_AM_73873428/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 19.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y'''' + 5y''' - 5y' - 2y = 18\,e^{x}(2x+5)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$2y'''' + 5y''' - 5y' - 2y = 0$$

The characteristic equation is

$$2\lambda^4 + 5\lambda^3 - 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2}$$
$$\lambda_3 = -2$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{-\frac{x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{x} \\ y_4 &= e^{-\frac{x}{2}} \end{aligned}$$

Now the particular solution to the given ODE is found

$$2y'''' + 5y''' - 5y' - 2y = 18\,e^{x}(2x + 5)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18\,e^{x}(2x + 5)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{x}, e^{-2x}, e^{-x}, e^{-\frac{x}{2}}\right\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}, x^2 e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 x^2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$18A_1 e^{x} + 54A_2 e^{x} + 36A_2 x\,e^{x} = 18\,e^{x}(2x + 5)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}}c_4\right) + \left(2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}}c_4 + 2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}}c_4 + 2x\,\mathrm{e}^{x} + x^2\mathrm{e}^{x}$$

Verified OK.

19.19.1 Maple step by step solution

Let's solve

$2y'''' + 5y''' - 5y' - 2y = 18\,\mathrm{e}^{x}(2x + 5)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = y - \frac{5y'''}{2} + \frac{5y'}{2} + 18x\,\mathrm{e}^{x} + 45\,\mathrm{e}^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{5y'''}{2} - \frac{5y'}{2} - y = 9\,\mathrm{e}^{x}(2x + 5)$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 18x\,e^x + 45\,e^x - \frac{5y_4(x)}{2} + \frac{5y_2(x)}{2} + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = 18x\,e^x + 45\,e^x - \frac{5y_4(x)}{2} + \frac{5y_2(x)}{2} + y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \frac{5}{2} & 0 & -\frac{5}{2} \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 18x\,e^x + 45\,e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 18x\,e^x + 45\,e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \frac{5}{2} & 0 & -\frac{5}{2} \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & -8\,e^{-\frac{x}{2}} & e^{x} \\ \frac{e^{-2x}}{4} & e^{-x} & 4\,e^{-\frac{x}{2}} & e^{x} \\ -\frac{e^{-2x}}{2} & -e^{-x} & -2\,e^{-\frac{x}{2}} & e^{x} \\ e^{-2x} & e^{-x} & e^{-\frac{x}{2}} & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & -8\,e^{-\frac{x}{2}} & e^{x} \\ \frac{e^{-2x}}{4} & e^{-x} & 4\,e^{-\frac{x}{2}} & e^{x} \\ -\frac{e^{-2x}}{2} & -e^{-x} & -2\,e^{-\frac{x}{2}} & e^{x} \\ e^{-2x} & e^{-x} & e^{-\frac{x}{2}} & e^{x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & -8 & 1 \\ \frac{1}{4} & 1 & 4 & 1 \\ -\frac{1}{2} & -1 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -\frac{\left(-e^{3x}-16\,e^{\frac{3x}{2}}+9\,e^{x}-1\right)e^{-2x}}{9} & -\frac{\left(-7\,e^{3x}-16\,e^{\frac{3x}{2}}+27\,e^{x}-4\right)e^{-2x}}{18} & \frac{\left(7\,e^{3x}-32\,e^{\frac{3x}{2}}+27\,e^{x}-2\right)e^{-2x}}{18} & \cdots \\ \frac{\left(e^{3x}-8\,e^{\frac{3x}{2}}+9\,e^{x}-2\right)e^{-2x}}{9} & \frac{\left(7\,e^{3x}-8\,e^{\frac{3x}{2}}+27\,e^{x}-8\right)e^{-2x}}{18} & -\frac{\left(-7\,e^{3x}-16\,e^{\frac{3x}{2}}+27\,e^{x}-4\right)e^{-2x}}{18} & \cdots \\ -\frac{\left(-e^{3x}-4\,e^{\frac{3x}{2}}+9\,e^{x}-4\right)e^{-2x}}{9} & -\frac{\left(-7\,e^{3x}-4\,e^{\frac{3x}{2}}+27\,e^{x}-16\right)e^{-2x}}{18} & \frac{\left(7\,e^{3x}-8\,e^{\frac{3x}{2}}+27\,e^{x}-8\right)e^{-2x}}{18} & \cdots \\ \frac{\left(e^{3x}-2\,e^{\frac{3x}{2}}+9\,e^{x}-8\right)e^{-2x}}{9} & \frac{\left(7\,e^{3x}-2\,e^{\frac{3x}{2}}+27\,e^{x}-32\right)e^{-2x}}{18} & -\frac{\left(-7\,e^{3x}-4\,e^{\frac{3x}{2}}+27\,e^{x}-16\right)e^{-2x}}{18} & \cdots \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{\left(-9x^2 \mathrm{e}^{3x}-18x\,\mathrm{e}^{3x}+40\,\mathrm{e}^{3x}-176\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-26\right)\mathrm{e}^{-2x}}{9} \\ \frac{\left(9x^2 \mathrm{e}^{3x}+36x\,\mathrm{e}^{3x}-22\,\mathrm{e}^{3x}-88\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-52\right)\mathrm{e}^{-2x}}{9} \\ -\frac{\left(-9x^2 \mathrm{e}^{3x}-54x\,\mathrm{e}^{3x}-14\,\mathrm{e}^{3x}-44\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-104\right)\mathrm{e}^{-2x}}{9} \\ \frac{\left(9x^2 \mathrm{e}^{3x}+72x\,\mathrm{e}^{3x}+68\,\mathrm{e}^{3x}-22\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-208\right)\mathrm{e}^{-2x}}{9} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + \left[\begin{array}{c} -\frac{\left(-9x^2 \mathrm{e}^{3x}-18x\,\mathrm{e}^{3x}+40\,\mathrm{e}^{3x}-176\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-26\right)\mathrm{e}^{-2x}}{9} \\ \frac{\left(9x^2 \mathrm{e}^{3x}+36x\,\mathrm{e}^{3x}-22\,\mathrm{e}^{3x}-88\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-52\right)\mathrm{e}^{-2x}}{9} \\ -\frac{\left(-9x^2 \mathrm{e}^{3x}-54x\,\mathrm{e}^{3x}-14\,\mathrm{e}^{3x}-44\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-104\right)\mathrm{e}^{-2x}}{9} \\ \frac{\left(9x^2 \mathrm{e}^{3x}+72x\,\mathrm{e}^{3x}+68\,\mathrm{e}^{3x}-22\,\mathrm{e}^{\frac{3x}{2}}+162\,\mathrm{e}^{x}-208\right)\mathrm{e}^{-2x}}{9} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \left(\left(-8c_3 + \frac{176}{9}\right)\mathrm{e}^{\frac{3x}{2}} + \left(c_4 + x^2 + 2x - \frac{40}{9}\right)\mathrm{e}^{3x} + \left(-c_2 - 18\right)\mathrm{e}^{x} - \frac{c_1}{8} + \frac{26}{9}\right)\mathrm{e}^{-2x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(2*diff(y(x),x$4)+5*diff(y(x),x$3)+0*diff(y(x),x$2)-5*diff(y(x),x)-2*y(x)=18*exp(x)*(5
```

$$y(x) = e^{-2x}\left(e^{\frac{3x}{2}}c_4 + \left(x^2 + c_1 + 2x\right)e^{3x} + c_3e^x + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 48

```
DSolve[2*y''''[x]+5*y'''[x]+0*y''[x]-5*y'[x]-2*y[x]==18*Exp[x]*(5+2*x),y[x],x,IncludeSingula
```

$$y(x) \to e^{-2x}\left(e^{3x}\left(x^2 + 2x - \frac{40}{9} + c_4\right) + c_1e^{3x/2} + c_3e^x + c_2\right)$$

19.20 problem section 9.3, problem 20

Internal problem ID [1517]
Internal file name [OUTPUT/1518_Sunday_June_05_2022_02_20_23_AM_71425407/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 20.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + y''' - 2y'' - 6y' - 4y = -\mathrm{e}^{2x}\left(15x^2 + 28x + 4\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + y''' - 2y'' - 6y' - 4y = 0$$

The characteristic equation is

$$\lambda^4 + \lambda^3 - 2\lambda^2 - 6\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{(-1-i)x} c_2 + c_3 e^{2x} + e^{(-1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{(-1-i)x}$$
$$y_3 = e^{2x}$$
$$y_4 = e^{(-1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + y''' - 2y'' - 6y' - 4y = -e^{2x}(15x^2 + 28x + 4)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{2x}(15x^2 + 28x + 4)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{2x}, x^2 e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{(-1-i)x}, e^{(-1+i)x}, e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{2x}, x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{2x} + A_2 x^2 e^{2x} + A_3 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$90A_3 e^{2x} x^2 + 168A_3 e^{2x} x + 60A_2 x e^{2x} + 56A_2 e^{2x} + 30A_1 e^{2x} + 54A_3 e^{2x} = -e^{2x}(15x^2 + 28x + 4)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{2x}}{6} - \frac{\mathrm{e}^{2x}x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \mathrm{e}^{(-1-i)x}c_2 + c_3\mathrm{e}^{2x} + \mathrm{e}^{(-1+i)x}c_4\right) + \left(\frac{x\,\mathrm{e}^{2x}}{6} - \frac{\mathrm{e}^{2x}x^3}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + \mathrm{e}^{(-1-i)x}c_2 + c_3\mathrm{e}^{2x} + \mathrm{e}^{(-1+i)x}c_4 + \frac{x\,\mathrm{e}^{2x}}{6} - \frac{\mathrm{e}^{2x}x^3}{6} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-x} + \mathrm{e}^{(-1-i)x}c_2 + c_3\mathrm{e}^{2x} + \mathrm{e}^{(-1+i)x}c_4 + \frac{x\,\mathrm{e}^{2x}}{6} - \frac{\mathrm{e}^{2x}x^3}{6}$$

Verified OK.

19.20.1 Maple step by step solution

Let's solve

$y'''' + y''' - 2y'' - 6y' - 4y = -\mathrm{e}^{2x}(15x^2 + 28x + 4)$

- Highest derivative means the order of the ODE is 4

  y''''

- ☐ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -15x^2 e^{2x} - 28x\,e^{2x} - 4\,e^{2x} - y_4(x) + 2y_3(x) + 6y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = -15x^2 e^{2x} - 28x\,e^{2x} - 4\,e^{2x} - y_4(x) + 2y_3($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 6 & 2 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -15x^2 e^{2x} - 28x\,e^{2x} - 4\,e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -15x^2 e^{2x} - 28x\,e^{2x} - 4\,e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 6 & 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[-1+\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ -\frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4}+\frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_4(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{-x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{4}-\frac{\sin(x)}{4}\right) \\ \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{-x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{4}-\frac{\sin(x)}{4}\right) \\ \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} -1 & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{4} & 0 & -\frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} \frac{2(10-3\cos(x)+6\sin(x))\mathrm{e}^{-x}}{15}+\frac{\mathrm{e}^{2x}}{15} & \frac{(10-12\cos(x)+9\sin(x))\mathrm{e}^{-x}}{15}+\frac{2\,\mathrm{e}^{2x}}{15} & \frac{(-\cos(x)-3\sin(x))\mathrm{e}^{-x}}{10}+ \\ \frac{2(-10+9\cos(x)-3\sin(x))\mathrm{e}^{-x}}{15}+\frac{2\,\mathrm{e}^{2x}}{15} & \frac{(-10+3\sin(x)+21\cos(x))\mathrm{e}^{-x}}{15}+\frac{4\,\mathrm{e}^{2x}}{15} & \frac{(2\sin(x)-\cos(x))\mathrm{e}^{-x}}{5}+ \\ \frac{4(5-3\sin(x)-6\cos(x))\mathrm{e}^{-x}}{15}+\frac{4\,\mathrm{e}^{2x}}{15} & \frac{2(5-9\cos(x)-12\sin(x))\mathrm{e}^{-x}}{15}+\frac{8\,\mathrm{e}^{2x}}{15} & \frac{(3\cos(x)-\sin(x))\mathrm{e}^{-x}}{5}+ \\ \frac{4(-5+3\cos(x)+9\sin(x))\mathrm{e}^{-x}}{15}+\frac{8\,\mathrm{e}^{2x}}{15} & \frac{2(-5-3\cos(x)+21\sin(x))\mathrm{e}^{-x}}{15}+\frac{16\,\mathrm{e}^{2x}}{15} & \frac{(-2\sin(x)-4\cos(x))\mathrm{e}^{-x}}{5}+ \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(-9\cos(x)+18\sin(x)+20)e^{-x}}{90} - \frac{\left(x^3-x+\frac{11}{15}\right)e^{2x}}{6} \\ \frac{(-30x^3-45x^2+30x-7)e^{2x}}{90} + \frac{3\left(\cos(x)-\frac{\sin(x)}{3}-\frac{20}{27}\right)e^{-x}}{10} \\ \frac{(-30x^3-90x^2-15x+8)e^{2x}}{45} - \frac{2\left(\cos(x)+\frac{\sin(x)}{2}-\frac{5}{9}\right)e^{-x}}{5} \\ \frac{(-60x^3-270x^2-210x+1)e^{2x}}{45} + \frac{\left(\cos(x)+3\sin(x)-\frac{10}{9}\right)e^{-x}}{5} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{(-9\cos(x)+18\sin(x)+20)e^{-x}}{90} - \frac{\left(x^3-x+\frac{11}{15}\right)e^{2x}}{6} \\ \frac{(-30x^3-45x^2+30x-7)e^{2x}}{90} + \frac{3\left(\cos(x)-\frac{\sin(x)}{3}-\frac{20}{27}\right)e^{-x}}{10} \\ \frac{(-30x^3-90x^2-15x+8)e^{2x}}{45} - \frac{2\left(\cos(x)+\frac{\sin(x)}{2}-\frac{5}{9}\right)e^{-x}}{5} \\ \frac{(-60x^3-270x^2-210x+1)e^{2x}}{45} + \frac{\left(\cos(x)+3\sin(x)-\frac{10}{9}\right)e^{-x}}{5} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((90c_3+90c_4-36)\cos(x)+(90c_3-90c_4+72)\sin(x)-360c_1+80)e^{-x}}{360} - \frac{\left(x^3-x-\frac{3}{4}c_2+\frac{11}{15}\right)e^{2x}}{6}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 36

```
dsolve(1*diff(y(x),x$4)+1*diff(y(x),x$3)-2*diff(y(x),x$2)-6*diff(y(x),x)-4*y(x)=-exp(2*x)*(4
```

$$y(x) = (\cos(x) c_3 + c_4 \sin(x) + c_1) e^{-x} - \frac{e^{2x}(x^3 - 6c_2 - x)}{6}$$

✓ Solution by Mathematica
Time used: 0.193 (sec). Leaf size: 65

```
DSolve[1*y''''[x]+1*y'''[x]-2*y''[x]-6*y'[x]-4*y[x]==-Exp[2*x]*(4+28*x+15*x^2),y[x],x,Includ
```

$$y(x) \to \frac{1}{90} e^{-x} \left(-15e^{3x}x^3 + 15e^{3x}x - 11e^{3x} + 90c_4e^{3x} + 90c_2\cos(x) + 90c_1\sin(x) + 90c_3\right)$$

19.21 problem section 9.3, problem 21

Internal problem ID [1518]
Internal file name [`OUTPUT/1519_Sunday_June_05_2022_02_20_26_AM_55030780/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 21.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y'''' + y''' - 2y' - y = 3\,\mathrm{e}^{-\frac{x}{2}}(1 - 6x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$2y'''' + y''' - 2y' - y = 0$$

The characteristic equation is

$$2\lambda^4 + \lambda^3 - 2\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2}$$
$$\lambda_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_4 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{-\frac{x}{2}} c_3 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{-\frac{x}{2}}$$
$$y_4 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$2y'''' + y''' - 2y' - y = 3\,e^{-\frac{x}{2}}(1 - 6x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,e^{-\frac{x}{2}}(1 - 6x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{x\,e^{-\frac{x}{2}}, e^{-\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}, e^{-\frac{x}{2}}\right\}$$

Since $e^{-\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x\,e^{-\frac{x}{2}}, e^{-\frac{x}{2}}x^2\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-\frac{x}{2}} + A_2 e^{-\frac{x}{2}} x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{9A_1 e^{-\frac{x}{2}}}{4} - \frac{9A_2 e^{-\frac{x}{2}} x}{2} + 3A_2 e^{-\frac{x}{2}} = 3\,e^{-\frac{x}{2}}(1-6x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4x\,e^{-\frac{x}{2}} + 4\,e^{-\frac{x}{2}} x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{-\frac{x}{2}} c_3 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4\right) + \left(4x\,e^{-\frac{x}{2}} + 4\,e^{-\frac{x}{2}} x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{-\frac{x}{2}} c_3 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4 + 4x\,e^{-\frac{x}{2}} + 4\,e^{-\frac{x}{2}} x^2 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{-\frac{x}{2}} c_3 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4 + 4x\,e^{-\frac{x}{2}} + 4\,e^{-\frac{x}{2}} x^2$$

Verified OK.

19.21.1 Maple step by step solution

Let's solve

$2y'''' + y''' - 2y' - y = 3\,e^{-\frac{x}{2}}(1-6x)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

$y'''' = \frac{y}{2} - \frac{y'''}{2} + y' - 9x\,e^{-\frac{x}{2}} + \frac{3\,e^{-\frac{x}{2}}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'''' + \frac{y'''}{2} - y' - \frac{y}{2} = -\frac{3\,e^{-\frac{x}{2}}(-1+6x)}{2}$

□ Convert linear ODE into a system of first order ODEs

- ○ Define new variable $y_1(x)$

$y_1(x) = y$

- ○ Define new variable $y_2(x)$

$y_2(x) = y'$

- ○ Define new variable $y_3(x)$

$y_3(x) = y''$

- ○ Define new variable $y_4(x)$

$y_4(x) = y'''$

- ○ Isolate for $y_4'(x)$ using original ODE

$y_4'(x) = -9x\,e^{-\frac{x}{2}} + \frac{3\,e^{-\frac{x}{2}}}{2} - \frac{y_4(x)}{2} + y_2(x) + \frac{y_1(x)}{2}$

Convert linear ODE into a system of first order ODEs

$\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -9x\,e^{-\frac{x}{2}} + \frac{3\,e^{-\frac{x}{2}}}{2} - \frac{y_4(x)}{2} + y_2(x) + \frac{y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -9x\,e^{-\frac{x}{2}} + \frac{3\,e^{-\frac{x}{2}}}{2} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -9x\,\mathrm{e}^{-\frac{x}{2}}+\frac{3\,\mathrm{e}^{-\frac{x}{2}}}{2} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A\cdot\vec{y}(x)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x}\cdot\left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -8 \\ 4 \\ -2 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_4(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & -8\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ e^{x} & 4\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & -2\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & -8\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ e^{x} & 4\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & -2\,e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} \cdot \frac{}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{x}}{9}+\frac{8\,e^{-\frac{x}{2}}}{9}+\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{e^{x}}{3}- \\ -\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9}+\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}-\frac{4\,e^{-\frac{x}{2}}}{9}+\frac{e^{x}}{9} & \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{e^{x}}{3} & \frac{e^{x}}{3}- \\ -\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9}-\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{2\,e^{-\frac{x}{2}}}{9}+\frac{e^{x}}{9} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{3} & \\ \frac{e^{x}}{9}-\frac{e^{-\frac{x}{2}}}{9}+\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9} & \frac{e^{x}}{3}-\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{e^{x}}{3}- \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{26\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - 2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3} + 4(x^2+x-2)\,\mathrm{e}^{-\frac{x}{2}} - \frac{2\,\mathrm{e}^{x}}{3} \\ -\frac{22\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{10\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} + 2(-x^2+3x+4)\,\mathrm{e}^{-\frac{x}{2}} - \frac{2\,\mathrm{e}^{x}}{3} \\ -\frac{4\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{16\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} + (x^2-7x+2)\,\mathrm{e}^{-\frac{x}{2}} - \frac{2\,\mathrm{e}^{x}}{3} \\ \frac{26\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - 2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3} + \frac{(-3x^2+33x-48)\,\mathrm{e}^{-\frac{x}{2}}}{6} - \frac{2\,\mathrm{e}^{x}}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{26\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - 2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3} + 4(x^2+x- \\ -\frac{22\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{10\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} + 2(-x^2+3x \\ -\frac{4\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{16\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} + (x^2-7x+ \\ \frac{26\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - 2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3} + \frac{(-3x^2+33x}{6} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{\mathrm{e}^{-\frac{x}{2}}(3c_3+26)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \mathrm{e}^{-\frac{x}{2}}(c_4+2\sqrt{3})\sin\left(\frac{\sqrt{3}\,x}{2}\right) + 4(x^2-2c_2+x-2)\,\mathrm{e}^{-\frac{x}{2}} + \frac{\mathrm{e}^{x}(3c_1-2)}{3}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 50

```
dsolve(2*diff(y(x),x$4)+1*diff(y(x),x$3)-0*diff(y(x),x$2)-2*diff(y(x),x)-1*y(x)=3*exp(-x/2)*
```

$$y(x) = c_3 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_4 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) + \left(4x^2 + c_2 + 4x\right) \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{x} c_1$$

✓ Solution by Mathematica
Time used: 0.635 (sec). Leaf size: 63

```
DSolve[2*y''''[x]+1*y'''[x]-0*y''[x]-2*y'[x]-1*y[x]==3*Exp[-x/2]*(1-6*x),y[x],x,IncludeSingu
```

$$y(x) \to e^{-x/2}\left(4x^2 + 4x + c_4 e^{3x/2} + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right) - 8 + c_3\right)$$

19.22 problem section 9.3, problem 22

Internal problem ID [1519]
Internal file name [`OUTPUT/1520_Sunday_June_05_2022_02_20_28_AM_83982495/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 22.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y'' + 4y = \mathrm{e}^{x}\left(-3x^{2} + x + 3\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{x}$$
$$y_4 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y'' + 4y = e^{x}\left(-3x^2 + x + 3\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}\left(-3x^2 + x + 3\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{x}, x^2 e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{-x}, e^{2x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{x}, x^2 e^{x}, e^{x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{x} + A_2 x^2 e^{x} + A_3 e^{x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{x} + 2A_2 e^{x} - 12A_2 x\, e^{x} - 18A_3 e^{x} x^2 + 6A_3 e^{x} x + 24A_3 e^{x} = e^{x}\left(-3x^2 + x + 3\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{6} + \frac{e^x x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x} + c_3 e^x + e^{2x} c_4\right) + \left(\frac{x\,e^x}{6} + \frac{e^x x^3}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^x + e^{2x} c_4 + \frac{x\,e^x}{6} + \frac{e^x x^3}{6} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^x + e^{2x} c_4 + \frac{x\,e^x}{6} + \frac{e^x x^3}{6}$$

Verified OK.

19.22.1 Maple step by step solution

Let's solve

$y'''' - 5y'' + 4y = e^x(-3x^2 + x + 3)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -4y - 3x^2 e^x + x\,e^x + 3\,e^x + 5y''$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - 5y'' + 4y = -e^x(3x^2 - x - 3)$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -3x^2 e^x + x e^x + 3 e^x + 5y_3(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3x^2 e^x + x e^x + 3 e^x + 5y_3(x) - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3x^2 e^x + x e^x + 3 e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3x^2 e^x + x e^x + 3 e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}$$

- Rewrite the system as

  $$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + \vec{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{x} & \frac{e^{2x}}{8} \\ \frac{e^{-2x}}{4} & e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{x} & \frac{e^{2x}}{8} \\ \frac{e^{-2x}}{4} & e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & 1 & \frac{1}{8} \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(-e^{4x}+4e^{3x}+4e^{x}-1)e^{-2x}}{6} & -\frac{(e^{4x}-8e^{3x}+8e^{x}-1)e^{-2x}}{12} & -\frac{(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{6} & \frac{(e^{4x}-2e^{3x}+2e^{x}-1)}{12} \\ -\frac{(e^{4x}-2e^{3x}+2e^{x}-1)e^{-2x}}{3} & \frac{(-e^{4x}+4e^{3x}+4e^{x}-1)e^{-2x}}{6} & \frac{(2e^{4x}-e^{3x}+e^{x}-2)e^{-2x}}{6} & -\frac{(-e^{4x}+e^{3x}+e^{x}-1}{6} \\ \frac{2(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{3} & -\frac{(e^{4x}-2e^{3x}+2e^{x}-1)e^{-2x}}{3} & -\frac{(-4e^{4x}+e^{3x}+e^{x}-4)e^{-2x}}{6} & \frac{(2e^{4x}-e^{3x}+e^{x}-2)e}{6} \\ -\frac{2(2e^{4x}-e^{3x}+e^{x}-2)e^{-2x}}{3} & \frac{2(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{3} & \frac{(8e^{4x}-e^{3x}+e^{x}-8)e^{-2x}}{6} & -\frac{(-4e^{4x}+e^{3x}+e^{x}-}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}^{\prime}(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}^{\prime}(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_{0}^{x}\frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_{p}(x)=\Phi(x)\cdot \left(\int_{0}^{x}\frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_{p}(x)=\left[\begin{array}{c}\frac{{\mathrm e}^{-2 x} \left(\left(6 x^{3}+6 x +7\right) {\mathrm e}^{3 x}-3 \,{\mathrm e}^{x}-6 \,{\mathrm e}^{4 x}+2\right)}{36} \\ \frac{\left(\left(x^{3}+3 x^{2}+x +\frac{13}{6}\right) {\mathrm e}^{3 x}+\frac{{\mathrm e}^{x}}{2}-2 \,{\mathrm e}^{4 x}-\frac{2}{3}\right) {\mathrm e}^{-2 x}}{6} \\ \frac{\left(\left(x^{3}+6 x^{2}+7 x +\frac{19}{6}\right) {\mathrm e}^{3 x}-\frac{{\mathrm e}^{x}}{2}-4 \,{\mathrm e}^{4 x}+\frac{4}{3}\right) {\mathrm e}^{-2 x}}{6} \\ \frac{{\mathrm e}^{-2 x} \left(\left(x^{3}+9 x^{2}+19 x +\frac{61}{6}\right) {\mathrm e}^{3 x}+\frac{{\mathrm e}^{x}}{2}-8 \,{\mathrm e}^{4 x}-\frac{8}{3}\right)}{6} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}+c_{3}\vec{y}_{3}+c_{4}\vec{y}_{4}+\left[\begin{array}{c}\frac{{\mathrm e}^{-2 x} \left(\left(6 x^{3}+6 x +7\right) {\mathrm e}^{3 x}-3 \,{\mathrm e}^{x}-6 \,{\mathrm e}^{4 x}+2\right)}{36} \\ \frac{\left(\left(x^{3}+3 x^{2}+x +\frac{13}{6}\right) {\mathrm e}^{3 x}+\frac{{\mathrm e}^{x}}{2}-2 \,{\mathrm e}^{4 x}-\frac{2}{3}\right) {\mathrm e}^{-2 x}}{6} \\ \frac{\left(\left(x^{3}+6 x^{2}+7 x +\frac{19}{6}\right) {\mathrm e}^{3 x}-\frac{{\mathrm e}^{x}}{2}-4 \,{\mathrm e}^{4 x}+\frac{4}{3}\right) {\mathrm e}^{-2 x}}{6} \\ \frac{{\mathrm e}^{-2 x} \left(\left(x^{3}+9 x^{2}+19 x +\frac{61}{6}\right) {\mathrm e}^{3 x}+\frac{{\mathrm e}^{x}}{2}-8 \,{\mathrm e}^{4 x}-\frac{8}{3}\right)}{6} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{\left(\left(x^{3}+x +6 c_{3}+\frac{7}{6}\right) {\mathrm e}^{3 x}+\left(\frac{3 c_{4}}{4}-1\right) {\mathrm e}^{4 x}+\left(-6 c_{2}-\frac{1}{2}\right) {\mathrm e}^{x}-\frac{3 c_{1}}{4}+\frac{1}{3}\right) {\mathrm e}^{-2 x}}{6}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 39

```
dsolve(1*diff(y(x),x$4)+0*diff(y(x),x$3)-5*diff(y(x),x$2)-0*diff(y(x),x)+4*y(x)=exp(x)*(3+x-
```

$$y(x) = \frac{\left(\left(x^3 + 6c_1 + x\right) e^{3x} + 6c_3 e^{x} + 6c_4 e^{4x} + 6c_2\right) e^{-2x}}{6}$$

✓ Solution by Mathematica
Time used: 0.111 (sec). Leaf size: 51

```
DSolve[1*y''''[x]+0*y'''[x]-5*y''[x]-0*y'[x]+4*y[x]==Exp[x]*(3+x-3*x^2),y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{36} e^{x} \left(6x^3 + 6x + 7 + 36c_3\right) + c_1 e^{-2x} + c_2 e^{-x} + c_4 e^{2x}$$

19.23 problem section 9.3, problem 23

Internal problem ID [1520]
Internal file name [`OUTPUT/1521_Sunday_June_05_2022_02_20_29_AM_88806639/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 23.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2y''' - 3y'' + 4y' + 4y = e^{2x}(18x^2 + 33x + 13)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' - 3y'' + 4y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{2x} + x\,e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\,e^{-x}$$
$$y_3 = e^{2x}$$
$$y_4 = x\,e^{2x}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' - 3y'' + 4y' + 4y = e^{2x}\left(18x^2 + 33x + 13\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}\left(18x^2 + 33x + 13\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{-x}, x\,e^{2x}, e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x} x^3\}]$$

Since $x\,e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x} x^3, e^{2x} x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x} + A_2 e^{2x} x^3 + A_3 e^{2x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$144A_3e^{2x}x + 54A_2e^{2x}x + 108A_3e^{2x}x^2 + 18A_1e^{2x} + 36A_2e^{2x} + 24A_3e^{2x} \\ = e^{2x}(18x^2 + 33x + 13)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{1}{6}, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2e^{2x}}{6} + \frac{e^{2x}x^3}{6} + \frac{e^{2x}x^4}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-x} + x\,e^{-x}c_2 + c_3e^{2x} + x\,e^{2x}c_4) + \left(\frac{x^2e^{2x}}{6} + \frac{e^{2x}x^3}{6} + \frac{e^{2x}x^4}{6}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2x + c_1)e^{-x} + (c_4x + c_3)e^{2x} + \frac{x^2e^{2x}}{6} + \frac{e^{2x}x^3}{6} + \frac{e^{2x}x^4}{6}$$

Summary

The solution(s) found are the following

$$y = (c_2x + c_1)e^{-x} + (c_4x + c_3)e^{2x} + \frac{x^2e^{2x}}{6} + \frac{e^{2x}x^3}{6} + \frac{e^{2x}x^4}{6} \tag{1}$$

Verification of solutions

$$y = (c_2x + c_1)e^{-x} + (c_4x + c_3)e^{2x} + \frac{x^2e^{2x}}{6} + \frac{e^{2x}x^3}{6} + \frac{e^{2x}x^4}{6}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve(1*diff(y(x),x$4)-2*diff(y(x),x$3)-3*diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=exp(2*x)*(13
```

$$y(x) = \frac{(x^4 + x^3 + 6c_4 x + x^2 + 6c_2)\, e^{2x}}{6} + e^{-x}(c_3 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 58

```
DSolve[1*y''''[x]-2*y'''[x]-3*y''[x]+4*y'[x]+4*y[x]==Exp[2*x]*(13+33*x+18*x^2),y[x],x,Includ
```

$$y(x) \to \frac{1}{54} e^{2x}(9x^4 + 9x^3 + 9x^2 + 18(-1 + 3c_4)x + 10 + 54c_3) + e^{-x}(c_2 x + c_1)$$

19.24 problem section 9.3, problem 24

Internal problem ID [1521]
Internal file name [`OUTPUT/1522_Sunday_June_05_2022_02_20_32_AM_57783203/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 24.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 3y''' + 4y' = e^{2x}(12x^2 + 26x + 15)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 4y' = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{2x} + x e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{2x}$$
$$y_4 = x e^{2x}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 4y' = e^{2x}(12x^2 + 26x + 15)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(12x^2 + 26x + 15)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{2x}, x^2 e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x e^{2x}, e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{2x}, x^2 e^{2x}, e^{2x} x^3\}]$$

Since $x e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x} x^3, e^{2x} x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x} + A_2 e^{2x} x^3 + A_3 e^{2x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&36A_2 e^{2x} x + 72A_3 e^{2x} x^2 + 120A_3 e^{2x} x + 30A_2 e^{2x} + 24A_3 e^{2x} + 12A_1 e^{2x} \\
&= e^{2x}\left(12x^2 + 26x + 15\right)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{6}, A_3 = \frac{1}{6}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{x^2 e^{2x}}{2} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{6}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + c_2 + c_3 e^{2x} + x\, e^{2x} c_4\right) + \left(\frac{x^2 e^{2x}}{2} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{6}\right)
\end{aligned}
$$

Which simplifies to

$$
y = (c_4 x + c_3) e^{2x} + c_1 e^{-x} + c_2 + \frac{x^2 e^{2x}}{2} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{6}
$$

Summary

The solution(s) found are the following

$$
y = (c_4 x + c_3) e^{2x} + c_1 e^{-x} + c_2 + \frac{x^2 e^{2x}}{2} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{6} \tag{1}
$$

Verification of solutions

$$
y = (c_4 x + c_3) e^{2x} + c_1 e^{-x} + c_2 + \frac{x^2 e^{2x}}{2} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{6}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = 12*exp(2*_a)*_a^2+2
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
dsolve(1*diff(y(x),x$4)-3*diff(y(x),x$3)-0*diff(y(x),x$2)+4*diff(y(x),x)+0*y(x)=exp(2*x)*(15
```

$$y(x) = \frac{(2x^4 + 2x^3 + 6x^2 + (6c_3 - 6)\,x + 6c_2 - 3c_3 + 3)\,\mathrm{e}^{2x}}{12} - \mathrm{e}^{-x}c_1 + c_4$$

✓ Solution by Mathematica

Time used: 0.189 (sec). Leaf size: 58

```
DSolve[1*y''''[x]-3*y'''[x]-0*y''[x]+4*y'[x]+0*y[x]==Exp[2*x]*(15+26*x+12*x^2),y[x],x,Includ
```

$$y(x) \to \frac{1}{12}e^{2x}\left(2x^4 + 2x^3 + 6x^2 + 6(-2 + c_3)x + 8 + 6c_2 - 3c_3\right) + c_1\left(-e^{-x}\right) + c_4$$

19.25 problem section 9.3, problem 25

Internal problem ID [1522]
Internal file name [`OUTPUT/1523_Sunday_June_05_2022_02_20_34_AM_18447264/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 25.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2y''' + 2y' - y = (x+1)\,e^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + 2y' - y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + x\,e^{x} c_3 + x^2 e^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{x} \\ y_3 &= x\,e^{x} \\ y_4 &= x^2 e^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + 2y' - y = (x+1)\,e^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(x+1)\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{x}, x^2 e^{x}, e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}, x^2 e^{x}\}]$$

Since $x\,e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{x}, e^{x} x^3\}]$$

Since $x^2 e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x} x^3, e^{x} x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{x} x^3 + A_2 e^{x} x^4$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^x + 48A_2 e^x x + 24A_2 e^x = (x+1)\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = \frac{1}{48}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^3}{24} + \frac{e^x x^4}{48}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + x\, e^x c_3 + x^2 e^x c_4\right) + \left(\frac{e^x x^3}{24} + \frac{e^x x^4}{48}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{-x} + e^x \left(c_4 x^2 + c_3 x + c_2\right) + \frac{e^x x^3}{24} + \frac{e^x x^4}{48}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^x \left(c_4 x^2 + c_3 x + c_2\right) + \frac{e^x x^3}{24} + \frac{e^x x^4}{48} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^x \left(c_4 x^2 + c_3 x + c_2\right) + \frac{e^x x^3}{24} + \frac{e^x x^4}{48}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(1*diff(y(x),x$4)-2*diff(y(x),x$3)-0*diff(y(x),x$2)+2*diff(y(x),x)-1*y(x)=exp(x)*(1+x)
```

$$y(x) = e^{-x} c_2 + \frac{e^{x}(x^4 + 48c_4x^2 + 2x^3 + 48c_3x + 48c_1)}{48}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 55

```
DSolve[1*y''''[x]-2*y'''[x]-0*y''[x]+2*y'[x]-1*y[x]==Exp[x]*(1+x),y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{96} e^{x} \left(2x^4 + 4x^3 + (-6 + 96c_4)x^2 + (6 + 96c_3)x - 3 + 96c_2\right) + c_1 e^{-x}$$

19.26 problem section 9.3, problem 26

Internal problem ID [1523]
Internal file name [`OUTPUT/1524_Sunday_June_05_2022_02_20_36_AM_21649587/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 26.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{2y'''' - 5y''' + 3y'' + y' - y = e^{x}(11 + 12x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$2y'''' - 5y''' + 3y'' + y' - y = 0$$

The characteristic equation is

$$2\lambda^4 - 5\lambda^3 + 3\lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + x^2 e^x c_3 + e^{-\frac{x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = x^2 e^x$$
$$y_4 = e^{-\frac{x}{2}}$$

Now the particular solution to the given ODE is found

$$2y'''' - 5y''' + 3y'' + y' - y = e^x (11 + 12x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x (11 + 12x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\, e^x, x^2 e^x, e^x, e^{-\frac{x}{2}}\right\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x, x^2 e^x\}]$$

Since $x\, e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x, e^x x^3\}]$$

Since $x^2 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x^3, e^x x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x^3 + A_2 e^x x^4$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$72A_2 e^x x + 48A_2 e^x + 18A_1 e^x = e^x(11 + 12x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^3}{6} + \frac{e^x x^4}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 x\, e^x + x^2 e^x c_3 + e^{-\frac{x}{2}} c_4\right) + \left(\frac{e^x x^3}{6} + \frac{e^x x^4}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-\frac{x}{2}} c_4 + e^x\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{6}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} c_4 + e^x\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{6} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{x}{2}} c_4 + e^x\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{6}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(2*diff(y(x),x$4)-5*diff(y(x),x$3)+3*diff(y(x),x$2)+1*diff(y(x),x)-1*y(x)=exp(x)*(11+1
```

$$y(x) = c_4 \mathrm{e}^{-\frac{x}{2}} + \frac{\mathrm{e}^{x} (x^4 + 6c_3 x^2 + x^3 + 6c_2 x + 6c_1)}{6}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 58

```
DSolve[2*y''''[x]-5*y'''[x]+3*y''[x]+1*y'[x]-1*y[x]==Exp[x]*(11+12*x),y[x],x,IncludeSingular
```

$$y(x) \to e^x \left(\frac{x^4}{6} + \frac{x^3}{6} + \left(-\frac{1}{3} + c_4 \right) x^2 + \left(\frac{4}{9} + c_3 \right) x - \frac{8}{27} + c_2 \right) + c_1 e^{-x/2}$$

19.27 problem section 9.3, problem 27

Internal problem ID [1524]
Internal file name [`OUTPUT/1525_Sunday_June_05_2022_02_20_38_AM_56632814/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 27.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 3y''' + 3y'' + y' = e^{-x}\left(10x^2 - 24x + 5\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y''' + 3y'' + y' = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 + 3\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + x^2\mathrm{e}^{-x}c_3 + c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= x^2\mathrm{e}^{-x} \\
y_4 &= 1
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y''' + 3y'' + y' = \mathrm{e}^{-x}\left(10x^2 - 24x + 5\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-x}\left(10x^2 - 24x + 5\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3, \mathrm{e}^{-x}x^4\}]$$

Since $x^2\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^5\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3, \mathrm{e}^{-x}x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^5 e^{-x} + A_2 e^{-x} x^3 + A_3 e^{-x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$120 A_1 x e^{-x} - 60 A_1 x^2 e^{-x} - 24 A_3 e^{-x} x + 24 A_3 e^{-x} - 6 A_2 e^{-x} = e^{-x} \left(10x^2 - 24x + 5\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}, A_2 = -\frac{1}{6}, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^5 e^{-x}}{6} - \frac{e^{-x} x^3}{6} + \frac{e^{-x} x^4}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x e^{-x} c_2 + x^2 e^{-x} c_3 + c_4\right) + \left(-\frac{x^5 e^{-x}}{6} - \frac{e^{-x} x^3}{6} + \frac{e^{-x} x^4}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x} \left(c_3 x^2 + c_2 x + c_1\right) + c_4 - \frac{x^5 e^{-x}}{6} - \frac{e^{-x} x^3}{6} + \frac{e^{-x} x^4}{6}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \left(c_3 x^2 + c_2 x + c_1\right) + c_4 - \frac{x^5 e^{-x}}{6} - \frac{e^{-x} x^3}{6} + \frac{e^{-x} x^4}{6} \tag{1}$$

Verification of solutions

$$y = e^{-x} \left(c_3 x^2 + c_2 x + c_1\right) + c_4 - \frac{x^5 e^{-x}}{6} - \frac{e^{-x} x^3}{6} + \frac{e^{-x} x^4}{6}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = 10*exp(-_a)*_a^2-24
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve(1*diff(y(x),x$4)+3*diff(y(x),x$3)+3*diff(y(x),x$2)+1*diff(y(x),x)-0*y(x)=exp(-x)*(5-2
```

$$y(x) = \frac{(-x^5 + x^4 - x^3 + (-6c_3 - 3)x^2 + (-6c_2 - 12c_3 - 6)x - 6c_1 - 6c_2 - 12c_3 - 6)e^{-x}}{6} + c_4$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 65

```
DSolve[1*y''''[x]+3*y'''[x]+3*y''[x]+1*y'[x]-0*y[x]==Exp[-x]*(5-24*x+10*x^2),y[x],x,IncludeS
```

$$y(x) \to \frac{1}{6}e^{-x}(-x^5 + x^4 - x^3 - 3(1 + 2c_3)x^2 - 6(1 + c_2 + 2c_3)x - 6(1 + c_1 + c_2 + 2c_3)) + c_4$$

19.28 problem section 9.3, problem 28

Internal problem ID [1525]
Internal file name [`OUTPUT/1526_Sunday_June_05_2022_02_20_40_AM_76968418/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 28.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 7y''' + 18y'' - 20y' + 8y = e^{2x}(-5x^2 - 8x + 3)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 7y''' + 18y'' - 20y' + 8y = 0$$

The characteristic equation is

$$\lambda^4 - 7\lambda^3 + 18\lambda^2 - 20\lambda + 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + c_3 x \,\mathrm{e}^{2x} + x^2 \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = x \,\mathrm{e}^{2x}$$
$$y_4 = x^2 \mathrm{e}^{2x}$$

Now the particular solution to the given ODE is found

$$y'''' - 7y''' + 18y'' - 20y' + 8y = \mathrm{e}^{2x}\left(-5x^2 - 8x + 3\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x}\left(-5x^2 - 8x + 3\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \,\mathrm{e}^{2x}, x^2 \mathrm{e}^{2x}, \mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \,\mathrm{e}^{2x}, x^2 \mathrm{e}^{2x}, \mathrm{e}^{x}, \mathrm{e}^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \,\mathrm{e}^{2x}, x^2 \mathrm{e}^{2x}, \mathrm{e}^{2x} x^3\}]$$

Since $x \,\mathrm{e}^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{2x}, \mathrm{e}^{2x} x^3, \mathrm{e}^{2x} x^4\}]$$

Since $x^2 \mathrm{e}^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^5 \mathrm{e}^{2x}, \mathrm{e}^{2x} x^3, \mathrm{e}^{2x} x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^5 e^{2x} + A_2 e^{2x} x^3 + A_3 e^{2x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$120 A_1 x\, e^{2x} + 60 A_1 x^2 e^{2x} + 24 A_3 e^{2x} x + 24 A_3 e^{2x} + 6 A_2 e^{2x} = e^{2x}\left(-5x^2 - 8x + 3\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}, A_2 = \frac{1}{6}, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^5 e^{2x}}{12} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{2x} + c_3 x\, e^{2x} + x^2 e^{2x} c_4\right) + \left(-\frac{x^5 e^{2x}}{12} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{12}\right) \end{aligned}$$

Which simplifies to

$$y = \left(c_4 x^2 + c_3 x + c_2\right) e^{2x} + c_1 e^x - \frac{x^5 e^{2x}}{12} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{12}$$

Summary

The solution(s) found are the following

$$y = \left(c_4 x^2 + c_3 x + c_2\right) e^{2x} + c_1 e^x - \frac{x^5 e^{2x}}{12} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{12} \tag{1}$$

Verification of solutions

$$y = \left(c_4 x^2 + c_3 x + c_2\right) e^{2x} + c_1 e^x - \frac{x^5 e^{2x}}{12} + \frac{e^{2x} x^3}{6} + \frac{e^{2x} x^4}{12}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(1*diff(y(x),x$4)-7*diff(y(x),x$3)+18*diff(y(x),x$2)-20*diff(y(x),x)+8*y(x)=exp(2*x)*(
```

$$y(x) = -\frac{e^{x}((x^{5} - x^{4} - 12c_{4}x^{2} - 2x^{3} - 12c_{3}x - 12c_{2})\,e^{x} - 12c_{1})}{12}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 59

```
DSolve[1*y''''[x]-7*y'''[x]+18*y''[x]-20*y'[x]+8*y[x]==Exp[2*x]*(3-8*x-5*x^2),y[x],x,Include
```

$$y(x) \to \frac{1}{12}e^{2x}\left(-x^{5} + x^{4} + 2x^{3} + 6(-1 + 2c_{4})x^{2} + 12(1 + c_{3})x + 12(-1 + c_{2})\right) + c_{1}e^{x}$$

19.29 problem section 9.3, problem 29

Internal problem ID [1526]
Internal file name [`OUTPUT/1527_Sunday_June_05_2022_02_20_43_AM_72947374/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 29.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - 4y' + 4y = \mathrm{e}^{-x}((16 + 10x)\cos(x) + (30 - 10x)\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - 4y' + 4y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - 4y' + 4y = e^{-x}((16+10x)\cos(x) + (30-10x)\sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}((16+10x)\cos(x) + (30-10x)\sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^{-x}, \sin(x) e^{-x}, x\cos(x) e^{-x}, \sin(x) x e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^{-x} + A_2 \sin(x) e^{-x} + A_3 x \cos(x) e^{-x} + A_4 \sin(x) x e^{-x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&10A_3 x \cos(x) e^{-x} - 2A_3 \cos(x) e^{-x} + 10A_4 \sin(x) x e^{-x} - 8A_4 \cos(x) e^{-x} \\ &- 2A_4 \sin(x) e^{-x} + 10A_2 \sin(x) e^{-x} + 10A_1 \cos(x) e^{-x} \\ &+ 8A_3 \sin(x) e^{-x} = e^{-x}((16+10x)\cos(x) + (30-10x)\sin(x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 1, A_4 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)\,\mathrm{e}^{-x} + 2\sin(x)\,\mathrm{e}^{-x} + x\cos(x)\,\mathrm{e}^{-x} - \sin(x)\,x\,\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{2x}\right) + \left(\cos(x)\,\mathrm{e}^{-x} + 2\sin(x)\,\mathrm{e}^{-x} + x\cos(x)\,\mathrm{e}^{-x} - \sin(x)\,x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{2x} + \cos(x)\,\mathrm{e}^{-x} + 2\sin(x)\,\mathrm{e}^{-x} + x\cos(x)\,\mathrm{e}^{-x} - \sin(x)\,x\,\mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{2x} + \cos(x)\,\mathrm{e}^{-x} + 2\sin(x)\,\mathrm{e}^{-x} + x\cos(x)\,\mathrm{e}^{-x} - \sin(x)\,x\,\mathrm{e}^{-x}$$

Verified OK.

19.29.1 Maple step by step solution

Let's solve

$y''' - y'' - 4y' + 4y = \mathrm{e}^{-x}((16 + 10x)\cos(x) + (30 - 10x)\sin(x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -4y + 10x\cos(x)\,\mathrm{e}^{-x} - 10\sin(x)\,x\,\mathrm{e}^{-x} + 16\cos(x)\,\mathrm{e}^{-x} + 30\sin(x)\,\mathrm{e}^{-x} + y'' + 4y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - y'' - 4y' + 4y = 2\,\mathrm{e}^{-x}(-5\sin(x)\,x + 5x\cos(x) + 15\sin(x) + 8\cos(x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 10x\cos(x)\,\mathrm{e}^{-x} - 10\sin(x)\,x\,\mathrm{e}^{-x} + 16\cos(x)\,\mathrm{e}^{-x} + 30\sin(x)\,\mathrm{e}^{-x} + y_3(x) + 4y_2(x) - 4y_1$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 10x\cos(x)\,\mathrm{e}^{-x} - 10\sin(x)\,x\,\mathrm{e}^{-x} + 16\cos(x)\,\mathrm{e}^{-x} + 30\sin$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -10\sin(x)\,x\,\mathrm{e}^{-x} + 10x\cos(x)\,\mathrm{e}^{-x} + 30\sin(x)\,\mathrm{e}^{-x} + 16\cos(x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -10\sin(x)\,x\,\mathrm{e}^{-x} + 10x\cos(x)\,\mathrm{e}^{-x} + 30\sin(x)\,\mathrm{e}^{-x} + 16\cos(x)\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]\right], \left[1, \left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c}\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^x & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^x & \frac{e^{2x}}{2} \\ e^{-2x} & e^x & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^x & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^x & \frac{e^{2x}}{2} \\ e^{-2x} & e^x & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{(3e^{4x}-8e^{3x}-1)e^{-2x}}{6} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{(3e^{4x}-4e^{3x}+1)e^{-2x}}{12} \\ -\frac{(3e^{4x}-4e^{3x}+1)e^{-2x}}{3} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & \frac{(3e^{4x}-2e^{3x}-1)e^{-2x}}{6} \\ -\frac{2(3e^{4x}-2e^{3x}-1)e^{-2x}}{3} & -e^{-2x} + e^{2x} & -\frac{(-3e^{4x}+e^{3x}-1)e^{-2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x)=\left[\begin{array}{c}(-4\,e^{3x}+2\,e^{4x}+1+((x+1)\cos(x)+(2-x)\sin(x))\,e^{x})\,e^{-2x}\\ -2(2\,e^{3x}-2\,e^{4x}+1+((x-1)\cos(x)+2\sin(x))\,e^{x})\,e^{-2x}\\ 2(-2\,e^{3x}+4\,e^{4x}+2+((x-4)\cos(x)+(x+1)\sin(x))\,e^{x})\,e^{-2x}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c}(-4\,e^{3x}+2\,e^{4x}+1+((x+1)\cos(x)+(2-x)\sin(x))\,e^{x})\,e^{-2x}\\ -2(2\,e^{3x}-2\,e^{4x}+1+((x-1)\cos(x)+2\sin(x))\,e^{x})\,e^{-2x}\\ 2(-2\,e^{3x}+4\,e^{4x}+2+((x-4)\cos(x)+(x+1)\sin(x))\,e^{x})\,e^{-2x}\end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=\left((c_2-4)\,e^{3x}+\left(\frac{c_3}{4}+2\right)e^{4x}+((x+1)\cos(x)-(-2+x)\sin(x))\,e^{x}+\frac{c_1}{4}+1\right)e^{-2x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(0*diff(y(x),x$4)+1*diff(y(x),x$3)-1*diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(-x)*((16
```

$$y(x) = \left(c_1 e^{3x} + c_3 e^{4x} + ((x+1)\cos(x) - \sin(x)(-2+x))e^x + c_2\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 49

```
DSolve[0*y''''[x]+1*y'''[x]-1*y''[x]-4*y'[x]+4*y[x]==Exp[-x]*((16+10*x)*Cos[x]+(30-10*x)*Sin
```

$$y(x) \to e^{-2x}\left(-e^x(x-2)\sin(x) + e^x(x+1)\cos(x) + c_2 e^{3x} + c_3 e^{4x} + c_1\right)$$

19.30 problem section 9.3, problem 30

Internal problem ID [1527]
Internal file name [`OUTPUT/1528_Sunday_June_05_2022_02_20_45_AM_76185064/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' - 4y' - 4y = e^{-x}((1 - 22x)\cos(2x) - (1 + 6x)\sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - 4y' - 4y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - 4y' - 4y = e^{-x}((1 - 22x)\cos(2x) - (1 + 6x)\sin(2x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}((1 - 22x)\cos(2x) - (1 + 6x)\sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x)\,e^{-x}, \sin(2x)\,e^{-x}, \cos(2x)\,e^{-x}x, \sin(2x)\,e^{-x}x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x)\,e^{-x} + A_2 \sin(2x)\,e^{-x} + A_3 \cos(2x)\,e^{-x}x + A_4 \sin(2x)\,e^{-x}x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&8A_4 \sin(2x)\,e^{-x}x + 8A_3 \cos(2x)\,e^{-x}x - 15A_4 \sin(2x)\,e^{-x} - 14A_4 \cos(2x)\,e^{-x}x \\
&- 15A_3 \cos(2x)\,e^{-x} + 14A_3 \sin(2x)\,e^{-x}x + 8A_3 \sin(2x)\,e^{-x} \\
&- 8A_4 \cos(2x)\,e^{-x} + 14A_1 \sin(2x)\,e^{-x} - 14A_2 \cos(2x)\,e^{-x} + 8A_1 \cos(2x)\,e^{-x} \\
&+ 8A_2 \sin(2x)\,e^{-x} = e^{-x}((1 - 22x)\cos(2x) - (1 + 6x)\sin(2x))
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = -1, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(2x)\,e^{-x} + \sin(2x)\,e^{-x} - \cos(2x)\,e^{-x}x + \sin(2x)\,e^{-x}x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{2x}\right) + \left(\cos(2x)\,e^{-x} + \sin(2x)\,e^{-x} - \cos(2x)\,e^{-x}x + \sin(2x)\,e^{-x}x\right)\end{aligned}$$

Which simplifies to

$$y = \left(e^{4x}c_3 + c_1 e^{x} + c_2\right)e^{-2x} + \cos(2x)\,e^{-x} + \sin(2x)\,e^{-x} - \cos(2x)\,e^{-x}x + \sin(2x)\,e^{-x}x$$

Summary

The solution(s) found are the following

$$\begin{aligned}y = {} & \left(e^{4x}c_3 + c_1 e^{x} + c_2\right)e^{-2x} + \cos(2x)\,e^{-x} \\ & + \sin(2x)\,e^{-x} - \cos(2x)\,e^{-x}x + \sin(2x)\,e^{-x}x\end{aligned} \tag{1}$$

Verification of solutions

$$y = \left(e^{4x}c_3 + c_1 e^{x} + c_2\right)e^{-2x} + \cos(2x)\,e^{-x} + \sin(2x)\,e^{-x} - \cos(2x)\,e^{-x}x + \sin(2x)\,e^{-x}x$$

Verified OK.

19.30.1 Maple step by step solution

Let's solve

$y''' + y'' - 4y' - 4y = e^{-x}((1-22x)\cos(2x) - (1+6x)\sin(2x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 4y - 22\cos(2x)\,e^{-x}x - 6\sin(2x)\,e^{-x}x + \cos(2x)\,e^{-x} - \sin(2x)\,e^{-x} - y'' + 4y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''' + y'' - 4y' - 4y = -\mathrm{e}^{-x}(6x\sin(2x) + 22x\cos(2x) + \sin(2x) - \cos(2x))$$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -22\cos(2x)\mathrm{e}^{-x}x - 6\sin(2x)\mathrm{e}^{-x}x + \cos(2x)\mathrm{e}^{-x} - \sin(2x)\mathrm{e}^{-x} - y_3(x) + 4y_2(x) + 4y_1($

    Convert linear ODE into a system of first order ODEs

    $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -22\cos(2x)\mathrm{e}^{-x}x - 6\sin(2x)\mathrm{e}^{-x}x + \cos(2x)\mathrm{e}^{-x} - \sin($
- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -6\sin(2x)\mathrm{e}^{-x}x - 22\cos(2x)\mathrm{e}^{-x}x - \sin(2x)\mathrm{e}^{-x} + \cos(2x)\mathrm{e}\end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -6\sin(2x)\mathrm{e}^{-x}x - 22\cos(2x)\mathrm{e}^{-x}x - \sin(2x)\mathrm{e}^{-x} + \cos(2x)\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(e^{4x}+8e^{x}-3)e^{-2x}}{6} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{(e^{4x}-4e^{x}+3)e^{-2x}}{12} \\ \frac{(e^{4x}-4e^{x}+3)e^{-2x}}{3} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & \frac{(e^{4x}+2e^{x}-3)e^{-2x}}{6} \\ \frac{2(e^{4x}+2e^{x}-3)e^{-2x}}{3} & -e^{-2x} + e^{2x} & -\frac{(-e^{4x}+e^{x}-3)e^{-2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\begin{bmatrix} -\left((x-1)\,\mathrm{e}^x\cos(2x)-(x+1)\,\mathrm{e}^x\sin(2x)+\frac{5\,\mathrm{e}^x}{3}+\frac{\mathrm{e}^{4x}}{12}-\frac{3}{4}\right)\mathrm{e}^{-2x} \\ \mathrm{e}^{-2x}\left(\mathrm{e}^x(-2+x)\sin(2x)+3\cos(2x)\,\mathrm{e}^x x+\frac{5\,\mathrm{e}^x}{3}-\frac{\mathrm{e}^{4x}}{6}-\frac{3}{2}\right) \\ -\left((x+1)\,\mathrm{e}^x\cos(2x)+\mathrm{e}^x(7x-3)\sin(2x)+\frac{5\,\mathrm{e}^x}{3}+\frac{\mathrm{e}^{4x}}{3}-3\right)\mathrm{e}^{-2x} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\begin{bmatrix} -\left((x-1)\,\mathrm{e}^x\cos(2x)-(x+1)\,\mathrm{e}^x\sin(2x)+\frac{5\,\mathrm{e}^x}{3}+\frac{\mathrm{e}^{4x}}{12}-\frac{3}{4}\right)\mathrm{e}^{-2 \\ \mathrm{e}^{-2x}\left(\mathrm{e}^x(-2+x)\sin(2x)+3\cos(2x)\,\mathrm{e}^x x+\frac{5\,\mathrm{e}^x}{3}-\frac{\mathrm{e}^{4x}}{6}-\frac{3}{2}\right) \\ -\left((x+1)\,\mathrm{e}^x\cos(2x)+\mathrm{e}^x(7x-3)\sin(2x)+\frac{5\,\mathrm{e}^x}{3}+\frac{\mathrm{e}^{4x}}{3}-3\right)\mathrm{e}^{- \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y=-\left((x-1)\,\mathrm{e}^x\cos(2x)+\left(-\frac{c_3}{4}+\frac{1}{12}\right)\mathrm{e}^{4x}-(x+1)\,\mathrm{e}^x\sin(2x)+\left(-c_2+\frac{5}{3}\right)\mathrm{e}^x-\frac{c_1}{4}-\frac{3}{4}\right)\mathrm{e}^{-2x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(1*diff(y(x),x$3)+1*diff(y(x),x$2)-4*diff(y(x),x)-4*y(x)=exp(-x)*((1-22*x)*cos(2*x)-(1
```

$$y(x) = -\left(e^{x}(x-1)\cos(2x) - (x+1)\,e^{x}\sin(2x) - c_3 e^{4x} + \left(-c_2 + \frac{5}{3}\right)e^{x} - c_1\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 51

```
DSolve[1*y'''[x]+1*y''[x]-4*y'[x]-4*y[x]==Exp[-x]*((1-22*x)*Cos[2*x]-(1+6*x)*Sin[2*x]),y[x],
```

$$y(x) \to e^{-2x}\left(e^{x}(x+1)\sin(2x) - e^{x}(x-1)\cos(2x) + c_2 e^{x} + c_3 e^{4x} + c_1\right)$$

19.31 problem section 9.3, problem 31

Internal problem ID [1528]
Internal file name [`OUTPUT/1529_Sunday_June_05_2022_02_20_48_AM_48538573/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' + 2y' - 2y = e^{2x}\left(\left(-x^2 + 5x + 27\right)\cos(x) + \left(9x^2 + 13x + 2\right)\sin(x)\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + 2y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 2\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i\sqrt{2}$$
$$\lambda_3 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{-i\sqrt{2}\,x}$$

$$y_3 = e^{i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + 2y' - 2y = e^{2x}\left(\left(-x^2 + 5x + 27\right)\cos(x) + \left(9x^2 + 13x + 2\right)\sin(x)\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}\left(\left(-x^2 + 5x + 27\right)\cos(x) + \left(9x^2 + 13x + 2\right)\sin(x)\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(x), e^{2x}\sin(x), e^{2x}\cos(x)\,x, e^{2x}\cos(x)\,x^2, e^{2x}\sin(x)\,x, e^{2x}\sin(x)\,x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{i\sqrt{2}\,x}, e^{-i\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned} y_p = {} & A_1 e^{2x}\cos(x) + A_2 e^{2x}\sin(x) + A_3 e^{2x}\cos(x)\,x \\ & + A_4 e^{2x}\cos(x)\,x^2 + A_5 e^{2x}\sin(x)\,x + A_6 e^{2x}\sin(x)\,x^2 \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & A_3 e^{2x}\cos(x)\,x - 9A_3 e^{2x}\sin(x)\,x - 9A_4 e^{2x}\sin(x)\,x^2 + 9A_5 e^{2x}\cos(x)\,x + 7A_5 e^{2x}\sin(x) \\ & + 10A_5 e^{2x}\cos(x) + 9A_6 e^{2x}\cos(x)\,x^2 - 10A_3 e^{2x}\sin(x) + 10A_6 e^{2x}\sin(x) + 7A_3 e^{2x}\cos(x) \\ & + 14A_4 e^{2x}\cos(x)\,x + 14A_6 e^{2x}\sin(x)\,x + 20A_6 e^{2x}\cos(x)\,x + 10A_4 e^{2x}\cos(x) \\ & + A_4 e^{2x}\cos(x)\,x^2 + A_5 e^{2x}\sin(x)\,x + A_6 e^{2x}\sin(x)\,x^2 - 20A_4 e^{2x}\sin(x)\,x \\ & + 6A_6 e^{2x}\cos(x) + A_2 e^{2x}\sin(x) + A_1 e^{2x}\cos(x) - 6A_4 e^{2x}\sin(x) + 9A_2 e^{2x}\cos(x) \\ & - 9A_1 e^{2x}\sin(x) = e^{2x}\left(\left(-x^2 + 5x + 27\right)\cos(x) + \left(9x^2 + 13x + 2\right)\sin(x)\right) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = 1, A_4 = -1, A_5 = 2, A_6 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x} \cos(x) + e^{2x} \sin(x) + e^{2x} \cos(x) x - e^{2x} \cos(x) x^2 + 2 e^{2x} \sin(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3\right) \\ &\quad + \left(e^{2x} \cos(x) + e^{2x} \sin(x) + e^{2x} \cos(x) x - e^{2x} \cos(x) x^2 + 2 e^{2x} \sin(x) x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 e^x + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3 + e^{2x} \cos(x) + e^{2x} \sin(x) \\ & + e^{2x} \cos(x) x - e^{2x} \cos(x) x^2 + 2 e^{2x} \sin(x) x \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & c_1 e^x + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3 + e^{2x} \cos(x) + e^{2x} \sin(x) \\ & + e^{2x} \cos(x) x - e^{2x} \cos(x) x^2 + 2 e^{2x} \sin(x) x \end{aligned}$$

Verified OK.

19.31.1 Maple step by step solution

Let's solve

$y''' - y'' + 2y' - 2y = e^{2x}((-x^2 + 5x + 27) \cos(x) + (9x^2 + 13x + 2) \sin(x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 2y - e^{2x} \cos(x) x^2 + 9 e^{2x} \sin(x) x^2 + 5 e^{2x} \cos(x) x + 13 e^{2x} \sin(x) x + 27 e^{2x} \cos(x) + 2 e^{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - y'' + 2y' - 2y = -e^{2x}(-9 \sin(x) x^2 + x^2 \cos(x) - 13 \sin(x) x - 5x \cos(x) - 2 \sin(x) - 27$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -e^{2x} \cos(x) x^2 + 9 e^{2x} \sin(x) x^2 + 5 e^{2x} \cos(x) x + 13 e^{2x} \sin(x) x + 27 e^{2x} \cos(x) + 2 e^{2x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -e^{2x} \cos(x) x^2 + 9 e^{2x} \sin(x) x^2 + 5 e^{2x} \cos(x) x + 13 e^{2x}$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 9 e^{2x} \sin(x) x^2 - e^{2x} \cos(x) x^2 + 13 e^{2x} \sin(x) x + 5 e^{2x} \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 9 e^{2x} \sin(x) x^2 - e^{2x} \cos(x) x^2 + 13 e^{2x} \sin(x) x + 5 e^{2x} \cos(x) x + 2 e^{2x} \sin(x) + 27 e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]\right], \left[\mathrm{I}\sqrt{2}, \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}\sqrt{2}\,x} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^{x} & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \mathrm{e}^{x} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ \mathrm{e}^{x} & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^{x} & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \mathrm{e}^{x} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ \mathrm{e}^{x} & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & 0 & \frac{\sqrt{2}}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2\,\mathrm{e}^{x}}{3} + \frac{\cos\left(\sqrt{2}\,x\right)}{3} - \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{3} & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\mathrm{e}^{x}}{3} - \frac{\cos\left(\sqrt{2}\,x\right)}{3} - \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{6} \\ \frac{2\,\mathrm{e}^{x}}{3} - \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{3} - \frac{2\cos\left(\sqrt{2}\,x\right)}{3} & \cos\left(\sqrt{2}\,x\right) & \frac{\mathrm{e}^{x}}{3} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{3} - \frac{\cos\left(\sqrt{2}\,x\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3} - \frac{2\cos\left(\sqrt{2}\,x\right)}{3} + \frac{2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{3} & -\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) & \frac{\mathrm{e}^{x}}{3} + \frac{2\cos\left(\sqrt{2}\,x\right)}{3} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{3} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

    $$\overrightarrow{y}_p(x) = \begin{bmatrix} 4\cos\left(\sqrt{2}\,x\right) + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + \left(\left(-x^2+x+1\right)\cos(x) + \sin(x)\left(1+2x\right)\right)\mathrm{e}^{2x} - 5 \\ \cos\left(\sqrt{2}\,x\right) - 4\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) + \left(\left(-2x^2+2x+4\right)\cos(x) + \left(x^2+3x+3\right)\sin(x)\right)\mathrm{e} \\ -8\cos\left(\sqrt{2}\,x\right) - \sqrt{2}\,\sin\left(\sqrt{2}\,x\right) + \left(\left(-3x^2+3x+13\right)\cos(x) + \left(4x^2+6x+5\right)\sin(x) \right. \end{bmatrix}$$
- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} 4\cos\left(\sqrt{2}\,x\right) + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + \left(\left(-x^2+x+1\right)\cos(\right. \\ \cos\left(\sqrt{2}\,x\right) - 4\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) + \left(\left(-2x^2+2x+4\right)\cos(\right. \\ -8\cos\left(\sqrt{2}\,x\right) - \sqrt{2}\,\sin\left(\sqrt{2}\,x\right) + \left(\left(-3x^2+3x+13\right)\cos\right. \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-c_2+8)\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\left(c_3+\sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)}{2} + \left(\left(-x^2+x+1\right)\cos(x) + \sin(x)(1+2x)\right)e^{2x} + e^x(-5+$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 179

```
dsolve(1*diff(y(x),x$3)-1*diff(y(x),x$2)+2*diff(y(x),x)-2*y(x)=exp(2*x)*((27+5*x-x^2)*cos(1*
```

$$y(x) = \frac{\sqrt{2}\left(\int e^{2x}\left(\sqrt{2}\,\cos\left(\sqrt{2}\,x\right) - \sin\left(\sqrt{2}\,x\right)\right)\left(\left(-9x^2-13x-2\right)\sin(x) + \left(x^2-5x-27\right)\cos(x)\right)dx\right)\cos\left(\sqrt{2}\right.}{6}$$
$$- \frac{\sqrt{2}\left(\int -e^{2x}\left(\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)\left(\left(-9x^2-13x-2\right)\sin(x) + \left(x^2-5x-27\right)\cos(x)\right)dx\right)\sin}{6}$$
$$+ c_2\cos\left(\sqrt{2}\,x\right) + c_3\sin\left(\sqrt{2}\,x\right)$$
$$+ \frac{\left(5\left(-x^2+x+3\right)\cos(x) + 4\left(x^2+\frac{5}{2}x+\frac{7}{4}\right)\sin(x)\right)e^{2x}}{3} + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 60

```
DSolve[1*y'''[x]-1*y''[x]+2*y'[x]-2*y[x]==Exp[2*x]*((27+5*x-x^2)*Cos[1*x]+(2+13*x+9*x^2)*Sin
```

$$y(x) \to e^{2x}\left(\left(-x^2+x+1\right)\cos(x)+(2x+1)\sin(x)\right)+c_3e^x+c_1\cos\left(\sqrt{2}x\right)+c_2\sin\left(\sqrt{2}x\right)$$

19.32 problem section 9.3, problem 32

Internal problem ID [1529]
Internal file name [OUTPUT/1530_Sunday_June_05_2022_02_20_50_AM_91271860/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 32.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' + y' - 2y = -e^{x}\left(\left(4x^{2} + 5x + 9\right)\cos\left(2x\right) - \left(-3x^{2} - 5x + 6\right)\sin\left(2x\right)\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' + y' - 2y = 0$$

The characteristic equation is

$$\lambda^{3} - 2\lambda^{2} + \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{-ix} + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{-ix}$$
$$y_3 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' + y' - 2y = -e^x \left(\left(4x^2 + 5x + 9\right) \cos(2x) - \left(-3x^2 - 5x + 6\right) \sin(2x) \right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^x \left(\left(4x^2 + 5x + 9\right) \cos(2x) - \left(-3x^2 - 5x + 6\right) \sin(2x) \right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x), \cos(2x) e^x x, \cos(2x) e^x x^2, \sin(2x) e^x x, \sin(2x) e^x x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{ix}, e^{2x}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned} y_p = {} & A_1 e^x \cos(2x) + A_2 e^x \sin(2x) + A_3 \cos(2x) e^x x \\ & + A_4 \cos(2x) e^x x^2 + A_5 \sin(2x) e^x x + A_6 \sin(2x) e^x x^2 \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & -8A_6 \cos(2x) e^x x^2 - 4A_3 \sin(2x) e^x - 6A_3 \cos(2x) e^x x + 2A_6 \sin(2x) e^x \\ & + 2A_4 \cos(2x) e^x + 8A_4 \sin(2x) e^x x^2 - 12A_5 \sin(2x) e^x - 8A_5 \cos(2x) e^x x \\ & - 12A_3 \cos(2x) e^x + 8A_3 \sin(2x) e^x x - 6A_4 \cos(2x) e^x x^2 - 6A_5 \sin(2x) e^x x \\ & - 6A_6 \sin(2x) e^x x^2 - 24A_6 \sin(2x) e^x x - 8A_4 \sin(2x) e^x x + 4A_5 \cos(2x) e^x \\ & + 8A_6 \cos(2x) e^x x - 24A_4 \cos(2x) e^x x - 6A_1 e^x \cos(2x) - 12A_4 \sin(2x) e^x \\ & + 8A_1 e^x \sin(2x) + 12A_6 \cos(2x) e^x - 8A_2 e^x \cos(2x) - 6A_2 e^x \sin(2x) = \\ & \quad -e^x \left(\left(4x^2 + 5x + 9\right) \cos(2x) - \left(-3x^2 - 5x + 6\right) \sin(2x) \right) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{61}{50}, A_2 = -\frac{27}{50}, A_3 = \frac{11}{10}, A_4 = 0, A_5 = \frac{3}{10}, A_6 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{61\,\mathrm{e}^x \cos(2x)}{50} - \frac{27\,\mathrm{e}^x \sin(2x)}{50} + \frac{11\cos(2x)\,\mathrm{e}^x x}{10} + \frac{3\sin(2x)\,\mathrm{e}^x x}{10} + \frac{\sin(2x)\,\mathrm{e}^x x^2}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3\right) \\
&\quad + \left(\frac{61\,\mathrm{e}^x \cos(2x)}{50} - \frac{27\,\mathrm{e}^x \sin(2x)}{50} + \frac{11\cos(2x)\,\mathrm{e}^x x}{10} + \frac{3\sin(2x)\,\mathrm{e}^x x}{10} + \frac{\sin(2x)\,\mathrm{e}^x x^2}{2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 + \frac{61\,\mathrm{e}^x \cos(2x)}{50} - \frac{27\,\mathrm{e}^x \sin(2x)}{50} \\
+ \frac{11\cos(2x)\,\mathrm{e}^x x}{10} + \frac{3\sin(2x)\,\mathrm{e}^x x}{10} + \frac{\sin(2x)\,\mathrm{e}^x x^2}{2}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 + \frac{61\,\mathrm{e}^x \cos(2x)}{50} - \frac{27\,\mathrm{e}^x \sin(2x)}{50} \\
+ \frac{11\cos(2x)\,\mathrm{e}^x x}{10} + \frac{3\sin(2x)\,\mathrm{e}^x x}{10} + \frac{\sin(2x)\,\mathrm{e}^x x^2}{2}
\end{aligned}$$

Verified OK.

19.32.1 Maple step by step solution

Let's solve

$y''' - 2y'' + y' - 2y = -\mathrm{e}^x\left((4x^2 + 5x + 9)\cos(2x) - (-3x^2 - 5x + 6)\sin(2x)\right)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

$y''' = 2y - 4\cos(2x)\,e^x x^2 - 3\sin(2x)\,e^x x^2 - 5\cos(2x)\,e^x x - 5\sin(2x)\,e^x x - 9\,e^x\cos(2x) + 6\,e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' - 2y'' + y' - 2y = -e^x(3\sin(2x)\,x^2 + 4x^2\cos(2x) + 5x\sin(2x) + 5x\cos(2x) - 6\sin(2x) + 9$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -4\cos(2x)\,e^x x^2 - 3\sin(2x)\,e^x x^2 - 5\cos(2x)\,e^x x - 5\sin(2x)\,e^x x - 9\,e^x\cos(2x) + 6\,e^x s$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -4\cos(2x)\,e^x x^2 - 3\sin(2x)\,e^x x^2 - 5\cos(2x)\,e^x x - 5\sin$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -3\sin(2x)\,e^x x^2 - 4\cos(2x)\,e^x x^2 - 5\sin(2x)\,e^x x - 5\cos(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -3\sin(2x)\,e^x x^2 - 4\cos(2x)\,e^x x^2 - 5\sin(2x)\,e^x x - 5\cos(2x)\,e^x x + 6\,e^x\sin(2x) - 9\,e \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}^{\prime}(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \left[\begin{array}{c} -\cos(x) \\ \sin(x) \\ \cos(x) \end{array}\right], \overrightarrow{y}_3(x) = \left[\begin{array}{c} \sin(x) \\ \cos(x) \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} \frac{e^{2x}}{4} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \sin(x) & \cos(x) \\ e^{2x} & \cos(x) & -\sin(x) \end{array}\right]$$

- ○ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- ○ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} \frac{e^{2x}}{4} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \sin(x) & \cos(x) \\ e^{2x} & \cos(x) & -\sin(x) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} \frac{1}{4} & -1 & 0 \\ \frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]}$$

- ○ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} \frac{e^{2x}}{5}+\frac{4\cos(x)}{5}-\frac{2\sin(x)}{5} & \sin(x) & \frac{e^{2x}}{5}-\frac{\cos(x)}{5}-\frac{2\sin(x)}{5} \\ \frac{2e^{2x}}{5}-\frac{4\sin(x)}{5}-\frac{2\cos(x)}{5} & \cos(x) & \frac{2e^{2x}}{5}+\frac{\sin(x)}{5}-\frac{2\cos(x)}{5} \\ \frac{4e^{2x}}{5}-\frac{4\cos(x)}{5}+\frac{2\sin(x)}{5} & -\sin(x) & \frac{4e^{2x}}{5}+\frac{\cos(x)}{5}+\frac{2\sin(x)}{5} \end{array}\right]$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x)\cdot\overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x) = A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x) = A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x)\cdot\overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{6e^{2x}}{25}+\frac{\left((110x+122)\cos(x)^2+(50x^2+30x-54)\sin(x)\cos(x)-55x-61\right)e^x}{50}-\frac{73\cos(x)}{50}-\frac{43\sin(x)}{25} \\ \frac{12e^{2x}}{25}+\frac{\left((100x^2+170x+124)\cos(x)^2+(50x^2-90x-268)\sin(x)\cos(x)-50x^2-85x-62\right)e^x}{50}-\frac{43\cos(x)}{25}+73 \\ \frac{24e^{2x}}{25}+\frac{\left((200x^2+190x-242)\cos(x)^2+(-150x^2-330x-606)\sin(x)\cos(x)-100x^2-95x+121\right)e^x}{50}+\frac{73\cos(x)}{50}+ \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} \frac{6\,e^{2x}}{25} + \frac{\left((110x+122)\cos(x)^2+(50x^2+30x-54)\sin(x)\cos(x)-55 \right.}{50} \\ \frac{12\,e^{2x}}{25} + \frac{\left((100x^2+170x+124)\cos(x)^2+(50x^2-90x-268)\sin(x)\cos(x)-5 \right.}{50} \\ \frac{24\,e^{2x}}{25} + \frac{\left((200x^2+190x-242)\cos(x)^2+(-150x^2-330x-606)\sin(x)\cos(x)- \right.}{50} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(25c_1+24)e^{2x}}{100} + \frac{(55x+61)e^x\cos(x)^2}{25} + \frac{\left(-73+50\left(x^2+\frac{3}{5}x-\frac{27}{25}\right)\sin(x)e^x-50c_2\right)\cos(x)}{50} + \frac{(-55x-61)e^x}{50} + \frac{\sin(x)(25c_3-4}{25}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve(1*diff(y(x),x$3)-2*diff(y(x),x$2)+1*diff(y(x),x)-2*y(x)=-exp(x)*((9+5*x+4*x^2)*cos(2*
```

$$y(x) = \frac{e^x\left(x^2+\frac{3}{5}x-\frac{27}{25}\right)\sin(2x)}{2} + \frac{(55x+61)e^x\cos(2x)}{50} + \cos(x)c_1 + \sin(x)c_2 + c_3e^{2x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 65

```
DSolve[1*y'''[x]-2*y''[x]+1*y'[x]-2*y[x]==Exp[2*x]*((9+5*x+4*x^2)*Cos[2*x]-(6-5*x-3*x^2)*Sin
```

$$y(x) \to -\frac{e^{2x}((6760x^2-17680x-29907)\sin(2x)+2(5915x^2+7345x+3928)\cos(2x))}{43940} + c_3e^{2x} + c_1\cos(x) + c_2\sin(x)$$

19.33 problem section 9.3, problem 33

Internal problem ID [1530]
Internal file name [`OUTPUT/1531_Sunday_June_05_2022_02_20_53_AM_84926883/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 33.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 3y'' + 4y' + 12y = 8\cos(2x) - 16\sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 4y' + 12y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 4\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-3x} \\ y_2 &= \mathrm{e}^{2ix} \\ y_3 &= \mathrm{e}^{-2ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 4y' + 12y = 8\cos(2x) - 16\sin(2x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{2ix} & \mathrm{e}^{-2ix} \\ -3\,\mathrm{e}^{-3x} & 2i\mathrm{e}^{2ix} & -2i\mathrm{e}^{-2ix} \\ 9\,\mathrm{e}^{-3x} & -4\,\mathrm{e}^{2ix} & -4\,\mathrm{e}^{-2ix} \end{bmatrix}$$

$$|W| = -52i\mathrm{e}^{-3x}\mathrm{e}^{2ix}\mathrm{e}^{-2ix}$$

The determinant simplifies to

$$|W| = -52i e^{-3x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{2ix} & e^{-2ix} \\ 2ie^{2ix} & -2ie^{-2ix} \end{bmatrix} \\ &= -4i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-3x} & e^{-2ix} \\ -3\,e^{-3x} & -2ie^{-2ix} \end{bmatrix} \\ &= (3-2i)\,e^{(-3-2i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-3x} & e^{2ix} \\ -3\,e^{-3x} & 2ie^{2ix} \end{bmatrix} \\ &= (3+2i)\,e^{(-3+2i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(8\cos(2x) - 16\sin(2x))(-4i)}{(1)(-52ie^{-3x})}\,dx \\ &= \int \frac{-4i(8\cos(2x) - 16\sin(2x))}{-52ie^{-3x}}\,dx \\ &= \int \left(\frac{8(\cos(2x) - 2\sin(2x))\,e^{3x}}{13}\right) dx \\ &= \frac{16(3\cos(x) + 2\sin(x))\,e^{3x}\cos(x)}{169} - \frac{24\,e^{3x}}{169} - \frac{16\,e^{3x}(3\sin(2x) - 2\cos(2x))}{169} \\ &= \frac{16(3\cos(x) + 2\sin(x))\,e^{3x}\cos(x)}{169} - \frac{24\,e^{3x}}{169} - \frac{16\,e^{3x}(3\sin(2x) - 2\cos(2x))}{169} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^{1} \int \frac{(8\cos(2x) - 16\sin(2x))\left((3-2i)\,\mathrm{e}^{(-3-2i)x}\right)}{(1)\left(-52i\mathrm{e}^{-3x}\right)}\, dx \\
&= -\int \frac{(3-2i)\,(8\cos(2x) - 16\sin(2x))\,\mathrm{e}^{(-3-2i)x}}{-52i\mathrm{e}^{-3x}}\, dx \\
&= -\int \left(\left(\frac{4}{13} + \frac{6i}{13}\right)(\cos(2x) - 2\sin(2x))\,\mathrm{e}^{-2ix}\right) dx \\
&= \frac{4x}{13} - \frac{7ix}{13} - \frac{\mathrm{e}^{-4ix}}{52} - \frac{2i\mathrm{e}^{-4ix}}{13} \\
&= \frac{4x}{13} - \frac{7ix}{13} - \frac{\mathrm{e}^{-4ix}}{52} - \frac{2i\mathrm{e}^{-4ix}}{13}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{(8\cos(2x) - 16\sin(2x))\left((3+2i)\,\mathrm{e}^{(-3+2i)x}\right)}{(1)\left(-52i\mathrm{e}^{-3x}\right)}\, dx \\
&= \int \frac{(3+2i)\,(8\cos(2x) - 16\sin(2x))\,\mathrm{e}^{(-3+2i)x}}{-52i\mathrm{e}^{-3x}}\, dx \\
&= \int \left(\left(-\frac{4}{13} + \frac{6i}{13}\right)(\cos(2x) - 2\sin(2x))\,\mathrm{e}^{2ix}\right) dx \\
&= \frac{4x}{13} + \frac{7ix}{13} - \frac{\mathrm{e}^{4ix}}{52} + \frac{2i\mathrm{e}^{4ix}}{13} \\
&= \frac{4x}{13} + \frac{7ix}{13} - \frac{\mathrm{e}^{4ix}}{52} + \frac{2i\mathrm{e}^{4ix}}{13}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(\frac{16(3\cos(x) + 2\sin(x))\,\mathrm{e}^{3x}\cos(x)}{169} - \frac{24\,\mathrm{e}^{3x}}{169} - \frac{16\,\mathrm{e}^{3x}(3\sin(2x) - 2\cos(2x))}{169}\right)\left(\mathrm{e}^{-3x}\right) \\
& + \left(\frac{4x}{13} - \frac{7ix}{13} - \frac{\mathrm{e}^{-4ix}}{52} - \frac{2i\mathrm{e}^{-4ix}}{13}\right)\left(\mathrm{e}^{2ix}\right) \\
& + \left(\frac{4x}{13} + \frac{7ix}{13} - \frac{\mathrm{e}^{4ix}}{52} + \frac{2i\mathrm{e}^{4ix}}{13}\right)\left(\mathrm{e}^{-2ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{(208x + 99)\cos(2x)}{338} + \frac{14(-6 + 13x)\sin(2x)}{169}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-3x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3\right) + \left(\frac{(208x+99)\cos(2x)}{338} + \frac{14(-6+13x)\sin(2x)}{169}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3 + \frac{(208x+99)\cos(2x)}{338} + \frac{14(-6+13x)\sin(2x)}{169} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3 + \frac{(208x+99)\cos(2x)}{338} + \frac{14(-6+13x)\sin(2x)}{169}
$$

Verified OK.

19.33.1 Maple step by step solution

Let's solve

$y''' + 3y'' + 4y' + 12y = 8\cos(2x) - 16\sin(2x)$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8\cos(2x) - 16\sin(2x) - 3y_3(x) - 4y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8\cos(2x) - 16\sin(2x) - 3y_3(x) - 4y_2(x) - 12y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -4 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 8\cos(2x) - 16\sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 8\cos(2x) - 16\sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -4 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos(2x) - \mathrm{I}\sin(2x)\right) \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}\left(\cos(2x) - \mathrm{I}\sin(2x)\right) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix}\frac{e^{-3x}}{9} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4}\\ -\frac{e^{-3x}}{3} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2}\\ e^{-3x} & \cos(2x) & -\sin(2x)\end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix}\frac{e^{-3x}}{9} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4}\\ -\frac{e^{-3x}}{3} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2}\\ e^{-3x} & \cos(2x) & -\sin(2x)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}\frac{1}{9} & -\frac{1}{4} & 0\\ -\frac{1}{3} & 0 & \frac{1}{2}\\ 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix}\frac{4e^{-3x}}{13}+\frac{9\cos(2x)}{13}+\frac{6\sin(2x)}{13} & \frac{\sin(2x)}{2} & \frac{e^{-3x}}{13}-\frac{\cos(2x)}{13}+\frac{3\sin(2x)}{26}\\ -\frac{12e^{-3x}}{13}-\frac{18\sin(2x)}{13}+\frac{12\cos(2x)}{13} & \cos(2x) & -\frac{3e^{-3x}}{13}+\frac{2\sin(2x)}{13}+\frac{3\cos(2x)}{13}\\ \frac{36e^{-3x}}{13}-\frac{36\cos(2x)}{13}-\frac{24\sin(2x)}{13} & -2\sin(2x) & \frac{9e^{-3x}}{13}+\frac{4\cos(2x)}{13}-\frac{6\sin(2x)}{13}\end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{8(7+13x)\cos(2x)}{169} + \frac{2(91x-68)\sin(2x)}{169} - \frac{56\,\mathrm{e}^{-3x}}{169} \\ \frac{28(-6+13x)\cos(2x)}{169} + \frac{2(-104x+35)\sin(2x)}{169} + \frac{168\,\mathrm{e}^{-3x}}{169} \\ \frac{8(-52x+63)\cos(2x)}{169} + \frac{8(-91x+16)\sin(2x)}{169} - \frac{504\,\mathrm{e}^{-3x}}{169} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \left[\begin{array}{c} \frac{8(7+13x)\cos(2x)}{169} + \frac{2(91x-68)\sin(2x)}{169} - \frac{56\,\mathrm{e}^{-3x}}{169} \\ \frac{28(-6+13x)\cos(2x)}{169} + \frac{2(-104x+35)\sin(2x)}{169} + \frac{168\,\mathrm{e}^{-3x}}{169} \\ \frac{8(-52x+63)\cos(2x)}{169} + \frac{8(-91x+16)\sin(2x)}{169} - \frac{504\,\mathrm{e}^{-3x}}{169} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(224+416x-169c_2)\cos(2x)}{676} + \frac{(-544+728x+169c_3)\sin(2x)}{676} + \frac{\mathrm{e}^{-3x}(169c_1-504)}{1521}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(1*diff(y(x),x$3)+3*diff(y(x),x$2)+4*diff(y(x),x)+12*y(x)=8*cos(2*x)-16*sin(2*x),y(x),
```

$$y(x) = \frac{(169c_1 + 104x + 56)\cos(2x)}{169} + \frac{(182x + 169c_3 - 136)\sin(2x)}{169} + e^{-3x}c_2$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 47

```
DSolve[1*y'''[x]+3*y''[x]+4*y'[x]+12*y[x]==8*Cos[2*x]-16*Sin[2*x],y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{1}{169}\left(169c_3e^{-3x} + (104x + 43 + 169c_1)\cos(2x) + (182x - 32 + 169c_2)\sin(2x)\right)$$

19.34 problem section 9.3, problem 34

Internal problem ID [1531]
Internal file name [`OUTPUT/1532_Sunday_June_05_2022_02_20_56_AM_8322504/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 34.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' + 2y = e^{x}((20 + 4x)\cos(x) - (12x + 12)\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + 2y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 2 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= 1 - i\\ \lambda_3 &= 1 + i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{(1-i)x} c_2 + \mathrm{e}^{(1+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{(1-i)x} \\ y_3 &= \mathrm{e}^{(1+i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + 2y = \mathrm{e}^{x}((20 + 4x) \cos(x) - (12x + 12) \sin(x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x} \\ -\mathrm{e}^{-x} & (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x} \\ \mathrm{e}^{-x} & -2i\mathrm{e}^{(1-i)x} & 2i\mathrm{e}^{(1+i)x} \end{bmatrix}$$

$$|W| = 10i\mathrm{e}^{-x} \mathrm{e}^{(1-i)x} \mathrm{e}^{(1+i)x}$$

The determinant simplifies to

$$|W| = 10i\mathrm{e}^{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det\begin{bmatrix}\mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x}\\ (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x}\end{bmatrix}\\ &= 2i\mathrm{e}^{2x}\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det\begin{bmatrix}\mathrm{e}^{-x} & \mathrm{e}^{(1+i)x}\\ -\mathrm{e}^{-x} & (1+i)\,\mathrm{e}^{(1+i)x}\end{bmatrix}\\ &= (2+i)\,\mathrm{e}^{ix}\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det\begin{bmatrix}\mathrm{e}^{-x} & \mathrm{e}^{(1-i)x}\\ -\mathrm{e}^{-x} & (1-i)\,\mathrm{e}^{(1-i)x}\end{bmatrix}\\ &= (2-i)\,\mathrm{e}^{-ix}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx\\ &= (-1)^{2}\int \frac{(\mathrm{e}^{x}((20+4x)\cos(x)-(12x+12)\sin(x)))\,(2i\mathrm{e}^{2x})}{(1)\,(10i\mathrm{e}^{x})}\,dx\\ &= \int \frac{2i\mathrm{e}^{x}((20+4x)\cos(x)-(12x+12)\sin(x))\,\mathrm{e}^{2x}}{10i\mathrm{e}^{x}}\,dx\\ &= \int \left(\frac{4((x+5)\cos(x)-3(x+1)\sin(x))\,\mathrm{e}^{2x}}{5}\right)\,dx\\ &= \frac{4\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\cos(x)}{5} - \frac{4\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\sin(x)}{5} - \frac{12\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\cos(x)}{5} - \frac{12\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\sin(x)}{5}\\ &= \frac{4\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\cos(x)}{5} - \frac{4\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\sin(x)}{5} - \frac{12\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\cos(x)}{5} - \frac{12\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\sin(x)}{5}\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\mathrm{e}^x((20+4x)\cos(x)-(12x+12)\sin(x)))\,((2+i)\,\mathrm{e}^{ix})}{(1)\,(10i\mathrm{e}^x)}\,dx \\
&= -\int \frac{(2+i)\,\mathrm{e}^x((20+4x)\cos(x)-(12x+12)\sin(x))\,\mathrm{e}^{ix}}{10i\mathrm{e}^x}\,dx \\
&= -\int \left(\left(\frac{2}{5}-\frac{4i}{5}\right)((x+5)\cos(x)-3(x+1)\sin(x))\,\mathrm{e}^{ix}\right)\,dx \\
&= \frac{x^2}{2}+\frac{x}{5}+\frac{13ix}{5}+\frac{ix^2}{2}+\left(-\frac{1}{100}+\frac{7i}{100}\right)(14-7i+10x)\,\mathrm{e}^{2ix} \\
&= \frac{x^2}{2}+\frac{x}{5}+\frac{13ix}{5}+\frac{ix^2}{2}+\left(-\frac{1}{100}+\frac{7i}{100}\right)(14-7i+10x)\,\mathrm{e}^{2ix}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\mathrm{e}^x((20+4x)\cos(x)-(12x+12)\sin(x)))\,((2-i)\,\mathrm{e}^{-ix})}{(1)\,(10i\mathrm{e}^x)}\,dx \\
&= \int \frac{(2-i)\,\mathrm{e}^x((20+4x)\cos(x)-(12x+12)\sin(x))\,\mathrm{e}^{-ix}}{10i\mathrm{e}^x}\,dx \\
&= \int \left(\left(-\frac{2}{5}-\frac{4i}{5}\right)((x+5)\cos(x)-3(x+1)\sin(x))\,\mathrm{e}^{-ix}\right)\,dx \\
&= \int \left(-\frac{2}{5}-\frac{4i}{5}\right)((x+5)\cos(x)-3(x+1)\sin(x))\,\mathrm{e}^{-ix}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{4\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\cos(x)}{5}-\frac{4\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\sin(x)}{5}-\frac{12\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\cos(x)}{5}-\frac{12\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\sin(x)}{5}\right. \\
&+\left(\frac{x^2}{2}+\frac{x}{5}+\frac{13ix}{5}+\frac{ix^2}{2}+\left(-\frac{1}{100}+\frac{7i}{100}\right)(14-7i+10x)\,\mathrm{e}^{2ix}\right)\left(\mathrm{e}^{(1-i)x}\right) \\
&+\left(\int\left(-\frac{2}{5}-\frac{4i}{5}\right)((x+5)\cos(x)-3(x+1)\sin(x))\,\mathrm{e}^{-ix}dx\right)\left(\mathrm{e}^{(1+i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{23\left(\left(\frac{20}{23}x^2+\frac{25}{23}+i+\frac{20}{23}x\right)\cos(x)-\frac{21\left(-\frac{20}{21}x^2+\frac{65}{21}+i-\frac{20}{7}x\right)\sin(x)}{23}\right)\mathrm{e}^x}{20}
$$

Which simplifies to

$$y_p = \frac{23\left(\left(\frac{20}{23}x^2 + \frac{25}{23} + i + \frac{20}{23}x\right)\cos(x) - \frac{21\left(-\frac{20}{21}x^2 + \frac{65}{21} + i - \frac{20}{7}x\right)\sin(x)}{23}\right)\mathrm{e}^x}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \mathrm{e}^{(1-i)x}c_2 + \mathrm{e}^{(1+i)x}c_3\right) \\ &\quad + \left(\frac{23\left(\left(\frac{20}{23}x^2 + \frac{25}{23} + i + \frac{20}{23}x\right)\cos(x) - \frac{21\left(-\frac{20}{21}x^2 + \frac{65}{21} + i - \frac{20}{7}x\right)\sin(x)}{23}\right)\mathrm{e}^x}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\mathrm{e}^{-x} + \mathrm{e}^{(1-i)x}c_2 + \mathrm{e}^{(1+i)x}c_3 \\ &\quad + \frac{23\left(\left(\frac{20}{23}x^2 + \frac{25}{23} + i + \frac{20}{23}x\right)\cos(x) - \frac{21\left(-\frac{20}{21}x^2 + \frac{65}{21} + i - \frac{20}{7}x\right)\sin(x)}{23}\right)\mathrm{e}^x}{20} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\mathrm{e}^{-x} + \mathrm{e}^{(1-i)x}c_2 + \mathrm{e}^{(1+i)x}c_3 \\ &\quad + \frac{23\left(\left(\frac{20}{23}x^2 + \frac{25}{23} + i + \frac{20}{23}x\right)\cos(x) - \frac{21\left(-\frac{20}{21}x^2 + \frac{65}{21} + i - \frac{20}{7}x\right)\sin(x)}{23}\right)\mathrm{e}^x}{20} \end{aligned}$$

Verified OK.

19.34.1 Maple step by step solution

Let's solve

$y''' - y'' + 2y = \mathrm{e}^x((20 + 4x)\cos(x) - (12x + 12)\sin(x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -2y + 4\,\mathrm{e}^x\cos(x)\,x - 12\,\mathrm{e}^x\sin(x)\,x + 20\cos(x)\,\mathrm{e}^x - 12\sin(x)\,\mathrm{e}^x + y''$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - y'' + 2y = 4\,\mathrm{e}^x(-3\sin(x)\,x + x\cos(x) - 3\sin(x) + 5\cos(x))$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 4\,\mathrm{e}^{x} \cos\left(x\right) x - 12\,\mathrm{e}^{x} \sin\left(x\right) x + 20 \cos\left(x\right) \mathrm{e}^{x} - 12 \sin\left(x\right) \mathrm{e}^{x} + y_3(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4\,\mathrm{e}^{x} \cos\left(x\right) x - 12\,\mathrm{e}^{x} \sin\left(x\right) x + 20 \cos\left(x\right) \mathrm{e}^{x} - 12 \sin\left(x\right) \mathrm{e}$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 1 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ -12\,\mathrm{e}^{x} \sin\left(x\right) x + 4\,\mathrm{e}^{x} \cos\left(x\right) x - 12 \sin\left(x\right) \mathrm{e}^{x} + 20 \cos\left(x\right) \mathrm{e}^{x} \end{array}\right.$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ -12\,\mathrm{e}^{x} \sin\left(x\right) x + 4\,\mathrm{e}^{x} \cos\left(x\right) x - 12 \sin\left(x\right) \mathrm{e}^{x} + 20 \cos\left(x\right) \mathrm{e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1-\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(\frac{1}{2} + \frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

□ Fundamental matrix

◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2} \\ -e^{-x} & e^{x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^{x} \\ e^{-x} & \cos(x)e^{x} & -\sin(x)e^{x} \end{bmatrix}$$

◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

◦ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2} \\ -e^{-x} & e^{x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^{x} \\ e^{-x} & \cos(x)e^{x} & -\sin(x)e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2e^{-x}}{5} + \frac{(3\cos(x)-\sin(x))e^{x}}{5} & -\frac{2e^{-x}}{5} + \frac{(2\cos(x)+\sin(x))e^{x}}{5} & \frac{e^{-x}}{5} + \frac{(2\sin(x)-\cos(x))e^{x}}{5} \\ -\frac{2e^{-x}}{5} + \frac{(-4\sin(x)+2\cos(x))e^{x}}{5} & \frac{2e^{-x}}{5} + \frac{(3\cos(x)-\sin(x))e^{x}}{5} & -\frac{e^{-x}}{5} + \frac{(\cos(x)+3\sin(x))e^{x}}{5} \\ \frac{2e^{-x}}{5} + \frac{(-2\cos(x)-6\sin(x))e^{x}}{5} & -\frac{2e^{-x}}{5} + \frac{(-4\sin(x)+2\cos(x))e^{x}}{5} & \frac{e^{-x}}{5} + \frac{(4\cos(x)+2\sin(x))e^{x}}{5} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -\frac{8\,\mathrm{e}^{-x}}{5}+\frac{\left(\left(5x^2+5x+8\right)\cos(x)+\left(5x^2+15x-21\right)\sin(x)\right)\mathrm{e}^{x}}{5} \\ \frac{8\,\mathrm{e}^{-x}}{5}+\frac{2\left(\left(5x^2+15x-4\right)\cos(x)+\left(-7+10x\right)\sin(x)\right)\mathrm{e}^{x}}{5} \\ -\frac{8\,\mathrm{e}^{-x}}{5}+\frac{2\left(\left(5x^2+35x+4\right)\cos(x)+\left(-5x^2-5x+7\right)\sin(x)\right)\mathrm{e}^{x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \left[\begin{array}{c} -\frac{8\,\mathrm{e}^{-x}}{5}+\frac{\left(\left(5x^2+5x+8\right)\cos(x)+\left(5x^2+15x-21\right)\sin(x)\right)\mathrm{e}^{x}}{5} \\ \frac{8\,\mathrm{e}^{-x}}{5}+\frac{2\left(\left(5x^2+15x-4\right)\cos(x)+\left(-7+10x\right)\sin(x)\right)\mathrm{e}^{x}}{5} \\ -\frac{8\,\mathrm{e}^{-x}}{5}+\frac{2\left(\left(5x^2+35x+4\right)\cos(x)+\left(-5x^2-5x+7\right)\sin(x)\right)\mathrm{e}^{x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-8+5c_1)\mathrm{e}^{-x}}{5} + \left(\left(\frac{1}{2}c_3 + x^2 + x + \frac{8}{5}\right)\cos(x) + \sin(x)\left(x^2 + 3x + \frac{1}{2}c_2 - \frac{21}{5}\right)\right)\mathrm{e}^{x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(1*diff(y(x),x$3)-1*diff(y(x),x$2)+0*diff(y(x),x)+2*y(x)=exp(x)*((20+4*x)*cos(x)-(12+1
```

$$y(x) = \mathrm{e}^{-x} c_1 + \mathrm{e}^{x} \left(\left(x^2 + x + c_2 + \frac{22}{5} \right) \cos\left(x\right) + \sin\left(x\right) \left(x^2 + 3x + \frac{1}{5} + c_3 \right) \right)$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 60

```
DSolve[1*y'''[x]-1*y''[x]+0*y'[x]+2*y[x]==Exp[x]*((20+4*x)*Cos[x]-(12+12*x)*Sin[x]),y[x],x,I
```

$$y(x) \to \frac{1}{10} e^x \left(10x^2 + 10x + 23 + 10c_2\right) \cos(x) + \frac{1}{10} e^x \left(10x^2 + 30x - 21 + 10c_1\right) \sin(x) + c_3 e^{-x}$$

19.35 problem section 9.3, problem 35

Internal problem ID [1532]
Internal file name [`OUTPUT/1533_Sunday_June_05_2022_02_21_00_AM_99028836/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 35.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$y''' - 7y'' + 20y' - 24y = -e^{2x}((13 - 8x)\cos(2x) - (8 - 4x)\sin(2x))$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 7y'' + 20y' - 24y = 0$$

The characteristic equation is

$$\lambda^3 - 7\lambda^2 + 20\lambda - 24 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 2 - 2i$$
$$\lambda_3 = 2 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{3x} + e^{(2-2i)x} c_2 + e^{(2+2i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{3x} \\ y_2 &= e^{(2-2i)x} \\ y_3 &= e^{(2+2i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 7y'' + 20y' - 24y = -e^{2x}((13 - 8x)\cos(2x) - (8 - 4x)\sin(2x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{3x} & e^{(2-2i)x} & e^{(2+2i)x} \\ 3\, e^{3x} & (2-2i)\, e^{(2-2i)x} & (2+2i)\, e^{(2+2i)x} \\ 9\, e^{3x} & -8i e^{(2-2i)x} & 8i e^{(2+2i)x} \end{bmatrix}$$

$$|W| = 20i e^{3x} e^{(2-2i)x} e^{(2+2i)x}$$

The determinant simplifies to

$$|W| = 20i\mathrm{e}^{7x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det\begin{bmatrix} \mathrm{e}^{(2-2i)x} & \mathrm{e}^{(2+2i)x} \\ (2-2i)\,\mathrm{e}^{(2-2i)x} & (2+2i)\,\mathrm{e}^{(2+2i)x} \end{bmatrix} \\ &= 4i\mathrm{e}^{4x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det\begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{(2+2i)x} \\ 3\,\mathrm{e}^{3x} & (2+2i)\,\mathrm{e}^{(2+2i)x} \end{bmatrix} \\ &= (-1+2i)\,\mathrm{e}^{(5+2i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det\begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{(2-2i)x} \\ 3\,\mathrm{e}^{3x} & (2-2i)\,\mathrm{e}^{(2-2i)x} \end{bmatrix} \\ &= (-1-2i)\,\mathrm{e}^{(5-2i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2\int \frac{\left(-\mathrm{e}^{2x}((13-8x)\cos(2x)-(8-4x)\sin(2x))\right)\left(4i\mathrm{e}^{4x}\right)}{(1)\left(20i\mathrm{e}^{7x}\right)}\,dx \\ &= \int \frac{-4i\mathrm{e}^{2x}((13-8x)\cos(2x)-(8-4x)\sin(2x))\,\mathrm{e}^{4x}}{20i\mathrm{e}^{7x}}\,dx \\ &= \int \left(\frac{(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)-13\cos(2x))\,\mathrm{e}^{-x}}{5}\right)dx \\ &= -\frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{4\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5} + \frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-}}{5} \\ &= -\frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{4\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5} + \frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-}}{5} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\,dx \\
&= (-1)^{1} \int \frac{\left(-\mathrm{e}^{2x}\left((13-8x)\cos(2x)-(8-4x)\sin(2x)\right)\right)\left((-1+2i)\,\mathrm{e}^{(5+2i)x}\right)}{(1)\left(20i\mathrm{e}^{7x}\right)}\,dx \\
&= -\int \frac{(1-2i)\,\mathrm{e}^{2x}\left((13-8x)\cos(2x)-(8-4x)\sin(2x)\right)\mathrm{e}^{(5+2i)x}}{20i\mathrm{e}^{7x}}\,dx \\
&= -\int \left(\left(\frac{1}{10}+\frac{i}{20}\right)\left(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)-13\cos(2x)\right)\mathrm{e}^{2ix}\right)dx \\
&= -\frac{x^2}{4}+\frac{17x}{20}-\frac{3ix}{40}+\left(-\frac{1}{100}+\frac{3i}{400}\right)(-17+i+10x)\,\mathrm{e}^{4ix} \\
&= -\frac{x^2}{4}+\frac{17x}{20}-\frac{3ix}{40}+\left(-\frac{1}{100}+\frac{3i}{400}\right)(-17+i+10x)\,\mathrm{e}^{4ix}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\,dx \\
&= (-1)^{0} \int \frac{\left(-\mathrm{e}^{2x}\left((13-8x)\cos(2x)-(8-4x)\sin(2x)\right)\right)\left((-1-2i)\,\mathrm{e}^{(5-2i)x}\right)}{(1)\left(20i\mathrm{e}^{7x}\right)}\,dx \\
&= \int \frac{(1+2i)\,\mathrm{e}^{2x}\left((13-8x)\cos(2x)-(8-4x)\sin(2x)\right)\mathrm{e}^{(5-2i)x}}{20i\mathrm{e}^{7x}}\,dx \\
&= \int \left(\left(-\frac{1}{10}+\frac{i}{20}\right)\left(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)-13\cos(2x)\right)\mathrm{e}^{-2ix}\right)dx \\
&= \frac{3ix}{40}-\frac{x^2}{4}+\frac{17x}{20}+\left(-\frac{1}{100}-\frac{3i}{400}\right)(-17-i+10x)\,\mathrm{e}^{-4ix} \\
&= \frac{3ix}{40}-\frac{x^2}{4}+\frac{17x}{20}+\left(-\frac{1}{100}-\frac{3i}{400}\right)(-17-i+10x)\,\mathrm{e}^{-4ix}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5}-\frac{4\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5}+\frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5}-\frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}}{5}\right. \\
&+\left(-\frac{x^2}{4}+\frac{17x}{20}-\frac{3ix}{40}+\left(-\frac{1}{100}+\frac{3i}{400}\right)(-17+i+10x)\,\mathrm{e}^{4ix}\right)\left(\mathrm{e}^{(2-2i)x}\right) \\
&+\left(\frac{3ix}{40}-\frac{x^2}{4}+\frac{17x}{20}+\left(-\frac{1}{100}-\frac{3i}{400}\right)(-17-i+10x)\,\mathrm{e}^{-4ix}\right)\left(\mathrm{e}^{(2+2i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{\left(\left(x^2-3x-\frac{21}{20}\right)\cos(2x)-\left(x-\frac{37}{20}\right)\sin(2x)\right)\mathrm{e}^{2x}}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} + e^{(2-2i)x} c_2 + e^{(2+2i)x} c_3\right) + \left(-\frac{\left(\left(x^2 - 3x - \frac{21}{20}\right)\cos(2x) - \left(x - \frac{37}{20}\right)\sin(2x)\right) e^{2x}}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{3x} + e^{(2-2i)x} c_2 + e^{(2+2i)x} c_3 - \frac{\left(\left(x^2 - 3x - \frac{21}{20}\right)\cos(2x) - \left(x - \frac{37}{20}\right)\sin(2x)\right) e^{2x}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{3x} + e^{(2-2i)x} c_2 + e^{(2+2i)x} c_3 - \frac{\left(\left(x^2 - 3x - \frac{21}{20}\right)\cos(2x) - \left(x - \frac{37}{20}\right)\sin(2x)\right) e^{2x}}{2}
$$

Verified OK.

19.35.1 Maple step by step solution

Let's solve

$y''' - 7y'' + 20y' - 24y = -e^{2x}((13 - 8x)\cos(2x) - (8 - 4x)\sin(2x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 24y + 8\cos(2x) e^{2x} x - 4\sin(2x) e^{2x} x - 13\cos(2x) e^{2x} + 8\sin(2x) e^{2x} + 7y'' - 20y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 7y'' + 20y' - 24y = e^{2x}(-4x\sin(2x) + 8x\cos(2x) + 8\sin(2x) - 13\cos(2x))$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 8\cos(2x)\,\mathrm{e}^{2x}x - 4\sin(2x)\,\mathrm{e}^{2x}x - 13\cos(2x)\,\mathrm{e}^{2x} + 8\sin(2x)\,\mathrm{e}^{2x} + 7y_3(x) - 20y_2(x) + 24$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8\cos(2x)\,\mathrm{e}^{2x}x - 4\sin(2x)\,\mathrm{e}^{2x}x - 13\cos(2x)\,\mathrm{e}^{2x} + 8\sin(2$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -20 & 7 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -4\sin(2x)\,\mathrm{e}^{2x}x + 8\cos(2x)\,\mathrm{e}^{2x}x + 8\sin(2x)\,\mathrm{e}^{2x} - 13\cos(2$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -4\sin(2x)\,\mathrm{e}^{2x}x + 8\cos(2x)\,\mathrm{e}^{2x}x + 8\sin(2x)\,\mathrm{e}^{2x} - 13\cos(2x)\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -20 & 7 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[2 - 2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ \frac{1}{4} + \frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right], \left[2 + 2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ \frac{1}{4} - \frac{\mathrm{I}}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2 - 2\,\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{8} \\ \frac{1}{4} + \frac{\mathrm{I}}{4} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-2\,\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{\mathrm{I}}{8} \\ \frac{1}{4} + \frac{\mathrm{I}}{4} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2x} \cdot (\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c} \frac{\mathrm{I}}{8} \\ \frac{1}{4} + \frac{\mathrm{I}}{4} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \left(\frac{1}{4} + \frac{\mathrm{I}}{4}\right)(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{\sin(2x)}{8} \\ \frac{\sin(2x)}{4} + \frac{\cos(2x)}{4} \\ \cos(2x) \end{array}\right], \overrightarrow{y}_3(x) = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{\cos(2x)}{8} \\ \frac{\cos(2x)}{4} - \frac{\sin(2x)}{4} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+\overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{ccc} \frac{e^{3x}}{9} & \frac{\sin(2x)e^{2x}}{8} & \frac{\cos(2x)e^{2x}}{8} \\ \frac{e^{3x}}{3} & e^{2x}\left(\frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\right) & e^{2x}\left(\frac{\cos(2x)}{4}-\frac{\sin(2x)}{4}\right) \\ e^{3x} & \cos(2x)e^{2x} & -\sin(2x)e^{2x} \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\left[\begin{array}{ccc} \frac{e^{3x}}{9} & \frac{\sin(2x)e^{2x}}{8} & \frac{\cos(2x)e^{2x}}{8} \\ \frac{e^{3x}}{3} & e^{2x}\left(\frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\right) & e^{2x}\left(\frac{\cos(2x)}{4}-\frac{\sin(2x)}{4}\right) \\ e^{3x} & \cos(2x)e^{2x} & -\sin(2x)e^{2x} \end{array}\right]\cdot\frac{1}{\left[\begin{array}{ccc} \frac{1}{9} & 0 & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\left[\begin{array}{ccc} \frac{(-3\cos(2x)-9\sin(2x))e^{2x}}{5}+\frac{8e^{3x}}{5} & \frac{(8\cos(2x)+9\sin(2x))e^{2x}}{10}-\frac{4e^{3x}}{5} & \frac{(-\sin(2x)-2\cos(2x))e^{2x}}{10}+\frac{e^{3x}}{5} \\ \frac{12(-\sin(2x)-2\cos(2x))e^{2x}}{5}+\frac{24e^{3x}}{5} & \frac{(17\cos(2x)+\sin(2x))e^{2x}}{5}-\frac{12e^{3x}}{5} & \frac{(-3\cos(2x)+\sin(2x))e^{2x}}{5}+\frac{3e^{3x}}{5} \\ \frac{24(-3\cos(2x)+\sin(2x))e^{2x}}{5}+\frac{72e^{3x}}{5} & \frac{4(9\cos(2x)-8\sin(2x))e^{2x}}{5}-\frac{36e^{3x}}{5} & \frac{(-4\cos(2x)+8\sin(2x))e^{2x}}{5}+\frac{9e^{3x}}{5} \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(\left(-10x^2+30x+4\right)\cos(2x)+\sin(2x)(10x-13)\right)e^{2x}}{20} - \frac{e^{3x}}{5} \\ \frac{\left(\left(-5x^2+15x+3\right)\cos(2x)+5\sin(2x)\left(x^2-2x-\frac{6}{5}\right)\right)e^{2x}}{5} - \frac{3e^{3x}}{5} \\ \frac{e^{2x}\left(4\left(5x^2-10x-7\right)\sin(2x)-9e^x+9\cos(2x)\right)}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \left[\begin{array}{c} \frac{\left(\left(-10x^2+30x+4\right)\cos(2x)+\sin(2x)(10x-13)\right)e^{2x}}{20} - \frac{e^{3x}}{5} \\ \frac{\left(\left(-5x^2+15x+3\right)\cos(2x)+5\sin(2x)\left(x^2-2x-\frac{6}{5}\right)\right)e^{2x}}{5} - \frac{3e^{3x}}{5} \\ \frac{e^{2x}\left(4\left(5x^2-10x-7\right)\sin(2x)-9e^x+9\cos(2x)\right)}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(-20x^2+5c_3+60x+8\right)\cos(2x)+20\sin(2x)\left(x+\frac{c_2}{4}-\frac{13}{10}\right)\right)e^{2x}}{40} + \frac{e^{3x}(5c_1-9)}{45}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve(1*diff(y(x),x$3)-7*diff(y(x),x$2)+20*diff(y(x),x)-24*y(x)=-exp(2*x)*((13-8*x)*cos(2*x
```

$$y(x) = \frac{\left(\left(-20x^2+40c_2+60x-83\right)\cos\left(2x\right)+20\sin\left(2x\right)\left(x+2c_3-\frac{47}{10}\right)\right)e^{2x}}{40}+c_1e^{3x}$$

✓ Solution by Mathematica

Time used: 0.955 (sec). Leaf size: 55

```
DSolve[1*y'''[x]-7*y''[x]+20*y'[x]-24*y[x]==-Exp[2*x]*((13-8*x)*Cos[2*x]-(8-4*x)*Sin[2*x]),y
```

$$y(x)\to \frac{1}{40}e^{2x}\left(\left(-20x^2+60x+21+40c_2\right)\cos(2x)+40c_3e^x+(20x-37+40c_1)\sin(2x)\right)$$

19.36 problem section 9.3, problem 36

Internal problem ID [1533]
Internal file name [`OUTPUT/1534_Sunday_June_05_2022_02_21_03_AM_38095410/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 36.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 6y'' + 18y' = -e^{3x}((-3x + 2)\cos(3x) - (3 + 3x)\sin(3x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 18y' = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 18\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3 + 3i$$
$$\lambda_3 = 3 - 3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{(3-3i)x}c_2 + e^{(3+3i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= e^{(3-3i)x} \\ y_3 &= e^{(3+3i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 18y' = -e^{3x}((-3x+2)\cos(3x) - (3+3x)\sin(3x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & e^{(3-3i)x} & e^{(3+3i)x} \\ 0 & (3-3i)\, e^{(3-3i)x} & (3+3i)\, e^{(3+3i)x} \\ 0 & -18i e^{(3-3i)x} & 18i e^{(3+3i)x} \end{bmatrix}$$

$$|W| = 108i e^{(3-3i)x} e^{(3+3i)x}$$

The determinant simplifies to

$$|W| = 108i\mathrm{e}^{6x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{(3-3i)x} & \mathrm{e}^{(3+3i)x} \\ (3-3i)\,\mathrm{e}^{(3-3i)x} & (3+3i)\,\mathrm{e}^{(3+3i)x} \end{bmatrix} \\ &= 6i\mathrm{e}^{6x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} 1 & \mathrm{e}^{(3+3i)x} \\ 0 & (3+3i)\,\mathrm{e}^{(3+3i)x} \end{bmatrix} \\ &= (3+3i)\,\mathrm{e}^{(3+3i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} 1 & \mathrm{e}^{(3-3i)x} \\ 0 & (3-3i)\,\mathrm{e}^{(3-3i)x} \end{bmatrix} \\ &= (3-3i)\,\mathrm{e}^{(3-3i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{\left(-\mathrm{e}^{3x}((-3x+2)\cos(3x) - (3+3x)\sin(3x))\right)\left(6i\mathrm{e}^{6x}\right)}{(1)\left(108i\mathrm{e}^{6x}\right)}\,dx \\ &= \int \frac{-6i\mathrm{e}^{3x}((-3x+2)\cos(3x) - (3+3x)\sin(3x))\,\mathrm{e}^{6x}}{108i\mathrm{e}^{6x}}\,dx \\ &= \int \left(-\frac{\mathrm{e}^{3x}((-3x+2)\cos(3x) - (3+3x)\sin(3x))}{18}\right)\,dx \\ &= \frac{\cos(3x)\,\mathrm{e}^{3x}x}{36} - \frac{\left(-\frac{3x}{2}+\frac{1}{2}\right)\mathrm{e}^{3x}\sin(3x)}{54} + \frac{\left(-\frac{3x}{2}+\frac{1}{2}\right)\mathrm{e}^{3x}\cos(3x)}{54} + \frac{\sin(3x)\,\mathrm{e}^{3x}x}{36} - \frac{5\cos(3x)\,\mathrm{e}^{3x}}{108} + \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx\\
&= (-1)^{1}\int \frac{\left(-\mathrm{e}^{3x}((-3x+2)\cos(3x)-(3+3x)\sin(3x))\right)\left((3+3i)\,\mathrm{e}^{(3+3i)x}\right)}{(1)\left(108i\mathrm{e}^{6x}\right)}\,dx\\
&= -\int \frac{(-3-3i)\,\mathrm{e}^{3x}((-3x+2)\cos(3x)-(3+3x)\sin(3x))\,\mathrm{e}^{(3+3i)x}}{108i\mathrm{e}^{6x}}\,dx\\
&= -\int \left(\left(\frac{1}{36}-\frac{i}{36}\right)(3x\sin(3x)+3x\cos(3x)+3\sin(3x)-2\cos(3x))\,\mathrm{e}^{3ix}\right)dx\\
&= -\frac{x^2}{24}-\frac{x}{72}-\frac{5ix}{72}+\frac{(1-4i+6x)\,\mathrm{e}^{6ix}}{432}\\
&= -\frac{x^2}{24}-\frac{x}{72}-\frac{5ix}{72}+\frac{(1-4i+6x)\,\mathrm{e}^{6ix}}{432}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3}\int \frac{F(x)W_3(x)}{aW(x)}\,dx\\
&= (-1)^{0}\int \frac{\left(-\mathrm{e}^{3x}((-3x+2)\cos(3x)-(3+3x)\sin(3x))\right)\left((3-3i)\,\mathrm{e}^{(3-3i)x}\right)}{(1)\left(108i\mathrm{e}^{6x}\right)}\,dx\\
&= \int \frac{(-3+3i)\,\mathrm{e}^{3x}((-3x+2)\cos(3x)-(3+3x)\sin(3x))\,\mathrm{e}^{(3-3i)x}}{108i\mathrm{e}^{6x}}\,dx\\
&= \int \left(\left(-\frac{1}{36}-\frac{i}{36}\right)(3x\sin(3x)+3x\cos(3x)+3\sin(3x)-2\cos(3x))\,\mathrm{e}^{-3ix}\right)dx\\
&= \frac{5ix}{72}-\frac{x^2}{24}-\frac{x}{72}+\frac{(1+4i+6x)\,\mathrm{e}^{-6ix}}{432}\\
&= \frac{5ix}{72}-\frac{x^2}{24}-\frac{x}{72}+\frac{(1+4i+6x)\,\mathrm{e}^{-6ix}}{432}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{\cos(3x)\,\mathrm{e}^{3x}x}{36}-\frac{\left(-\frac{3x}{2}+\frac{1}{2}\right)\mathrm{e}^{3x}\sin(3x)}{54}+\frac{\left(-\frac{3x}{2}+\frac{1}{2}\right)\mathrm{e}^{3x}\cos(3x)}{54}+\frac{\sin(3x)\,\mathrm{e}^{3x}x}{36}-\frac{5\cos(3x)\,\mathrm{e}^{3x}}{108}+\right.\\
&+\left(-\frac{x^2}{24}-\frac{x}{72}-\frac{5ix}{72}+\frac{(1-4i+6x)\,\mathrm{e}^{6ix}}{432}\right)\left(\mathrm{e}^{(3-3i)x}\right)\\
&+\left(\frac{5ix}{72}-\frac{x^2}{24}-\frac{x}{72}+\frac{(1+4i+6x)\,\mathrm{e}^{-6ix}}{432}\right)\left(\mathrm{e}^{(3+3i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{\mathrm{e}^{3x}(18\cos(3x)\,x^2+18x\sin(3x)+7\cos(3x)-4\sin(3x))}{216}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \mathrm{e}^{(3-3i)x} c_2 + \mathrm{e}^{(3+3i)x} c_3\right) \\
&\quad + \left(-\frac{\mathrm{e}^{3x}(18\cos(3x)\, x^2 + 18x\sin(3x) + 7\cos(3x) - 4\sin(3x))}{216}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 + \mathrm{e}^{(3-3i)x} c_2 + \mathrm{e}^{(3+3i)x} c_3 \\
&\quad - \frac{\mathrm{e}^{3x}(18\cos(3x)\, x^2 + 18x\sin(3x) + 7\cos(3x) - 4\sin(3x))}{216}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 + \mathrm{e}^{(3-3i)x} c_2 + \mathrm{e}^{(3+3i)x} c_3 - \frac{\mathrm{e}^{3x}(18\cos(3x)\, x^2 + 18x\sin(3x) + 7\cos(3x) - 4\sin(3x))}{216}
$$

Verified OK.

19.36.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 18y' = -\mathrm{e}^{3x}((-3x+2)\cos(3x) - (3+3x)\sin(3x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 3\cos(3x)\,\mathrm{e}^{3x} x + 3\sin(3x)\,\mathrm{e}^{3x} x - 2\cos(3x)\,\mathrm{e}^{3x} + 3\sin(3x)\,\mathrm{e}^{3x} + 6y'' - 18y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 6y'' + 18y' = \mathrm{e}^{3x}(3x\sin(3x) + 3x\cos(3x) + 3\sin(3x) - 2\cos(3x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

$y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 3\cos(3x)\,\mathrm{e}^{3x}x + 3\sin(3x)\,\mathrm{e}^{3x}x - 2\cos(3x)\,\mathrm{e}^{3x} + 3\sin(3x)\,\mathrm{e}^{3x} + 6y_3(x) - 18y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3\cos(3x)\,\mathrm{e}^{3x}x + 3\sin(3x)\,\mathrm{e}^{3x}x - 2\cos(3x)\,\mathrm{e}^{3x} + 3\sin(3x$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -18 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3\sin(3x)\,\mathrm{e}^{3x}x + 3\cos(3x)\,\mathrm{e}^{3x}x + 3\sin(3x)\,\mathrm{e}^{3x} - 2\cos(3x)\,\mathrm{e} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3\sin(3x)\,\mathrm{e}^{3x}x + 3\cos(3x)\,\mathrm{e}^{3x}x + 3\sin(3x)\,\mathrm{e}^{3x} - 2\cos(3x)\,\mathrm{e}^{3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -18 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3 - 3\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{18} \\ \frac{1}{6} + \frac{\mathrm{I}}{6} \\ 1 \end{bmatrix}\right], \left[3 + 3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{18} \\ \frac{1}{6} - \frac{\mathrm{I}}{6} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - 3\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{18} \\ \frac{1}{6} + \frac{\mathrm{I}}{6} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-3\,\mathrm{I})x} \cdot \begin{bmatrix} \frac{\mathrm{I}}{18} \\ \frac{1}{6} + \frac{\mathrm{I}}{6} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{3x} \cdot (\cos(3x) - \mathrm{I}\sin(3x)) \cdot \begin{bmatrix} \frac{\mathrm{I}}{18} \\ \frac{1}{6} + \frac{\mathrm{I}}{6} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{\mathrm{I}}{18}(\cos(3x) - \mathrm{I}\sin(3x)) \\ \left(\frac{1}{6} + \frac{\mathrm{I}}{6}\right)(\cos(3x) - \mathrm{I}\sin(3x)) \\ \cos(3x) - \mathrm{I}\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{3x} \cdot \begin{bmatrix} \frac{\sin(3x)}{18} \\ \frac{\cos(3x)}{6} + \frac{\sin(3x)}{6} \\ \cos(3x) \end{bmatrix}, \vec{y}_3(x) = e^{3x} \cdot \begin{bmatrix} \frac{\cos(3x)}{18} \\ -\frac{\sin(3x)}{6} + \frac{\cos(3x)}{6} \\ -\sin(3x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & \frac{\sin(3x)e^{3x}}{18} & \frac{\cos(3x)e^{3x}}{18} \\ 0 & e^{3x}\left(\frac{\cos(3x)}{6} + \frac{\sin(3x)}{6}\right) & e^{3x}\left(-\frac{\sin(3x)}{6} + \frac{\cos(3x)}{6}\right) \\ 0 & \cos(3x)e^{3x} & -\sin(3x)e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & \frac{\sin(3x)e^{3x}}{18} & \frac{\cos(3x)e^{3x}}{18} \\ 0 & e^{3x}\left(\frac{\cos(3x)}{6} + \frac{\sin(3x)}{6}\right) & e^{3x}\left(-\frac{\sin(3x)}{6} + \frac{\cos(3x)}{6}\right) \\ 0 & \cos(3x)e^{3x} & -\sin(3x)e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 & \frac{1}{18} \\ 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{1}{3} + \frac{\cos(3x)e^{3x}}{3} & \frac{1}{18} + \frac{(\sin(3x)-\cos(3x))e^{3x}}{18} \\ 0 & e^{3x}(-\sin(3x) + \cos(3x)) & \frac{\sin(3x)e^{3x}}{3} \\ 0 & -6\sin(3x)e^{3x} & e^{3x}(\sin(3x) + \cos(3x)) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{1}{27}+\frac{e^{3x}\left(-9\cos(3x)x^2-9x\sin(3x)-4\cos(3x)+4\sin(3x)\right)}{108} \\ -\frac{\left(\left(-x^2+x-\frac{5}{9}\right)\sin(3x)+\cos(3x)x\left(x+\frac{5}{3}\right)\right)e^{3x}}{4} \\ \frac{3\left(\left(x+\frac{1}{3}\right)^2\sin(3x)-\frac{5x\cos(3x)}{3}\right)e^{3x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \left[\begin{array}{c} \frac{1}{27}+\frac{e^{3x}\left(-9\cos(3x)x^2-9x\sin(3x)-4\cos(3x)+4\sin(3x)\right)}{108} \\ -\frac{\left(\left(-x^2+x-\frac{5}{9}\right)\sin(3x)+\cos(3x)x\left(x+\frac{5}{3}\right)\right)e^{3x}}{4} \\ \frac{3\left(\left(x+\frac{1}{3}\right)^2\sin(3x)-\frac{5x\cos(3x)}{3}\right)e^{3x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y = \frac{\left(\left(-9x^2+6c_3-4\right)\cos(3x)-9\left(x-\frac{2c_2}{3}-\frac{4}{9}\right)\sin(3x)\right)e^{3x}}{108} + c_1 + \frac{1}{27}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 3*exp(3*_a)*cos(3*_a)*_a+3*ex
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(1*diff(y(x),x$3)-6*diff(y(x),x$2)+18*diff(y(x),x)-0*y(x)=-exp(3*x)*((2-3*x)*cos(3*x)-
```

$$y(x) = \frac{\left(\left(-3x^2+6c_1-6c_2-1\right)\cos\left(3x\right)-3\sin\left(3x\right)\left(x-2c_1-2c_2+\frac{1}{9}\right)\right)e^{3x}}{36}+c_3$$

✓ Solution by Mathematica

Time used: 3.765 (sec). Leaf size: 57

```
DSolve[1*y'''[x]-6*y''[x]+18*y'[x]-0*y[x]==-Exp[3*x]*((2-3*x)*Cos[3*x]-(3+3*x)*Sin[3*x]),y[x
```

$$y(x) \to c_3 - \frac{1}{216}e^{3x}\left(6\left(3x^2+1+6c_1-6c_2\right)\cos(3x)+\left(18x+1-36c_1-36c_2\right)\sin(3x)\right)$$

19.37 problem section 9.3, problem 37

Internal problem ID [1534]
Internal file name [`OUTPUT/1535_Sunday_June_05_2022_02_21_07_AM_21645714/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 37.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y''' - 2y'' - 8y' - 8y = e^{x}(8\cos(x) + 16\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' - 2y'' - 8y' - 8y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 2\lambda^2 - 8\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + e^{(-1-i)x} c_2 + c_3 e^{2x} + e^{(-1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^{(-1-i)x} \\ y_3 &= e^{2x} \\ y_4 &= e^{(-1+i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' - 2y'' - 8y' - 8y = e^x (8\cos(x) + 16\sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x (8\cos(x) + 16\sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^x, \sin(x) e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{(-1-i)x}, e^{(-1+i)x}, e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^x + A_2 \sin(x) e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-24A_1 \cos(x) e^x - 24A_2 \sin(x) e^x + 8A_1 \sin(x) e^x - 8A_2 \cos(x) e^x \\ &= e^x (8\cos(x) + 16\sin(x)) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = -\frac{7}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, e^x}{10} - \frac{7\sin(x)\, e^x}{10}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2x} + e^{(-1-i)x} c_2 + c_3 e^{2x} + e^{(-1+i)x} c_4\right) + \left(-\frac{\cos(x)\, e^x}{10} - \frac{7\sin(x)\, e^x}{10}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + e^{(-1-i)x} c_2 + c_3 e^{2x} + e^{(-1+i)x} c_4 - \frac{\cos(x)\, e^x}{10} - \frac{7\sin(x)\, e^x}{10} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + e^{(-1-i)x} c_2 + c_3 e^{2x} + e^{(-1+i)x} c_4 - \frac{\cos(x)\, e^x}{10} - \frac{7\sin(x)\, e^x}{10}$$

Verified OK.

19.37.1 Maple step by step solution

Let's solve

$y'''' + 2y''' - 2y'' - 8y' - 8y = e^x(8\cos(x) + 16\sin(x))$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = 8y + 8\cos(x)\, e^x + 16\sin(x)\, e^x - 2y''' + 2y'' + 8y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 2y''' - 2y'' - 8y' - 8y = 8\, e^x(\cos(x) + 2\sin(x))$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 8\cos(x)\,e^x + 16\sin(x)\,e^x - 2y_4(x) + 2y_3(x) + 8y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = 8\cos(x)\,e^x + 16\sin(x)\,e^x - 2y_4(x) + 2y_3($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 8 & 2 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16\sin(x)\,e^x + 8\cos(x)\,e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16\sin(x)\,e^x + 8\cos(x)\,e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 8 & 2 & -2 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}, \begin{bmatrix} \frac{1}{4} + \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}, \begin{bmatrix} \frac{1}{4} - \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ -\frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{\mathrm{e}^{-2x}}{8} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) \\ \frac{\mathrm{e}^{-2x}}{4} & \frac{\mathrm{e}^{2x}}{4} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2} \\ -\frac{\mathrm{e}^{-2x}}{2} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ \mathrm{e}^{-2x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{\mathrm{e}^{-2x}}{8} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) \\ \frac{\mathrm{e}^{-2x}}{4} & \frac{\mathrm{e}^{2x}}{4} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2} \\ -\frac{\mathrm{e}^{-2x}}{2} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ \mathrm{e}^{-2x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(\mathrm{e}^{4x}+4\cos(x)\mathrm{e}^{x}+12\sin(x)\mathrm{e}^{x}+5)\mathrm{e}^{-2x}}{10} & -\frac{2\left(-\frac{3\,\mathrm{e}^{4x}}{8}-\frac{5}{8}+(-2\sin(x)+\cos(x))\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{5} & \frac{(-\cos(x)-3\sin(x))\mathrm{e}^{-x}}{10} \\ \frac{(\mathrm{e}^{4x}-5+(4\cos(x)-8\sin(x))\mathrm{e}^{x})\mathrm{e}^{-2x}}{5} & \frac{(3\,\mathrm{e}^{4x}-5+(12\cos(x)-4\sin(x))\mathrm{e}^{x})\mathrm{e}^{-2x}}{10} & \frac{(2\sin(x)-\cos(x))\mathrm{e}^{-x}}{5} + \\ \frac{2(5+\mathrm{e}^{4x}+2(\sin(x)-3\cos(x))\mathrm{e}^{x})\mathrm{e}^{-2x}}{5} & \frac{(3\,\mathrm{e}^{4x}+5+(-8\cos(x)-4\sin(x))\mathrm{e}^{x})\mathrm{e}^{-2x}}{5} & \frac{(3\cos(x)-\sin(x))\mathrm{e}^{-x}}{5} + \\ \frac{4(-5+\mathrm{e}^{4x}+2(2\cos(x)+\sin(x))\mathrm{e}^{x})\mathrm{e}^{-2x}}{5} & \frac{4\left(\frac{3\,\mathrm{e}^{4x}}{2}-\frac{5}{2}+(\cos(x)+3\sin(x))\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{5} & \frac{(-4\cos(x)-2\sin(x))\mathrm{e}^{-x}}{5} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{e^{-2x} \left(-(\cos(x)+7\sin(x))e^{3x}-3\cos(x)e^{x}+\sin(x)e^{x}+3\,e^{4x}+1\right)}{10} \\ \frac{e^{-2x} \left((-3\sin(x)-4\cos(x))e^{3x}+2\cos(x)e^{x}+\sin(x)e^{x}+3\,e^{4x}-1\right)}{5} \\ -\frac{\left((7\cos(x)-\sin(x))e^{3x}+\cos(x)e^{x}+3\sin(x)e^{x}-6\,e^{4x}-2\right)e^{-2x}}{5} \\ -\frac{2\left((3\cos(x)-4\sin(x))e^{3x}+\cos(x)e^{x}-2\sin(x)e^{x}-6\,e^{4x}+2\right)e^{-2x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{e^{-2x}\left(-(\cos(x)+7\sin(x))e^{3x}-3\cos(x)e^{x}+\sin(x)e^{x}+3e^{4x}+1\right)}{10} \\ \frac{e^{-2x}\left((-3\sin(x)-4\cos(x))e^{3x}+2\cos(x)e^{x}+\sin(x)e^{x}+3e^{4x}-1\right)}{5} \\ -\frac{\left((7\cos(x)-\sin(x))e^{3x}+\cos(x)e^{x}+3\sin(x)e^{x}-6e^{4x}-2\right)e^{-2x}}{5} \\ -\frac{2\left((3\cos(x)-4\sin(x))e^{3x}+\cos(x)e^{x}-2\sin(x)e^{x}-6e^{4x}+2\right)e^{-2x}}{5} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left((\cos(x)+7\sin(x))e^{3x}+\left(-\frac{5c_2}{4}-3\right)e^{4x}+\left(\left(-\frac{5c_3}{2}-\frac{5c_4}{2}+3\right)\cos(x)-\frac{5\left(c_3-c_4+\frac{2}{5}\right)\sin(x)}{2}\right)e^{x}+\frac{5c_1}{4}-1\right)e^{-2x}}{10}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve(1*diff(y(x),x$4)+2*diff(y(x),x$3)-2*diff(y(x),x$2)-8*diff(y(x),x)-8*y(x)=exp(x)*(8*co
```

$$y(x) = -\frac{\left((7\sin(x)+\cos(x))e^{3x}-10c_3\cos(x)e^{x}-10c_4\sin(x)e^{x}-10c_2e^{4x}-10c_1\right)e^{-2x}}{10}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 56

```
DSolve[1*y''''[x]+2*y'''[x]-2*y''[x]-8*y'[x]-8*y[x]==Exp[x]*(8*Cos[x]+16*Sin[x]),y[x],x,Incl
```

$$y(x) \to c_3e^{-2x} + c_4e^{2x} - \frac{1}{10}e^{x}(7\sin(x)+\cos(x)) + c_2e^{-x}\cos(x) + c_1e^{-x}\sin(x)$$

19.38 problem section 9.3, problem 38

Internal problem ID [1535]
Internal file name [`OUTPUT/1536_Sunday_June_05_2022_02_21_09_AM_72758574/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 38.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 3y''' + 2y'' + 2y' - 4y = e^{x}(-\sin(2x) + 2\cos(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 2y'' + 2y' - 4y = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 2\lambda^2 + 2\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$
$$\lambda_3 = 1 - i$$
$$\lambda_4 = 1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + e^{(1-i)x} c_3 + e^{(1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{(1-i)x}$$
$$y_4 = e^{(1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 2y'' + 2y' - 4y = e^{x}(-\sin(2x) + 2\cos(2x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}(-\sin(2x) + 2\cos(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\cos(2x), e^{x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{(1-i)x}, e^{(1+i)x}, e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}\cos(2x) + A_2 e^{x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$18A_1 e^{x}\cos(2x) + 6A_1 e^{x}\sin(2x) + 18A_2 e^{x}\sin(2x) - 6A_2 e^{x}\cos(2x) \\ = e^{x}(-\sin(2x) + 2\cos(2x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}, A_2 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x \cos(2x)}{12} - \frac{e^x \sin(2x)}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{2x} + e^{(1-i)x} c_3 + e^{(1+i)x} c_4\right) + \left(\frac{e^x \cos(2x)}{12} - \frac{e^x \sin(2x)}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 + \frac{e^x \cos(2x)}{12} - \frac{e^x \sin(2x)}{12} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{2x} + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 + \frac{e^x \cos(2x)}{12} - \frac{e^x \sin(2x)}{12}$$

Verified OK.

19.38.1 Maple step by step solution

Let's solve

$y'''' - 3y''' + 2y'' + 2y' - 4y = e^x(-\sin(2x) + 2\cos(2x))$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 2\,\mathrm{e}^x \cos(2x) - \mathrm{e}^x \sin(2x) + 3y_4(x) - 2y_3(x) - 2y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2\,\mathrm{e}^x \cos(2x) - \mathrm{e}^x \sin(2x) + 3y_4(x) - 2y_3($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & -2 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\,\mathrm{e}^x \cos(2x) - \mathrm{e}^x \sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\,\mathrm{e}^x \cos(2x) - \mathrm{e}^x \sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & -2 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} -\frac{1}{4} + \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} -\frac{1}{4} - \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_4(x) = e^{x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4}+\frac{\cos(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & \mathrm{e}^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) \\ \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & \frac{\sin(x)\mathrm{e}^{x}}{2} & \frac{\cos(x)\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)\mathrm{e}^{x} \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{x} & -\sin(x)\,\mathrm{e}^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{8} & \mathrm{e}^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & \mathrm{e}^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) \\ \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & \frac{\sin(x)\mathrm{e}^{x}}{2} & \frac{\cos(x)\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \mathrm{e}^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)\mathrm{e}^{x} \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \cos(x)\,\mathrm{e}^{x} & -\sin(x)\,\mathrm{e}^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{4} & 0 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{4\,\mathrm{e}^{-x}}{15}+\frac{\mathrm{e}^{2x}}{3}+\frac{2(-2\sin(x)+\cos(x))\mathrm{e}^{x}}{5} & -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{(2\cos(x)+\sin(x))\mathrm{e}^{x}}{5} & \frac{4\,\mathrm{e}^{-x}}{15}-\frac{\mathrm{e}^{2x}}{6}+(-\cos(x)\cdots \\ -\frac{4\,\mathrm{e}^{-x}}{15}+\frac{2\,\mathrm{e}^{2x}}{3}+\frac{2(-\cos(x)-3\sin(x))\mathrm{e}^{x}}{5} & \frac{2\,\mathrm{e}^{-x}}{5}+\frac{(3\cos(x)-\sin(x))\mathrm{e}^{x}}{5} & -\frac{4\,\mathrm{e}^{-x}}{15}-\frac{\mathrm{e}^{2x}}{3}+(3\cos(x\cdots \\ \frac{4\,\mathrm{e}^{-x}}{15}+\frac{4\,\mathrm{e}^{2x}}{3}+\frac{4(-2\cos(x)-\sin(x))\mathrm{e}^{x}}{5} & -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{(-4\sin(x)+2\cos(x))\mathrm{e}^{x}}{5} & \frac{4\,\mathrm{e}^{-x}}{15}-\frac{2\,\mathrm{e}^{2x}}{3}+(7\cos(x\cdots \\ -\frac{4\,\mathrm{e}^{-x}}{15}+\frac{8\,\mathrm{e}^{2x}}{3}+\frac{4(\sin(x)-3\cos(x))\mathrm{e}^{x}}{5} & \frac{2\,\mathrm{e}^{-x}}{5}+\frac{(-2\cos(x)-6\sin(x))\mathrm{e}^{x}}{5} & -\frac{4\,\mathrm{e}^{-x}}{15}-\frac{4\,\mathrm{e}^{2x}}{3}+2(4\cos(\cdots \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{-x}}{20} + \frac{\left(10\cos(x)^2+(-10\sin(x)-8)\cos(x)+16\sin(x)-5\right)e^x}{60} \\ -\frac{e^{-x}}{20} + \frac{\left(-10\cos(x)^2+(-30\sin(x)+8)\cos(x)+24\sin(x)+5\right)e^x}{60} \\ \frac{e^{-x}}{20} + \frac{\left(-70\cos(x)^2+(-10\sin(x)+32)\cos(x)+16\sin(x)+35\right)e^x}{60} \\ -\frac{e^{-x}}{20} + \frac{\left(-90\cos(x)^2+(130\sin(x)+48)\cos(x)-16\sin(x)+45\right)e^x}{60} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{e^{-x}}{20} + \frac{\left(10\cos(x)^2+(-10\sin(x)-8)\cos(x)+16\sin(x)-5\right)e^x}{60} \\ -\frac{e^{-x}}{20} + \frac{\left(-10\cos(x)^2+(-30\sin(x)+8)\cos(x)+24\sin(x)+5\right)e^x}{60} \\ \frac{e^{-x}}{20} + \frac{\left(-70\cos(x)^2+(-10\sin(x)+32)\cos(x)+16\sin(x)+35\right)e^x}{60} \\ -\frac{e^{-x}}{20} + \frac{\left(-90\cos(x)^2+(130\sin(x)+48)\cos(x)-16\sin(x)+45\right)e^x}{60} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(1-20c_1)e^{-x}}{20} + \frac{c_2 e^{2x}}{8} - \frac{\left(-\frac{2\cos(x)^2}{3}+\left(c_3-c_4+\frac{2\sin(x)}{3}+\frac{8}{15}\right)\cos(x)+\frac{1}{3}+\left(-c_3-c_4-\frac{16}{15}\right)\sin(x)\right)e^x}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(1*diff(y(x),x$4)-3*diff(y(x),x$3)+2*diff(y(x),x$2)+2*diff(y(x),x)-4*y(x)=exp(x)*(2*co
```

$$y(x) = \mathrm{e}^{-x} c_1 + c_2 \mathrm{e}^{2x} + \mathrm{e}^{x} \left(\frac{\cos(x)^2}{6} + \left(c_3 - \frac{\sin(x)}{6}\right)\cos(x) + c_4 \sin(x) - \frac{1}{12}\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 56

```
DSolve[1*y''''[x]-3*y'''[x]+2*y''[x]+2*y'[x]-4*y[x]==Exp[x]*(2*Cos[2*x]-Sin[2*x]),y[x],x,Inc
```

$$y(x) \to c_3 e^{-x} + c_4 e^{2x} + \frac{1}{12} e^{x} (\cos(2x) - \sin(2x)) + c_2 e^{x} \cos(x) + c_1 e^{x} \sin(x)$$

19.39 problem section 9.3, problem 39

Internal problem ID [1536]
Internal file name [`OUTPUT/1537_Sunday_June_05_2022_02_21_12_AM_48519893/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 39.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 8y''' + 24y'' - 32y' + 15y = e^{2x}(15x\cos(2x) + 32\sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 8y''' + 24y'' - 32y' + 15y = 0$$

The characteristic equation is

$$\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 15 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2 + i$$
$$\lambda_4 = 2 - i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{3x} + e^{(2+i)x} c_3 + e^{(2-i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{3x}$$
$$y_3 = e^{(2+i)x}$$
$$y_4 = e^{(2-i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 8y''' + 24y'' - 32y' + 15y = e^{2x}(15x \cos(2x) + 32 \sin(2x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(15x \cos(2x) + 32 \sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x) e^{2x}, \sin(2x) e^{2x}, \cos(2x) e^{2x} x, \sin(2x) e^{2x} x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{(2-i)x}, e^{(2+i)x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) e^{2x} + A_2 \sin(2x) e^{2x} + A_3 \cos(2x) e^{2x} x + A_4 \sin(2x) e^{2x} x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&15A_4 \sin(2x) e^{2x} x + 15A_3 \cos(2x) e^{2x} x - 32A_4 \cos(2x) e^{2x} + 32A_3 \sin(2x) e^{2x} \\ &+ 15A_1 \cos(2x) e^{2x} + 15A_2 \sin(2x) e^{2x} = e^{2x}(15x \cos(2x) + 32 \sin(2x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(2x)\,e^{2x}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{3x} + e^{(2+i)x} c_3 + e^{(2-i)x} c_4\right) + \left(\cos(2x)\,e^{2x}x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{3x} + e^{(2+i)x} c_3 + e^{(2-i)x} c_4 + \cos(2x)\,e^{2x}x \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{3x} + e^{(2+i)x} c_3 + e^{(2-i)x} c_4 + \cos(2x)\,e^{2x}x$$

Verified OK.

19.39.1 Maple step by step solution

Let's solve

$y'''' - 8y''' + 24y'' - 32y' + 15y = e^{2x}(15x\cos(2x) + 32\sin(2x))$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 15\cos(2x)\,\mathrm{e}^{2x}x + 32\sin(2x)\,\mathrm{e}^{2x} + 8y_4(x) - 24y_3(x) + 32y_2(x) - 15y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 15\cos(2x)\,\mathrm{e}^{2x}x + 32\sin(2x)\,\mathrm{e}^{2x} + 8y_4(x)$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 32 & -24 & 8 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 15\cos(2x)\,\mathrm{e}^{2x}x + 32\sin(2x)\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 15\cos(2x)\,\mathrm{e}^{2x}x + 32\sin(2x)\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 32 & -24 & 8 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right], \left[3, \left[\begin{array}{c}\frac{1}{27}\\\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]\right], \left[2-\mathrm{I}, \left[\begin{array}{c}\frac{2}{125}+\frac{11\,\mathrm{I}}{125}\\\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\\\frac{2}{5}+\frac{\mathrm{I}}{5}\\1\end{array}\right]\right], \left[2+\mathrm{I}, \left[\begin{array}{c}\frac{2}{125}-\frac{11\,\mathrm{I}}{125}\\\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\\frac{2}{5}-\frac{\mathrm{I}}{5}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{27}\\\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c}\frac{1}{27}\\\frac{1}{9}\\\frac{1}{3}\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-\mathrm{I}, \left[\begin{array}{c} \frac{2}{125}+\frac{11\,\mathrm{I}}{125} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{2}{125}+\frac{11\,\mathrm{I}}{125} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2x} \cdot \left(\cos\left(x\right)-\mathrm{I}\sin\left(x\right)\right) \cdot \left[\begin{array}{c} \frac{2}{125}+\frac{11\,\mathrm{I}}{125} \\ \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{2x} \cdot \left[\begin{array}{c} \left(\frac{2}{125}+\frac{11\,\mathrm{I}}{125}\right)\left(\cos\left(x\right)-\mathrm{I}\sin\left(x\right)\right) \\ \left(\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\right)\left(\cos\left(x\right)-\mathrm{I}\sin\left(x\right)\right) \\ \left(\frac{2}{5}+\frac{\mathrm{I}}{5}\right)\left(\cos\left(x\right)-\mathrm{I}\sin\left(x\right)\right) \\ \cos\left(x\right)-\mathrm{I}\sin\left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{2\cos(x)}{125}+\frac{11\sin(x)}{125} \\ \frac{3\cos(x)}{25}+\frac{4\sin(x)}{25} \\ \frac{\sin(x)}{5}+\frac{2\cos(x)}{5} \\ \cos\left(x\right) \end{array}\right], \vec{y}_4(x)=\mathrm{e}^{2x} \cdot \left[\begin{array}{c} -\frac{2\sin(x)}{125}+\frac{11\cos(x)}{125} \\ -\frac{3\sin(x)}{25}+\frac{4\cos(x)}{25} \\ -\frac{2\sin(x)}{5}+\frac{\cos(x)}{5} \\ -\sin\left(x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & \frac{e^{3x}}{27} & e^{2x}\left(\frac{2\cos(x)}{125}+\frac{11\sin(x)}{125}\right) & e^{2x}\left(-\frac{2\sin(x)}{125}+\frac{11\cos(x)}{125}\right) \\ e^{x} & \frac{e^{3x}}{9} & e^{2x}\left(\frac{3\cos(x)}{25}+\frac{4\sin(x)}{25}\right) & e^{2x}\left(-\frac{3\sin(x)}{25}+\frac{4\cos(x)}{25}\right) \\ e^{x} & \frac{e^{3x}}{3} & e^{2x}\left(\frac{\sin(x)}{5}+\frac{2\cos(x)}{5}\right) & e^{2x}\left(-\frac{2\sin(x)}{5}+\frac{\cos(x)}{5}\right) \\ e^{x} & e^{3x} & e^{2x}\cos(x) & -e^{2x}\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & \frac{e^{3x}}{27} & e^{2x}\left(\frac{2\cos(x)}{125}+\frac{11\sin(x)}{125}\right) & e^{2x}\left(-\frac{2\sin(x)}{125}+\frac{11\cos(x)}{125}\right) \\ e^{x} & \frac{e^{3x}}{9} & e^{2x}\left(\frac{3\cos(x)}{25}+\frac{4\sin(x)}{25}\right) & e^{2x}\left(-\frac{3\sin(x)}{25}+\frac{4\cos(x)}{25}\right) \\ e^{x} & \frac{e^{3x}}{3} & e^{2x}\left(\frac{\sin(x)}{5}+\frac{2\cos(x)}{5}\right) & e^{2x}\left(-\frac{2\sin(x)}{5}+\frac{\cos(x)}{5}\right) \\ e^{x} & e^{3x} & e^{2x}\cos(x) & -e^{2x}\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{27} & \frac{2}{125} & \frac{11}{125} \\ 1 & \frac{1}{9} & \frac{3}{25} & \frac{4}{25} \\ 1 & \frac{1}{3} & \frac{2}{5} & \frac{1}{5} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{3(2\sin(x)-\cos(x))e^{2x}}{2}+\frac{15\,e^{x}}{4}-\frac{5\,e^{3x}}{4} & \frac{(8\cos(x)-22\sin(x))e^{2x}}{4}-\frac{17\,e^{x}}{4}+\frac{9\,e^{3x}}{4} & \frac{(-\cos(x)+6\,s}{2} \\ \frac{15\,e^{2x}\sin(x)}{2}+\frac{15\,e^{x}}{4}-\frac{15\,e^{3x}}{4} & \frac{(-6\cos(x)-52\sin(x))e^{2x}}{4}-\frac{17\,e^{x}}{4}+\frac{27\,e^{3x}}{4} & \frac{(8\cos(x)+26\,s}{4} \\ \frac{15(\cos(x)+2\sin(x))e^{2x}}{2}+\frac{15\,e^{x}}{4}-\frac{45\,e^{3x}}{4} & \frac{(-64\cos(x)-98\sin(x))e^{2x}}{4}-\frac{17\,e^{x}}{4}+\frac{81\,e^{3x}}{4} & \frac{(42\cos(x)+44}{4} \\ \frac{15(4\cos(x)+3\sin(x))e^{2x}}{2}+\frac{15\,e^{x}}{4}-\frac{135\,e^{3x}}{4} & \frac{(-113\cos(x)-66\sin(x))e^{2x}}{2}-\frac{17\,e^{x}}{4}+\frac{243\,e^{3x}}{4} & \frac{(64\cos(x)+23\,s}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{\mathrm{e}^{x}\left(8\,\mathrm{e}^{x}\cos(x)^{2}x-26\sin(x)\mathrm{e}^{x}-4x\,\mathrm{e}^{x}+11\,\mathrm{e}^{2x}-11\right)}{4}\\ 4\left(\frac{33\,\mathrm{e}^{2x}}{16}-\frac{11}{16}+\left(\left(x+\frac{1}{2}\right)\cos(x)^{2}+\left(-\sin(x)\,x-\frac{13}{8}\right)\cos(x)-\frac{x}{2}-\frac{13\sin(x)}{4}-\frac{1}{4}\right)\mathrm{e}^{x}\right)\\ -16\left(-\frac{99\,\mathrm{e}^{2x}}{64}+\frac{11}{64}+\left(-\frac{\cos(x)^{2}}{2}+\left(\frac{13}{8}+\left(x+\frac{1}{2}\right)\sin(x)\right)\cos(x)+\frac{39\sin(x)}{32}+\frac{1}{4}\right)\mathrm{e}^{x}\right)\mathrm{e}^{x}\\ -32\left(-\frac{297\,\mathrm{e}^{2x}}{128}+\frac{11}{128}+\left(\cos(x)^{2}x+\left(\frac{143}{64}+\left(x+\frac{3}{2}\right)\sin(x)\right)\cos(x)-\frac{x}{2}+\frac{13\sin(x)}{32}\right)\mathrm{e}^{x}\right)\end{array}\right.$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3(x)+c_4\overrightarrow{y}_4(x)+\left[\begin{array}{c}\frac{\mathrm{e}^{x}\left(8\,\mathrm{e}^{x}\cos(x)^{2}x-26\sin(x)\mathrm{e}^{x}-4\right.}{4}\\ 4\left(\frac{33\,\mathrm{e}^{2x}}{16}-\frac{11}{16}+\left(\left(x+\frac{1}{2}\right)\cos(x)^{2}+\left(-\sin(x)\,x-\right.\right.\right.\\ -16\left(-\frac{99\,\mathrm{e}^{2x}}{64}+\frac{11}{64}+\left(-\frac{\cos(x)^{2}}{2}+\left(\frac{13}{8}+\left(x+\frac{1}{2}\right)\mathrm{s}\right.\right.\right.\\ -32\left(-\frac{297\,\mathrm{e}^{2x}}{128}+\frac{11}{128}+\left(\cos(x)^{2}x+\left(\frac{143}{64}+\left(x+\frac{3}{2}\right.\right.\right.\right.\end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y=\frac{\left(27000\cos(x)^{2}x+(216c_3+1188c_4)\cos(x)+(1188c_3-216c_4-87750)\sin(x)-13500x\right)\mathrm{e}^{2x}}{13500}+\frac{(500c_2+37125)\mathrm{e}^{3x}}{13500}+\left(c_1-\frac{11}{4}\right.$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(1*diff(y(x),x$4)-8*diff(y(x),x$3)+24*diff(y(x),x$2)-32*diff(y(x),x)+15*y(x)=exp(2*x)*
```

$$y(x) = e^{x}\left(2\,e^{x}x\cos\left(x\right)^{2}+c_{4}\sin\left(x\right)e^{x}+c_{3}\cos\left(x\right)e^{x}-x\,e^{x}+c_{2}e^{2x}+c_{1}\right)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 45

```
DSolve[1*y''''[x]-8*y'''[x]+24*y''[x]-32*y'[x]+15*y[x]==Exp[2*x]*(15*x*Cos[2*x]+32*Sin[2*x])
```

$$y(x)\to e^{x}\left(e^{x}x\cos(2x)+c_{4}e^{2x}+c_{2}e^{x}\cos(x)+c_{1}e^{x}\sin(x)+c_{3}\right)$$

19.40 problem section 9.3, problem 40

Internal problem ID [1537]
Internal file name [`OUTPUT/1538_Sunday_June_05_2022_02_21_14_AM_13504486/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 40.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 6y''' + 13y'' + 12y' + 4y = e^{-x}((-x + 4)\cos(x) - (x + 5)\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 6y''' + 13y'' + 12y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 6\lambda^3 + 13\lambda^2 + 12\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{-2x} c_3 + x\,\mathrm{e}^{-2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= \mathrm{e}^{-2x} \\ y_4 &= \mathrm{e}^{-2x} x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 6y''' + 13y'' + 12y' + 4y = \mathrm{e}^{-x}((-x+4)\cos(x) - (x+5)\sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-x}((-x+4)\cos(x) - (x+5)\sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,\mathrm{e}^{-x}, \sin(x)\,\mathrm{e}^{-x}, x\cos(x)\,\mathrm{e}^{-x}, \sin(x)\,x\,\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-2x} x, \mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\,\mathrm{e}^{-x} + A_2 \sin(x)\,\mathrm{e}^{-x} + A_3 x \cos(x)\,\mathrm{e}^{-x} + A_4 \sin(x)\,x\,\mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-2A_4 \cos(x)\,\mathrm{e}^{-x} + 2A_3 x \sin(x)\,\mathrm{e}^{-x} - 2A_4 \cos(x)\,x\,\mathrm{e}^{-x} \\ &- 6A_4 \sin(x)\,\mathrm{e}^{-x} - 6A_3 \cos(x)\,\mathrm{e}^{-x} - 2A_2 \cos(x)\,\mathrm{e}^{-x} + 2A_3 \sin(x)\,\mathrm{e}^{-x} \\ &+ 2A_1 \sin(x)\,\mathrm{e}^{-x} = \mathrm{e}^{-x}((-x+4)\cos(x) - (x+5)\sin(x)) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -1, A_3 = -\frac{1}{2}, A_4 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \sin(x)\,\mathrm{e}^{-x} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} + \frac{\sin(x)\,x\,\mathrm{e}^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{-2x} c_3 + x\,\mathrm{e}^{-2x} c_4\right) \\ &\quad + \left(-\frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \sin(x)\,\mathrm{e}^{-x} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} + \frac{\sin(x)\,x\,\mathrm{e}^{-x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + (c_2 x + c_1)\,\mathrm{e}^{-x} - \frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \sin(x)\,\mathrm{e}^{-x} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} + \frac{\sin(x)\,x\,\mathrm{e}^{-x}}{2}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + (c_2 x + c_1)\,\mathrm{e}^{-x} - \frac{\cos(x)\,\mathrm{e}^{-x}}{2} \\ - \sin(x)\,\mathrm{e}^{-x} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} + \frac{\sin(x)\,x\,\mathrm{e}^{-x}}{2} \end{aligned} \tag{1}$$

Verification of solutions

$$y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + (c_2 x + c_1)\,\mathrm{e}^{-x} - \frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \sin(x)\,\mathrm{e}^{-x} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} + \frac{\sin(x)\,x\,\mathrm{e}^{-x}}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(1*diff(y(x),x$4)+6*diff(y(x),x$3)+13*diff(y(x),x$2)+12*diff(y(x),x)+4*y(x)=exp(-1*x)*
```

$$y(x) = \frac{((-x-1)\cos(x) + \sin(x)(-2+x) + 2c_3x + 2c_2)\,\mathrm{e}^{-x}}{2} + \mathrm{e}^{-2x}(c_4x + c_1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 56

```
DSolve[1*y''''[x]+6*y'''[x]+13*y''[x]+12*y'[x]+4*y[x]==Exp[-1*x]*((4-x)*Cos[x]-(5+x)*Sin[x])
```

$$y(x) \to \frac{1}{2}e^{-2x}(e^x(x-2)\sin(x) - e^x(x+1)\cos(x) + 2(c_2x + c_3e^x + c_4e^xx + c_1))$$

19.41 problem section 9.3, problem 41

Internal problem ID [1538]
Internal file name [`OUTPUT/1539_Sunday_June_05_2022_02_21_17_AM_67691959/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 41.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos(x) - \sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y''' + 2y'' - 2y' - 4y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 + 2\lambda^2 - 2\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{(-1-i)x} c_3 + \mathrm{e}^{(-1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$

$$y_2 = \mathrm{e}^{x}$$

$$y_3 = \mathrm{e}^{(-1-i)x}$$

$$y_4 = \mathrm{e}^{(-1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y''' + 2y'' - 2y' - 4y = -\mathrm{e}^{-x}(\cos(x) - \sin(x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} & \mathrm{e}^{(-1-i)x} & \mathrm{e}^{(-1+i)x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & (-1-i)\,\mathrm{e}^{(-1-i)x} & (-1+i)\,\mathrm{e}^{(-1+i)x} \\ 4\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & 2i\mathrm{e}^{(-1-i)x} & -2i\mathrm{e}^{(-1+i)x} \\ -8\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & (2-2i)\,\mathrm{e}^{(-1-i)x} & (2+2i)\,\mathrm{e}^{(-1+i)x} \end{bmatrix}$$

$$|W| = 60i\mathrm{e}^{-2x}\mathrm{e}^{x}\mathrm{e}^{(-1-i)x}\mathrm{e}^{(-1+i)x}$$

The determinant simplifies to

$$|W| = 60i\mathrm{e}^{-3x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{(-1-i)x} & \mathrm{e}^{(-1+i)x} \\ \mathrm{e}^{x} & (-1-i)\,\mathrm{e}^{(-1-i)x} & (-1+i)\,\mathrm{e}^{(-1+i)x} \\ \mathrm{e}^{x} & 2i\mathrm{e}^{(-1-i)x} & -2i\mathrm{e}^{(-1+i)x} \end{bmatrix}$$
$$= 10i\mathrm{e}^{-x}$$

$$W_2(x) = \det\begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{(-1-i)x} & \mathrm{e}^{(-1+i)x} \\ -2\,\mathrm{e}^{-2x} & (-1-i)\,\mathrm{e}^{(-1-i)x} & (-1+i)\,\mathrm{e}^{(-1+i)x} \\ 4\,\mathrm{e}^{-2x} & 2i\mathrm{e}^{(-1-i)x} & -2i\mathrm{e}^{(-1+i)x} \end{bmatrix}$$
$$= 4i\mathrm{e}^{-4x}$$

$$W_3(x) = \det\begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} & \mathrm{e}^{(-1+i)x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & (-1+i)\,\mathrm{e}^{(-1+i)x} \\ 4\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & -2i\mathrm{e}^{(-1+i)x} \end{bmatrix}$$
$$= (-9-3i)\,\mathrm{e}^{(-2+i)x}$$

$$W_4(x) = \det\begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} & \mathrm{e}^{(-1-i)x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & (-1-i)\,\mathrm{e}^{(-1-i)x} \\ 4\,\mathrm{e}^{-2x} & \mathrm{e}^{x} & 2i\mathrm{e}^{(-1-i)x} \end{bmatrix}$$
$$= (-9+3i)\,\mathrm{e}^{(-2-i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^{3}\int \frac{\left(-\mathrm{e}^{-x}(\cos(x)-\sin(x))\right)\left(10i\mathrm{e}^{-x}\right)}{(1)\left(60i\mathrm{e}^{-3x}\right)}\,dx \\ &= -\int \frac{-10i\mathrm{e}^{-2x}(\cos(x)-\sin(x))}{60i\mathrm{e}^{-3x}}\,dx \\ &= -\int \left(\frac{(-\cos(x)+\sin(x))\,\mathrm{e}^{x}}{6}\right)dx \\ &= \frac{\cos(x)\,\mathrm{e}^{x}}{6} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(-\mathrm{e}^{-x}(\cos(x)-\sin(x)))\,(4i\mathrm{e}^{-4x})}{(1)\,(60i\mathrm{e}^{-3x})}\,dx \\
&= \int \frac{-4i\mathrm{e}^{-x}(\cos(x)-\sin(x))\,\mathrm{e}^{-4x}}{60i\mathrm{e}^{-3x}}\,dx \\
&= \int \left(\frac{(-\cos(x)+\sin(x))\,\mathrm{e}^{-2x}}{15}\right)dx \\
&= \frac{\mathrm{e}^{-2x}\cos(x)}{75} - \frac{\mathrm{e}^{-2x}\sin(x)}{25} \\
&= \frac{\mathrm{e}^{-2x}\cos(x)}{75} - \frac{\mathrm{e}^{-2x}\sin(x)}{25}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(-\mathrm{e}^{-x}(\cos(x)-\sin(x)))\,((-9-3i)\,\mathrm{e}^{(-2+i)x})}{(1)\,(60i\mathrm{e}^{-3x})}\,dx \\
&= -\int \frac{(9+3i)\,\mathrm{e}^{-x}(\cos(x)-\sin(x))\,\mathrm{e}^{(-2+i)x}}{60i\mathrm{e}^{-3x}}\,dx \\
&= -\int \left(\left(-\frac{1}{20}+\frac{3i}{20}\right)(-\cos(x)+\sin(x))\,\mathrm{e}^{ix}\right)dx \\
&= \frac{x}{20} + \frac{ix}{10} + \frac{\mathrm{e}^{2ix}}{40} + \frac{i\mathrm{e}^{2ix}}{20} \\
&= \frac{x}{20} + \frac{ix}{10} + \frac{\mathrm{e}^{2ix}}{40} + \frac{i\mathrm{e}^{2ix}}{20}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(-\mathrm{e}^{-x}(\cos(x)-\sin(x)))\,((-9+3i)\,\mathrm{e}^{(-2-i)x})}{(1)\,(60i\mathrm{e}^{-3x})}\,dx \\
&= \int \frac{(9-3i)\,\mathrm{e}^{-x}(\cos(x)-\sin(x))\,\mathrm{e}^{(-2-i)x}}{60i\mathrm{e}^{-3x}}\,dx \\
&= \int \left(\left(\frac{1}{20}+\frac{3i}{20}\right)(-\cos(x)+\sin(x))\,\mathrm{e}^{-ix}\right)dx \\
&= \int \left(\frac{1}{20}+\frac{3i}{20}\right)(-\cos(x)+\sin(x))\,\mathrm{e}^{-ix}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$
\begin{aligned}
y_p &= \left(\frac{\cos(x)\,\mathrm{e}^{x}}{6}\right)\left(\mathrm{e}^{-2x}\right) \\
&+ \left(\frac{\mathrm{e}^{-2x}\cos(x)}{75} - \frac{\mathrm{e}^{-2x}\sin(x)}{25}\right)\left(\mathrm{e}^{x}\right) \\
&+ \left(\frac{x}{20} + \frac{ix}{10} + \frac{\mathrm{e}^{2ix}}{40} + \frac{i\mathrm{e}^{2ix}}{20}\right)\left(\mathrm{e}^{(-1-i)x}\right) \\
&+ \left(\int \left(\frac{1}{20} + \frac{3i}{20}\right)\left(-\cos(x) + \sin(x)\right)\mathrm{e}^{-ix}dx\right)\left(\mathrm{e}^{(-1+i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\left(\frac{41}{10} + i + 2x\right)\cos(x) - \frac{\left(\frac{38}{5} + i - 8x\right)\sin(x)}{2}\right)\mathrm{e}^{-x}}{20}
$$

Which simplifies to

$$
y_p = \frac{\left(\left(\frac{41}{10} + i + 2x\right)\cos(x) - \frac{\left(\frac{38}{5} + i - 8x\right)\sin(x)}{2}\right)\mathrm{e}^{-x}}{20}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_3 + \mathrm{e}^{(-1+i)x}c_4\right) \\
&\quad + \left(\frac{\left(\left(\frac{41}{10} + i + 2x\right)\cos(x) - \frac{\left(\frac{38}{5} + i - 8x\right)\sin(x)}{2}\right)\mathrm{e}^{-x}}{20}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_3 + \mathrm{e}^{(-1+i)x}c_4 \\
&\quad + \frac{\left(\left(\frac{41}{10} + i + 2x\right)\cos(x) - \frac{\left(\frac{38}{5} + i - 8x\right)\sin(x)}{2}\right)\mathrm{e}^{-x}}{20}
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_3 + \mathrm{e}^{(-1+i)x}c_4 + \frac{\left(\left(\frac{41}{10} + i + 2x\right)\cos(x) - \frac{\left(\frac{38}{5} + i - 8x\right)\sin(x)}{2}\right)\mathrm{e}^{-x}}{20}
$$

Verified OK.

19.41.1 Maple step by step solution

Let's solve

$y'''' + 3y''' + 2y'' - 2y' - 4y = -e^{-x}(\cos(x) - \sin(x))$

- Highest derivative means the order of the ODE is 4

y''''

- Isolate 4th derivative

$y'''' = 4y + \sin(x)\, e^{-x} - \cos(x)\, e^{-x} - 3y''' - 2y'' + 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'''' + 3y''' + 2y'' - 2y' - 4y = e^{-x}(-\cos(x) + \sin(x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

  $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \sin(x)\, e^{-x} - \cos(x)\, e^{-x} - 3y_4(x) - 2y_3(x) + 2y_2(x) + 4y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \sin(x)\, e^{-x} - \cos(x)\, e^{-x} - 3y_4(x) - 2y_3(x$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 2 & -2 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\,\mathrm{e}^{-x} - \cos(x)\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\,\mathrm{e}^{-x} - \cos(x)\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 2 & -2 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}, \begin{bmatrix} \frac{1}{4} + \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-1-\mathrm{I})x} \cdot \begin{bmatrix} \frac{1}{4} + \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{1}{4} + \frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} \left(\frac{1}{4} + \frac{I}{4}\right)(\cos(x) - I\sin(x)) \\ -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2} + \frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-x} \cdot \begin{bmatrix} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & e^x & e^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & e^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) \\ \frac{e^{-2x}}{4} & e^x & -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} \\ -\frac{e^{-2x}}{2} & e^x & e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ e^{-2x} & e^x & \cos(x)e^{-x} & -\sin(x)e^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & e^{x} & e^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & e^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) \\ \frac{e^{-2x}}{4} & e^{x} & -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} \\ -\frac{e^{-2x}}{2} & e^{x} & e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) \\ e^{-2x} & e^{x} & \cos(x)\,e^{-x} & -\sin(x)\,e^{-x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2\left(\frac{2e^{3x}}{3}+\frac{5}{6}+e^{x}(\cos(x)+2\sin(x))\right)e^{-2x}}{5} & \frac{(-2\cos(x)+\sin(x))e^{-x}}{5} + \frac{2e^{x}}{5} & -\frac{e^{-2x}\left(-\frac{8e^{3x}}{3}+\frac{5}{3}+(\cos(x)+7\sin(x \cdots\right.}{10} \\ \frac{2\left(\frac{2e^{3x}}{3}-\frac{5}{3}+(\cos(x)-3\sin(x))e^{x}\right)e^{-2x}}{5} & \frac{(3\cos(x)+\sin(x))e^{-x}}{5} + \frac{2e^{x}}{5} & -\frac{3e^{-2x}\left(-\frac{4e^{3x}}{9}-\frac{5}{9}+\left(\cos(x)-\frac{4\sin(\cdots}{3}\right.\right.}{5} \\ \frac{4(5+e^{3x}+3(-2\cos(x)+\sin(x))e^{x})e^{-2x}}{15} & \frac{(-2\cos(x)-4\sin(x))e^{-x}}{5} + \frac{2e^{x}}{5} & \frac{(4e^{3x}-10+(-3\sin(x)+21\cos(x))e^{x} \cdots}{15} \\ \frac{4(-10+e^{3x}+3(3\cos(x)+\sin(x))e^{x})e^{-2x}}{15} & \frac{(-2\cos(x)+6\sin(x))e^{-x}}{5} + \frac{2e^{x}}{5} & -\frac{8\left(-\frac{e^{3x}}{6}-\frac{5}{6}+\left(\cos(x)+\frac{3\sin(x)}{4}\right)e^{x} \cdots\right.}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der…

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{\left(-\frac{5}{3}-\frac{2\,\mathrm{e}^{3x}}{15}+\left(\left(x+\frac{9}{5}\right)\cos(x)+2\left(x-\frac{6}{5}\right)\sin(x)\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{10} \\ \frac{\left(-\frac{2\,\mathrm{e}^{3x}}{15}+\frac{10}{3}+\left(\left(x-\frac{16}{5}\right)\cos(x)+\left(-3x+\frac{13}{5}\right)\sin(x)\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{10} \\ -\frac{\left(-15\,\mathrm{e}^{x}\sin(x)x+30\,\mathrm{e}^{x}\cos(x)x+\mathrm{e}^{3x}+18\sin(x)\mathrm{e}^{x}-51\cos(x)\mathrm{e}^{x}+50\right)\mathrm{e}^{-2x}}{75} \\ \frac{3\left(-\frac{\mathrm{e}^{3x}}{45}+\frac{20}{9}+\left(\left(x-\frac{11}{5}\right)\cos(x)+\frac{\left(x-\frac{6}{5}\right)\sin(x)}{3}\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{5} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3(x) + c_4\overrightarrow{y}_4(x) + \begin{bmatrix} \frac{\left(-\frac{5}{3}-\frac{2\,\mathrm{e}^{3x}}{15}+\left(\left(x+\frac{9}{5}\right)\cos(x)+2\left(x-\frac{6}{5}\right)\sin(x)\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{10} \\ \frac{\left(-\frac{2\,\mathrm{e}^{3x}}{15}+\frac{10}{3}+\left(\left(x-\frac{16}{5}\right)\cos(x)+\left(-3x+\frac{13}{5}\right)\sin(x)\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{10} \\ -\frac{\left(-15\,\mathrm{e}^{x}\sin(x)x+30\,\mathrm{e}^{x}\cos(x)x+\mathrm{e}^{3x}+18\sin(x)\mathrm{e}^{x}-51\cos(x)\mathrm{e}^{x}+50\right.}{75} \\ \frac{3\left(-\frac{\mathrm{e}^{3x}}{45}+\frac{20}{9}+\left(\left(x-\frac{11}{5}\right)\cos(x)+\frac{\left(x-\frac{6}{5}\right)\sin(x)}{3}\right)\mathrm{e}^{x}\right)\mathrm{e}^{-2x}}{5} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(10c_2-\frac{2}{15}\right)\mathrm{e}^{3x}+\left(\left(x+\frac{5c_3}{2}+\frac{5c_4}{2}+\frac{9}{5}\right)\cos(x)+2\left(x+\frac{5c_3}{4}-\frac{5c_4}{4}-\frac{6}{5}\right)\sin(x)\right)\mathrm{e}^{x}-\frac{5c_1}{4}-\frac{5}{3}\right)\mathrm{e}^{-2x}}{10}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(1*diff(y(x),x$4)+3*diff(y(x),x$3)+2*diff(y(x),x$2)-2*diff(y(x),x)-4*y(x)=-exp(-1*x)*(
```

$$y(x) = \frac{\left(10c_1 e^{3x} + \left(\left(x + 10c_3 + \frac{14}{5}\right)\cos(x) + 2\sin(x)\left(x + 5c_4 - \frac{1}{5}\right)\right)e^{x} + 10c_2\right)e^{-2x}}{10}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 58

```
DSolve[1*y''''[x]+3*y'''[x]+2*y''[x]-2*y'[x]-4*y[x]==-Exp[-1*x]*(Cos[x]-Sin[x]),y[x],x,Inclu
```

$$y(x) \to \frac{1}{50}e^{-2x}\left(50\left(c_4 e^{3x} + c_3\right) + e^{x}(5x + 14 + 50c_2)\cos(x) + e^{x}(10x - 7 + 50c_1)\sin(x)\right)$$

19.42 problem section 9.3, problem 42

Internal problem ID [1539]
Internal file name [`OUTPUT/1540_Sunday_June_05_2022_02_21_21_AM_71404590/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 42.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y''' + 13y'' - 19y' + 10y = \mathrm{e}^{x}(\cos(2x) + \sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y''' + 13y'' - 19y' + 10y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^3 + 13\lambda^2 - 19\lambda + 10 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 1 - 2i$$
$$\lambda_4 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{x} + c_2 e^{2x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{(1+2i)x}$$
$$y_4 = e^{(1-2i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y''' + 13y'' - 19y' + 10y = e^{x}(\cos(2x) + \sin(2x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{x} & e^{2x} & e^{(1+2i)x} & e^{(1-2i)x} \\ e^{x} & 2 e^{2x} & (1+2i) e^{(1+2i)x} & (1-2i) e^{(1-2i)x} \\ e^{x} & 4 e^{2x} & (-3+4i) e^{(1+2i)x} & (-3-4i) e^{(1-2i)x} \\ e^{x} & 8 e^{2x} & (-11-2i) e^{(1+2i)x} & (-11+2i) e^{(1-2i)x} \end{bmatrix}$$

$$|W| = -80i e^{x} e^{2x} e^{(1+2i)x} e^{(1-2i)x}$$

The determinant simplifies to

$$|W| = -80ie^{5x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{2x} & e^{(1+2i)x} & e^{(1-2i)x} \\ 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -20ie^{4x}$$

$$W_2(x) = \det \begin{bmatrix} e^{x} & e^{(1+2i)x} & e^{(1-2i)x} \\ e^{x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ e^{x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -16ie^{3x}$$

$$W_3(x) = \det \begin{bmatrix} e^{x} & e^{2x} & e^{(1-2i)x} \\ e^{x} & 2\,e^{2x} & (1-2i)\,e^{(1-2i)x} \\ e^{x} & 4\,e^{2x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= (-4+2i)\,e^{(4-2i)x}$$

$$W_4(x) = \det \begin{bmatrix} e^{x} & e^{2x} & e^{(1+2i)x} \\ e^{x} & 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} \\ e^{x} & 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} \end{bmatrix}$$
$$= (-4-2i)\,e^{(4+2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{\left(e^x(\cos(2x)+\sin(2x))\right)\left(-20ie^{4x}\right)}{(1)\left(-80ie^{5x}\right)}\,dx \\ &= -\int \frac{-20ie^x(\cos(2x)+\sin(2x))\,e^{4x}}{-80ie^{5x}}\,dx \\ &= -\int \left(\frac{\sin(2x)}{4}+\frac{\cos(2x)}{4}\right)dx \\ &= -\frac{\sin(2x)}{8}+\frac{\cos(2x)}{8} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(\mathrm{e}^x(\cos(2x)+\sin(2x)))\,(-16i\mathrm{e}^{3x})}{(1)\,(-80i\mathrm{e}^{5x})}\,dx \\
&= \int \frac{-16i\mathrm{e}^x(\cos(2x)+\sin(2x))\,\mathrm{e}^{3x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= \int \left(\frac{(\cos(2x)+\sin(2x))\,\mathrm{e}^{-x}}{5}\right)dx \\
&= \frac{\mathrm{e}^{-x}(-\sin(2x)-2\cos(2x))}{25} + \frac{2(2\sin(x)-\cos(x))\,\mathrm{e}^{-x}\cos(x)}{25} + \frac{\mathrm{e}^{-x}}{25} \\
&= \frac{\mathrm{e}^{-x}(-\sin(2x)-2\cos(2x))}{25} + \frac{2(2\sin(x)-\cos(x))\,\mathrm{e}^{-x}\cos(x)}{25} + \frac{\mathrm{e}^{-x}}{25}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\mathrm{e}^x(\cos(2x)+\sin(2x)))\,\left((-4+2i)\,\mathrm{e}^{(4-2i)x}\right)}{(1)\,(-80i\mathrm{e}^{5x})}\,dx \\
&= -\int \frac{(-4+2i)\,\mathrm{e}^x(\cos(2x)+\sin(2x))\,\mathrm{e}^{(4-2i)x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= -\int \left(\left(-\frac{1}{40}-\frac{i}{20}\right)(\cos(2x)+\sin(2x))\,\mathrm{e}^{-2ix}\right)dx \\
&= \frac{3x}{80} + \frac{ix}{80} - \frac{3\,\mathrm{e}^{-4ix}}{320} - \frac{i\mathrm{e}^{-4ix}}{320} \\
&= \frac{3x}{80} + \frac{ix}{80} - \frac{3\,\mathrm{e}^{-4ix}}{320} - \frac{i\mathrm{e}^{-4ix}}{320}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\mathrm{e}^x(\cos(2x)+\sin(2x)))\,\left((-4-2i)\,\mathrm{e}^{(4+2i)x}\right)}{(1)\,(-80i\mathrm{e}^{5x})}\,dx \\
&= \int \frac{(-4-2i)\,\mathrm{e}^x(\cos(2x)+\sin(2x))\,\mathrm{e}^{(4+2i)x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= \int \left(\left(\frac{1}{40}-\frac{i}{20}\right)(\cos(2x)+\sin(2x))\,\mathrm{e}^{2ix}\right)dx \\
&= \frac{3x}{80} - \frac{ix}{80} - \frac{3\,\mathrm{e}^{4ix}}{320} + \frac{i\mathrm{e}^{4ix}}{320} \\
&= \frac{3x}{80} - \frac{ix}{80} - \frac{3\,\mathrm{e}^{4ix}}{320} + \frac{i\mathrm{e}^{4ix}}{320}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Hence

$$\begin{aligned} y_p = &\left(-\frac{\sin(2x)}{8} + \frac{\cos(2x)}{8}\right)(\mathrm{e}^{x}) \\ &+ \left(\frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{25} + \frac{2(2\sin(x) - \cos(x))\,\mathrm{e}^{-x}\cos(x)}{25} + \frac{\mathrm{e}^{-x}}{25}\right)(\mathrm{e}^{2x}) \\ &+ \left(\frac{3x}{80} + \frac{ix}{80} - \frac{3\,\mathrm{e}^{-4ix}}{320} - \frac{i\mathrm{e}^{-4ix}}{320}\right)\left(\mathrm{e}^{(1+2i)x}\right) \\ &+ \left(\frac{3x}{80} - \frac{ix}{80} - \frac{3\,\mathrm{e}^{4ix}}{320} + \frac{i\mathrm{e}^{4ix}}{320}\right)\left(\mathrm{e}^{(1-2i)x}\right) \end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{\mathrm{e}^{x}(60x\cos(2x) - 20x\sin(2x) - 11\cos(2x) - 73\sin(2x))}{800}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{(1+2i)x}c_3 + \mathrm{e}^{(1-2i)x}c_4\right) \\ &\quad + \left(\frac{\mathrm{e}^{x}(60x\cos(2x) - 20x\sin(2x) - 11\cos(2x) - 73\sin(2x))}{800}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{(1+2i)x}c_3 + \mathrm{e}^{(1-2i)x}c_4 \\ &+ \frac{\mathrm{e}^{x}(60x\cos(2x) - 20x\sin(2x) - 11\cos(2x) - 73\sin(2x))}{800} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{(1+2i)x}c_3 + \mathrm{e}^{(1-2i)x}c_4 \\ &+ \frac{\mathrm{e}^{x}(60x\cos(2x) - 20x\sin(2x) - 11\cos(2x) - 73\sin(2x))}{800} \end{aligned}$$

Verified OK.

19.42.1 Maple step by step solution

Let's solve

$y'''' - 5y''' + 13y'' - 19y' + 10y = e^{x}(\cos(2x) + \sin(2x))$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = e^{x}\cos(2x) + e^{x}\sin(2x) + 5y_4(x) - 13y_3(x) + 19y_2(x) - 10y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^{x}\cos(2x) + e^{x}\sin(2x) + 5y_4(x) - 13y_3(x$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 19 & -13 & 5 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{x}\cos(2x) + e^{x}\sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{x} \cos(2x) + e^{x} \sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 19 & -13 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1-2\,\mathrm{I}, \begin{bmatrix} -\frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} -\frac{11}{125} + \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-2\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot \left(\cos\left(2x\right)-\mathrm{I}\sin\left(2x\right)\right) \cdot \left[\begin{array}{c} -\frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \left(-\frac{11}{125}-\frac{2\,\mathrm{I}}{125}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \left(-\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \left(\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \cos(2x)-\mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} -\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = e^{x} \cdot \begin{bmatrix} \frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{8} & e^{x}\left(-\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\right) & e^{x}\left(\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\right) \\ e^{x} & \frac{e^{2x}}{4} & e^{x}\left(-\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25}\right) & e^{x}\left(\frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25}\right) \\ e^{x} & \frac{e^{2x}}{2} & e^{x}\left(\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\right) & e^{x}\left(-\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\right) \\ e^{x} & e^{2x} & e^{x}\cos(2x) & -e^{x}\sin(2x) \end{bmatrix}$$

◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

◦ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{8} & e^{x}\left(-\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\right) & e^{x}\left(\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\right) \\ e^{x} & \frac{e^{2x}}{4} & e^{x}\left(-\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25}\right) & e^{x}\left(\frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25}\right) \\ e^{x} & \frac{e^{2x}}{2} & e^{x}\left(\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\right) & e^{x}\left(-\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\right) \\ e^{x} & e^{2x} & e^{x}\cos(2x) & -e^{x}\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{8} & -\frac{11}{125} & -\frac{2}{125} \\ 1 & \frac{1}{4} & -\frac{3}{25} & \frac{4}{25} \\ 1 & \frac{1}{2} & \frac{1}{5} & \frac{2}{5} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{e^{x}\cos(2x)}{2} + \frac{5e^{x}}{2} - e^{2x} & \frac{17e^{x}\cos(2x)}{20} - \frac{e^{x}\sin(2x)}{5} - \frac{9e^{x}}{4} + \frac{7e^{2x}}{5} & -\frac{2e^{x}}{\cdots} \\ -\frac{e^{x}\cos(2x)}{2} + e^{x}\sin(2x) + \frac{5e^{x}}{2} - 2e^{2x} & \frac{9e^{x}\cos(2x)}{20} - \frac{19e^{x}\sin(2x)}{10} - \frac{9e^{x}}{4} + \frac{14e^{2x}}{5} & \frac{e^{x}\cos\cdots}{5} \\ \frac{3e^{x}\cos(2x)}{2} + 2e^{x}\sin(2x) + \frac{5e^{x}}{2} - 4e^{2x} & -\frac{67e^{x}\cos(2x)}{20} - \frac{14e^{x}\sin(2x)}{5} - \frac{9e^{x}}{4} + \frac{28e^{2x}}{5} & \frac{12e^{x}\cos\cdots}{5} \\ \frac{5e^{x}}{2} - 8e^{2x} + \frac{11e^{x}\cos(2x)}{2} - e^{x}\sin(2x) & -\frac{9e^{x}}{4} + \frac{56e^{2x}}{5} - \frac{179e^{x}\cos(2x)}{20} + \frac{39e^{x}\sin(2x)}{10} & e^{x} - \cdots \end{bmatrix}$$

- [ ] Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
  - Integrate to solve for $\vec{v}(x)$

    $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
  - Plug $\vec{v}(x)$ into the equation for the particular solution

    $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\left(30\cos(x)^2 x-10x\cos(x)\sin(x)+2\cos(x)^2-39\cos(x)\sin(x)+24\,e^x-15x-26\right)e^x}{200} \\ \frac{\left(10\cos(x)^2 x-70x\cos(x)\sin(x)-46\cos(x)^2-53\cos(x)\sin(x)+48\,e^x-5x-2\right)e^x}{200} \\ \frac{(-130x-142)e^x\cos(x)^2}{200} - \frac{9\sin(x)\left(x+\frac{31}{90}\right)e^x\cos(x)}{20} - \frac{(-65x-96\,e^x-46)e^x}{200} \\ \frac{(-310x-334)e^x\cos(x)^2}{200} + \frac{17\left(x+\frac{163}{170}\right)\sin(x)e^x\cos(x)}{20} - \frac{(-155x-192\,e^x-142)e^x}{200} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{\left(30\cos(x)^2 x-10x\cos(x)\sin(x)+2\cos(x)^2-39\cos(x)\sin(x)+24\,e^x\right.}{200} \\ \frac{\left(10\cos(x)^2 x-70x\cos(x)\sin(x)-46\cos(x)^2-53\cos(x)\sin(x)+48\right.}{200} \\ \frac{(-130x-142)e^x\cos(x)^2}{200} - \frac{9\sin(x)\left(x+\frac{31}{90}\right)e^x\cos(x)}{20} - \frac{(-65x-}{} \\ \frac{(-310x-334)e^x\cos(x)^2}{200} + \frac{17\left(x+\frac{163}{170}\right)\sin(x)e^x\cos(x)}{20} - \frac{(-155x}{} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{3\left(x-\frac{88c_3}{75}-\frac{16c_4}{75}+\frac{1}{15}\right)e^x\cos(2x)}{40} - \frac{\left(x+\frac{16c_3}{25}-\frac{88c_4}{25}+\frac{39}{10}\right)e^x\sin(2x)}{40} + \frac{(24+25c_2)e^{2x}}{200} + \frac{e^x(-1+8c_1)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve(1*diff(y(x),x$4)-5*diff(y(x),x$3)+13*diff(y(x),x$2)-19*diff(y(x),x)+10*y(x)=exp(x)*(c
```

$$y(x) = \frac{3\,e^x\left(x+\frac{40c_3}{3}+\frac{1}{15}\right)\cos(2x)}{40} - \frac{e^x\left(x-40c_4+\frac{39}{10}\right)\sin(2x)}{40} + c_2e^{2x} + \frac{e^x(8c_1+1)}{8}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 53

```
DSolve[1*y''''[x]-5*y'''[x]+13*y''[x]-19*y'[x]+10*y[x]==Exp[x]*(Cos[2*x]+Sin[2*x]),y[x],x,In
```

$$y(x) \to \frac{1}{400} e^x (400(c_4 e^x + c_3) + (30x - 13 + 400c_2)\cos(2x) - 2(5x + 17 - 200c_1)\sin(2x))$$

19.43 problem section 9.3, problem 43

Internal problem ID [1540]
Internal file name [`OUTPUT/1541_Sunday_June_05_2022_02_21_25_AM_68185977/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 43.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 8y''' + 32y'' + 64y' + 39y = \mathrm{e}^{-2x}((4 - 15x)\cos(3x) - (4 + 15x)\sin(3x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 8y''' + 32y'' + 64y' + 39y = 0$$

The characteristic equation is

$$\lambda^4 + 8\lambda^3 + 32\lambda^2 + 64\lambda + 39 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -1$$
$$\lambda_3 = -2 - 3i$$
$$\lambda_4 = -2 + 3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2 + \mathrm{e}^{(-2+3i)x} c_3 + \mathrm{e}^{(-2-3i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-3x}$$
$$y_3 = \mathrm{e}^{(-2+3i)x}$$
$$y_4 = \mathrm{e}^{(-2-3i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 8y''' + 32y'' + 64y' + 39y = \mathrm{e}^{-2x}((4 - 15x)\cos(3x) - (4 + 15x)\sin(3x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-3x} & \mathrm{e}^{(-2+3i)x} & \mathrm{e}^{(-2-3i)x} \\ -\mathrm{e}^{-x} & -3\,\mathrm{e}^{-3x} & (-2+3i)\,\mathrm{e}^{(-2+3i)x} & (-2-3i)\,\mathrm{e}^{(-2-3i)x} \\ \mathrm{e}^{-x} & 9\,\mathrm{e}^{-3x} & (-5-12i)\,\mathrm{e}^{(-2+3i)x} & (-5+12i)\,\mathrm{e}^{(-2-3i)x} \\ -\mathrm{e}^{-x} & -27\,\mathrm{e}^{-3x} & (46+9i)\,\mathrm{e}^{(-2+3i)x} & (46-9i)\,\mathrm{e}^{(-2-3i)x} \end{bmatrix}$$

$$|W| = 1200i \mathrm{e}^{-x} \mathrm{e}^{-3x} \mathrm{e}^{(-2+3i)x} \mathrm{e}^{(-2-3i)x}$$

The determinant simplifies to

$$|W| = 1200i\mathrm{e}^{-8x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{(-2+3i)x} & \mathrm{e}^{(-2-3i)x} \\ -3\,\mathrm{e}^{-3x} & (-2+3i)\,\mathrm{e}^{(-2+3i)x} & (-2-3i)\,\mathrm{e}^{(-2-3i)x} \\ 9\,\mathrm{e}^{-3x} & (-5-12i)\,\mathrm{e}^{(-2+3i)x} & (-5+12i)\,\mathrm{e}^{(-2-3i)x} \end{bmatrix} \\ &= -60i\mathrm{e}^{-7x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{(-2+3i)x} & \mathrm{e}^{(-2-3i)x} \\ -\mathrm{e}^{-x} & (-2+3i)\,\mathrm{e}^{(-2+3i)x} & (-2-3i)\,\mathrm{e}^{(-2-3i)x} \\ \mathrm{e}^{-x} & (-5-12i)\,\mathrm{e}^{(-2+3i)x} & (-5+12i)\,\mathrm{e}^{(-2-3i)x} \end{bmatrix} \\ &= -60i\mathrm{e}^{-5x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-3x} & \mathrm{e}^{(-2-3i)x} \\ -\mathrm{e}^{-x} & -3\,\mathrm{e}^{-3x} & (-2-3i)\,\mathrm{e}^{(-2-3i)x} \\ \mathrm{e}^{-x} & 9\,\mathrm{e}^{-3x} & (-5+12i)\,\mathrm{e}^{(-2-3i)x} \end{bmatrix} \\ &= 20\,\mathrm{e}^{(-6-3i)x} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-3x} & \mathrm{e}^{(-2+3i)x} \\ -\mathrm{e}^{-x} & -3\,\mathrm{e}^{-3x} & (-2+3i)\,\mathrm{e}^{(-2+3i)x} \\ \mathrm{e}^{-x} & 9\,\mathrm{e}^{-3x} & (-5-12i)\,\mathrm{e}^{(-2+3i)x} \end{bmatrix} \\ &= 20\,\mathrm{e}^{(-6+3i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$U_1 = (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx$$
$$= (-1)^3 \int \frac{\left(e^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\right)\left(-60ie^{-7x}\right)}{(1)\left(1200ie^{-8x}\right)}\,dx$$
$$= -\int \frac{-60ie^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\,e^{-7x}}{1200ie^{-8x}}\,dx$$
$$= -\int \left(\frac{(15x\sin(3x) + 15x\cos(3x) + 4\sin(3x) - 4\cos(3x))\,e^{-x}}{20}\right)dx$$
$$= -\frac{3\left(-\frac{3x}{10} - \frac{3}{50}\right)e^{-x}\cos(3x)}{4} - \frac{3\left(-\frac{x}{10} + \frac{2}{25}\right)e^{-x}\sin(3x)}{4} - \frac{3\left(-\frac{x}{10} + \frac{2}{25}\right)e^{-x}\cos(3x)}{4} + \frac{3\left(-\frac{3x}{10} - \frac{3}{50}\right)e}{4}$$
$$= -\frac{3\left(-\frac{3x}{10} - \frac{3}{50}\right)e^{-x}\cos(3x)}{4} - \frac{3\left(-\frac{x}{10} + \frac{2}{25}\right)e^{-x}\sin(3x)}{4} - \frac{3\left(-\frac{x}{10} + \frac{2}{25}\right)e^{-x}\cos(3x)}{4} + \frac{3\left(-\frac{3x}{10} - \frac{3}{50}\right)e}{4}$$

$$U_2 = (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx$$
$$= (-1)^2 \int \frac{\left(e^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\right)\left(-60ie^{-5x}\right)}{(1)\left(1200ie^{-8x}\right)}\,dx$$
$$= \int \frac{-60ie^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\,e^{-5x}}{1200ie^{-8x}}\,dx$$
$$= \int \left(\frac{(15x\sin(3x) + 15x\cos(3x) + 4\sin(3x) - 4\cos(3x))\,e^{x}}{20}\right)dx$$
$$= \frac{3\left(-\frac{3x}{10} + \frac{3}{50}\right)e^{x}\cos(3x)}{4} + \frac{3\left(\frac{x}{10} + \frac{2}{25}\right)e^{x}\sin(3x)}{4} + \frac{3\left(\frac{x}{10} + \frac{2}{25}\right)e^{x}\cos(3x)}{4} - \frac{3\left(-\frac{3x}{10} + \frac{3}{50}\right)e^{x}\sin(3x)}{4}$$
$$= \frac{3\left(-\frac{3x}{10} + \frac{3}{50}\right)e^{x}\cos(3x)}{4} + \frac{3\left(\frac{x}{10} + \frac{2}{25}\right)e^{x}\sin(3x)}{4} + \frac{3\left(\frac{x}{10} + \frac{2}{25}\right)e^{x}\cos(3x)}{4} - \frac{3\left(-\frac{3x}{10} + \frac{3}{50}\right)e^{x}\sin(3x)}{4}$$

$$U_3 = (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx$$
$$= (-1)^1 \int \frac{\left(e^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\right)\left(20\,e^{(-6-3i)x}\right)}{(1)\left(1200ie^{-8x}\right)}\,dx$$
$$= -\int \frac{20\,e^{-2x}((4-15x)\cos(3x) - (4+15x)\sin(3x))\,e^{(-6-3i)x}}{1200ie^{-8x}}\,dx$$
$$= -\int \left(\frac{i(15x\sin(3x) + 15x\cos(3x) + 4\sin(3x) - 4\cos(3x))\,e^{-3ix}}{60}\right)dx$$
$$= \frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} - \frac{ix^2}{16} + \left(\frac{1}{480} + \frac{i}{480}\right)(i+10x)\,e^{-6ix}$$
$$= \frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} - \frac{ix^2}{16} + \left(\frac{1}{480} + \frac{i}{480}\right)(i+10x)\,e^{-6ix}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\mathrm{e}^{-2x}((4-15x)\cos(3x)-(4+15x)\sin(3x))\right)\left(20\,\mathrm{e}^{(-6+3i)x}\right)}{(1)\left(1200 i \mathrm{e}^{-8x}\right)}\, dx \\
&= \int \frac{20\,\mathrm{e}^{-2x}((4-15x)\cos(3x)-(4+15x)\sin(3x))\,\mathrm{e}^{(-6+3i)x}}{1200 i \mathrm{e}^{-8x}}\, dx \\
&= \int \left(\frac{i(15x\sin(3x)+15x\cos(3x)+4\sin(3x)-4\cos(3x))\,\mathrm{e}^{3ix}}{60}\right) dx \\
&= -\frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} + \frac{ix^2}{16} + \left(\frac{1}{480} - \frac{i}{480}\right)(-i+10x)\,\mathrm{e}^{6ix} \\
&= -\frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} + \frac{ix^2}{16} + \left(\frac{1}{480} - \frac{i}{480}\right)(-i+10x)\,\mathrm{e}^{6ix}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{3\left(-\frac{3x}{10}-\frac{3}{50}\right)\mathrm{e}^{-x}\cos(3x)}{4} - \frac{3\left(-\frac{x}{10}+\frac{2}{25}\right)\mathrm{e}^{-x}\sin(3x)}{4} - \frac{3\left(-\frac{x}{10}+\frac{2}{25}\right)\mathrm{e}^{-x}\cos(3x)}{4} + \frac{3\left(-\frac{3x}{10}-\frac{3}{50}\right)}{4}\right. \\
&+ \left(\frac{3\left(-\frac{3x}{10}+\frac{3}{50}\right)\mathrm{e}^{x}\cos(3x)}{4} + \frac{3\left(\frac{x}{10}+\frac{2}{25}\right)\mathrm{e}^{x}\sin(3x)}{4} + \frac{3\left(\frac{x}{10}+\frac{2}{25}\right)\mathrm{e}^{x}\cos(3x)}{4} - \frac{3\left(-\frac{3x}{10}+\frac{3}{50}\right)\mathrm{e}^{x}\sin(3x}{4}\right. \\
&+ \left(\frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} - \frac{ix^2}{16} + \left(\frac{1}{480}+\frac{i}{480}\right)(i+10x)\,\mathrm{e}^{-6ix}\right)\left(\mathrm{e}^{(-2+3i)x}\right) \\
&+ \left(-\frac{ix}{30} - \frac{x^2}{16} - \frac{x}{30} + \frac{ix^2}{16} + \left(\frac{1}{480}-\frac{i}{480}\right)(-i+10x)\,\mathrm{e}^{6ix}\right)\left(\mathrm{e}^{(-2-3i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{\left(\left(x^2-x-\frac{11}{30}\right)\cos(3x)-\left(x^2+x-\frac{11}{30}\right)\sin(3x)\right)\mathrm{e}^{-2x}}{8}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2 + \mathrm{e}^{(-2+3i)x} c_3 + \mathrm{e}^{(-2-3i)x} c_4\right) \\
&\quad + \left(-\frac{\left(\left(x^2-x-\frac{11}{30}\right)\cos(3x)-\left(x^2+x-\frac{11}{30}\right)\sin(3x)\right)\mathrm{e}^{-2x}}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2 + \mathrm{e}^{(-2+3i)x} c_3 + \mathrm{e}^{(-2-3i)x} c_4 - \frac{\left(\left(x^2 - x - \frac{11}{30}\right)\cos(3x) - \left(x^2 + x - \frac{11}{30}\right)\sin(3x)\right)\mathrm{e}^{-2x}}{8} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-3x} c_2 + \mathrm{e}^{(-2+3i)x} c_3 + \mathrm{e}^{(-2-3i)x} c_4 - \frac{\left(\left(x^2 - x - \frac{11}{30}\right)\cos(3x) - \left(x^2 + x - \frac{11}{30}\right)\sin(3x)\right)\mathrm{e}^{-2x}}{8}$$

Verified OK.

19.43.1 Maple step by step solution

Let's solve

$y'''' + 8y''' + 32y'' + 64y' + 39y = \mathrm{e}^{-2x}((4 - 15x)\cos(3x) - (4 + 15x)\sin(3x))$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -39y - 15\cos(3x)\mathrm{e}^{-2x}x - 15\sin(3x)\mathrm{e}^{-2x}x + 4\cos(3x)\mathrm{e}^{-2x} - 4\sin(3x)\mathrm{e}^{-2x} - 8y''' - 32$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 8y''' + 32y'' + 64y' + 39y = -\mathrm{e}^{-2x}(15x\sin(3x) + 15x\cos(3x) + 4\sin(3x) - 4\cos(3x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -15\cos(3x)\mathrm{e}^{-2x}x - 15\sin(3x)\mathrm{e}^{-2x}x + 4\cos(3x)\mathrm{e}^{-2x} - 4\sin(3x)\mathrm{e}^{-2x} - 8y_4(x) - 32y_3$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -15\cos(3x)\,\mathrm{e}^{-2x}x - 15\sin(3x)\,\mathrm{e}^{-2x}x + 4$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -39 & -64 & -32 & -8 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -15\sin(3x)\,\mathrm{e}^{-2x}x - 15\cos(3x)\,\mathrm{e}^{-2x}x - 4\sin(3x \end{array}\right.$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -15\sin(3x)\,\mathrm{e}^{-2x}x - 15\cos(3x)\,\mathrm{e}^{-2x}x - 4\sin(3x)\,\mathrm{e}^{-2x} + 4\cos(3x)\,\mathrm{e}^{-2x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -39 & -64 & -32 & -8 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-2-3\,\mathrm{I}, \left[\begin{array}{c} \frac{46}{2197}+\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{2}{13}+\frac{3\,\mathrm{I}}{13} \\ 1 \end{array}\right]\right], \left[-2+3\,\mathrm{I}, \left[\begin{array}{c} \frac{46}{2197}-\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}+\frac{12\,\mathrm{I}}{169} \\ -\frac{2}{13}-\frac{3\,\mathrm{I}}{13} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-3\,\mathrm{I}, \begin{bmatrix} \frac{46}{2197}+\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{2}{13}+\frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-3\,\mathrm{I})x}\cdot\begin{bmatrix} \frac{46}{2197}+\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{2}{13}+\frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-2x}\cdot(\cos(3x)-\mathrm{I}\sin(3x))\cdot\begin{bmatrix} \frac{46}{2197}+\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{2}{13}+\frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-2x}\cdot\begin{bmatrix} \left(\frac{46}{2197}+\frac{9\,\mathrm{I}}{2197}\right)(\cos(3x)-\mathrm{I}\sin(3x)) \\ \left(-\frac{5}{169}-\frac{12\,\mathrm{I}}{169}\right)(\cos(3x)-\mathrm{I}\sin(3x)) \\ \left(-\frac{2}{13}+\frac{3\,\mathrm{I}}{13}\right)(\cos(3x)-\mathrm{I}\sin(3x)) \\ \cos(3x)-\mathrm{I}\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\mathrm{e}^{-2x}\cdot\begin{bmatrix} \frac{46\cos(3x)}{2197}+\frac{9\sin(3x)}{2197} \\ -\frac{5\cos(3x)}{169}-\frac{12\sin(3x)}{169} \\ -\frac{2\cos(3x)}{13}+\frac{3\sin(3x)}{13} \\ \cos(3x) \end{bmatrix}, \vec{y}_4(x)=\mathrm{e}^{-2x}\cdot\begin{bmatrix} -\frac{46\sin(3x)}{2197}+\frac{9\cos(3x)}{2197} \\ \frac{5\sin(3x)}{169}-\frac{12\cos(3x)}{169} \\ \frac{2\sin(3x)}{13}+\frac{3\cos(3x)}{13} \\ -\sin(3x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} -\frac{e^{-3x}}{27} & -e^{-x} & e^{-2x}\left(\frac{46\cos(3x)}{2197}+\frac{9\sin(3x)}{2197}\right) & e^{-2x}\left(-\frac{46\sin(3x)}{2197}+\frac{9\cos(3x)}{2197}\right) \\ \frac{e^{-3x}}{9} & e^{-x} & e^{-2x}\left(-\frac{5\cos(3x)}{169}-\frac{12\sin(3x)}{169}\right) & e^{-2x}\left(\frac{5\sin(3x)}{169}-\frac{12\cos(3x)}{169}\right) \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{-2x}\left(-\frac{2\cos(3x)}{13}+\frac{3\sin(3x)}{13}\right) & e^{-2x}\left(\frac{2\sin(3x)}{13}+\frac{3\cos(3x)}{13}\right) \\ e^{-3x} & e^{-x} & \cos(3x)\,e^{-2x} & -\sin(3x)\,e^{-2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} -\frac{e^{-3x}}{27} & -e^{-x} & e^{-2x}\left(\frac{46\cos(3x)}{2197}+\frac{9\sin(3x)}{2197}\right) & e^{-2x}\left(-\frac{46\sin(3x)}{2197}+\frac{9\cos(3x)}{2197}\right) \\ \frac{e^{-3x}}{9} & e^{-x} & e^{-2x}\left(-\frac{5\cos(3x)}{169}-\frac{12\sin(3x)}{169}\right) & e^{-2x}\left(\frac{5\sin(3x)}{169}-\frac{12\cos(3x)}{169}\right) \\ -\frac{e^{-3x}}{3} & -e^{-x} & e^{-2x}\left(-\frac{2\cos(3x)}{13}+\frac{3\sin(3x)}{13}\right) & e^{-2x}\left(\frac{2\sin(3x)}{13}+\frac{3\cos(3x)}{13}\right) \\ e^{-3x} & e^{-x} & \cos(3x)\,e^{-2x} & -\sin(3x)\,e^{-2x} \end{bmatrix}\cdot\frac{}{\begin{bmatrix} -\frac{1}{27} & -1 \\ \frac{1}{9} & 1 \\ -\frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} \frac{(-6\cos(3x)-4\sin(3x))e^{-2x}}{20}-\frac{13e^{-3x}}{20}+\frac{39e^{-x}}{20} & \frac{(-24\cos(3x)-22\sin(3x))e^{-2x}}{60}-\frac{17e^{-3x}}{20}+\frac{5e^{-x}}{4} \\ \frac{13\sin(3x)e^{-2x}}{10}-\frac{39e^{-x}}{20}+\frac{39e^{-3x}}{20} & \frac{(-9\cos(3x)+58\sin(3x))e^{-2x}}{30}+\frac{51e^{-3x}}{20}-\frac{5e^{-x}}{4} \\ \frac{13(3\cos(3x)-2\sin(3x))e^{-2x}}{10}-\frac{117e^{-3x}}{20}+\frac{39e^{-x}}{20} & \frac{(192\cos(3x)-89\sin(3x))e^{-2x}}{30}-\frac{153e^{-3x}}{20}+\frac{5e^{-x}}{4} \\ \frac{13(-12\cos(3x)-5\sin(3x))e^{-2x}}{10}+\frac{351e^{-3x}}{20}-\frac{39e^{-x}}{20} & \frac{(-651\cos(3x)-398\sin(3x))e^{-2x}}{30}+\frac{459e^{-3x}}{20}-\frac{5e^{-x}}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}'_p(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{\left(\left(-5x^2+5x+2\right)\cos(3x)+5\left(x^2+x-\frac{1}{3}\right)\sin(3x)\right)e^{-2x}}{40}-\frac{e^{-3x}}{40}-\frac{e^{-x}}{40}\\ \frac{\left(\left(25x^2-5x-4\right)\cos(3x)+5\left(x^2-3x+\frac{7}{15}\right)\sin(3x)\right)e^{-2x}}{40}+\frac{3\,e^{-3x}}{40}+\frac{e^{-x}}{40}\\ \frac{\left(\left(-105x^2+45x+30\right)\cos(3x)+\left(-255x^2+165x-23\right)\sin(3x)\right)e^{-2x}}{120}-\frac{9\,e^{-3x}}{40}-\frac{e^{-x}}{40}\\ \frac{\left(\left(-185x^2+65x-28\right)\cos(3x)+275\left(x^2-\frac{13}{11}x+\frac{11}{75}\right)\sin(3x)\right)e^{-2x}}{40}+\frac{27\,e^{-3x}}{40}+\frac{e^{-x}}{40}\end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3(x)+c_4\overrightarrow{y}_4(x)+\left[\begin{array}{c}\frac{\left(\left(-5x^2+5x+2\right)\cos(3x)+5\left(x^2+x-\frac{1}{3}\right)\sin(3x)\right)e^{-2x}}{40}-\frac{e^{-3x}}{40}\\ \frac{\left(\left(25x^2-5x-4\right)\cos(3x)+5\left(x^2-3x+\frac{7}{15}\right)\sin(3x)\right)e^{-2x}}{40}+\frac{3\,e^{-}}{40}\\ \frac{\left(\left(-105x^2+45x+30\right)\cos(3x)+\left(-255x^2+165x-23\right)\sin(3x)\right)e^{-2x}}{120}-\\ \frac{\left(\left(-185x^2+65x-28\right)\cos(3x)+275\left(x^2-\frac{13}{11}x+\frac{11}{75}\right)\sin(3x)\right)e^{-2x}}{40}+\end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y=\frac{\left(\left(-10985x^2+1840c_3+360c_4+10985x+4394\right)\cos(3x)+10985\sin(3x)\left(x^2+x+\frac{72}{2197}c_3-\frac{368}{2197}c_4-\frac{1}{3}\right)\right)e^{-2x}}{87880}+\frac{(-27-40c_1)e^{-3x}}{1080}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 57

```
dsolve(1*diff(y(x),x$4)+8*diff(y(x),x$3)+32*diff(y(x),x$2)+64*diff(y(x),x)+39*y(x)=exp(-2*x)
```

$$y(x) = \frac{\left(\left(-30x^2+240c_3+30x+11\right)\cos\left(3x\right)+30\left(x^2+x+8c_4-\frac{11}{30}\right)\sin\left(3x\right)\right)\mathrm{e}^{-2x}}{240} + c_1\mathrm{e}^{-3x}+\mathrm{e}^{-x}c_2$$

✓ Solution by Mathematica

Time used: 0.841 (sec). Leaf size: 73

```
DSolve[1*y''''[x]+8*y'''[x]+32*y''[x]+64*y'[x]+39*y[x]==Exp[-2*x]*((4-15*x)*Cos[3*x]-(4+15*x
```

$$y(x) \to \frac{1}{720}e^{-3x}\left(-5e^x\left(18x^2-18x-5-144c_2\right)\cos(3x) + e^x\left(90x^2+90x-41+720c_1\right)\sin(3x)+720\left(c_4e^{2x}+c_3\right)\right)$$

19.44 problem section 9.3, problem 44

Internal problem ID [1541]
Internal file name [`OUTPUT/1542_Sunday_June_05_2022_02_21_29_AM_55063930/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 44.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y''' + 13y'' - 19y' + 10y = e^{x}((8x + 7)\cos(2x) + (8 - 4x)\sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y''' + 13y'' - 19y' + 10y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^3 + 13\lambda^2 - 19\lambda + 10 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 1 - 2i$$
$$\lambda_4 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

$$y_3 = e^{(1+2i)x}$$

$$y_4 = e^{(1-2i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y''' + 13y'' - 19y' + 10y = e^x((8x + 7)\cos(2x) + (8 - 4x)\sin(2x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^x & e^{2x} & e^{(1+2i)x} & e^{(1-2i)x} \\ e^x & 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ e^x & 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \\ e^x & 8\,e^{2x} & (-11-2i)\,e^{(1+2i)x} & (-11+2i)\,e^{(1-2i)x} \end{bmatrix}$$

$$|W| = -80i e^x e^{2x} e^{(1+2i)x} e^{(1-2i)x}$$

The determinant simplifies to

$$|W| = -80i e^{5x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{2x} & e^{(1+2i)x} & e^{(1-2i)x} \\ 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -20i e^{4x}$$

$$W_2(x) = \det \begin{bmatrix} e^{x} & e^{(1+2i)x} & e^{(1-2i)x} \\ e^{x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ e^{x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -16i e^{3x}$$

$$W_3(x) = \det \begin{bmatrix} e^{x} & e^{2x} & e^{(1-2i)x} \\ e^{x} & 2\,e^{2x} & (1-2i)\,e^{(1-2i)x} \\ e^{x} & 4\,e^{2x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= (-4+2i)\,e^{(4-2i)x}$$

$$W_4(x) = \det \begin{bmatrix} e^{x} & e^{2x} & e^{(1+2i)x} \\ e^{x} & 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} \\ e^{x} & 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} \end{bmatrix}$$
$$= (-4-2i)\,e^{(4+2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\right)\left(-20i\mathrm{e}^{4x}\right)}{(1)\left(-80i\mathrm{e}^{5x}\right)}\,dx \\
&= -\int \frac{-20i\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\,\mathrm{e}^{4x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= -\int \left(\frac{(8x+7)\cos(2x)}{4} - (-2+x)\sin(2x)\right)dx \\
&= \frac{\cos(2x)}{2} - x\sin(2x) - \frac{5\sin(2x)}{8} - \frac{x\cos(2x)}{2} \\
&= \frac{\cos(2x)}{2} - x\sin(2x) - \frac{5\sin(2x)}{8} - \frac{x\cos(2x)}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\right)\left(-16i\mathrm{e}^{3x}\right)}{(1)\left(-80i\mathrm{e}^{5x}\right)}\,dx \\
&= \int \frac{-16i\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\,\mathrm{e}^{3x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= \int \left(\frac{(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)+7\cos(2x))\,\mathrm{e}^{-x}}{5}\right)dx \\
&= \frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5} + \frac{7\,\mathrm{e}^{-x}}{25} - \frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{4\left(-\frac{x}{5}+\right.}{} \\
&= \frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5} + \frac{7\,\mathrm{e}^{-x}}{25} - \frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5} - \frac{4\left(-\frac{x}{5}+\right.}{}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\right)\left((-4+2i)\,\mathrm{e}^{(4-2i)x}\right)}{(1)\left(-80i\mathrm{e}^{5x}\right)}\,dx \\
&= -\int \frac{(-4+2i)\,\mathrm{e}^x((8x+7)\cos(2x)+(8-4x)\sin(2x))\,\mathrm{e}^{(4-2i)x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= -\int \left(\left(-\frac{1}{40}-\frac{i}{20}\right)(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)+7\cos(2x))\,\mathrm{e}^{-2ix}\right)dx \\
&= \frac{23x}{80} + \frac{ix^2}{8} + \frac{3ix}{40} + \left(-\frac{3}{800}+\frac{i}{200}\right)(3+9i+10x)\,\mathrm{e}^{-4ix} \\
&= \frac{23x}{80} + \frac{ix^2}{8} + \frac{3ix}{40} + \left(-\frac{3}{800}+\frac{i}{200}\right)(3+9i+10x)\,\mathrm{e}^{-4ix}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^{0} \int \frac{\left(\mathrm{e}^{x}\left((8x+7)\cos(2x)+(8-4x)\sin(2x)\right)\right)\left((-4-2i)\,\mathrm{e}^{(4+2i)x}\right)}{(1)\left(-80i\mathrm{e}^{5x}\right)}\,dx \\
&= \int \frac{(-4-2i)\,\mathrm{e}^{x}\left((8x+7)\cos(2x)+(8-4x)\sin(2x)\right)\mathrm{e}^{(4+2i)x}}{-80i\mathrm{e}^{5x}}\,dx \\
&= \int \left(\left(\frac{1}{40}-\frac{i}{20}\right)\left(-4x\sin(2x)+8x\cos(2x)+8\sin(2x)+7\cos(2x)\right)\mathrm{e}^{2ix}\right)dx \\
&= \frac{23x}{80}-\frac{ix^2}{8}-\frac{3ix}{40}+\left(-\frac{3}{800}-\frac{i}{200}\right)(3-9i+10x)\,\mathrm{e}^{4ix} \\
&= \frac{23x}{80}-\frac{ix^2}{8}-\frac{3ix}{40}+\left(-\frac{3}{800}-\frac{i}{200}\right)(3-9i+10x)\,\mathrm{e}^{4ix}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1+U_2y_2+U_3y_3+U_4y_4
$$

Hence

$$
\begin{aligned}
y_p &= \left(\frac{\cos(2x)}{2}-x\sin(2x)-\frac{5\sin(2x)}{8}-\frac{x\cos(2x)}{2}\right)(\mathrm{e}^{x}) \\
&+ \left(\frac{8\left(-\frac{x}{5}+\frac{3}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5}-\frac{8\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\sin(2x)}{5}+\frac{7\,\mathrm{e}^{-x}}{25}-\frac{4\left(-\frac{2x}{5}-\frac{4}{25}\right)\mathrm{e}^{-x}\cos(2x)}{5}-\frac{4\left(-\frac{x}{5}\right.}{}\right. \\
&+ \left(\frac{23x}{80}+\frac{ix^2}{8}+\frac{3ix}{40}+\left(-\frac{3}{800}+\frac{i}{200}\right)(3+9i+10x)\,\mathrm{e}^{-4ix}\right)\left(\mathrm{e}^{(1+2i)x}\right) \\
&+ \left(\frac{23x}{80}-\frac{ix^2}{8}-\frac{3ix}{40}+\left(-\frac{3}{800}-\frac{i}{200}\right)(3-9i+10x)\,\mathrm{e}^{4ix}\right)\left(\mathrm{e}^{(1-2i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{\left(\left(x^2+x+\frac{21}{20}\right)\sin(2x)+\frac{17\cos(2x)}{20}\right)\mathrm{e}^{x}}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{x}+c_2\mathrm{e}^{2x}+\mathrm{e}^{(1+2i)x}c_3+\mathrm{e}^{(1-2i)x}c_4\right)+\left(-\frac{\left(\left(x^2+x+\frac{21}{20}\right)\sin(2x)+\frac{17\cos(2x)}{20}\right)\mathrm{e}^{x}}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4 - \frac{\left(\left(x^2 + x + \frac{21}{20}\right)\sin(2x) + \frac{17\cos(2x)}{20}\right)e^x}{4} \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4 - \frac{\left(\left(x^2 + x + \frac{21}{20}\right)\sin(2x) + \frac{17\cos(2x)}{20}\right)e^x}{4}$$

Verified OK.

19.44.1 Maple step by step solution

Let's solve

$y'''' - 5y''' + 13y'' - 19y' + 10y = e^x((8x + 7)\cos(2x) + (8 - 4x)\sin(2x))$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -10y + 8\cos(2x) e^x x - 4\sin(2x) e^x x + 7 e^x \cos(2x) + 8 e^x \sin(2x) + 5y''' - 13y'' + 19y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - 5y''' + 13y'' - 19y' + 10y = e^x(-4x\sin(2x) + 8x\cos(2x) + 8\sin(2x) + 7\cos(2x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 8\cos(2x) e^x x - 4\sin(2x) e^x x + 7 e^x \cos(2x) + 8 e^x \sin(2x) + 5y_4(x) - 13y_3(x) + 19y_2(x$

  Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 8\cos(2x)\,\mathrm{e}^x x - 4\sin(2x)\,\mathrm{e}^x x + 7\,\mathrm{e}^x \cos(2$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 19 & -13 & 5 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4\sin(2x)\,\mathrm{e}^x x + 8\cos(2x)\,\mathrm{e}^x x + 8\,\mathrm{e}^x \sin(2x) + 7\,\mathrm{e}^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4\sin(2x)\,\mathrm{e}^x x + 8\cos(2x)\,\mathrm{e}^x x + 8\,\mathrm{e}^x \sin(2x) + 7\,\mathrm{e}^x \cos(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 19 & -13 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{array}\right]\right], \left[1-2\,\mathrm{I}, \left[\begin{array}{c}-\frac{11}{125}-\frac{2\,\mathrm{I}}{125}\\-\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\\\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]\right], \left[1+2\,\mathrm{I}, \left[\begin{array}{c}-\frac{11}{125}+\frac{2\,\mathrm{I}}{125}\\-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\\frac{1}{5}-\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c}\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})x}\cdot\begin{bmatrix} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{x}\cdot\left(\cos(2x)-\mathrm{I}\sin(2x)\right)\cdot\begin{bmatrix} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{x}\cdot\begin{bmatrix} \left(-\frac{11}{125}-\frac{2\,\mathrm{I}}{125}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \left(-\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \left(\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\right)\left(\cos(2x)-\mathrm{I}\sin(2x)\right) \\ \cos(2x)-\mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=\mathrm{e}^{x}\cdot\begin{bmatrix} -\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x)=\mathrm{e}^{x}\cdot\begin{bmatrix} \frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{x} & \frac{e^{2x}}{8} & e^{x}\left(-\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\right) & e^{x}\left(\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\right) \\ e^{x} & \frac{e^{2x}}{4} & e^{x}\left(-\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25}\right) & e^{x}\left(\frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25}\right) \\ e^{x} & \frac{e^{2x}}{2} & e^{x}\left(\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\right) & e^{x}\left(-\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\right) \\ e^{x} & e^{2x} & e^{x}\cos(2x) & -e^{x}\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} e^{x} & \frac{e^{2x}}{8} & e^{x}\left(-\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\right) & e^{x}\left(\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\right) \\ e^{x} & \frac{e^{2x}}{4} & e^{x}\left(-\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25}\right) & e^{x}\left(\frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25}\right) \\ e^{x} & \frac{e^{2x}}{2} & e^{x}\left(\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\right) & e^{x}\left(-\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\right) \\ e^{x} & e^{2x} & e^{x}\cos(2x) & -e^{x}\sin(2x) \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & \frac{1}{8} & -\frac{11}{125} & -\frac{2}{125} \\ 1 & \frac{1}{4} & -\frac{3}{25} & \frac{4}{25} \\ 1 & \frac{1}{2} & \frac{1}{5} & \frac{2}{5} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -\frac{e^{x}\cos(2x)}{2}+\frac{5e^{x}}{2}-e^{2x} & \frac{17e^{x}\cos(2x)}{20}-\frac{e^{x}\sin(2x)}{5}-\frac{9e^{x}}{4}+\frac{7e^{2x}}{5} \\ -\frac{e^{x}\cos(2x)}{2}+e^{x}\sin(2x)+\frac{5e^{x}}{2}-2e^{2x} & \frac{9e^{x}\cos(2x)}{20}-\frac{19e^{x}\sin(2x)}{10}-\frac{9e^{x}}{4}+\frac{14e^{2x}}{5} \\ \frac{3e^{x}\cos(2x)}{2}+2e^{x}\sin(2x)+\frac{5e^{x}}{2}-4e^{2x} & -\frac{67e^{x}\cos(2x)}{20}-\frac{14e^{x}\sin(2x)}{5}-\frac{9e^{x}}{4}+\frac{28e^{2x}}{5} \\ \frac{5e^{x}}{2}-8e^{2x}+\frac{11e^{x}\cos(2x)}{2}-e^{x}\sin(2x) & -\frac{9e^{x}}{4}+\frac{56e^{2x}}{5}-\frac{179e^{x}\cos(2x)}{20}+\frac{39e^{x}\sin(2x)}{10} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}^{\prime}(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- Cancel like terms

  $\Phi(x)\cdot\vec{v}^{\prime}(x)=\vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}^{\prime}(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_{0}^{x}\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_{p}(x)=\Phi(x)\cdot\left(\int_{0}^{x}\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_{p}(x)=\left[\begin{array}{c}-\frac{\left(\left(x^{2}+x+\frac{6}{5}\right)\sin(2x)-\frac{12\,\mathrm{e}^{x}}{5}+\frac{2\cos(2x)}{5}+2\right)\mathrm{e}^{x}}{4}\\ -\frac{\left(x^{2}+x+\frac{7}{5}\right)\mathrm{e}^{x}\cos(2x)}{2}-\frac{\left(x^{2}+3x+\frac{7}{5}\right)\mathrm{e}^{x}\sin(2x)}{4}-\frac{\mathrm{e}^{x}}{2}+\frac{6\,\mathrm{e}^{2x}}{5}\\ -\left(x^{2}+3x+\frac{19}{10}\right)\mathrm{e}^{x}\cos\left(2x\right)+\frac{\mathrm{e}^{x}\left(15x^{2}-5x+6\right)\sin(2x)}{20}-\frac{\mathrm{e}^{x}}{2}+\frac{12\,\mathrm{e}^{2x}}{5}\\ \frac{\left(x^{2}-11x-\frac{43}{5}\right)\mathrm{e}^{x}\cos(2x)}{2}+\frac{11\left(x^{2}+\frac{29}{11}x+\frac{7}{5}\right)\mathrm{e}^{x}\sin(2x)}{4}-\frac{\mathrm{e}^{x}}{2}+\frac{24\,\mathrm{e}^{2x}}{5}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)+\left[\begin{array}{c}-\frac{\left(\left(x^{2}+x+\frac{6}{5}\right)\sin(2x)-\frac{12\,\mathrm{e}^{x}}{5}+\frac{2\cos(2x)}{5}+2\right)\mathrm{e}^{x}}{4}\\ -\frac{\left(x^{2}+x+\frac{7}{5}\right)\mathrm{e}^{x}\cos(2x)}{2}-\frac{\left(x^{2}+3x+\frac{7}{5}\right)\mathrm{e}^{x}\sin(2x)}{4}-\frac{\mathrm{e}^{x}}{2}\\ -\left(x^{2}+3x+\frac{19}{10}\right)\mathrm{e}^{x}\cos\left(2x\right)+\frac{\mathrm{e}^{x}\left(15x^{2}-5x+6\right)\sin(2x)}{20}\\ \frac{\left(x^{2}-11x-\frac{43}{5}\right)\mathrm{e}^{x}\cos(2x)}{2}+\frac{11\left(x^{2}+\frac{29}{11}x+\frac{7}{5}\right)\mathrm{e}^{x}\sin(2x)}{4}-\frac{\mathrm{e}^{x}}{2}\end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y=-\frac{\left(x^{2}+x+\frac{8}{125}c_{3}-\frac{44}{125}c_{4}+\frac{6}{5}\right)\mathrm{e}^{x}\sin(2x)}{4}-\frac{11\left(c_{3}+\frac{2c_{4}}{11}+\frac{25}{22}\right)\mathrm{e}^{x}\cos(2x)}{125}+\frac{(24+5c_{2})\mathrm{e}^{2x}}{40}+\frac{\mathrm{e}^{x}(-1+2c_{1})}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve(1*diff(y(x),x$4)-5*diff(y(x),x$3)+13*diff(y(x),x$2)-19*diff(y(x),x)+10*y(x)=exp(x)*((
```

$$y(x) = -\frac{e^{x}\left(x^{2}+x-4c_{4}+\frac{23}{4}\right)\sin\left(2x\right)}{4}+\frac{\left(4c_{3}+3\right)e^{x}\cos\left(2x\right)}{4}+c_{2}e^{2x}+\frac{e^{x}\left(2c_{1}+7\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 172

```
DSolve[1*y''''[x]-5*y'''[x]+13*y''[x]-19*y'[x]-10*y[x]==Exp[x]*((7+8*x)*Cos[2*x]+(8-4*x)*Sin
```

$$\begin{aligned} y(x) \to\ & c_1 \exp\left(x\text{Root}\left[\#1^4-5\#1^3+13\#1^2-19\#1-10\&,1\right]\right) \\ & + c_3 \exp\left(x\text{Root}\left[\#1^4-5\#1^3+13\#1^2-19\#1-10\&,3\right]\right) \\ & + c_4 \exp\left(x\text{Root}\left[\#1^4-5\#1^3+13\#1^2-19\#1-10\&,4\right]\right) \\ & + c_2 \exp\left(x\text{Root}\left[\#1^4-5\#1^3+13\#1^2-19\#1-10\&,2\right]\right) \\ & - \frac{1}{100}e^x(-20x\sin(2x)+64\sin(2x)+40x\cos(2x)+67\cos(2x)) \end{aligned}$$

19.45 problem section 9.3, problem 45

Internal problem ID [1542]
Internal file name [`OUTPUT/1543_Sunday_June_05_2022_02_21_34_AM_62348847/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 45.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y''' + 8y'' + 8y' + 4y = -2\,\mathrm{e}^{x}(\cos(x) - \sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' + 8y'' + 8y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -1 - i$$
$$\lambda_2 = -1 + i$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(-1-i)x}c_1 + x\,e^{(-1-i)x}c_2 + e^{(-1+i)x}c_3 + x\,e^{(-1+i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{(-1-i)x}$$
$$y_2 = e^{(-1-i)x}x$$
$$y_3 = e^{(-1+i)x}$$
$$y_4 = x\,e^{(-1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y''' + 8y'' + 8y' + 4y = -2\,e^x(\cos(x) - \sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2\,e^x(\cos(x) - \sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,e^x, \sin(x)\,e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{(-1+i)x}, e^{(-1-i)x}x, e^{(-1-i)x}, e^{(-1+i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x)\,e^x + A_2\sin(x)\,e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-32A_1\sin(x)\,e^x + 32A_2\cos(x)\,e^x = -2\,e^x(\cos(x) - \sin(x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{16}, A_2 = -\frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,e^x}{16} - \frac{\sin(x)\,e^x}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{(-1-i)x} c_1 + x\,e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3 + x\,e^{(-1+i)x} c_4\right) + \left(-\frac{\cos(x)\,e^x}{16} - \frac{\sin(x)\,e^x}{16}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 x + c_1)\,e^{(-1-i)x} + e^{(-1+i)x}(c_4 x + c_3) - \frac{\cos(x)\,e^x}{16} - \frac{\sin(x)\,e^x}{16}$$

Summary

The solution(s) found are the following

$$y = (c_2 x + c_1)\,e^{(-1-i)x} + e^{(-1+i)x}(c_4 x + c_3) - \frac{\cos(x)\,e^x}{16} - \frac{\sin(x)\,e^x}{16} \tag{1}$$

Verification of solutions

$$y = (c_2 x + c_1)\,e^{(-1-i)x} + e^{(-1+i)x}(c_4 x + c_3) - \frac{\cos(x)\,e^x}{16} - \frac{\sin(x)\,e^x}{16}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(1*diff(y(x),x$4)+4*diff(y(x),x$3)+8*diff(y(x),x$2)+8*diff(y(x),x)+4*y(x)=-2*exp(x)*(c
```

$$y(x) = ((c_3x + c_1)\cos(x) + \sin(x)(c_4x + c_2))\,\mathrm{e}^{-x} - \frac{(\sin(x) + \cos(x))\,\mathrm{e}^{x}}{16}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 54

```
DSolve[1*y''''[x]+4*y'''[x]+8*y''[x]+8*y'[x]+4*y[x]==-2*Exp[x]*(Cos[1*x]-Sin[1*x]),y[x],x,In
```

$$y(x) \to \frac{1}{16}e^{-x}\left(\left(-e^{2x} + 16(c_4x + c_3)\right)\cos(x) - \left(e^{2x} - 16(c_2x + c_1)\right)\sin(x)\right)$$

19.46 problem section 9.3, problem 46

Internal problem ID [1543]
Internal file name [`OUTPUT/1544_Sunday_June_05_2022_02_21_37_AM_81282288/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 46.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 8y''' + 32y'' - 64y' + 64y = e^{2x}(\cos(2x) - \sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 8y''' + 32y'' - 64y' + 64y = 0$$

The characteristic equation is

$$\lambda^4 - 8\lambda^3 + 32\lambda^2 - 64\lambda + 64 = 0$$

The roots of the above equation are

$$\lambda_1 = 2 - 2i$$
$$\lambda_2 = 2 + 2i$$
$$\lambda_3 = 2 - 2i$$
$$\lambda_4 = 2 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(2-2i)x}c_1 + x\,\mathrm{e}^{(2-2i)x}c_2 + \mathrm{e}^{(2+2i)x}c_3 + x\,\mathrm{e}^{(2+2i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{(2-2i)x} \\ y_2 &= x\,\mathrm{e}^{(2-2i)x} \\ y_3 &= \mathrm{e}^{(2+2i)x} \\ y_4 &= x\,\mathrm{e}^{(2+2i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 8y''' + 32y'' - 64y' + 64y = \mathrm{e}^{2x}(\cos(2x) - \sin(2x))$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{(2-2i)x} & x\,\mathrm{e}^{(2-2i)x} & \mathrm{e}^{(2+2i)x} & x\,\mathrm{e}^{(2+2i)x} \\ (2-2i)\,\mathrm{e}^{(2-2i)x} & -2\left(-\frac{1}{2} + (-1+i)\,x\right)\mathrm{e}^{(2-2i)x} & (2+2i)\,\mathrm{e}^{(2+2i)x} & (1+(2+2i)\,x)\,\mathrm{e}^{(2+ \\ -8i\mathrm{e}^{(2-2i)x} & (-8ix - 4i + 4)\,\mathrm{e}^{(2-2i)x} & 8i\mathrm{e}^{(2+2i)x} & (8ix + 4i + 4)\,\mathrm{e}^{(2+2 \\ (-16-16i)\,\mathrm{e}^{(2-2i)x} & -16\left(\frac{3i}{2} + (1+i)\,x\right)\mathrm{e}^{(2-2i)x} & (-16+16i)\,\mathrm{e}^{(2+2i)x} & 16\left(\frac{3i}{2} + (-1+i)\,x\right)\mathrm{e}^{(\end{bmatrix}$$

$$|W| = 256\,\mathrm{e}^{(4+4i)x}\mathrm{e}^{(4-4i)x}$$

The determinant simplifies to

$$|W| = 256\,\mathrm{e}^{8x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} x\,\mathrm{e}^{(2-2i)x} & \mathrm{e}^{(2+2i)x} & x\,\mathrm{e}^{(2+2i)x} \\ -2\left(-\frac{1}{2}+(-1+i)\,x\right)\mathrm{e}^{(2-2i)x} & (2+2i)\,\mathrm{e}^{(2+2i)x} & (1+(2+2i)\,x)\,\mathrm{e}^{(2+2i)x} \\ (-8ix-4i+4)\,\mathrm{e}^{(2-2i)x} & 8i\mathrm{e}^{(2+2i)x} & (8ix+4i+4)\,\mathrm{e}^{(2+2i)x} \end{bmatrix}$$
$$= -8\,\mathrm{e}^{(6+2i)x}(i+2x)$$

$$W_2(x) = \det\begin{bmatrix} \mathrm{e}^{(2-2i)x} & \mathrm{e}^{(2+2i)x} & x\,\mathrm{e}^{(2+2i)x} \\ (2-2i)\,\mathrm{e}^{(2-2i)x} & (2+2i)\,\mathrm{e}^{(2+2i)x} & (1+(2+2i)\,x)\,\mathrm{e}^{(2+2i)x} \\ -8i\mathrm{e}^{(2-2i)x} & 8i\mathrm{e}^{(2+2i)x} & (8ix+4i+4)\,\mathrm{e}^{(2+2i)x} \end{bmatrix}$$
$$= -16\,\mathrm{e}^{(6+2i)x}$$

$$W_3(x) = \det\begin{bmatrix} \mathrm{e}^{(2-2i)x} & x\,\mathrm{e}^{(2-2i)x} & x\,\mathrm{e}^{(2+2i)x} \\ (2-2i)\,\mathrm{e}^{(2-2i)x} & -2\left(-\frac{1}{2}+(-1+i)\,x\right)\mathrm{e}^{(2-2i)x} & (1+(2+2i)\,x)\,\mathrm{e}^{(2+2i)x} \\ -8i\mathrm{e}^{(2-2i)x} & (-8ix-4i+4)\,\mathrm{e}^{(2-2i)x} & (8ix+4i+4)\,\mathrm{e}^{(2+2i)x} \end{bmatrix}$$
$$= 8\,\mathrm{e}^{(6-2i)x}(i-2x)$$

$$W_4(x) = \det\begin{bmatrix} \mathrm{e}^{(2-2i)x} & x\,\mathrm{e}^{(2-2i)x} & \mathrm{e}^{(2+2i)x} \\ (2-2i)\,\mathrm{e}^{(2-2i)x} & -2\left(-\frac{1}{2}+(-1+i)\,x\right)\mathrm{e}^{(2-2i)x} & (2+2i)\,\mathrm{e}^{(2+2i)x} \\ -8i\mathrm{e}^{(2-2i)x} & (-8ix-4i+4)\,\mathrm{e}^{(2-2i)x} & 8i\mathrm{e}^{(2+2i)x} \end{bmatrix}$$
$$= -16\,\mathrm{e}^{(6-2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(\mathrm{e}^{2x}(\cos(2x) - \sin(2x))\right)\left(-8\,\mathrm{e}^{(6+2i)x}(i+2x)\right)}{(1)\left(256\,\mathrm{e}^{8x}\right)}\,dx \\
&= -\int \frac{-8\,\mathrm{e}^{2x}(\cos(2x)-\sin(2x))\,\mathrm{e}^{(6+2i)x}(i+2x)}{256\,\mathrm{e}^{8x}}\,dx \\
&= -\int \left(-\frac{(i+2x)(\cos(2x)-\sin(2x))\,\mathrm{e}^{2ix}}{32}\right)dx \\
&= -\left(\int -\frac{(i+2x)(\cos(2x)-\sin(2x))\,\mathrm{e}^{2ix}}{32}dx\right) \\
&= -\left(\int -\frac{(i+2x)(\cos(2x)-\sin(2x))\,\mathrm{e}^{2ix}}{32}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(\mathrm{e}^{2x}(\cos(2x) - \sin(2x))\right)\left(-16\,\mathrm{e}^{(6+2i)x}\right)}{(1)\left(256\,\mathrm{e}^{8x}\right)}\,dx \\
&= \int \frac{-16\,\mathrm{e}^{2x}(\cos(2x)-\sin(2x))\,\mathrm{e}^{(6+2i)x}}{256\,\mathrm{e}^{8x}}\,dx \\
&= \int \left(-\frac{(\cos(2x)-\sin(2x))\,\mathrm{e}^{2ix}}{16}\right)dx \\
&= -\frac{x}{32} + \frac{ix}{32} - \frac{\mathrm{e}^{4ix}}{128} + \frac{i\mathrm{e}^{4ix}}{128} \\
&= -\frac{x}{32} + \frac{ix}{32} - \frac{\mathrm{e}^{4ix}}{128} + \frac{i\mathrm{e}^{4ix}}{128}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\mathrm{e}^{2x} (\cos(2x) - \sin(2x))\right) \left(8\,\mathrm{e}^{(6-2i)x} (i - 2x)\right)}{(1) \left(256\,\mathrm{e}^{8x}\right)}\, dx \\
&= -\int \frac{8\,\mathrm{e}^{2x} (\cos(2x) - \sin(2x))\,\mathrm{e}^{(6-2i)x} (i - 2x)}{256\,\mathrm{e}^{8x}}\, dx \\
&= -\int \left(\frac{(i - 2x) (\cos(2x) - \sin(2x))\,\mathrm{e}^{-2ix}}{32}\right) dx \\
&= -\left(\int \frac{(i - 2x) (\cos(2x) - \sin(2x))\,\mathrm{e}^{-2ix}}{32} dx\right) \\
&= -\left(\int \frac{(i - 2x) (\cos(2x) - \sin(2x))\,\mathrm{e}^{-2ix}}{32} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\mathrm{e}^{2x} (\cos(2x) - \sin(2x))\right) \left(-16\,\mathrm{e}^{(6-2i)x}\right)}{(1) \left(256\,\mathrm{e}^{8x}\right)}\, dx \\
&= \int \frac{-16\,\mathrm{e}^{2x} (\cos(2x) - \sin(2x))\,\mathrm{e}^{(6-2i)x}}{256\,\mathrm{e}^{8x}}\, dx \\
&= \int \left(-\frac{(\cos(2x) - \sin(2x))\,\mathrm{e}^{-2ix}}{16}\right) dx \\
&= -\frac{x}{32} - \frac{ix}{32} - \frac{\mathrm{e}^{-4ix}}{128} - \frac{i\mathrm{e}^{-4ix}}{128} \\
&= -\frac{x}{32} - \frac{ix}{32} - \frac{\mathrm{e}^{-4ix}}{128} - \frac{i\mathrm{e}^{-4ix}}{128}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\left(\int -\frac{(i + 2x) (\cos(2x) - \sin(2x))\,\mathrm{e}^{2ix}}{32} dx\right)\right) \left(\mathrm{e}^{(2-2i)x}\right) \\
&+ \left(-\frac{x}{32} + \frac{ix}{32} - \frac{\mathrm{e}^{4ix}}{128} + \frac{i\mathrm{e}^{4ix}}{128}\right) \left(x\,\mathrm{e}^{(2-2i)x}\right) \\
&+ \left(-\left(\int \frac{(i - 2x) (\cos(2x) - \sin(2x))\,\mathrm{e}^{-2ix}}{32} dx\right)\right) \left(\mathrm{e}^{(2+2i)x}\right) \\
&+ \left(-\frac{x}{32} - \frac{ix}{32} - \frac{\mathrm{e}^{-4ix}}{128} - \frac{i\mathrm{e}^{-4ix}}{128}\right) \left(x\,\mathrm{e}^{(2+2i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = -\frac{\left(\int (i-2x)\left(\cos(2x)-\sin(2x)\right)\mathrm{e}^{-2ix}dx\right)\mathrm{e}^{(2+2i)x}}{32}+\frac{\left(\int (i+2x)\left(\cos(2x)-\sin(2x)\right)\mathrm{e}^{2ix}dx\right)\mathrm{e}^{(2-2i)x}}{32}$$

Which simplifies to

$$y_p = -\frac{\left(-\left(\int (\cos(2x)-\sin(2x))\,(2x\cos(2x)-\sin(2x))\,dx\right)\cos(2x)-\left(\int (\cos(2x)-\sin(2x))\,(\cos(2x)\right.}{16}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{(2-2i)x}c_1 + x\,\mathrm{e}^{(2-2i)x}c_2 + \mathrm{e}^{(2+2i)x}c_3 + x\,\mathrm{e}^{(2+2i)x}c_4\right) \\
&\quad + \left(-\frac{\left(-\left(\int (\cos(2x)-\sin(2x))\,(2x\cos(2x)-\sin(2x))\,dx\right)\cos(2x)-\left(\int (\cos(2x)-\sin(2x))\,(\cos(2\right.}{16}\right.
\end{aligned}$$

Which simplifies to

$$y = (c_2x+c_1)\,\mathrm{e}^{(2-2i)x}+\mathrm{e}^{(2+2i)x}(c_4x+c_3) - \frac{\left(-\left(\int (\cos(2x)-\sin(2x))\,(2x\cos(2x)-\sin(2x))\,dx\right)\cos(2x)-\left(\int (\cos(2x)-\sin(2x))\,(\cos(2x)+\right.}{16}$$

Summary

The solution(s) found are the following

$$y = (c_2x+c_1)\,\mathrm{e}^{(2-2i)x}+\mathrm{e}^{(2+2i)x}(c_4x+c_3) - \frac{\left(-\left(\int (\cos(2x)-\sin(2x))\,(2x\cos(2x)-\sin(2x))\,dx\right)\cos(2x)-\left(\int (\cos(2x)-\sin(2x))\,(\cos(2x)+\right.}{16} \tag{1}$$

Verification of solutions

$$y = (c_2x+c_1)\,\mathrm{e}^{(2-2i)x}+\mathrm{e}^{(2+2i)x}(c_4x+c_3) - \frac{\left(-\left(\int (\cos(2x)-\sin(2x))\,(2x\cos(2x)-\sin(2x))\,dx\right)\cos(2x)-\left(\int (\cos(2x)-\sin(2x))\,(\cos(2x)+\right.}{16}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve(1*diff(y(x),x$4)-8*diff(y(x),x$3)+32*diff(y(x),x$2)-64*diff(y(x),x)+64*y(x)=exp(2*x)*
```

$$y(x) = -\frac{e^{2x}\left(\left(x^2+\left(-32c_3-\frac{3}{2}\right)x-32c_1-\frac{41}{4}\right)\cos(2x)-\sin(2x)\left(x^2+\left(32c_4+\frac{1}{2}\right)x+32c_2-\frac{517}{18}\right)\right)}{32}$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 65

```
DSolve[1*y''''[x]-8*y'''[x]+32*y''[x]-64*y'[x]+64*y[x]==Exp[2*x]*(Cos[2*x]-Sin[2*x]),y[x],x,
```

$$y(x) \to \frac{1}{256}e^{2x}\left(\left(-8x^2+4(1+64c_4)x+5+256c_3\right)\cos(2x) + \left(8x^2+8(1+32c_2)x-1+256c_1\right)\sin(2x)\right)$$

19.47 problem section 9.3, problem 47

Internal problem ID [1544]
Internal file name [OUTPUT/1545_Sunday_June_05_2022_02_21_41_AM_51848980/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 47.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 8y''' + 26y'' - 40y' + 25y = \mathrm{e}^{2x}(3\cos(x) - (3x+1)\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 8y''' + 26y'' - 40y' + 25y = 0$$

The characteristic equation is

$$\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 25 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2 - i\\ \lambda_2 &= 2 + i\\ \lambda_3 &= 2 - i\\ \lambda_4 &= 2 + i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(2+i)x}c_1 + x\,e^{(2+i)x}c_2 + e^{(2-i)x}c_3 + x\,e^{(2-i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{(2+i)x} \\ y_2 &= x\,e^{(2+i)x} \\ y_3 &= e^{(2-i)x} \\ y_4 &= x\,e^{(2-i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 8y''' + 26y'' - 40y' + 25y = e^{2x}(3\cos(x) - (3x+1)\sin(x))$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{(2+i)x} & x\,e^{(2+i)x} & e^{(2-i)x} & x\,e^{(2-i)x} \\ (2+i)\,e^{(2+i)x} & e^{(2+i)x}(1+(2+i)\,x) & (2-i)\,e^{(2-i)x} & -(-1+(-2+i)\,x)\,e^{(2-i)x} \\ (3+4i)\,e^{(2+i)x} & (4+2i+(3+4i)\,x)\,e^{(2+i)x} & (3-4i)\,e^{(2-i)x} & -4\,e^{(2-i)x}\left(-1+\frac{i}{2}+\left(-\frac{3}{4}+i \right.\right. \\ (2+11i)\,e^{(2+i)x} & (9+12i+(2+11i)\,x)\,e^{(2+i)x} & (2-11i)\,e^{(2-i)x} & (9-12i+(2-11i)\,x)\,e^{(2-i} \end{bmatrix}$$

$$|W| = 16\,e^{(4+2i)x}e^{(4-2i)x}$$

The determinant simplifies to

$$|W| = 16\,\mathrm{e}^{8x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} x\,\mathrm{e}^{(2+i)x} & \mathrm{e}^{(2-i)x} & x\,\mathrm{e}^{(2-i)x} \\ \mathrm{e}^{(2+i)x}(1+(2+i)\,x) & (2-i)\,\mathrm{e}^{(2-i)x} & -(-1+(-2+i)\,x)\,\mathrm{e}^{(2-i)x} \\ (4+2i+(3+4i)\,x)\,\mathrm{e}^{(2+i)x} & (3-4i)\,\mathrm{e}^{(2-i)x} & -4\,\mathrm{e}^{(2-i)x}\left(-1+\frac{i}{2}+\left(-\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{(6-i)x}(x-i)$$

$$W_2(x) = \det\begin{bmatrix} \mathrm{e}^{(2+i)x} & \mathrm{e}^{(2-i)x} & x\,\mathrm{e}^{(2-i)x} \\ (2+i)\,\mathrm{e}^{(2+i)x} & (2-i)\,\mathrm{e}^{(2-i)x} & -(-1+(-2+i)\,x)\,\mathrm{e}^{(2-i)x} \\ (3+4i)\,\mathrm{e}^{(2+i)x} & (3-4i)\,\mathrm{e}^{(2-i)x} & -4\,\mathrm{e}^{(2-i)x}\left(-1+\frac{i}{2}+\left(-\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{(6-i)x}$$

$$W_3(x) = \det\begin{bmatrix} \mathrm{e}^{(2+i)x} & x\,\mathrm{e}^{(2+i)x} & x\,\mathrm{e}^{(2-i)x} \\ (2+i)\,\mathrm{e}^{(2+i)x} & \mathrm{e}^{(2+i)x}(1+(2+i)\,x) & -(-1+(-2+i)\,x)\,\mathrm{e}^{(2-i)x} \\ (3+4i)\,\mathrm{e}^{(2+i)x} & (4+2i+(3+4i)\,x)\,\mathrm{e}^{(2+i)x} & -4\,\mathrm{e}^{(2-i)x}\left(-1+\frac{i}{2}+\left(-\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= -4\,\mathrm{e}^{(6+i)x}(x+i)$$

$$W_4(x) = \det\begin{bmatrix} \mathrm{e}^{(2+i)x} & x\,\mathrm{e}^{(2+i)x} & \mathrm{e}^{(2-i)x} \\ (2+i)\,\mathrm{e}^{(2+i)x} & \mathrm{e}^{(2+i)x}(1+(2+i)\,x) & (2-i)\,\mathrm{e}^{(2-i)x} \\ (3+4i)\,\mathrm{e}^{(2+i)x} & (4+2i+(3+4i)\,x)\,\mathrm{e}^{(2+i)x} & (3-4i)\,\mathrm{e}^{(2-i)x} \end{bmatrix}$$
$$= -4\,\mathrm{e}^{(6+i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3\int \frac{\left(\mathrm{e}^{2x}(3\cos(x)-(3x+1)\sin(x))\right)\left(-4\,\mathrm{e}^{(6-i)x}(x-i)\right)}{(1)\left(16\,\mathrm{e}^{8x}\right)}\,dx \\ &= -\int \frac{-4\,\mathrm{e}^{2x}(3\cos(x)-(3x+1)\sin(x))\,\mathrm{e}^{(6-i)x}(x-i)}{16\,\mathrm{e}^{8x}}\,dx \\ &= -\int \left(-\frac{(-x+i)\,(3\sin(x)\,x+\sin(x)-3\cos(x))\,\mathrm{e}^{-ix}}{4}\right)dx \\ &= -\left(\int -\frac{(-x+i)\,(3\sin(x)\,x+\sin(x)-3\cos(x))\,\mathrm{e}^{-ix}}{4}dx\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(e^{2x}(3\cos(x) - (3x+1)\sin(x))\right)\left(-4\, e^{(6-i)x}\right)}{(1)\left(16\, e^{8x}\right)}\, dx \\
&= \int \frac{-4\, e^{2x}(3\cos(x) - (3x+1)\sin(x))\, e^{(6-i)x}}{16\, e^{8x}}\, dx \\
&= \int \left(\frac{((3x+1)\sin(x) - 3\cos(x))\, e^{-ix}}{4}\right) dx \\
&= \int \frac{((3x+1)\sin(x) - 3\cos(x))\, e^{-ix}}{4} dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(e^{2x}(3\cos(x) - (3x+1)\sin(x))\right)\left(-4\, e^{(6+i)x}(x+i)\right)}{(1)\left(16\, e^{8x}\right)}\, dx \\
&= -\int \frac{-4\, e^{2x}(3\cos(x) - (3x+1)\sin(x))\, e^{(6+i)x}(x+i)}{16\, e^{8x}}\, dx \\
&= -\int \left(\frac{(3\sin(x)\, x + \sin(x) - 3\cos(x))\, e^{ix}(x+i)}{4}\right) dx \\
&= -\left(\int \frac{(3\sin(x)\, x + \sin(x) - 3\cos(x))\, e^{ix}(x+i)}{4} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(e^{2x}(3\cos(x) - (3x+1)\sin(x))\right)\left(-4\, e^{(6+i)x}\right)}{(1)\left(16\, e^{8x}\right)}\, dx \\
&= \int \frac{-4\, e^{2x}(3\cos(x) - (3x+1)\sin(x))\, e^{(6+i)x}}{16\, e^{8x}}\, dx \\
&= \int \left(\frac{((3x+1)\sin(x) - 3\cos(x))\, e^{ix}}{4}\right) dx \\
&= -\frac{3x}{8} + \frac{3ix^2}{16} + \frac{ix}{8} + \frac{\left(-\frac{1}{4} + \frac{3i}{8} - \frac{3x}{4}\right) e^{2ix}}{4} \\
&= -\frac{3x}{8} + \frac{3ix^2}{16} + \frac{ix}{8} + \frac{\left(-\frac{1}{4} + \frac{3i}{8} - \frac{3x}{4}\right) e^{2ix}}{4}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$y_p = \left(-\left(\int -\frac{(-x+i)\left(3\sin(x)x+\sin(x)-3\cos(x)\right)e^{-ix}}{4}dx\right)\right)\left(e^{(2+i)x}\right) + \left(\int \frac{((3x+1)\sin(x)-3\cos(x))e^{-ix}}{4}dx\right)\left(x\,e^{(2+i)x}\right) + \left(-\left(\int \frac{(3\sin(x)x+\sin(x)-3\cos(x))e^{ix}(x+i)}{4}dx\right)\right)\left(e^{(2-i)x}\right) + \left(-\frac{3x}{8}+\frac{3ix^2}{16}+\frac{ix}{8}+\frac{\left(-\frac{1}{4}+\frac{3i}{8}-\frac{3x}{4}\right)e^{2ix}}{4}\right)\left(x\,e^{(2-i)x}\right)$$

Therefore the particular solution is

$$y_p = -\frac{\left(\int -(-x+i)\left(3\sin(x)x+\sin(x)-3\cos(x)\right)e^{-ix}dx\right)e^{(2+i)x}}{4} + \frac{\left(\int -(3\sin(x)x+\sin(x)-3\cos(x\right.}{4}$$

Which simplifies to

$$y_p = \frac{9\left(-\frac{8\left(\int\left(-\sin(x)^2+3\left(x^2+\frac{1}{3}x+1\right)\cos(x)\sin(x)-3x\right)dx\right)(i\sin(x)+\cos(x))}{9} + \frac{8(-i\sin(x)+\cos(x))\left(\int(-3\sin(x)x-\sin(x)+3\cos(x))(x\,c\right.}{9}\right.}{}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{(2+i)x}c_1 + x\,e^{(2+i)x}c_2 + e^{(2-i)x}c_3 + x\,e^{(2-i)x}c_4\right) \\ &\quad + \left(\frac{9\left(-\frac{8\left(\int\left(-\sin(x)^2+3\left(x^2+\frac{1}{3}x+1\right)\cos(x)\sin(x)-3x\right)dx\right)(i\sin(x)+\cos(x))}{9} + \frac{8(-i\sin(x)+\cos(x))\left(\int(-3\sin(x)x-\sin(x)+3\cos(x)\right)}{9}\right.}{}\right. \end{aligned}$$

Which simplifies to

$$y = (c_4x+c_3)e^{(2-i)x} + e^{(2+i)x}(c_2x+c_1) + \frac{9\left(-\frac{8\left(\int\left(-\sin(x)^2+3\left(x^2+\frac{1}{3}x+1\right)\cos(x)\sin(x)-3x\right)dx\right)(i\sin(x)+\cos(x))}{9} + \frac{8(-i\sin(x)+\cos(x))\left(\int(-3\sin(x)x-\sin(x)+3\cos(x))(x\,c\right.}{9}\right.}{}$$

<u>Summary</u>

The solution(s) found are the following

$$y = (c_4x+c_3)e^{(2-i)x} + e^{(2+i)x}(c_2x+c_1) + \frac{9\left(-\frac{8\left(\int\left(-\sin(x)^2+3\left(x^2+\frac{1}{3}x+1\right)\cos(x)\sin(x)-3x\right)dx\right)(i\sin(x)+\cos(x))}{9} + \frac{8(-i\sin(x)+\cos(x))\left(\int(-3\sin(x)x-\sin(x)+3\cos(x))(x\,c\right.}{9}\right.}{} \tag{1}$$

Verification of solutions

$$y = (c_4x + c_3)\,e^{(2-i)x} + e^{(2+i)x}(c_2x + c_1) + \frac{9\left(-\frac{8\left(\int\left(-\sin(x)^2+3\left(x^2+\frac{1}{3}x+1\right)\cos(x)\sin(x)-3x\right)dx\right)(i\sin(x)+\cos(x))}{9} + \frac{8(-i\sin(x)+\cos(x))\left(\int(-3\sin(x)x-\sin(x)+3\cos(x))(x\,\mathrm{co}\right.}{9}\right.}{}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(1*diff(y(x),x$4)-8*diff(y(x),x$3)+26*diff(y(x),x$2)-40*diff(y(x),x)+25*y(x)=exp(2*x)*
```

$$y(x) = \frac{e^{2x}\left(\left(x^3 + x^2 + (8c_4 + 3)\,x + 8c_2 - 2\right)\sin(x) + 8\cos(x)\left(\left(c_3 + \frac{1}{4}\right)x + c_1 + \frac{9}{16}\right)\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 60

```
DSolve[1*y''''[x]-8*y'''[x]+26*y''[x]-40*y'[x]+25*y[x]==Exp[2*x]*(3*Cos[1*x]-(1+3*x)*Sin[1*x
```

$$y(x) \to \frac{1}{16}e^{2x}\left(\left(2x^3 + 2x^2 + (9 + 16c_2)x - 1 + 16c_1\right)\sin(x) + \left(2(1 + 8c_4)x + 3 + 16c_3\right)\cos(x)\right)$$

19.48 problem section 9.3, problem 48

Internal problem ID [1545]
Internal file name [`OUTPUT/1546_Sunday_June_05_2022_02_21_46_AM_7938600/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 48.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 4y'' + 5y' - 2y = e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + 5y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x e^x + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^x \\ y_2 &= x e^x \\ y_3 &= e^{2x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + 5y' - 2y = e^{2x} - 4 e^x - 2 \cos(x) + 4 \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} - 4 e^x - 2 \cos(x) + 4 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x\}, \{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

Since $x e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}, \{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}, \{x e^{2x}\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^x + A_2 x e^{2x} + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{x} + A_2\mathrm{e}^{2x} - 4A_3\sin(x) + 4A_4\cos(x) + 2A_3\cos(x) + 2A_4\sin(x) \\ = \mathrm{e}^{2x} - 4\,\mathrm{e}^{x} - 2\cos(x) + 4\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2\mathrm{e}^{x} + x\,\mathrm{e}^{2x} - \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2x\,\mathrm{e}^{x} + c_3\mathrm{e}^{2x}\right) + \left(2x^2\mathrm{e}^{x} + x\,\mathrm{e}^{2x} - \cos(x)\right) \end{aligned}$$

Which simplifies to

$$y = c_3\mathrm{e}^{2x} + \mathrm{e}^{x}(c_2x + c_1) + 2x^2\mathrm{e}^{x} + x\,\mathrm{e}^{2x} - \cos(x)$$

Summary

The solution(s) found are the following

$$y = c_3\mathrm{e}^{2x} + \mathrm{e}^{x}(c_2x + c_1) + 2x^2\mathrm{e}^{x} + x\,\mathrm{e}^{2x} - \cos(x) \tag{1}$$

Verification of solutions

$$y = c_3\mathrm{e}^{2x} + \mathrm{e}^{x}(c_2x + c_1) + 2x^2\mathrm{e}^{x} + x\,\mathrm{e}^{2x} - \cos(x)$$

Verified OK.

19.48.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 5y' - 2y = e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x) + 4y_3(x) - 5y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x) + 4y_3(x) - 5y_2(x) + 2y_1($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^{2x} - 4\,e^{x} - 2\cos(x) + 4\sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

  $$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

  $$\vec{y}_2(x) = \mathrm{e}^{x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{x} & (x-1)\,e^{x} & \frac{e^{2x}}{4} \\ e^{x} & x\,e^{x} & \frac{e^{2x}}{2} \\ e^{x} & x\,e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} e^{x} & (x-1)\,e^{x} & \frac{e^{2x}}{4} \\ e^{x} & x\,e^{x} & \frac{e^{2x}}{2} \\ e^{x} & x\,e^{x} & e^{2x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -(x-1)\,e^{x} & -\frac{e^{2x}}{2}+\frac{(3x+1)e^{x}}{2} & \frac{e^{2x}}{2}+\frac{e^{x}(-x-1)}{2} \\ -x\,e^{x} & 2\,e^{x}+\frac{3x\,e^{x}}{2}-e^{2x} & -e^{x}-\frac{x\,e^{x}}{2}+e^{2x} \\ -x\,e^{x} & 2\,e^{x}+\frac{3x\,e^{x}}{2}-2\,e^{2x} & -e^{x}-\frac{x\,e^{x}}{2}+2\,e^{2x} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

$\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(x-6)e^{2x}}{2} + \frac{(2x^2+4x+7)e^x}{2} - \frac{\cos(x)}{2} \\ \frac{(2x-11)e^{2x}}{2} + \frac{(2x^2+8x+11)e^x}{2} + \frac{\sin(x)}{2} \\ \frac{(4x-19)e^{2x}}{2} + \frac{(2x^2+8x+19)e^x}{2} - \frac{3\sin(x)}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + \begin{bmatrix} \frac{(x-6)e^{2x}}{2} + \frac{(2x^2+4x+7)e^x}{2} - \frac{\cos(x)}{2} \\ \frac{(2x-11)e^{2x}}{2} + \frac{(2x^2+8x+11)e^x}{2} + \frac{\sin(x)}{2} \\ \frac{(4x-19)e^{2x}}{2} + \frac{(2x^2+8x+19)e^x}{2} - \frac{3\sin(x)}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(c_3+2x-12)e^{2x}}{4} + \frac{(7+2x^2+2(c_2+2)x+2c_1-2c_2)e^x}{2} - \frac{\cos(x)}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(1*diff(y(x),x$3)-4*diff(y(x),x$2)+5*diff(y(x),x)-2*y(x)=exp(2*x)-4*exp(x)-2*cos(x)+4*
```

$$y(x) = (x + c_2 - 2)\,\mathrm{e}^{2x} + \left(2x^2 + (c_3 + 4)\,x + c_1 + 4\right)\mathrm{e}^{x} - \cos(x)$$

✓ Solution by Mathematica

Time used: 0.373 (sec). Leaf size: 38

```
DSolve[1*y'''[x]-4*y''[x]+5*y'[x]-2*y[x]==Exp[2*x]-4*Exp[x]-2*Cos[x]+4*Sin[x],y[x],x,Include
```

$$y(x) \to -\cos(x) + e^x\left(2x^2 + (4 + c_2)x + e^x(x - 2 + c_3) + 4 + c_1\right)$$

19.49 problem section 9.3, problem 49

Internal problem ID [1546]
Internal file name [OUTPUT/1547_Sunday_June_05_2022_02_21_48_AM_59185175/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 49.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' + y' - y = 5\,e^{2x} + 2\,e^{x} - 4\cos(x) + 4\sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + y' - y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= \mathrm{e}^{-ix} \\ y_3 &= \mathrm{e}^{ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + y' - y = 5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ \mathrm{e}^{x} & -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \\ \mathrm{e}^{x} & -\mathrm{e}^{-ix} & -\mathrm{e}^{ix} \end{bmatrix}$$

$$|W| = 4i\mathrm{e}^{x}\mathrm{e}^{-ix}\mathrm{e}^{ix}$$

The determinant simplifies to

$$|W| = 4i\mathrm{e}^{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \end{bmatrix} \\ &= 2i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{ix} \\ \mathrm{e}^{x} & i\mathrm{e}^{ix} \end{bmatrix} \\ &= (-1+i)\,\mathrm{e}^{(1+i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{-ix} \\ \mathrm{e}^{x} & -i\mathrm{e}^{-ix} \end{bmatrix} \\ &= (-1-i)\,\mathrm{e}^{(1-i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x))\,(2i)}{(1)\,(4i\mathrm{e}^{x})}\, dx \\ &= \int \frac{2i(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x))}{4i\mathrm{e}^{x}}\, dx \\ &= \int \left(1 + 2\,\mathrm{e}^{-x}(-\cos(x) + \sin(x)) + \frac{5\,\mathrm{e}^{x}}{2}\right) dx \\ &= x - 2\sin(x)\,\mathrm{e}^{-x} + \frac{5\,\mathrm{e}^{x}}{2} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\left((-1+i)\,\mathrm{e}^{(1+i)x}\right)}{(1)\,(4i\mathrm{e}^{x})}\, dx \\
&= -\int \frac{(-1+i)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{(1+i)x}}{4i\mathrm{e}^{x}}\, dx \\
&= -\int \left(\left(\frac{1}{4} + \frac{i}{4}\right)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{ix}\right) dx \\
&= x + \frac{\mathrm{e}^{2ix}}{2} - \frac{\mathrm{e}^{(1+i)x}}{2} - \frac{3\,\mathrm{e}^{(2+i)x}}{4} - \frac{i\mathrm{e}^{(2+i)x}}{4} \\
&= x + \frac{\mathrm{e}^{2ix}}{2} - \frac{\mathrm{e}^{(1+i)x}}{2} - \frac{3\,\mathrm{e}^{(2+i)x}}{4} - \frac{i\mathrm{e}^{(2+i)x}}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\left((-1-i)\,\mathrm{e}^{(1-i)x}\right)}{(1)\,(4i\mathrm{e}^{x})}\, dx \\
&= \int \frac{(-1-i)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{(1-i)x}}{4i\mathrm{e}^{x}}\, dx \\
&= \int \left(\left(-\frac{1}{4} + \frac{i}{4}\right)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{-ix}\right) dx \\
&= \int \left(-\frac{1}{4} + \frac{i}{4}\right)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{-ix} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(x - 2\sin(x)\,\mathrm{e}^{-x} + \frac{5\,\mathrm{e}^{x}}{2}\right)(\mathrm{e}^{x}) \\
& + \left(x + \frac{\mathrm{e}^{2ix}}{2} - \frac{\mathrm{e}^{(1+i)x}}{2} - \frac{3\,\mathrm{e}^{(2+i)x}}{4} - \frac{i\mathrm{e}^{(2+i)x}}{4}\right)(\mathrm{e}^{-ix}) \\
& + \left(\int \left(-\frac{1}{4} + \frac{i}{4}\right)\left(5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x)\right)\mathrm{e}^{-ix} dx\right)(\mathrm{e}^{ix})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \mathrm{e}^{2x} + \frac{(4x+1)\cos(x)}{2} + (x-1)\,\mathrm{e}^{x} + \left(-2 - \frac{i}{2}\right)\sin(x)
$$

Which simplifies to

$$y_p = e^{2x} + \frac{(4x+1)\cos(x)}{2} + (x-1)e^x + \left(-2 - \frac{i}{2}\right)\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-ix} + e^{ix} c_3\right) + \left(e^{2x} + \frac{(4x+1)\cos(x)}{2} + (x-1)e^x + \left(-2 - \frac{i}{2}\right)\sin(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-ix} + e^{ix} c_3 + e^{2x} + \frac{(4x+1)\cos(x)}{2} + (x-1)e^x + \left(-2 - \frac{i}{2}\right)\sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2 e^{-ix} + e^{ix} c_3 + e^{2x} + \frac{(4x+1)\cos(x)}{2} + (x-1)e^x + \left(-2 - \frac{i}{2}\right)\sin(x)$$

Verified OK.

19.49.1 Maple step by step solution

Let's solve

$y''' - y'' + y' - y = 5e^{2x} + 2e^x - 4\cos(x) + 4\sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = 5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x) + y_3(x) - y_2(x) + y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x) + y_3(x) - y_2(x) + y_1(x$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 5\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} - 4\cos(x) + 4\sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(-\mathrm{I}\sin(x) + \cos(x)) \cdot \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \mathrm{I}\sin(x) - \cos(x) \\ \mathrm{I}(-\mathrm{I}\sin(x) + \cos(x)) \\ -\mathrm{I}\sin(x) + \cos(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\cos(x) \\ \sin(x) \\ \cos(x) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \sin(x) \\ \cos(x) \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} e^x & -\cos(x) & \sin(x) \\ e^x & \sin(x) & \cos(x) \\ e^x & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} e^x & -\cos(x) & \sin(x) \\ e^x & \sin(x) & \cos(x) \\ e^x & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \sin(x) & \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \cos(x) & \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} & -\sin(x) & \frac{e^x}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - Cancel like terms

$$\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c} -\frac{3\sin(x)}{2}+e^{2x}+x\,e^{x}+2x\cos(x)-\frac{7\,e^{x}}{2}+\frac{5\cos(x)}{2} \\ -\frac{5\sin(x)}{2}-2\sin(x)\,x+2\,e^{2x}+x\,e^{x}-\frac{5\,e^{x}}{2}+\frac{\cos(x)}{2} \\ -\frac{5\sin(x)}{2}+4\,e^{2x}+x\,e^{x}-2x\cos(x)-\frac{3\,e^{x}}{2}-\frac{5\cos(x)}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\left[\begin{array}{c} -\frac{3\sin(x)}{2}+e^{2x}+x\,e^{x}+2x\cos(x)-\frac{7\,e^{x}}{2}+\frac{5\cos(x)}{2} \\ -\frac{5\sin(x)}{2}-2\sin(x)\,x+2\,e^{2x}+x\,e^{x}-\frac{5\,e^{x}}{2}+\frac{\cos(x)}{2} \\ -\frac{5\sin(x)}{2}+4\,e^{2x}+x\,e^{x}-2x\cos(x)-\frac{3\,e^{x}}{2}-\frac{5\cos(x)}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=e^{2x}+\frac{(4x-2c_2+5)\cos(x)}{2}+\frac{(2x+2c_1-7)e^{x}}{2}+\frac{(2c_3-3)\sin(x)}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(1*diff(y(x),x$3)-1*diff(y(x),x$2)+1*diff(y(x),x)-1*y(x)=5*exp(2*x)+2*exp(x)-4*cos(x)+
```

$$y(x) = e^{2x} + (2x + c_1 + 2)\cos(x) + (c_2 + x - 1)e^x + (c_3 - 2)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.282 (sec). Leaf size: 35

```
DSolve[1*y'''[x]-1*y''[x]+1*y'[x]-1*y[x]==5*Exp[2*x]+2*Exp[x]-4*Cos[x]+4*Sin[x],y[x],x,Inclu
```

$$y(x) \to e^x(x + e^x - 1 + c_3) + (2x + 1 + c_1)\cos(x) + (-2 + c_2)\sin(x)$$

19.50 problem section 9.3, problem 50

Internal problem ID [1547]
Internal file name [`OUTPUT/1548_Sunday_June_05_2022_02_21_52_AM_69407526/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 50.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y' = -2 - 2x + 4\,\mathrm{e}^{x} - 6\,\mathrm{e}^{-x} + 96\,\mathrm{e}^{3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= 1 \\ y_3 &= e^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y' = -2 - 2x + 4 e^{x} - 6 e^{-x} + 96 e^{3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2 - 2x + 4 e^{x} - 6 e^{-x} + 96 e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{-x}\}, \{e^{3x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x e^{-x}\}, \{e^{3x}\}, \{1, x\}]$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x e^{-x}\}, \{e^{3x}\}, \{x, x^2\}]$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{x}\}, \{x e^{-x}\}, \{e^{3x}\}, \{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{x} + A_2 x e^{-x} + A_3 e^{3x} + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} + 2A_2\mathrm{e}^{-x} + 24A_3\mathrm{e}^{3x} - A_4 - 2A_5x = -2 - 2x + 4\,\mathrm{e}^{x} - 6\,\mathrm{e}^{-x} + 96\,\mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -3, A_3 = 4, A_4 = 2, A_5 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,\mathrm{e}^{x} - 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{3x} + 2x + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^{x}\right) + \left(2x\,\mathrm{e}^{x} - 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{3x} + 2x + x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^{x} + 2x\,\mathrm{e}^{x} - 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{3x} + 2x + x^2 \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^{x} + 2x\,\mathrm{e}^{x} - 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{3x} + 2x + x^2$$

Verified OK.

19.50.1 Maple step by step solution

Let's solve

$y''' - y' = -2 - 2x + 4\,\mathrm{e}^{x} - 6\,\mathrm{e}^{-x} + 96\,\mathrm{e}^{3x}$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -2 - 2x + 4\,e^{x} - 6\,e^{-x} + 96\,e^{3x} + y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -2 - 2x + 4\,e^{x} - 6\,e^{-x} + 96\,e^{3x} + y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -2 - 2x + 4\,e^{x} - 6\,e^{-x} + 96\,e^{3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -2 - 2x + 4\,e^{x} - 6\,e^{-x} + 96\,e^{3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right], \left[0, \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]\right], \left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} e^{-x} & 1 & e^{x} \\ -e^{-x} & 0 & e^{x} \\ e^{-x} & 0 & e^{x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & e^{x} \\ -e^{-x} & 0 & e^{x} \\ e^{-x} & 0 & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{2} - 1 + \frac{e^{x}}{2} \\ 0 & \frac{e^{x}}{2} + \frac{e^{-x}}{2} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} \\ 0 & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{x}}{2} + \frac{e^{-x}}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - Cancel like terms

$$\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$$

- Integrate to solve for $\overrightarrow{v}(x)$

$$\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

$$\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -3x\,\mathrm{e}^{-x} + 2x + 2x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{3x} + 44 - \frac{35\,\mathrm{e}^{-x}}{2} - \frac{61\,\mathrm{e}^{x}}{2} + x^2 \\ 3x\,\mathrm{e}^{-x} + 2x + 2x\,\mathrm{e}^{x} + 12\,\mathrm{e}^{3x} + 2 + \frac{29\,\mathrm{e}^{-x}}{2} - \frac{57\,\mathrm{e}^{x}}{2} \\ -3x\,\mathrm{e}^{-x} + 2x\,\mathrm{e}^{x} + 36\,\mathrm{e}^{3x} + 2 - \frac{23\,\mathrm{e}^{-x}}{2} - \frac{53\,\mathrm{e}^{x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} -3x\,\mathrm{e}^{-x} + 2x + 2x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{3x} + 44 - \frac{35\,\mathrm{e}^{-x}}{2} - \frac{61\,\mathrm{e}^{x}}{2} + x^2 \\ 3x\,\mathrm{e}^{-x} + 2x + 2x\,\mathrm{e}^{x} + 12\,\mathrm{e}^{3x} + 2 + \frac{29\,\mathrm{e}^{-x}}{2} - \frac{57\,\mathrm{e}^{x}}{2} \\ -3x\,\mathrm{e}^{-x} + 2x\,\mathrm{e}^{x} + 36\,\mathrm{e}^{3x} + 2 - \frac{23\,\mathrm{e}^{-x}}{2} - \frac{53\,\mathrm{e}^{x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-6x+2c_1-35)\mathrm{e}^{-x}}{2} + 4\,\mathrm{e}^{3x} + \frac{(4x+2c_3-61)\mathrm{e}^{x}}{2} + x^2 + 2x + c_2 + 44$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _b(_a)-2*_a-2+4*exp(_a)-6*exp
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(1*diff(y(x),x$3)-0*diff(y(x),x$2)-1*diff(y(x),x)-0*y(x)=-2*(1+x)+4*exp(x)-6*exp(-x)+9
```

$$y(x) = \frac{(-6x - 2c_2 - 9)\,e^{-x}}{2} + 4\,e^{3x} + (2x - 3 + c_1)\,e^{x} + x^2 + 2x + c_3$$

✓ Solution by Mathematica

Time used: 1.399 (sec). Leaf size: 49

```
DSolve[1*y'''[x]-0*y''[x]-1*y'[x]-0*y[x]==-2*(1+x)+4*Exp[x]-6*Exp[-x]+96*Exp[3*x],y[x],x,Inc
```

$$y(x) \to x(x+2) + 4e^{3x} + e^x(2x - 3 + c_1) - \frac{1}{2}e^{-x}(6x + 9 + 2c_2) + c_3$$

19.51 problem section 9.3, problem 51

Internal problem ID [1548]
Internal file name [OUTPUT/1549_Sunday_June_05_2022_02_21_54_AM_74796529/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 51.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 4y'' + 9y' - 10y = 10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + 9y' - 10y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 9\lambda - 10 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 1 - 2i$$
$$\lambda_3 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{(1+2i)x}$$
$$y_3 = e^{(1-2i)x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + 9y' - 10y = 10\,e^{2x} + 20\,e^{x} \sin(2x) - 10$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{2x} & e^{(1+2i)x} & e^{(1-2i)x} \\ 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \\ 4\,e^{2x} & (-3+4i)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$|W| = -20i e^{2x} e^{(1+2i)x} e^{(1-2i)x}$$

The determinant simplifies to

$$|W| = -20ie^{4x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{(1+2i)x} & e^{(1-2i)x} \\ (1+2i)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -4ie^{2x}$$

$$W_2(x) = \det \begin{bmatrix} e^{2x} & e^{(1-2i)x} \\ 2\,e^{2x} & (1-2i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= (-1-2i)\,e^{(3-2i)x}$$

$$W_3(x) = \det \begin{bmatrix} e^{2x} & e^{(1+2i)x} \\ 2\,e^{2x} & (1+2i)\,e^{(1+2i)x} \end{bmatrix}$$
$$= (-1+2i)\,e^{(3+2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(10\,e^{2x} + 20\,e^{x}\sin(2x) - 10\right)\left(-4ie^{2x}\right)}{(1)\left(-20ie^{4x}\right)}\,dx \\
&= \int \frac{-4i(10\,e^{2x} + 20\,e^{x}\sin(2x) - 10)\,e^{2x}}{-20ie^{4x}}\,dx \\
&= \int \left(\left(4\,e^{x}\sin(2x) + 2\,e^{2x} - 2\right)e^{-2x}\right)\,dx \\
&= \frac{4\,e^{-x}(-\sin(2x) - 2\cos(2x))}{5} + 2x + e^{-2x}
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^{1} \int \frac{\left(10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10\right)\left((-1-2i)\,\mathrm{e}^{(3-2i)x}\right)}{(1)\left(-20i\mathrm{e}^{4x}\right)}\, dx \\
&= -\int \frac{(-1-2i)\left(10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10\right)\mathrm{e}^{(3-2i)x}}{-20i\mathrm{e}^{4x}}\, dx \\
&= -\int \left(\left(1 - \frac{i}{2}\right)\left(-1 + \mathrm{e}^{2x} + 2\,\mathrm{e}^{x} \sin(2x)\right)\mathrm{e}^{(-1-2i)x}\right) dx \\
&= \frac{i\mathrm{e}^{(-1-2i)x}}{2} - \frac{2\,\mathrm{e}^{(1-2i)x}}{5} - \frac{3i\mathrm{e}^{(1-2i)x}}{10} + \frac{x}{2} + ix + \frac{\mathrm{e}^{-4ix}}{4} - \frac{i\mathrm{e}^{-4ix}}{8} \\
&= \frac{i\mathrm{e}^{(-1-2i)x}}{2} - \frac{2\,\mathrm{e}^{(1-2i)x}}{5} - \frac{3i\mathrm{e}^{(1-2i)x}}{10} + \frac{x}{2} + ix + \frac{\mathrm{e}^{-4ix}}{4} - \frac{i\mathrm{e}^{-4ix}}{8}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{\left(10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10\right)\left((-1+2i)\,\mathrm{e}^{(3+2i)x}\right)}{(1)\left(-20i\mathrm{e}^{4x}\right)}\, dx \\
&= \int \frac{(-1+2i)\left(10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10\right)\mathrm{e}^{(3+2i)x}}{-20i\mathrm{e}^{4x}}\, dx \\
&= \int \left(\left(-1 - \frac{i}{2}\right)\left(-1 + \mathrm{e}^{2x} + 2\,\mathrm{e}^{x} \sin(2x)\right)\mathrm{e}^{(-1+2i)x}\right) dx \\
&= -\frac{i\mathrm{e}^{(-1+2i)x}}{2} - \frac{2\,\mathrm{e}^{(1+2i)x}}{5} + \frac{3i\mathrm{e}^{(1+2i)x}}{10} + \frac{\mathrm{e}^{4ix}}{4} + \frac{i\mathrm{e}^{4ix}}{8} + \frac{x}{2} - ix \\
&= -\frac{i\mathrm{e}^{(-1+2i)x}}{2} - \frac{2\,\mathrm{e}^{(1+2i)x}}{5} + \frac{3i\mathrm{e}^{(1+2i)x}}{10} + \frac{\mathrm{e}^{4ix}}{4} + \frac{i\mathrm{e}^{4ix}}{8} + \frac{x}{2} - ix
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{4\,\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{5} + 2x + \mathrm{e}^{-2x}\right)\left(\mathrm{e}^{2x}\right) \\
&+ \left(\frac{i\mathrm{e}^{(-1-2i)x}}{2} - \frac{2\,\mathrm{e}^{(1-2i)x}}{5} - \frac{3i\mathrm{e}^{(1-2i)x}}{10} + \frac{x}{2} + ix + \frac{\mathrm{e}^{-4ix}}{4} - \frac{i\mathrm{e}^{-4ix}}{8}\right)\left(\mathrm{e}^{(1+2i)x}\right) \\
&+ \left(-\frac{i\mathrm{e}^{(-1+2i)x}}{2} - \frac{2\,\mathrm{e}^{(1+2i)x}}{5} + \frac{3i\mathrm{e}^{(1+2i)x}}{10} + \frac{\mathrm{e}^{4ix}}{4} + \frac{i\mathrm{e}^{4ix}}{8} + \frac{x}{2} - ix\right)\left(\mathrm{e}^{(1-2i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = 1 + \frac{(10x - 11)\,\mathrm{e}^{x} \cos(2x)}{10} + \frac{2(-2 + 5x)\,\mathrm{e}^{2x}}{5} + \frac{(-40x - 21)\,\mathrm{e}^{x} \sin(2x)}{20}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3\right) \\
&\quad + \left(1 + \frac{(10x-11)\,e^{x}\cos(2x)}{10} + \frac{2(-2+5x)\,e^{2x}}{5} + \frac{(-40x-21)\,e^{x}\sin(2x)}{20}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 e^{2x} + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3 + 1 + \frac{(10x-11)\,e^{x}\cos(2x)}{10} \\
& + \frac{2(-2+5x)\,e^{2x}}{5} + \frac{(-40x-21)\,e^{x}\sin(2x)}{20}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 e^{2x} + e^{(1+2i)x} c_2 + e^{(1-2i)x} c_3 + 1 + \frac{(10x-11)\,e^{x}\cos(2x)}{10} \\
& + \frac{2(-2+5x)\,e^{2x}}{5} + \frac{(-40x-21)\,e^{x}\sin(2x)}{20}
\end{aligned}
$$

Verified OK.

19.51.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 9y' - 10y = 10\,e^{2x} + 20\,e^{x}\sin(2x) - 10$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 20\,\mathrm{e}^{x} \sin(2x) + 10\,\mathrm{e}^{2x} + 4 y_3(x) - 9 y_2(x) + 10 y_1(x) - 10$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 20\,\mathrm{e}^{x} \sin(2x) + 10\,\mathrm{e}^{2x} + 4 y_3(x) - 9 y_2(x) + 10 y_1(x) - 10]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -9 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 10\,\mathrm{e}^{2x} + 20\,\mathrm{e}^{x} \sin(2x) - 10 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -9 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1 - 2\,\mathrm{I}, \begin{bmatrix} -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[1 + 2\,\mathrm{I}, \begin{bmatrix} -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - 2\,\mathrm{I}, \left[\begin{array}{c} -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{x} \cdot \left(\cos\left(2x\right) - \mathrm{I}\sin\left(2x\right)\right) \cdot \left[\begin{array}{c} -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{x} \cdot \left[\begin{array}{c} \left(-\frac{3}{25} + \frac{4\,\mathrm{I}}{25}\right)\left(\cos\left(2x\right) - \mathrm{I}\sin\left(2x\right)\right) \\ \left(\frac{1}{5} + \frac{2\,\mathrm{I}}{5}\right)\left(\cos\left(2x\right) - \mathrm{I}\sin\left(2x\right)\right) \\ \cos\left(2x\right) - \mathrm{I}\sin\left(2x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} -\frac{3\cos(2x)}{25} + \frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5} \\ \cos\left(2x\right) \end{array}\right], \overrightarrow{y}_3(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} \frac{3\sin(2x)}{25} + \frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5} \\ -\sin\left(2x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^x\left(-\frac{3\cos(2x)}{25} + \frac{4\sin(2x)}{25}\right) & e^x\left(\frac{3\sin(2x)}{25} + \frac{4\cos(2x)}{25}\right) \\ \frac{e^{2x}}{2} & e^x\left(\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right) & e^x\left(-\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5}\right) \\ e^{2x} & e^x\cos(2x) & -e^x\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^x\left(-\frac{3\cos(2x)}{25} + \frac{4\sin(2x)}{25}\right) & e^x\left(\frac{3\sin(2x)}{25} + \frac{4\cos(2x)}{25}\right) \\ \frac{e^{2x}}{2} & e^x\left(\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right) & e^x\left(-\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5}\right) \\ e^{2x} & e^x\cos(2x) & -e^x\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & -\frac{3}{25} & \frac{4}{25} \\ \frac{1}{2} & \frac{1}{5} & \frac{2}{5} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -e^x\sin(2x) + e^{2x} & \frac{7e^x\sin(2x)}{10} + \frac{2e^x\cos(2x)}{5} - \frac{2e^{2x}}{5} & -\frac{e^x\sin(2x)}{10} - \frac{e^x \text{co}}{} \\ -e^x\sin(2x) - 2e^x\cos(2x) + 2e^{2x} & -\frac{e^x\sin(2x)}{10} + \frac{9e^x\cos(2x)}{5} - \frac{4e^{2x}}{5} & \frac{2e^{2x}}{5} - \frac{2e^x\cos(2x)}{5} - \\ 4e^{2x} - 4e^x\cos(2x) + 3e^x\sin(2x) & -\frac{8e^{2x}}{5} + \frac{8e^x\cos(2x)}{5} - \frac{37e^x\sin(2x)}{10} & \frac{4e^{2x}}{5} + \frac{e^x\cos(2x)}{5} + \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

  $$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

  $$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

  $$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \left[\begin{array}{c} 1+\frac{(5x-4)\mathrm{e}^{x}\cos(2x)}{5}+\frac{(10x-1)\mathrm{e}^{2x}}{5}+\frac{(-20x-9)\mathrm{e}^{x}\sin(2x)}{10} \\ \frac{\mathrm{e}^{x}(40x\,\mathrm{e}^{x}-40x\sin(2x)-30x\cos(2x)+16\,\mathrm{e}^{x}-13\sin(2x)-16\cos(2x))}{10} \\ \frac{\mathrm{e}^{x}(80x\,\mathrm{e}^{x}+20x\sin(2x)-110x\cos(2x)+72\,\mathrm{e}^{x}-21\sin(2x)-72\cos(2x))}{10} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} 1+\frac{(5x-4)\mathrm{e}^{x}\cos(2x)}{5}+\frac{(10x-1)\mathrm{e}^{2x}}{5}+\frac{(-20x-9)\mathrm{e}^{x}\sin(2x)}{10} \\ \frac{\mathrm{e}^{x}(40x\,\mathrm{e}^{x}-40x\sin(2x)-30x\cos(2x)+16\,\mathrm{e}^{x}-13\sin(2x)-16\cos(2x))}{10} \\ \frac{\mathrm{e}^{x}(80x\,\mathrm{e}^{x}+20x\sin(2x)-110x\cos(2x)+72\,\mathrm{e}^{x}-21\sin(2x)-72\cos(2x))}{10} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y = 1 + \left(x - \frac{3c_2}{25} + \frac{4c_3}{25} - \frac{4}{5}\right)\mathrm{e}^{x}\cos(2x) - 2\left(x - \frac{2c_2}{25} - \frac{3c_3}{50} + \frac{9}{20}\right)\mathrm{e}^{x}\sin(2x) + \frac{(-4+40x+5c_1)\mathrm{e}^{2x}}{20}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(1*diff(y(x),x$3)-4*diff(y(x),x$2)+9*diff(y(x),x)-10*y(x)=10*exp(2*x)+20*exp(x)*sin(2*
```

$$y(x) = 1 + e^{x}\left(x - \frac{8}{5} + c_2\right)\cos(2x) + \frac{(-4 + 10x + 5c_1)\,e^{2x}}{5} - 2\left(x - \frac{c_3}{2} + \frac{13}{20}\right)e^{x}\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.992 (sec). Leaf size: 72

```
DSolve[1*y'''[x]-4*y''[x]+9*y'[x]-10*y[x]==10*Exp[2*x]+20*Exp[x]*Sin[2*x]-10,y[x],x,IncludeS
```

$$y(x) \to 2e^{2x}x - \frac{4e^{2x}}{5} + c_3e^{2x} + \frac{1}{10}e^{x}(10x - 11 + 10c_2)\cos(2x) - \frac{1}{20}e^{x}(40x + 21 - 20c_1)\sin(2x) + 1$$

19.52 problem section 9.3, problem 52

Internal problem ID [1549]
Internal file name [`OUTPUT/1550_Sunday_June_05_2022_02_21_57_AM_96974772/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 52.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 3y'' + 3y' + y = 12\,\mathrm{e}^{-x} + 9\cos(2x) - 13\sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= x^2 \mathrm{e}^{-x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 3y' + y = 12\,\mathrm{e}^{-x} + 9\cos(2x) - 13\sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{-x} + 9\cos(2x) - 13\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\}, \{\cos(2x), \sin(2x)\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-x}\}, \{\cos(2x), \sin(2x)\}]$$

Since $x^2 \mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{-x} x^3\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^{-x} x^3 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&6A_1\mathrm{e}^{-x} + 2A_2 \sin(2x) - 2A_3 \cos(2x) - 11A_2 \cos(2x) - 11A_3 \sin(2x) \\&= 12\,\mathrm{e}^{-x} + 9\cos(2x) - 13\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-x}x^3 - \cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\&= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + x^2\mathrm{e}^{-x}c_3\right) + \left(2\,\mathrm{e}^{-x}x^3 - \cos(2x) + \sin(2x)\right)\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + 2\,\mathrm{e}^{-x}x^3 - \cos(2x) + \sin(2x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + 2\,\mathrm{e}^{-x}x^3 - \cos(2x) + \sin(2x) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}\left(c_3x^2 + c_2x + c_1\right) + 2\,\mathrm{e}^{-x}x^3 - \cos(2x) + \sin(2x)$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 35

```
dsolve(1*diff(y(x),x$3)+3*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=12*exp(-x)+9*cos(2*x)-13*sin(2*
```

$$y(x) = \left(c_3 x^2 + 2x^3 + c_2 x + c_1\right) e^{-x} - \cos(2x) + \sin(2x)$$

✓ Solution by Mathematica
Time used: 0.295 (sec). Leaf size: 46

```
DSolve[1*y'''[x]+3*y''[x]+3*y'[x]+1*y[x]==12*Exp[-x]+9*Cos[2*x]-13*Sin[2*x],y[x],x,IncludeSi
```

$$y(x) \to e^{-x}\left(2x^3 + c_3 x^2 + e^x \sin(2x) - e^x \cos(2x) + c_2 x + c_1\right)$$

19.53 problem section 9.3, problem 53

Internal problem ID [1550]
Internal file name [`OUTPUT/1551_Sunday_June_05_2022_02_22_00_AM_73103628/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 53.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' - y' - y = 4\,\mathrm{e}^{-x}(1 - 6x) - 2x\cos(x) + 2(x + 1)\sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - y' - y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= \mathrm{e}^{x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - y' - y = 4\,\mathrm{e}^{-x}(1 - 6x) - 2x\cos(x) + 2(x + 1)\sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}(1 - 6x) - 2x\cos(x) + 2(x + 1)\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{x\cos(x), \sin(x)\,x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}\}, \{x\cos(x), \sin(x)\,x, \cos(x), \sin(x)\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3\}, \{x\cos(x), \sin(x)\,x, \cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x} + A_2 \mathrm{e}^{-x} x^3 + A_3 x\cos(x) + A_4 \sin(x)\,x + A_5 \cos(x) + A_6 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-12A_2\mathrm{e}^{-x}x - 4A_4\sin(x) - 2A_6\cos(x) + 2A_3x\sin(x) - 2A_3\sin(x) - 2A_5\cos(x) \\
&- 2A_6\sin(x) - 2A_3x\cos(x) + 2A_4\cos(x) - 4A_3\cos(x) + 2A_5\sin(x) - 2A_4\cos(x)\,x \\
&+ 6A_2\mathrm{e}^{-x} - 4A_1\mathrm{e}^{-x} - 2A_4\sin(x)\,x = 4\,\mathrm{e}^{-x}(1-6x) - 2x\cos(x) + 2(x+1)\sin(x)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 2, A_2 = 2, A_3 = 1, A_4 = 0, A_5 = 0, A_6 = -2]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = 2x^2\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^3 + x\cos(x) - 2\sin(x)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + c_3\mathrm{e}^{x}\right) + \left(2x^2\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^3 + x\cos(x) - 2\sin(x)\right)
\end{aligned}
$$

Which simplifies to

$$
y = (c_2x + c_1)\,\mathrm{e}^{-x} + c_3\mathrm{e}^{x} + 2x^2\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^3 + x\cos(x) - 2\sin(x)
$$

Summary

The solution(s) found are the following

$$
y = (c_2x + c_1)\,\mathrm{e}^{-x} + c_3\mathrm{e}^{x} + 2x^2\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^3 + x\cos(x) - 2\sin(x) \tag{1}
$$

Verification of solutions

$$
y = (c_2x + c_1)\,\mathrm{e}^{-x} + c_3\mathrm{e}^{x} + 2x^2\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}x^3 + x\cos(x) - 2\sin(x)
$$

Verified OK.

19.53.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = 4\,\mathrm{e}^{-x}(1-6x) - 2x\cos(x) + 2(x+1)\sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = y - 24x\,\mathrm{e}^{-x} - 2x\cos(x) + 2\sin(x)\,x + 4\,\mathrm{e}^{-x} + 2\sin(x) - y'' + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + y'' - y' - y = 2\sin(x)\,x - 2x\cos(x) - 24x\,\mathrm{e}^{-x} + 2\sin(x) + 4\,\mathrm{e}^{-x}$

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -24x\,\mathrm{e}^{-x} - 2x\cos(x) + 2\sin(x)\,x + 4\,\mathrm{e}^{-x} + 2\sin(x) - y_3(x) + y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -24x\,\mathrm{e}^{-x} - 2x\cos(x) + 2\sin(x)\,x + 4\,\mathrm{e}^{-x} + 2\sin(x) - y$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}\,'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2\sin(x)\,x - 2x\cos(x) - 24x\,\mathrm{e}^{-x} + 2\sin(x) + 4\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2\sin(x)\,x - 2x\cos(x) - 24x\,\mathrm{e}^{-x} + 2\sin(x) + 4\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{array}\right]-(-1)\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}1\\ -1\\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c}1\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue -1

  $$\overrightarrow{y}_2(x)=\mathrm{e}^{-x}\cdot\left(x\cdot\left[\begin{array}{c}1\\ -1\\ 1\end{array}\right]+\left[\begin{array}{c}1\\ 0\\ 0\end{array}\right]\right)$$

- Consider eigenpair

  $$\left[1,\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_3=\mathrm{e}^{x}\cdot\left[\begin{array}{c}1\\ 1\\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & (x+1)\,\mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & -x\,\mathrm{e}^{-x} & \mathrm{e}^{x} \\ \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & (x+1)\,\mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & -x\,\mathrm{e}^{-x} & \mathrm{e}^{x} \\ \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} (x+1)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} & -\frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x}}{2} \\ -x\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2} & -\frac{\mathrm{e}^{-x}}{2} + x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x}}{2} \\ x\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}(4x^3+4x^2+4x+1)\,\mathrm{e}^{-x}+2x\cos(x)-\mathrm{e}^{x}-4\sin(x)\\ (-4x^3+8x^2+4x+3)\,\mathrm{e}^{-x}-2\sin(x)\,x-2\cos(x)-\mathrm{e}^{x}\\ (4x^3-8x^2+8x+1)\,\mathrm{e}^{-x}-\mathrm{e}^{x}-2\sin(x)\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3+\left[\begin{array}{c}(4x^3+4x^2+4x+1)\,\mathrm{e}^{-x}+2x\cos(x)-\mathrm{e}^{x}-4\sin(x)\\ (-4x^3+8x^2+4x+3)\,\mathrm{e}^{-x}-2\sin(x)\,x-2\cos(x)-\mathrm{e}^{x}\\ (4x^3-8x^2+8x+1)\,\mathrm{e}^{-x}-\mathrm{e}^{x}-2\sin(x)\end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=(4x^3+4x^2+(c_2+4)\,x+c_1+c_2+1)\,\mathrm{e}^{-x}+(c_3-1)\,\mathrm{e}^{x}+2x\cos(x)-4\sin(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(1*diff(y(x),x$3)+1*diff(y(x),x$2)-1*diff(y(x),x)-1*y(x)=4*exp(-x)*(1-6*x)-2*x*cos(x)+
```

$$y(x) = \left(2x^3 + 2x^2 + (c_3 + 2)\, x + c_2 + 1\right) \mathrm{e}^{-x} + x\cos(x) + \mathrm{e}^{x} c_1 - 2\sin(x)$$

✓ Solution by Mathematica

Time used: 0.627 (sec). Leaf size: 54

```
DSolve[1*y'''[x]+1*y''[x]-1*y'[x]-1*y[x]==4*Exp[-x]*(1-6*x)-2*x*Cos[x]+2*(1+x)*Sin[x],y[x],x
```

$$y(x) \to e^{-x}\left(2x^3 + 2x^2 + 2x - 2e^x \sin(x) + e^x x \cos(x) + c_2 x + c_3 e^{2x} + 1 + c_1\right)$$

19.54 problem section 9.3, problem 54

Internal problem ID [1551]
Internal file name [`OUTPUT/1552_Sunday_June_05_2022_02_22_03_AM_53523338/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 54.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y'' + 4y = -12\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} + 10\cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{x} \\ y_4 &= e^{2x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y'' + 4y = -12\,e^{x} + 6\,e^{-x} + 10\cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-12\,e^{x} + 6\,e^{-x} + 10\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{-x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{-x}, e^{2x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x\,e^{-x}\}, \{\cos(x), \sin(x)\}]$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\}, \{x\,e^{-x}\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 x\,e^{-x} + A_3\cos(x) + A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1e^x + 6A_2e^{-x} + 10A_3\cos(x) + 10A_4\sin(x) = -12\,e^x + 6\,e^{-x} + 10\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,e^x + x\,e^{-x} + \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-x} + c_2e^{-2x} + c_3e^x + e^{2x}c_4\right) + \left(2x\,e^x + x\,e^{-x} + \cos(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + c_2e^{-2x} + c_3e^x + e^{2x}c_4 + 2x\,e^x + x\,e^{-x} + \cos(x) \tag{1}$$

Verification of solutions

$$y = c_1e^{-x} + c_2e^{-2x} + c_3e^x + e^{2x}c_4 + 2x\,e^x + x\,e^{-x} + \cos(x)$$

Verified OK.

19.54.1 Maple step by step solution

Let's solve

$y'''' - 5y'' + 4y = -12\,e^x + 6\,e^{-x} + 10\cos(x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -12\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} + 10\cos(x) + 5y_3(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -12\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} + 10\cos(x) + 5y_3(x) - 4y_1$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} + 10\cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} + 10\cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + \vec{y}_p(x)$

□ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{x} & \frac{e^{2x}}{8} \\ \frac{e^{-2x}}{4} & e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t…

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} -\frac{e^{-2x}}{8} & -e^{-x} & e^{x} & \frac{e^{2x}}{8} \\ \frac{e^{-2x}}{4} & e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{-x} & e^{x} & e^{2x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & 1 & \frac{1}{8} \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} \frac{(-e^{4x}+4e^{3x}+4e^{x}-1)e^{-2x}}{6} & -\frac{(e^{4x}-8e^{3x}+8e^{x}-1)e^{-2x}}{12} & -\frac{(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{6} & \frac{(e^{4x}-2e^{3x}+2e^{x}-1)\ldots}{12} \\ -\frac{(e^{4x}-2e^{3x}+2e^{x}-1)e^{-2x}}{3} & \frac{(-e^{4x}+4e^{3x}+4e^{x}-1)e^{-2x}}{6} & \frac{(2e^{4x}-e^{3x}+e^{x}-2)e^{-2x}}{6} & -\frac{(-e^{4x}+e^{3x}+e^{x}-1\ldots}{6} \\ \frac{2(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{3} & -\frac{(e^{4x}-2e^{3x}+2e^{x}-1)e^{-2x}}{3} & -\frac{(-4e^{4x}+e^{3x}+e^{x}-4)e^{-2x}}{6} & \frac{(2e^{4x}-e^{3x}+e^{x}-2)\ldots}{6} \\ -\frac{2(2e^{4x}-e^{3x}+e^{x}-2)e^{-2x}}{3} & \frac{2(-e^{4x}+e^{3x}+e^{x}-1)e^{-2x}}{3} & \frac{(8e^{4x}-e^{3x}+e^{x}-8)e^{-2x}}{6} & -\frac{(-4e^{4x}+e^{3x}+e^{x}-\ldots}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der…

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{-2x}\left(-e^{4x}+4x\,e^{3x}-2\,e^{3x}+2\,e^{2x}\cos(x)+2x\,e^{x}+1\right)}{2} \\ -\left((-1-2x)\,e^{3x}+e^{2x}\sin(x)+e^{4x}+1+(x-1)\,e^{x}\right)e^{-2x} \\ \left((2x+3)\,e^{3x}-e^{2x}\cos(x)-2\,e^{4x}+2+(-2+x)\,e^{x}\right)e^{-2x} \\ -\left((-2x-5)\,e^{3x}-e^{2x}\sin(x)+4\,e^{4x}+4+(x-3)\,e^{x}\right)e^{-2x} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + \begin{bmatrix} \frac{e^{-2x}\left(-e^{4x}+4x\,e^{3x}-2\,e^{3x}+2\,e^{2x}\cos(x)+2x\,e^{x}+1\right)}{2} \\ -\left((-1-2x)\,e^{3x}+e^{2x}\sin(x)+e^{4x}+1+(x-1)\,e^{x}\right)e^{-2x} \\ \left((2x+3)\,e^{3x}-e^{2x}\cos(x)-2\,e^{4x}+2+(-2+x)\,e^{x}\right)e^{-2x} \\ -\left((-2x-5)\,e^{3x}-e^{2x}\sin(x)+4\,e^{4x}+4+(x-3)\,e^{x}\right)e^{-2x} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = e^{-2x}\left((2x+c_3-1)\,e^{3x} + x\,e^{x} + \frac{c_4 e^{4x}}{8} - c_2 e^{x} + e^{2x}\cos(x) - \frac{e^{4x}}{2} - \frac{c_1}{8} + \frac{1}{2}\right)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$4)-0*diff(y(x),x$3)-5*diff(y(x),x$2)-0*diff(y(x),x)+4*y(x)=-12*exp(x)+6*e
```

$$y(x) = \mathrm{e}^{-2x}\left(\frac{(6x+3c_1+1)\,\mathrm{e}^{3x}}{3}+\cos{(x)}\,\mathrm{e}^{2x}+c_4\mathrm{e}^{4x}+\left(x+c_3-\frac{1}{6}\right)\mathrm{e}^{x}+c_2\right)$$

✓ Solution by Mathematica

Time used: 0.19 (sec). Leaf size: 58

```
DSolve[y''''[x]-0*y'''[x]-5*y''[x]-0*y'[x]+4*y[x]==-12*Exp[x]+6*Exp[-x]+10*Cos[x],y[x],x,Inc
```

$$y(x)\to \cos(x)+c_1e^{-2x}+\frac{1}{6}e^{-x}(6x-1+6c_2)+\frac{1}{3}e^{x}(6x+1+3c_3)+c_4e^{2x}$$

19.55 problem section 9.3, problem 55

Internal problem ID [1552]
Internal file name [`OUTPUT/1553_Sunday_June_05_2022_02_22_05_AM_61086094/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 55.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 4y''' + 11y'' - 14y' + 10y = -e^{x}(\sin(x) + 2\cos(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 11y'' - 14y' + 10y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 11\lambda^2 - 14\lambda + 10 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1 - 2i$$
$$\lambda_4 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(1+2i)x} c_1 + \mathrm{e}^{(1-i)x} c_2 + \mathrm{e}^{(1+i)x} c_3 + \mathrm{e}^{(1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{(1+2i)x}$$
$$y_2 = \mathrm{e}^{(1-i)x}$$
$$y_3 = \mathrm{e}^{(1+i)x}$$
$$y_4 = \mathrm{e}^{(1-2i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 4y''' + 11y'' - 14y' + 10y = -\mathrm{e}^{x}(\sin(x) + 2\cos(2x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{(1+2i)x} & \mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x} & \mathrm{e}^{(1-2i)x} \\ (1+2i)\,\mathrm{e}^{(1+2i)x} & (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x} & (1-2i)\,\mathrm{e}^{(1-2i)x} \\ (-3+4i)\,\mathrm{e}^{(1+2i)x} & -2i\mathrm{e}^{(1-i)x} & 2i\mathrm{e}^{(1+i)x} & (-3-4i)\,\mathrm{e}^{(1-2i)x} \\ (-11-2i)\,\mathrm{e}^{(1+2i)x} & (-2-2i)\,\mathrm{e}^{(1-i)x} & (-2+2i)\,\mathrm{e}^{(1+i)x} & (-11+2i)\,\mathrm{e}^{(1-2i)x} \end{bmatrix}$$

$$|W| = 72\,\mathrm{e}^{(1+2i)x}\mathrm{e}^{(1-i)x}\mathrm{e}^{(1+i)x}\mathrm{e}^{(1-2i)x}$$

The determinant simplifies to

$$|W| = 72\,e^{4x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} e^{(1-i)x} & e^{(1+i)x} & e^{(1-2i)x} \\ (1-i)\,e^{(1-i)x} & (1+i)\,e^{(1+i)x} & (1-2i)\,e^{(1-2i)x} \\ -2ie^{(1-i)x} & 2ie^{(1+i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= -6ie^{(3-2i)x}$$

$$W_2(x) = \det\begin{bmatrix} e^{(1+2i)x} & e^{(1+i)x} & e^{(1-2i)x} \\ (1+2i)\,e^{(1+2i)x} & (1+i)\,e^{(1+i)x} & (1-2i)\,e^{(1-2i)x} \\ (-3+4i)\,e^{(1+2i)x} & 2ie^{(1+i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= 12ie^{(3+i)x}$$

$$W_3(x) = \det\begin{bmatrix} e^{(1+2i)x} & e^{(1-i)x} & e^{(1-2i)x} \\ (1+2i)\,e^{(1+2i)x} & (1-i)\,e^{(1-i)x} & (1-2i)\,e^{(1-2i)x} \\ (-3+4i)\,e^{(1+2i)x} & -2ie^{(1-i)x} & (-3-4i)\,e^{(1-2i)x} \end{bmatrix}$$
$$= 12ie^{(3-i)x}$$

$$W_4(x) = \det\begin{bmatrix} e^{(1+2i)x} & e^{(1-i)x} & e^{(1+i)x} \\ (1+2i)\,e^{(1+2i)x} & (1-i)\,e^{(1-i)x} & (1+i)\,e^{(1+i)x} \\ (-3+4i)\,e^{(1+2i)x} & -2ie^{(1-i)x} & 2ie^{(1+i)x} \end{bmatrix}$$
$$= -6ie^{(3+2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3\int \frac{\left(-e^x(\sin(x)+2\cos(2x))\right)\left(-6ie^{(3-2i)x}\right)}{(1)\,(72\,e^{4x})}\,dx \\ &= -\int \frac{6ie^x(\sin(x)+2\cos(2x))\,e^{(3-2i)x}}{72\,e^{4x}}\,dx \\ &= -\int \left(\frac{i(\sin(x)+2\cos(2x))\,e^{-2ix}}{12}\right)dx \\ &= -\left(\int \frac{i(\sin(x)+2\cos(2x))\,e^{-2ix}}{12}dx\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(-\mathrm{e}^x(\sin(x)+2\cos(2x))\right)\left(12i\mathrm{e}^{(3+i)x}\right)}{(1)\left(72\,\mathrm{e}^{4x}\right)}\,dx \\
&= \int \frac{-12i\mathrm{e}^x(\sin(x)+2\cos(2x))\,\mathrm{e}^{(3+i)x}}{72\,\mathrm{e}^{4x}}\,dx \\
&= \int \left(-\frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{ix}}{6}\right)dx \\
&= \int -\frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{ix}}{6}dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(-\mathrm{e}^x(\sin(x)+2\cos(2x))\right)\left(12i\mathrm{e}^{(3-i)x}\right)}{(1)\left(72\,\mathrm{e}^{4x}\right)}\,dx \\
&= -\int \frac{-12i\mathrm{e}^x(\sin(x)+2\cos(2x))\,\mathrm{e}^{(3-i)x}}{72\,\mathrm{e}^{4x}}\,dx \\
&= -\int \left(-\frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{-ix}}{6}\right)dx \\
&= -\left(\int -\frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{-ix}}{6}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(-\mathrm{e}^x(\sin(x)+2\cos(2x))\right)\left(-6i\mathrm{e}^{(3+2i)x}\right)}{(1)\left(72\,\mathrm{e}^{4x}\right)}\,dx \\
&= \int \frac{6i\mathrm{e}^x(\sin(x)+2\cos(2x))\,\mathrm{e}^{(3+2i)x}}{72\,\mathrm{e}^{4x}}\,dx \\
&= \int \left(\frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{2ix}}{12}\right)dx \\
&= \int \frac{i(\sin(x)+2\cos(2x))\,\mathrm{e}^{2ix}}{12}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$y_p = \left(-\left(\int \frac{i(\sin(x)+2\cos(2x))\,e^{-2ix}}{12}dx\right)\right)\left(e^{(1+2i)x}\right) + \left(\int -\frac{i(\sin(x)+2\cos(2x))\,e^{ix}}{6}dx\right)\left(e^{(1-i)x}\right) + \left(-\left(\int -\frac{i(\sin(x)+2\cos(2x))\,e^{-ix}}{6}dx\right)\right)\left(e^{(1+i)x}\right) + \left(\int \frac{i(\sin(x)+2\cos(2x))\,e^{2ix}}{12}dx\right)\left(e^{(1-2i)x}\right)$$

Therefore the particular solution is

$$y_p = -\frac{i\left(2\left(\int(\sin(x)+2\cos(2x))\,e^{ix}dx\right)e^{(1-i)x} - \left(\int(\sin(x)+2\cos(2x))\,e^{2ix}dx\right)e^{(1-2i)x} + \left(\int(\sin(x)+\right.\right.}{12}$$

Which simplifies to

$$y_p = -\frac{e^{x}\left(\left(\int(\sin(x)+2\cos(2x))\sin(2x)\,dx\right)\cos(2x) + 2\left(\int(\sin(x)+2\cos(2x))\cos(x)\,dx\right)\sin(x) - 2\left(\right.\right.}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{(1+2i)x}c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3 + e^{(1-2i)x}c_4\right) \\ &\quad + \left(-\frac{e^{x}\left(\left(\int(\sin(x)+2\cos(2x))\sin(2x)\,dx\right)\cos(2x) + 2\left(\int(\sin(x)+2\cos(2x))\cos(x)\,dx\right)\sin(x) - \right.}{6}\right. \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = e^{(1+2i)x}c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3 + e^{(1-2i)x}c_4 - \frac{e^{x}\left(\left(\int(\sin(x)+2\cos(2x))\sin(2x)\,dx\right)\cos(2x) + 2\left(\int(\sin(x)+2\cos(2x))\cos(x)\,dx\right)\sin(x) - 2\left(\int\right.\right.}{6} \quad (1)$$

<u>Verification of solutions</u>

$$y = e^{(1+2i)x}c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3 + e^{(1-2i)x}c_4 - \frac{e^{x}\left(\left(\int(\sin(x)+2\cos(2x))\sin(2x)\,dx\right)\cos(2x) + 2\left(\int(\sin(x)+2\cos(2x))\cos(x)\,dx\right)\sin(x) - 2\left(\int\right.\right.}{6}$$

Verified OK.

19.55.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 11y'' - 14y' + 10y = -e^{x}(\sin(x) + 2\cos(2x))$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2\,e^{x}\cos(2x) - \sin(x)\,e^{x} + 4y_4(x) - 11y_3(x) + 14y_2(x) - 10y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2\,e^{x}\cos(2x) - \sin(x)\,e^{x} + 4y_4(x) - 11y_3$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 14 & -11 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2\,e^{x}\cos(2x) - \sin(x)\,e^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -2\,{\mathrm e}^{x} \cos\left(2 x\right)-\sin\left(x\right) {\mathrm e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 14 & -11 & 4 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}^{\prime}(x) = A\cdot \vec{y}(x)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[1-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125}+ \\ -\frac{3}{25}- \\ \frac{1}{5}-\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$${\mathrm e}^{\left(1-2\,\mathrm{I}\right) x}\cdot \left[\begin{array}{c} -\frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{11}{125} - \frac{2I}{125} \\ -\frac{3}{25} + \frac{4I}{25} \\ \frac{1}{5} + \frac{2I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \left(-\frac{11}{125} - \frac{2I}{125}\right)(\cos(2x) - I\sin(2x)) \\ \left(-\frac{3}{25} + \frac{4I}{25}\right)(\cos(2x) - I\sin(2x)) \\ \left(\frac{1}{5} + \frac{2I}{5}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{x} \cdot \begin{bmatrix} -\frac{11\cos(2x)}{125} - \frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25} + \frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} \frac{11\sin(2x)}{125} - \frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25} + \frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I, \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot \left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ \frac{\mathrm{I}}{2}\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ -\mathrm{I}\sin\left(x\right)+\cos\left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=e^{x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos\left(x\right) \end{array}\right], \vec{y}_{4}(x)=e^{x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4}+\frac{\cos(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2} \\ -\sin\left(x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_{p}($

$$\vec{y}(x)=c_{1}\vec{y}_{1}(x)+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)+\vec{y}_{p}(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{cccc} e^{x}\left(-\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\right) & e^{x}\left(\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\right) & e^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & e^{x}\left(\frac{\sin(x)}{4}+\frac{\mathrm{co}}{}\right. \\ e^{x}\left(-\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25}\right) & e^{x}\left(\frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25}\right) & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2} \\ e^{x}\left(\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\right) & e^{x}\left(-\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\right) & e^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x}{2}\right. \\ e^{x}\cos\left(2x\right) & -e^{x}\sin\left(2x\right) & \cos\left(x\right)e^{x} & -\sin\left(x\right)e \end{array}\right.$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^x\left(-\frac{11\cos(2x)}{125} - \frac{2\sin(2x)}{125}\right) & e^x\left(\frac{11\sin(2x)}{125} - \frac{2\cos(2x)}{125}\right) & e^x\left(-\frac{\cos(x)}{4} + \frac{\sin(x)}{4}\right) & e^x\left(\frac{\sin(x)}{4} + \frac{\mathrm{co}}{}\right. \\ e^x\left(-\frac{3\cos(2x)}{25} + \frac{4\sin(2x)}{25}\right) & e^x\left(\frac{3\sin(2x)}{25} + \frac{4\cos(2x)}{25}\right) & \frac{\sin(x)e^x}{2} & \frac{\cos(x)e^x}{2} \\ e^x\left(\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right) & e^x\left(-\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5}\right) & e^x\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x}{2}\right. \\ e^x\cos(2x) & -e^x\sin(2x) & \cos(x)\,e^x & -\sin(x)\,e \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^x(-2\cos(2x)-5\sin(x)+5\cos(x)+\sin(2x))}{3} & \frac{e^x(2\cos(2x)-2\sin(2x)-2\cos(x)+7\sin(x))}{3} & \frac{(-2\cos(x)+3\,\mathrm{s}}{} \\ -\frac{5e^x(-\sin(2x)+2\sin(x))}{3} & -\frac{e^x(2\cos(2x)+6\sin(2x)-9\sin(x)-5\cos(x))}{3} & \frac{e^x(4\cos(2x)+7\,\mathrm{si}}{} \\ \frac{10(\cos(x)-1)(2\cos(x)+\sin(x)+1)e^x}{3} & -\frac{2(\cos(x)-1)(14\cos(x)+2\sin(x)+7)e^x}{3} & -\frac{e^x(-18\cos(2x)+\mathrm{s}}{} \\ \frac{5e^x(4\cos(2x)-3\sin(2x)-4\cos(x))}{3} & -\frac{2e^x(9\cos(2x)-13\sin(2x)-9\cos(x)+5\sin(x))}{3} & \frac{e^x(16\cos(2x)-37\,\mathrm{s}}{} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(6x\cos(x)\sin(x)-2\cos(x)\sin(x)+8\cos(x)^2+3x\cos(x)-\sin(x)-4\cos(x)-4\right)e^x}{18} \\ \frac{(12x+4)e^x\cos(x)^2}{18} + \frac{(6\sin(x)x+3x-12\sin(x)-2)e^x\cos(x)}{18} + \frac{(-3\sin(x)x-6x+3\sin(x)-2)e^x}{18} \\ \frac{(12x-4)e^x\cos(x)^2}{9} + \frac{((-9x-7)e^x\sin(x)+2e^x)\cos(x)}{9} - \frac{\left(x-\frac{1}{3}\right)(\sin(x)+2)e^x}{3} \\ \frac{e^x\left(-2(x+1)\cos(x)^2-\frac{(33\sin(x)x+3x+8\sin(x)-3)\cos(x)}{3}-\sin(x)x+x-\frac{4\sin(x)}{3}+1\right)}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{\left(6x\cos(x)\sin(x)-2\cos(x)\sin(x)+8\cos(x)^2+3x\cos\right.}{18} \\ \frac{(12x+4)e^x\cos(x)^2}{18} + \frac{(6\sin(x)x+3x-12\sin(x)-2)e^x\cos(}{18} \\ \frac{(12x-4)e^x\cos(x)^2}{9} + \frac{((-9x-7)e^x\sin(x)+2e^x)\cos}{9} \\ \frac{e^x\left(-2(x+1)\cos(x)^2-\frac{(33\sin(x)x+3x+8\sin(x)-3)\cos}{3}\right.}{3} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^x\left(\left(-\frac{66c_1}{125}-\frac{12c_2}{125}+\frac{4}{3}\right)\cos(x)^2+\left(\left(x-\frac{12c_1}{125}+\frac{66c_2}{125}-\frac{1}{3}\right)\sin(x)+\frac{x}{2}-\frac{3c_3}{4}+\frac{3c_4}{4}-\frac{2}{3}\right)\cos(x)+\left(\frac{3c_3}{4}+\frac{3c_4}{4}-\frac{1}{6}\right)\sin(x)+\frac{33c_1}{125}+\frac{6c_2}{125}-\frac{2}{3}}{3}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+11*diff(y(x),x$2)-14*diff(y(x),x)+10*y(x)=-exp(x)*(si
```

$$y(x) = \frac{e^{x}\left(\left(6c_3 + \frac{7}{3}\right)\cos(2x) + (x + 6c_4)\sin(2x) + (x + 6c_1)\cos(x) + 6\left(\frac{1}{9} + c_2\right)\sin(x)\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 53

```
DSolve[y''''[x]-4*y'''[x]+11*y''[x]-14*y'[x]+10*y[x]==-Exp[x]*(Sin[x]+2*Cos[2*x]),y[x],x,Inc
```

$$y(x) \to \frac{1}{36}e^{x}((11+36c_2)\cos(2x)+(1+36c_3)\sin(x)+6\cos(x)(x+2(x+6c_1)\sin(x)+6c_4))$$

19.56 problem section 9.3, problem 56

Internal problem ID [1553]
Internal file name [`OUTPUT/1554_Sunday_June_05_2022_02_22_09_AM_77937519/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 56.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y''' - 3y'' - 4y' + 4y = 2(x+1)\,\mathrm{e}^{x} + \mathrm{e}^{-2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' - 3y'' - 4y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -2 \\
\lambda_3 &= 1 \\
\lambda_4 &= 1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + c_3 \mathrm{e}^{x} + c_4 x\,\mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-2x} x$$
$$y_3 = \mathrm{e}^{x}$$
$$y_4 = x\,\mathrm{e}^{x}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' - 3y'' - 4y' + 4y = 2(x+1)\,\mathrm{e}^{x} + \mathrm{e}^{-2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2(x+1)\,\mathrm{e}^{x} + \mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x}, \mathrm{e}^{-2x} x, \mathrm{e}^{x}, \mathrm{e}^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{-2x} x\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

Since $\mathrm{e}^{-2x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{x}, x^2 \mathrm{e}^{x}\}]$$

Since $x\,\mathrm{e}^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-2x}\}, \{x^2 \mathrm{e}^{x}, \mathrm{e}^{x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-2x} + A_2 x^2 \mathrm{e}^{x} + A_3 \mathrm{e}^{x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$54A_3 \mathrm{e}^{x} x + 18A_2 \mathrm{e}^{x} + 36A_3 \mathrm{e}^{x} + 18A_1 \mathrm{e}^{-2x} = 2(x+1)\,\mathrm{e}^{x} + \mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = \frac{1}{27}, A_3 = \frac{1}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \mathrm{e}^{-2x}}{18} + \frac{x^2 \mathrm{e}^{x}}{27} + \frac{\mathrm{e}^{x} x^3}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + c_3 \mathrm{e}^{x} + c_4 x\,\mathrm{e}^{x}\right) + \left(\frac{x^2 \mathrm{e}^{-2x}}{18} + \frac{x^2 \mathrm{e}^{x}}{27} + \frac{\mathrm{e}^{x} x^3}{27}\right) \end{aligned}$$

Which simplifies to

$$y = \left((c_4 x + c_3)\,\mathrm{e}^{3x} + c_2 x + c_1\right)\mathrm{e}^{-2x} + \frac{x^2 \mathrm{e}^{-2x}}{18} + \frac{x^2 \mathrm{e}^{x}}{27} + \frac{\mathrm{e}^{x} x^3}{27}$$

Summary

The solution(s) found are the following

$$y = \left((c_4 x + c_3)\,\mathrm{e}^{3x} + c_2 x + c_1\right)\mathrm{e}^{-2x} + \frac{x^2 \mathrm{e}^{-2x}}{18} + \frac{x^2 \mathrm{e}^{x}}{27} + \frac{\mathrm{e}^{x} x^3}{27} \tag{1}$$

Verification of solutions

$$y = \left((c_4 x + c_3)\,\mathrm{e}^{3x} + c_2 x + c_1\right)\mathrm{e}^{-2x} + \frac{x^2 \mathrm{e}^{-2x}}{18} + \frac{x^2 \mathrm{e}^{x}}{27} + \frac{\mathrm{e}^{x} x^3}{27}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=2*exp(x)*(1+x)
```

$$y(x) = \frac{e^{-2x}\left(\left(x^3+x^2+(27c_3-2)\,x+27c_1+\frac{10}{9}\right)e^{3x}+\frac{3x^2}{2}+(27c_4+2)\,x+27c_2+1\right)}{27}$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 66

```
DSolve[y''''[x]+2*y'''[x]-3*y''[x]-4*y'[x]+4*y[x]==2*Exp[x]*(1+x)+Exp[-2*x],y[x],x,IncludeSi
```

$$y(x) \to \frac{1}{54}e^{-2x}\left(3x^2+(4+54c_2)x+2+54c_1\right) + \frac{1}{243}e^{x}\left(9x^3+9x^2+9(-2+27c_4)x+10+243c_3\right)$$

19.57 problem section 9.3, problem 57

Internal problem ID [1554]
Internal file name [OUTPUT/1555_Sunday_June_05_2022_02_22_11_AM_36190050/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 57.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y = \sinh(x)\cos(x) - \cosh(x)\sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = -1 - i$$
$$\lambda_4 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(-1-i)x} c_1 + \mathrm{e}^{(-1+i)x} c_2 + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{(-1-i)x}$$
$$y_2 = \mathrm{e}^{(-1+i)x}$$
$$y_3 = \mathrm{e}^{(1-i)x}$$
$$y_4 = \mathrm{e}^{(1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y = \sinh(x)\cos(x) - \cosh(x)\sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{(-1-i)x} & \mathrm{e}^{(-1+i)x} & \mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x} \\ (-1-i)\,\mathrm{e}^{(-1-i)x} & (-1+i)\,\mathrm{e}^{(-1+i)x} & (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x} \\ 2i\mathrm{e}^{(-1-i)x} & -2i\mathrm{e}^{(-1+i)x} & -2i\mathrm{e}^{(1-i)x} & 2i\mathrm{e}^{(1+i)x} \\ (2-2i)\,\mathrm{e}^{(-1-i)x} & (2+2i)\,\mathrm{e}^{(-1+i)x} & (-2-2i)\,\mathrm{e}^{(1-i)x} & (-2+2i)\,\mathrm{e}^{(1+i)x} \end{bmatrix}$$

$$|W| = -128\,\mathrm{e}^{(-1-i)x}\mathrm{e}^{(-1+i)x}\mathrm{e}^{(1+i)x}\mathrm{e}^{(1-i)x}$$

The determinant simplifies to

$$|W| = -128$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{(-1+i)x} & e^{(1-i)x} & e^{(1+i)x} \\ (-1+i)\, e^{(-1+i)x} & (1-i)\, e^{(1-i)x} & (1+i)\, e^{(1+i)x} \\ -2ie^{(-1+i)x} & -2ie^{(1-i)x} & 2ie^{(1+i)x} \end{bmatrix}$$
$$= (8+8i)\, e^{(1+i)x}$$

$$W_2(x) = \det \begin{bmatrix} e^{(-1-i)x} & e^{(1-i)x} & e^{(1+i)x} \\ (-1-i)\, e^{(-1-i)x} & (1-i)\, e^{(1-i)x} & (1+i)\, e^{(1+i)x} \\ 2ie^{(-1-i)x} & -2ie^{(1-i)x} & 2ie^{(1+i)x} \end{bmatrix}$$
$$= (-8+8i)\, e^{(1-i)x}$$

$$W_3(x) = \det \begin{bmatrix} e^{(-1-i)x} & e^{(-1+i)x} & e^{(1+i)x} \\ (-1-i)\, e^{(-1-i)x} & (-1+i)\, e^{(-1+i)x} & (1+i)\, e^{(1+i)x} \\ 2ie^{(-1-i)x} & -2ie^{(-1+i)x} & 2ie^{(1+i)x} \end{bmatrix}$$
$$= (-8+8i)\, e^{(-1+i)x}$$

$$W_4(x) = \det \begin{bmatrix} e^{(-1-i)x} & e^{(-1+i)x} & e^{(1-i)x} \\ (-1-i)\, e^{(-1-i)x} & (-1+i)\, e^{(-1+i)x} & (1-i)\, e^{(1-i)x} \\ 2ie^{(-1-i)x} & -2ie^{(-1+i)x} & -2ie^{(1-i)x} \end{bmatrix}$$
$$= (8+8i)\, e^{(-1-i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x) W_1(x)}{a W(x)}\, dx \\ &= (-1)^3 \int \frac{(\sinh(x)\cos(x) - \cosh(x)\sin(x))\left((8+8i)\, e^{(1+i)x}\right)}{(1)\,(-128)}\, dx \\ &= -\int \frac{(8+8i)\,(\sinh(x)\cos(x) - \cosh(x)\sin(x))\, e^{(1+i)x}}{-128}\, dx \\ &= -\int \left(\left(-\frac{1}{16} - \frac{i}{16}\right)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\, e^{(1+i)x}\right) dx \\ &= \frac{ie^{2ix}}{64} + \frac{e^{(2+2i)x}}{128} + \frac{ie^{(2+2i)x}}{128} + \frac{e^{2x}}{64} - \frac{ix}{32} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^2 \int \frac{(\sinh(x)\cos(x) - \cosh(x)\sin(x))\left((-8+8i)\,\mathrm{e}^{(1-i)x}\right)}{(1)(-128)}\, dx \\
&= \int \frac{(-8+8i)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(1-i)x}}{-128}\, dx \\
&= \int \left(\left(\frac{1}{16} - \frac{i}{16}\right)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(1-i)x}\right) dx \\
&= \frac{\mathrm{e}^{2x}}{64} + \frac{ix}{32} - \frac{i\mathrm{e}^{-2ix}}{64} + \frac{\mathrm{e}^{(2-2i)x}}{128} - \frac{i\mathrm{e}^{(2-2i)x}}{128}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(\sinh(x)\cos(x) - \cosh(x)\sin(x))\left((-8+8i)\,\mathrm{e}^{(-1+i)x}\right)}{(1)(-128)}\, dx \\
&= -\int \frac{(-8+8i)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(-1+i)x}}{-128}\, dx \\
&= -\int \left(\left(\frac{1}{16} - \frac{i}{16}\right)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(-1+i)x}\right) dx \\
&= -\frac{\mathrm{e}^{(-2+2i)x}}{128} + \frac{i\mathrm{e}^{2ix}}{64} + \frac{i\mathrm{e}^{(-2+2i)x}}{128} + \frac{ix}{32} - \frac{\mathrm{e}^{-2x}}{64}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\sinh(x)\cos(x) - \cosh(x)\sin(x))\left((8+8i)\,\mathrm{e}^{(-1-i)x}\right)}{(1)(-128)}\, dx \\
&= \int \frac{(8+8i)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(-1-i)x}}{-128}\, dx \\
&= \int \left(\left(-\frac{1}{16} - \frac{i}{16}\right)(\sinh(x)\cos(x) - \cosh(x)\sin(x))\,\mathrm{e}^{(-1-i)x}\right) dx \\
&= -\frac{ix}{32} - \frac{\mathrm{e}^{-2x}}{64} - \frac{\mathrm{e}^{(-2-2i)x}}{128} - \frac{i\mathrm{e}^{-2ix}}{64} - \frac{i\mathrm{e}^{(-2-2i)x}}{128}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$y_p = \left(\frac{i e^{2ix}}{64} + \frac{e^{(2+2i)x}}{128} + \frac{i e^{(2+2i)x}}{128} + \frac{e^{2x}}{64} - \frac{ix}{32}\right)\left(e^{(-1-i)x}\right)$$
$$+ \left(\frac{e^{2x}}{64} + \frac{ix}{32} - \frac{i e^{-2ix}}{64} + \frac{e^{(2-2i)x}}{128} - \frac{i e^{(2-2i)x}}{128}\right)\left(e^{(-1+i)x}\right)$$
$$+ \left(-\frac{e^{(-2+2i)x}}{128} + \frac{i e^{2ix}}{64} + \frac{i e^{(-2+2i)x}}{128} + \frac{ix}{32} - \frac{e^{-2x}}{64}\right)\left(e^{(1-i)x}\right)$$
$$+ \left(-\frac{ix}{32} - \frac{e^{-2x}}{64} - \frac{e^{(-2-2i)x}}{128} - \frac{i e^{-2ix}}{64} - \frac{i e^{(-2-2i)x}}{128}\right)\left(e^{(1+i)x}\right)$$

Therefore the particular solution is

$$y_p = \frac{(-4ix - 3i - 3)\, e^{(-1-i)x}}{128} + \frac{(4ix - 3i + 3)\, e^{(1-i)x}}{128} + \frac{(4ix + 3i - 3)\, e^{(-1+i)x}}{128} - \frac{\left(ix - \frac{3}{4} - \frac{3}{4}i\right) e^{(1+i)x}}{32}$$

Which simplifies to

$$y_p = \frac{((-4x - 3)\sin(x) - 3\cos(x))\, e^{-x}}{64} + \frac{\left(\left(x - \frac{3}{4}\right)\sin(x) + \frac{3\cos(x)}{4}\right) e^{x}}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{(-1-i)x} c_1 + e^{(-1+i)x} c_2 + e^{(1-i)x} c_3 + e^{(1+i)x} c_4\right) \\ &\quad + \left(\frac{((-4x - 3)\sin(x) - 3\cos(x))\, e^{-x}}{64} + \frac{\left(\left(x - \frac{3}{4}\right)\sin(x) + \frac{3\cos(x)}{4}\right) e^{x}}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= e^{(-1-i)x} c_1 + e^{(-1+i)x} c_2 + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 \\ &\quad + \frac{((-4x - 3)\sin(x) - 3\cos(x))\, e^{-x}}{64} + \frac{\left(\left(x - \frac{3}{4}\right)\sin(x) + \frac{3\cos(x)}{4}\right) e^{x}}{16} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= e^{(-1-i)x} c_1 + e^{(-1+i)x} c_2 + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 \\ &\quad + \frac{((-4x - 3)\sin(x) - 3\cos(x))\, e^{-x}}{64} + \frac{\left(\left(x - \frac{3}{4}\right)\sin(x) + \frac{3\cos(x)}{4}\right) e^{x}}{16} \end{aligned}$$

Verified OK.

19.57.1 Maple step by step solution

Let's solve

$y'''' + 4y = \sinh(x)\cos(x) - \cosh(x)\sin(x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -\cosh(x)\sin(x) + \sinh(x)\cos(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -\cosh(x)\sin(x) + \sinh(x)\cos(x) - 4y_1($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sinh(x)\cos(x) - \cosh(x)\sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sinh(x)\cos(x) - \cosh(x)\sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1-\mathrm{I}, \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+\mathrm{I}, \begin{bmatrix} -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot (-I\sin(x) + \cos(x)) \cdot \begin{bmatrix} \frac{1}{4} + \frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} \left(\frac{1}{4} + \frac{I}{4}\right)(-I\sin(x) + \cos(x)) \\ -\frac{I}{2}(-I\sin(x) + \cos(x)) \\ \left(-\frac{1}{2} + \frac{I}{2}\right)(-I\sin(x) + \cos(x)) \\ -I\sin(x) + \cos(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-x} \cdot \begin{bmatrix} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_2(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I, \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot \left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ \frac{\mathrm{I}}{2}\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)\left(-\mathrm{I}\sin\left(x\right)+\cos\left(x\right)\right) \\ -\mathrm{I}\sin\left(x\right)+\cos\left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=e^{x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos\left(x\right) \end{array}\right], \vec{y}_{4}(x)=e^{x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4}+\frac{\cos(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2} \\ -\sin\left(x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_{p}($

$$\vec{y}(x)=c_{1}\vec{y}_{1}(x)+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)+\vec{y}_{p}(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{cccc} e^{-x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) & e^{-x}\left(\frac{\cos(x)}{4}-\frac{\sin(x)}{4}\right) & e^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & e^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) \\ -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2} \\ e^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & e^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)e^{x} \\ \cos\left(x\right)e^{-x} & -\sin\left(x\right)e^{-x} & \cos\left(x\right)e^{x} & -\sin\left(x\right)e^{x} \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$$\Phi(x)=\phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{cccc} \mathrm{e}^{-x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{4}-\frac{\sin(x)}{4}\right) & \mathrm{e}^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & \mathrm{e}^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right) \\ -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2} & \frac{\sin(x)\mathrm{e}^{x}}{2} & \frac{\cos(x)\mathrm{e}^{x}}{2} \\ \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)\mathrm{e}^{x} \\ \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x} & \cos(x)\,\mathrm{e}^{x} & -\sin(x)\,\mathrm{e}^{x} \end{array}\right]$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} \frac{\cos(x)\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)}{2} & \frac{\mathrm{e}^{-x}(-\cos(x)+\sin(x))}{4}+\frac{\mathrm{e}^{x}(\cos(x)+\sin(x))}{4} & \\ \frac{(-\cos(x)-\sin(x))\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{2} & \frac{\cos(x)\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)}{2} & \frac{\mathrm{e}^{-x}(-}{} \\ -\sin(x)\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right) & \frac{(-\cos(x)-\sin(x))\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}(\cos(x)-\sin(x))}{2} & \\ \mathrm{e}^{-x}(\cos(x)-\sin(x))-\mathrm{e}^{x}(\cos(x)+\sin(x)) & -\sin(x)\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right) & \frac{(-\cos}{} \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x)\cdot\overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x) = A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \Phi(x)\cdot\overrightarrow{v}'(x) = A\cdot\Phi(x)\cdot\overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{((-3-2x)\sin(x)-3\cos(x))\,\mathrm{e}^{-x}}{32} + \frac{\left(\left(x-\frac{3}{2}\right)\sin(x)+\frac{3\cos(x)}{2}\right)\mathrm{e}^{x}}{16} \\ \frac{((2+x)\sin(x)-x\cos(x))\,\mathrm{e}^{-x}}{16} + \frac{\mathrm{e}^{x}((-2+x)\sin(x)+x\cos(x))}{16} \\ \frac{((1+2x)\cos(x)-\sin(x))\,\mathrm{e}^{-x}}{16} + \frac{\left(\left(x-\frac{1}{2}\right)\cos(x)-\frac{\sin(x)}{2}\right)\mathrm{e}^{x}}{8} \\ \frac{((-\cos(x)-\sin(x))\,\mathrm{e}^{-x}+\mathrm{e}^{x}(\cos(x)-\sin(x)))\,x}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{((-3-2x)\sin(x)-3\cos(x))\,\mathrm{e}^{-x}}{32} + \frac{\left(\left(x-\frac{3}{2}\right)\sin(x)+\frac{3\cos(x)}{2}\right)\mathrm{e}^{x}}{16} \\ \frac{((2+x)\sin(x)-x\cos(x))\,\mathrm{e}^{-x}}{16} + \frac{\mathrm{e}^{x}((-2+x)\sin(x)+x\cos(x))}{16} \\ \frac{((1+2x)\cos(x)-\sin(x))\,\mathrm{e}^{-x}}{16} + \frac{\left(\left(x-\frac{1}{2}\right)\cos(x)-\frac{\sin(x)}{2}\right)\mathrm{e}^{x}}{8} \\ \frac{((-\cos(x)-\sin(x))\,\mathrm{e}^{-x}+\mathrm{e}^{x}(\cos(x)-\sin(x)))\,x}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((-2x+8c_1-8c_2-3)\sin(x)+8\left(c_1+c_2-\frac{3}{8}\right)\cos(x)\right)\mathrm{e}^{-x}}{32} + \frac{\mathrm{e}^{x}\left(\left(x+4c_3+4c_4-\frac{3}{2}\right)\sin(x)-4\left(c_3-c_4-\frac{3}{8}\right)\cos(x)\right)}{16}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$4)+0*diff(y(x),x$3)-0*diff(y(x),x$2)-0*diff(y(x),x)+4*y(x)=sinh(x)*cos(x)
```

$$y(x) = \frac{((-4x + 64c_4 - 3)\sin(x) + \cos(x)(64c_3 - 3))\,\mathrm{e}^{-x}}{64} + \frac{\mathrm{e}^{x}\left(\left(x + 16c_2 - \frac{3}{4}\right)\sin(x) + \cos(x)\left(16c_1 + \frac{3}{4}\right)\right)}{16}$$

✓ Solution by Mathematica

Time used: 0.676 (sec). Leaf size: 63

```
DSolve[y''''[x]+0*y'''[x]-0*y''[x]-0*y'[x]+4*y[x]==Sinh[x]*Cos[x]-Cosh[x]*Sin[x],y[x],x,Incl
```

$$y(x) \to \frac{1}{64}e^{-x}\left(((3 + 64c_4)e^{2x} - 3 + 64c_1)\cos(x) + \left(-4x + e^{2x}(4x - 3 + 64c_3) - 3 + 64c_2\right)\sin(x)\right)$$

19.58 problem section 9.3, problem 58

Internal problem ID [1555]
Internal file name [`OUTPUT/1556_Sunday_June_05_2022_02_22_15_AM_95328679/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 58.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 5y''' + 9y'' + 7y' + 2y = \mathrm{e}^{-x}(30 + 24x) - \mathrm{e}^{-2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 5y''' + 9y'' + 7y' + 2y = 0$$

The characteristic equation is

$$\lambda^4 + 5\lambda^3 + 9\lambda^2 + 7\lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + x\,e^{-x} c_2 + x^2 e^{-x} c_3 + e^{-2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= x\,e^{-x} \\
y_3 &= x^2 e^{-x} \\
y_4 &= e^{-2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x) - e^{-2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}(30 + 24x) - e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{-x}, x^2 e^{-x}, e^{-2x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}\}, \{x\,e^{-x}, x^2 e^{-x}\}]$$

Since $x\,e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}\}, \{x^2 e^{-x}, e^{-x} x^3\}]$$

Since $x^2 e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}\}, \{e^{-x} x^3, e^{-x} x^4\}]$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x} x\}, \{e^{-x} x^3, e^{-x} x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^{-2x} x + A_2 \mathrm{e}^{-x} x^3 + A_3 \mathrm{e}^{-x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_3 \mathrm{e}^{-x} x - A_1 \mathrm{e}^{-2x} + 24A_3 \mathrm{e}^{-x} + 6A_2 \mathrm{e}^{-x} = \mathrm{e}^{-x}(30 + 24x) - \mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-2x} x + \mathrm{e}^{-x} x^3 + \mathrm{e}^{-x} x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + x^2 \mathrm{e}^{-x} c_3 + \mathrm{e}^{-2x} c_4\right) + \left(\mathrm{e}^{-2x} x + \mathrm{e}^{-x} x^3 + \mathrm{e}^{-x} x^4\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \mathrm{e}^{-2x} c_4 + \mathrm{e}^{-2x} x + \mathrm{e}^{-x} x^3 + \mathrm{e}^{-x} x^4$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \mathrm{e}^{-2x} c_4 + \mathrm{e}^{-2x} x + \mathrm{e}^{-x} x^3 + \mathrm{e}^{-x} x^4 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}\left(c_3 x^2 + c_2 x + c_1\right) + \mathrm{e}^{-2x} c_4 + \mathrm{e}^{-2x} x + \mathrm{e}^{-x} x^3 + \mathrm{e}^{-x} x^4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$4)+5*diff(y(x),x$3)+9*diff(y(x),x$2)+7*diff(y(x),x)+2*y(x)=exp(-x)*(30+24
```

$$y(x) = \left(x^4 + x^3 + (c_4 - 3)\, x^2 + (c_3 + 6)\, x + c_2 - 6\right) \mathrm{e}^{-x} + \mathrm{e}^{-2x}(x + c_1 + 3)$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 44

```
DSolve[y''''[x]+5*y'''[x]+9*y''[x]+7*y'[x]+2*y[x]==Exp[-x]*(30+24*x)-Exp[-2*x],y[x],x,Includ
```

$$y(x) \to e^{-2x}\left(e^x\left(x^4 + x^3 + (-3 + c_4)x^2 + (6 + c_3)x - 6 + c_2\right) + x + 3 + c_1\right)$$

19.59 problem section 9.3, problem 59

Internal problem ID [1556]
Internal file name [OUTPUT/1557_Sunday_June_05_2022_02_22_18_AM_88952084/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 59.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 4y''' + 7y'' - 6y' + 2y = \mathrm{e}^{x}(12x - 2\cos(x) + 2\sin(x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 7y'' - 6y' + 2y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 7\lambda^2 - 6\lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{(1-i)x}$$
$$y_4 = e^{(1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 4y''' + 7y'' - 6y' + 2y = e^x(12x - 2\cos(x) + 2\sin(x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^x & x\, e^x & e^{(1-i)x} & e^{(1+i)x} \\ e^x & (x+1)\, e^x & (1-i)\, e^{(1-i)x} & (1+i)\, e^{(1+i)x} \\ e^x & e^x(2+x) & -2i e^{(1-i)x} & 2i e^{(1+i)x} \\ e^x & e^x(x+3) & (-2-2i)\, e^{(1-i)x} & (-2+2i)\, e^{(1+i)x} \end{bmatrix}$$

$$|W| = 2i e^{2x} e^{(1+i)x} e^{(1-i)x}$$

The determinant simplifies to

$$|W| = 2i\mathrm{e}^{4x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\,\mathrm{e}^{x} & \mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x} \\ (x+1)\,\mathrm{e}^{x} & (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x} \\ \mathrm{e}^{x}(2+x) & -2i\mathrm{e}^{(1-i)x} & 2i\mathrm{e}^{(1+i)x} \end{bmatrix}$$
$$= 2ix\,\mathrm{e}^{3x}$$

$$W_2(x) = \det \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{(1-i)x} & \mathrm{e}^{(1+i)x} \\ \mathrm{e}^{x} & (1-i)\,\mathrm{e}^{(1-i)x} & (1+i)\,\mathrm{e}^{(1+i)x} \\ \mathrm{e}^{x} & -2i\mathrm{e}^{(1-i)x} & 2i\mathrm{e}^{(1+i)x} \end{bmatrix}$$
$$= 2i\mathrm{e}^{3x}$$

$$W_3(x) = \det \begin{bmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} & \mathrm{e}^{(1+i)x} \\ \mathrm{e}^{x} & (x+1)\,\mathrm{e}^{x} & (1+i)\,\mathrm{e}^{(1+i)x} \\ \mathrm{e}^{x} & \mathrm{e}^{x}(2+x) & 2i\mathrm{e}^{(1+i)x} \end{bmatrix}$$
$$= -\mathrm{e}^{(3+i)x}$$

$$W_4(x) = \det \begin{bmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} & \mathrm{e}^{(1-i)x} \\ \mathrm{e}^{x} & (x+1)\,\mathrm{e}^{x} & (1-i)\,\mathrm{e}^{(1-i)x} \\ \mathrm{e}^{x} & \mathrm{e}^{x}(2+x) & -2i\mathrm{e}^{(1-i)x} \end{bmatrix}$$
$$= -\mathrm{e}^{(3-i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^{3} \int \frac{\left(\mathrm{e}^{x}(12x - 2\cos(x) + 2\sin(x))\right)\left(2ix\,\mathrm{e}^{3x}\right)}{(1)\left(2i\mathrm{e}^{4x}\right)}\,dx \\
&= -\int \frac{2i\mathrm{e}^{x}(12x - 2\cos(x) + 2\sin(x))\,x\,\mathrm{e}^{3x}}{2i\mathrm{e}^{4x}}\,dx \\
&= -\int \left(x(12x - 2\cos(x) + 2\sin(x))\right)\,dx \\
&= -4x^{3} + 2\cos(x) + 2\sin(x)\,x - 2\sin(x) + 2x\cos(x) \\
&= -4x^{3} + 2\cos(x) + 2\sin(x)\,x - 2\sin(x) + 2x\cos(x)
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\right)\left(2i\mathrm{e}^{3x}\right)}{(1)\left(2i\mathrm{e}^{4x}\right)}\,dx \\
&= \int \frac{2i\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\,\mathrm{e}^{3x}}{2i\mathrm{e}^{4x}}\,dx \\
&= \int (12x - 2\cos(x) + 2\sin(x))\,\,dx \\
&= 6x^2 - 2\sin(x) - 2\cos(x)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\right)\left(-\mathrm{e}^{(3+i)x}\right)}{(1)\left(2i\mathrm{e}^{4x}\right)}\,dx \\
&= -\int \frac{-\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\,\mathrm{e}^{(3+i)x}}{2i\mathrm{e}^{4x}}\,dx \\
&= -\int \left(i(6x - \cos(x) + \sin(x))\,\mathrm{e}^{ix}\right)\,dx \\
&= -\frac{12i\mathrm{e}^{ix} + \left(\frac{11}{2} - \frac{i}{2}\right)x\,\mathrm{e}^{ix} + (13-i)\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right) + (-1+i)\,x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right) + \left(\frac{13}{2} + \frac{i}{2}\right)x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2}{1 + \tan\left(\frac{x}{2}\right)^2} \\
&= -\frac{12i\mathrm{e}^{ix} + \left(\frac{11}{2} - \frac{i}{2}\right)x\,\mathrm{e}^{ix} + (13-i)\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right) + (-1+i)\,x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right) + \left(\frac{13}{2} + \frac{i}{2}\right)x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2}{1 + \tan\left(\frac{x}{2}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\right)\left(-\mathrm{e}^{(3-i)x}\right)}{(1)\left(2i\mathrm{e}^{4x}\right)}\,dx \\
&= \int \frac{-\mathrm{e}^x(12x - 2\cos(x) + 2\sin(x))\,\mathrm{e}^{(3-i)x}}{2i\mathrm{e}^{4x}}\,dx \\
&= \int \left(i(6x - \cos(x) + \sin(x))\,\mathrm{e}^{-ix}\right)\,dx \\
&= \frac{12i\mathrm{e}^{-ix} + \left(-\frac{11}{2} - \frac{i}{2}\right)x\,\mathrm{e}^{-ix} + (-13-i)\,\mathrm{e}^{-ix}\tan\left(\frac{x}{2}\right) + (1+i)\,x\,\mathrm{e}^{-ix}\tan\left(\frac{x}{2}\right) + \left(-\frac{13}{2} + \frac{i}{2}\right)x\,\mathrm{e}^{-ix}\tan\left(\right.}{1 + \tan\left(\frac{x}{2}\right)^2} \\
&= \frac{12i\mathrm{e}^{-ix} + \left(-\frac{11}{2} - \frac{i}{2}\right)x\,\mathrm{e}^{-ix} + (-13-i)\,\mathrm{e}^{-ix}\tan\left(\frac{x}{2}\right) + (1+i)\,x\,\mathrm{e}^{-ix}\tan\left(\frac{x}{2}\right) + \left(-\frac{13}{2} + \frac{i}{2}\right)x\,\mathrm{e}^{-ix}\tan\left(\right.}{1 + \tan\left(\frac{x}{2}\right)^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Hence

$$\begin{aligned} y_p = & \left(-4x^3 + 2\cos(x) + 2\sin(x)x - 2\sin(x) + 2x\cos(x)\right)(e^x) \\ & + \left(6x^2 - 2\sin(x) - 2\cos(x)\right)(x\,e^x) \\ & + \left(-\frac{12i e^{ix} + \left(\frac{11}{2} - \frac{i}{2}\right) x\,e^{ix} + (13 - i)\,e^{ix}\tan\left(\frac{x}{2}\right) + (-1 + i)\,x\,e^{ix}\tan\left(\frac{x}{2}\right) + \left(\frac{13}{2} + \frac{i}{2}\right) x\,e^{ix}\tan\left(\frac{x}{2}\right)^2}{1 + \tan\left(\frac{x}{2}\right)^2}\right) (\\ & + \left(\frac{12i e^{-ix} + \left(-\frac{11}{2} - \frac{i}{2}\right) x\,e^{-ix} + (-13 - i)\,e^{-ix}\tan\left(\frac{x}{2}\right) + (1 + i)\,x\,e^{-ix}\tan\left(\frac{x}{2}\right) + \left(-\frac{13}{2} + \frac{i}{2}\right) x\,e^{-ix}\tan}{1 + \tan\left(\frac{x}{2}\right)^2}\right. \end{aligned}$$

Therefore the particular solution is

$$y_p = e^x\left(x\cos(x) - 12x + \sin(x)x - 15\sin(x) + 2x^3 + 2\cos(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 x\,e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4\right) \\ &\quad + \left(e^x\left(x\cos(x) - 12x + \sin(x)x - 15\sin(x) + 2x^3 + 2\cos(x)\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y = & e^{(1-i)x} c_3 + e^{(1+i)x} c_4 + e^x(c_2 x + c_1) \\ & + e^x\left(x\cos(x) - 12x + \sin(x)x - 15\sin(x) + 2x^3 + 2\cos(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = & e^{(1-i)x} c_3 + e^{(1+i)x} c_4 + e^x(c_2 x + c_1) \\ & + e^x\left(x\cos(x) - 12x + \sin(x)x - 15\sin(x) + 2x^3 + 2\cos(x)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = & e^{(1-i)x} c_3 + e^{(1+i)x} c_4 + e^x(c_2 x + c_1) \\ & + e^x\left(x\cos(x) - 12x + \sin(x)x - 15\sin(x) + 2x^3 + 2\cos(x)\right) \end{aligned}$$

Verified OK.

19.59.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 7y'' - 6y' + 2y = e^x(12x - 2\cos(x) + 2\sin(x))$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -2y - 2\cos(x)\,e^x + 2\sin(x)\,e^x + 12x\,e^x + 4y''' - 7y'' + 6y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - 4y''' + 7y'' - 6y' + 2y = -2\,e^x(-6x + \cos(x) - \sin(x))$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2\cos(x)\,e^x + 2\sin(x)\,e^x + 12x\,e^x + 4y_4(x) - 7y_3(x) + 6y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2\cos(x)\,e^x + 2\sin(x)\,e^x + 12x\,e^x + 4y_4($

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 2\sin(x)\,\mathrm{e}^{x} - 2\cos(x)\,\mathrm{e}^{x} + 12x\,\mathrm{e}^{x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 2\sin(x)\,\mathrm{e}^{x} - 2\cos(x)\,\mathrm{e}^{x} + 12x\,\mathrm{e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1 - \mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} + \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1 + \mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} - \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_1(x) = \mathrm{e}^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 6 & -7 & 4 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I, \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^x \cdot (-I\sin(x) + \cos(x)) \cdot \begin{bmatrix} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^x \cdot \begin{bmatrix} \left(-\frac{1}{4} + \frac{I}{4}\right)(-I\sin(x) + \cos(x)) \\ \frac{I}{2}(-I\sin(x) + \cos(x)) \\ \left(\frac{1}{2} + \frac{I}{2}\right)(-I\sin(x) + \cos(x)) \\ -I\sin(x) + \cos(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^x \cdot \left[\begin{array}{c} -\frac{\cos(x)}{4} + \frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_4(x) = e^x \cdot \left[\begin{array}{c} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{cccc} e^x & (x-1)e^x & e^x\left(-\frac{\cos(x)}{4} + \frac{\sin(x)}{4}\right) & e^x\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) \\ e^x & x e^x & \frac{\sin(x)e^x}{2} & \frac{\cos(x)e^x}{2} \\ e^x & x e^x & e^x\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^x \\ e^x & x e^x & \cos(x)e^x & -\sin(x)e^x \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{cccc} e^x & (x-1)e^x & e^x\left(-\frac{\cos(x)}{4} + \frac{\sin(x)}{4}\right) & e^x\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) \\ e^x & x e^x & \frac{\sin(x)e^x}{2} & \frac{\cos(x)e^x}{2} \\ e^x & x e^x & e^x\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^x \\ e^x & x e^x & \cos(x)e^x & -\sin(x)e^x \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cccc} 1 & -1 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{cccc} -(x-1)e^x & e^x(2x - \sin(x)) & \frac{e^x(-1-3x+\cos(x)+3\sin(x))}{2} & -\frac{e^x(-1-x+\cos(x)+}{2} \\ -x e^x & e^x(2 + 2x - \sin(x) - \cos(x)) & \frac{e^x(-4-3x+2\sin(x)+4\cos(x))}{2} & \frac{e^x(2+x-2\cos(x}{2} \\ -x e^x & -2e^x(-1 - x + \cos(x)) & \frac{e^x(-4-3x+6\cos(x)-2\sin(x))}{2} & \frac{e^x(2+x-2\cos(x)+2}{2} \\ -x e^x & 2e^x(1 + x - \cos(x) + \sin(x)) & \frac{e^x(-4-3x+4\cos(x)-8\sin(x))}{2} & \frac{e^x(2+x+4\sin(x}{2} \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \left(\left(x+6\right) \cos \left(x\right)+x^{3}+3 x^{2}-5 x+4 \sin \left(x\right)-6\right) {\mathrm e}^{x} \\ \left(\left(x+11\right) \cos \left(x\right)+\left(-x-2\right) \sin \left(x\right)+x^{3}+6 x^{2}+x-11\right) {\mathrm e}^{x} \\ \left(\left(-2 x-13\right) \sin \left(x\right)+x^{3}+6 x^{2}+11 \cos \left(x\right)+13 x-11\right) {\mathrm e}^{x} \\ \left(\left(-1-2 x\right) \cos \left(x\right)+\left(-2 x-25\right) \sin \left(x\right)+x^{3}+6 x^{2}+25 x+1\right) {\mathrm e}^{x} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + c_4 \overrightarrow{y}_4(x) + \left[\begin{array}{c} \left(\left(x+6\right) \cos \left(x\right)+x^{3}+3 x^{2}-5 x+4 \\ \left(\left(x+11\right) \cos \left(x\right)+\left(-x-2\right) \sin \left(x\right)+x^{3} \\ \left(\left(-2 x-13\right) \sin \left(x\right)+x^{3}+6 x^{2}+11 \cos \\ \left(\left(-1-2 x\right) \cos \left(x\right)+\left(-2 x-25\right) \sin \left(x\right)+ \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \left(\left(-\frac{c_3}{4} + \frac{c_4}{4} + x + 6\right)\cos(x) + \left(\frac{c_3}{4} + \frac{c_4}{4} + 4\right)\sin(x) + x^3 + 3x^2 + (-5 + c_2)x + c_1 - c_2 - 6\right)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+7*diff(y(x),x$2)-6*diff(y(x),x)+2*y(x)=exp(x)*(12*x-2
```

$$y(x) = e^x\left((x + c_3 + 3)\cos(x) + (c_4 + x - 2)\sin(x) + 2x^3 + (c_2 - 12)x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 40

```
DSolve[y''''[x]-4*y'''[x]+7*y''[x]-6*y'[x]+2*y[x]==Exp[x]*(12*x-2*Cos[x]+2*Sin[x]),y[x],x,In
```

$$y(x) \to e^x\left(2x^3 - 12x + c_4x + (x + 3 + c_2)\cos(x) + (x - 2 + c_1)\sin(x) + c_3\right)$$

19.60 problem section 9.3, problem 60

Internal problem ID [1557]
Internal file name [`OUTPUT/1558_Sunday_June_05_2022_02_22_21_AM_93531774/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 60.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - y' + y = e^{2x}(10 + 3x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - y' + y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + x\,e^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = x\,e^{x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - y' + y = e^{2x}(10 + 3x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(10 + 3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{x}, e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 e^{2x} + 3A_1 x\,e^{2x} + 3A_2 e^{2x} = e^{2x}(10 + 3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{2x} + e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{x} + x\,e^{x} c_3\right) + \left(x\,e^{2x} + e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{-x} + e^{x}(c_3 x + c_2) + x\,e^{2x} + e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{x}(c_3 x + c_2) + x\,e^{2x} + e^{2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{x}(c_3 x + c_2) + x\,e^{2x} + e^{2x}$$

Verified OK.

19.60.1 Maple step by step solution

Let's solve

$y''' - y'' - y' + y = e^{2x}(10 + 3x)$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 3x\,\mathrm{e}^{2x} + y_3(x) + y_2(x) - y_1(x) + 10\,\mathrm{e}^{2x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3x\,\mathrm{e}^{2x} + y_3(x) + y_2(x) - y_1(x) + 10\,\mathrm{e}^{2x}]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3x\,\mathrm{e}^{2x} + 10\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3x\,\mathrm{e}^{2x} + 10\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_{3}(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{3}(x)$ into the homogeneous system

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \mathrm{e}^{\lambda x} \left(\lambda x \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{3}(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{array}\right] - 1 \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 1

$$\overrightarrow{y}_3(x) = \mathrm{e}^x \cdot \left(x \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} \mathrm{e}^{-x} & \mathrm{e}^x & (x-1)\,\mathrm{e}^x \\ -\mathrm{e}^{-x} & \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^{-x} & \mathrm{e}^x & x\,\mathrm{e}^x \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} \mathrm{e}^{-x} & \mathrm{e}^x & (x-1)\,\mathrm{e}^x \\ -\mathrm{e}^{-x} & \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^{-x} & \mathrm{e}^x & x\,\mathrm{e}^x \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -(x-1)\,\mathrm{e}^{x} & -\frac{\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{-x}}{2}-\frac{\mathrm{e}^{x}}{2}+x\,\mathrm{e}^{x} \\ -x\,\mathrm{e}^{x} & \frac{\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2} & -\frac{\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}}{2}+x\,\mathrm{e}^{x} \\ -x\,\mathrm{e}^{x} & -\frac{\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{-x}}{2}+\frac{\mathrm{e}^{x}}{2}+x\,\mathrm{e}^{x} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x)=\begin{bmatrix} 2\,\mathrm{e}^{2x}(x+1)-\frac{3\,\mathrm{e}^{-x}}{2}+\frac{(-14x-1)\mathrm{e}^{x}}{2} \\ 6\,\mathrm{e}^{2x}+\frac{3\,\mathrm{e}^{-x}}{2}-\frac{15\,\mathrm{e}^{x}}{2}-7x\,\mathrm{e}^{x}+4x\,\mathrm{e}^{2x} \\ (9+5x)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{-x}}{2}+\frac{(-14x-15)\mathrm{e}^{x}}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} 2\,e^{2x}(x+1) - \frac{3\,e^{-x}}{2} + \frac{(-14x-1)e^{x}}{2} \\ 6\,e^{2x} + \frac{3\,e^{-x}}{2} - \frac{15\,e^{x}}{2} - 7x\,e^{x} + 4x\,e^{2x} \\ (9+5x)\,e^{2x} - \frac{3\,e^{-x}}{2} + \frac{(-14x-15)e^{x}}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(2c_1-3)e^{-x}}{2} + 2\,e^{2x}(x+1) + \left((c_3-7)\,x + c_2 - c_3 - \frac{1}{2}\right)e^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)-1*diff(y(x),x$2)-1*diff(y(x),x)+1*y(x)=exp(2*x)*(10+3*x),y(x), singsol
```

$$y(x) = (x+1)\,e^{2x} + e^{-x}c_2 + e^{x}(c_3x + c_1)$$

✓ Solution by Mathematica
Time used: 0.006 (sec). Leaf size: 34

```
DSolve[y'''[x]-1*y''[x]-1*y'[x]+1*y[x]==Exp[2*x]*(10+3*x),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{2x}(x+1) + c_1e^{-x} + e^{x}(c_3x + c_2)$$

19.61 problem section 9.3, problem 61

Internal problem ID [1558]
Internal file name [`OUTPUT/1559_Sunday_June_05_2022_02_22_23_AM_96200015/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 61.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' + y'' - 2y = -e^{3x}(17x^2 + 67x + 9)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - 2y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1 - i$$
$$\lambda_3 = -1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{(-1-i)x} c_2 + e^{(-1+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^x \\ y_2 &= e^{(-1-i)x} \\ y_3 &= e^{(-1+i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - 2y = -e^{3x}(17x^2 + 67x + 9)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{3x}(17x^2 + 67x + 9)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{3x}, x^2 e^{3x}, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{(-1-i)x}, e^{(-1+i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x e^{3x} + A_2 x^2 e^{3x} + A_3 e^{3x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &33A_1 e^{3x} + 34A_1 x e^{3x} + 20A_2 e^{3x} + 66A_2 x e^{3x} + 34A_2 x^2 e^{3x} + 34A_3 e^{3x} \\ &= -e^{3x}(17x^2 + 67x + 9) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = -\frac{1}{2}, A_3 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{3x} - \frac{x^2\mathrm{e}^{3x}}{2} + \mathrm{e}^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_2 + \mathrm{e}^{(-1+i)x}c_3\right) + \left(-x\,\mathrm{e}^{3x} - \frac{x^2\mathrm{e}^{3x}}{2} + \mathrm{e}^{3x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_2 + \mathrm{e}^{(-1+i)x}c_3 - x\,\mathrm{e}^{3x} - \frac{x^2\mathrm{e}^{3x}}{2} + \mathrm{e}^{3x} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{x} + \mathrm{e}^{(-1-i)x}c_2 + \mathrm{e}^{(-1+i)x}c_3 - x\,\mathrm{e}^{3x} - \frac{x^2\mathrm{e}^{3x}}{2} + \mathrm{e}^{3x}$$

Verified OK.

19.61.1 Maple step by step solution

Let's solve

$y''' + y'' - 2y = -\mathrm{e}^{3x}(17x^2 + 67x + 9)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -17x^2 e^{3x} - 67x\,e^{3x} - y_3(x) + 2y_1(x) - 9\,e^{3x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -17x^2 e^{3x} - 67x\,e^{3x} - y_3(x) + 2y_1(x) - 9\,e^{3x}]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -17x^2 e^{3x} - 67x\,e^{3x} - 9\,e^{3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -17x^2 e^{3x} - 67x\,e^{3x} - 9\,e^{3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-1-\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[-1+\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (-\mathrm{I}\sin(x) + \cos(x)) \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{2}(-\mathrm{I}\sin(x) + \cos(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(-\mathrm{I}\sin(x) + \cos(x)) \\ -\mathrm{I}\sin(x) + \cos(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=\mathrm{e}^{-x}\cdot\left[\begin{array}{c}-\frac{\sin(x)}{2}\\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ \cos(x)\end{array}\right], \vec{y}_3(x)=\mathrm{e}^{-x}\cdot\left[\begin{array}{c}-\frac{\cos(x)}{2}\\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ -\sin(x)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{ccc}\mathrm{e}^{x} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2}\\ \mathrm{e}^{x} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right)\\ \mathrm{e}^{x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x}\end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\left[\begin{array}{ccc}\mathrm{e}^{x} & -\frac{\sin(x)\mathrm{e}^{-x}}{2} & -\frac{\cos(x)\mathrm{e}^{-x}}{2}\\ \mathrm{e}^{x} & \mathrm{e}^{-x}\left(-\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \mathrm{e}^{-x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right)\\ \mathrm{e}^{x} & \cos(x)\,\mathrm{e}^{-x} & -\sin(x)\,\mathrm{e}^{-x}\end{array}\right]\cdot\frac{1}{\left[\begin{array}{ccc}1 & 0 & -\frac{1}{2}\\ 1 & -\frac{1}{2} & \frac{1}{2}\\ 1 & 1 & 0\end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\left[\begin{array}{ccc}\frac{(3\cos(x)+\sin(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(-2\cos(x)+\sin(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(-\cos(x)-2\sin(x))\mathrm{e}^{-x}}{5}+\frac{\mathrm{e}^{x}}{5}\\ \frac{(-4\sin(x)-2\cos(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(3\cos(x)+\sin(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(-\cos(x)+3\sin(x))\mathrm{e}^{-x}}{5}+\frac{\mathrm{e}^{x}}{5}\\ \frac{(-2\cos(x)+6\sin(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(-4\sin(x)-2\cos(x))\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5} & \frac{(4\cos(x)-2\sin(x))\mathrm{e}^{-x}}{5}+\frac{\mathrm{e}^{x}}{5}\end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(-x^{2}-2x+2\right) e^{3x}}{2}+\frac{\left(3\cos\left(x\right)+\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \\ \frac{\left(-3x^{2}-8x+4\right) e^{3x}}{2}+\frac{2\left(-\cos\left(x\right)-2\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \\ \frac{\left(-9x^{2}-30x+4\right) e^{3x}}{2}+\frac{2\left(-\cos\left(x\right)+3\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \left[\begin{array}{c} \frac{\left(-x^{2}-2x+2\right) e^{3x}}{2}+\frac{\left(3\cos\left(x\right)+\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \\ \frac{\left(-3x^{2}-8x+4\right) e^{3x}}{2}+\frac{2\left(-\cos\left(x\right)-2\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \\ \frac{\left(-9x^{2}-30x+4\right) e^{3x}}{2}+\frac{2\left(-\cos\left(x\right)+3\sin\left(x\right)\right) e^{-x}}{5}-\frac{8\,e^{x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(-5c_3+6\right)\cos\left(x\right)+\left(-5c_2+2\right)\sin\left(x\right)\right) e^{-x}}{10}+\frac{\left(-x^{2}-2x+2\right) e^{3x}}{2}+\frac{e^{x}\left(5c_1-8\right)}{5}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)+1*diff(y(x),x$2)-0*diff(y(x),x)-2*y(x)=-exp(3*x)*(9+67*x+17*x^2),y(x),
```

$$y(x) = -\frac{e^{-x}((x^2 + 2x - 2)\, e^{4x} - 2c_1 e^{2x} - 2c_2 \cos(x) - 2c_3 \sin(x))}{2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 49

```
DSolve[y'''[x]+1*y''[x]-0*y'[x]-2*y[x]==-Exp[3*x]*(9+67*x+17*x^2),y[x],x,IncludeSingularSolu
```

$$y(x) \to -\frac{1}{2}e^{3x}(x^2 + 2x - 2) + c_3 e^x + c_2 e^{-x}\cos(x) + c_1 e^{-x}\sin(x)$$

19.62 problem section 9.3, problem 62

Internal problem ID [1559]
Internal file name [`OUTPUT/1560_Sunday_June_05_2022_02_22_26_AM_7042313/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 62.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 6y'' + 11y' - 6y = e^{2x}\left(-3x^2 - 4x + 5\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 11y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 11y' - 6y = e^{2x}(-3x^2 - 4x + 5)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(-3x^2 - 4x + 5)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 x^2 e^{2x} + A_3 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2 x\,e^{2x} - 3A_3 e^{2x} x^2 + 6A_3 e^{2x} - A_1 e^{2x} = e^{2x}(-3x^2 - 4x + 5)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{2x} + 2x^2e^{2x} + e^{2x}x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^x + c_2e^{2x} + c_3e^{3x}\right) + \left(x\,e^{2x} + 2x^2e^{2x} + e^{2x}x^3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + x\,e^{2x} + 2x^2e^{2x} + e^{2x}x^3 \tag{1}$$

Verification of solutions

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + x\,e^{2x} + 2x^2e^{2x} + e^{2x}x^3$$

Verified OK.

19.62.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 11y' - 6y = e^{2x}(-3x^2 - 4x + 5)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 6y - 3x^2e^{2x} - 4x\,e^{2x} + 5\,e^{2x} + 6y'' - 11y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 6y'' + 11y' - 6y = -e^{2x}(3x^2 + 4x - 5)$

- ☐ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -3x^2 e^{2x} - 4x\,e^{2x} + 5\,e^{2x} + 6y_3(x) - 11y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -3x^2 e^{2x} - 4x\,e^{2x} + 5\,e^{2x} + 6y_3(x) - 11y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -3x^2 e^{2x} - 4x\,e^{2x} + 5\,e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -3x^2 e^{2x} - 4x\,e^{2x} + 5\,e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 3\,e^{x} - 3\,e^{2x} + e^{3x} & -\frac{5\,e^{x}}{2} + 4\,e^{2x} - \frac{3\,e^{3x}}{2} & \frac{e^{x}}{2} - e^{2x} + \frac{e^{3x}}{2} \\ 3\,e^{x} - 6\,e^{2x} + 3\,e^{3x} & -\frac{5\,e^{x}}{2} + 8\,e^{2x} - \frac{9\,e^{3x}}{2} & \frac{e^{x}}{2} - 2\,e^{2x} + \frac{3\,e^{3x}}{2} \\ 3\,e^{x} - 12\,e^{2x} + 9\,e^{3x} & -\frac{5\,e^{x}}{2} + 16\,e^{2x} - \frac{27\,e^{3x}}{2} & \frac{e^{x}}{2} - 4\,e^{2x} + \frac{9\,e^{3x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot \left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c} (x^3+2x^2+x+4)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{5\,\mathrm{e}^{3x}}{2} \\ (2x^3+7x^2+6x+9)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{15\,\mathrm{e}^{3x}}{2} \\ 2(2x^3+10x^2+13x+12)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{45\,\mathrm{e}^{3x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c} (x^3+2x^2+x+4)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{5\,\mathrm{e}^{3x}}{2} \\ (2x^3+7x^2+6x+9)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{15\,\mathrm{e}^{3x}}{2} \\ 2(2x^3+10x^2+13x+12)\,\mathrm{e}^{2x}-\frac{3\,\mathrm{e}^{x}}{2}-\frac{45\,\mathrm{e}^{3x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{(4x^3+8x^2+c_2+4x+16)\,\mathrm{e}^{2x}}{4}+\frac{(-45+2c_3)\,\mathrm{e}^{3x}}{18}+\frac{\mathrm{e}^{x}(2c_1-3)}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=exp(2*x)*(5-4*x-3*x^2),y(x), s
```

$$y(x) = e^{x}\left(c_3 e^{2x} + \left(x^3 + 2x^2 + c_2 + x\right) e^{x} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 37

```
DSolve[y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==Exp[2*x]*(5-4*x-3*x^2),y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{x}\left(e^{x}\left(x^3 + 2x^2 + x + 4 + c_2\right) + c_3 e^{2x} + c_1\right)$$

19.63 problem section 9.3, problem 63

Internal problem ID [1560]
Internal file name [`OUTPUT/1561_Sunday_June_05_2022_02_22_28_AM_73161/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 63.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y'' + y' = -2\,\mathrm{e}^{-x}(6x^2 - 18x + 7)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' + y' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2 + c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= 1
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + 2y'' + y' = -2\,\mathrm{e}^{-x}\left(6x^2 - 18x + 7\right)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-2\,\mathrm{e}^{-x}\left(6x^2 - 18x + 7\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3\}]
$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3, \mathrm{e}^{-x}x^4\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1x^2\mathrm{e}^{-x} + A_2\mathrm{e}^{-x}x^3 + A_3\mathrm{e}^{-x}x^4
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
24A_3\mathrm{e}^{-x}x - 6A_2\mathrm{e}^{-x}x - 12A_3\mathrm{e}^{-x}x^2 - 2A_1\mathrm{e}^{-x} + 6A_2\mathrm{e}^{-x} = -2\,\mathrm{e}^{-x}\left(6x^2 - 18x + 7\right)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{-x} - 2\,e^{-x} x^3 + e^{-x} x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\,e^{-x} c_2 + c_3\right) + \left(x^2 e^{-x} - 2\,e^{-x} x^3 + e^{-x} x^4\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 x + c_1)\,e^{-x} + c_3 + x^2 e^{-x} - 2\,e^{-x} x^3 + e^{-x} x^4$$

Summary

The solution(s) found are the following

$$y = (c_2 x + c_1)\,e^{-x} + c_3 + x^2 e^{-x} - 2\,e^{-x} x^3 + e^{-x} x^4 \tag{1}$$

Verification of solutions

$$y = (c_2 x + c_1)\,e^{-x} + c_3 + x^2 e^{-x} - 2\,e^{-x} x^3 + e^{-x} x^4$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -12*exp(-_a)*_a^2+36*_a*exp(-
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)+1*diff(y(x),x)-0*y(x)=-2*exp(-x)*(7-18*x+6*x^2),y(x),
```

$$y(x) = \left(x^4 - 2x^3 + x^2 + (-c_1 + 2)\,x - c_1 - c_2 + 2\right) e^{-x} + c_3$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 42

```
DSolve[y'''[x]+2*y''[x]+1*y'[x]-0*y[x]==-2*Exp[-x]*(7-18*x+6*x^2),y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{-x}\left(x^4 - 2x^3 + x^2 - (-2 + c_2)x + 2 - c_1 - c_2\right) + c_3$$

19.64 problem section 9.3, problem 64

Internal problem ID [1561]
Internal file name [`OUTPUT/1562_Sunday_June_05_2022_02_22_30_AM_22651286/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 64.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 3y'' + 3y' - y = (x+1)\,e^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + x^2 e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = x^2 e^x$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 3y' - y = (x + 1)\, e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(x + 1)\, e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, x^2 e^x, e^x\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x, x^2 e^x\}]$$

Since $x\, e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x, e^x x^3\}]$$

Since $x^2 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x^3, e^x x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x^3 + A_2 e^x x^4$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^x + 24A_2 e^x x = (x+1) e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{1}{24}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^3}{6} + \frac{e^x x^4}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 x\, e^x + x^2 e^x c_3\right) + \left(\frac{e^x x^3}{6} + \frac{e^x x^4}{24}\right) \end{aligned}$$

Which simplifies to

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{24}$$

Summary

The solution(s) found are the following

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{24} \tag{1}$$

Verification of solutions

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} + \frac{e^x x^4}{24}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=exp(x)*(1+x),y(x), singsol=all)
```

$$y(x) = \frac{e^x (x^4 + 4x^3 + (24c_3 + 3)\, x^2 + 24c_2 x + 24c_1)}{24}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 37

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]-1*y[x]==Exp[x]*(1+x),y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{24} e^x \left(x^4 + 4x^3 + 24c_3 x^2 + 24c_2 x + 24c_1\right)$$

19.65 problem section 9.3, problem 65

Internal problem ID [1562]
Internal file name [`OUTPUT/1563_Sunday_June_05_2022_02_22_32_AM_42329436/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 65.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' - 2y'' + y = -\mathrm{e}^{-x}(3x^2 - 9x + 4)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x} + c_4 x\,\mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= \mathrm{e}^{x} \\ y_4 &= x\,\mathrm{e}^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y'' + y = -\mathrm{e}^{-x}\left(3x^2 - 9x + 4\right)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\mathrm{e}^{-x}\left(3x^2 - 9x + 4\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x}, x\,\mathrm{e}^{-x}, \mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3, \mathrm{e}^{-x}x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x} + A_2 \mathrm{e}^{-x} x^3 + A_3 \mathrm{e}^{-x} x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&8A_1 e^{-x} + 24A_2 e^{-x} x - 24A_2 e^{-x} + 48A_3 e^{-x} x^2 - 96A_3 e^{-x} x + 24A_3 e^{-x} \\
&= -e^{-x}\left(3x^2 - 9x + 4\right)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{1}{16}, A_2 = \frac{1}{8}, A_3 = -\frac{1}{16}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{x^2 e^{-x}}{16} + \frac{e^{-x} x^3}{8} - \frac{e^{-x} x^4}{16}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + c_4 x\,e^{x}\right) + \left(\frac{x^2 e^{-x}}{16} + \frac{e^{-x} x^3}{8} - \frac{e^{-x} x^4}{16}\right)
\end{aligned}
$$

Which simplifies to

$$
y = (c_2 x + c_1)\,e^{-x} + e^{x}(c_4 x + c_3) + \frac{x^2 e^{-x}}{16} + \frac{e^{-x} x^3}{8} - \frac{e^{-x} x^4}{16}
$$

Summary

The solution(s) found are the following

$$
y = (c_2 x + c_1)\,e^{-x} + e^{x}(c_4 x + c_3) + \frac{x^2 e^{-x}}{16} + \frac{e^{-x} x^3}{8} - \frac{e^{-x} x^4}{16} \tag{1}
$$

Verification of solutions

$$
y = (c_2 x + c_1)\,e^{-x} + e^{x}(c_4 x + c_3) + \frac{x^2 e^{-x}}{16} + \frac{e^{-x} x^3}{8} - \frac{e^{-x} x^4}{16}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$4)-0*diff(y(x),x$3)-2*diff(y(x),x$2)+0*diff(y(x),x)+y(x)=-exp(-x)*(4-9*x+
```

$$y(x) = \frac{(-x^4 + 2x^3 + 16c_4x + x^2 + 16c_2)\, e^{-x}}{16} + e^{x}(c_3x + c_1)$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 61

```
DSolve[y''''[x]-0*y'''[x]-2*y''[x]+0*y'[x]+1*y[x]==-Exp[-x]*(4-9*x+3*x^2),y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{32}e^{-x}\left(-2x^4 + 4x^3 + 2x^2 + x\left(32c_4e^{2x} - 2 + 32c_2\right) + 32c_3e^{2x} - 3 + 32c_1\right)$$

19.66 problem section 9.3, problem 66

Internal problem ID [1563]
Internal file name [`OUTPUT/1564_Sunday_June_05_2022_02_22_34_AM_71224203/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 66.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$
\boxed{y''' + 2y'' - y' - 2y = \mathrm{e}^{-2x}((23 - 2x)\cos(x) + (8 - 9x)\sin(x))}
$$

This is higher order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$
y''' + 2y'' - y' - 2y = 0
$$

The characteristic equation is

$$
\lambda^3 + 2\lambda^2 - \lambda - 2 = 0
$$

The roots of the above equation are

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2 \\
\lambda_3 &= -1
\end{aligned}
$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' - y' - 2y = e^{-2x}((23 - 2x)\cos(x) + (8 - 9x)\sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}((23 - 2x)\cos(x) + (8 - 9x)\sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\cos(x), e^{-2x}\sin(x), e^{-2x}\cos(x)x, e^{-2x}\sin(x)x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}\cos(x) + A_2 e^{-2x}\sin(x) + A_3 e^{-2x}\cos(x)x + A_4 e^{-2x}\sin(x)x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&4A_4 e^{-2x}\sin(x)x + 4A_3 e^{-2x}\cos(x)x - 8A_4 e^{-2x}\cos(x) - 2A_3 e^{-2x}\sin(x)x \\ &+ 2A_4 e^{-2x}\cos(x)x + 8A_3 e^{-2x}\sin(x) + 2A_2 e^{-2x}\cos(x) - 2A_1 e^{-2x}\sin(x) \\ &+ 4A_1 e^{-2x}\cos(x) + 4A_2 e^{-2x}\sin(x) = e^{-2x}((23 - 2x)\cos(x) + (8 - 9x)\sin(x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{3}{2}, A_3 = \frac{1}{2}, A_4 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-2x} \cos(x) + \frac{3 e^{-2x} \sin(x)}{2} + \frac{e^{-2x} \cos(x) x}{2} - 2 e^{-2x} \sin(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x}\right) + \left(e^{-2x} \cos(x) + \frac{3 e^{-2x} \sin(x)}{2} + \frac{e^{-2x} \cos(x) x}{2} - 2 e^{-2x} \sin(x) x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{-2x} \cos(x) \\ & + \frac{3 e^{-2x} \sin(x)}{2} + \frac{e^{-2x} \cos(x) x}{2} - 2 e^{-2x} \sin(x) x \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{-2x} \cos(x) + \frac{3 e^{-2x} \sin(x)}{2} + \frac{e^{-2x} \cos(x) x}{2} - 2 e^{-2x} \sin(x) x$$

Verified OK.

19.66.1 Maple step by step solution

Let's solve

$y''' + 2y'' - y' - 2y = e^{-2x}((23 - 2x) \cos(x) + (8 - 9x) \sin(x))$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 2y - 2 e^{-2x} \cos(x) x - 9 e^{-2x} \sin(x) x + 23 e^{-2x} \cos(x) + 8 e^{-2x} \sin(x) - 2y'' + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + 2y'' - y' - 2y = -e^{-2x}(9 \sin(x) x + 2x \cos(x) - 8 \sin(x) - 23 \cos(x))$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -2\,\mathrm{e}^{-2x}\cos(x)\,x - 9\,\mathrm{e}^{-2x}\sin(x)\,x + 23\,\mathrm{e}^{-2x}\cos(x) + 8\,\mathrm{e}^{-2x}\sin(x) - 2y_3(x) + y_2(x) + 2$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -2\,\mathrm{e}^{-2x}\cos(x)\,x - 9\,\mathrm{e}^{-2x}\sin(x)\,x + 23\,\mathrm{e}^{-2x}\cos(x) + 8\,\mathrm{e}^{-}$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -9\,\mathrm{e}^{-2x}\sin(x)\,x - 2\,\mathrm{e}^{-2x}\cos(x)\,x + 8\,\mathrm{e}^{-2x}\sin(x) + 23\,\mathrm{e}^{-2x}\mathrm{c}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -9\,\mathrm{e}^{-2x}\sin(x)\,x - 2\,\mathrm{e}^{-2x}\cos(x)\,x + 8\,\mathrm{e}^{-2x}\sin(x) + 23\,\mathrm{e}^{-2x}\cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & e^x \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^x \\ e^{-2x} & e^{-x} & e^x \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & e^x \\ -\frac{e^{-2x}}{2} & -e^{-x} & e^x \\ e^{-2x} & e^{-x} & e^x \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(e^{3x}+3e^x-1)e^{-2x}}{3} & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{(e^{3x}-3e^x+2)e^{-2x}}{6} \\ \frac{(e^{3x}-3e^x+2)e^{-2x}}{3} & \frac{e^x}{2} + \frac{e^{-x}}{2} & \frac{(e^{3x}+3e^x-4)e^{-2x}}{6} \\ \frac{(e^{3x}+3e^x-4)e^{-2x}}{3} & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{(e^{3x}-3e^x+8)e^{-2x}}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{\left(7\,e^{3x}+(3x+6)\cos(x)+(-12x+9)\sin(x)-33\,e^{x}+20\right)e^{-2x}}{6}\\ \frac{\left(7\,e^{3x}+(21x-36)\sin(x)-18x\cos(x)+33\,e^{x}-40\right)e^{-2x}}{6}\\ \frac{\left(7\,e^{3x}+(57x-54)\cos(x)+(-24x+93)\sin(x)-33\,e^{x}+80\right)e^{-2x}}{6}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3+\left[\begin{array}{c}\frac{\left(7\,e^{3x}+(3x+6)\cos(x)+(-12x+9)\sin(x)-33\,e^{x}+20\right)e^{-2x}}{6}\\ \frac{\left(7\,e^{3x}+(21x-36)\sin(x)-18x\cos(x)+33\,e^{x}-40\right)e^{-2x}}{6}\\ \frac{\left(7\,e^{3x}+(57x-54)\cos(x)+(-24x+93)\sin(x)-33\,e^{x}+80\right)e^{-2x}}{6}\end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y=\frac{\left(12c_3e^{3x}+14\,e^{3x}+6x\cos(x)-24\sin(x)x+12c_2e^{x}+12\cos(x)+18\sin(x)-66\,e^{x}+3c_1+40\right)e^{-2x}}{12}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=exp(-2*x)*((23-2*x)*cos(x)+(8-9*x
```

$$y(x) = \frac{(2c_1 e^{3x} + (2+x)\cos(x) + (-4x+3)\sin(x) + 2c_3 e^{x} + 2c_2)\, e^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 48

```
DSolve[y'''[x]+2*y''[x]-y'[x]-2*y[x]==Exp[-2*x]*((23-2*x)*Cos[x]+(8-9*x)*Sin[x]),y[x],x,Incl
```

$$y(x) \to \frac{1}{2}e^{-2x}\left((3-4x)\sin(x) + (x+2)\cos(x) + 2\left(c_2 e^{x} + c_3 e^{3x} + c_1\right)\right)$$

19.67 problem section 9.3, problem 67

Internal problem ID [1564]
Internal file name [OUTPUT/1565_Sunday_June_05_2022_02_22_36_AM_67589364/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 67.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 3y''' + 4y'' - 2y' = \mathrm{e}^{x}((28 + 6x)\cos(2x) + (11 - 12x)\sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 4y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 4\lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 1\\ \lambda_3 &= 1 - i\\ \lambda_4 &= 1 + i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{(1-i)x}$$
$$y_4 = e^{(1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 4y'' - 2y' = e^x((28 + 6x)\cos(2x) + (11 - 12x)\sin(2x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x((28 + 6x)\cos(2x) + (11 - 12x)\sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x), \cos(2x) e^x x, \sin(2x) e^x x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{(1-i)x}, e^{(1+i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x) + A_3 \cos(2x) e^x x + A_4 \sin(2x) e^x x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_3 \cos(2x) e^x x - 11A_4 \sin(2x) e^x - 6A_4 \cos(2x) e^x x - 28A_4 \cos(2x) e^x - 11A_3 \cos(2x) e^x + 6A_3 \sin(2x) e^x x + 28A_3 \sin(2x) e^x + 12A_4 \sin(2x) e^x x - 6A_2 e^x \cos(2x) + 6A_1 e^x \sin(2x) + 12A_1 e^x \cos(2x) + 12A_2 e^x \sin(2x) = e^x((28 + 6x)\cos(2x) + (11 - 12x)\sin(2x))$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 0, A_4 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\sin(2x)\,e^x x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4\right) + \left(-\sin(2x)\,e^x x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 - \sin(2x)\,e^x x \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + e^{(1-i)x} c_3 + e^{(1+i)x} c_4 - \sin(2x)\,e^x x$$

Verified OK.

19.67.1 Maple step by step solution

Let's solve

$y'''' - 3y''' + 4y'' - 2y' = e^x\left((28 + 6x)\cos(2x) + (11 - 12x)\sin(2x)\right)$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = 6\cos(2x)\,e^x x - 12\sin(2x)\,e^x x + 28\,e^x \cos(2x) + 11\,e^x \sin(2x) + 3y''' - 4y'' + 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - 3y''' + 4y'' - 2y' = e^x\left(-12x\sin(2x) + 6x\cos(2x) + 11\sin(2x) + 28\cos(2x)\right)$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 6\cos(2x)\,e^x x - 12\sin(2x)\,e^x x + 28\,e^x\cos(2x) + 11\,e^x\sin(2x) + 3y_4(x) - 4y_3(x) + 2y_2($

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 6\cos(2x)\,e^x x - 12\sin(2x)\,e^x x + 28\,e^x \cos$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -4 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\sin(2x)\,e^x x + 6\cos(2x)\,e^x x + 11\,e^x\sin(2x) + 28\,e^x \text{co} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\sin(2x)\,e^x x + 6\cos(2x)\,e^x x + 11\,e^x\sin(2x) + 28\,e^x\cos(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} -\frac{1}{4} + \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} -\frac{1}{4} - \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}, \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-\mathrm{I})x} \cdot \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (-\mathrm{I}\sin(x)+\cos(x)) \cdot \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(-\mathrm{I}\sin(x)+\cos(x)) \\ \frac{\mathrm{I}}{2}(-\mathrm{I}\sin(x)+\cos(x)) \\ \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(-\mathrm{I}\sin(x)+\cos(x)) \\ -\mathrm{I}\sin(x)+\cos(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=e^{x}\cdot\begin{bmatrix}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\\ \frac{\sin(x)}{2}\\ \frac{\cos(x)}{2}+\frac{\sin(x)}{2}\\ \cos(x)\end{bmatrix},\ \vec{y}_4(x)=e^{x}\cdot\begin{bmatrix}\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\\ \frac{\cos(x)}{2}\\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2}\\ -\sin(x)\end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($…

$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst…

$$\phi(x)=\begin{bmatrix}1 & e^{x} & e^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & e^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right)\\ 0 & e^{x} & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2}\\ 0 & e^{x} & e^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)e^{x}\\ 0 & e^{x} & \cos(x)e^{x} & -\sin(x)e^{x}\end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t…

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix}1 & e^{x} & e^{x}\left(-\frac{\cos(x)}{4}+\frac{\sin(x)}{4}\right) & e^{x}\left(\frac{\sin(x)}{4}+\frac{\cos(x)}{4}\right)\\ 0 & e^{x} & \frac{\sin(x)e^{x}}{2} & \frac{\cos(x)e^{x}}{2}\\ 0 & e^{x} & e^{x}\left(\frac{\cos(x)}{2}+\frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2}-\frac{\sin(x)}{2}\right)e^{x}\\ 0 & e^{x} & \cos(x)e^{x} & -\sin(x)e^{x}\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}1 & 1 & -\frac{1}{4} & \frac{1}{4}\\ 0 & 1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix}1 & -2+(-\sin(x)+2)e^{x} & \frac{3}{2}+\frac{(-4+\cos(x)+3\sin(x))e^{x}}{2} & -\frac{1}{2}+\frac{e^{x}(2-\cos(x)-\sin(\dots}{2}\\ 0 & -e^{x}(\cos(x)+\sin(x)-2) & e^{x}(-2+2\cos(x)+\sin(x)) & -e^{x}(\cos(x)-1)\\ 0 & -2e^{x}(\cos(x)-1) & e^{x}(-2+3\cos(x)-\sin(x)) & -e^{x}(\cos(x)-\sin(x)\dots\\ 0 & -2e^{x}(\cos(x)-\sin(x)-1) & 2e^{x}(-1-2\sin(x)+\cos(x)) & e^{x}(1+2\sin(x))\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -2(\sin(x)\, x + 2)\, \mathrm{e}^{x} \cos(x) + 4\, \mathrm{e}^{x} \\ -2\left(2\cos(x)^{2} x + x\cos(x)\sin(x) + \cos(x)\sin(x) + 2\cos(x) - 2\sin(x) - x - 2\right) \mathrm{e}^{x} \\ -2\left(4\cos(x)^{2} x - 3x\cos(x)\sin(x) + 2\cos(x)\sin(x) - 4\sin(x)^{2} - 4\sin(x) - 2x\right) \mathrm{e}^{x} \\ 4\left(4 + (x-6)\cos(x)^{2} + \left(2 + \frac{(11x+9)\sin(x)}{2}\right)\cos(x) - \frac{x}{2} + 2\sin(x)\right) \mathrm{e}^{x} \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} -2(\sin(x)\, x + 2)\, \mathrm{e}^{x} \cos(\\ -2\left(2\cos(x)^{2} x + x\cos(x)\sin(x) + \cos(x)\sin(x \right. \\ -2\left(4\cos(x)^{2} x - 3x\cos(x)\sin(x) + 2\cos(x)\sin \right. \\ 4\left(4 + (x-6)\cos(x)^{2} + \left(2 + \frac{(11x+9)\sin(x)}{2}\right.\right. \end{array}\right.$$

- First component of the vector is the solution to the ODE

  $y = \frac{((-8\sin(x)x-c_3+c_4-16)\cos(x)+(c_3+c_4)\sin(x)+4c_2+16)e^x}{4} + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = -12*sin(2*_a)*exp(_
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)+4*diff(y(x),x$2)-2*diff(y(x),x)-0*y(x)=exp(x)*((28+6*
```

$$y(x) = \frac{((-4x\sin(x)+c_2-c_3)\cos(x)+(c_2+c_3)\sin(x)+2c_1+3)\,e^x}{2} + c_4$$

✓ Solution by Mathematica

Time used: 0.715 (sec). Leaf size: 43

```
DSolve[y''''[x]-3*y'''[x]+4*y''[x]-2*y'[x]-0*y[x]==Exp[x]*((28+6*x)*Cos[2*x]+(11-12*x)*Sin[2
```

$$y(x) \to \frac{1}{2}e^x((c_1+c_2)\sin(x)+\cos(x)(-4x\sin(x)-c_1+c_2)+2c_3)+c_4$$

19.68 problem section 9.3, problem 68

Internal problem ID [1565]
Internal file name [`OUTPUT/1566_Sunday_June_05_2022_02_22_39_AM_45324078/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 68.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 4y''' + 14y'' - 20y' + 25y = e^{x}((6x + 2)\cos(2x) + 3\sin(2x))}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 14y'' - 20y' + 25y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 14\lambda^2 - 20\lambda + 25 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - 2i$$
$$\lambda_2 = 1 + 2i$$
$$\lambda_3 = 1 - 2i$$
$$\lambda_4 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(1+2i)x}c_1 + x\,e^{(1+2i)x}c_2 + e^{(1-2i)x}c_3 + x\,e^{(1-2i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{(1+2i)x} \\ y_2 &= x\,e^{(1+2i)x} \\ y_3 &= e^{(1-2i)x} \\ y_4 &= x\,e^{(1-2i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 4y''' + 14y'' - 20y' + 25y = e^x((6x+2)\cos(2x) + 3\sin(2x))$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{(1+2i)x} & x\,e^{(1+2i)x} & e^{(1-2i)x} & x\,e^{(1-2i)x} \\ (1+2i)\,e^{(1+2i)x} & (1+(1+2i)\,x)\,e^{(1+2i)x} & (1-2i)\,e^{(1-2i)x} & e^{(1-2i)x}(-2ix+ \\ (-3+4i)\,e^{(1+2i)x} & (2+4i+(-3+4i)\,x)\,e^{(1+2i)x} & (-3-4i)\,e^{(1-2i)x} & -4\,e^{(1-2i)x}\left(-\frac{1}{2}+i+\right. \\ (-11-2i)\,e^{(1+2i)x} & -2\left(\frac{9}{2}-6i+\left(\frac{11}{2}+i\right)x\right)e^{(1+2i)x} & (-11+2i)\,e^{(1-2i)x} & (-9-12i+(-11+ \end{bmatrix}$$

$$|W| = 256\,e^{(2+4i)x}e^{(2-4i)x}$$

The determinant simplifies to

$$|W| = 256\,\mathrm{e}^{4x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} x\,\mathrm{e}^{(1+2i)x} & \mathrm{e}^{(1-2i)x} & x\,\mathrm{e}^{(1-2i)x} \\ (1+(1+2i)\,x)\,\mathrm{e}^{(1+2i)x} & (1-2i)\,\mathrm{e}^{(1-2i)x} & \mathrm{e}^{(1-2i)x}(-2ix+x+1) \\ (2+4i+(-3+4i)\,x)\,\mathrm{e}^{(1+2i)x} & (-3-4i)\,\mathrm{e}^{(1-2i)x} & -4\,\mathrm{e}^{(1-2i)x}\left(-\frac{1}{2}+i+\left(\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= 8\,\mathrm{e}^{(3-2i)x}(i-2x)$$

$$W_2(x) = \det\begin{bmatrix} \mathrm{e}^{(1+2i)x} & \mathrm{e}^{(1-2i)x} & x\,\mathrm{e}^{(1-2i)x} \\ (1+2i)\,\mathrm{e}^{(1+2i)x} & (1-2i)\,\mathrm{e}^{(1-2i)x} & \mathrm{e}^{(1-2i)x}(-2ix+x+1) \\ (-3+4i)\,\mathrm{e}^{(1+2i)x} & (-3-4i)\,\mathrm{e}^{(1-2i)x} & -4\,\mathrm{e}^{(1-2i)x}\left(-\frac{1}{2}+i+\left(\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= -16\,\mathrm{e}^{(3-2i)x}$$

$$W_3(x) = \det\begin{bmatrix} \mathrm{e}^{(1+2i)x} & x\,\mathrm{e}^{(1+2i)x} & x\,\mathrm{e}^{(1-2i)x} \\ (1+2i)\,\mathrm{e}^{(1+2i)x} & (1+(1+2i)\,x)\,\mathrm{e}^{(1+2i)x} & \mathrm{e}^{(1-2i)x}(-2ix+x+1) \\ (-3+4i)\,\mathrm{e}^{(1+2i)x} & (2+4i+(-3+4i)\,x)\,\mathrm{e}^{(1+2i)x} & -4\,\mathrm{e}^{(1-2i)x}\left(-\frac{1}{2}+i+\left(\frac{3}{4}+i\right)x\right) \end{bmatrix}$$
$$= -8\,\mathrm{e}^{(3+2i)x}(i+2x)$$

$$W_4(x) = \det\begin{bmatrix} \mathrm{e}^{(1+2i)x} & x\,\mathrm{e}^{(1+2i)x} & \mathrm{e}^{(1-2i)x} \\ (1+2i)\,\mathrm{e}^{(1+2i)x} & (1+(1+2i)\,x)\,\mathrm{e}^{(1+2i)x} & (1-2i)\,\mathrm{e}^{(1-2i)x} \\ (-3+4i)\,\mathrm{e}^{(1+2i)x} & (2+4i+(-3+4i)\,x)\,\mathrm{e}^{(1+2i)x} & (-3-4i)\,\mathrm{e}^{(1-2i)x} \end{bmatrix}$$
$$= -16\,\mathrm{e}^{(3+2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(\mathrm{e}^{x}((6x+2)\cos(2x)+3\sin(2x))\right)\left(8\,\mathrm{e}^{(3-2i)x}(i-2x)\right)}{(1)\left(256\,\mathrm{e}^{4x}\right)}\,dx \\
&= -\int \frac{8\,\mathrm{e}^{x}((6x+2)\cos(2x)+3\sin(2x))\,\mathrm{e}^{(3-2i)x}(i-2x)}{256\,\mathrm{e}^{4x}}\,dx \\
&= -\int \left(\frac{(i-2x)\,(6x\cos(2x)+3\sin(2x)+2\cos(2x))\,\mathrm{e}^{-2ix}}{32}\right)dx \\
&= -\left(\int \frac{(i-2x)\,(6x\cos(2x)+3\sin(2x)+2\cos(2x))\,\mathrm{e}^{-2ix}}{32}dx\right) \\
&= -\left(\int \frac{(i-2x)\,(6x\cos(2x)+3\sin(2x)+2\cos(2x))\,\mathrm{e}^{-2ix}}{32}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(\mathrm{e}^{x}((6x+2)\cos(2x)+3\sin(2x))\right)\left(-16\,\mathrm{e}^{(3-2i)x}\right)}{(1)\left(256\,\mathrm{e}^{4x}\right)}\,dx \\
&= \int \frac{-16\,\mathrm{e}^{x}((6x+2)\cos(2x)+3\sin(2x))\,\mathrm{e}^{(3-2i)x}}{256\,\mathrm{e}^{4x}}\,dx \\
&= \int \left(\frac{((-6x-2)\cos(2x)-3\sin(2x))\,\mathrm{e}^{-2ix}}{16}\right)dx \\
&= \frac{3ix}{32} - \frac{3x^2}{32} - \frac{x}{16} - \frac{i(4+3i+12x)\,\mathrm{e}^{-4ix}}{256} \\
&= \frac{3ix}{32} - \frac{3x^2}{32} - \frac{x}{16} - \frac{i(4+3i+12x)\,\mathrm{e}^{-4ix}}{256}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\mathrm{e}^x((6x+2)\cos(2x)+3\sin(2x))\right)\left(-8\,\mathrm{e}^{(3+2i)x}(i+2x)\right)}{(1)\left(256\,\mathrm{e}^{4x}\right)}\, dx \\
&= -\int \frac{-8\,\mathrm{e}^x((6x+2)\cos(2x)+3\sin(2x))\,\mathrm{e}^{(3+2i)x}(i+2x)}{256\,\mathrm{e}^{4x}}\, dx \\
&= -\int \left(-\frac{(i+2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{2ix}}{32}\right) dx \\
&= -\left(\int -\frac{(i+2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{2ix}}{32} dx\right) \\
&= -\left(\int -\frac{(i+2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{2ix}}{32} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\mathrm{e}^x((6x+2)\cos(2x)+3\sin(2x))\right)\left(-16\,\mathrm{e}^{(3+2i)x}\right)}{(1)\left(256\,\mathrm{e}^{4x}\right)}\, dx \\
&= \int \frac{-16\,\mathrm{e}^x((6x+2)\cos(2x)+3\sin(2x))\,\mathrm{e}^{(3+2i)x}}{256\,\mathrm{e}^{4x}}\, dx \\
&= \int \left(\frac{((-6x-2)\cos(2x)-3\sin(2x))\,\mathrm{e}^{2ix}}{16}\right) dx \\
&= -\frac{3ix}{32}-\frac{3x^2}{32}-\frac{x}{16}+\frac{i(4-3i+12x)\,\mathrm{e}^{4ix}}{256} \\
&= -\frac{3ix}{32}-\frac{3x^2}{32}-\frac{x}{16}+\frac{i(4-3i+12x)\,\mathrm{e}^{4ix}}{256}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\left(\int \frac{(i-2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{-2ix}}{32} dx\right)\right)\left(\mathrm{e}^{(1+2i)x}\right) \\
&+ \left(\frac{3ix}{32}-\frac{3x^2}{32}-\frac{x}{16}-\frac{i(4+3i+12x)\,\mathrm{e}^{-4ix}}{256}\right)\left(x\,\mathrm{e}^{(1+2i)x}\right) \\
&+ \left(-\left(\int -\frac{(i+2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{2ix}}{32} dx\right)\right)\left(\mathrm{e}^{(1-2i)x}\right) \\
&+ \left(-\frac{3ix}{32}-\frac{3x^2}{32}-\frac{x}{16}+\frac{i(4-3i+12x)\,\mathrm{e}^{4ix}}{256}\right)\left(x\,\mathrm{e}^{(1-2i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = -\frac{\left(\int (i-2x)\left(6x\cos(2x)+3\sin(2x)+2\cos(2x)\right)\mathrm{e}^{-2ix}dx\right)\mathrm{e}^{(1+2i)x}}{32} + \frac{\left(\int (i+2x)\left(6x\cos(2x)+3\sin\right.\right.}{3}$$

Which simplifies to

$$y_p = -\frac{\mathrm{e}^{x}\left(24x^3\cos(2x)+16x^2\cos(2x)+4i\left(\int \left((-12x^2-4x)\cos(2x)^2+2\cos(2x)\sin(2x)+3\sin(2x)^2\right)\right.\right.}{}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{(1+2i)x}c_1 + x\,\mathrm{e}^{(1+2i)x}c_2 + \mathrm{e}^{(1-2i)x}c_3 + x\,\mathrm{e}^{(1-2i)x}c_4\right) \\ &\quad + \left(-\frac{\mathrm{e}^{x}\left(24x^3\cos(2x)+16x^2\cos(2x)+4i\left(\int \left((-12x^2-4x)\cos(2x)^2+2\cos(2x)\sin(2x)+3\sin(2x\right.\right.\right.}{}\right. \end{aligned}$$

Which simplifies to

$$\begin{aligned} y &= (c_4x+c_3)\,\mathrm{e}^{(1-2i)x} + \mathrm{e}^{(1+2i)x}(c_2x+c_1) \\ &\quad - \frac{\mathrm{e}^{x}\left(24x^3\cos(2x)+16x^2\cos(2x)+4i\left(\int \left((-12x^2-4x)\cos(2x)^2+2\cos(2x)\sin(2x)+3\sin(2x)^2\right)dx\right.\right.}{} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= (c_4x+c_3)\,\mathrm{e}^{(1-2i)x} + \mathrm{e}^{(1+2i)x}(c_2x+c_1) \\ &\quad - \frac{\mathrm{e}^{x}\left(24x^3\cos(2x)+16x^2\cos(2x)+4i\left(\int \left((-12x^2-4x)\cos(2x)^2+2\cos(2x)\sin(2x)+3\sin(2x)^2\right)dx\right.\right.}{} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= (c_4x+c_3)\,\mathrm{e}^{(1-2i)x} + \mathrm{e}^{(1+2i)x}(c_2x+c_1) \\ &\quad - \frac{\mathrm{e}^{x}\left(24x^3\cos(2x)+16x^2\cos(2x)+4i\left(\int \left((-12x^2-4x)\cos(2x)^2+2\cos(2x)\sin(2x)+3\sin(2x)^2\right)dx\right.\right.}{} \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+14*diff(y(x),x$2)-20*diff(y(x),x)+25*y(x)=exp(x)*((2+
```

$$y(x) = -\frac{e^{x}\left(\left(x^{3}+x^{2}+\left(-16c_{3}+\frac{63}{2}\right)x-16c_{1}-\frac{41}{4}\right)\cos\left(2x\right)-16\left(\left(c_{4}+\frac{1}{32}\right)x+c_{2}+\frac{1111}{192}\right)\sin\left(2x\right)\right)}{16}$$

✓ Solution by Mathematica

Time used: 0.674 (sec). Leaf size: 62

```
DSolve[y''''[x]-4*y'''[x]+14*y''[x]-20*y'[x]+25*y[x]==Exp[x]*((2+6*x)*Cos[2*x]+3*Sin[2*x]),y
```

$$y(x) \to \frac{1}{256}e^{x}\left(\left(-16x^{3}-16x^{2}+(6+256c_{4})x+6+256c_{3}\right)\cos(2x) + (8(3+32c_{2})x+3+256c_{1})\sin(2x)\right)$$

19.69 problem section 9.3, problem 69

Internal problem ID [1566]
Internal file name [`OUTPUT/1567_Sunday_June_05_2022_02_22_44_AM_43420082/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 69.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' - 5y' + 6y = 2\,\mathrm{e}^{x}(1 - 6x)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 7, y''(0) = 9]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' - 5y' + 6y = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^x$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' - 5y' + 6y = 2\,e^x(1 - 6x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,e^x(1 - 6x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}, e^{3x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x, x^2 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x + A_2 x^2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1e^x + 2A_2e^x - 12A_2x\,e^x = 2\,e^x(1-6x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-2x} + c_2e^x + c_3e^{3x}\right) + \left(x^2e^x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1e^{-2x} + c_2e^x + c_3e^{3x} + x^2e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1e^{-2x} + c_2e^x + 3c_3e^{3x} + 2x\,e^x + x^2e^x$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -2c_1 + c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 4c_1e^{-2x} + c_2e^x + 9c_3e^{3x} + 4x\,e^x + 2\,e^x + x^2e^x$$

substituting $y'' = 9$ and $x = 0$ in the above gives

$$9 = 4c_1 + c_2 + 9c_3 + 2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 2$$
$$c_3 = 1$$

Substituting these values back in above solution results in

$$y = x^2 e^x - e^{-2x} + e^{3x} + 2 e^x$$

Summary

The solution(s) found are the following

$$y = x^2 e^x - e^{-2x} + e^{3x} + 2 e^x \tag{1}$$

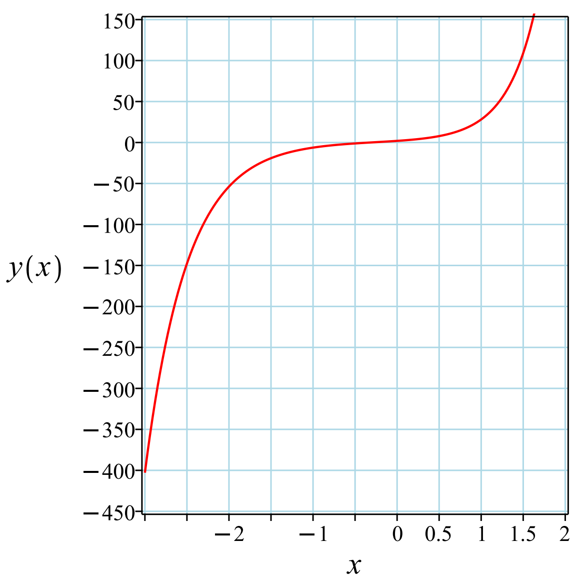


Figure 553: Solution plot

Verification of solutions

$$y = x^2 e^x - e^{-2x} + e^{3x} + 2 e^x$$

Verified OK.

19.69.1 Maple step by step solution

Let's solve

$$\left[y''' - 2y'' - 5y' + 6y = 2\,\mathrm{e}^{x}(1-6x)\,, y(0) = 2, y'\Big|_{\{x=0\}} = 7, y''\Big|_{\{x=0\}} = 9\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -6y - 12x\,\mathrm{e}^{x} + 2y'' + 5y' + 2\,\mathrm{e}^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 2y'' - 5y' + 6y = -2\,\mathrm{e}^{x}(-1+6x)$

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -12x\,\mathrm{e}^{x} + 2y_3(x) + 5y_2(x) - 6y_1(x) + 2\,\mathrm{e}^{x}$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -12x\,\mathrm{e}^{x} + 2y_3(x) + 5y_2(x) - 6y_1(x) + 2\,\mathrm{e}^{x}]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -12x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ -12x\,\mathrm{e}^{x}+2\,\mathrm{e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A\cdot \vec{y}(x)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{1} = \mathrm{e}^{-2x}\cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{3x}}{9} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{3x}}{3} \\ e^{-2x} & e^{x} & e^{3x} \end{bmatrix}$$

  ◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  ◦ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{3x}}{9} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{3x}}{3} \\ e^{-2x} & e^{x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{9} \\ -\frac{1}{2} & 1 & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

  ◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{(e^{5x}-5e^{3x}-1)e^{-2x}}{5} & \frac{(3e^{5x}+5e^{3x}-8)e^{-2x}}{30} & \frac{(3e^{5x}-5e^{3x}+2)e^{-2x}}{30} \\ -\frac{(3e^{5x}-5e^{3x}+2)e^{-2x}}{5} & \frac{(9e^{5x}+5e^{3x}+16)e^{-2x}}{30} & \frac{(9e^{5x}-5e^{3x}-4)e^{-2x}}{30} \\ -\frac{(9e^{5x}-5e^{3x}-4)e^{-2x}}{5} & \frac{(27e^{5x}+5e^{3x}-32)e^{-2x}}{30} & \frac{(27e^{5x}-5e^{3x}+8)e^{-2x}}{30} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

    $$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{(-3e^{5x}+15x^2e^{3x}+5e^{3x}-2)e^{-2x}}{15} \\ \frac{(15x^2+30x+5)e^{-2x}e^{3x}}{15} + \frac{(-9e^{5x}+4)e^{-2x}}{15} \\ \frac{(15x^2+60x+35)e^{-2x}e^{3x}}{15} + \frac{(-27e^{5x}-8)e^{-2x}}{15} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} \frac{\left(-3\,\mathrm{e}^{5x}+15x^2\mathrm{e}^{3x}+5\,\mathrm{e}^{3x}-2\right)\mathrm{e}^{-2x}}{15} \\ \frac{\left(15x^2+30x+5\right)\mathrm{e}^{-2x}\mathrm{e}^{3x}}{15} + \frac{\left(-9\,\mathrm{e}^{5x}+4\right)\mathrm{e}^{-2x}}{15} \\ \frac{\left(15x^2+60x+35\right)\mathrm{e}^{-2x}\mathrm{e}^{3x}}{15} + \frac{\left(-27\,\mathrm{e}^{5x}-8\right)\mathrm{e}^{-2x}}{15} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{\left(180x^2+180c_2+60\right)\mathrm{e}^{-2x}\mathrm{e}^{3x}}{180} + \frac{\left(20c_3\mathrm{e}^{5x}+45c_1-36\,\mathrm{e}^{5x}-24\right)\mathrm{e}^{-2x}}{180}$

- Use the initial condition $y(0) = 2$

  $2 = c_2 + \frac{c_3}{9} + \frac{c_1}{4}$

- Calculate the 1st derivative of the solution

  $y' = 2x\,\mathrm{e}^{-2x}\mathrm{e}^{3x} + \frac{\left(180x^2+180c_2+60\right)\mathrm{e}^{-2x}\mathrm{e}^{3x}}{180} + \frac{\left(100c_3\mathrm{e}^{5x}-180\,\mathrm{e}^{5x}\right)\mathrm{e}^{-2x}}{180} - \frac{\left(20c_3\mathrm{e}^{5x}+45c_1-36\,\mathrm{e}^{5x}-24\right)\mathrm{e}^{-2x}}{90}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 7$

  $7 = c_2 + \frac{c_3}{3} - \frac{c_1}{2}$

- Calculate the 2nd derivative of the solution

  $y'' = 2\,\mathrm{e}^{-2x}\mathrm{e}^{3x} + 4x\,\mathrm{e}^{-2x}\mathrm{e}^{3x} + \frac{\left(180x^2+180c_2+60\right)\mathrm{e}^{-2x}\mathrm{e}^{3x}}{180} + \frac{\left(500c_3\mathrm{e}^{5x}-900\,\mathrm{e}^{5x}\right)\mathrm{e}^{-2x}}{180} - \frac{\left(100c_3\mathrm{e}^{5x}-180\,\mathrm{e}^{5x}\right)\mathrm{e}^{-2x}}{45} +$

- Use the initial condition $y''\Big|_{\{x=0\}} = 9$

  $9 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = -\frac{52}{15}, c_2 = \frac{5}{3}, c_3 = \frac{54}{5}\right\}$

- Solution to the IVP

  $y = \left(\mathrm{e}^{5x} + x^2\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x} - 1\right)\mathrm{e}^{-2x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve([diff(y(x),x$3)-2*diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=2*exp(x)*(1-6*x),y(0) = 2, D(y
```

$$y(x) = \left(e^{5x} + x^2 e^{3x} + 2\, e^{3x} - 1\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 27

```
DSolve[{y'''[x]-2*y''[x]-5*y'[x]+6*y[x]==2*Exp[x]*(1-6*x),{y[0]==2,y'[0]==7,y''[0]==9}},y[x]
```

$$y(x) \to e^x \left(x^2 + 2\right) - e^{-2x} + e^{3x}$$

19.70 problem section 9.3, problem 70

Internal problem ID [1567]
Internal file name [`OUTPUT/1568_Sunday_June_05_2022_02_22_47_AM_66778376/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 70.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - y' + y = -e^{-x}(4 - 8x)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0, y''(0) = 0]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - y' + y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + x e^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = x e^{x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - y' + y = -e^{-x}(4 - 8x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{-x}(4 - 8x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{x}, e^{x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-x}, x^2 e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{-x} + A_2 x^2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{-x} - 8A_2\mathrm{e}^{-x} + 8A_2x\,\mathrm{e}^{-x} = -\mathrm{e}^{-x}(4-8x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + x\,\mathrm{e}^{x}c_3\right) + \left(x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = c_1\mathrm{e}^{-x} + \mathrm{e}^{x}(c_3x + c_2) + x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-x} + \mathrm{e}^{x}(c_3x + c_2) + x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1\mathrm{e}^{-x} + \mathrm{e}^{x}(c_3x + c_2) + c_3\mathrm{e}^{x} + \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} - x^2\mathrm{e}^{-x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 1 - c_1 + c_2 + c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = c_1 e^{-x} + e^{x}(c_3 x + c_2) + 2c_3 e^{x} - 3x\, e^{-x} + x^2 e^{-x}$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + 2c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1 \\ c_3 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2 e^{-x} + x\, e^{-x} - x\, e^{x} + e^{-x} + e^{x}$$

Which simplifies to

$$y = (x^2 + x + 1)\, e^{-x} - (x - 1)\, e^{x}$$

Summary

The solution(s) found are the following

$$y = (x^2 + x + 1)\, e^{-x} - (x - 1)\, e^{x} \tag{1}$$

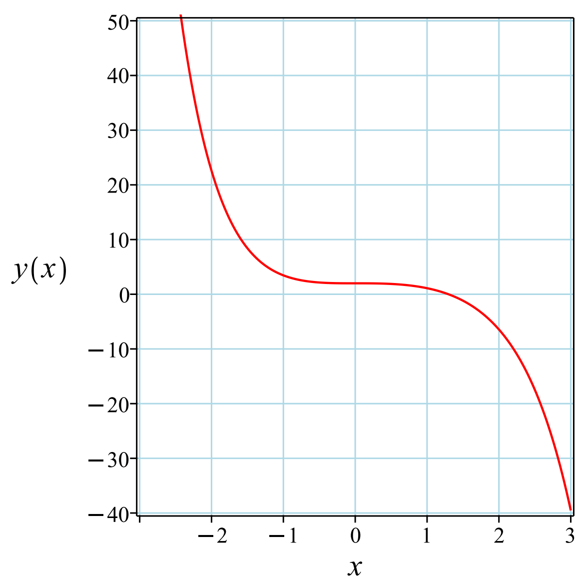


Figure 554: Solution plot

Verification of solutions

$$y = (x^2 + x + 1) e^{-x} - (x - 1) e^{x}$$

Verified OK.

19.70.1 Maple step by step solution

Let's solve

$$\left[y''' - y'' - y' + y = -e^{-x}(4 - 8x), y(0) = 2, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -y + 8x e^{-x} - 4 e^{-x} + y'' + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - y'' - y' + y = 4 e^{-x}(2x - 1)$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8x e^{-x} - 4 e^{-x} + y_3(x) + y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8x e^{-x} - 4 e^{-x} + y_3(x) + y_2(x) - y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 8x\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 8x\,\mathrm{e}^{-x} - 4\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_{3}(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{3}(x)$ into the homogeneous system

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \mathrm{e}^{\lambda x} \left(\lambda x \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{3}(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{array}\right] - 1 \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_3(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & e^x & (x-1)\,e^x \\ -e^{-x} & e^x & x\,e^x \\ e^{-x} & e^x & x\,e^x \end{bmatrix}$$

- ◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- ◦ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & e^x & (x-1)\,e^x \\ -e^{-x} & e^x & x\,e^x \\ e^{-x} & e^x & x\,e^x \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- ◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -(x-1)\,e^x & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{e^{-x}}{2} - \frac{e^x}{2} + x\,e^x \\ -x\,e^x & \frac{e^x}{2} + \frac{e^{-x}}{2} & -\frac{e^{-x}}{2} + \frac{e^x}{2} + x\,e^x \\ -x\,e^x & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{e^{-x}}{2} + \frac{e^x}{2} + x\,e^x \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- ◦ Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \left[\begin{array}{c} (2x^2+2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \\ (-2x^2+2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \\ (2x^2-2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} (2x^2+2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \\ (-2x^2+2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \\ (2x^2-2x+1)\,\mathrm{e}^{-x} - \mathrm{e}^{x} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = (2x^2 + c_1 + 2x + 1)\,\mathrm{e}^{-x} + \mathrm{e}^{x}(c_3 x + c_2 - c_3 - 1)$

- Use the initial condition $y(0) = 2$

  $2 = c_1 + c_2 - c_3$

- Calculate the 1st derivative of the solution

  $y' = (4x+2)\,e^{-x} - (2x^2 + c_1 + 2x + 1)\,e^{-x} + e^x(c_3 x + c_2 - c_3 - 1) + c_3 e^x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -c_1 + c_2$

- Calculate the 2nd derivative of the solution

  $y'' = 4\,e^{-x} - 2(4x+2)\,e^{-x} + (2x^2 + c_1 + 2x + 1)\,e^{-x} + e^x(c_3 x + c_2 - c_3 - 1) + 2c_3 e^x$

- Use the initial condition $y''\Big|_{\{x=0\}} = 0$

  $0 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{1}{2}, c_2 = \frac{1}{2}, c_3 = -1\right\}$

- Solution to the IVP

  $y = \frac{(4x^2+4x+3)\,e^{-x}}{2} + \frac{(1-2x)\,e^x}{2}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$3)-1*diff(y(x),x$2)-1*diff(y(x),x)+1*y(x)=-exp(-x)*(4-8*x),y(0) = 2, D(y
```

$$y(x) = (x^2 + x + 1)\,e^{-x} - (x-1)\,e^x$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 27

```
DSolve[{y'''[x]-1*y''[x]-1*y'[x]+1*y[x]==-Exp[-x]*(4-8*x),{y[0]==2,y'[0]==0,y''[0]==0}},y[x]
```

$$y(x) \to e^{-x}\left(x^2 + x - e^{2x}(x-1) + 1\right)$$

19.71 problem section 9.3, problem 71

Internal problem ID [1568]
Internal file name [`OUTPUT/1569_Sunday_June_05_2022_02_22_49_AM_98419638/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 71.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y''' - 3y' - y = e^{-\frac{x}{2}}(-3x + 2)}$$

With initial conditions

$$[y(0) = -1, y'(0) = 15, y''(0) = -17]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$4y''' - 3y' - y = 0$$

The characteristic equation is

$$4\lambda^3 - 3\lambda - 1 = 0$$

The roots of the above equation are

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -\frac{1}{2} \\
\lambda_3 &= -\frac{1}{2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
y_h(x) = c_1 e^{x} + c_2 e^{-\frac{x}{2}} + c_3 x \, e^{-\frac{x}{2}}
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{x} \\
y_2 &= e^{-\frac{x}{2}} \\
y_3 &= x \, e^{-\frac{x}{2}}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
4y''' - 3y' - y = e^{-\frac{x}{2}} (-3x + 2)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{-\frac{x}{2}} (-3x + 2)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x \, e^{-\frac{x}{2}}, e^{-\frac{x}{2}}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x \, e^{-\frac{x}{2}}, e^{x}, e^{-\frac{x}{2}}\}
$$

Since $e^{-\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x \, e^{-\frac{x}{2}}, e^{-\frac{x}{2}} x^2\}]
$$

Since $x \, e^{-\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{e^{-\frac{x}{2}} x^2, e^{-\frac{x}{2}} x^3\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-\frac{x}{2}} x^2 + A_2 e^{-\frac{x}{2}} x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 e^{-\frac{x}{2}} - 36A_2 e^{-\frac{x}{2}} x + 24A_2 e^{-\frac{x}{2}} = e^{-\frac{x}{2}}(-3x + 2)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-\frac{x}{2}} x^3}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-\frac{x}{2}} + c_3 x\, e^{-\frac{x}{2}}\right) + \left(\frac{e^{-\frac{x}{2}} x^3}{12}\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\, e^{-\frac{x}{2}} + c_1 e^x + \frac{e^{-\frac{x}{2}} x^3}{12}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_3 x + c_2)\, e^{-\frac{x}{2}} + c_1 e^x + \frac{e^{-\frac{x}{2}} x^3}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{-\frac{x}{2}} c_3 - \frac{(c_3 x + c_2) e^{-\frac{x}{2}}}{2} + c_1 e^x - \frac{e^{-\frac{x}{2}} x^3}{24} + \frac{e^{-\frac{x}{2}} x^2}{4}$$

substituting $y' = 15$ and $x = 0$ in the above gives

$$15 = c_3 - \frac{c_2}{2} + c_1 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = -e^{-\frac{x}{2}} c_3 + \frac{(c_3 x + c_2) e^{-\frac{x}{2}}}{4} + c_1 e^x + \frac{e^{-\frac{x}{2}} x^3}{48} - \frac{e^{-\frac{x}{2}} x^2}{4} + \frac{x e^{-\frac{x}{2}}}{2}$$

substituting $y'' = -17$ and $x = 0$ in the above gives

$$-17 = -c_3 + \frac{c_2}{4} + c_1 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 0 \\ c_3 &= 16 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 16x e^{-\frac{x}{2}} - e^x + \frac{e^{-\frac{x}{2}} x^3}{12}$$

Which simplifies to

$$y = \frac{(x^3 + 192x) e^{-\frac{x}{2}}}{12} - e^x$$

Summary

The solution(s) found are the following

$$y = \frac{(x^3 + 192x) e^{-\frac{x}{2}}}{12} - e^x \tag{1}$$

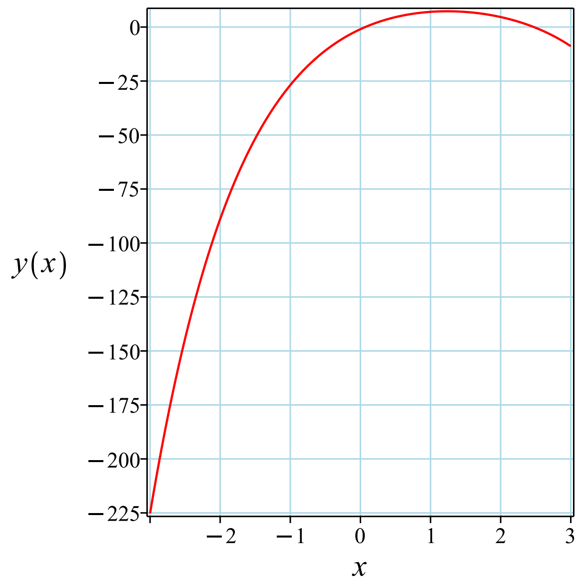


Figure 555: Solution plot

Verification of solutions

$$y = \frac{(x^3 + 192x)\,e^{-\frac{x}{2}}}{12} - e^x$$

Verified OK.

19.71.1 Maple step by step solution

Let's solve

$$\left[4y''' - 3y' - y = e^{-\frac{x}{2}}(-3x+2)\,, y(0) = -1, y'\Big|_{\{x=0\}} = 15, y''\Big|_{\{x=0\}} = -17\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y}{4} + \frac{3y'}{4} - \frac{3x\,e^{-\frac{x}{2}}}{4} + \frac{e^{-\frac{x}{2}}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y'}{4} - \frac{y}{4} = -\frac{e^{-\frac{x}{2}}(3x-2)}{4}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

$y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -\frac{3x\,\mathrm{e}^{-\frac{x}{2}}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}}{2} + \frac{3y_2(x)}{4} + \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = -\frac{3x\,\mathrm{e}^{-\frac{x}{2}}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}}{2} + \frac{3y_2(x)}{4} + \frac{y_1(x)}{4}\right]$

- Define vector

  $$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

  $$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ -\frac{3x\,\mathrm{e}^{-\frac{x}{2}}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}}{2} \end{array}\right]$$

- Define the forcing function

  $$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ -\frac{3x\,\mathrm{e}^{-\frac{x}{2}}}{4} + \frac{\mathrm{e}^{-\frac{x}{2}}}{2} \end{array}\right]$$

- Define the coefficient matrix

  $$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{array}\right]$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue $-\frac{1}{2}$

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ -2 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -\frac{1}{2}$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} - -\frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $-\frac{1}{2}$

$$\vec{y}_2(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 4\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{-\frac{x}{2}}(8+4x) & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-\frac{x}{2}} & -2x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{x} \\ \mathrm{e}^{-\frac{x}{2}} & x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 4\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{-\frac{x}{2}}(8+4x) & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-\frac{x}{2}} & -2x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{x} \\ \mathrm{e}^{-\frac{x}{2}} & x\,\mathrm{e}^{-\frac{x}{2}} & \mathrm{e}^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 4 & 8 & 1 \\ -2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{-\frac{x}{2}}(2+x)}{2} & \frac{\mathrm{e}^{-\frac{x}{2}}(3x-2)}{6} + \frac{\mathrm{e}^{x}}{3} & -\frac{2\,\mathrm{e}^{-\frac{x}{2}}}{3} - x\,\mathrm{e}^{-\frac{x}{2}} + \frac{2\,\mathrm{e}^{x}}{3} \\ -\frac{x\,\mathrm{e}^{-\frac{x}{2}}}{4} & \frac{(-3x+8)\mathrm{e}^{-\frac{x}{2}}}{12} + \frac{\mathrm{e}^{x}}{3} & \frac{(3x-4)\mathrm{e}^{-\frac{x}{2}}}{6} + \frac{2\,\mathrm{e}^{x}}{3} \\ \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8} & \frac{(3x-8)\mathrm{e}^{-\frac{x}{2}}}{24} + \frac{\mathrm{e}^{x}}{3} & \frac{(4-3x)\mathrm{e}^{-\frac{x}{2}}}{12} + \frac{2\,\mathrm{e}^{x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{-\frac{x}{2}} x^3}{8} \\ -\frac{e^{-\frac{x}{2}} x^2 (x-6)}{16} \\ \frac{e^{-\frac{x}{2}} x(x^2-6x+16)}{32} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + \begin{bmatrix} \frac{e^{-\frac{x}{2}} x^3}{8} \\ -\frac{e^{-\frac{x}{2}} x^2 (x-6)}{16} \\ \frac{e^{-\frac{x}{2}} x(x^2-6x+16)}{32} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(32(2+x)c_2 + x^3 + 32c_1) e^{-\frac{x}{2}}}{8} + c_3 e^x$$

- Use the initial condition $y(0) = -1$

$$-1 = 8c_2 + 4c_1 + c_3$$

- Calculate the 1st derivative of the solution

$$y' = \frac{(3x^2 + 32c_2) e^{-\frac{x}{2}}}{8} - \frac{(32(2+x)c_2 + x^3 + 32c_1) e^{-\frac{x}{2}}}{16} + c_3 e^x$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 15$

$$15 = -2c_1 + c_3$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{3x e^{-\frac{x}{2}}}{4} - \frac{(3x^2 + 32c_2) e^{-\frac{x}{2}}}{8} + \frac{(32(2+x)c_2 + x^3 + 32c_1) e^{-\frac{x}{2}}}{32} + c_3 e^x$$

- Use the initial condition $y''\Big|_{\{x=0\}} = -17$

$$-17 = -2c_2 + c_1 + c_3$$

- Solve for the unknown coefficients

$$\{c_1 = -8, c_2 = 4, c_3 = -1\}$$

- Solution to the IVP

$$y = \frac{(x^3 + 128x) e^{-\frac{x}{2}}}{8} - e^x$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 22

```
dsolve([4*diff(y(x),x$3)-0*diff(y(x),x$2)-3*diff(y(x),x)-1*y(x)=exp(-x/2)*(2-3*x),y(0) = -1,
```

$$y(x) = \frac{(x^3 + 192x)\,e^{-\frac{x}{2}}}{12} - e^{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 35

```
DSolve[{4*y'''[x]-0*y''[x]-3*y'[x]-1*y[x]==Exp[-x/2]*(2-3*x),{y[0]==2,y'[0]==0,y''[0]==0}},y
```

$$y(x) \to \frac{1}{36}e^{-x/2}\left(3x^3 + 24x + 8e^{3x/2} + 64\right)$$

19.72 problem section 9.3, problem 72

Internal problem ID [1569]
Internal file name [`OUTPUT/1570_Sunday_June_05_2022_02_22_52_AM_59473195/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 72.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y''' + 2y'' + 2y' + y = \mathrm{e}^{-x}(20 - 12x)}$$

With initial conditions

$$[y(0) = 3, y'(0) = -4, y''(0) = 7, y'''(0) = -22]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' + 2y'' + 2y' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{-ix}$$
$$y_4 = \mathrm{e}^{ix}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' + 2y'' + 2y' + y = \mathrm{e}^{-x}(20 - 12x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-x}(20 - 12x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{ix}, \mathrm{e}^{-x}, \mathrm{e}^{-ix}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{-x} + A_2 e^{-x} x^3$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_2 e^{-x} x - 12A_2 e^{-x} + 4A_1 e^{-x} = e^{-x}(20 - 12x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 e^{-x} - e^{-x} x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\, e^{-x} c_2 + e^{-ix} c_3 + e^{ix} c_4\right) + \left(2x^2 e^{-x} - e^{-x} x^3\right) \end{aligned}$$

Which simplifies to

$$y = e^{-ix} c_3 + e^{ix} c_4 + (c_2 x + c_1)\, e^{-x} + 2x^2 e^{-x} - e^{-x} x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-ix} c_3 + e^{ix} c_4 + (c_2 x + c_1)\, e^{-x} + 2x^2 e^{-x} - e^{-x} x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_3 + c_4 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -i e^{-ix} c_3 + i e^{ix} c_4 + c_2 e^{-x} - (c_2 x + c_1) e^{-x} + 4x e^{-x} - 5x^2 e^{-x} + e^{-x} x^3$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$-4 = -c_3 i + c_4 i - c_1 + c_2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = -e^{-ix} c_3 - e^{ix} c_4 - 2c_2 e^{-x} + (c_2 x + c_1) e^{-x} + 4 e^{-x} - 14x e^{-x} + 8x^2 e^{-x} - e^{-x} x^3$$

substituting $y'' = 7$ and $x = 0$ in the above gives

$$7 = -c_3 - c_4 - 2c_2 + c_1 + 4 \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = i e^{-ix} c_3 - i e^{ix} c_4 + 3c_2 e^{-x} - (c_2 x + c_1) e^{-x} - 18 e^{-x} + 30x e^{-x} - 11x^2 e^{-x} + e^{-x} x^3$$

substituting $y''' = -22$ and $x = 0$ in the above gives

$$-22 = c_3 i - c_4 i - c_1 + 3c_2 - 18 \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -1 \\ c_3 &= \frac{1}{2} - \frac{i}{2} \\ c_4 &= \frac{1}{2} + \frac{i}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -e^{-x} x^3 + 2x^2 e^{-x} - x e^{-x} + 2 e^{-x} + \cos(x) - \sin(x)$$

Which simplifies to

$$y = (-x^3 + 2x^2 - x + 2) e^{-x} + \cos(x) - \sin(x)$$

Summary

The solution(s) found are the following

$$y = (-x^3 + 2x^2 - x + 2) e^{-x} + \cos(x) - \sin(x) \tag{1}$$

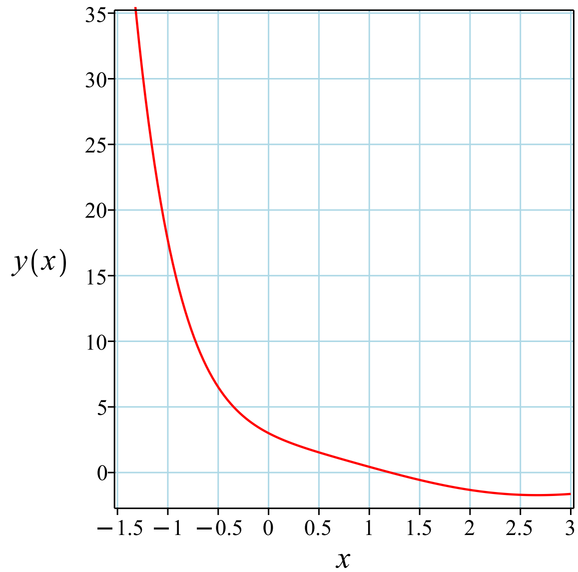


Figure 556: Solution plot

Verification of solutions

$$y = \left(-x^3 + 2x^2 - x + 2\right) e^{-x} + \cos(x) - \sin(x)$$

Verified OK.

19.72.1 Maple step by step solution

Let's solve

$$\left[y'''' + 2y''' + 2y'' + 2y' + y = e^{-x}(20 - 12x), y(0) = 3, y'\Big|_{\{x=0\}} = -4, y''\Big|_{\{x=0\}} = 7, y'''\Big|_{\{x=0\}} = \right.$$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -y - 12x e^{-x} + 20 e^{-x} - 2y''' - 2y'' - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + 2y''' + 2y'' + 2y' + y = -4 e^{-x}(3x - 5)$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = -12x\,\mathrm{e}^{-x} + 20\,\mathrm{e}^{-x} - 2y_4(x) - 2y_3(x) - 2y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = -12x\,\mathrm{e}^{-x} + 20\,\mathrm{e}^{-x} - 2y_4(x) - 2y_3(x) - 2y$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12x\,\mathrm{e}^{-x} + 20\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12x\,\mathrm{e}^{-x} + 20\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A\cdot \vec{y}(x)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x}\cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccc}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & -2 & -2 & -2\end{array}\right]-(-1)\cdot\left[\begin{array}{cccc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c}-1\\ 1\\ -1\\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c}-1\\ 0\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x)=\mathrm{e}^{-x}\cdot\left(x\cdot\left[\begin{array}{c}-1\\ 1\\ -1\\ 1\end{array}\right]+\left[\begin{array}{c}-1\\ 0\\ 0\\ 0\end{array}\right]\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I},\left[\begin{array}{c}-\mathrm{I}\\ -1\\ \mathrm{I}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(-I\sin(x) + \cos(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(-I\sin(x) + \cos(x)) \\ I\sin(x) - \cos(x) \\ I(-I\sin(x) + \cos(x)) \\ -I\sin(x) + \cos(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{-x}(-x-1) & -\sin(x) & -\cos(x) \\ e^{-x} & x\,e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & -x\,e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & x\,e^{-x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- ○ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{-x}(-x-1) & -\sin(x) & -\cos(x) \\ e^{-x} & x\,e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & -x\,e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & x\,e^{-x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (x+1)\,e^{-x} & x\,e^{-x} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} + \frac{e^{-x}}{2} & x\,e^{-x} - \cos(x) + e^{-x} & x\,e^{-x} - \frac{\cos(x)}{2} - \frac{\sin \\ -x\,e^{-x} & \frac{e^{-x}}{2} - x\,e^{-x} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & -x\,e^{-x} + \sin(x) & \frac{e^{-x}}{2} - x\,e^{-x} - \frac{\cos(\\ x\,e^{-x} & -\frac{e^{-x}}{2} + x\,e^{-x} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2} & x\,e^{-x} + \cos(x) & -\frac{e^{-x}}{2} + x\,e^{-x} + \frac{\sin(\\ -x\,e^{-x} & \frac{e^{-x}}{2} - x\,e^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x\,e^{-x} - \sin(x) & \frac{e^{-x}}{2} - x\,e^{-x} + \frac{\cos(\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

$\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\left(-2x^3+7x^2+10x+3\right)\mathrm{e}^{-x}-3\cos(x)-7\sin(x)\\ \left(2x^3-13x^2+4x+7\right)\mathrm{e}^{-x}-7\cos(x)+3\sin(x)\\ \left(-2x^3+13x^2-10x-3\right)\mathrm{e}^{-x}+3\cos(x)+7\sin(x)\\ \left(2x^3-13x^2+16x-7\right)\mathrm{e}^{-x}+7\cos(x)-3\sin(x)\end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1(x)+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+c_4\overrightarrow{y}_4(x)+\left[\begin{array}{c}\left(-2x^3+7x^2+10x+3\right)\mathrm{e}^{-x}-3\cos(x)-\\ \left(2x^3-13x^2+4x+7\right)\mathrm{e}^{-x}-7\cos(x)+3\\ \left(-2x^3+13x^2-10x-3\right)\mathrm{e}^{-x}+3\cos(x)+\\ \left(2x^3-13x^2+16x-7\right)\mathrm{e}^{-x}+7\cos(x)-\end{array}\right.$$

- First component of the vector is the solution to the ODE

  $y=\left(-2x^3+7x^2+\left(-c_2+10\right)x-c_1-c_2+3\right)\mathrm{e}^{-x}+\left(-c_4-3\right)\cos(x)-\sin(x)\left(c_3+7\right)$

- Use the initial condition $y(0)=3$

  $3=-c_1-c_2-c_4$

- Calculate the 1st derivative of the solution

  $y'=\left(-6x^2-c_2+14x+10\right)\mathrm{e}^{-x}-\left(-2x^3+7x^2+\left(-c_2+10\right)x-c_1-c_2+3\right)\mathrm{e}^{-x}-\left(-c_4-3\right)\mathrm{s}$

- Use the initial condition $y'\Big|_{\{x=0\}}=-4$

  $-4=c_1-c_3$

- Calculate the 2nd derivative of the solution

  $y''=\left(-12x+14\right)\mathrm{e}^{-x}-2\left(-6x^2-c_2+14x+10\right)\mathrm{e}^{-x}+\left(-2x^3+7x^2+\left(-c_2+10\right)x-c_1-c_2+\right.$

- Use the initial condition $y''\Big|_{\{x=0\}}=7$

  $7=c_2-c_1+c_4$

- Calculate the 3rd derivative of the solution

  $y'''=-12\,\mathrm{e}^{-x}-3\left(-12x+14\right)\mathrm{e}^{-x}+3\left(-6x^2-c_2+14x+10\right)\mathrm{e}^{-x}-\left(-2x^3+7x^2+\left(-c_2+10\right)\right.$

- Use the initial condition $y'''\Big|_{\{x=0\}} = -22$

  $-22 = -20 - 2c_2 + c_1 + c_3$
- Solve for the unknown coefficients

  $\{c_1 = -5, c_2 = -2, c_3 = -1, c_4 = 4\}$
- Solution to the IVP

  $y = (-2x^3 + 7x^2 + 12x + 10)\,\mathrm{e}^{-x} - 7\cos(x) - 6\sin(x)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 31

```
dsolve([diff(y(x),x$4)+2*diff(y(x),x$3)+2*diff(y(x),x$2)+2*diff(y(x),x)+1*y(x)=exp(-x)*(20-1
```

$$y(x) = (-x^3 + 2x^2 - x + 2)\,\mathrm{e}^{-x} + \cos(x) - \sin(x)$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 39

```
DSolve[{y''''[x]+2*y'''[x]+2*y''[x]+2*y'[x]+1*y[x]==Exp[-x]*(20-12*x),{y[0]==3,y'[0]==-4,y''
```

$$y(x) \to e^{-x}(-x^3 + 2x^2 - x - e^x \sin(x) + e^x \cos(x) + 2)$$

19.73 problem section 9.3, problem 73

Internal problem ID [1570]
Internal file name [`OUTPUT/1571_Sunday_June_05_2022_02_22_55_AM_60291558/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 73.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 2y'' + y' + 2y = 30\cos(x) - 10\sin(x)}$$

With initial conditions

$$[y(0) = 3, y'(0) = -4, y''(0) = 16]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' + y' + 2y = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 + \lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-ix}$$
$$y_3 = \mathrm{e}^{ix}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' + y' + 2y = 30\cos(x) - 10\sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-2x} & e^{-ix} & e^{ix} \\ -2\,e^{-2x} & -ie^{-ix} & ie^{ix} \\ 4\,e^{-2x} & -e^{-ix} & -e^{ix} \end{bmatrix}$$
$$|W| = 10ie^{-2x}e^{-ix}e^{ix}$$

The determinant simplifies to

$$|W| = 10ie^{-2x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{-ix} & e^{ix} \\ -ie^{-ix} & ie^{ix} \end{bmatrix}$$
$$= 2i$$

$$W_2(x) = \det \begin{bmatrix} e^{-2x} & e^{ix} \\ -2\,e^{-2x} & ie^{ix} \end{bmatrix}$$
$$= (2+i)\,e^{(-2+i)x}$$

$$W_3(x) = \det \begin{bmatrix} e^{-2x} & e^{-ix} \\ -2\,e^{-2x} & -ie^{-ix} \end{bmatrix}$$
$$= (2-i)\,e^{(-2-i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(30\cos(x) - 10\sin(x))\,(2i)}{(1)\,(10ie^{-2x})}\,dx \\ &= \int \frac{2i(30\cos(x) - 10\sin(x))}{10ie^{-2x}}\,dx \\ &= \int \left((6\cos(x) - 2\sin(x))\,e^{2x}\right)\,dx \\ &= \frac{14\,e^{2x}\cos(x)}{5} + \frac{2\,e^{2x}\sin(x)}{5} \\ &= \frac{14\,e^{2x}\cos(x)}{5} + \frac{2\,e^{2x}\sin(x)}{5} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx\\
&= (-1)^{1}\int \frac{\left(30\cos\left(x\right)-10\sin\left(x\right)\right)\left(\left(2+i\right){\mathrm e}^{\left(-2+i\right)x}\right)}{\left(1\right)\left(10i{\mathrm e}^{-2x}\right)}\,dx\\
&= -\int \frac{\left(2+i\right)\left(30\cos\left(x\right)-10\sin\left(x\right)\right){\mathrm e}^{\left(-2+i\right)x}}{10i{\mathrm e}^{-2x}}\,dx\\
&= -\int \left(\left(-1+2i\right)\left(-3\cos\left(x\right)+\sin\left(x\right)\right){\mathrm e}^{ix}\right)\,dx\\
&= -\frac{\left(\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{ix}+\left(5-5i\right){\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-7-i\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}\\
&= -\frac{\left(\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{ix}+\left(5-5i\right){\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-7-i\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3}\int \frac{F(x)W_3(x)}{aW(x)}\,dx\\
&= (-1)^{0}\int \frac{\left(30\cos\left(x\right)-10\sin\left(x\right)\right)\left(\left(2-i\right){\mathrm e}^{\left(-2-i\right)x}\right)}{\left(1\right)\left(10i{\mathrm e}^{-2x}\right)}\,dx\\
&= \int \frac{\left(2-i\right)\left(30\cos\left(x\right)-10\sin\left(x\right)\right){\mathrm e}^{\left(-2-i\right)x}}{10i{\mathrm e}^{-2x}}\,dx\\
&= \int \left(\left(1+2i\right)\left(-3\cos\left(x\right)+\sin\left(x\right)\right){\mathrm e}^{-ix}\right)\,dx\\
&= \frac{\left(-\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}+\left(-5-5i\right){\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(7-i\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}\\
&= \frac{\left(-\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}+\left(-5-5i\right){\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(7-i\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1+U_2y_2+U_3y_3
$$

Hence

$$
\begin{aligned}
y_p &= \left(\frac{14\,{\mathrm e}^{2x}\cos\left(x\right)}{5}+\frac{2\,{\mathrm e}^{2x}\sin\left(x\right)}{5}\right)\left({\mathrm e}^{-2x}\right)\\
&\quad+\left(-\frac{\left(\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{ix}+\left(5-5i\right){\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-7-i\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)+\left(-\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}\right)\left({\mathrm e}^{-ix}\right)\\
&\quad+\left(\frac{\left(-\frac{1}{2}-\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}+\left(-5-5i\right){\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(7-i\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)+\left(\frac{1}{2}+\frac{7i}{2}\right)x\,{\mathrm e}^{-ix}\tan\left(\frac{x}{2}\right)^{2}}{1+\tan\left(\frac{x}{2}\right)^{2}}\right)\left({\mathrm e}^{ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = \frac{(-5x+14)\cos(x)}{5} + \frac{\sin(x)(35x-23)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3\right) + \left(\frac{(-5x+14)\cos(x)}{5} + \frac{\sin(x)(35x-23)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 + \frac{(-5x+14)\cos(x)}{5} + \frac{\sin(x)(35x-23)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 + c_3 + \frac{14}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2x} - ic_2 \mathrm{e}^{-ix} + i\mathrm{e}^{ix} c_3 - \cos(x) - \frac{(-5x+14)\sin(x)}{5} + \frac{\cos(x)(35x-23)}{5} + 7\sin(x)$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$-4 = ic_3 - ic_2 - 2c_1 - \frac{28}{5} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 4c_1 \mathrm{e}^{-2x} - c_2 \mathrm{e}^{-ix} - \mathrm{e}^{ix} c_3 + 2\sin(x) - \frac{(-5x+14)\cos(x)}{5} - \frac{\sin(x)(35x-23)}{5} + 14\cos(x)$$

substituting $y'' = 16$ and $x = 0$ in the above gives

$$16 = -c_3 - c_2 + 4c_1 + \frac{56}{5} \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -\frac{2}{5} + \frac{9i}{5} \\ c_3 &= -\frac{2}{5} - \frac{9i}{5} \end{aligned}$$

Substituting these values back in above solution results in

$$y = e^{-2x} - x\cos(x) + 2\cos(x) + 7\sin(x)\, x - \sin(x)$$

Which simplifies to

$$y = e^{-2x} + (2 - x)\cos(x) + (7x - 1)\sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-2x} + (2 - x)\cos(x) + (7x - 1)\sin(x) \tag{1}$$

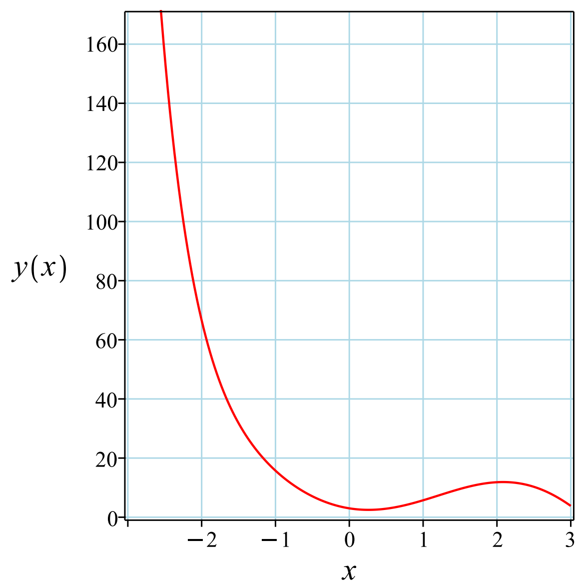


Figure 557: Solution plot

Verification of solutions

$$y = e^{-2x} + (2 - x)\cos(x) + (7x - 1)\sin(x)$$

Verified OK.

19.73.1 Maple step by step solution

Let's solve

$$\left[y''' + 2y'' + y' + 2y = 30\cos(x) - 10\sin(x), y(0) = 3, y'\Big|_{\{x=0\}} = -4, y''\Big|_{\{x=0\}} = 16\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 30\cos(x) - 10\sin(x) - 2y_3(x) - y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 30\cos(x) - 10\sin(x) - 2y_3(x) - y_2(x) - 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 30\cos(x) - 10\sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 30\cos(x) - 10\sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(-\mathrm{I}\sin(x)+\cos(x))\cdot\begin{bmatrix}-1\\ \mathrm{I}\\ 1\end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix}\mathrm{I}\sin(x)-\cos(x)\\ \mathrm{I}(-\mathrm{I}\sin(x)+\cos(x))\\ -\mathrm{I}\sin(x)+\cos(x)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=\begin{bmatrix}-\cos(x)\\ \sin(x)\\ \cos(x)\end{bmatrix},\vec{y}_3(x)=\begin{bmatrix}\sin(x)\\ \cos(x)\\ -\sin(x)\end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix}\frac{\mathrm{e}^{-2x}}{4} & -\cos(x) & \sin(x)\\ -\frac{\mathrm{e}^{-2x}}{2} & \sin(x) & \cos(x)\\ \mathrm{e}^{-2x} & \cos(x) & -\sin(x)\end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix}\frac{\mathrm{e}^{-2x}}{4} & -\cos(x) & \sin(x)\\ -\frac{\mathrm{e}^{-2x}}{2} & \sin(x) & \cos(x)\\ \mathrm{e}^{-2x} & \cos(x) & -\sin(x)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}\frac{1}{4} & -1 & 0\\ -\frac{1}{2} & 0 & 1\\ 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{5} + \frac{4\cos(x)}{5} + \frac{2\sin(x)}{5} & \sin(x) & \frac{e^{-2x}}{5} - \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} \\ -\frac{2e^{-2x}}{5} - \frac{4\sin(x)}{5} + \frac{2\cos(x)}{5} & \cos(x) & -\frac{2e^{-2x}}{5} + \frac{\sin(x)}{5} + \frac{2\cos(x)}{5} \\ \frac{4e^{-2x}}{5} - \frac{4\cos(x)}{5} - \frac{2\sin(x)}{5} & -\sin(x) & \frac{4e^{-2x}}{5} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{23\sin(x)}{5} + 7\sin(x)\,x + \frac{14\cos(x)}{5} - x\cos(x) - \frac{14\,e^{-2x}}{5} \\ \frac{28\,e^{-2x}}{5} + \frac{7(5x-4)\cos(x)}{5} + \frac{(5x+21)\sin(x)}{5} \\ \frac{33\sin(x)}{5} - 7\sin(x)\,x + \frac{56\cos(x)}{5} + x\cos(x) - \frac{56\,e^{-2x}}{5} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \left[\begin{array}{c} -\frac{23 \sin(x)}{5} + 7 \sin(x) x + \frac{14 \cos(x)}{5} - x \cos(x) - \frac{14 \,\mathrm{e}^{-2x}}{5} \\ \frac{28 \,\mathrm{e}^{-2x}}{5} + \frac{7(5x-4) \cos(x)}{5} + \frac{(5x+21) \sin(x)}{5} \\ \frac{33 \sin(x)}{5} - 7 \sin(x) x + \frac{56 \cos(x)}{5} + x \cos(x) - \frac{56 \,\mathrm{e}^{-2x}}{5} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(5c_1-56)\mathrm{e}^{-2x}}{20} + \frac{(14-5x-5c_2)\cos(x)}{5} + 7\left(x + \frac{c_3}{7} - \frac{23}{35}\right)\sin(x)$

- Use the initial condition $y(0) = 3$

  $3 = \frac{c_1}{4} - c_2$

- Calculate the 1st derivative of the solution

  $y' = -\frac{(5c_1-56)\mathrm{e}^{-2x}}{10} - \cos(x) - \frac{(14-5x-5c_2)\sin(x)}{5} + 7\sin(x) + 7\left(x + \frac{c_3}{7} - \frac{23}{35}\right)\cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = -4$

  $-4 = -\frac{c_1}{2} + c_3$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{(5c_1-56)\mathrm{e}^{-2x}}{5} + 2\sin(x) - \frac{(14-5x-5c_2)\cos(x)}{5} + 14\cos(x) - 7\left(x + \frac{c_3}{7} - \frac{23}{35}\right)\sin(x)$

- Use the initial condition $y''\Big|_{\{x=0\}} = 16$

  $16 = c_1 + c_2$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{76}{5}, c_2 = \frac{4}{5}, c_3 = \frac{18}{5}\right\}$

- Solution to the IVP

  $y = \mathrm{e}^{-2x} + (2-x)\cos(x) + (7x-1)\sin(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([0*diff(y(x),x$4)+1*diff(y(x),x$3)+2*diff(y(x),x$2)+1*diff(y(x),x)+2*y(x)=30*cos(x)-1
```

$$y(x) = \mathrm{e}^{-2x} + (2 - x)\cos(x) + (7x - 1)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 26

```
DSolve[{0*y''''[x]+1*y'''[x]+2*y''[x]+1*y'[x]+2*y[x]==30*Cos[x]-10*Sin[x],{y[0]==3,y'[0]==-4
```

$$y(x) \to e^{-2x} + (7x - 1)\sin(x) - ((x - 2)\cos(x))$$

19.74 problem section 9.3, problem 74

Internal problem ID [1571]
Internal file name [`OUTPUT/1572_Sunday_June_05_2022_02_22_59_AM_20540918/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.3. Undetermined Coefficients for Higher Order Equations. Page 495
Problem number: section 9.3, problem 74.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' - 3y''' + 5y'' - 2y' = -2\,\mathrm{e}^{x}(\cos(x) - \sin(x))}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0, y''(0) = -1, y'''(0) = -5]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 5y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 5\lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1$$

$$\lambda_3 = \frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_4 = \frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{17\right.}}\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_3 = \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1\right)x}$$

$$y_4 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 5y'' - 2y' = -2\,\mathrm{e}^{x}(\cos(x) - \sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2\,e^{x}(\cos (x)-\sin (x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos (x)\,e^{x}, \sin (x)\,e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6}+\frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}+1\right)x}, e^{\left(\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{12}-\frac{2}{(108+12\sqrt{177})^{\frac{1}{3}}}+1-\frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6}-\frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)}{2}\right)x}, e\right.$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos (x)\,e^{x} + A_2 \sin (x)\,e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin (x)\,e^{x} + 2A_2 \cos (x)\,e^{x} = -2\,e^{x}(\cos (x)-\sin (x))$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos (x)\,e^{x} - \sin (x)\,e^{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 \right. \\
&\quad + \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1\right)x} c_3 \\
&\quad \left. + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_4 \right) \\
&\quad + \left(-\cos\left(x\right)\mathrm{e}^{x} - \sin\left(x\right)\mathrm{e}^{x}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)}\right.} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = -1 + c_1 + c_2 + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = \left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right) e^{\left(\frac{\left(108+12\sqrt{}\right.}{12}\right.}$$

substituting $y'=0$ and $x=0$ in the above gives

$$0 = \frac{\left(108+12\sqrt{177}\right)^{\frac{2}{3}}\left(-i(c_2-c_4)\sqrt{3}+c_2-2c_3+c_4\right) - 12(2-c_2-c_3-c_4)\left(108+12\sqrt{177}\right)^{\frac{1}{3}} - 24i(c_2}{12\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)^2 e^{\left(\frac{\left(108+12\right.}{}\right.}$$

substituting $y''=-1$ and $x=0$ in the above gives

$$-1 = \frac{\frac{11\left(-\frac{(c_2-2c_3+c_4)\sqrt{177}}{33} - \frac{i(c_2-c_4)\sqrt{59}}{11} - i(c_2-c_4)\sqrt{3} - c_2+2c_3-c_4\right)\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{2} - \left(2+\frac{c_2}{3}+\frac{c_3}{3}+\frac{c_4}{3}\right)\left(108+12\sqrt{177}\right)}{\left(108+12\sqrt{177}\right.} \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = \left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + 1 + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)^3 e^{\left(\frac{\left(108+1\right.}{}\right.}$$

substituting $y'''=-5$ and $x=0$ in the above gives

$$-5 = \frac{2\left(108+12\sqrt{177}\right)^{\frac{1}{3}}\left((c_2-2c_3+c_4)\sqrt{177}+3i(-c_2+c_4)\sqrt{59}+15i(c_2-c_4)\sqrt{3}-15c_2+30c_3-15\right.}{} \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = 2$$

$$c_2 = -\frac{144\left(-1067\left(108+12\sqrt{177}\right)^{\frac{2}{3}}\sqrt{177}+1575\left(108+12\sqrt{177}\right)^{\frac{1}{3}}\sqrt{177}+28362\sqrt{177}-13971\left(\right.\right.}{\left(\left(108+12\sqrt{177}\right)^{\frac{2}{3}}-6\left(108+12\sqrt{177}\right)^{\frac{1}{3}}-24\right)\left(43\sqrt{59}+177\sqrt{3}\right)\left(-3\left(108+12\sqrt{177}\right)^{\frac{2}{3}}\sqrt{59}-\right.}$$

$$c_3 = \frac{11895552\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{59}+52607232\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{3}-361827648}{\left(96\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{59}+\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{3}\,\sqrt{59}-540\sqrt{3}\,\sqrt{59}+1248\left(108+12\sqrt{3}\right.\right.}$$

$$c_4 = \frac{10i\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{3}\,\sqrt{59}-153i\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{59}+1050i\sqrt{3}\,\sqrt{59}+97\left(108+12\right.}{}$$

Substituting these values back in above solution results in

$$y = \text{Expression too large to display}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = -2*exp(_a)*cos(_a)+
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 4.203 (sec). Leaf size: 1300

```
dsolve([1*diff(y(x),x$4)-3*diff(y(x),x$3)+5*diff(y(x),x$2)-2*diff(y(x),x)+0*y(x)=-2*exp(x)*(
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 3484

```
DSolve[{1*y''''[x]-3*y'''[x]+5*y''[x]-2*y'[x]+0*y[x]==-2*Exp[x]*(Cos[x]-Sin[x]),{y[0]==2,y'[
```

Too large to display

20 Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503

20.1 problem section 9.4, problem 3

Internal problem ID [1572]
Internal file name [`OUTPUT/1573_Sunday_June_05_2022_02_23_07_AM_11611486/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 3x^2y'' + 6y'x - 6y = 2x}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - 3x^2y'' + 6y'x - 6y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - 3x^2y'' + 6y'x - 6y = 2x$$

gives

$$6x\lambda\, x^{\lambda-1}-3x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}-6x^{\lambda}=0$$

Which simplifies to

$$6\lambda\, x^{\lambda}-3\lambda(\lambda-1)\, x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}-6x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$6\lambda-3\lambda(\lambda-1)+\lambda(\lambda-1)\,(\lambda-2)-6=0$$

Simplifying gives the characteristic equation as

$$\lambda^{3}-6\lambda^{2}+11\lambda-6=0$$

Solving the above gives the following roots

$$\begin{aligned}\lambda_1&=1\\ \lambda_2&=2\\ \lambda_3&=3\end{aligned}$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^{2}\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=c_3x^{3}+c_2x^{2}+c_1x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Now the particular solution to the given ODE is found

$$x^3y''' - 3x^2y'' + 6y'x - 6y = 2x$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix}$$
$$|W| = 2x^3$$

The determinant simplifies to

$$|W| = 2x^3$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} \\ &= x^4 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} x & x^3 \\ 1 & 3x^2 \end{bmatrix} \\ &= 2x^3 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} \\ &= x^2 \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)} \, dx \\ &= (-1)^2 \int \frac{(2x)(x^4)}{(x^3)(2x^3)} \, dx \\ &= \int \frac{2x^5}{2x^6} \, dx \\ &= \int \left(\frac{1}{x}\right) \, dx \\ &= \ln(x) \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\ &= (-1)^1 \int \frac{(2x)(2x^3)}{(x^3)(2x^3)} \, dx \\ &= -\int \frac{4x^4}{2x^6} \, dx \\ &= -\int \left(\frac{2}{x^2}\right) \, dx \\ &= \frac{2}{x} \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{(2x)\left(x^2\right)}{\left(x^3\right)\left(2x^3\right)}\, dx \\
&= \int \frac{2x^3}{2x^6}\, dx \\
&= \int \left(\frac{1}{x^3}\right) dx \\
&= -\frac{1}{2x^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = & (\ln(x))(x) \\
& + \left(\frac{2}{x}\right)\left(x^2\right) \\
& + \left(-\frac{1}{2x^2}\right)\left(x^3\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \left(\ln(x) + \frac{3}{2}\right) x
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_3 x^3 + c_2 x^2 + c_1 x\right) + \left(\left(\ln(x) + \frac{3}{2}\right) x\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_3 x^2 + c_2 x + c_1\right) + \left(\ln(x) + \frac{3}{2}\right) x
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = x\left(c_3 x^2 + c_2 x + c_1\right) + \left(\ln(x) + \frac{3}{2}\right) x \tag{1}
$$

Verification of solutions

$$y = x(c_3x^2 + c_2x + c_1) + \left(\ln(x) + \frac{3}{2}\right)x$$

Verified OK.

20.1.1 Maple step by step solution

Let's solve

$x^3y''' - 3x^2y'' + 6y'x - 6y = 2x$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)-6*y(x)=2*x,y(x), singsol=all
```

$$y(x) = \frac{x(2c_3x^2 + 2c_2x + 2\ln(x) + 2c_1 + 3)}{2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[x^3*y'''[x]-3*x^2*y''[x]+6*x*y'[x]-6*y[x]==2*x,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to x\left(c_3x^2 + \log(x) + c_2x + \frac{3}{2} + c_1\right)$$

20.2 problem section 9.4, problem 8

Internal problem ID [1573]
Internal file name [`OUTPUT/1574_Sunday_June_05_2022_02_23_09_AM_4417829/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{4x^3y''' + 4x^2y'' - 5y'x + 2y = 30x^2}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$4x^3y''' + 4x^2y'' - 5y'x + 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$4x^3y''' + 4x^2y'' - 5y'x + 2y = 30x^2$$

gives

$$-5x\lambda\, x^{\lambda-1}+4x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+4x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}+2x^{\lambda}=0$$

Which simplifies to

$$-5\lambda\, x^{\lambda}+4\lambda(\lambda-1)\, x^{\lambda}+4\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+2x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$-5\lambda+4\lambda(\lambda-1)+4\lambda(\lambda-1)\,(\lambda-2)+2=0$$

Simplifying gives the characteristic equation as

$$4\lambda^{3}-8\lambda^{2}-\lambda+2=0$$

Solving the above gives the following roots

$$\begin{aligned}\lambda_{1}&=2\\ \lambda_{2}&=-\frac{1}{2}\\ \lambda_{3}&=\frac{1}{2}\end{aligned}$$

This table summarises the result

root	multiplicity	type of root
2	1	real root
$-\frac{1}{2}$	1	real root
$\frac{1}{2}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^{2} x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=c_1x^{2}+\frac{c_2}{\sqrt{x}}+c_3\sqrt{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \frac{1}{\sqrt{x}} \\
y_3 &= \sqrt{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$4x^3y''' + 4x^2y'' - 5y'x + 2y = 30x^2$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} x^2 & \frac{1}{\sqrt{x}} & \sqrt{x} \\ 2x & -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} \\ 2 & \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}} \end{bmatrix} \\
|W| &= \frac{15}{4x}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{15}{4x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det \begin{bmatrix} \frac{1}{\sqrt{x}} & \sqrt{x} \\ -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} \end{bmatrix} \\ &= \frac{1}{x}\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det \begin{bmatrix} x^2 & \sqrt{x} \\ 2x & \frac{1}{2\sqrt{x}} \end{bmatrix} \\ &= -\frac{3x^{\frac{3}{2}}}{2}\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det \begin{bmatrix} x^2 & \frac{1}{\sqrt{x}} \\ 2x & -\frac{1}{2x^{\frac{3}{2}}} \end{bmatrix} \\ &= -\frac{5\sqrt{x}}{2}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(30x^2)\left(\frac{1}{x}\right)}{(4x^3)\left(\frac{15}{4x}\right)}\, dx \\ &= \int \frac{30x}{15x^2}\, dx \\ &= \int \left(\frac{2}{x}\right) dx \\ &= 2\ln(x)\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(30x^2)\left(-\frac{3x^{\frac{3}{2}}}{2}\right)}{(4x^3)\left(\frac{15}{4x}\right)}\,dx \\
&= -\int \frac{-45x^{\frac{7}{2}}}{15x^2}\,dx \\
&= -\int \left(-3x^{\frac{3}{2}}\right)\,dx \\
&= \frac{6x^{\frac{5}{2}}}{5}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(30x^2)\left(-\frac{5\sqrt{x}}{2}\right)}{(4x^3)\left(\frac{15}{4x}\right)}\,dx \\
&= \int \frac{-75x^{\frac{5}{2}}}{15x^2}\,dx \\
&= \int \left(-5\sqrt{x}\right)\,dx \\
&= -\frac{10x^{\frac{3}{2}}}{3}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p = &(2\ln(x))\left(x^2\right) \\
&+ \left(\frac{6x^{\frac{5}{2}}}{5}\right)\left(\frac{1}{\sqrt{x}}\right) \\
&+ \left(-\frac{10x^{\frac{3}{2}}}{3}\right)\left(\sqrt{x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = 2x^2\left(-\frac{16}{15} + \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x^2 + \frac{c_2}{\sqrt{x}} + c_3\sqrt{x}\right) + \left(2x^2\left(-\frac{16}{15} + \ln\left(x\right)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_1 x^{\frac{5}{2}} + c_3 x + c_2}{\sqrt{x}} + 2x^2\left(-\frac{16}{15} + \ln\left(x\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{5}{2}} + c_3 x + c_2}{\sqrt{x}} + 2x^2\left(-\frac{16}{15} + \ln\left(x\right)\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{5}{2}} + c_3 x + c_2}{\sqrt{x}} + 2x^2\left(-\frac{16}{15} + \ln\left(x\right)\right)
$$

Verified OK.

20.2.1 Maple step by step solution

Let's solve

$4x^3 y''' + 4x^2 y'' - 5y'x + 2y = 30x^2$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(4*x^3*diff(y(x),x$3)+4*x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+2*y(x)=30*x^2,y(x), singso
```

$$y(x) = \frac{\left(c_1 + 2\ln\left(x\right) - \frac{32}{15}\right) x^{\frac{5}{2}} + c_3 x + c_2}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 38

```
DSolve[4*x^3*y'''[x]+4*x^2*y''[x]-5*x*y'[x]+2*y[x]==30*x^2,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to 2x^2 \log(x) + \frac{\left(-\frac{32}{15} + c_3\right) x^{5/2} + c_2 x + c_1}{\sqrt{x}}$$

20.3 problem section 9.4, problem 11

Internal problem ID [1574]
Internal file name [`OUTPUT/1575_Sunday_June_05_2022_02_23_12_AM_68532501/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + x^2y'' - 2y'x + 2y = x^2}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + x^2y'' - 2y'x + 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + x^2y'' - 2y'x + 2y = x^2$$

gives

$$-2x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 2x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda + \lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 2 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x \\
y_3 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' + x^2 y'' - 2y'x + 2y = x^2
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix} \\
|W| &= \frac{6}{x}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{6}{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} \\ &= x^2 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{bmatrix} \\ &= 3 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{bmatrix} \\ &= \frac{2}{x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(x^2)\,(x^2)}{(x^3)\left(\frac{6}{x}\right)}\,dx \\ &= \int \frac{x^4}{6x^2}\,dx \\ &= \int \left(\frac{x^2}{6}\right)\,dx \\ &= \frac{x^3}{18} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(x^2)(3)}{(x^3)\left(\frac{6}{x}\right)} \, dx \\
&= -\int \frac{3x^2}{6x^2} \, dx \\
&= -\int \left(\frac{1}{2}\right) \, dx \\
&= -\frac{x}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(x^2)\left(\frac{2}{x}\right)}{(x^3)\left(\frac{6}{x}\right)} \, dx \\
&= \int \frac{2x}{6x^2} \, dx \\
&= \int \left(\frac{1}{3x}\right) \, dx \\
&= \frac{\ln(x)}{3}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^3}{18}\right)\left(\frac{1}{x}\right) \\
&+ \left(-\frac{x}{2}\right)(x) \\
&+ \left(\frac{\ln(x)}{3}\right)(x^2)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^2(-4 + 3\ln(x))}{9}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + c_3 x^2\right) + \left(\frac{x^2(-4 + 3\ln(x))}{9}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2 + \frac{x^2(-4 + 3\ln(x))}{9} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2 + \frac{x^2(-4 + 3\ln(x))}{9}$$

Verified OK.

20.3.1 Maple step by step solution

Let's solve

$x^3 y''' + x^2 y'' - 2y'x + 2y = x^2$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-2*_a*_b(_a)+2*(diff(_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^3*diff(y(x),x$3)+x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{2x^3 \ln(x) + 6c_3x^3 + 6c_2x^2 + c_1}{6x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 35

```
DSolve[x^3*y'''[x]+x^2*y''[x]-2*x*y'[x]+2*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}x^2 \log(x) + \left(-\frac{4}{9} + c_3\right)x^2 + c_2x + \frac{c_1}{x}$$

20.4 problem section 9.4, problem 14

Internal problem ID [1575]
Internal file name [`OUTPUT/1576_Sunday_June_05_2022_02_23_14_AM_15472858/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{16x^4y'''' + 96x^3y''' + 72x^2y'' - 24y'x + 9y = 96x^{\frac{5}{2}}}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$16x^4y'''' + 96x^3y''' + 72x^2y'' - 24y'x + 9y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$16x^4y'''' + 96x^3y''' + 72x^2y'' - 24y'x + 9y = 96x^{\frac{5}{2}}$$

gives

$$-24x\lambda\, x^{\lambda-1}+72x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+96x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}$$
$$+16x^{4}\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}+9x^{\lambda}=0$$

Which simplifies to

$$-24\lambda\, x^{\lambda}+72\lambda(\lambda-1)\, x^{\lambda}+96\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+16\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda}+9x^{\lambda}$$
$$=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$-24\lambda+72\lambda(\lambda-1)+96\lambda(\lambda-1)\,(\lambda-2)+16\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)+9=0$$

Simplifying gives the characteristic equation as

$$16\lambda^{4}-40\lambda^{2}+9=0$$

Solving the above gives the following roots

$$\lambda_1=-\frac{3}{2}$$
$$\lambda_2=\frac{3}{2}$$
$$\lambda_3=-\frac{1}{2}$$
$$\lambda_4=\frac{1}{2}$$

This table summarises the result

root	multiplicity	type of root
$-\frac{1}{2}$	1	real root
$\frac{1}{2}$	1	real root
$-\frac{3}{2}$	1	real root
$\frac{3}{2}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and

$c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{\sqrt{x}} + \sqrt{x}\, c_2 + \frac{c_3}{x^{\frac{3}{2}}} + c_4 x^{\frac{3}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{\sqrt{x}}$$
$$y_2 = \sqrt{x}$$
$$y_3 = \frac{1}{x^{\frac{3}{2}}}$$
$$y_4 = x^{\frac{3}{2}}$$

Now the particular solution to the given ODE is found

$$16x^4y'''' + 96x^3y''' + 72x^2y'' - 24y'x + 9y = 96x^{\frac{5}{2}}$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
W = \begin{bmatrix} \frac{1}{\sqrt{x}} & \sqrt{x} & \frac{1}{x^{\frac{3}{2}}} & x^{\frac{3}{2}} \\ -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} & -\frac{3}{2x^{\frac{5}{2}}} & \frac{3\sqrt{x}}{2} \\ \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}} & \frac{15}{4x^{\frac{7}{2}}} & \frac{3}{4\sqrt{x}} \\ -\frac{15}{8x^{\frac{7}{2}}} & \frac{3}{8x^{\frac{5}{2}}} & -\frac{105}{8x^{\frac{9}{2}}} & -\frac{3}{8x^{\frac{3}{2}}} \end{bmatrix}
$$

$$
|W| = \frac{12}{x^6}
$$

The determinant simplifies to

$$
|W| = \frac{12}{x^6}
$$

Now we determine W_i for each U_i.

$$
\begin{aligned}
W_1(x) &= \det \begin{bmatrix} \sqrt{x} & \frac{1}{x^{\frac{3}{2}}} & x^{\frac{3}{2}} \\ \frac{1}{2\sqrt{x}} & -\frac{3}{2x^{\frac{5}{2}}} & \frac{3\sqrt{x}}{2} \\ -\frac{1}{4x^{\frac{3}{2}}} & \frac{15}{4x^{\frac{7}{2}}} & \frac{3}{4\sqrt{x}} \end{bmatrix} \\
&= -\frac{6}{x^{\frac{5}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} \frac{1}{\sqrt{x}} & \frac{1}{x^{\frac{3}{2}}} & x^{\frac{3}{2}} \\ -\frac{1}{2x^{\frac{3}{2}}} & -\frac{3}{2x^{\frac{5}{2}}} & \frac{3\sqrt{x}}{2} \\ \frac{3}{4x^{\frac{5}{2}}} & \frac{15}{4x^{\frac{7}{2}}} & \frac{3}{4\sqrt{x}} \end{bmatrix} \\
&= -\frac{6}{x^{\frac{7}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} \frac{1}{\sqrt{x}} & \sqrt{x} & x^{\frac{3}{2}} \\ -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} & \frac{3\sqrt{x}}{2} \\ \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}} & \frac{3}{4\sqrt{x}} \end{bmatrix} \\
&= \frac{2}{x^{\frac{3}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
W_4(x) &= \det \begin{bmatrix} \frac{1}{\sqrt{x}} & \sqrt{x} & \frac{1}{x^{\frac{3}{2}}} \\ -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} & -\frac{3}{2x^{\frac{5}{2}}} \\ \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}} & \frac{15}{4x^{\frac{7}{2}}} \end{bmatrix} \\
&= \frac{2}{x^{\frac{9}{2}}}
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{\left(96x^{\frac{5}{2}}\right)\left(-\frac{6}{x^{\frac{5}{2}}}\right)}{(16x^4)\left(\frac{12}{x^6}\right)}\, dx \\
&= -\int \frac{-576}{\frac{192}{x^2}}\, dx \\
&= -\int \left(-3x^2\right)\, dx \\
&= x^3
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(96x^{\frac{5}{2}}\right)\left(-\frac{6}{x^{\frac{7}{2}}}\right)}{(16x^4)\left(\frac{12}{x^6}\right)}\, dx \\
&= \int \frac{-\frac{576}{x}}{\frac{192}{x^2}}\, dx \\
&= \int (-3x)\, dx \\
&= -\frac{3x^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(96x^{\frac{5}{2}}\right)\left(\frac{2}{x^{\frac{3}{2}}}\right)}{\left(16x^4\right)\left(\frac{12}{x^6}\right)}\,dx \\
&= -\int \frac{192x}{\frac{192}{x^2}}\,dx \\
&= -\int \left(x^3\right)\,dx \\
&= -\frac{x^4}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(96x^{\frac{5}{2}}\right)\left(\frac{2}{x^{\frac{9}{2}}}\right)}{\left(16x^4\right)\left(\frac{12}{x^6}\right)}\,dx \\
&= \int \frac{\frac{192}{x^2}}{\frac{192}{x^2}}\,dx \\
&= \int (1)\,dx \\
&= x
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(x^3\right)\left(\frac{1}{\sqrt{x}}\right) \\
&+ \left(-\frac{3x^2}{2}\right)\left(\sqrt{x}\right) \\
&+ \left(-\frac{x^4}{4}\right)\left(\frac{1}{x^{\frac{3}{2}}}\right) \\
&+ (x)\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^{\frac{5}{2}}}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{\sqrt{x}} + \sqrt{x}\, c_2 + \frac{c_3}{x^{\frac{3}{2}}} + c_4 x^{\frac{3}{2}}\right) + \left(\frac{x^{\frac{5}{2}}}{4}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_4 x^3 + c_2 x^2 + c_1 x + c_3}{x^{\frac{3}{2}}} + \frac{x^{\frac{5}{2}}}{4}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_4 x^3 + c_2 x^2 + c_1 x + c_3}{x^{\frac{3}{2}}} + \frac{x^{\frac{5}{2}}}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_4 x^3 + c_2 x^2 + c_1 x + c_3}{x^{\frac{3}{2}}} + \frac{x^{\frac{5}{2}}}{4}
$$

Verified OK.

20.4.1 Maple step by step solution

Let's solve

$16x^4 y'''' + 96x^3 y''' + 72x^2 y'' - 24y'x + 9y = 96x^{\frac{5}{2}}$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(16*x^4*diff(y(x),x$4)+96*x^3*diff(y(x),x$3)+72*x^2*diff(y(x),x$2)-24*x*diff(y(x),x)+9
```

$$y(x) = \frac{4c_4x^3 + x^4 + 4c_3x^2 + 4c_2x + 4c_1}{4x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 41

```
DSolve[16*x^4*y''''[x]+96*x^3*y'''[x]+72*x^2*y''[x]-24*x*y'[x]+9*y[x]==96*x^(5/2),y[x],x,Inc
```

$$y(x) \to \frac{x^4 + 4c_4x^3 + 4c_3x^2 + 4c_2x + 4c_1}{4x^{3/2}}$$

20.5 problem section 9.4, problem 16

Internal problem ID [1576]
Internal file name [`OUTPUT/1577_Sunday_June_05_2022_02_23_17_AM_73330081/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 16.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{x^4y'''' - 4x^3y''' + 12x^2y'' - 24y'x + 24y = x^4}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4y'''' - 4x^3y''' + 12x^2y'' - 24y'x + 24y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' - 4x^3y''' + 12x^2y'' - 24y'x + 24y = x^4$$

gives

$$
\begin{aligned}
&-24x\lambda\, x^{\lambda-1}+12x^{2}\lambda(\lambda-1)\, x^{\lambda-2}-4x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}\\
&+x^{4}\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}+24x^{\lambda}=0
\end{aligned}
$$

Which simplifies to

$$
-24\lambda\, x^{\lambda}+12\lambda(\lambda-1)\, x^{\lambda}-4\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda}+24x^{\lambda}=0
$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$
-24\lambda+12\lambda(\lambda-1)-4\lambda(\lambda-1)\,(\lambda-2)+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)+24=0
$$

Simplifying gives the characteristic equation as

$$
\lambda^{4}-10\lambda^{3}+35\lambda^{2}-50\lambda+24=0
$$

Solving the above gives the following roots

$$
\begin{aligned}
\lambda_{1}&=1\\
\lambda_{2}&=2\\
\lambda_{3}&=3\\
\lambda_{4}&=4
\end{aligned}
$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
3	1	real root
4	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$

$$y_2 = x^2$$

$$y_3 = x^3$$

$$y_4 = x^4$$

Now the particular solution to the given ODE is found

$$x^4y'''' - 4x^3y''' + 12x^2y'' - 24y'x + 24y = x^4$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{bmatrix}$$

$$|W| = 12x^4$$

The determinant simplifies to

$$|W| = 12x^4$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x^2 & x^3 & x^4 \\ 2x & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{bmatrix}$$
$$= 2x^6$$

$$W_2(x) = \det \begin{bmatrix} x & x^3 & x^4 \\ 1 & 3x^2 & 4x^3 \\ 0 & 6x & 12x^2 \end{bmatrix}$$
$$= 6x^5$$

$$W_3(x) = \det \begin{bmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{bmatrix}$$
$$= 6x^4$$

$$W_4(x) = \det \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix}$$
$$= 2x^3$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x) W_1(x)}{a W(x)} \, dx \\ &= (-1)^3 \int \frac{(x^4)(2x^6)}{(x^4)(12x^4)} \, dx \\ &= -\int \frac{2x^{10}}{12x^8} \, dx \\ &= -\int \left(\frac{x^2}{6}\right) \, dx \\ &= -\frac{x^3}{18} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(x^4)(6x^5)}{(x^4)(12x^4)} \, dx \\
&= \int \frac{6x^9}{12x^8} \, dx \\
&= \int \left(\frac{x}{2}\right) \, dx \\
&= \frac{x^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(x^4)(6x^4)}{(x^4)(12x^4)} \, dx \\
&= -\int \frac{6x^8}{12x^8} \, dx \\
&= -\int \left(\frac{1}{2}\right) \, dx \\
&= -\frac{x}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(x^4)(2x^3)}{(x^4)(12x^4)} \, dx \\
&= \int \frac{2x^7}{12x^8} \, dx \\
&= \int \left(\frac{1}{6x}\right) \, dx \\
&= \frac{\ln(x)}{6}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{x^3}{18}\right)(x) \\
&+\left(\frac{x^2}{4}\right)(x^2) \\
&+\left(-\frac{x}{2}\right)(x^3) \\
&+\left(\frac{\ln(x)}{6}\right)(x^4)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^4(-11+6\ln(x))}{36}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_4x^4 + c_3x^3 + c_2x^2 + c_1x\right) + \left(\frac{x^4(-11+6\ln(x))}{36}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_4x^3 + c_3x^2 + c_2x + c_1\right) + \frac{x^4(-11+6\ln(x))}{36}
$$

Summary

The solution(s) found are the following

$$
y = x\left(c_4x^3 + c_3x^2 + c_2x + c_1\right) + \frac{x^4(-11+6\ln(x))}{36} \tag{1}
$$

Verification of solutions

$$
y = x\left(c_4x^3 + c_3x^2 + c_2x + c_1\right) + \frac{x^4(-11+6\ln(x))}{36}
$$

Verified OK.

20.5.1 Maple step by step solution

Let's solve

$x^4y'''' - 4x^3y''' + 12x^2y'' - 24y'x + 24y = x^4$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(x^4*diff(y(x),x$4)-4*x^3*diff(y(x),x$3)+12*x^2*diff(y(x),x$2)-24*x*diff(y(x),x)+24*y(
```

$$y(x) = \left(\frac{x^3 \ln (x)}{6} + \left(c_4 - \frac{11}{36}\right) x^3 + c_3x^2 + c_2x + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 40

```
DSolve[x^4*y''''[x]-4*x^3*y'''[x]+12*x^2*y''[x]-24*x*y'[x]+24*y[x]==x^4,y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{6}x^4 \log(x) + x\left(\left(-\frac{11}{36} + c_4\right) x^3 + c_3x^2 + c_2x + c_1\right)$$

20.6 problem section 9.4, problem 18

Internal problem ID [1577]
Internal file name [`OUTPUT/1578_Sunday_June_05_2022_02_23_19_AM_87539480/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = 12x^2}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = 12x^2$$

gives

$$-4x\lambda\, x^{\lambda-1}+2x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+6x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ +x^{4}\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}+4x^{\lambda}=0$$

Which simplifies to

$$-4\lambda\, x^{\lambda}+2\lambda(\lambda-1)\, x^{\lambda}+6\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda}+4x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$-4\lambda+2\lambda(\lambda-1)+6\lambda(\lambda-1)\,(\lambda-2)+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)+4=0$$

Simplifying gives the characteristic equation as

$$\lambda^{4}-5\lambda^{2}+4=0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
-2	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^{2}\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{1}{x^2}$$
$$y_3 = x$$
$$y_4 = x^2$$

Now the particular solution to the given ODE is found

$$x^4 y'''' + 6x^3 y''' + 2x^2 y'' - 4y' x + 4y = 12x^2$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x & x^2 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 & 2x \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 & 2 \\ -\frac{6}{x^4} & -\frac{24}{x^5} & 0 & 0 \end{bmatrix}$$

$$|W| = -\frac{72}{x^6}$$

The determinant simplifies to

$$|W| = -\frac{72}{x^6}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \frac{1}{x^2} & x & x^2 \\ -\frac{2}{x^3} & 1 & 2x \\ \frac{6}{x^4} & 0 & 2 \end{bmatrix} \\ &= \frac{12}{x^2} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix} \\ &= \frac{6}{x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x^2 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 2x \\ \frac{2}{x^3} & \frac{6}{x^4} & 2 \end{bmatrix} \\ &= -\frac{12}{x^4} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 \end{bmatrix} \\ &= -\frac{6}{x^5} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{(12x^2)\left(\frac{12}{x^2}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= -\int \frac{144}{-\frac{72}{x^2}}\, dx \\
&= -\int \left(-2x^2\right)\, dx \\
&= \frac{2x^3}{3}
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(12x^2)\left(\frac{6}{x}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= \int \frac{72x}{-\frac{72}{x^2}}\, dx \\
&= \int \left(-x^3\right)\, dx \\
&= -\frac{x^4}{4}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{(12x^2)\left(-\frac{12}{x^4}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= -\int \frac{-\frac{144}{x^2}}{-\frac{72}{x^2}}\, dx \\
&= -\int (2)\, dx \\
&= -2x
\end{aligned}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(12x^2)\left(-\frac{6}{x^5}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\,dx \\
&= \int \frac{-\frac{72}{x^3}}{-\frac{72}{x^2}}\,dx \\
&= \int \left(\frac{1}{x}\right)\,dx \\
&= \ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{2x^3}{3}\right)\left(\frac{1}{x}\right) \\
&+ \left(-\frac{x^4}{4}\right)\left(\frac{1}{x^2}\right) \\
&+ (-2x)(x) \\
&+ (\ln(x))\left(x^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^2\left(-\frac{19}{12} + \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + \frac{c_2}{x^2} + c_3x + c_4x^2\right) + \left(x^2\left(-\frac{19}{12} + \ln(x)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3x + c_4x^2 + x^2\left(-\frac{19}{12} + \ln(x)\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3x + c_4x^2 + x^2\left(-\frac{19}{12} + \ln(x)\right)
$$

Verified OK.

20.6.1 Maple step by step solution

Let's solve

$x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = 12x^2$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1-4*_a*_b(_a)+4
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x^4*diff(y(x),x$4)+6*x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+4*y(x)=
```

$$y(x) = \frac{12x^4 \ln(x) + (12c_2 - 15)x^4 + 12c_3x^3 + 2c_1x + 12c_4}{12x^2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 38

```
DSolve[x^4*y''''[x]+6*x^3*y'''[x]+2*x^2*y''[x]-4*x*y'[x]+4*y[x]==12*x^2,y[x],x,IncludeSingul
```

$$y(x) \to \frac{x^4 \log(x) + \left(-\frac{19}{12} + c_4\right)x^4 + c_3x^3 + c_2x + c_1}{x^2}$$

20.7 problem section 9.4, problem 22

Internal problem ID [1578]
Internal file name [`OUTPUT/1579_Sunday_June_05_2022_02_23_21_AM_46587098/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 2x^2y'' + 3y'x - 3y = 4x}$$

With initial conditions

$$[y(1) = 4, y'(1) = 4, y''(1) = 2]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - 2x^2y'' + 3y'x - 3y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}\end{aligned}$$

Substituting these back into

$$x^3y''' - 2x^2y'' + 3y'x - 3y = 4x$$

gives

$$3x\lambda\, x^{\lambda-1} - 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 3x^{\lambda} = 0$$

Which simplifies to

$$3\lambda\, x^{\lambda} - 2\lambda(\lambda-1)\, x^{\lambda} + \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda} - 3x^{\lambda} = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$3\lambda - 2\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) - 3 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-3)\,(\lambda-1)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
1	2	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1x + \ln(x)\, c_2x + c_3x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x \ln(x) \\
y_3 &= x^3
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$x^3 y''' - 2x^2 y'' + 3y'x - 3y = 4x$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} x & x \ln(x) & x^3 \\ 1 & \ln(x) + 1 & 3x^2 \\ 0 & \frac{1}{x} & 6x \end{bmatrix} \\
|W| &= 4x^2
\end{aligned}
$$

The determinant simplifies to

$$|W| = 4x^2$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det\begin{bmatrix} x\ln(x) & x^3 \\ \ln(x)+1 & 3x^2 \end{bmatrix} \\ &= x^3(2\ln(x)-1)\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det\begin{bmatrix} x & x^3 \\ 1 & 3x^2 \end{bmatrix} \\ &= 2x^3\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det\begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\ &= x\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2\int \frac{(4x)\left(x^3(2\ln(x)-1)\right)}{(x^3)(4x^2)}\,dx \\ &= \int \frac{4x^4(2\ln(x)-1)}{4x^5}\,dx \\ &= \int \left(\frac{2\ln(x)-1}{x}\right)dx \\ &= \ln(x)^2-\ln(x)\end{aligned}$$

$$\begin{aligned}U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx \\ &= (-1)^1\int \frac{(4x)(2x^3)}{(x^3)(4x^2)}\,dx \\ &= -\int \frac{8x^4}{4x^5}\,dx \\ &= -\int \left(\frac{2}{x}\right)dx \\ &= -2\ln(x)\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{(4x)\,(x)}{(x^3)\,(4x^2)}\, dx \\
&= \int \frac{4x^2}{4x^5}\, dx \\
&= \int \left(\frac{1}{x^3}\right) dx \\
&= -\frac{1}{2x^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\ln(x)^2 - \ln(x)\right)(x) \\
&+ (-2\ln(x))\,(x\ln(x)) \\
&+ \left(-\frac{1}{2x^2}\right)(x^3)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x\left(-\frac{1}{2} - \ln(x)^2 - \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x + \ln(x)\, c_2 x + c_3 x^3\right) + \left(x\left(-\frac{1}{2} - \ln(x)^2 - \ln(x)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_3 x^2 + c_2 \ln(x) + c_1\right) + x\left(-\frac{1}{2} - \ln(x)^2 - \ln(x)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x\left(c_3 x^2 + c_2 \ln(x) + c_1\right) + x\left(-\frac{1}{2} - \ln(x)^2 - \ln(x)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = c_3 + c_1 - \frac{1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3 x^2 + c_2 \ln(x) + c_1 + x\left(2c_3 x + \frac{c_2}{x}\right) - \frac{1}{2} - \ln(x)^2 - \ln(x) + x\left(-\frac{2\ln(x)}{x} - \frac{1}{x}\right)$$

substituting $y' = 4$ and $x = 1$ in the above gives

$$4 = c_1 + c_2 + 3c_3 - \frac{3}{2} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 4c_3 x + \frac{2c_2}{x} + x\left(2c_3 - \frac{c_2}{x^2}\right) - \frac{4\ln(x)}{x} - \frac{2}{x} + x\left(-\frac{1}{x^2} + \frac{2\ln(x)}{x^2}\right)$$

substituting $y'' = 2$ and $x = 1$ in the above gives

$$2 = c_2 + 6c_3 - 3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{7}{2} \\ c_2 &= -1 \\ c_3 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x\ln(x)^2 + x^3 - 2x\ln(x) + 3x$$

Which simplifies to

$$y = \left(x^2 - \ln(x)^2 - 2\ln(x) + 3\right) x$$

Summary
The solution(s) found are the following

$$y = \left(x^2 - \ln(x)^2 - 2\ln(x) + 3\right) x \tag{1}$$

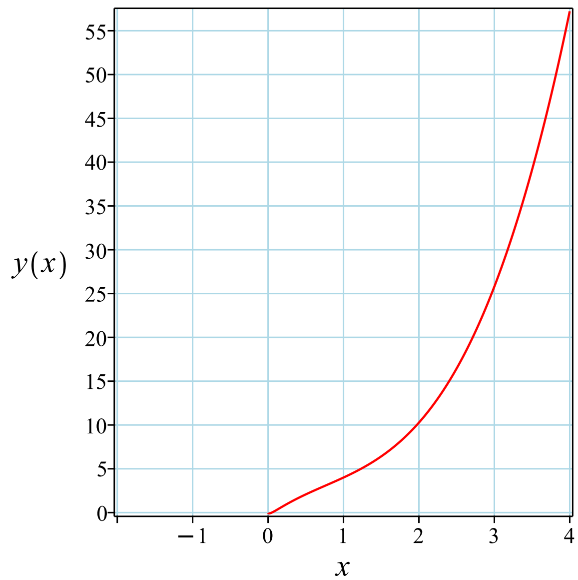


Figure 558: Solution plot

Verification of solutions

$$y = \left(x^2 - \ln(x)^2 - 2\ln(x) + 3\right)x$$

Verified OK.

20.7.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - 2x^2 y'' + 3y'x - 3y = 4x, y(1) = 4, y'\Big|_{\{x=1\}} = 4, y''\Big|_{\{x=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve([x^3*diff(y(x),x$3)-2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-3*y(x)=4*x,y(1) = 4, D(y)(1
```

$$y(x) = x\left(-\ln\left(x\right)^{2} + x^{2} - 2\ln\left(x\right) + 3\right)$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 22

```
DSolve[{x^3*y'''[x]-2*x^2*y''[x]+3*x*y'[x]-3*y[x]==4*x,{y[1]==4,y'[1]==4,y''[1]==2}},y[x],x,
```

$$y(x) \to x\left(x^2 - \log^2(x) - 2\log(x) + 3\right)$$

20.8 problem section 9.4, problem 23

Internal problem ID [1579]
Internal file name [OUTPUT/1580_Sunday_June_05_2022_02_23_24_AM_875933/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 23.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 5x^2y'' + 14y'x - 18y = x^3}$$

With initial conditions

$$[y(1) = 0, y'(1) = 1, y''(1) = 7]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - 5x^2y'' + 14y'x - 18y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - 5x^2y'' + 14y'x - 18y = x^3$$

gives

$$14x\lambda\, x^{\lambda-1} - 5x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 18x^\lambda = 0$$

Which simplifies to

$$14\lambda\, x^\lambda - 5\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 18x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$14\lambda - 5\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) - 18 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-2)\,(\lambda-3)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = 3$$

This table summarises the result

root	multiplicity	type of root
2	1	real root
3	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_2 x^3 + c_1 x^2 + c_3 \ln(x)\, x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^3 \\ y_3 &= x^3 \ln(x) \end{aligned}$$

Now the particular solution to the given ODE is found

$$x^3 y''' - 5x^2 y'' + 14y'x - 18y = x^3$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x^2 & x^3 & x^3 \ln(x) \\ 2x & 3x^2 & x^2(1 + 3\ln(x)) \\ 2 & 6x & x(6\ln(x) + 5) \end{bmatrix}$$

$$|W| = x^5$$

The determinant simplifies to

$$|W| = x^5$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x^3 & x^3 \ln(x) \\ 3x^2 & x^2(1 + 3\ln(x)) \end{bmatrix} \\ &= x^5 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} x^2 & x^3 \ln(x) \\ 2x & x^2(1 + 3\ln(x)) \end{bmatrix} \\ &= x^4(\ln(x) + 1) \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} \\ &= x^4 \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(x^3)(x^5)}{(x^3)(x^5)}\, dx \\ &= \int \frac{x^8}{x^8}\, dx \\ &= \int (1)\, dx \\ &= x \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\ &= (-1)^1 \int \frac{(x^3)(x^4(\ln(x) + 1))}{(x^3)(x^5)}\, dx \\ &= -\int \frac{x^7(\ln(x) + 1)}{x^8}\, dx \\ &= -\int \left(\frac{\ln(x) + 1}{x}\right) dx \\ &= -\frac{\ln(x)^2}{2} - \ln(x) \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(x^3)(x^4)}{(x^3)(x^5)}\, dx \\
&= \int \frac{x^7}{x^8}\, dx \\
&= \int \left(\frac{1}{x}\right) dx \\
&= \ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p &= (x)(x^2) \\
&+ \left(-\frac{\ln(x)^2}{2} - \ln(x)\right)(x^3) \\
&+ (\ln(x))(x^3 \ln(x))
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^3 \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_2 x^3 + c_1 x^2 + c_3 \ln(x) x^3) + \left(x^3 \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = x^2(\ln(x) c_3 x + c_2 x + c_1) + x^3 \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^2(\ln(x) c_3 x + c_2 x + c_1) + x^3 \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_2 + c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2x(\ln(x) c_3 x + c_2 x + c_1) + x^2(c_3 + \ln(x) c_3 + c_2) + 3x^2 \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right) + x^3 \left(\frac{\ln(x)}{x} - \frac{1}{x}\right)$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 3c_2 + 2c_1 + c_3 + 2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 2\ln(x) c_3 x + 2c_2 x + 2c_1 + 4x(c_3 + \ln(x) c_3 + c_2) + c_3 x + 6x \left(1 + \frac{\ln(x)^2}{2} - \ln(x)\right) + 6x^2 \left(\frac{\ln(x)}{x} -$$

substituting $y'' = 7$ and $x = 1$ in the above gives

$$7 = 6c_2 + 2c_1 + 5c_3 + 2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -2 \\ c_3 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{\ln(x)^2 x^3}{2} + 2x^3 \ln(x) - x^3 + x^2$$

Which simplifies to

$$y = \frac{(x \ln(x)^2 + 4x \ln(x) - 2x + 2) x^2}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(x \ln \left(x\right)^{2}+4 x \ln \left(x\right)-2 x +2\right) x^{2}}{2} \tag{1}$$

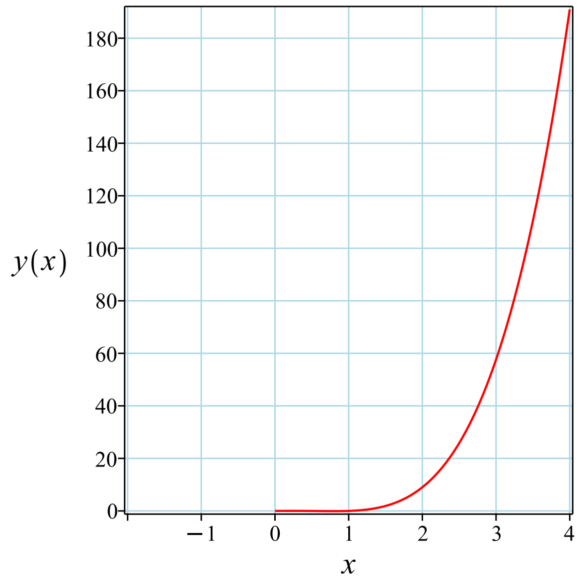


Figure 559: Solution plot

Verification of solutions

$$y = \frac{\left(x \ln \left(x\right)^{2}+4 x \ln \left(x\right)-2 x +2\right) x^{2}}{2}$$

Verified OK.

20.8.1 Maple step by step solution

Let's solve

$$\left[x^{3} y'''-5 x^{2} y''+14 y' x -18 y =x^{3}, y \left(1\right)=0, y'\Big|_{\left\{x =1\right\}}=1, y''\Big|_{\left\{x =1\right\}}=7\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve([x^3*diff(y(x),x$3)-5*x^2*diff(y(x),x$2)+14*x*diff(y(x),x)-18*y(x)=x^3,y(1) = 0, D(y)
```

$$y(x) = \frac{x^2 (\ln(x)^2 x + 4x \ln(x) - 2x + 2)}{2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 28

```
DSolve[{x^3*y'''[x]-5*x^2*y''[x]+14*x*y'[x]-18*y[x]==x^3,{y[1]==0,y'[1]==1,y''[1]==7}},y[x],
```

$$y(x) \to \frac{1}{2} x^2 \left(-2x + x \log^2(x) + 4x \log(x) + 2\right)$$

20.9 problem section 9.4, problem 25

Internal problem ID [1580]
Internal file name [`OUTPUT/1581_Sunday_June_05_2022_02_23_27_AM_91236329/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 6x^2y'' + 16y'x - 16y = 9x^4}$$

With initial conditions

$$[y(1) = 2, y'(1) = 1, y''(1) = 5]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - 6x^2y'' + 16y'x - 16y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - 6x^2y'' + 16y'x - 16y = 9x^4$$

gives

$$16x\lambda\, x^{\lambda-1} - 6x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 16x^{\lambda} = 0$$

Which simplifies to

$$16\lambda\, x^{\lambda} - 6\lambda(\lambda-1)\, x^{\lambda} + \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda} - 16x^{\lambda} = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$16\lambda - 6\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) - 16 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-1)\,(\lambda-4)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 4$$
$$\lambda_3 = 4$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
4	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^{\alpha}(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_2x^4 + c_1x + c_3 \ln (x) x^4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= x \\ y_2 &= x^4 \\ y_3 &= \ln (x) x^4 \end{aligned}$$

Now the particular solution to the given ODE is found

$$x^3y''' - 6x^2y'' + 16y'x - 16y = 9x^4$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & x^4 & \ln (x) x^4 \\ 1 & 4x^3 & x^3(1 + 4\ln (x)) \\ 0 & 12x^2 & x^2(7 + 12\ln (x)) \end{bmatrix}$$

$$|W| = 9x^6$$

The determinant simplifies to

$$|W| = 9x^6$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x^4 & \ln(x)\, x^4 \\ 4x^3 & x^3(1+4\ln(x)) \end{bmatrix}$$
$$= x^7$$

$$W_2(x) = \det \begin{bmatrix} x & \ln(x)\, x^4 \\ 1 & x^3(1+4\ln(x)) \end{bmatrix}$$
$$= x^4(1+3\ln(x))$$

$$W_3(x) = \det \begin{bmatrix} x & x^4 \\ 1 & 4x^3 \end{bmatrix}$$
$$= 3x^4$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(9x^4)(x^7)}{(x^3)(9x^6)}\, dx \\ &= \int \frac{9x^{11}}{9x^9}\, dx \\ &= \int (x^2)\, dx \\ &= \frac{x^3}{3} \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\ &= (-1)^1 \int \frac{(9x^4)(x^4(1+3\ln(x)))}{(x^3)(9x^6)}\, dx \\ &= -\int \frac{9x^8(1+3\ln(x))}{9x^9}\, dx \\ &= -\int \left(\frac{1+3\ln(x)}{x}\right) dx \\ &= -\frac{3\ln(x)^2}{2} - \ln(x) \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{(9x^4)(3x^4)}{(x^3)(9x^6)}\, dx \\
&= \int \frac{27x^8}{9x^9}\, dx \\
&= \int \left(\frac{3}{x}\right) dx \\
&= 3\ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^3}{3}\right)(x) \\
&+ \left(-\frac{3\ln(x)^2}{2} - \ln(x)\right)(x^4) \\
&+ (3\ln(x))\left(\ln(x)\, x^4\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^4(9\ln(x)^2 - 6\ln(x) + 2)}{6}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_2x^4 + c_1x + c_3\ln(x)\, x^4\right) + \left(\frac{x^4(9\ln(x)^2 - 6\ln(x) + 2)}{6}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2x^4 + c_1x + c_3\ln(x)\, x^4 + \frac{x^4(9\ln(x)^2 - 6\ln(x) + 2)}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$

in the above gives

$$2 = c_2 + c_1 + \frac{1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_2x^3 + c_1 + c_3x^3 + 4c_3 \ln(x)\, x^3 + \frac{2x^3 (9 \ln(x)^2 - 6 \ln(x) + 2)}{3} + \frac{x^4 \left(\frac{18 \ln(x)}{x} - \frac{6}{x}\right)}{6}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 4c_2 + c_1 + c_3 + \frac{1}{3} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 12c_2x^2 + 7c_3x^2 + 12c_3 \ln(x)\, x^2 + 2x^2 (9 \ln(x)^2 - 6 \ln(x) + 2) + \frac{4x^3 \left(\frac{18 \ln(x)}{x} - \frac{6}{x}\right)}{3} + \frac{x^4 \left(\frac{24}{x^2} - \frac{18 \ln(x)}{x^2}\right)}{6}$$

substituting $y'' = 5$ and $x = 1$ in the above gives

$$5 = 12c_2 + 7c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -\frac{4}{3}$$
$$c_3 = 3$$

Substituting these values back in above solution results in

$$y = \frac{3 \ln(x)^2 x^4}{2} + 2 \ln(x)\, x^4 - x^4 + 3x$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{3 \ln(x)^2 x^4}{2} + 2 \ln(x)\, x^4 - x^4 + 3x \tag{1}$$

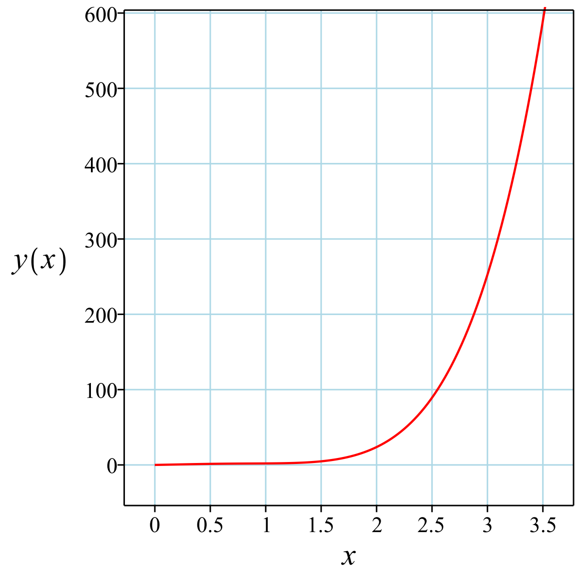


Figure 560: Solution plot

Verification of solutions

$$y = \frac{3\ln(x)^2 x^4}{2} + 2\ln(x)\, x^4 - x^4 + 3x$$

Verified OK.

20.9.1 Maple step by step solution

Let's solve

$$\left[x^3 y''' - 6x^2 y'' + 16y' x - 16y = 9x^4, y(1) = 2, y'\Big|_{\{x=1\}} = 1, y''\Big|_{\{x=1\}} = 5\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve([x^3*diff(y(x),x$3)-6*x^2*diff(y(x),x$2)+16*x*diff(y(x),x)-16*y(x)=9*x^4,y(1) = 2, D(
```

$$y(x) = -x^4 + \frac{3\ln(x)^2 x^4}{2} + 2x^4\ln(x) + 3x$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 32

```
DSolve[{x^3*y'''[x]-6*x^2*y''[x]+16*x*y'[x]-16*y[x]==9*x^4,{y[1]==2,y'[1]==1,y''[1]==5}},y[x
```

$$y(x) \to -x^4 + \frac{3}{2}x^4\log^2(x) + 2x^4\log(x) + 3x$$

20.10 problem section 9.4, problem 27

Internal problem ID [1581]
Internal file name [`OUTPUT/1582_Sunday_June_05_2022_02_23_30_AM_91497692/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 27.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + x^2y'' - 2y'x + 2y = x(x+1)}$$

With initial conditions

$$\left[y(-1) = -6, y'(-1) = \frac{43}{6}, y''(-1) = -\frac{5}{2}\right]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + x^2y'' - 2y'x + 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + x^2y'' - 2y'x + 2y = x(x+1)$$

gives

$$-2x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 2x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda + \lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 2 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = \frac{c_1}{x} + c_2 x + c_3 x^2
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x \\
y_3 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' + x^2 y'' - 2y'x + 2y = x(x+1)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix} \\
|W| &= \frac{6}{x}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{6}{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} \\ &= x^2 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{bmatrix} \\ &= 3 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{bmatrix} \\ &= \frac{2}{x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(x(x+1))\,(x^2)}{(x^3)\left(\frac{6}{x}\right)}\, dx \\ &= \int \frac{x^3(x+1)}{6x^2}\, dx \\ &= \int \left(\frac{x(x+1)}{6}\right) dx \\ &= \frac{1}{18}x^3 + \frac{1}{12}x^2 \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(x(x+1))\,(3)}{(x^3)\left(\frac{6}{x}\right)}\, dx \\
&= -\int \frac{3x(x+1)}{6x^2}\, dx \\
&= -\int \left(\frac{x+1}{2x}\right) dx \\
&= -\frac{x}{2} - \frac{\ln(x)}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(x(x+1))\left(\frac{2}{x}\right)}{(x^3)\left(\frac{6}{x}\right)}\, dx \\
&= \int \frac{2+2x}{6x^2}\, dx \\
&= \int \left(\frac{x+1}{3x^2}\right) dx \\
&= \frac{\ln(x)}{3} - \frac{1}{3x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{1}{18}x^3 + \frac{1}{12}x^2\right)\left(\frac{1}{x}\right) \\
&+ \left(-\frac{x}{2} - \frac{\ln(x)}{2}\right)(x) \\
&+ \left(\frac{\ln(x)}{3} - \frac{1}{3x}\right)(x^2)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x(12x\ln(x) - 18\ln(x) - 16x - 9)}{36}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + c_3 x^2\right) + \left(\frac{x(12x\ln(x) - 18\ln(x) - 16x - 9)}{36}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + c_2 x + c_3 x^2 + \frac{x(12x \ln(x) - 18\ln(x) - 16x - 9)}{36} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -6$ and $x = -1$ in the above gives

$$-6 = \frac{5i\pi}{6} - c_1 - c_2 + c_3 - \frac{7}{36} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + c_2 + 2c_3 x + \frac{x\ln(x)}{3} - \frac{\ln(x)}{2} - \frac{4x}{9} - \frac{1}{4} + \frac{x\left(12\ln(x) - 4 - \frac{18}{x}\right)}{36}$$

substituting $y' = \frac{43}{6}$ and $x = -1$ in the above gives

$$\frac{43}{6} = -\frac{7i\pi}{6} - c_1 + c_2 - 2c_3 - \frac{7}{36} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + 2c_3 + \frac{2\ln(x)}{3} - \frac{2}{9} - \frac{1}{x} + \frac{x\left(\frac{12}{x} + \frac{18}{x^2}\right)}{36}$$

substituting $y'' = -\frac{5}{2}$ and $x = -1$ in the above gives

$$-\frac{5}{2} = \frac{2i\pi}{3} - 2c_1 + 2c_3 + \frac{11}{18} \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{i\pi}{2} + \frac{17}{4} \\ c_3 &= -\frac{i\pi}{3} - \frac{14}{9} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{i x^2 \pi}{3} + \frac{\ln(x) x^2}{3} + \frac{i x \pi}{2} - \frac{x\ln(x)}{2} - 2x^2 + 4x$$

Summary

The solution(s) found are the following

$$y = \frac{(-2ix\pi + 2x\ln(x) + 3i\pi - 3\ln(x) - 12x + 24)\, x}{6} \tag{1}$$

Verification of solutions

$$y = \frac{(-2ix\pi + 2x\ln(x) + 3i\pi - 3\ln(x) - 12x + 24)\,x}{6}$$

Verified OK.

20.10.1 Maple step by step solution

Let's solve

$$\left[x^3y''' + x^2y'' - 2y'x + 2y = x(x+1)\,, y(-1) = -6, y'\Big|_{\{x=-1\}} = \tfrac{43}{6}, y''\Big|_{\{x=-1\}} = -\tfrac{5}{2}\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-2*_a*_b(_a)+2*(diff(_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 30

```
dsolve([x^3*diff(y(x),x$3)+x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=x*(x+1),y(-1) = -6, D(
```

$$y(x) = \frac{x(-2i\pi x + 2x\ln(x) + 3i\pi - 3\ln(x) - 12x + 24)}{6}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 34

```
DSolve[{x^3*y'''[x]+x^2*y''[x]-2*x*y'[x]+2*y[x]==x*(x+1),{y[-1]==-6,y'[-1]==43/6,y''[-1]==-5
```

$$y(x) \to \frac{1}{6}x(-2i\pi x - 12x + (2x-3)\log(x) + 3i\pi + 24)$$

20.11 problem section 9.4, problem 30

Internal problem ID [1582]
Internal file name [`OUTPUT/1583_Sunday_June_05_2022_02_23_33_AM_87938238/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 30.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^4 y'''' + 3x^3 y''' - x^2 y'' + 2y'x - 2y = 9x^2}$$

With initial conditions

$$[y(1) = -7, y'(1) = -11, y''(1) = -5, y'''(1) = 6]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4 y'''' + 3x^3 y''' - x^2 y'' + 2y'x - 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' + 3x^3y''' - x^2y'' + 2y'x - 2y = 9x^2$$

gives

$$\begin{aligned}2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + 3x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ + x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} - 2x^\lambda = 0\end{aligned}$$

Which simplifies to

$$2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + 3\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda - 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$2\lambda - \lambda(\lambda-1) + 3\lambda(\lambda-1)\,(\lambda-2) + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3) - 2 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda+1)\,(\lambda-2)\,(\lambda-1)^2 = 0$$

Solving the above gives the following roots

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= 2\\ \lambda_3 &= 1\\ \lambda_4 &= 1\end{aligned}$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	2	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$

basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$
basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x + \ln(x) c_3 x + c_4 x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \frac{1}{x} \\ y_2 &= x \\ y_3 &= x \ln(x) \\ y_4 &= x^2 \end{aligned}$$

Now the particular solution to the given ODE is found

$$x^4 y'''' + 3x^3 y''' - x^2 y'' + 2y'x - 2y = 9x^2$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{x} & x & x\ln(x) & x^2 \\ -\frac{1}{x^2} & 1 & \ln(x)+1 & 2x \\ \frac{2}{x^3} & 0 & \frac{1}{x} & 2 \\ -\frac{6}{x^4} & 0 & -\frac{1}{x^2} & 0 \end{bmatrix}$$

$$|W| = \frac{12}{x^3}$$

The determinant simplifies to

$$|W| = \frac{12}{x^3}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x & x\ln(x) & x^2 \\ 1 & \ln(x)+1 & 2x \\ 0 & \frac{1}{x} & 2 \end{bmatrix}$$

$$= x$$

$$W_2(x) = \det \begin{bmatrix} \frac{1}{x} & x\ln(x) & x^2 \\ -\frac{1}{x^2} & \ln(x)+1 & 2x \\ \frac{2}{x^3} & \frac{1}{x} & 2 \end{bmatrix}$$

$$= \frac{6\ln(x)-3}{x}$$

$$W_3(x) = \det \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix}$$

$$= \frac{6}{x}$$

$$W_4(x) = \det \begin{bmatrix} \frac{1}{x} & x & x\ln(x) \\ -\frac{1}{x^2} & 1 & \ln(x)+1 \\ \frac{2}{x^3} & 0 & \frac{1}{x} \end{bmatrix}$$

$$= \frac{4}{x^2}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)} \, dx \\
&= (-1)^3 \int \frac{(9x^2)\,(x)}{(x^4)\left(\frac{12}{x^3}\right)} \, dx \\
&= -\int \frac{9x^3}{12x} \, dx \\
&= -\int \left(\frac{3x^2}{4}\right) dx \\
&= -\frac{x^3}{4}
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)} \, dx \\
&= (-1)^2 \int \frac{(9x^2)\left(\frac{6\ln(x)-3}{x}\right)}{(x^4)\left(\frac{12}{x^3}\right)} \, dx \\
&= \int \frac{9x(6\ln(x)-3)}{12x} \, dx \\
&= \int \left(\frac{9\ln(x)}{2} - \frac{9}{4}\right) dx \\
&= -\frac{27x}{4} + \frac{9x\ln(x)}{2}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)} \, dx \\
&= (-1)^1 \int \frac{(9x^2)\left(\frac{6}{x}\right)}{(x^4)\left(\frac{12}{x^3}\right)} \, dx \\
&= -\int \frac{54x}{12x} \, dx \\
&= -\int \left(\frac{9}{2}\right) dx \\
&= -\frac{9x}{2}
\end{aligned}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(9x^2)\left(\frac{4}{x^2}\right)}{(x^4)\left(\frac{12}{x^3}\right)}\, dx \\
&= \int \frac{36}{12x}\, dx \\
&= \int \left(\frac{3}{x}\right) dx \\
&= 3\ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{x^3}{4}\right)\left(\frac{1}{x}\right) \\
&+ \left(-\frac{27x}{4} + \frac{9x\ln(x)}{2}\right)(x) \\
&+ \left(-\frac{9x}{2}\right)(x\ln(x)) \\
&+ (3\ln(x))\left(x^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^2(-7 + 3\ln(x))
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + \ln(x) c_3 x + c_4 x^2\right) + \left(x^2(-7 + 3\ln(x))\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x} + c_2 x + \ln(x) c_3 x + c_4 x^2 + x^2(-7 + 3\ln(x)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -7$ and $x = 1$ in the above gives

$$
-7 = c_1 + c_2 + c_4 - 7 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + c_2 + c_3 + \ln(x)\, c_3 + 2c_4 x + 2x(-7 + 3\ln(x)) + 3x$$

substituting $y' = -11$ and $x = 1$ in the above gives

$$-11 = -c_1 + c_2 + c_3 + 2c_4 - 11 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + \frac{c_3}{x} + 2c_4 - 5 + 6\ln(x)$$

substituting $y'' = -5$ and $x = 1$ in the above gives

$$-5 = 2c_1 + c_3 + 2c_4 - 5 \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = -\frac{6c_1}{x^4} - \frac{c_3}{x^2} + \frac{6}{x}$$

substituting $y''' = 6$ and $x = 1$ in the above gives

$$6 = -6c_1 - c_3 + 6 \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 3\ln(x)\, x^2 - 7x^2$$

Which simplifies to

$$y = x^2(-7 + 3\ln(x))$$

Summary

The solution(s) found are the following

$$y = x^2(-7 + 3\ln(x)) \tag{1}$$

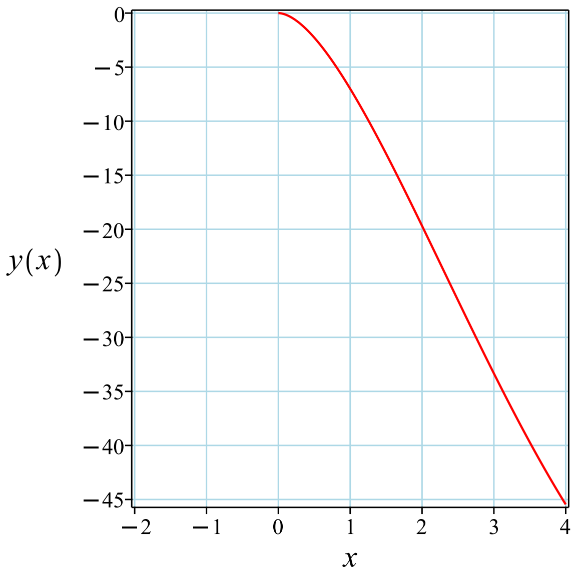


Figure 561: Solution plot

Verification of solutions

$$y = x^2(-7 + 3\ln(x))$$

Verified OK.

20.11.1 Maple step by step solution

Let's solve

$$\left[x^4 y'''' + 3x^3 y''' - x^2 y'' + 2y'x - 2y = 9x^2, y(1) = -7, y'\Big|_{\{x=1\}} = -11, y''\Big|_{\{x=1\}} = -5, y'''\Big|_{\{x=1\}} =$$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1+2*_a*_b(_a)-2
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve([x^4*diff(y(x),x$4)+3*x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=9
```

$$y(x) = x^2(-7 + 3\ln(x))$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 15

```
DSolve[{x^4*y''''[x]+3*x^3*y'''[x]-x^2*y''[x]+2*x*y'[x]-2*y[x]==9*x^2,{y[1]==-7,y'[1]==-11,y
```

$$y(x) \to x^2(3\log(x) - 7)$$

20.12 problem section 9.4, problem 32

Internal problem ID [1583]
Internal file name [OUTPUT/1584_Sunday_June_05_2022_02_23_36_AM_54639454/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 32.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{4x^4y'''' + 24x^3y''' + 23x^2y'' - y'x + y = 6x}$$

With initial conditions

$$\left[y(1) = 2, y'(1) = 0, y''(1) = 4, y'''(1) = -\frac{37}{4}\right]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$4x^4y'''' + 24x^3y''' + 23x^2y'' - y'x + y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} \end{aligned}$$

Substituting these back into

$$4x^4y'''' + 24x^3y''' + 23x^2y'' - y'x + y = 6x$$

gives

$$-x\lambda\, x^{\lambda-1} + 23x^2\lambda(\lambda-1)\, x^{\lambda-2} + 24x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 4x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} + x^\lambda = 0$$

Which simplifies to

$$-\lambda\, x^\lambda + 23\lambda(\lambda-1)\, x^\lambda + 24\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-\lambda + 23\lambda(\lambda-1) + 24\lambda(\lambda-1)\,(\lambda-2) + 4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3) + 1 = 0$$

Simplifying gives the characteristic equation as

$$4\lambda^4 - 5\lambda^2 + 1 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -\frac{1}{2}$$
$$\lambda_4 = \frac{1}{2}$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
$-\frac{1}{2}$	1	real root
$\frac{1}{2}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and

$c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2 x + \frac{c_3}{\sqrt{x}} + c_4 \sqrt{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \frac{1}{x} \\ y_2 &= x \\ y_3 &= \frac{1}{\sqrt{x}} \\ y_4 &= \sqrt{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$4x^4 y'''' + 24x^3 y''' + 23x^2 y'' - y'x + y = 6x$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W=\begin{bmatrix}\frac{1}{x} & x & \frac{1}{\sqrt{x}} & \sqrt{x}\\ -\frac{1}{x^2} & 1 & -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}}\\ \frac{2}{x^3} & 0 & \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}}\\ -\frac{6}{x^4} & 0 & -\frac{15}{8x^{\frac{7}{2}}} & \frac{3}{8x^{\frac{5}{2}}}\end{bmatrix}$$
$$|W|=\frac{9}{8x^6}$$

The determinant simplifies to

$$|W|=\frac{9}{8x^6}$$

Now we determine W_i for each U_i.

$$W_1(x)=\det\begin{bmatrix}x & \frac{1}{\sqrt{x}} & \sqrt{x}\\ 1 & -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}}\\ 0 & \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}}\end{bmatrix}$$
$$=\frac{3}{4x^2}$$

$$W_2(x)=\det\begin{bmatrix}\frac{1}{x} & \frac{1}{\sqrt{x}} & \sqrt{x}\\ -\frac{1}{x^2} & -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}}\\ \frac{2}{x^3} & \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}}\end{bmatrix}$$
$$=\frac{3}{4x^4}$$

$$W_3(x)=\det\begin{bmatrix}\frac{1}{x} & x & \sqrt{x}\\ -\frac{1}{x^2} & 1 & \frac{1}{2\sqrt{x}}\\ \frac{2}{x^3} & 0 & -\frac{1}{4x^{\frac{3}{2}}}\end{bmatrix}$$
$$=-\frac{3}{2x^{\frac{5}{2}}}$$

$$W_4(x) = \det \begin{bmatrix} \frac{1}{x} & x & \frac{1}{\sqrt{x}} \\ -\frac{1}{x^2} & 1 & -\frac{1}{2x^{\frac{3}{2}}} \\ \frac{2}{x^3} & 0 & \frac{3}{4x^{\frac{5}{2}}} \end{bmatrix}$$
$$= -\frac{3}{2x^{\frac{7}{2}}}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^3 \int \frac{(6x)\left(\frac{3}{4x^2}\right)}{(4x^4)\left(\frac{9}{8x^6}\right)}\, dx \\ &= -\int \frac{\frac{9}{2x}}{\frac{9}{2x^2}}\, dx \\ &= -\int (x)\, dx \\ &= -\frac{x^2}{2} \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(6x)\left(\frac{3}{4x^4}\right)}{(4x^4)\left(\frac{9}{8x^6}\right)}\, dx \\ &= \int \frac{\frac{9}{2x^3}}{\frac{9}{2x^2}}\, dx \\ &= \int \left(\frac{1}{x}\right)\, dx \\ &= \ln(x) \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(6x)\left(-\frac{3}{2x^{\frac{5}{2}}}\right)}{(4x^4)\left(\frac{9}{8x^6}\right)}\, dx \\
&= -\int \frac{-\frac{9}{x^{\frac{3}{2}}}}{\frac{9}{2x^2}}\, dx \\
&= -\int \left(-2\sqrt{x}\right) dx \\
&= \frac{4x^{\frac{3}{2}}}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(6x)\left(-\frac{3}{2x^{\frac{7}{2}}}\right)}{(4x^4)\left(\frac{9}{8x^6}\right)}\, dx \\
&= \int \frac{-\frac{9}{x^{\frac{5}{2}}}}{\frac{9}{2x^2}}\, dx \\
&= \int \left(-\frac{2}{\sqrt{x}}\right) dx \\
&= -4\sqrt{x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{x^2}{2}\right)\left(\frac{1}{x}\right) \\
&+ (\ln(x))(x) \\
&+ \left(\frac{4x^{\frac{3}{2}}}{3}\right)\left(\frac{1}{\sqrt{x}}\right) \\
&+ \left(-4\sqrt{x}\right)\left(\sqrt{x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x\left(-\frac{19}{6} + \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + \frac{c_3}{\sqrt{x}} + c_4\sqrt{x}\right) + \left(x\left(-\frac{19}{6} + \ln(x)\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x} + c_2 x + \frac{c_3}{\sqrt{x}} + c_4\sqrt{x} + x\left(-\frac{19}{6} + \ln(x)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$
2 = c_1 + c_2 - \frac{19}{6} + c_3 + c_4 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{c_1}{x^2} + c_2 - \frac{c_3}{2x^{\frac{3}{2}}} + \frac{c_4}{2\sqrt{x}} - \frac{13}{6} + \ln(x)
$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$
0 = -c_1 + c_2 - \frac{13}{6} - \frac{c_3}{2} + \frac{c_4}{2} \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = \frac{2c_1}{x^3} + \frac{3c_3}{4x^{\frac{5}{2}}} - \frac{c_4}{4x^{\frac{3}{2}}} + \frac{1}{x}
$$

substituting $y'' = 4$ and $x = 1$ in the above gives

$$
4 = 2c_1 + 1 + \frac{3c_3}{4} - \frac{c_4}{4} \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -\frac{6c_1}{x^4} - \frac{15c_3}{8x^{\frac{7}{2}}} + \frac{3c_4}{8x^{\frac{5}{2}}} - \frac{1}{x^2}
$$

substituting $y''' = -\frac{37}{4}$ and $x = 1$ in the above gives

$$
-\frac{37}{4} = -6c_1 - 1 - \frac{15c_3}{8} + \frac{3c_4}{8} \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{25}{6}$$
$$c_3 = 1$$
$$c_4 = -1$$

Substituting these values back in above solution results in

$$y = \frac{x^{\frac{5}{2}} \ln(x) - x^2 + x^{\frac{5}{2}} + \sqrt{x} + x}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{\frac{5}{2}} \ln(x) - x^2 + x^{\frac{5}{2}} + \sqrt{x} + x}{x^{\frac{3}{2}}} \tag{1}$$

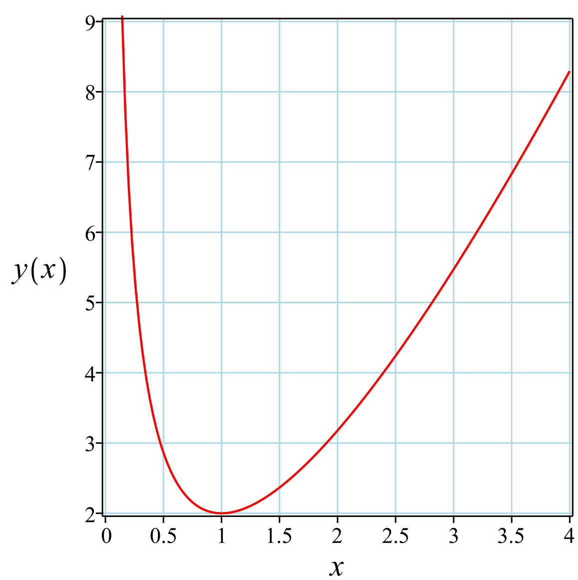


Figure 562: Solution plot

Verification of solutions

$$y = \frac{x^{\frac{5}{2}} \ln(x) - x^2 + x^{\frac{5}{2}} + \sqrt{x} + x}{x^{\frac{3}{2}}}$$

Verified OK.

20.12.1 Maple step by step solution

Let's solve

$$\left[4x^4y'''' + 24x^3y''' + 23x^2y'' - y'x + y = 6x, y(1) = 2, y'\Big|_{\{x=1\}} = 0, y''\Big|_{\{x=1\}} = 4, y'''\Big|_{\{x=1\}} = -\frac{37}{4}\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (1/4)*(c__1-_a*_b(_
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   Euler equation successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve([4*x^4*diff(y(x),x$4)+24*x^3*diff(y(x),x$3)+23*x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)
```

$$y(x) = \frac{\ln(x)\, x^{\frac{5}{2}} - x^2 + x^{\frac{5}{2}} + \sqrt{x} + x}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 26

```
DSolve[{4*x^4*y''''[x]+24*x^3*y'''[x]+23*x^2*y''[x]-x*y'[x]+y[x]==6*x,{y[1]==2,y'[1]==0,y''[
```

$$y(x) \to x - \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{x} + x \log(x)$$

20.13 problem section 9.4, problem 33

Internal problem ID [1584]
Internal file name [`OUTPUT/1585_Sunday_June_05_2022_02_23_39_AM_65608186/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 33.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^4y'''' + 5x^3y''' - 3x^2y'' - 6y'x + 6y = 40x^3}$$

With initial conditions

$$[y(-1) = -1, y'(-1) = -7, y''(-1) = -1, y'''(-1) = -31]$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4y'''' + 5x^3y''' - 3x^2y'' - 6y'x + 6y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' + 5x^3y''' - 3x^2y'' - 6y'x + 6y = 40x^3$$

gives

$$-6x\lambda\, x^{\lambda-1} - 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + 5x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ + x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} + 6x^\lambda = 0$$

Which simplifies to

$$-6\lambda\, x^\lambda - 3\lambda(\lambda-1)\, x^\lambda + 5\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda + 6x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-6\lambda - 3\lambda(\lambda-1) + 5\lambda(\lambda-1)\,(\lambda-2) + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3) + 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^4 - \lambda^3 - 7\lambda^2 + \lambda + 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$
$$\lambda_4 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
-2	1	real root
1	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$

basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$
basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \frac{1}{x} \\ y_2 &= \frac{1}{x^2} \\ y_3 &= x \\ y_4 &= x^3 \end{aligned}$$

Now the particular solution to the given ODE is found

$$x^4 y'''' + 5x^3 y''' - 3x^2 y'' - 6y'x + 6y = 40x^3$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x & x^3 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 & 3x^2 \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 & 6x \\ -\frac{6}{x^4} & -\frac{24}{x^5} & 0 & 6 \end{bmatrix}$$
$$|W| = -\frac{240}{x^5}$$

The determinant simplifies to

$$|W| = -\frac{240}{x^5}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} \frac{1}{x^2} & x & x^3 \\ -\frac{2}{x^3} & 1 & 3x^2 \\ \frac{6}{x^4} & 0 & 6x \end{bmatrix}$$
$$= \frac{30}{x}$$

$$W_2(x) = \det \begin{bmatrix} \frac{1}{x} & x & x^3 \\ -\frac{1}{x^2} & 1 & 3x^2 \\ \frac{2}{x^3} & 0 & 6x \end{bmatrix}$$
$$= 16$$

$$W_3(x) = \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x^3 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 3x^2 \\ \frac{2}{x^3} & \frac{6}{x^4} & 6x \end{bmatrix}$$
$$= -\frac{20}{x^3}$$

$$W_4(x) = \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 \end{bmatrix}$$
$$= -\frac{6}{x^5}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{(40x^3)\left(\frac{30}{x}\right)}{(x^4)\left(-\frac{240}{x^5}\right)}\,dx \\
&= -\int \frac{1200x^2}{-\frac{240}{x}}\,dx \\
&= -\int \left(-5x^3\right)\,dx \\
&= \frac{5x^4}{4}
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(40x^3)(16)}{(x^4)\left(-\frac{240}{x^5}\right)}\,dx \\
&= \int \frac{640x^3}{-\frac{240}{x}}\,dx \\
&= \int \left(-\frac{8x^4}{3}\right)\,dx \\
&= -\frac{8x^5}{15}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(40x^3)\left(-\frac{20}{x^3}\right)}{(x^4)\left(-\frac{240}{x^5}\right)}\,dx \\
&= -\int \frac{-800}{-\frac{240}{x}}\,dx \\
&= -\int \left(\frac{10x}{3}\right)\,dx \\
&= -\frac{5x^2}{3}
\end{aligned}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(40x^3)\left(-\frac{6}{x^5}\right)}{(x^4)\left(-\frac{240}{x^5}\right)}\, dx \\
&= \int \frac{-\frac{240}{x^2}}{-\frac{240}{x}}\, dx \\
&= \int \left(\frac{1}{x}\right) dx \\
&= \ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{5x^4}{4}\right)\left(\frac{1}{x}\right) \\
&+ \left(-\frac{8x^5}{15}\right)\left(\frac{1}{x^2}\right) \\
&+ \left(-\frac{5x^2}{3}\right)(x) \\
&+ (\ln(x))\left(x^3\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^3\left(-\frac{19}{20} + \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^3\right) + \left(x^3\left(-\frac{19}{20} + \ln(x)\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^3 + x^3\left(-\frac{19}{20} + \ln(x)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = -1$ in the above gives

$$-1 = \frac{19}{20} - c_1 + c_2 - c_3 - c_4 - i\pi \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} - \frac{2c_2}{x^3} + c_3 + 3c_4x^2 + 3x^2\left(-\frac{19}{20} + \ln(x)\right) + x^2$$

substituting $y' = -7$ and $x = -1$ in the above gives

$$-7 = -\frac{37}{20} - c_1 + 2c_2 + c_3 + 3c_4 + 3i\pi \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = \frac{2c_1}{x^3} + \frac{6c_2}{x^4} + 6c_4x + 6x\left(-\frac{19}{20} + \ln(x)\right) + 5x$$

substituting $y'' = -1$ and $x = -1$ in the above gives

$$-1 = -6i\pi - 2c_1 + 6c_2 - 6c_4 + \frac{7}{10} \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = -\frac{6c_1}{x^4} - \frac{24c_2}{x^5} + 6c_4 + \frac{53}{10} + 6\ln(x)$$

substituting $y''' = -31$ and $x = -1$ in the above gives

$$-31 = 6i\pi - 6c_1 + 24c_2 + 6c_4 + \frac{53}{10} \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -1 \\ c_3 &= 1 \\ c_4 &= -i\pi - \frac{21}{20} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^5\ln(x) - ix^5\pi - 2x^5 + x^3 + x - 1}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^5 \ln(x) - 1 + (-i\pi - 2) x^5 + x^3 + x}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{x^5 \ln(x) - 1 + (-i\pi - 2) x^5 + x^3 + x}{x^2}$$

Verified OK.

20.13.1 Maple step by step solution

Let's solve

$$\left[x^4 y'''' + 5x^3 y''' - 3x^2 y'' - 6y'x + 6y = 40x^3, y(-1) = -1, y'\Big|_{\{x=-1\}} = -7, y''\Big|_{\{x=-1\}} = -1, y'''\Big|_{\{x=\right.$$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1-6*_a*_b(_a)+6
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   Euler equation successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve([x^4*diff(y(x),x$4)+5*x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)-6*x*diff(y(x),x)+6*y(x)
```

$$y(x) = \frac{x^5 \ln(x) - 1 + (-i\pi - 2) x^5 + x^3 + x}{x^2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 32

```
DSolve[{x^4*y''''[x]+5*x^3*y'''[x]-3*x^2*y''[x]-6*x*y'[x]+6*y[x]==40*x^3,{y[-1]==-1,y'[-1]==
```

$$y(x) \to \frac{(-2 - i\pi)x^5 + x^5 \log(x) + x^3 + x - 1}{x^2}$$

20.14 problem section 9.4, problem 35

Internal problem ID [1585]
Internal file name [`OUTPUT/1586_Sunday_June_05_2022_02_23_42_AM_43286464/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 35.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 2y'' - y' - 2y = F(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' - y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' - y' - 2y = F(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & e^{-2x} & e^{x} \\ -e^{-x} & -2\,e^{-2x} & e^{x} \\ e^{-x} & 4\,e^{-2x} & e^{x} \end{bmatrix}$$

$$|W| = -6\,e^{-x} e^{-2x} e^{x}$$

The determinant simplifies to

$$|W| = -6\,e^{-2x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix} \\ &= 3\,e^{-x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix} \\ &= 2 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2\,e^{-2x} \end{bmatrix} \\ &= -e^{-3x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)}\,dx \\ &= (-1)^2 \int \frac{(F(x))\,(3\,e^{-x})}{(1)\,(-6\,e^{-2x})}\,dx \\ &= \int \frac{3F(x)\,e^{-x}}{-6\,e^{-2x}}\,dx \\ &= \int \left(-\frac{F(x)\,e^{x}}{2}\right)\,dx \\ &= \int -\frac{F(x)\,e^{x}}{2}\,dx \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(F(x))\,(2)}{(1)\,(-6\,\mathrm{e}^{-2x})}\, dx \\
&= -\int \frac{2F(x)}{-6\,\mathrm{e}^{-2x}}\, dx \\
&= -\int \left(-\frac{F(x)\,\mathrm{e}^{2x}}{3}\right) dx \\
&= -\left(\int -\frac{F(x)\,\mathrm{e}^{2x}}{3} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(F(x))\,(-\mathrm{e}^{-3x})}{(1)\,(-6\,\mathrm{e}^{-2x})}\, dx \\
&= \int \frac{-F(x)\,\mathrm{e}^{-3x}}{-6\,\mathrm{e}^{-2x}}\, dx \\
&= \int \left(\frac{F(x)\,\mathrm{e}^{-x}}{6}\right) dx \\
&= \int \frac{F(x)\,\mathrm{e}^{-x}}{6} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\int -\frac{F(x)\,\mathrm{e}^{x}}{2} dx\right) (\mathrm{e}^{-x}) \\
&+ \left(-\left(\int -\frac{F(x)\,\mathrm{e}^{2x}}{3} dx\right)\right) (\mathrm{e}^{-2x}) \\
&+ \left(\int \frac{F(x)\,\mathrm{e}^{-x}}{6} dx\right) (\mathrm{e}^{x})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\left(\int F(x)\,\mathrm{e}^{-x} dx\right) \mathrm{e}^{3x} - 3\left(\int F(x)\,\mathrm{e}^{x} dx\right) \mathrm{e}^{x} + 2\left(\int F(x)\,\mathrm{e}^{2x} dx\right)\right) \mathrm{e}^{-2x}}{6}
$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + c_3 \mathrm{e}^{x}\right) + \left(\frac{\left(\left(\int F(x)\,\mathrm{e}^{-x} dx\right) \mathrm{e}^{3x} - 3\left(\int F(x)\,\mathrm{e}^{x} dx\right) \mathrm{e}^{x} + 2\left(\int F(x)\,\mathrm{e}^{2x} dx\right)\right) \mathrm{e}^{-2x}}{6}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + c_3 \mathrm{e}^{x} + \frac{\left(\left(\int F(x)\,\mathrm{e}^{-x} dx\right) \mathrm{e}^{3x} - 3\left(\int F(x)\,\mathrm{e}^{x} dx\right) \mathrm{e}^{x} + 2\left(\int F(x)\,\mathrm{e}^{2x} dx\right)\right) \mathrm{e}^{-2x}}{6} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + c_3 \mathrm{e}^{x} + \frac{\left(\left(\int F(x)\,\mathrm{e}^{-x} dx\right) \mathrm{e}^{3x} - 3\left(\int F(x)\,\mathrm{e}^{x} dx\right) \mathrm{e}^{x} + 2\left(\int F(x)\,\mathrm{e}^{2x} dx\right)\right) \mathrm{e}^{-2x}}{6}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(\int \mathrm{e}^{-x} F(x)\,dx\right) \mathrm{e}^{3x} + 6c_1 \mathrm{e}^{3x} - 3\left(\int \mathrm{e}^{x} F(x)\,dx\right) \mathrm{e}^{x} + 6c_3 \mathrm{e}^{x} + 2\left(\int F(x)\,\mathrm{e}^{2x} dx\right) + 6c_2\right) \mathrm{e}^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 96

```
DSolve[y'''[x]+2*y''[x]-y'[x]-2*y[x]==f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(\int_1^x \frac{1}{3}e^{2K[1]}f(K[1])dK[1] + e^x\int_1^x -\frac{1}{2}e^{K[2]}f(K[2])dK[2] + e^{3x}\int_1^x \frac{1}{6}e^{-K[3]}f(K[3])dK[3] + c_2e^x + c_3e^{3x} + c_1\right)$$

20.15 problem section 9.4, problem 36

Internal problem ID [1586]
Internal file name [OUTPUT/1587_Sunday_June_05_2022_02_23_44_AM_23991318/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 36.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + x^2y'' - 2y'x + 2y = F(x)}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + x^2y'' - 2y'x + 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + x^2y'' - 2y'x + 2y = F(x)$$

gives

$$-2x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 2x^\lambda = 0$$

Which simplifies to

$$-2\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-2\lambda + \lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 2 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2x + c_3x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x \\
y_3 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' + x^2 y'' - 2y' x + 2y = F(x)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix} \\
|W| &= \frac{6}{x}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{6}{x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} \\ &= x^2\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det\begin{bmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{bmatrix} \\ &= 3\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det\begin{bmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{bmatrix} \\ &= \frac{2}{x}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2\int \frac{(F(x))\,(x^2)}{(x^3)\left(\frac{6}{x}\right)}\,dx \\ &= \int \frac{F(x)\,x^2}{6x^2}\,dx \\ &= \int \left(\frac{F(x)}{6}\right)\,dx \\ &= \int \frac{F(x)}{6}dx\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(F(x))\,(3)}{(x^3)\left(\frac{6}{x}\right)}\, dx \\
&= -\int \frac{3F(x)}{6x^2}\, dx \\
&= -\int \left(\frac{F(x)}{2x^2}\right) dx \\
&= -\left(\int \frac{F(x)}{2x^2} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(F(x))\left(\frac{2}{x}\right)}{(x^3)\left(\frac{6}{x}\right)}\, dx \\
&= \int \frac{\frac{2F(x)}{x}}{6x^2}\, dx \\
&= \int \left(\frac{F(x)}{3x^3}\right) dx \\
&= \int \frac{F(x)}{3x^3} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\int \frac{F(x)}{6} dx\right)\left(\frac{1}{x}\right) \\
&+ \left(-\left(\int \frac{F(x)}{2x^2} dx\right)\right)(x) \\
&+ \left(\int \frac{F(x)}{3x^3} dx\right)\left(x^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{2\left(\int \frac{F(x)}{x^3} dx\right) x^3 - 3\left(\int \frac{F(x)}{x^2} dx\right) x^2 + \int F(x)\, dx}{6x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + c_3 x^2\right) + \left(\frac{2\left(\int \frac{F(x)}{x^3}dx\right)x^3 - 3\left(\int \frac{F(x)}{x^2}dx\right)x^2 + \int F(x)\,dx}{6x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + c_2 x + c_3 x^2 + \frac{2\left(\int \frac{F(x)}{x^3}dx\right)x^3 - 3\left(\int \frac{F(x)}{x^2}dx\right)x^2 + \int F(x)\,dx}{6x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + c_2 x + c_3 x^2 + \frac{2\left(\int \frac{F(x)}{x^3}dx\right)x^3 - 3\left(\int \frac{F(x)}{x^2}dx\right)x^2 + \int F(x)\,dx}{6x}
$$

Verified OK.

20.15.1 Maple step by step solution

Let's solve

$x^3 y''' + x^2 y'' - 2y'x + 2y = F(x)$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-2*_a*_b(_a)+2*(diff(_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(x^3*diff(y(x),x$3)+x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = \left(c_3 + \int \frac{c_2 + \int \frac{c_1 + \int F(x)dx}{x^3}dx}{x^2}dx\right)x^2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 82

```
DSolve[x^3*y'''[x]+x^2*y''[x]-2*x*y'[x]+2*y[x]==f[x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{x^3 \int_1^x \frac{f(K[3])}{3K[3]^3}dK[3] + x^2 \int_1^x -\frac{f(K[2])}{2K[2]^2}dK[2] + \int_1^x \frac{1}{6}f(K[1])dK[1] + c_3x^3 + c_2x^2 + c_1}{x}$$

20.16 problem section 9.4, problem 39

Internal problem ID [1587]
Internal file name [`OUTPUT/1588_Sunday_June_05_2022_02_23_47_AM_30120502/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 39.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y'' + 4y = F(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\\ \lambda_3 &= 1\\ \lambda_4 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{-2x} \\ y_3 &= e^{x} \\ y_4 &= e^{2x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y'' + 4y = F(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & e^{-2x} & e^{x} & e^{2x} \\ -e^{-x} & -2\,e^{-2x} & e^{x} & 2\,e^{2x} \\ e^{-x} & 4\,e^{-2x} & e^{x} & 4\,e^{2x} \\ -e^{-x} & -8\,e^{-2x} & e^{x} & 8\,e^{2x} \end{bmatrix}$$

$$|W| = -72\,e^{-x} e^{-2x} e^{x} e^{2x}$$

The determinant simplifies to

$$|W| = -72$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{-2x} & e^{x} & e^{2x} \\ -2\,e^{-2x} & e^{x} & 2\,e^{2x} \\ 4\,e^{-2x} & e^{x} & 4\,e^{2x} \end{bmatrix} \\ &= 12\,e^{x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2\,e^{2x} \\ e^{-x} & e^{x} & 4\,e^{2x} \end{bmatrix} \\ &= 6\,e^{2x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-x} & e^{-2x} & e^{2x} \\ -e^{-x} & -2\,e^{-2x} & 2\,e^{2x} \\ e^{-x} & 4\,e^{-2x} & 4\,e^{2x} \end{bmatrix} \\ &= -12\,e^{-x} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} e^{-x} & e^{-2x} & e^{x} \\ -e^{-x} & -2\,e^{-2x} & e^{x} \\ e^{-x} & 4\,e^{-2x} & e^{x} \end{bmatrix} \\ &= -6\,e^{-2x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x) W_1(x)}{a W(x)}\,dx \\ &= (-1)^3 \int \frac{(F(x))\,(12\,e^{x})}{(1)\,(-72)}\,dx \\ &= -\int \frac{12 F(x)\,e^{x}}{-72}\,dx \\ &= -\int \left(-\frac{F(x)\,e^{x}}{6}\right)\,dx \\ &= -\left(\int -\frac{F(x)\,e^{x}}{6}\,dx\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(F(x))\left(6\,\mathrm{e}^{2x}\right)}{(1)\,(-72)} \, dx \\
&= \int \frac{6 F(x)\,\mathrm{e}^{2x}}{-72} \, dx \\
&= \int \left(-\frac{F(x)\,\mathrm{e}^{2x}}{12}\right) dx \\
&= \int -\frac{F(x)\,\mathrm{e}^{2x}}{12} dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(F(x))\left(-12\,\mathrm{e}^{-x}\right)}{(1)\,(-72)} \, dx \\
&= -\int \frac{-12 F(x)\,\mathrm{e}^{-x}}{-72} \, dx \\
&= -\int \left(\frac{F(x)\,\mathrm{e}^{-x}}{6}\right) dx \\
&= -\left(\int \frac{F(x)\,\mathrm{e}^{-x}}{6} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(F(x))\left(-6\,\mathrm{e}^{-2x}\right)}{(1)\,(-72)} \, dx \\
&= \int \frac{-6 F(x)\,\mathrm{e}^{-2x}}{-72} \, dx \\
&= \int \left(\frac{F(x)\,\mathrm{e}^{-2x}}{12}\right) dx \\
&= \int \frac{F(x)\,\mathrm{e}^{-2x}}{12} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = & \left(-\left(\int -\frac{F(x)\,\mathrm{e}^{x}}{6}dx\right)\right)\left(\mathrm{e}^{-x}\right) \\
& + \left(\int -\frac{F(x)\,\mathrm{e}^{2x}}{12}dx\right)\left(\mathrm{e}^{-2x}\right) \\
& + \left(-\left(\int \frac{F(x)\,\mathrm{e}^{-x}}{6}dx\right)\right)\left(\mathrm{e}^{x}\right) \\
& + \left(\int \frac{F(x)\,\mathrm{e}^{-2x}}{12}dx\right)\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\left(\int F(x)\,\mathrm{e}^{-2x}dx\right)\mathrm{e}^{4x} - 2\left(\int F(x)\,\mathrm{e}^{-x}dx\right)\mathrm{e}^{3x} + 2\left(\int F(x)\,\mathrm{e}^{x}dx\right)\mathrm{e}^{x} - \left(\int F(x)\,\mathrm{e}^{2x}dx\right)\right)\mathrm{e}^{-2x}}{12}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{2x}c_4\right) \\
&\quad + \left(\frac{\left(\left(\int F(x)\,\mathrm{e}^{-2x}dx\right)\mathrm{e}^{4x} - 2\left(\int F(x)\,\mathrm{e}^{-x}dx\right)\mathrm{e}^{3x} + 2\left(\int F(x)\,\mathrm{e}^{x}dx\right)\mathrm{e}^{x} - \left(\int F(x)\,\mathrm{e}^{2x}dx\right)\right)\mathrm{e}^{-2x}}{12}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{2x}c_4 \\
& + \frac{\left(\left(\int F(x)\,\mathrm{e}^{-2x}dx\right)\mathrm{e}^{4x} - 2\left(\int F(x)\,\mathrm{e}^{-x}dx\right)\mathrm{e}^{3x} + 2\left(\int F(x)\,\mathrm{e}^{x}dx\right)\mathrm{e}^{x} - \left(\int F(x)\,\mathrm{e}^{2x}dx\right)\right)\mathrm{e}^{-2x}}{12}
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \mathrm{e}^{2x}c_4 \\
& + \frac{\left(\left(\int F(x)\,\mathrm{e}^{-2x}dx\right)\mathrm{e}^{4x} - 2\left(\int F(x)\,\mathrm{e}^{-x}dx\right)\mathrm{e}^{3x} + 2\left(\int F(x)\,\mathrm{e}^{x}dx\right)\mathrm{e}^{x} - \left(\int F(x)\,\mathrm{e}^{2x}dx\right)\right)\mathrm{e}^{-2x}}{12}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 84

```
dsolve(diff(y(x),x$4)-5*diff(y(x),x$2)+4*y(x)=F(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\left(\int e^{-x} F(x)\,dx\right) e^{3x} - 6c_1 e^{3x} - \frac{\left(\int F(x) e^{-2x} dx\right) e^{4x}}{2} - 6c_4 e^{4x} - \left(\int e^{x} F(x)\,dx\right) e^{x} - 6c_3 e^{x} + \frac{\left(\int F(x) e^{2x} dx\right)}{2} -}{6}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 130

```
DSolve[y''''[x]-5*y''[x]+4*y[x]==f[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(\int_1^x -\frac{1}{12}e^{2K[1]}f(K[1])dK[1] + e^x \int_1^x \frac{1}{6}e^{K[2]}f(K[2])dK[2] + e^{3x}\int_1^x -\frac{1}{6}e^{-K[3]}f(K[3])dK[3] + e^{4x}\int_1^x \frac{1}{12}e^{-2K[4]}f(K[4])dK[4] + c_2e^x + c_3e^{3x} + c_4e^{4x} + c_1\right)$$

20.17 problem section 9.4, problem 41

Internal problem ID [1588]
Internal file name [`OUTPUT/1589_Sunday_June_05_2022_02_23_49_AM_85162777/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 9 Introduction to Linear Higher Order Equations. Section 9.4. Variation of Parameters for Higher Order Equations. Page 503
Problem number: section 9.4, problem 41.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = F(x)}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\ y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} \end{aligned}$$

Substituting these back into

$$x^4y'''' + 6x^3y''' + 2x^2y'' - 4y'x + 4y = F(x)$$

gives

$$
-4x\lambda\, x^{\lambda-1}+2x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+6x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
+x^{4}\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}+4x^{\lambda}=0
$$

Which simplifies to

$$
-4\lambda\, x^{\lambda}+2\lambda(\lambda-1)\, x^{\lambda}+6\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda}+4x^{\lambda}=0
$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$
-4\lambda+2\lambda(\lambda-1)+6\lambda(\lambda-1)\,(\lambda-2)+\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)+4=0
$$

Simplifying gives the characteristic equation as

$$
\lambda^{4}-5\lambda^{2}+4=0
$$

Solving the above gives the following roots

$$
\begin{aligned}
\lambda_1 &= 2\\
\lambda_2 &= -2\\
\lambda_3 &= 1\\
\lambda_4 &= -1
\end{aligned}
$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
-2	1	real root
1	1	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^2
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= \frac{1}{x^2} \\
y_3 &= x \\
y_4 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^4 y'''' + 6x^3 y''' + 2x^2 y'' - 4y'x + 4y = F(x)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x & x^2 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 & 2x \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 & 2 \\ -\frac{6}{x^4} & -\frac{24}{x^5} & 0 & 0 \end{bmatrix}$$

$$|W| = -\frac{72}{x^6}$$

The determinant simplifies to

$$|W| = -\frac{72}{x^6}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} \frac{1}{x^2} & x & x^2 \\ -\frac{2}{x^3} & 1 & 2x \\ \frac{6}{x^4} & 0 & 2 \end{bmatrix}$$

$$= \frac{12}{x^2}$$

$$W_2(x) = \det \begin{bmatrix} \frac{1}{x} & x & x^2 \\ -\frac{1}{x^2} & 1 & 2x \\ \frac{2}{x^3} & 0 & 2 \end{bmatrix}$$

$$= \frac{6}{x}$$

$$W_3(x) = \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x^2 \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 2x \\ \frac{2}{x^3} & \frac{6}{x^4} & 2 \end{bmatrix}$$

$$= -\frac{12}{x^4}$$

$$W_4(x) = \det \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} & x \\ -\frac{1}{x^2} & -\frac{2}{x^3} & 1 \\ \frac{2}{x^3} & \frac{6}{x^4} & 0 \end{bmatrix}$$

$$= -\frac{6}{x^5}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{(F(x))\left(\frac{12}{x^2}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= -\int \frac{\frac{12F(x)}{x^2}}{-\frac{72}{x^2}}\, dx \\
&= -\int \left(-\frac{F(x)}{6}\right)\, dx \\
&= -\left(\int -\frac{F(x)}{6} dx\right)
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(F(x))\left(\frac{6}{x}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= \int \frac{\frac{6F(x)}{x}}{-\frac{72}{x^2}}\, dx \\
&= \int \left(-\frac{F(x)\,x}{12}\right)\, dx \\
&= \int -\frac{F(x)\,x}{12} dx
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{(F(x))\left(-\frac{12}{x^4}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= -\int \frac{-\frac{12F(x)}{x^4}}{-\frac{72}{x^2}}\, dx \\
&= -\int \left(\frac{F(x)}{6x^2}\right)\, dx \\
&= -\left(\int \frac{F(x)}{6x^2} dx\right)
\end{aligned}$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(F(x)) \left(-\frac{6}{x^5}\right)}{(x^4)\left(-\frac{72}{x^6}\right)}\, dx \\
&= \int \frac{-\frac{6F(x)}{x^5}}{-\frac{72}{x^2}}\, dx \\
&= \int \left(\frac{F(x)}{12x^3}\right) dx \\
&= \int \frac{F(x)}{12x^3} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\left(\int -\frac{F(x)}{6} dx\right)\right)\left(\frac{1}{x}\right) \\
&+ \left(\int -\frac{F(x)\, x}{12} dx\right)\left(\frac{1}{x^2}\right) \\
&+ \left(-\left(\int \frac{F(x)}{6x^2} dx\right)\right)(x) \\
&+ \left(\int \frac{F(x)}{12x^3} dx\right)(x^2)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(\int \frac{F(x)}{x^3} dx\right) x^4 - 2\left(\int \frac{F(x)}{x^2} dx\right) x^3 + 2\left(\int F(x)\, dx\right) x - \left(\int F(x)\, x dx\right)}{12x^2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x + c_4 x^2\right) \\
&\quad + \left(\frac{\left(\int \frac{F(x)}{x^3} dx\right) x^4 - 2\left(\int \frac{F(x)}{x^2} dx\right) x^3 + 2\left(\int F(x)\, dx\right) x - \left(\int F(x)\, x dx\right)}{12x^2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \frac{c_1}{x}+\frac{c_2}{x^2}+c_3 x+c_4 x^2 \\
&+ \frac{\left(\int \frac{F(x)}{x^3}dx\right)x^4-2\left(\int \frac{F(x)}{x^2}dx\right)x^3+2\left(\int F(x)\,dx\right)x-\left(\int F(x)\,x dx\right)}{12x^2} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \frac{c_1}{x}+\frac{c_2}{x^2}+c_3 x+c_4 x^2 \\
&+ \frac{\left(\int \frac{F(x)}{x^3}dx\right)x^4-2\left(\int \frac{F(x)}{x^2}dx\right)x^3+2\left(\int F(x)\,dx\right)x-\left(\int F(x)\,x dx\right)}{12x^2}
\end{align*}
$$

Verified OK.

20.17.1 Maple step by step solution

Let's solve

$x^4 y'''' + 6x^3 y''' + 2x^2 y'' - 4y'x + 4y = F(x)$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1-4*_a*_b(_a)+4
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x^4*diff(y(x),x$4)+6*x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+4*y(x)=
```

$$y(x) = \frac{c_4 + \int \left(2c_2 x + c_3 + \int \int \frac{c_1 + \int F(x)dx}{x^4} dx dx\right) x^2 dx}{x^2}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 104

```
DSolve[x^4*y''''[x]+6*x^3*y'''[x]+2*x^2*y''[x]-4*x*y'[x]+4*y[x]==f[x],y[x],x,IncludeSingular
```

$$y(x) \to \frac{x^4 \int_1^x \frac{f(K[4])}{12K[4]^3} dK[4] + x^3 \int_1^x -\frac{f(K[3])}{6K[3]^2} dK[3] + x \int_1^x \frac{1}{6} f(K[2]) dK[2] + \int_1^x -\frac{1}{12} f(K[1]) K[1] dK[1] + c_4 x^4 + c}{x^2}$$

21 Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540

21.1 problem section 10.4, problem 1

Internal problem ID [1589]
Internal file name [`OUTPUT/1590_Sunday_June_05_2022_02_23_52_AM_91386351/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= y_1(t) + 2y_2(t) \\
y_2'(t) &= 2y_1(t) + y_2(t)
\end{aligned}
$$

21.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{3t}}{2} & \frac{\mathrm{e}^{3t}}{2} - \frac{\mathrm{e}^{-t}}{2} \\ \frac{\mathrm{e}^{3t}}{2} - \frac{\mathrm{e}^{-t}}{2} & \frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{3t}}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^{3t}}{2} & \frac{e^{3t}}{2} - \frac{e^{-t}}{2} \\ \frac{e^{3t}}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_1 + \left(\frac{e^{3t}}{2} - \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{e^{3t}}{2} - \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} + \frac{e^{3t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-c_2)e^{-t}}{2} + \frac{e^{3t}(c_1+c_2)}{2} \\ \frac{(-c_1+c_2)e^{-t}}{2} + \frac{e^{3t}(c_1+c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{3t} - c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

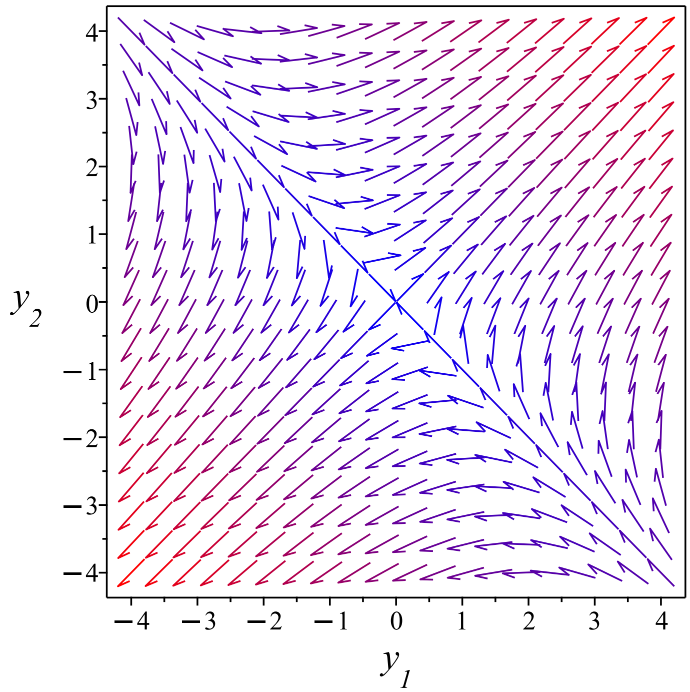


Figure 563: Phase plot

21.1.3 Maple step by step solution

Let's solve

$$[y_1'(t) = y_1(t) + 2y_2(t)\,, y_2'(t) = 2y_1(t) + y_2(t)]$$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} + c_2 e^{3t} \\ c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{y_1(t) = -c_1 e^{-t} + c_2 e^{3t}, y_2(t) = c_1 e^{-t} + c_2 e^{3t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(y__1(t),t)=y__1(t)+2*y__2(t),diff(y__2(t),t)=2*y__1(t)+1*y__2(t)],singsol=all)
```

$$y_1(t) = c_1 e^{3t} + c_2 e^{-t}$$
$$y_2(t) = c_1 e^{3t} - c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 68

```
DSolve[{y1'[t]==y1[t]+2*y2[t],y2'[t]==2*y1[t]+y2[t]},{y1[t],y2[t]},t,IncludeSingularSolution
```

$$\text{y1}(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(e^{4t} + 1\right) + c_2 \left(e^{4t} - 1\right)\right)$$
$$\text{y2}(t) \to \frac{1}{2} e^{-t} \left(c_1 \left(e^{4t} - 1\right) + c_2 \left(e^{4t} + 1\right)\right)$$

21.2 problem section 10.4, problem 2

Internal problem ID [1590]
Internal file name [`OUTPUT/1591_Sunday_June_05_2022_02_23_53_AM_12529775/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -\frac{5y_1(t)}{4} + \frac{3y_2(t)}{4} \\
y_2'(t) &= \frac{3y_1(t)}{4} - \frac{5y_2(t)}{4}
\end{aligned}
$$

21.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{-\frac{t}{2}}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} & \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} \\ \frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2}\right)c_1+\left(\frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2}\right)c_2 \\ \left(\frac{e^{-\frac{t}{2}}}{2}-\frac{e^{-2t}}{2}\right)c_1+\left(\frac{e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-c_2)e^{-2t}}{2}+\frac{e^{-\frac{t}{2}}(c_1+c_2)}{2} \\ \frac{e^{-\frac{t}{2}}(c_1+c_2)}{2}-\frac{(c_1-c_2)e^{-2t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{5}{4}-\lambda & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{5}{2}\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{3}{4} & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & -\frac{3}{4} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}$ is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-\frac{t}{2}}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{t}{2}}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-\frac{t}{2}} \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-\frac{t}{2}} - c_2 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-\frac{t}{2}} + c_2 \mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

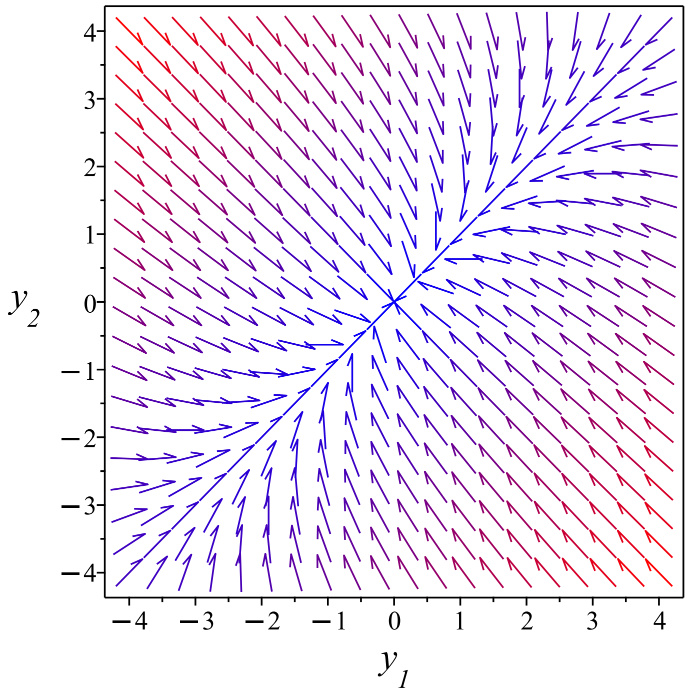


Figure 564: Phase plot

21.2.3 Maple step by step solution

Let's solve

$$\left[y_1'(t) = -\frac{5y_1(t)}{4} + \frac{3y_2(t)}{4}, y_2'(t) = \frac{3y_1(t)}{4} - \frac{5y_2(t)}{4}\right]$$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = e^{-\frac{t}{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{t}{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}} \\ c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{y_1(t) = -c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}}, y_2(t) = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(y__1(t),t)=-5/4*y__1(t)+3/4*y__2(t),diff(y__2(t),t)=3/4*y__1(t)-5/4*y__2(t)],si
```

$$y_1(t) = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}}$$
$$y_2(t) = -c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 76

```
DSolve[{y1'[t]==-5/4*y1[t]+3/4*y2[t],y2'[t]==3/4*y1[t]-5/4*y2[t]},{y1[t],y2[t]},t,IncludeSin
```

$$\mathrm{y1}(t) \to \frac{1}{2} e^{-2t} \left(c_1 \left(e^{3t/2} + 1\right) + c_2 \left(e^{3t/2} - 1\right)\right)$$
$$\mathrm{y2}(t) \to \frac{1}{2} e^{-2t} \left(c_1 \left(e^{3t/2} - 1\right) + c_2 \left(e^{3t/2} + 1\right)\right)$$

21.3 problem section 10.4, problem 3

Internal problem ID [1591]
Internal file name [`OUTPUT/1592_Sunday_June_05_2022_02_23_55_AM_37212796/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -\frac{4y_1(t)}{5} + \frac{3y_2(t)}{5} \\
y_2'(t) &= -\frac{2y_1(t)}{5} - \frac{11y_2(t)}{5}
\end{aligned}
$$

21.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -\frac{e^{-2t}}{5} + \frac{6\,e^{-t}}{5} & \frac{3\,e^{-t}}{5} - \frac{3\,e^{-2t}}{5} \\ -\frac{2\,e^{-t}}{5} + \frac{2\,e^{-2t}}{5} & \frac{6\,e^{-2t}}{5} - \frac{e^{-t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-2t}}{5} + \frac{6\,e^{-t}}{5} & \frac{3\,e^{-t}}{5} - \frac{3\,e^{-2t}}{5} \\ -\frac{2\,e^{-t}}{5} + \frac{2\,e^{-2t}}{5} & \frac{6\,e^{-2t}}{5} - \frac{e^{-t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-2t}}{5} + \frac{6\,e^{-t}}{5}\right) c_1 + \left(\frac{3\,e^{-t}}{5} - \frac{3\,e^{-2t}}{5}\right) c_2 \\ \left(-\frac{2\,e^{-t}}{5} + \frac{2\,e^{-2t}}{5}\right) c_1 + \left(\frac{6\,e^{-2t}}{5} - \frac{e^{-t}}{5}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1-3c_2)e^{-2t}}{5} + \frac{6\left(c_1+\frac{c_2}{2}\right)e^{-t}}{5} \\ \frac{(2c_1+6c_2)e^{-2t}}{5} - \frac{2\left(c_1+\frac{c_2}{2}\right)e^{-t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -\frac{4}{5} - \lambda & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{6}{5} & \frac{3}{5} & 0 \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} \frac{6}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{5} & \frac{3}{5} & 0 \\ -\frac{2}{5} & -\frac{6}{5} & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \implies \left[\begin{array}{cc|c} \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\
&= \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -3\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-2t}}{2} - 3 c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} \end{bmatrix}
$$

The following is the phase plot of the system.

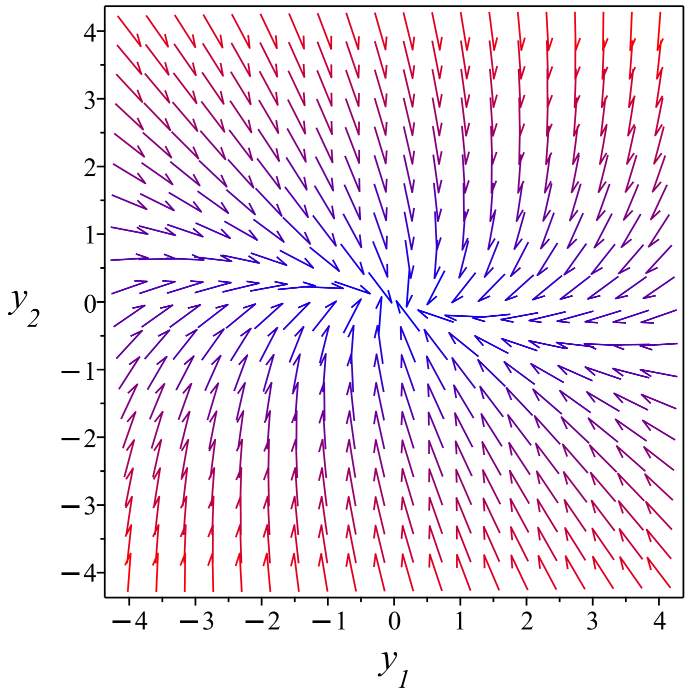


Figure 565: Phase plot

21.3.3 Maple step by step solution

Let's solve

$$\left[y_1'(t) = -\frac{4y_1(t)}{5} + \frac{3y_2(t)}{5}, y_2'(t) = -\frac{2y_1(t)}{5} - \frac{11y_2(t)}{5}\right]$$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -\frac{4}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{11}{5} \end{array}\right]$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -3 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -3 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-2t}}{2} - 3c_2 e^{-t} \\ c_1 e^{-2t} + c_2 e^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = -\frac{c_1 e^{-2t}}{2} - 3c_2 e^{-t}, y_2(t) = c_1 e^{-2t} + c_2 e^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve([diff(y__1(t),t)=-4/5*y__1(t)+3/5*y__2(t),diff(y__2(t),t)=-2/5*y__1(t)-11/5*y__2(t)],
```

$$y_1(t) = e^{-t} c_1 + c_2 e^{-2t}$$
$$y_2(t) = -\frac{e^{-t} c_1}{3} - 2c_2 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
DSolve[{y1'[t]==-4/5*y1[t]+3/5*y2[t],y2'[t]==-2/5*y1[t]-11/5*y2[t]},{y1[t],y2[t]},t,IncludeS
```

$$\text{y1}(t) \to \frac{1}{5} e^{-2t} \left(c_1 \left(6e^t - 1\right) + 3c_2 \left(e^t - 1\right)\right)$$
$$\text{y2}(t) \to \frac{1}{5} e^{-2t} \left(-2c_1 \left(e^t - 1\right) - c_2 \left(e^t - 6\right)\right)$$

21.4 problem section 10.4, problem 4

Internal problem ID [1592]
Internal file name [`OUTPUT/1593_Sunday_June_05_2022_02_23_56_AM_1394617/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$y_1'(t) = -y_1(t) - 4y_2(t)$$
$$y_2'(t) = -y_1(t) - y_2(t)$$

21.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & -(e^{4t}-1)\,e^{-3t} \\ -\frac{(e^{4t}-1)e^{-3t}}{4} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}}{2} & -(e^{4t}-1)e^{-3t} \\ -\frac{(e^{4t}-1)e^{-3t}}{4} & \frac{(e^{4t}+1)e^{-3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{4t}+1)e^{-3t}c_1}{2} - (e^{4t}-1)e^{-3t}c_2 \\ -\frac{(e^{4t}-1)e^{-3t}c_1}{4} + \frac{(e^{4t}+1)e^{-3t}c_2}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{((c_1-2c_2)e^{4t}+c_1+2c_2)e^{-3t}}{2} \\ -\frac{((c_1-2c_2)e^{4t}-c_1-2c_2)e^{-3t}}{4} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -4 & 0 \\ -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^t$$
$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2(c_1 \mathrm{e}^{4t} - c_2)\,\mathrm{e}^{-3t} \\ (c_1 \mathrm{e}^{4t} + c_2)\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

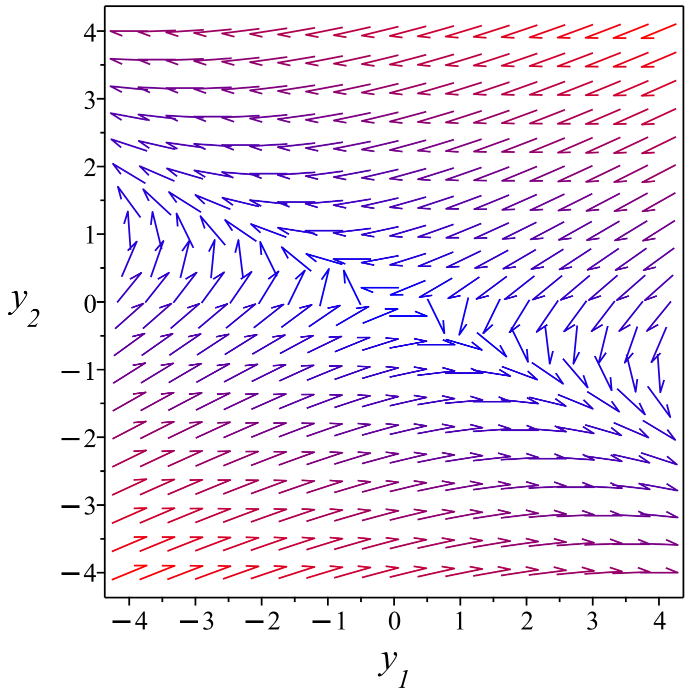


Figure 566: Phase plot

21.4.3 Maple step by step solution

Let's solve

$[y_1'(t) = -y_1(t) - 4y_2(t), y_2'(t) = -y_1(t) - y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = e^{t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2(-c_2 \mathrm{e}^{4t} + c_1) \mathrm{e}^{-3t} \\ (c_2 \mathrm{e}^{4t} + c_1) \mathrm{e}^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{y_1(t) = 2(-c_2 \mathrm{e}^{4t} + c_1) \mathrm{e}^{-3t}, y_2(t) = (c_2 \mathrm{e}^{4t} + c_1) \mathrm{e}^{-3t}\}$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve([diff(y__1(t),t)=-1*y__1(t)-4*y__2(t),diff(y__2(t),t)=-1*y__1(t)-1*y__2(t)],singsol=a
```

$$y_1(t) = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-3t}$$
$$y_2(t) = -\frac{c_1 \mathrm{e}^{t}}{2} + \frac{c_2 \mathrm{e}^{-3t}}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 71

```
DSolve[{y1'[t]==-1*y1[t]-4*y2[t],y2'[t]==-1*y1[t]-1*y2[t]},{y1[t],y2[t]},t,IncludeSingularSo
```

$$\mathrm{y1}(t) \to \frac{1}{2} e^{-3t} \left(c_1 \left(e^{4t} + 1\right) - 2c_2 \left(e^{4t} - 1\right)\right)$$
$$\mathrm{y2}(t) \to \frac{1}{4} e^{-3t} \left(2c_2 \left(e^{4t} + 1\right) - c_1 \left(e^{4t} - 1\right)\right)$$

21.5 problem section 10.4, problem 5

Internal problem ID [1593]
Internal file name [`OUTPUT/1594_Sunday_June_05_2022_02_23_58_AM_28622100/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$y_1'(t) = 2y_1(t) - 4y_2(t)$$
$$y_2'(t) = -y_1(t) - y_2(t)$$

21.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(4\,e^{5t}+1)e^{-2t}}{5} & -\frac{4(e^{5t}-1)e^{-2t}}{5} \\ -\frac{(e^{5t}-1)e^{-2t}}{5} & \frac{(e^{5t}+4)e^{-2t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(4\,\mathrm{e}^{5t}+1)\mathrm{e}^{-2t}}{5} & -\frac{4(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}}{5} \\ -\frac{(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}}{5} & \frac{(\mathrm{e}^{5t}+4)\mathrm{e}^{-2t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4\,\mathrm{e}^{5t}+1)\mathrm{e}^{-2t}c_1}{5} - \frac{4(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}c_2}{5} \\ -\frac{(\mathrm{e}^{5t}-1)\mathrm{e}^{-2t}c_1}{5} + \frac{(\mathrm{e}^{5t}+4)\mathrm{e}^{-2t}c_2}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4\left((c_1-c_2)\mathrm{e}^{5t}+\frac{c_1}{4}+c_2\right)\mathrm{e}^{-2t}}{5} \\ -\frac{\mathrm{e}^{-2t}\left((c_1-c_2)\mathrm{e}^{5t}-c_1-4c_2\right)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - \lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ -1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -4 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -4\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (-4c_1 \mathrm{e}^{5t} + c_2)\,\mathrm{e}^{-2t} \\ (c_1 \mathrm{e}^{5t} + c_2)\,\mathrm{e}^{-2t} \end{bmatrix}$$

The following is the phase plot of the system.

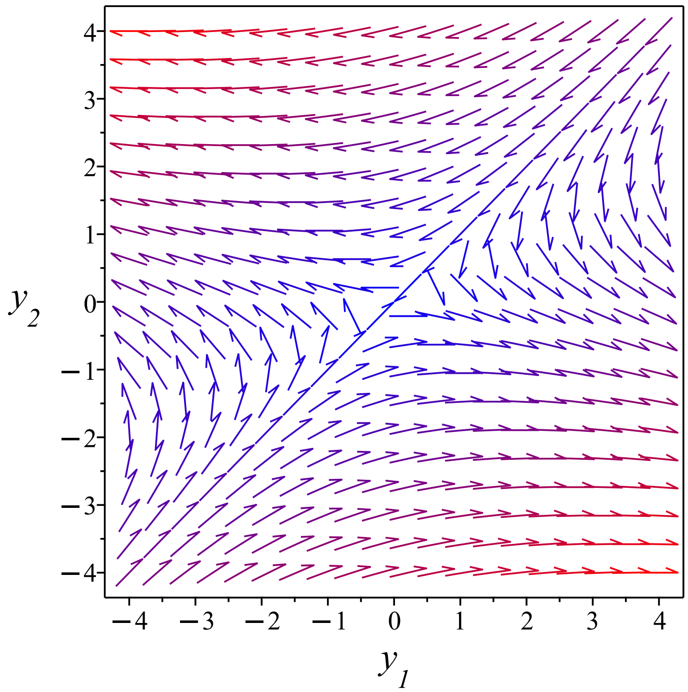


Figure 567: Phase plot

21.5.3 Maple step by step solution

Let's solve

$[y_1'(t) = 2y_1(t) - 4y_2(t), y_2'(t) = -y_1(t) - y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{3t} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{3t} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \left(-4c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-2t} \\ \left(c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs
  $\{y_1(t) = \left(-4c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-2t}, y_2(t) = \left(c_2 \mathrm{e}^{5t} + c_1\right) \mathrm{e}^{-2t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(y__1(t),t)=2*y__1(t)-4*y__2(t),diff(y__2(t),t)=-1*y__1(t)-1*y__2(t)],singsol=al
```

$$y_1(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-2t}$$
$$y_2(t) = -\frac{c_1 \mathrm{e}^{3t}}{4} + c_2 \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 72

```
DSolve[{y1'[t]==2*y1[t]-4*y2[t],y2'[t]==-1*y1[t]-1*y2[t]},{y1[t],y2[t]},t,IncludeSingularSol
```

$$\text{y1}(t) \rightarrow \frac{1}{5} e^{-2t} \left(c_1 \left(4e^{5t} + 1\right) - 4c_2 \left(e^{5t} - 1\right)\right)$$
$$\text{y2}(t) \rightarrow \frac{1}{5} e^{-2t} \left(c_2 \left(e^{5t} + 4\right) - c_1 \left(e^{5t} - 1\right)\right)$$

21.6 problem section 10.4, problem 6

Internal problem ID [1594]
Internal file name [`OUTPUT/1595_Sunday_June_05_2022_02_24_00_AM_97349432/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 4y_1(t) - 3y_2(t) \\
y_2'(t) &= 2y_1(t) - y_2(t)
\end{aligned}
$$

21.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{t} + 3\,\mathrm{e}^{2t} & -3\,\mathrm{e}^{2t} + 3\,\mathrm{e}^{t} \\ 2\,\mathrm{e}^{2t} - 2\,\mathrm{e}^{t} & 3\,\mathrm{e}^{t} - 2\,\mathrm{e}^{2t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,e^{t}+3\,e^{2t} & -3\,e^{2t}+3\,e^{t} \\ 2\,e^{2t}-2\,e^{t} & 3\,e^{t}-2\,e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,e^{t}+3\,e^{2t})\,c_1+(-3\,e^{2t}+3\,e^{t})\,c_2 \\ (2\,e^{2t}-2\,e^{t})\,c_1+(3\,e^{t}-2\,e^{2t})\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-3c_2+3c_1)\,e^{2t}-2\left(c_1-\frac{3c_2}{2}\right)e^{t} \\ (2c_1-2c_2)\,e^{2t}-2\left(c_1-\frac{3c_2}{2}\right)e^{t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{2t}}{2} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3c_1\mathrm{e}^{2t}}{2} + c_2\mathrm{e}^{t} \\ c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{t} \end{bmatrix}
$$

The following is the phase plot of the system.

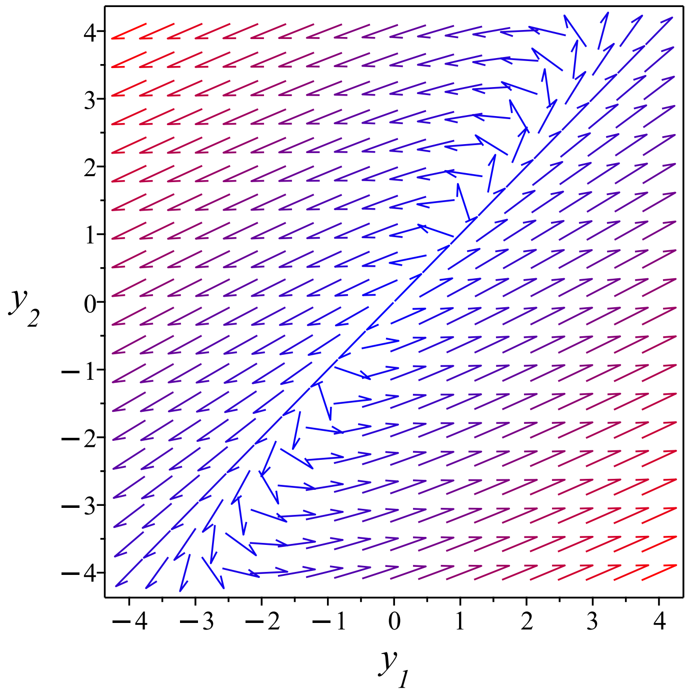


Figure 568: Phase plot

21.6.3 Maple step by step solution

Let's solve

$[y_1'(t) = 4y_1(t) - 3y_2(t), y_2'(t) = 2y_1(t) - y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + \frac{3c_2 e^{2t}}{2} \\ c_1 e^t + c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{y_1(t) = c_1 e^t + \frac{3c_2 e^{2t}}{2}, y_2(t) = c_1 e^t + c_2 e^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve([diff(y__1(t),t)=4*y__1(t)-3*y__2(t),diff(y__2(t),t)=2*y__1(t)-1*y__2(t)],singsol=all
```

$$y_1(t) = c_1 e^t + c_2 e^{2t}$$
$$y_2(t) = c_1 e^t + \frac{2c_2 e^{2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 56

```
DSolve[{y1'[t]==4*y1[t]-3*y2[t],y2'[t]==2*y1[t]-1*y2[t]},{y1[t],y2[t]},t,IncludeSingularSolu
```

$$\text{y1}(t) \to e^t(c_1(3e^t - 2) - 3c_2(e^t - 1))$$
$$\text{y2}(t) \to e^t(2c_1(e^t - 1) + c_2(3 - 2e^t))$$

21.7 problem section 10.4, problem 7

Internal problem ID [1595]
Internal file name [`OUTPUT/1596_Sunday_June_05_2022_02_24_02_AM_64170985/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -6y_1(t) - 3y_2(t) \\
y_2'(t) &= y_1(t) - 2y_2(t)
\end{aligned}
$$

21.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,\mathrm{e}^{-5t}}{2} - \frac{\mathrm{e}^{-3t}}{2} & -\frac{3\,\mathrm{e}^{-3t}}{2} + \frac{3\,\mathrm{e}^{-5t}}{2} \\ \frac{\mathrm{e}^{-3t}}{2} - \frac{\mathrm{e}^{-5t}}{2} & -\frac{\mathrm{e}^{-5t}}{2} + \frac{3\,\mathrm{e}^{-3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-5t}}{2}-\frac{\mathrm{e}^{-3t}}{2} & -\frac{3\,\mathrm{e}^{-3t}}{2}+\frac{3\,\mathrm{e}^{-5t}}{2} \\ \frac{\mathrm{e}^{-3t}}{2}-\frac{\mathrm{e}^{-5t}}{2} & -\frac{\mathrm{e}^{-5t}}{2}+\frac{3\,\mathrm{e}^{-3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{3\,\mathrm{e}^{-5t}}{2}-\frac{\mathrm{e}^{-3t}}{2}\right)c_1+\left(-\frac{3\,\mathrm{e}^{-3t}}{2}+\frac{3\,\mathrm{e}^{-5t}}{2}\right)c_2 \\ \left(\frac{\mathrm{e}^{-3t}}{2}-\frac{\mathrm{e}^{-5t}}{2}\right)c_1+\left(-\frac{\mathrm{e}^{-5t}}{2}+\frac{3\,\mathrm{e}^{-3t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3c_1+3c_2)\mathrm{e}^{-5t}}{2}-\frac{\mathrm{e}^{-3t}(c_1+3c_2)}{2} \\ \frac{(-c_1-c_2)\mathrm{e}^{-5t}}{2}+\frac{\mathrm{e}^{-3t}(c_1+3c_2)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 15 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -3 & 0 \\ 1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-3t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-5t}$$
$$= \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} -3\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-3t} - 3c_2 \mathrm{e}^{-5t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

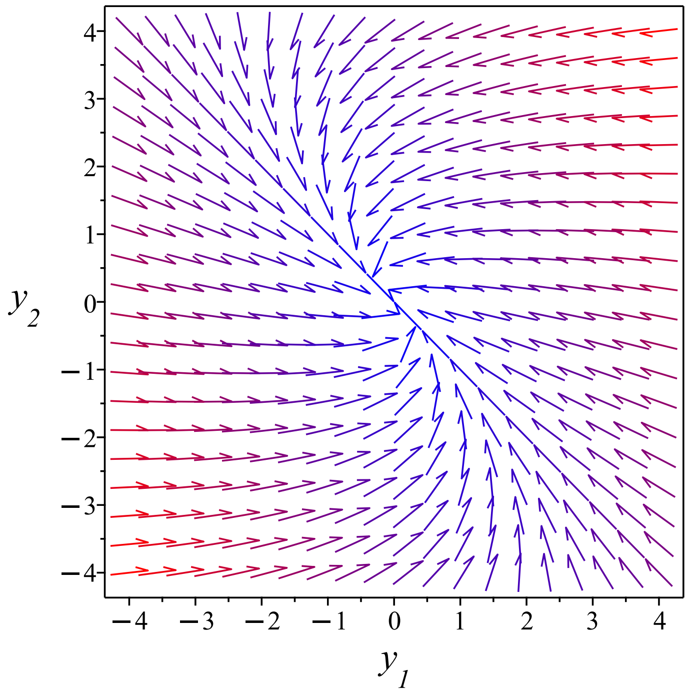


Figure 569: Phase plot

21.7.3 Maple step by step solution

Let's solve

$[y_1'(t) = -6y_1(t) - 3y_2(t), y_2'(t) = y_1(t) - 2y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-5t} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = e^{-3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-5t} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} + e^{-3t} c_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -3c_1 e^{-5t} - e^{-3t} c_2 \\ c_1 e^{-5t} + e^{-3t} c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $\{y_1(t) = -3c_1 e^{-5t} - e^{-3t} c_2, y_2(t) = c_1 e^{-5t} + e^{-3t} c_2\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(y__1(t),t)=-6*y__1(t)-3*y__2(t),diff(y__2(t),t)=1*y__1(t)-2*y__2(t)],singsol=al
```

$$y_1(t) = c_1 e^{-3t} + c_2 e^{-5t}$$
$$y_2(t) = -c_1 e^{-3t} - \frac{c_2 e^{-5t}}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 72

```
DSolve[{y1'[t]==-6*y1[t]-3*y2[t],y2'[t]==1*y1[t]-2*y2[t]},{y1[t],y2[t]},t,IncludeSingularSol
```

$$\text{y1}(t) \to \frac{1}{2} e^{-5t} \left(-\left(c_1 \left(e^{2t} - 3\right)\right) - 3c_2 \left(e^{2t} - 1\right)\right)$$
$$\text{y2}(t) \to \frac{1}{2} e^{-5t} \left(c_1 \left(e^{2t} - 1\right) + c_2 \left(3e^{2t} - 1\right)\right)$$

21.8 problem section 10.4, problem 8

Internal problem ID [1596]
Internal file name [`OUTPUT/1597_Sunday_June_05_2022_02_24_03_AM_23896703/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= y_1(t) - y_2(t) - 2y_3(t)\\
y_2'(t) &= y_1(t) - 2y_2(t) - 3y_3(t)\\
y_3'(t) &= -4y_1(t) + y_2(t) - y_3(t)
\end{aligned}
$$

21.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t)\\ y_2'(t)\\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2\\ 1 & -2 & -3\\ -4 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t)\\ y_2(t)\\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-1)\,\mathrm{e}^{-3t}}{2} & -\frac{\mathrm{e}^{2t}}{3}+\frac{\mathrm{e}^{-t}}{3} & -\frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}}{6} \\ (\mathrm{e}^{5t}-2\,\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t} & \frac{4\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{3} & -\frac{(\mathrm{e}^{5t}+2\,\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}}{3} \\ \frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t}}{2} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+3)\,\mathrm{e}^{-3t}}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-1)\,\mathrm{e}^{-3t}}{2} & -\frac{\mathrm{e}^{2t}}{3}+\frac{\mathrm{e}^{-t}}{3} & -\frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}}{6} \\ (\mathrm{e}^{5t}-2\,\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t} & \frac{4\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{3} & -\frac{(\mathrm{e}^{5t}+2\,\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}}{3} \\ \frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t}}{2} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+3)\,\mathrm{e}^{-3t}}{6} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-1)\,\mathrm{e}^{-3t}c_1}{2} + \left(-\frac{\mathrm{e}^{2t}}{3}+\frac{\mathrm{e}^{-t}}{3}\right)c_2 - \frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}c_3}{6} \\ (\mathrm{e}^{5t}-2\,\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t}c_1 + \left(\frac{4\,\mathrm{e}^{-t}}{3}-\frac{\mathrm{e}^{2t}}{3}\right)c_2 - \frac{(\mathrm{e}^{5t}+2\,\mathrm{e}^{2t}-3)\,\mathrm{e}^{-3t}c_3}{3} \\ \frac{(-2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+1)\,\mathrm{e}^{-3t}c_1}{2} + \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2 + \frac{(2\,\mathrm{e}^{5t}+\mathrm{e}^{2t}+3)\,\mathrm{e}^{-3t}c_3}{6} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{2t}+\left(-2c_1+\frac{2c_2}{3}+\frac{2c_3}{3}\right)\mathrm{e}^{5t}-c_1-c_3\right)\mathrm{e}^{-3t}}{2} \\ -2\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{2t}+\left(-\frac{c_1}{2}+\frac{c_2}{6}+\frac{c_3}{6}\right)\mathrm{e}^{5t}-\frac{c_1}{2}-\frac{c_3}{2}\right)\mathrm{e}^{-3t} \\ \frac{\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{2t}+\left(-2c_1+\frac{2c_2}{3}+\frac{2c_3}{3}\right)\mathrm{e}^{5t}+c_1+c_3\right)\mathrm{e}^{-3t}}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & -2 \\ 1 & -2-\lambda & -3 \\ -4 & 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -2 \\ 1 & 1 & -3 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -1 & -2 & 0 \\ 1 & 1 & -3 & 0 \\ -4 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & -2 & 0 \\ 0 & \frac{5}{4} & -\frac{5}{2} & 0 \\ -4 & 1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & -2 & 0 \\ 0 & \frac{5}{4} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 & -2 \\ 0 & \frac{5}{4} & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -2 \\ 1 & -1 & -3 \\ -4 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 1 & -1 & -3 & 0 \\ -4 & 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & -\frac{1}{2} & -2 & 0 \\ -4 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & -\frac{1}{2} & -2 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$$

$$
R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & -\frac{1}{2} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -1 & -2 \\ 0 & -\frac{1}{2} & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -4t\}$

Hence the solution is

$$
\begin{bmatrix} -t \\ -4t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -4t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -t \\ -4t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} -t \\ -4t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}
$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -2 \\ 1 & -4 & -3 \\ -4 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 1 & -4 & -3 & 0 \\ -4 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ -4 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & -2 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{2t}$$
$$= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ -4\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-3t} \\ 2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{2t} \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -(c_3\mathrm{e}^{5t} + c_1\mathrm{e}^{2t} - c_2)\,\mathrm{e}^{-3t} \\ (-c_3\mathrm{e}^{5t} - 4c_1\mathrm{e}^{2t} + 2c_2)\,\mathrm{e}^{-3t} \\ (c_3\mathrm{e}^{5t} + c_1\mathrm{e}^{2t} + c_2)\,\mathrm{e}^{-3t} \end{bmatrix}$$

21.8.3 Maple step by step solution

Let's solve

$[y_1'(t) = y_1(t) - y_2(t) - 2y_3(t)\,, y_2'(t) = y_1(t) - 2y_2(t) - 3y_3(t)\,, y_3'(t) = -4y_1(t) + y_2(t) - y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \cdot \underline{\vec{y}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \cdot \underline{\vec{y}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-3t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{2} = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{3} = \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2} + c_3 \underline{\vec{y}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-3t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \left(-c_3 \mathrm{e}^{5t} - c_2 \mathrm{e}^{2t} + c_1\right) \mathrm{e}^{-3t} \\ \left(-c_3 \mathrm{e}^{5t} - 4 c_2 \mathrm{e}^{2t} + 2 c_1\right) \mathrm{e}^{-3t} \\ \left(c_1 + c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{5t}\right) \mathrm{e}^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = \left(-c_3 \mathrm{e}^{5t} - c_2 \mathrm{e}^{2t} + c_1\right) \mathrm{e}^{-3t}, y_2(t) = \left(-c_3 \mathrm{e}^{5t} - 4 c_2 \mathrm{e}^{2t} + 2 c_1\right) \mathrm{e}^{-3t}, y_3(t) = \left(c_1 + c_2 \mathrm{e}^{2t} + c_3 \mathrm{e}^{5t}\right)$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 72

```
dsolve([diff(y__1(t),t)=1*y__1(t)-1*y__2(t)-2*y__3(t),diff(y__2(t),t)=1*y__1(t)-2*y__2(t)-3*
```

$$y_1(t) = e^{-t}c_1 + c_2e^{2t} + c_3e^{-3t}$$
$$y_2(t) = 4\,e^{-t}c_1 + c_2e^{2t} + 2c_3e^{-3t}$$
$$y_3(t) = -e^{-t}c_1 - c_2e^{2t} + c_3e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 102

```
DSolve[{y1'[t]==1*y1[t]-1*y2[t]-2*y3[t],y2'[t]==1*y1[t]-2*y2[t]-3*y3[t],y1'[t]==-4*y1[t]+1*y
```

$$\text{y1}(t) \to \frac{1}{576}e^{-3t}\left(c_1\left(63 - 128e^t\right) + c_2\left(64e^t - 27\right)\right)$$
$$\text{y2}(t) \to \frac{1}{864}e^{-3t}\left(c_2\left(224e^t - 81\right) - 7c_1\left(64e^t - 27\right)\right)$$
$$\text{y3}(t) \to \frac{e^{-3t}(c_1(189 - 128e^t) + c_2(64e^t - 81))}{1728}$$

21.9 problem section 10.4, problem 9

Internal problem ID [1597]
Internal file name [`OUTPUT/1598_Sunday_June_05_2022_02_24_05_AM_1459569/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -6y_1(t) - 4y_2(t) - 8y_3(t) \\
y_2'(t) &= -4y_1(t) - 4y_3(t) \\
y_3'(t) &= -8y_1(t) - 4y_2(t) - 6y_3(t)
\end{aligned}
$$

21.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(5\,e^{18t}+4)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} & -\frac{4(e^{18t}-1)e^{-16t}}{9} \\ -\frac{2(e^{18t}-1)e^{-16t}}{9} & \frac{(8\,e^{18t}+1)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} \\ -\frac{4(e^{18t}-1)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} & \frac{(5\,e^{18t}+4)e^{-16t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(5\,e^{18t}+4)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} & -\frac{4(e^{18t}-1)e^{-16t}}{9} \\ -\frac{2(e^{18t}-1)e^{-16t}}{9} & \frac{(8\,e^{18t}+1)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} \\ -\frac{4(e^{18t}-1)e^{-16t}}{9} & -\frac{2(e^{18t}-1)e^{-16t}}{9} & \frac{(5\,e^{18t}+4)e^{-16t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(5\,e^{18t}+4)e^{-16t}c_1}{9} - \frac{2(e^{18t}-1)e^{-16t}c_2}{9} - \frac{4(e^{18t}-1)e^{-16t}c_3}{9} \\ -\frac{2(e^{18t}-1)e^{-16t}c_1}{9} + \frac{(8\,e^{18t}+1)e^{-16t}c_2}{9} - \frac{2(e^{18t}-1)e^{-16t}c_3}{9} \\ -\frac{4(e^{18t}-1)e^{-16t}c_1}{9} - \frac{2(e^{18t}-1)e^{-16t}c_2}{9} + \frac{(5\,e^{18t}+4)e^{-16t}c_3}{9} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-16t}((5c_1-2c_2-4c_3)e^{18t}+4c_1+2c_2+4c_3)}{9} \\ -\frac{2((c_1-4c_2+c_3)e^{18t}-c_1-\frac{c_2}{2}-c_3)e^{-16t}}{9} \\ -\frac{4\left(\left(c_1+\frac{c_2}{2}-\frac{5c_3}{4}\right)e^{18t}-c_1-\frac{c_2}{2}-c_3\right)e^{-16t}}{9} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -6-\lambda & -4 & -8 \\ -4 & -\lambda & -4 \\ -8 & -4 & -6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 12\lambda^2 - 60\lambda + 64 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -16$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
-16	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -16$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} - (-16)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -4 & -8 \\ -4 & 16 & -4 \\ -8 & -4 & 10 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 10 & -4 & -8 & 0 \\ -4 & 16 & -4 & 0 \\ -8 & -4 & 10 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 10 & -4 & -8 & 0 \\ 0 & \frac{72}{5} & -\frac{36}{5} & 0 \\ -8 & -4 & 10 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 10 & -4 & -8 & 0 \\ 0 & \frac{72}{5} & -\frac{36}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 10 & -4 & -8 & 0 \\ 0 & \frac{72}{5} & -\frac{36}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 10 & -4 & -8 \\ 0 & \frac{72}{5} & -\frac{36}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -4 & -8 \\ -4 & -2 & -4 \\ -8 & -4 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & -4 & -8 & 0 \\ -4 & -2 & -4 & 0 \\ -8 & -4 & -8 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & -4 & -8 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & -4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2} - s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{2} - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-16	1	1	No	$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$
2	2	2	No	$\begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -16 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-16t} \\ &= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{-16t} \end{aligned}$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

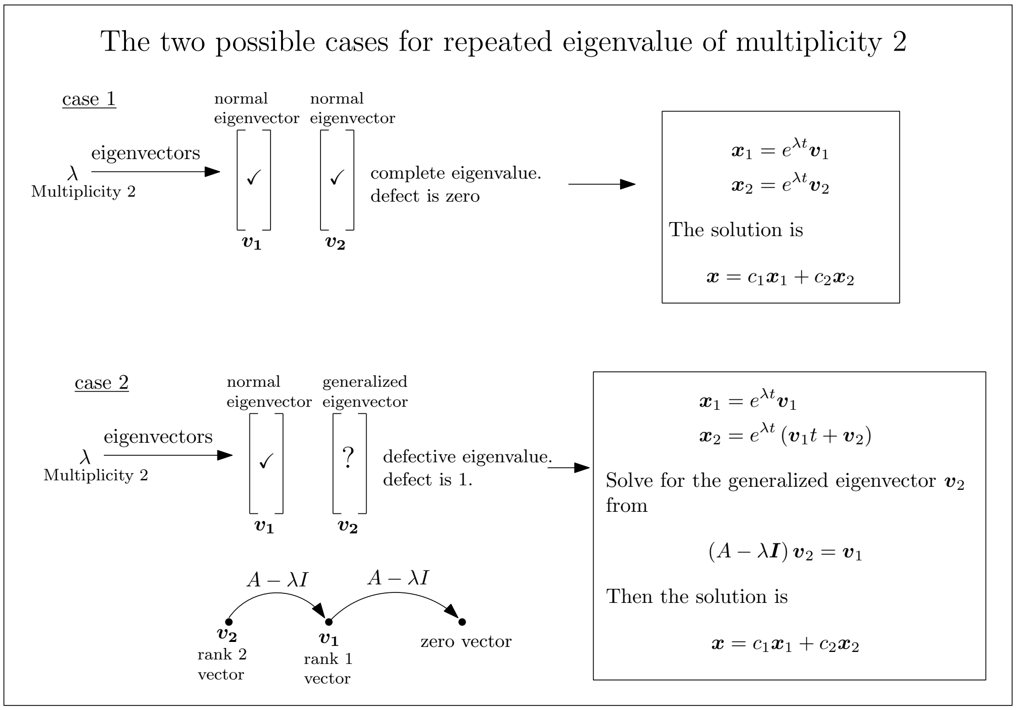


Figure 570: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-16t} \\ \frac{\mathrm{e}^{-16t}}{2} \\ \mathrm{e}^{-16t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \left(\left(-c_2 - \frac{c_3}{2}\right) \mathrm{e}^{18t} + c_1\right) \mathrm{e}^{-16t} \\ \frac{(2c_3\mathrm{e}^{18t} + c_1)\mathrm{e}^{-16t}}{2} \\ (c_2\mathrm{e}^{18t} + c_1)\,\mathrm{e}^{-16t} \end{bmatrix}$$

21.9.3 Maple step by step solution

Let's solve

$[y_1'(t) = -6y_1(t) - 4y_2(t) - 8y_3(t), y_2'(t) = -4y_1(t) - 4y_3(t), y_3'(t) = -8y_1(t) - 4y_2(t) - 6y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix}$$

- Rewrite the system as

  $\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-16, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-16, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\underline{y}^{\rightarrow}_{1} = e^{-16t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\underline{y}^{\rightarrow}_{2}(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

$\underline{\vec{y}}_3(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{y}}_3(t)$ into the homogeneous system
  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A
  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation
  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I
  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\underline{\vec{y}}_3(t)$ to be a solution to the homogeneous system
  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2
  $$\left(\begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
- Choice of $\vec{p}$
  $$\vec{p} = \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue 2
  $$\underline{\vec{y}}_3(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs
  $\underline{\vec{y}} = c_1\underline{\vec{y}}_1 + c_2\underline{\vec{y}}_2(t) + c_3\underline{\vec{y}}_3(t)$
- Substitute solutions into the general solution

$$\underline{\overrightarrow{y}} = c_1 e^{-16t} \cdot \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\left(\left(\left(t - \frac{1}{8}\right) c_3 + c_2\right) e^{18t} - c_1\right) e^{-16t} \\ \frac{c_1 e^{-16t}}{2} \\ \left((c_3 t + c_2) e^{18t} + c_1\right) e^{-16t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = -\left(\left(\left(t - \frac{1}{8}\right) c_3 + c_2\right) e^{18t} - c_1\right) e^{-16t}, y_2(t) = \frac{c_1 e^{-16t}}{2}, y_3(t) = \left((c_3 t + c_2) e^{18t} + c_1\right) e^{-16t}\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 67

```
dsolve([diff(y__1(t),t)=-6*y__1(t)-4*y__2(t)-8*y__3(t),diff(y__2(t),t)=-4*y__1(t)-0*y__2(t)-
```

$$y_1(t) = 2c_2 e^{-16t} + 2c_3 e^{2t} + c_1 e^{2t}$$
$$y_2(t) = c_2 e^{-16t} + c_3 e^{2t}$$
$$y_3(t) = 2c_2 e^{-16t} - \frac{5c_3 e^{2t}}{2} - c_1 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 116

```
DSolve[{y1'[t]==-6*y1[t]-4*y2[t]-8*y3[t],y2'[t]==-4*y1[t]-0*y2[t]-4*y3[t],y1'[t]==-8*y1[t]-4
```

$$\text{y1}(t) \to \frac{e^{-16t}(c_1(8 - 4913e^{18t}) + 2c_2(4913e^{18t} + 1))}{44217}$$
$$\text{y2}(t) \to \frac{e^{-16t}(4c_1(4913e^{18t} + 1) + c_2(1 - 39304e^{18t}))}{44217}$$
$$\text{y3}(t) \to \frac{e^{-16t}(c_1(8 - 4913e^{18t}) + 2c_2(4913e^{18t} + 1))}{44217}$$

21.10 problem section 10.4, problem 10

Internal problem ID [1598]
Internal file name [`OUTPUT/1599_Sunday_June_05_2022_02_24_08_AM_41318160/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 3y_1(t) + 5y_2(t) + 8y_3(t) \\
y_2'(t) &= y_1(t) - y_2(t) - 2y_3(t) \\
y_3'(t) &= -y_1(t) - y_2(t) - y_3(t)
\end{aligned}
$$

21.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{12} & \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-13)\mathrm{e}^{-2t}}{12} & \frac{(7\,\mathrm{e}^{4t}-4\,\mathrm{e}^{3t}-3)\mathrm{e}^{-2t}}{2} \\ \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+1)\mathrm{e}^{-2t}}{12} & \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+13)\mathrm{e}^{-2t}}{12} & \frac{(5\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}+3)\mathrm{e}^{-2t}}{2} \\ -\mathrm{e}^{2t}+\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & 3\,\mathrm{e}^{t}-2\,\mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{12} & \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-13)\mathrm{e}^{-2t}}{12} & \frac{(7\,\mathrm{e}^{4t}-4\,\mathrm{e}^{3t}-3)\mathrm{e}^{-2t}}{2} \\ \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+1)\mathrm{e}^{-2t}}{12} & \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+13)\mathrm{e}^{-2t}}{12} & \frac{(5\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}+3)\mathrm{e}^{-2t}}{2} \\ -\mathrm{e}^{2t}+\mathrm{e}^{t} & -\mathrm{e}^{2t}+\mathrm{e}^{t} & 3\,\mathrm{e}^{t}-2\,\mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_1}{12} + \frac{(21\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}-13)\mathrm{e}^{-2t}c_2}{12} + \frac{(7\,\mathrm{e}^{4t}-4\,\mathrm{e}^{3t}-3)\mathrm{e}^{-2t}c_3}{2} \\ \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+1)\mathrm{e}^{-2t}c_1}{12} + \frac{(15\,\mathrm{e}^{4t}-16\,\mathrm{e}^{3t}+13)\mathrm{e}^{-2t}c_2}{12} + \frac{(5\,\mathrm{e}^{4t}-8\,\mathrm{e}^{3t}+3)\mathrm{e}^{-2t}c_3}{2} \\ (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_1 + (-\mathrm{e}^{2t}+\mathrm{e}^{t})\,c_2 + (3\,\mathrm{e}^{t}-2\,\mathrm{e}^{2t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2\left((c_1+c_2+3c_3)\mathrm{e}^{3t}+\left(-\frac{21c_1}{8}-\frac{21c_2}{8}-\frac{21c_3}{4}\right)\mathrm{e}^{4t}+\frac{c_1}{8}+\frac{13c_2}{8}+\frac{9c_3}{4}\right)\mathrm{e}^{-2t}}{3} \\ -\frac{4\left((c_1+c_2+3c_3)\mathrm{e}^{3t}+\left(-\frac{15c_1}{16}-\frac{15c_2}{16}-\frac{15c_3}{8}\right)\mathrm{e}^{4t}-\frac{c_1}{16}-\frac{13c_2}{16}-\frac{9c_3}{8}\right)\mathrm{e}^{-2t}}{3} \\ (-c_1-c_2-2c_3)\,\mathrm{e}^{2t}+\mathrm{e}^{t}(c_1+c_2+3c_3) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 5 & 8 \\ 1 & -1-\lambda & -2 \\ -1 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= 1 \\ \lambda_3 &= 2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 8 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 5 & 8 & 0 \\ 1 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 5 & 8 & 0 \\ 0 & 0 & -\frac{18}{5} & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 5 & 8 & 0 \\ 0 & 0 & -\frac{18}{5} & 0 \\ 0 & 0 & \frac{13}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{13R_2}{18} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 5 & 8 & 0 \\ 0 & 0 & -\frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 5 & 8 \\ 0 & 0 & -\frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 8 \\ 1 & -2 & -2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 5 & 8 & 0 \\ 1 & -2 & -2 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 2 & 5 & 8 & 0 \\ 0 & -\frac{9}{2} & -6 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 2 & 5 & 8 & 0 \\ 0 & -\frac{9}{2} & -6 & 0 \\ 0 & \frac{3}{2} & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 5 & 8 & 0 \\ 0 & -\frac{9}{2} & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 5 & 8 \\ 0 & -\frac{9}{2} & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}, v_2 = -\frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 8 \\ 1 & -3 & -2 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 1 & -3 & -2 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -8 & -10 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -8 & -10 & 0 \\ 0 & 4 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -8 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 5 & 8 \\ 0 & -8 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{4}, v_2 = -\frac{5t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{4} \\ -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{4} \\ -\frac{5t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{4} \\ -\frac{5t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{4} \\ -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{4} \\ -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{t}}{3} \\ -\frac{4\,\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{7\,\mathrm{e}^{2t}}{4} \\ -\frac{5\,\mathrm{e}^{2t}}{4} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{(21c_3\mathrm{e}^{4t}+8c_2\mathrm{e}^{3t}+12c_1)\mathrm{e}^{-2t}}{12} \\ -\frac{(15c_3\mathrm{e}^{4t}+16c_2\mathrm{e}^{3t}-12c_1)\mathrm{e}^{-2t}}{12} \\ c_2\mathrm{e}^{t} + c_3\mathrm{e}^{2t} \end{bmatrix}
$$

21.10.3 Maple step by step solution

Let's solve

$[y_1'(t) = 3y_1(t) + 5y_2(t) + 8y_3(t)\,, y_2'(t) = y_1(t) - y_2(t) - 2y_3(t)\,, y_3'(t) = -y_1(t) - y_2(t) - y_3(t)]$

- Define vector

$$
\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{1} = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{2} = e^{t} \cdot \left[\begin{array}{c} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{3} = e^{2t} \cdot \left[\begin{array}{c} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1} + c_2 \underline{y}^{\rightarrow}{}_{2} + c_3 \underline{y}^{\rightarrow}{}_{3}$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 e^{-2t} \cdot \left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right] + c_2 e^{t} \cdot \left[\begin{array}{c} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{array}\right] + c_3 e^{2t} \cdot \left[\begin{array}{c} -\frac{7}{4} \\ -\frac{5}{4} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right] = \left[\begin{array}{c} -\frac{(21c_3 e^{4t} + 8c_2 e^{3t} + 12c_1) e^{-2t}}{12} \\ -\frac{(15c_3 e^{4t} + 16c_2 e^{3t} - 12c_1) e^{-2t}}{12} \\ c_2 e^{t} + c_3 e^{2t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(t) = -\frac{(21c_3 e^{4t} + 8c_2 e^{3t} + 12c_1) e^{-2t}}{12}, y_2(t) = -\frac{(15c_3 e^{4t} + 16c_2 e^{3t} - 12c_1) e^{-2t}}{12}, y_3(t) = c_2 e^{t} + c_3 e^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve([diff(y__1(t),t)=3*y__1(t)+5*y__2(t)+8*y__3(t),diff(y__2(t),t)=1*y__1(t)-1*y__2(t)-2*
```

$$y_1(t) = c_1 e^t + c_2 e^{-2t} + c_3 e^{2t}$$
$$y_2(t) = 2c_1 e^t - c_2 e^{-2t} + \frac{5c_3 e^{2t}}{7}$$
$$y_3(t) = -\frac{3c_1 e^t}{2} - \frac{4c_3 e^{2t}}{7}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 193

```
DSolve[{y1'[t]==3*y1[t]+5*y2[t]+8*y3[t],y2'[t]==1*y1[t]-1*y2[t]-2*y3[t],y1'[t]==-1*y1[t]-1*y
```

$$\text{y1}(t) \to \frac{e^{-t/9}\left(\sqrt{35}(2c_2 - 121c_1)\sin\left(\frac{\sqrt{35}t}{9}\right) - 7(74c_1 + 53c_2)\cos\left(\frac{\sqrt{35}t}{9}\right)\right)}{1575}$$
$$\text{y2}(t) \to \frac{e^{-t/9}\left(7(901c_1 + 202c_2)\cos\left(\frac{\sqrt{35}t}{9}\right) - \sqrt{35}(34c_1 + 379c_2)\sin\left(\frac{\sqrt{35}t}{9}\right)\right)}{4725}$$
$$\text{y3}(t) \to \frac{e^{-t/9}\left(2\sqrt{35}(92c_1 + 125c_2)\sin\left(\frac{\sqrt{35}t}{9}\right) - 14(251c_1 + 32c_2)\cos\left(\frac{\sqrt{35}t}{9}\right)\right)}{4725}$$

21.11 problem section 10.4, problem 11

Internal problem ID [1599]
Internal file name [`OUTPUT/1600_Sunday_June_05_2022_02_24_10_AM_9599456/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= y_1(t) - y_2(t) + 2y_3(t) \\
y_2'(t) &= 12y_1(t) - 4y_2(t) + 10y_3(t) \\
y_3'(t) &= -6y_1(t) + y_2(t) - 7y_3(t)
\end{aligned}
$$

21.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 3\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-5t} & -\mathrm{e}^{-2t} + \mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t} + 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-5t} \\ 6\,\mathrm{e}^{-3t} - 6\,\mathrm{e}^{-5t} & -\mathrm{e}^{-2t} + 2\,\mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t} + 8\,\mathrm{e}^{-3t} - 6\,\mathrm{e}^{-5t} \\ -3\,\mathrm{e}^{-3t} + 3\,\mathrm{e}^{-5t} & \mathrm{e}^{-2t} - \mathrm{e}^{-3t} & 3\,\mathrm{e}^{-5t} + 2\,\mathrm{e}^{-2t} - 4\,\mathrm{e}^{-3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 3\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-5t} & -\mathrm{e}^{-2t}+\mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t}+4\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-5t} \\ 6\,\mathrm{e}^{-3t}-6\,\mathrm{e}^{-5t} & -\mathrm{e}^{-2t}+2\,\mathrm{e}^{-3t} & -2\,\mathrm{e}^{-2t}+8\,\mathrm{e}^{-3t}-6\,\mathrm{e}^{-5t} \\ -3\,\mathrm{e}^{-3t}+3\,\mathrm{e}^{-5t} & \mathrm{e}^{-2t}-\mathrm{e}^{-3t} & 3\,\mathrm{e}^{-5t}+2\,\mathrm{e}^{-2t}-4\,\mathrm{e}^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (3\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-5t})\,c_1+(-\mathrm{e}^{-2t}+\mathrm{e}^{-3t})\,c_2+(-2\,\mathrm{e}^{-2t}+4\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-5t})\,c_3 \\ (6\,\mathrm{e}^{-3t}-6\,\mathrm{e}^{-5t})\,c_1+(-\mathrm{e}^{-2t}+2\,\mathrm{e}^{-3t})\,c_2+(-2\,\mathrm{e}^{-2t}+8\,\mathrm{e}^{-3t}-6\,\mathrm{e}^{-5t})\,c_3 \\ (-3\,\mathrm{e}^{-3t}+3\,\mathrm{e}^{-5t})\,c_1+(\mathrm{e}^{-2t}-\mathrm{e}^{-3t})\,c_2+(3\,\mathrm{e}^{-5t}+2\,\mathrm{e}^{-2t}-4\,\mathrm{e}^{-3t})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (3c_1+c_2+4c_3)\,\mathrm{e}^{-3t}+(-2c_1-2c_3)\,\mathrm{e}^{-5t}-\mathrm{e}^{-2t}(c_2+2c_3) \\ (6c_1+2c_2+8c_3)\,\mathrm{e}^{-3t}+(-6c_1-6c_3)\,\mathrm{e}^{-5t}-\mathrm{e}^{-2t}(c_2+2c_3) \\ (-3c_1-c_2-4c_3)\,\mathrm{e}^{-3t}+(3c_1+3c_3)\,\mathrm{e}^{-5t}+\mathrm{e}^{-2t}(c_2+2c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 2 \\ 12 & -4-\lambda & 10 \\ -6 & 1 & -7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 10\lambda^2 + 31\lambda + 30 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-3	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 & 2 \\ 12 & 1 & 10 \\ -6 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & -1 & 2 & 0 \\ 12 & 1 & 10 & 0 \\ -6 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & -1 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ -6 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & -1 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -1 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}, v_2 = -2t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 2 \\ 12 & -1 & 10 \\ -6 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 12 & -1 & 10 & 0 \\ -6 & 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ -6 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -\frac{1}{2} & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ 12 & -2 & 10 \\ -6 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 12 & -2 & 10 & 0 \\ -6 & 1 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -6 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{-5t}$$
$$= \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix} e^{-5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ -2\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-5t}}{3} \\ -2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-3t} - c_2\mathrm{e}^{-2t} - \frac{2c_3\mathrm{e}^{-5t}}{3} \\ -2c_1\mathrm{e}^{-3t} - c_2\mathrm{e}^{-2t} - 2c_3\mathrm{e}^{-5t} \\ c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-2t} + c_3\mathrm{e}^{-5t} \end{bmatrix}$$

21.11.3 Maple step by step solution

Let's solve

$[y_1'(t) = y_1(t) - y_2(t) + 2y_3(t)\,, y_2'(t) = 12y_1(t) - 4y_2(t) + 10y_3(t)\,, y_3'(t) = -6y_1(t) + y_2(t) - 7y_3(t$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-5t} \cdot \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -3, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = e^{-3t} \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_3 = e^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2 + c_3 \underline{\vec{y}}_3$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-5t} \cdot \begin{bmatrix} -\frac{2}{3} \\ -2 \\ 1 \end{bmatrix} + e^{-3t} c_2 \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-5t}}{3} - e^{-3t}c_2 - c_3 e^{-2t} \\ -2c_1 e^{-5t} - 2\,e^{-3t}c_2 - c_3 e^{-2t} \\ c_1 e^{-5t} + e^{-3t}c_2 + c_3 e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = -\frac{2c_1 e^{-5t}}{3} - e^{-3t}c_2 - c_3 e^{-2t}, y_2(t) = -2c_1 e^{-5t} - 2\,e^{-3t}c_2 - c_3 e^{-2t}, y_3(t) = c_1 e^{-5t} + e^{-3t}c_2 +\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 73

```
dsolve([diff(y__1(t),t)=1*y__1(t)-1*y__2(t)+2*y__3(t),diff(y__2(t),t)=12*y__1(t)-4*y__2(t)+1
```

$$y_1(t) = c_1 e^{-2t} + c_2 e^{-3t} + c_3 e^{-5t}$$
$$y_2(t) = c_1 e^{-2t} + 2c_2 e^{-3t} + 3c_3 e^{-5t}$$
$$y_3(t) = -c_1 e^{-2t} - c_2 e^{-3t} - \frac{3c_3 e^{-5t}}{2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 193

```
DSolve[{y1'[t]==1*y1[t]-1*y2[t]+2*y3[t],y2'[t]==12*y1[t]-4*y2[t]+10*y3[t],y1'[t]==-6*y1[t]+1
```

$$\text{y1}(t) \to \frac{e^{-7t/6}\left(71(77c_1 - 109c_2)\cos\left(\frac{\sqrt{71}t}{6}\right) + \sqrt{71}(143c_2 - 2479c_1)\sin\left(\frac{\sqrt{71}t}{6}\right)\right)}{340800}$$
$$\text{y2}(t) \to \frac{e^{-7t/6}\left(71(2071c_1 - 407c_2)\cos\left(\frac{\sqrt{71}t}{6}\right) - \sqrt{71}(2717c_1 + 5411c_2)\sin\left(\frac{\sqrt{71}t}{6}\right)\right)}{852000}$$
$$\text{y3}(t) \to \frac{e^{-7t/6}\left(639(23c_1 + 9c_2)\cos\left(\frac{\sqrt{71}t}{6}\right) + 3\sqrt{71}(937c_1 - 329c_2)\sin\left(\frac{\sqrt{71}t}{6}\right)\right)}{568000}$$

21.12 problem section 10.4, problem 12

Internal problem ID [1600]
Internal file name [`OUTPUT/1601_Sunday_June_05_2022_02_24_12_AM_85778873/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 4y_1(t) - y_2(t) - 4y_3(t) \\
y_2'(t) &= 4y_1(t) - 3y_2(t) - 2y_3(t) \\
y_3'(t) &= y_1(t) - y_2(t) - y_3(t)
\end{aligned}
$$

21.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(11\,e^{5t}+5\,e^{t}-6)e^{-2t}}{10} & -e^{-t}+e^{-2t} & -\frac{(11\,e^{5t}-15\,e^{t}+4)e^{-2t}}{10} \\ \frac{(7\,e^{5t}+5\,e^{t}-12)e^{-2t}}{10} & -e^{-t}+2\,e^{-2t} & -\frac{(7\,e^{5t}-15\,e^{t}+8)e^{-2t}}{10} \\ \frac{(e^{5t}+5\,e^{t}-6)e^{-2t}}{10} & -e^{-t}+e^{-2t} & -\frac{(e^{5t}-15\,e^{t}+4)e^{-2t}}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(11\,e^{5t}+5\,e^{t}-6)e^{-2t}}{10} & -e^{-t}+e^{-2t} & -\frac{(11\,e^{5t}-15\,e^{t}+4)e^{-2t}}{10} \\ \frac{(7\,e^{5t}+5\,e^{t}-12)e^{-2t}}{10} & -e^{-t}+2\,e^{-2t} & -\frac{(7\,e^{5t}-15\,e^{t}+8)e^{-2t}}{10} \\ \frac{(e^{5t}+5\,e^{t}-6)e^{-2t}}{10} & -e^{-t}+e^{-2t} & -\frac{(e^{5t}-15\,e^{t}+4)e^{-2t}}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(11\,e^{5t}+5\,e^{t}-6)e^{-2t}c_1}{10} + \left(-e^{-t}+e^{-2t}\right)c_2 - \frac{(11\,e^{5t}-15\,e^{t}+4)e^{-2t}c_3}{10} \\ \frac{(7\,e^{5t}+5\,e^{t}-12)e^{-2t}c_1}{10} + \left(-e^{-t}+2\,e^{-2t}\right)c_2 - \frac{(7\,e^{5t}-15\,e^{t}+8)e^{-2t}c_3}{10} \\ \frac{(e^{5t}+5\,e^{t}-6)e^{-2t}c_1}{10} + \left(-e^{-t}+e^{-2t}\right)c_2 - \frac{(e^{5t}-15\,e^{t}+4)e^{-2t}c_3}{10} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(\frac{11(c_1-c_3)e^{5t}}{5}+(c_1-2c_2+3c_3)e^{t}-\frac{6c_1}{5}+2c_2-\frac{4c_3}{5}\right)e^{-2t}}{2} \\ \frac{\left(\frac{7(c_1-c_3)e^{5t}}{5}+(c_1-2c_2+3c_3)e^{t}-\frac{12c_1}{5}+4c_2-\frac{8c_3}{5}\right)e^{-2t}}{2} \\ \frac{\left(\frac{(c_1-c_3)e^{5t}}{5}+(c_1-2c_2+3c_3)e^{t}-\frac{6c_1}{5}+2c_2-\frac{4c_3}{5}\right)e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -1 & -4 \\ 4 & -3-\lambda & -2 \\ 1 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 7\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 & -4 \\ 4 & -1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & -1 & -4 & 0 \\ 4 & -1 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \implies \left[\begin{array}{ccc|c} 6 & -1 & -4 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{6} \implies \left[\begin{array}{ccc|c} 6 & -1 & -4 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{5}{6} & \frac{5}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_2}{2} \implies \left[\begin{array}{ccc|c} 6 & -1 & -4 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & -1 & -4 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & -4 \\ 4 & -2 & -2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -1 & -4 & 0 \\ 4 & -2 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & -4 & 0 \\ 0 & -\frac{6}{5} & \frac{6}{5} & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & -4 & 0 \\ 0 & -\frac{6}{5} & \frac{6}{5} & 0 \\ 0 & -\frac{4}{5} & \frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & -4 & 0 \\ 0 & -\frac{6}{5} & \frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -1 & -4 \\ 0 & -\frac{6}{5} & \frac{6}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -4 \\ 4 & -6 & -2 \\ 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 4 & -6 & -2 & 0 \\ 1 & -1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & -2 & 14 & 0 \\ 1 & -1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & -2 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & -4 \\ 0 & -2 & 14 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 11t, v_2 = 7t\}$

Hence the solution is

$$\begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix} = \begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix} = t \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 11t \\ 7t \\ t \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\ &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t} \\ &= \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-2t} \\ 2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} 11\,\mathrm{e}^{3t} \\ 7\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (11c_3\mathrm{e}^{5t} + c_1\mathrm{e}^{t} + c_2)\,\mathrm{e}^{-2t} \\ (7c_3\mathrm{e}^{5t} + c_1\mathrm{e}^{t} + 2c_2)\,\mathrm{e}^{-2t} \\ (c_3\mathrm{e}^{5t} + c_1\mathrm{e}^{t} + c_2)\,\mathrm{e}^{-2t} \end{bmatrix}$$

21.12.3 Maple step by step solution

Let's solve

$[y_1'(t) = 4y_1(t) - y_2(t) - 4y_3(t), y_2'(t) = 4y_1(t) - 3y_2(t) - 2y_3(t), y_3'(t) = y_1(t) - y_2(t) - y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-2t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_3 = e^{3t} \cdot \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2 + c_3 \underline{\vec{y}}_3$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{3t} \cdot \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (11c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t} \\ (7c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + 2c_1) \mathrm{e}^{-2t} \\ (c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = (11c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t}, y_2(t) = (7c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + 2c_1) \mathrm{e}^{-2t}, y_3(t) = (c_3 \mathrm{e}^{5t} + c_2 \mathrm{e}^{t} + c_1) \mathrm{e}^{-2t}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 71

```
dsolve([diff(y__1(t),t)=4*y__1(t)-1*y__2(t)-4*y__3(t),diff(y__2(t),t)=4*y__1(t)-3*y__2(t)-2*
```

$$y_1(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{-2t}$$
$$y_2(t) = \frac{7c_1 \mathrm{e}^{3t}}{11} + c_2 \mathrm{e}^{-t} + 2c_3 \mathrm{e}^{-2t}$$
$$y_3(t) = \frac{c_1 \mathrm{e}^{3t}}{11} + c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 102

```
DSolve[{y1'[t]==4*y1[t]-1*y2[t]-4*y3[t],y2'[t]==4*y1[t]-3*y2[t]-2*y3[t],y1'[t]==1*y1[t]-1*y2
```

$$\mathrm{y1}(t) \to \frac{1}{216} e^{-2t} \left(c_1 \left(54e^t - 8\right) + c_2 \left(8 - 27e^t\right)\right)$$
$$\mathrm{y2}(t) \to \frac{1}{216} e^{-2t} \left(2c_1 \left(27e^t - 8\right) + c_2 \left(16 - 27e^t\right)\right)$$
$$\mathrm{y3}(t) \to \frac{1}{216} e^{-2t} \left(c_1 \left(54e^t - 8\right) + c_2 \left(8 - 27e^t\right)\right)$$

21.13 problem section 10.4, problem 13

Internal problem ID [1601]
Internal file name [`OUTPUT/1602_Sunday_June_05_2022_02_24_15_AM_69405403/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -2y_1(t) + 2y_2(t) - 6y_3(t) \\
y_2'(t) &= 2y_1(t) + 6y_2(t) + 2y_3(t) \\
y_3'(t) &= -2y_1(t) - 2y_2(t) + 2y_3(t)
\end{aligned}
$$

21.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{10t}+4)e^{-4t}}{5} & e^{6t} - e^{4t} & \frac{(e^{10t}-5e^{8t}+4)e^{-4t}}{5} \\ \frac{(e^{10t}-1)e^{-4t}}{5} & e^{6t} & \frac{(e^{10t}-1)e^{-4t}}{5} \\ -\frac{(e^{10t}-1)e^{-4t}}{5} & -e^{6t} + e^{4t} & -\frac{(e^{10t}-5e^{8t}-1)e^{-4t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(e^{10t}+4)e^{-4t}}{5} & e^{6t} - e^{4t} & \frac{(e^{10t}-5e^{8t}+4)e^{-4t}}{5} \\ \frac{(e^{10t}-1)e^{-4t}}{5} & e^{6t} & \frac{(e^{10t}-1)e^{-4t}}{5} \\ -\frac{(e^{10t}-1)e^{-4t}}{5} & -e^{6t} + e^{4t} & -\frac{(e^{10t}-5e^{8t}-1)e^{-4t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(e^{10t}+4)e^{-4t}c_1}{5} + (e^{6t} - e^{4t})\, c_2 + \frac{(e^{10t}-5e^{8t}+4)e^{-4t}c_3}{5} \\ \frac{(e^{10t}-1)e^{-4t}c_1}{5} + e^{6t}c_2 + \frac{(e^{10t}-1)e^{-4t}c_3}{5} \\ -\frac{(e^{10t}-1)e^{-4t}c_1}{5} + (-e^{6t} + e^{4t})\, c_2 - \frac{(e^{10t}-5e^{8t}-1)e^{-4t}c_3}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{((c_1+5c_2+c_3)e^{10t}+(-5c_2-5c_3)e^{8t}+4c_1+4c_3)e^{-4t}}{5} \\ \frac{((c_1+5c_2+c_3)e^{10t}-c_1-c_3)e^{-4t}}{5} \\ -\frac{((c_1+5c_2+c_3)e^{10t}+(-5c_2-5c_3)e^{8t}-c_1-c_3)e^{-4t}}{5} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-2 & 2 & -6\\ 2 & 6 & 2\\ -2 & -2 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-2-\lambda & 2 & -6\\ 2 & 6-\lambda & 2\\ -2 & -2 & 2-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 - 16\lambda + 96 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= -4\\ \lambda_2 &= 4\\ \lambda_3 &= 6\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
4	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & 2 & -6\\ 2 & 6 & 2\\ -2 & -2 & 2\end{bmatrix} - (-4)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2 & 2 & -6\\ 2 & 10 & 2\\ -2 & -2 & 6\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & -6 & 0 \\ 2 & 10 & 2 & 0 \\ -2 & -2 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -6 & 0 \\ 0 & 8 & 8 & 0 \\ -2 & -2 & 6 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -6 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & -6 \\ 0 & 8 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 4t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 4t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 4t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2 & -6 \\ 2 & 2 & 2 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -6 & 2 & -6 & 0 \\ 2 & 2 & 2 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} -6 & 2 & -6 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} -6 & 2 & -6 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} -6 & 2 & -6 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 2 & -6 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & -6 \\ 2 & 0 & 2 \\ -2 & -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & 2 & -6 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & -2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \implies \left[\begin{array}{ccc|c} -8 & 2 & -6 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & -2 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \implies \left[\begin{array}{ccc|c} -8 & 2 & -6 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_2 \implies \left[\begin{array}{ccc|c} -8 & 2 & -6 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & 2 & -6 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-4t} \\ &= \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} e^{-4t}\end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{6t} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{6t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 4\,\mathrm{e}^{-4t} \\ -\mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{4t} \\ 0 \\ \mathrm{e}^{4t} \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{6t} \\ -\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (-c_3\mathrm{e}^{10t} - c_2\mathrm{e}^{8t} + 4c_1)\,\mathrm{e}^{-4t} \\ -(c_3\mathrm{e}^{10t} + c_1)\,\mathrm{e}^{-4t} \\ (c_3\mathrm{e}^{10t} + c_2\mathrm{e}^{8t} + c_1)\,\mathrm{e}^{-4t} \end{bmatrix}$$

21.13.3 Maple step by step solution

Let's solve

$[y_1'(t) = -2y_1(t) + 2y_2(t) - 6y_3(t), y_2'(t) = 2y_1(t) + 6y_2(t) + 2y_3(t), y_3'(t) = -2y_1(t) - 2y_2(t) + 2y$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c}4 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{1} = e^{-4t} \cdot \left[\begin{array}{c}4 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c}-1 \\ 0 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{2} = e^{4t} \cdot \left[\begin{array}{c}-1 \\ 0 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[6, \left[\begin{array}{c}-1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{3} = e^{6t} \cdot \left[\begin{array}{c}-1 \\ -1 \\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2} + c_3 \underline{\vec{y}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{y}} = c_1 e^{-4t} \cdot \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{6t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (-c_3 e^{10t} - c_2 e^{8t} + 4c_1) e^{-4t} \\ -(c_3 e^{10t} + c_1) e^{-4t} \\ (c_3 e^{10t} + c_2 e^{8t} + c_1) e^{-4t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = (-c_3 e^{10t} - c_2 e^{8t} + 4c_1) e^{-4t}, y_2(t) = -(c_3 e^{10t} + c_1) e^{-4t}, y_3(t) = (c_3 e^{10t} + c_2 e^{8t} + c_1) e^{-4t}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 66

```
dsolve([diff(y__1(t),t)=-2*y__1(t)+2*y__2(t)-6*y__3(t),diff(y__2(t),t)=2*y__1(t)+6*y__2(t)+2
```

$$y_1(t) = c_1 e^{4t} + c_2 e^{-4t} + c_3 e^{6t}$$
$$y_2(t) = -\frac{c_2 e^{-4t}}{4} + c_3 e^{6t}$$
$$y_3(t) = -c_1 e^{4t} + \frac{c_2 e^{-4t}}{4} - c_3 e^{6t}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 257

```
DSolve[{y1'[t]==-2*y1[t]+2*y2[t]-6*y3[t],y2'[t]==2*y1[t]+6*y2[t]+2*y3[t],y1'[t]==-2*y1[t]-2*
```

$$y1(t) \to -\frac{e^{-\frac{1}{2}(\sqrt{73}-5)t}\left(2c_1\left(\left(841\sqrt{73}-7227\right)e^{\sqrt{73}t}-7227-841\sqrt{73}\right)+c_2\left(\left(171\sqrt{73}-1825\right)e^{\sqrt{73}t}-1825-\right.\right.}{598016}$$

$$y2(t) \to \frac{e^{-\frac{1}{2}(\sqrt{73}-5)t}\left(c_1\left(\left(342\sqrt{73}-3650\right)e^{\sqrt{73}t}-3650-342\sqrt{73}\right)-c_2\left(\left(1971+143\sqrt{73}\right)e^{\sqrt{73}t}+1971-143\right.\right.}{598016}$$

$$y3(t) \to \frac{e^{-\frac{1}{2}(\sqrt{73}-5)t}\left(c_1\left(\left(342\sqrt{73}-3650\right)e^{\sqrt{73}t}-3650-342\sqrt{73}\right)-c_2\left(\left(1971+143\sqrt{73}\right)e^{\sqrt{73}t}+1971-143\right.\right.}{1196032}$$

21.14 problem section 10.4, problem 14

Internal problem ID [1602]
Internal file name [`OUTPUT/1603_Sunday_June_05_2022_02_24_17_AM_90430065/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 3y_1(t) + 2y_2(t) - 2y_3(t) \\
y_2'(t) &= -2y_1(t) + 7y_2(t) - 2y_3(t) \\
y_3'(t) &= -10y_1(t) + 10y_2(t) - 5y_3(t)
\end{aligned}
$$

21.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4\,e^{5t}}{5} + \frac{e^{-5t}}{5} & \frac{e^{5t}}{5} - \frac{e^{-5t}}{5} & -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} \\ -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} & \frac{6\,e^{5t}}{5} - \frac{e^{-5t}}{5} & -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} \\ -e^{5t} + e^{-5t} & e^{5t} - e^{-5t} & e^{-5t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{4\,e^{5t}}{5} + \frac{e^{-5t}}{5} & \frac{e^{5t}}{5} - \frac{e^{-5t}}{5} & -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} \\ -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} & \frac{6\,e^{5t}}{5} - \frac{e^{-5t}}{5} & -\frac{e^{5t}}{5} + \frac{e^{-5t}}{5} \\ -e^{5t} + e^{-5t} & e^{5t} - e^{-5t} & e^{-5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{4\,e^{5t}}{5} + \frac{e^{-5t}}{5}\right) c_1 + \left(\frac{e^{5t}}{5} - \frac{e^{-5t}}{5}\right) c_2 + \left(-\frac{e^{5t}}{5} + \frac{e^{-5t}}{5}\right) c_3 \\ \left(-\frac{e^{5t}}{5} + \frac{e^{-5t}}{5}\right) c_1 + \left(\frac{6\,e^{5t}}{5} - \frac{e^{-5t}}{5}\right) c_2 + \left(-\frac{e^{5t}}{5} + \frac{e^{-5t}}{5}\right) c_3 \\ \left(-e^{5t} + e^{-5t}\right) c_1 + \left(e^{5t} - e^{-5t}\right) c_2 + e^{-5t} c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1 - c_2 + c_3)e^{-5t}}{5} + \frac{4\left(c_1 + \frac{c_2}{4} - \frac{c_3}{4}\right)e^{5t}}{5} \\ \frac{(c_1 - c_2 + c_3)e^{-5t}}{5} - \frac{e^{5t}(c_1 - 6c_2 + c_3)}{5} \\ (c_1 - c_2 + c_3)\,e^{-5t} - e^{5t}(c_1 - c_2) \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & -2 \\ -2 & 7-\lambda & -2 \\ -10 & 10 & -5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 - 25\lambda + 125 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-5	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 2 & -2 \\ -2 & 12 & -2 \\ -10 & 10 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ -2 & 12 & -2 & 0 \\ -10 & 10 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 0 & \frac{25}{2} & -\frac{5}{2} & 0 \\ -10 & 10 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 0 & \frac{25}{2} & -\frac{5}{2} & 0 \\ 0 & \frac{25}{2} & -\frac{5}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 0 & \frac{25}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & 2 & -2 \\ 0 & \frac{25}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{5}, v_2 = \frac{t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{5} \\ \frac{t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{5} \\ \frac{t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ -2 & 2 & -2 \\ -10 & 10 & -10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ -10 & 10 & -10 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 10 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 - 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t - s\}$

Hence the solution is

$$\begin{bmatrix} t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	2	No	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

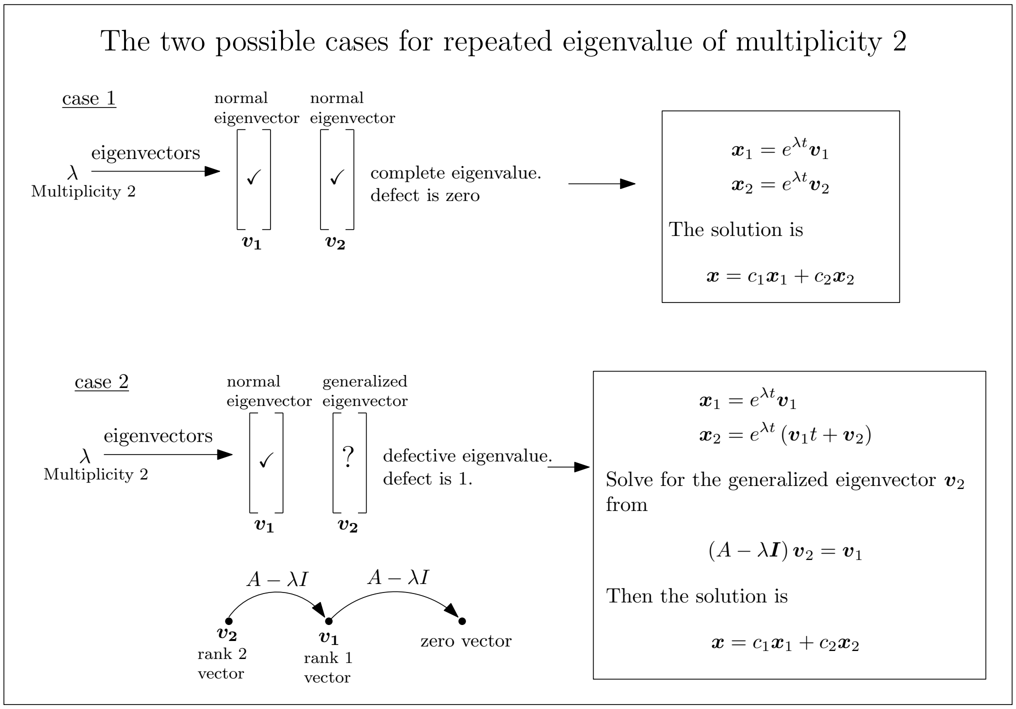


Figure 571: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{5t} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{5t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{5t}
\end{aligned}
$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{-5t}$$
$$= \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} e^{-5t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{5t} \\ 0 \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t} \\ \mathrm{e}^{5t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{-5t}}{5} \\ \frac{\mathrm{e}^{-5t}}{5} \\ \mathrm{e}^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (-c_1 + c_2)\,\mathrm{e}^{5t} + \frac{c_3\mathrm{e}^{-5t}}{5} \\ c_2\mathrm{e}^{5t} + \frac{c_3\mathrm{e}^{-5t}}{5} \\ c_1\mathrm{e}^{5t} + c_3\mathrm{e}^{-5t} \end{bmatrix}$$

21.14.3 Maple step by step solution

Let's solve

$[y_1'(t) = 3y_1(t) + 2y_2(t) - 2y_3(t)\,, y_2'(t) = -2y_1(t) + 7y_2(t) - 2y_3(t)\,, y_3'(t) = -10y_1(t) + 10y_2(t) -$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow \prime}(t) = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow \prime}(t) = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow \prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}_{1} = \mathrm{e}^{-5t} \cdot \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[5, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 5

$$\vec{y}_2(t) = e^{5t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

$$\vec{y}_3(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{y}_3(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} - 5 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 5

$$\underline{y}^{\rightarrow}{}_3(t) = \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_1 + c_2 \underline{y}^{\rightarrow}{}_2(t) + c_3 \underline{y}^{\rightarrow}{}_3(t)$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 \mathrm{e}^{-5t} \cdot \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{5t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((1-2t)c_3 - 2c_2)\mathrm{e}^{5t}}{2} + \frac{c_1 \mathrm{e}^{-5t}}{5} \\ \frac{c_1 \mathrm{e}^{-5t}}{5} \\ (c_3 t + c_2)\, \mathrm{e}^{5t} + c_1 \mathrm{e}^{-5t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ y_1(t) = \frac{((1-2t)c_3 - 2c_2)\mathrm{e}^{5t}}{2} + \frac{c_1 \mathrm{e}^{-5t}}{5}, y_2(t) = \frac{c_1 \mathrm{e}^{-5t}}{5}, y_3(t) = (c_3 t + c_2)\, \mathrm{e}^{5t} + c_1 \mathrm{e}^{-5t} \right\}$$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 57

```
dsolve([diff(y__1(t),t)=3*y__1(t)+2*y__2(t)-2*y__3(t),diff(y__2(t),t)=-2*y__1(t)+7*y__2(t)-2
```

$$y_1(t) = c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{-5t}$$
$$y_2(t) = c_2 \mathrm{e}^{5t} + c_3 \mathrm{e}^{-5t} + \mathrm{e}^{5t} c_1$$
$$y_3(t) = 5c_3 \mathrm{e}^{-5t} + \mathrm{e}^{5t} c_1$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 126

```
DSolve[{y1'[t]==3*y1[t]+2*y2[t]-2*y3[t],y2'[t]==-2*y1[t]+7*y2[t]-2*y3[t],y1'[t]==-10*y1[t]+1
```

$$y1(t) \to -\frac{e^{5t}\left(c_1\left(432e^{10t/3}-1331\right)+c_2\left(1331-216e^{10t/3}\right)\right)}{85184}$$
$$y2(t) \to -\frac{e^{5t}\left(c_1\left(216e^{10t/3}-1331\right)+c_2\left(1331-108e^{10t/3}\right)\right)}{42592}$$
$$y3(t) \to \frac{e^{5t}\left(c_1\left(720e^{10t/3}+1331\right)-c_2\left(360e^{10t/3}+1331\right)\right)}{85184}$$

21.15 problem section 10.4, problem 15

Internal problem ID [1603]
Internal file name [`OUTPUT/1604_Sunday_June_05_2022_02_24_19_AM_66657072/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.4, constant coefficient homogeneous system. Page 540
Problem number: section 10.4, problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 3y_1(t) + y_2(t) - y_3(t) \\
y_2'(t) &= 3y_1(t) + 5y_2(t) + y_3(t) \\
y_3'(t) &= -6y_1(t) + 2y_2(t) + 4y_3(t)
\end{aligned}
$$

21.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2}+\frac{e^{6t}}{2} & \frac{e^{6t}}{6}-\frac{1}{6} & \frac{1}{6}-\frac{e^{6t}}{6} \\ -\frac{1}{2}+\frac{e^{6t}}{2} & \frac{1}{6}+\frac{5e^{6t}}{6} & \frac{e^{6t}}{6}-\frac{1}{6} \\ 1-e^{6t} & -\frac{1}{3}+\frac{e^{6t}}{3} & \frac{1}{3}+\frac{2e^{6t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{1}{2}+\frac{e^{6t}}{2} & \frac{e^{6t}}{6}-\frac{1}{6} & \frac{1}{6}-\frac{e^{6t}}{6} \\ -\frac{1}{2}+\frac{e^{6t}}{2} & \frac{1}{6}+\frac{5e^{6t}}{6} & \frac{e^{6t}}{6}-\frac{1}{6} \\ 1-e^{6t} & -\frac{1}{3}+\frac{e^{6t}}{3} & \frac{1}{3}+\frac{2e^{6t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{6t}}{2}\right)c_1 + \left(\frac{e^{6t}}{6}-\frac{1}{6}\right)c_2 + \left(\frac{1}{6}-\frac{e^{6t}}{6}\right)c_3 \\ \left(-\frac{1}{2}+\frac{e^{6t}}{2}\right)c_1 + \left(\frac{1}{6}+\frac{5e^{6t}}{6}\right)c_2 + \left(\frac{e^{6t}}{6}-\frac{1}{6}\right)c_3 \\ (1-e^{6t})c_1 + \left(-\frac{1}{3}+\frac{e^{6t}}{3}\right)c_2 + \left(\frac{1}{3}+\frac{2e^{6t}}{3}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(3c_1+c_2-c_3)e^{6t}}{6} + \frac{c_1}{2} - \frac{c_2}{6} + \frac{c_3}{6} \\ \frac{(3c_1+5c_2+c_3)e^{6t}}{6} - \frac{c_1}{2} + \frac{c_2}{6} - \frac{c_3}{6} \\ \frac{(-3c_1+c_2+2c_3)e^{6t}}{3} + c_1 - \frac{c_2}{3} + \frac{c_3}{3} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

21.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 & -1 \\ 3 & 5-\lambda & 1 \\ -6 & 2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 12\lambda^2 + 36\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 3 & 5 & 1 & 0 \\ -6 & 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ -6 & 2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -1 \\ 3 & -1 & 1 \\ -6 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 3 & -1 & 1 & 0 \\ -6 & 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 2 & -2 & 0 \end{array}\right]$$

$$
R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -3 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3} - \frac{s}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix}
$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$
\begin{aligned}
\begin{bmatrix} \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} &= \begin{bmatrix} \frac{t}{3} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s}{3} \\ 0 \\ s \end{bmatrix} \\
&= t \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$
\begin{bmatrix} \frac{t}{3} - \frac{s}{3} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}
$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
6	2	2	No	$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{0}$$
$$= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{0}$$

eigenvalue 6 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

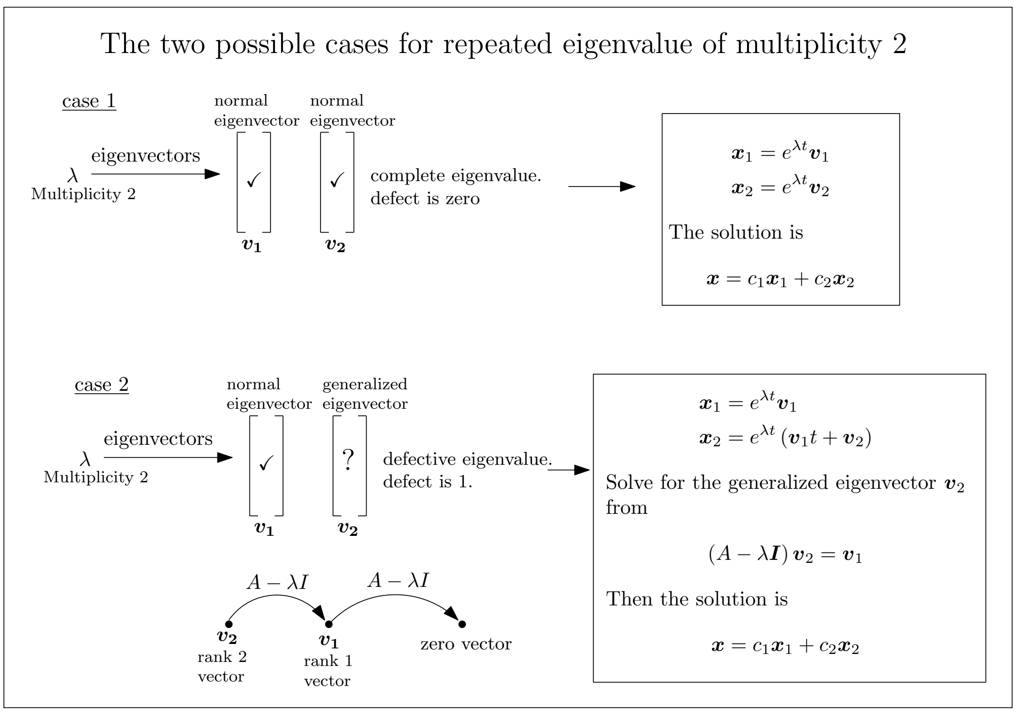


Figure 572: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\vec{x}_2(t) = \vec{v}_2 e^{6t}$$
$$= \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} e^{6t}$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{6t} \\
&= \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} e^{6t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{6t}}{3} \\ e^{6t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{e^{6t}}{3} \\ 0 \\ e^{6t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(2c_2-2c_3)e^{6t}}{6} + \frac{c_1}{2} \\ -\frac{c_1}{2} + c_2 e^{6t} \\ c_1 + c_3 e^{6t} \end{bmatrix}
$$

21.15.3 Maple step by step solution

Let's solve

$[y_1'(t) = 3y_1(t) + y_2(t) - y_3(t)\,, y_2'(t) = 3y_1(t) + 5y_2(t) + y_3(t)\,, y_3'(t) = -6y_1(t) + 2y_2(t) + 4y_3(t)]$

- Define vector

$$
\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$
\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}_{1} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[6, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 6

$$\overrightarrow{y}_2(t) = \mathrm{e}^{6t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 6$ is the eigenvalue, and

$$\overrightarrow{y}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{y}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 6

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} - 6 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} \frac{1}{9} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 6

$$\vec{y}_3(t) = e^{6t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{9} \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(t) + c_3 \vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{6t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + c_3 e^{6t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{9} \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} \frac{c_1}{2} \\ -\frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((1-3t)c_3-3c_2)e^{6t}}{9} + \frac{c_1}{2} \\ -\frac{c_1}{2} \\ (c_3 t + c_2)\, e^{6t} + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ y_1(t) = \frac{((1-3t)c_3-3c_2)e^{6t}}{9} + \frac{c_1}{2}, y_2(t) = -\frac{c_1}{2}, y_3(t) = (c_3 t + c_2)\, e^{6t} + c_1 \right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 53

```
dsolve([diff(y__1(t),t)=3*y__1(t)+1*y__2(t)-1*y__3(t),diff(y__2(t),t)=3*y__1(t)+5*y__2(t)+1*
```

$$y_1(t) = c_2 + c_3 e^{6t}$$
$$y_2(t) = -c_2 - c_3 e^{6t} + e^{6t} c_1$$
$$y_3(t) = -4c_3 e^{6t} + 2c_2 + e^{6t} c_1$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 93

```
DSolve[{y1'[t]==3*y1[t]+1*y2[t]-1*y3[t],y2'[t]==3*y1[t]+5*y2[t]+1*y3[t],y1'[t]==-6*y1[t]+2*y
```

$$y1(t) \to \frac{1}{625}\left(-\left(c_1\left(e^{6t}-500\right)\right)-c_2\left(e^{6t}+125\right)\right)$$
$$y2(t) \to \frac{1}{625}\left(c_2\left(125-4e^{6t}\right)-4c_1\left(e^{6t}+125\right)\right)$$
$$y3(t) \to \frac{1}{625}\left(-\left(c_1\left(e^{6t}-1000\right)\right)-c_2\left(e^{6t}+250\right)\right)$$

22 Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555

22.1 problem section 10.5, problem 1

Internal problem ID [1604]
Internal file name [`OUTPUT/1605_Sunday_June_05_2022_02_24_21_AM_69324230/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(t) &= 3y_1(t) + 4y_2(t) \\ y_2'(t) &= -y_1(t) + 7y_2(t) \end{aligned}$$

22.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{5t}(1-2t) & 4t\,e^{5t} \\ -t\,e^{5t} & e^{5t}(1+2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{5t}(1-2t) & 4t\,e^{5t} \\ -t\,e^{5t} & e^{5t}(1+2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(1-2t)\,c_1 + 4t\,e^{5t}c_2 \\ -t\,e^{5t}c_1 + e^{5t}(1+2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) + 4c_2t)\,e^{5t} \\ e^{5t}(-tc_1 + 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

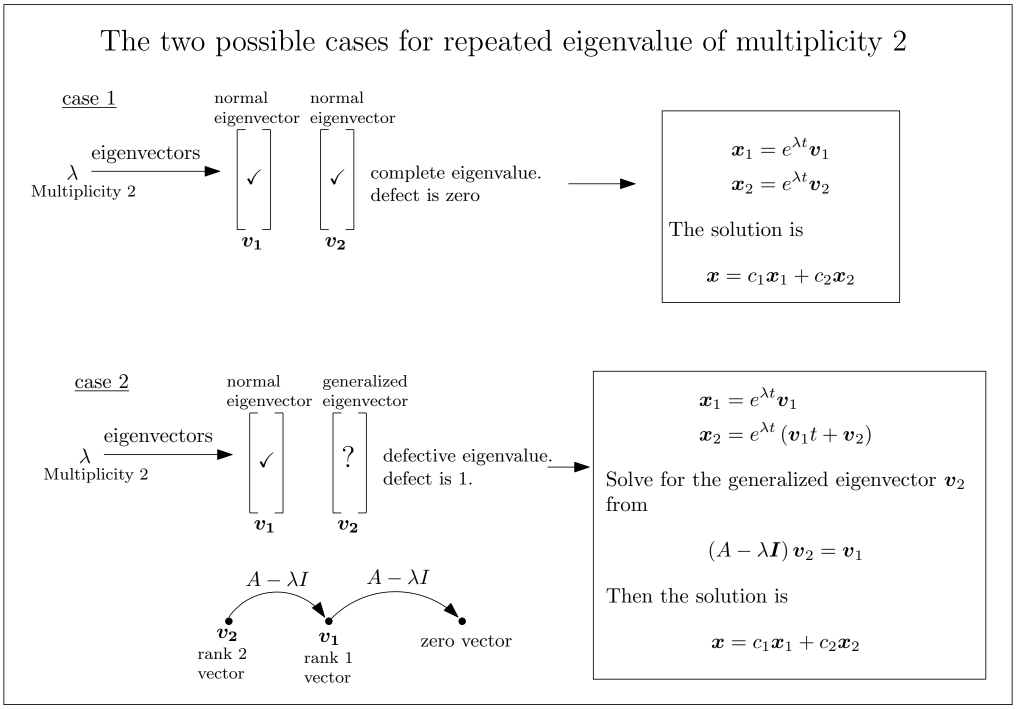


Figure 573: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{5t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{5t} \\
&= \begin{bmatrix} \mathrm{e}^{5t}(1+2t) \\ \mathrm{e}^{5t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t}(1+2t) \\ \mathrm{e}^{5t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{5t}(2c_2 t + 2c_1 + c_2) \\ \mathrm{e}^{5t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

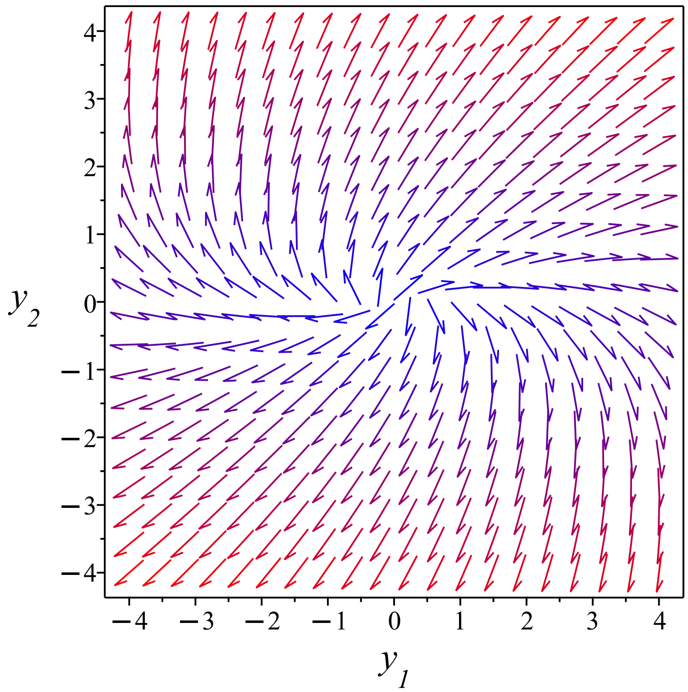


Figure 574: Phase plot

22.1.3 Maple step by step solution

Let's solve

$[y_1'(t) = 3y_1(t) + 4y_2(t), y_2'(t) = -y_1(t) + 7y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 3 & 4 \\ -1 & 7 \end{array}\right]$$

- Rewrite the system as

  $$\underline{y}^{\rightarrow \prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[5, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right], \left[5, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[5, \left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 5

  $$\underline{y}^{\rightarrow}_{1}(t) = e^{5t} \cdot \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

  $$\underline{y}^{\rightarrow}_{2}(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{y}^{\rightarrow}_{2}(t)$ into the homogeneous system

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{y}^{\rightarrow}_{2}(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\left[\begin{array}{cc} 3 & 4 \\ -1 & 7 \end{array}\right]-5\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} -1 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 5

$$\overrightarrow{y}_2(t)=\mathrm{e}^{5t}\cdot\left(t\cdot\left[\begin{array}{c} 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y}=c_1\overrightarrow{y}_1(t)+c_2\overrightarrow{y}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{y}=c_1\mathrm{e}^{5t}\cdot\left[\begin{array}{c} 2 \\ 1 \end{array}\right]+c_2\mathrm{e}^{5t}\cdot\left(t\cdot\left[\begin{array}{c} 2 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right]=\left[\begin{array}{c} \left(\left(2t-1\right)c_2+2c_1\right)\mathrm{e}^{5t} \\ \mathrm{e}^{5t}\left(c_2t+c_1\right) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\left(\left(2t-1\right)c_2+2c_1\right)\mathrm{e}^{5t}, y_2(t)=\mathrm{e}^{5t}\left(c_2t+c_1\right)\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve([diff(y__1(t),t)=3*y__1(t)+4*y__2(t),diff(y__2(t),t)=-1*y__1(t)+7*y__2(t)],singsol=al
```

$$y_1(t)=\mathrm{e}^{5t}(c_2t+c_1)$$
$$y_2(t)=\frac{\mathrm{e}^{5t}(2c_2t+2c_1+c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{y1'[t]==3*y1[t]+4*y2[t],y2'[t]==-1*y1[t]+7*y2[t]},{y1[t],y2[t]},t,IncludeSingularSol
```

$$\mathrm{y1}(t) \to e^{5t}(-2c_1t + 4c_2t + c_1)$$
$$\mathrm{y2}(t) \to e^{5t}(c_1(-t) + 2c_2t + c_2)$$

22.2 problem section 10.5, problem 2

Internal problem ID [1605]
Internal file name [`OUTPUT/1606_Sunday_June_05_2022_02_24_23_AM_23986191/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_2(t) \\
y_2'(t) &= y_1(t) - 2y_2(t)
\end{aligned}
$$

22.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(t+1) & -t\,\mathrm{e}^{-t} \\ t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(t+1) & -t\,e^{-t} \\ t\,e^{-t} & e^{-t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(t+1)\,c_1 - t\,e^{-t}c_2 \\ t\,e^{-t}c_1 + e^{-t}(1-t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(tc_1 - c_2t + c_1) \\ e^{-t}(tc_1 - c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

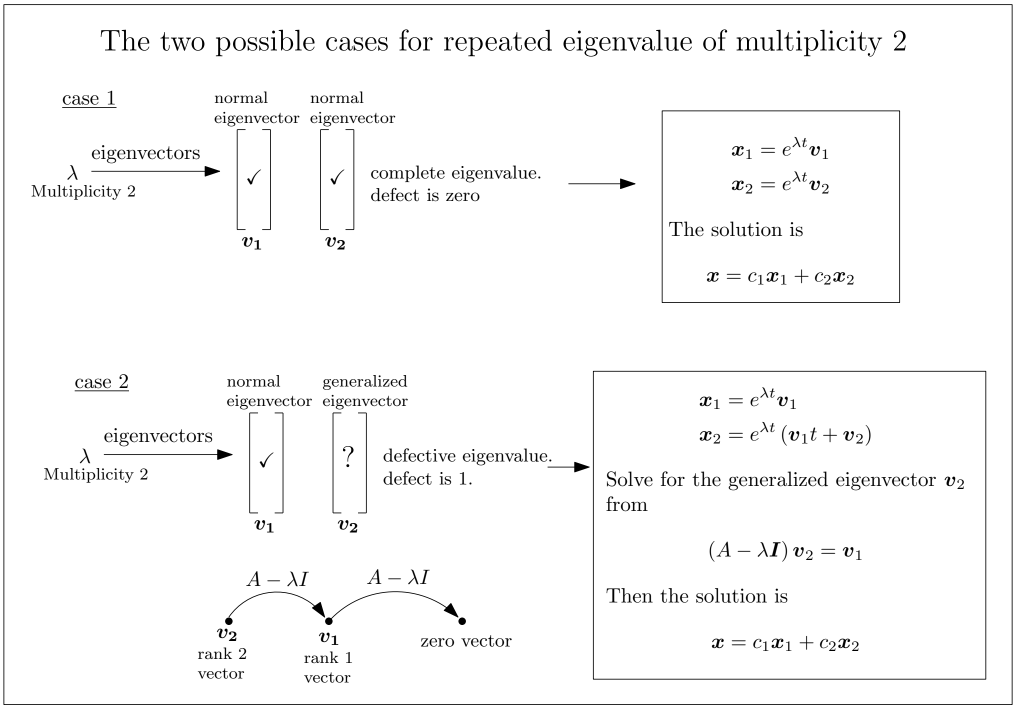


Figure 575: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(2+t) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(2+t) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ((2+t)\, c_2 + c_1)\, \mathrm{e}^{-t} \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

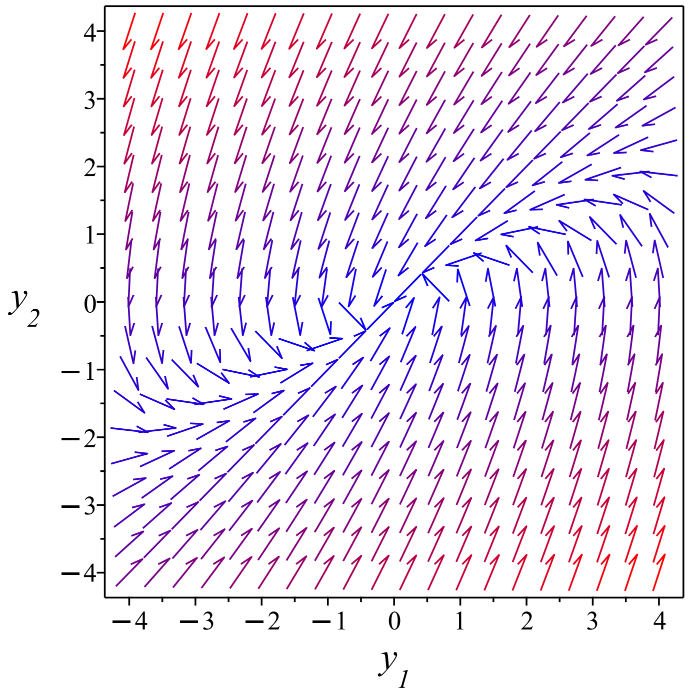


Figure 576: Phase plot

22.2.3 Maple step by step solution

Let's solve

$[y_1'(t) = -y_2(t), y_2'(t) = y_1(t) - 2y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

  $$\underline{y}^{\rightarrow \prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

  $$\underline{y}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $$\underline{y}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{y}^{\rightarrow}{}_{2}(t)$ into the homogeneous system

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{y}^{\rightarrow}{}_{2}(t)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\underline{\vec{y}}_2(t) = e^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1(t) + c_2 \underline{\vec{y}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t}(c_2 t + c_1 + c_2) \\ (c_2 t + c_1)\, e^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = e^{-t}(c_2 t + c_1 + c_2)\, , y_2(t) = (c_2 t + c_1)\, e^{-t}\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve([diff(y__1(t),t)=0*y__1(t)-1*y__2(t),diff(y__2(t),t)=1*y__1(t)-2*y__2(t)],singsol=all
```

$$y_1(t) = e^{-t}(c_2 t + c_1)$$
$$y_2(t) = e^{-t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 44

```
DSolve[{y1'[t]==0*y1[t]-1*y2[t],y2'[t]==1*y1[t]-2*y2[t]},{y1[t],y2[t]},t,IncludeSingularSolu
```

$$
\begin{aligned}
\text{y1}(t) &\rightarrow e^{-t}(c_1(t+1) - c_2 t) \\
\text{y2}(t) &\rightarrow e^{-t}((c_1 - c_2)t + c_2)
\end{aligned}
$$

22.3 problem section 10.5, problem 3

Internal problem ID [1606]
Internal file name [`OUTPUT/1607_Sunday_June_05_2022_02_24_24_AM_75763991/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$y_1'(t) = -7y_1(t) + 4y_2(t)$$
$$y_2'(t) = -y_1(t) - 11y_2(t)$$

22.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-9t}(1+2t) & 4t\,\mathrm{e}^{-9t} \\ -t\,\mathrm{e}^{-9t} & \mathrm{e}^{-9t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-9t}(1+2t) & 4t\,e^{-9t} \\ -t\,e^{-9t} & e^{-9t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-9t}(1+2t)\,c_1 + 4t\,e^{-9t}c_2 \\ -t\,e^{-9t}c_1 + e^{-9t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-9t}(2tc_1 + 4c_2t + c_1) \\ e^{-9t}(-tc_1 - 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 4 \\ -1 & -11-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 18\lambda + 81 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} - (-9)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-9	2	1	Yes	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -9 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

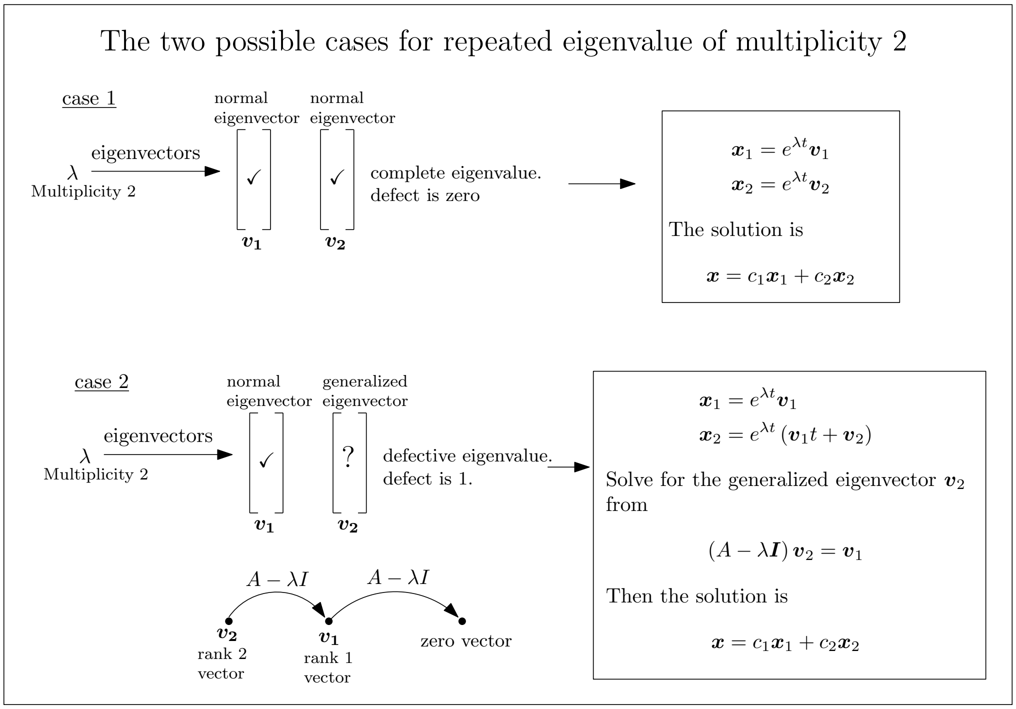


Figure 577: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} - (-9)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -9. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mathrm{e}^{-9t} \\ &= \begin{bmatrix} -2\,\mathrm{e}^{-9t} \\ \mathrm{e}^{-9t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-9t} \\ &= \begin{bmatrix} \mathrm{e}^{-9t}(-2t-3) \\ \mathrm{e}^{-9t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-9t} \\ \mathrm{e}^{-9t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-9t}(-2t-3) \\ \mathrm{e}^{-9t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-9t}(-2c_2 t - 2c_1 - 3c_2) \\ \mathrm{e}^{-9t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

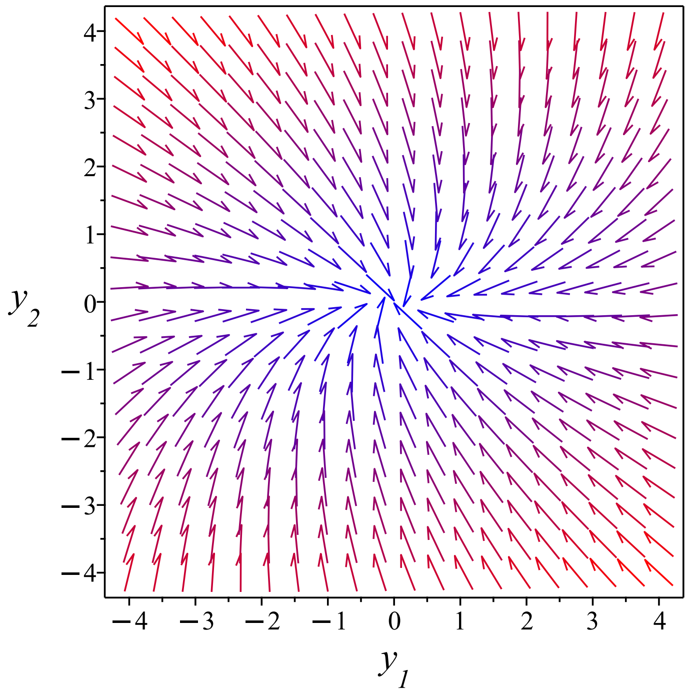


Figure 578: Phase plot

22.3.3 Maple step by step solution

Let's solve

$[y_1'(t) = -7y_1(t) + 4y_2(t), y_2'(t) = -y_1(t) - 11y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix}$$

- Rewrite the system as

  $\underline{y}^{\rightarrow \prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-9, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[-9, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-9, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -9

  $$\underline{y}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{-9t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -9$ is the eigenvalue, an

  $\underline{y}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{y}^{\rightarrow}{}_{2}(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{y}^{\rightarrow}{}_{2}(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -9

$$\left(\left[\begin{array}{cc}-7 & 4 \\ -1 & -11\end{array}\right]-(-9)\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}-2 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-1 \\ 0\end{array}\right]$$

- Second solution from eigenvalue -9

$$\overrightarrow{y}_{2}(t)=\mathrm{e}^{-9t}\cdot\left(t\cdot\left[\begin{array}{c}-2 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y}=c_1\overrightarrow{y}_{1}(t)+c_2\overrightarrow{y}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{y}=c_1\mathrm{e}^{-9t}\cdot\left[\begin{array}{c}-2 \\ 1\end{array}\right]+c_2\mathrm{e}^{-9t}\cdot\left(t\cdot\left[\begin{array}{c}-2 \\ 1\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}y_1(t) \\ y_2(t)\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{-9t}(-2c_2t-2c_1-c_2) \\ \mathrm{e}^{-9t}(c_2t+c_1)\end{array}\right]$$

- Solution to the system of ODEs

$$\{y_1(t)=\mathrm{e}^{-9t}(-2c_2t-2c_1-c_2),y_2(t)=\mathrm{e}^{-9t}(c_2t+c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(y__1(t),t)=-7*y__1(t)+4*y__2(t),diff(y__2(t),t)=-1*y__1(t)-11*y__2(t)],singsol=
```

$$y_1(t)=\mathrm{e}^{-9t}(c_2t+c_1)$$
$$y_2(t)=-\frac{\mathrm{e}^{-9t}(2c_2t+2c_1-c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{y1'[t]==-7*y1[t]+4*y2[t],y2'[t]==-1*y1[t]-11*y2[t]},{y1[t],y2[t]},t,IncludeSingularS
```

$$\mathrm{y1}(t) \rightarrow e^{-9t}(2c_1t + 4c_2t + c_1)$$
$$\mathrm{y2}(t) \rightarrow e^{-9t}(c_2 - (c_1 + 2c_2)t)$$

22.4 problem section 10.5, problem 4

Internal problem ID [1607]
Internal file name [`OUTPUT/1608_Sunday_June_05_2022_02_24_26_AM_61519766/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(t) &= 3y_1(t) + y_2(t) \\ y_2'(t) &= -y_1(t) + y_2(t) \end{aligned}$$

22.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(t+1) & e^{2t}t \\ -e^{2t}t & e^{2t}(1-t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(t+1) & e^{2t}t \\ -e^{2t}t & e^{2t}(1-t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(t+1)\, c_1 + e^{2t}tc_2 \\ -e^{2t}tc_1 + e^{2t}(1-t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(tc_1 + c_2 t + c_1) \\ -((-1+t)\, c_2 + tc_1)\, e^{2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

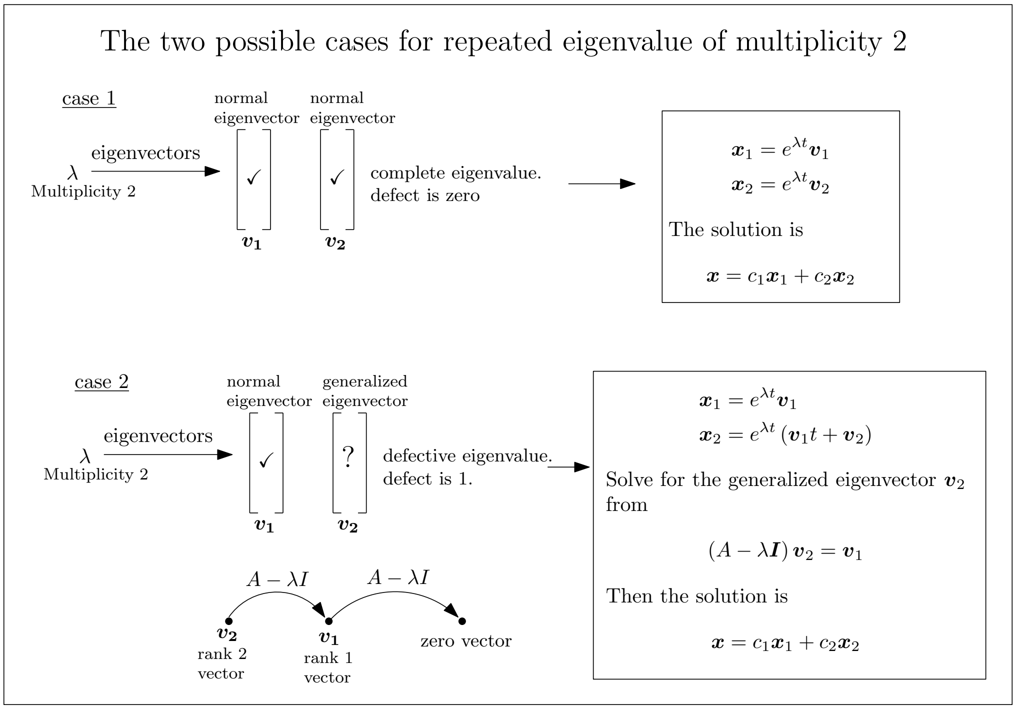


Figure 579: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} -\mathrm{e}^{2t}(2+t) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}(-t-2) \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -((2+t)\, c_2 + c_1)\, \mathrm{e}^{2t} \\ \mathrm{e}^{2t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

The following is the phase plot of the system.

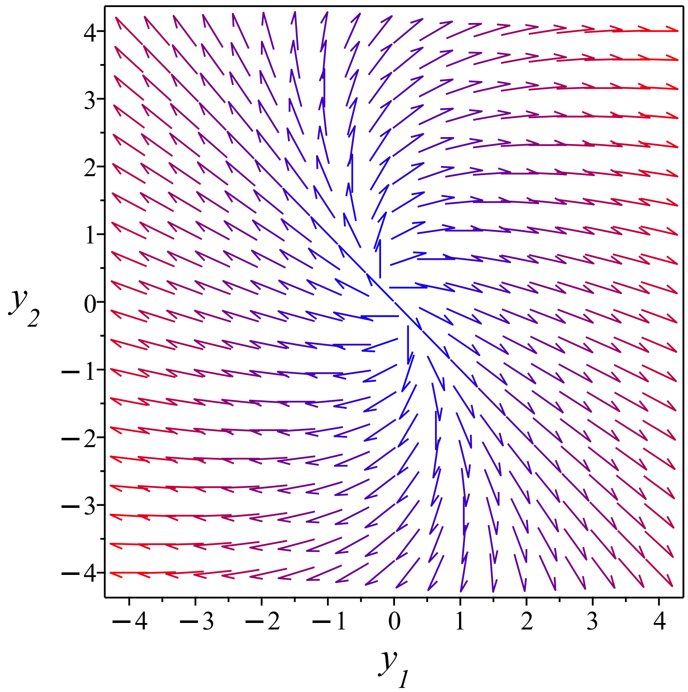


Figure 580: Phase plot

22.4.3 Maple step by step solution

Let's solve

$[y_1'(t) = 3y_1(t) + y_2(t), y_2'(t) = -y_1(t) + y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

  $$\underline{\vec{y}}_1(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and

  $\underline{\vec{y}}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{y}}_2(t)$ into the homogeneous system

  $\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{\vec{y}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\underline{\vec{y}}_2(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1(t) + c_2 \underline{\vec{y}}_2(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(-c_2 t - c_1 - c_2) \\ e^{2t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = e^{2t}(-c_2 t - c_1 - c_2), y_2(t) = e^{2t}(c_2 t + c_1)\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve([diff(y__1(t),t)=3*y__1(t)+1*y__2(t),diff(y__2(t),t)=-1*y__1(t)+1*y__2(t)],singsol=al
```

$$y_1(t) = e^{2t}(c_2 t + c_1)$$
$$y_2(t) = -e^{2t}(c_2 t + c_1 - c_2)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 42

```
DSolve[{y1'[t]==3*y1[t]+1*y2[t],y2'[t]==-1*y1[t]+1*y2[t]},{y1[t],y2[t]},t,IncludeSingularSol
```

$$\mathrm{y1}(t) \to e^{2t}(c_1(t+1)+c_2t)$$
$$\mathrm{y2}(t) \to e^{2t}(c_2-(c_1+c_2)t)$$

22.5 problem section 10.5, problem 5

Internal problem ID [1608]
Internal file name [`OUTPUT/1609_Sunday_June_05_2022_02_24_27_AM_13734712/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 4y_1(t) + 12y_2(t) \\
y_2'(t) &= -3y_1(t) - 8y_2(t)
\end{aligned}
$$

22.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-2t}(1+6t) & 12t\,\mathrm{e}^{-2t} \\ -3t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(1-6t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t}(1+6t) & 12t\,e^{-2t} \\ -3t\,e^{-2t} & e^{-2t}(1-6t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(1+6t)\,c_1 + 12t\,e^{-2t}c_2 \\ -3t\,e^{-2t}c_1 + e^{-2t}(1-6t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(6tc_1 + 12c_2t + c_1) \\ (c_2(1-6t) - 3tc_1)\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 12 \\ -3 & -8-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 12 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 12 & 0 \\ -3 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 6 & 12 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 12 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	2	1	Yes	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

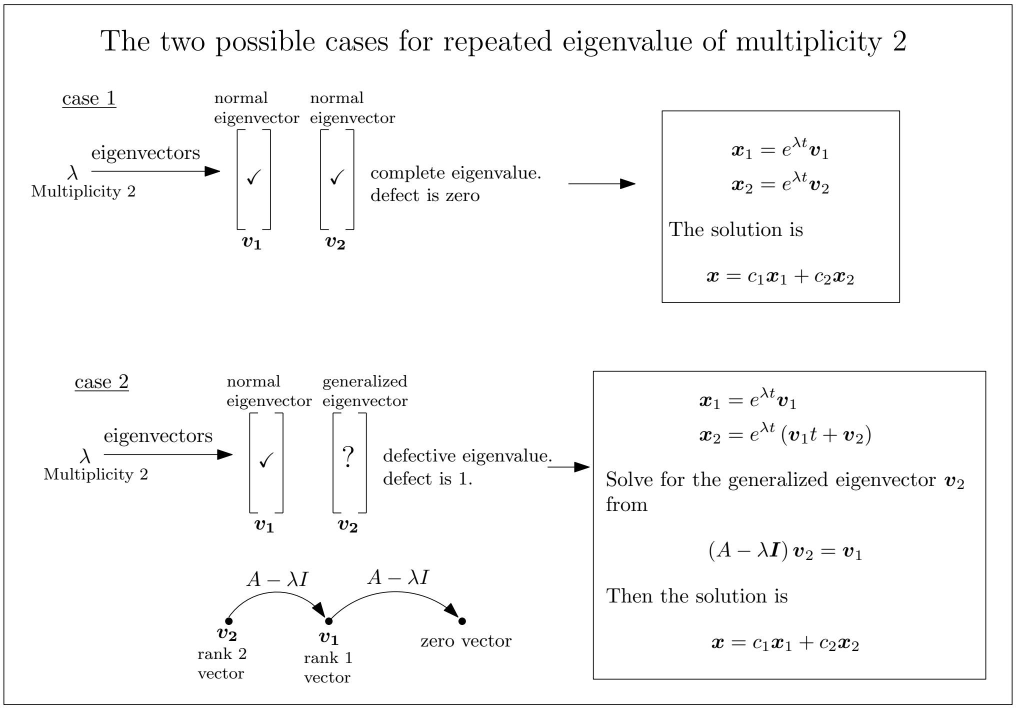


Figure 581: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$
(A-\lambda I)\,\vec{v}_2=\vec{v}_1
$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$
\begin{aligned}
\left(\begin{bmatrix}4 & 12\\ -3 & -8\end{bmatrix}-(-2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}-2\\ 1\end{bmatrix}\\
\begin{bmatrix}6 & 12\\ -3 & -6\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} &= \begin{bmatrix}-2\\ 1\end{bmatrix}
\end{aligned}
$$

Solving for $\vec{v}_2$ gives

$$
\vec{v}_2=\begin{bmatrix}-\frac{7}{3}\\ 1\end{bmatrix}
$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mathrm{e}^{-2t} \\ &= \begin{bmatrix} -2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-2t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(7+6t)}{3} \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} -2\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-2t}\left(-2t - \frac{7}{3}\right) \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-2t}\left(-2c_1 - 2c_2 t - \frac{7}{3} c_2\right) \\ \mathrm{e}^{-2t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

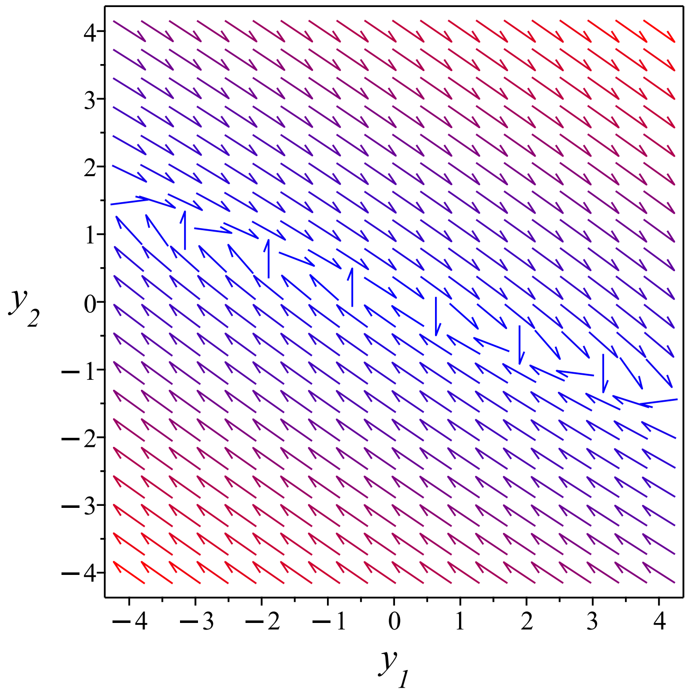


Figure 582: Phase plot

22.5.3 Maple step by step solution

Let's solve

$[y_1'(t) = 4y_1(t) + 12y_2(t), y_2'(t) = -3y_1(t) - 8y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -2

$$\underline{\vec{y}}_1(t) = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, an

$$\underline{\vec{y}}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{y}}_2(t)$ into the homogeneous system

$$\lambda \,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{\vec{y}}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

$$\left(\begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} - (-2) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -2

$$\overrightarrow{y\_\_}_2(t) = \mathrm{e}^{-2t} \cdot \left(t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y\_\_} = c_1 \overrightarrow{y\_\_}_1(t) + c_2 \overrightarrow{y\_\_}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{y\_\_} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \mathrm{e}^{-2t} c_2 \cdot \left(t \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-2t}\left(-2c_1 - 2c_2 t - \frac{1}{3}c_2\right) \\ \mathrm{e}^{-2t}(c_2 t + c_1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = \mathrm{e}^{-2t}\left(-2c_1 - 2c_2 t - \frac{1}{3}c_2\right), y_2(t) = \mathrm{e}^{-2t}(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(y__1(t),t)=4*y__1(t)+12*y__2(t),diff(y__2(t),t)=-3*y__1(t)-8*y__2(t)],singsol=a
```

$$y_1(t) = \mathrm{e}^{-2t}(c_2 t + c_1)$$
$$y_2(t) = -\frac{\mathrm{e}^{-2t}(6c_2 t + 6c_1 - c_2)}{12}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{y1'[t]==4*y1[t]+12*y2[t],y2'[t]==-3*y1[t]-8*y2[t]},{y1[t],y2[t]},t,IncludeSingularSo
```

$$\text{y1}(t) \rightarrow e^{-2t}(6c_1t + 12c_2t + c_1)$$
$$\text{y2}(t) \rightarrow e^{-2t}(c_2 - 3(c_1 + 2c_2)t)$$

22.6 problem section 10.5, problem 6

Internal problem ID [1609]
Internal file name [`OUTPUT/1610_Sunday_June_05_2022_02_24_29_AM_78869279/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -10y_1(t) + 9y_2(t) \\
y_2'(t) &= -4y_1(t) + 2y_2(t)
\end{aligned}
$$

22.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-4t}(1-6t) & 9t\,\mathrm{e}^{-4t} \\ -4t\,\mathrm{e}^{-4t} & \mathrm{e}^{-4t}(1+6t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-4t}(1-6t) & 9t\,e^{-4t} \\ -4t\,e^{-4t} & e^{-4t}(1+6t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-4t}(1-6t)\,c_1 + 9t\,e^{-4t}c_2 \\ -4t\,e^{-4t}c_1 + e^{-4t}(1+6t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-6t) + 9c_2t)\,e^{-4t} \\ (c_2(1+6t) - 4tc_1)\,e^{-4t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -10-\lambda & 9 \\ -4 & 2-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 & 9 & 0 \\ -4 & 6 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -6 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	2	1	Yes	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

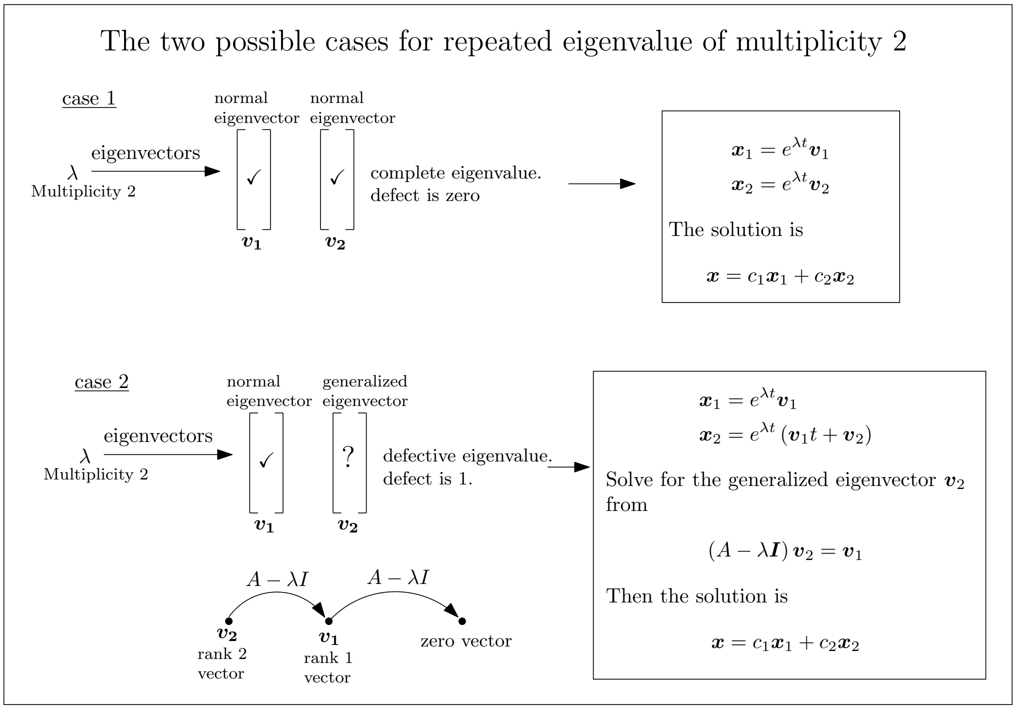


Figure 583: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}
\left(\begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \\
\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}
\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \mathrm{e}^{-4t} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-4t}}{2} \\ \mathrm{e}^{-4t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-4t}(6t+5)}{4} \\ \mathrm{e}^{-4t}(t+1) \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{3\,\mathrm{e}^{-4t}}{2} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-4t}\left(\frac{3t}{2} + \frac{5}{4}\right) \\ \mathrm{e}^{-4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{((6t+5)c_2+6c_1)\mathrm{e}^{-4t}}{4} \\ \mathrm{e}^{-4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

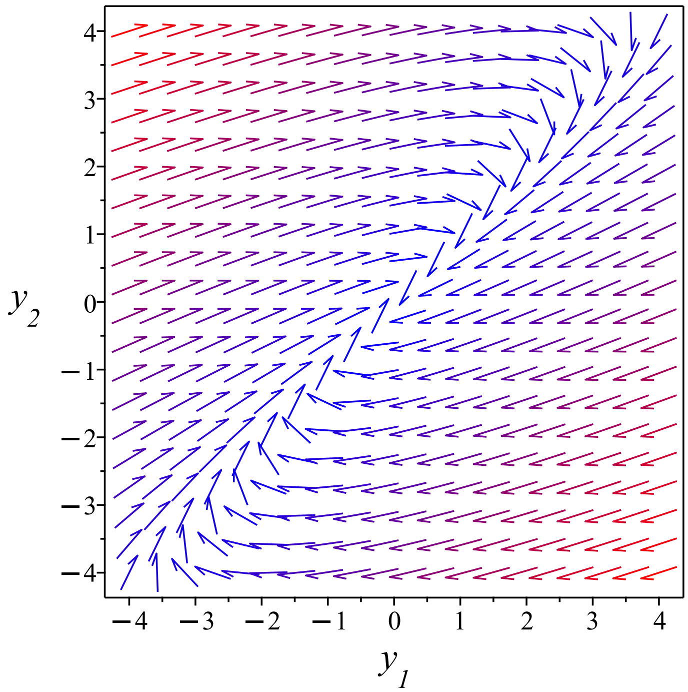


Figure 584: Phase plot

22.6.3 Maple step by step solution

Let's solve

$[y_1'(t) = -10y_1(t) + 9y_2(t), y_2'(t) = -4y_1(t) + 2y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-4, \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -4

$$\underline{y}^{\rightarrow}{}_1(t) = e^{-4t} \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -4$ is the eigenvalue, an

$$\underline{y}^{\rightarrow}{}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{y}^{\rightarrow}{}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{y}^{\rightarrow}{}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -4

$$\left(\left[\begin{array}{cc} -10 & 9 \\ -4 & 2 \end{array}\right]-(-4)\cdot\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} -\frac{1}{4} \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -4

$$\overrightarrow{y}_{2}(t)=e^{-4t}\cdot\left(t\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{4} \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y}=c_1\overrightarrow{y}_{1}(t)+c_2\overrightarrow{y}_{2}(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{y}=c_1e^{-4t}\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+c_2e^{-4t}\cdot\left(t\cdot\left[\begin{array}{c} \frac{3}{2} \\ 1 \end{array}\right]+\left[\begin{array}{c} -\frac{1}{4} \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right]=\left[\begin{array}{c} \frac{((6t-1)c_2+6c_1)e^{-4t}}{4} \\ e^{-4t}(c_2t+c_1) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\frac{((6t-1)c_2+6c_1)e^{-4t}}{4}, y_2(t)=e^{-4t}(c_2t+c_1)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(y__1(t),t)=-10*y__1(t)+9*y__2(t),diff(y__2(t),t)=-4*y__1(t)+2*y__2(t)],singsol=
```

$$y_1(t)=e^{-4t}(c_2t+c_1)$$
$$y_2(t)=\frac{e^{-4t}(6c_2t+6c_1+c_2)}{9}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{y1'[t]==-10*y1[t]+9*y2[t],y2'[t]==-4*y1[t]+2*y2[t]},{y1[t],y2[t]},t,IncludeSingularS
```

$$y1(t) \to e^{-4t}(-6c_1t + 9c_2t + c_1)$$
$$y2(t) \to e^{-4t}(-4c_1t + 6c_2t + c_2)$$

22.7 problem section 10.5, problem 7

Internal problem ID [1610]
Internal file name [`OUTPUT/1611_Sunday_June_05_2022_02_24_31_AM_80051039/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -13y_1(t) + 16y_2(t) \\
y_2'(t) &= -9y_1(t) + 11y_2(t)
\end{aligned}
$$

22.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(1-12t) & 16t\,\mathrm{e}^{-t} \\ -9t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+12t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(1-12t) & 16t\,e^{-t} \\ -9t\,e^{-t} & e^{-t}(1+12t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(1-12t)\,c_1 + 16t\,e^{-t}c_2 \\ -9t\,e^{-t}c_1 + e^{-t}(1+12t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-12t) + 16c_2t)\,e^{-t} \\ (c_2(1+12t) - 9tc_1)\,e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -13-\lambda & 16 \\ -9 & 11-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 16 \\ -9 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -12 & 16 & 0 \\ -9 & 12 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -12 & 16 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12 & 16 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	2	1	Yes	$\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

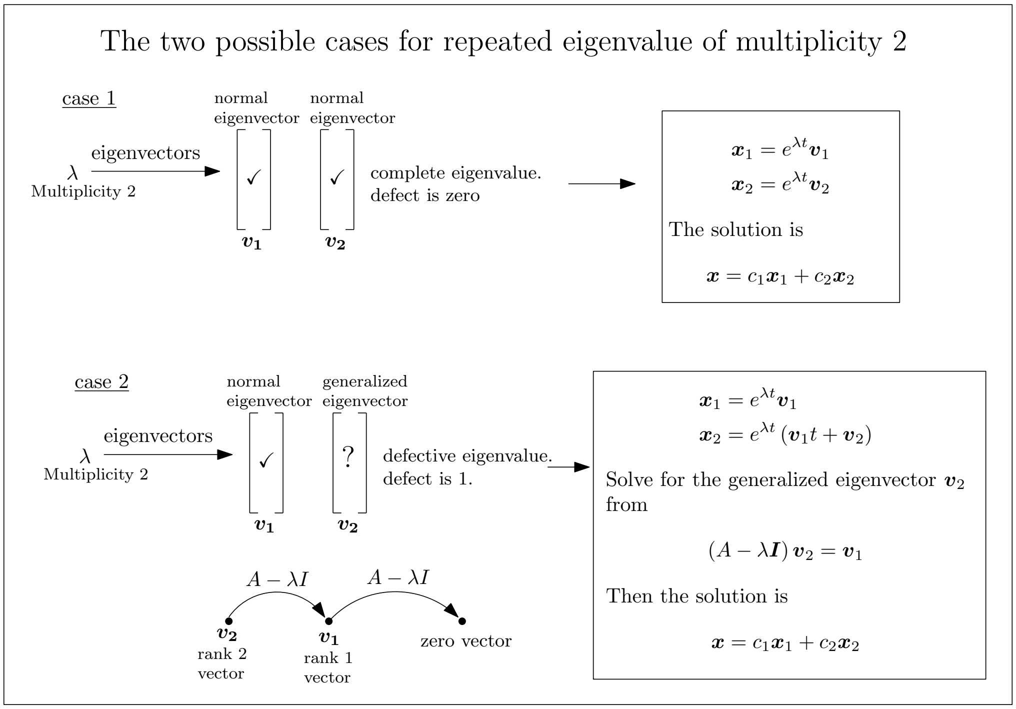


Figure 585: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 16 \\ -9 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{11}{9} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{11}{9} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-t}(12t+11)}{9} \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{4\,\mathrm{e}^{-t}}{3} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}\left(\frac{4t}{3} + \frac{11}{9}\right) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{((12t+11)c_2+12c_1)\mathrm{e}^{-t}}{9} \\ \mathrm{e}^{-t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

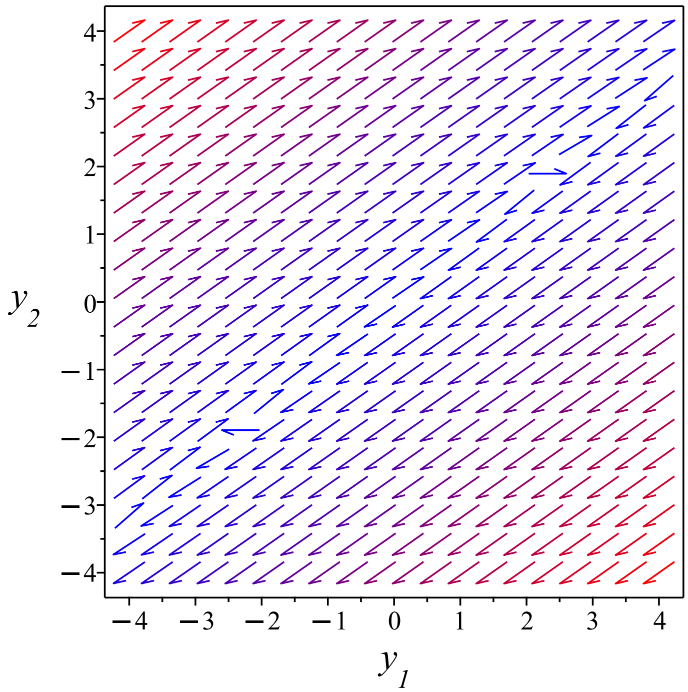


Figure 586: Phase plot

22.7.3 Maple step by step solution

Let's solve

$[y_1'(t) = -13y_1(t) + 16y_2(t), y_2'(t) = -9y_1(t) + 11y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\overrightarrow{y}}'(t) = A \cdot \underline{\overrightarrow{y}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

  $$\underline{\overrightarrow{y}}_1(t) = e^{-t} \cdot \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

  $\underline{\overrightarrow{y}}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\overrightarrow{y}}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\underline{\overrightarrow{y}}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cc}-13 & 16 \\ -9 & 11\end{array}\right]-(-1)\cdot\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}\frac{4}{3} \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c}-\frac{1}{9} \\ 0\end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{y}_2(t)=\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c}\frac{4}{3} \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{9} \\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y}=c_1\overrightarrow{y}_1(t)+c_2\overrightarrow{y}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{y}=c_1\mathrm{e}^{-t}\cdot\left[\begin{array}{c}\frac{4}{3} \\ 1\end{array}\right]+c_2\mathrm{e}^{-t}\cdot\left(t\cdot\left[\begin{array}{c}\frac{4}{3} \\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{9} \\ 0\end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}y_1(t) \\ y_2(t)\end{array}\right]=\left[\begin{array}{c}\frac{((-1+12t)c_2+12c_1)\mathrm{e}^{-t}}{9} \\ (c_2t+c_1)\,\mathrm{e}^{-t}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\frac{((-1+12t)c_2+12c_1)\mathrm{e}^{-t}}{9}, y_2(t)=(c_2t+c_1)\,\mathrm{e}^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(y__1(t),t)=-13*y__1(t)+16*y__2(t),diff(y__2(t),t)=-9*y__1(t)+11*y__2(t)],singso
```

$$y_1(t)=\mathrm{e}^{-t}(c_2t+c_1)$$
$$y_2(t)=\frac{\mathrm{e}^{-t}(12c_2t+12c_1+c_2)}{16}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 46

```
DSolve[{y1'[t]==-13*y1[t]+16*y2[t],y2'[t]==-9*y1[t]+11*y2[t]},{y1[t],y2[t]},t,IncludeSingula
```

$$\text{y1}(t) \to e^{-t}(-12c_1t + 16c_2t + c_1)$$
$$\text{y2}(t) \to e^{-t}(-9c_1t + 12c_2t + c_2)$$

22.8 problem section 10.5, problem 8

Internal problem ID [1611]
Internal file name [`OUTPUT/1612_Sunday_June_05_2022_02_24_32_AM_82817633/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(t) &= 2y_2(t) + y_3(t) \\ y_2'(t) &= -4y_1(t) + 6y_2(t) + y_3(t) \\ y_3'(t) &= 4y_2(t) + 2y_3(t) \end{aligned}$$

22.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2} - 2t\,\mathrm{e}^{4t} + \frac{\mathrm{e}^{4t}}{2} & 2t\,\mathrm{e}^{4t} & \frac{\mathrm{e}^{4t}}{4} - \frac{1}{4} \\ -\frac{\mathrm{e}^{4t}}{2} + \frac{1}{2} - 2t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(1+2t) & \frac{\mathrm{e}^{4t}}{4} - \frac{1}{4} \\ -4t\,\mathrm{e}^{4t} + \mathrm{e}^{4t} - 1 & 4t\,\mathrm{e}^{4t} & \frac{1}{2} + \frac{\mathrm{e}^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{1}{2} - 2t\,\mathrm{e}^{4t} + \frac{\mathrm{e}^{4t}}{2} & 2t\,\mathrm{e}^{4t} & \frac{\mathrm{e}^{4t}}{4} - \frac{1}{4} \\ -\frac{\mathrm{e}^{4t}}{2} + \frac{1}{2} - 2t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(1+2t) & \frac{\mathrm{e}^{4t}}{4} - \frac{1}{4} \\ -4t\,\mathrm{e}^{4t} + \mathrm{e}^{4t} - 1 & 4t\,\mathrm{e}^{4t} & \frac{1}{2} + \frac{\mathrm{e}^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1}{2} - 2t\,\mathrm{e}^{4t} + \frac{\mathrm{e}^{4t}}{2}\right) c_1 + 2t\,\mathrm{e}^{4t} c_2 + \left(\frac{\mathrm{e}^{4t}}{4} - \frac{1}{4}\right) c_3 \\ \left(-\frac{\mathrm{e}^{4t}}{2} + \frac{1}{2} - 2t\,\mathrm{e}^{4t}\right) c_1 + \mathrm{e}^{4t}(1+2t)\, c_2 + \left(\frac{\mathrm{e}^{4t}}{4} - \frac{1}{4}\right) c_3 \\ \left(-4t\,\mathrm{e}^{4t} + \mathrm{e}^{4t} - 1\right) c_1 + 4t\,\mathrm{e}^{4t} c_2 + \left(\frac{1}{2} + \frac{\mathrm{e}^{4t}}{2}\right) c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((2-8t)c_1+8c_2t+c_3)\mathrm{e}^{4t}}{4} + \frac{c_1}{2} - \frac{c_3}{4} \\ \frac{((-8t-2)c_1+(8t+4)c_2+c_3)\mathrm{e}^{4t}}{4} + \frac{c_1}{2} - \frac{c_3}{4} \\ \frac{((2-8t)c_1+8c_2t+c_3)\mathrm{e}^{4t}}{2} - c_1 + \frac{c_3}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 & 1 \\ -4 & 6-\lambda & 1 \\ 0 & 4 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 8\lambda^2 + 16\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ -4 & 6 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -4 & 6 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 6 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & 1 \\ -4 & 2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ -4 & 2 & 1 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
4	2	1	Yes	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^0 \\
&= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^0
\end{aligned}
$$

eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

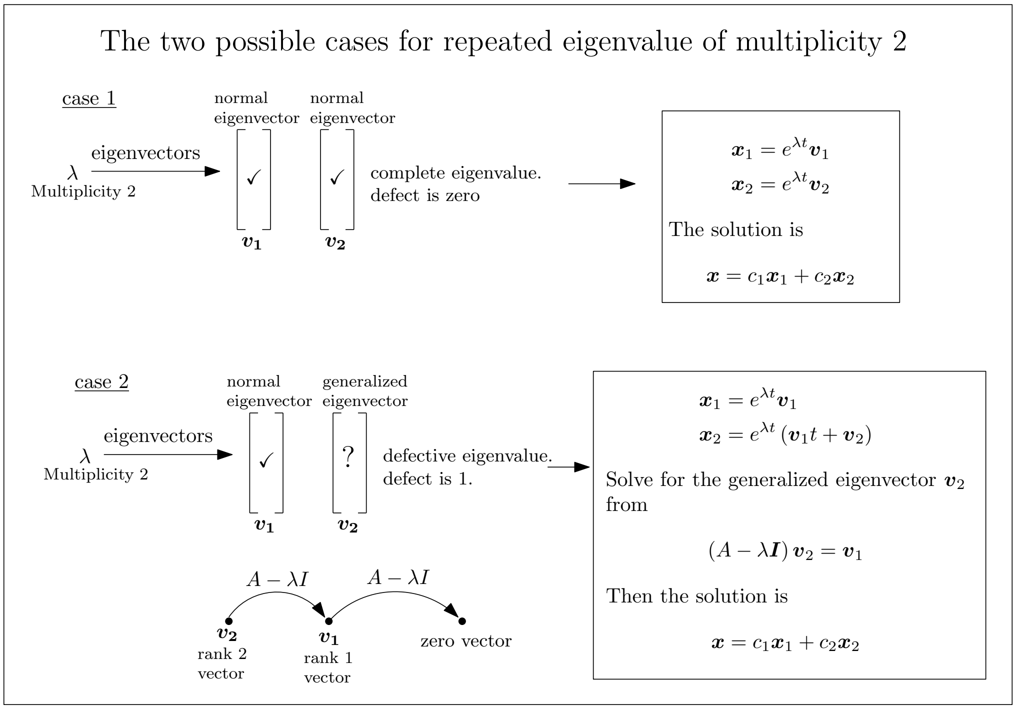


Figure 587: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix}0 & 2 & 1\\ -4 & 6 & 1\\ 0 & 4 & 2\end{bmatrix}-(4)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 2 & 1\\ -4 & 2 & 1\\ 0 & 4 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{4} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{4t}}{2} \\ \frac{\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{4} \\ 1 \\ \frac{3}{2} \end{bmatrix} \right) \mathrm{e}^{4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{4t}(2t+3)}{4} \\ \frac{\mathrm{e}^{4t}(2+t)}{2} \\ \frac{\mathrm{e}^{4t}(2t+3)}{2} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{4t}}{2} \\ \frac{\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{4t}\left(\frac{t}{2} + \frac{3}{4}\right) \\ \mathrm{e}^{4t}\left(\frac{t}{2} + 1\right) \\ \mathrm{e}^{4t}\left(t + \frac{3}{2}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((2t+3)c_3+2c_2)e^{4t}}{4} - \frac{c_1}{2} \\ \frac{((2+t)c_3+c_2)e^{4t}}{2} - \frac{c_1}{2} \\ \frac{((2t+3)c_3+2c_2)e^{4t}}{2} + c_1 \end{bmatrix}$$

22.8.3 Maple step by step solution

Let's solve

$[y_1'(t) = 2y_2(t) + y_3(t), y_2'(t) = -4y_1(t) + 6y_2(t) + y_3(t), y_3'(t) = 4y_2(t) + 2y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[4, \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[4, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[4, \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 4

$$\overrightarrow{y}_{2}(t) = e^{4t} \cdot \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

$$\overrightarrow{y}_{3}(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{y}_{3}(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\underline{y}^{\rightarrow}{}_{3}(t)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 4

$$\underline{y}^{\rightarrow}{}_{3}(t) = e^{4t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1} + c_2 \underline{y}^{\rightarrow}{}_{2}(t) + c_3 \underline{y}^{\rightarrow}{}_{3}(t)$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_2 e^{4t} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{4t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} -\frac{c_1}{2} \\ -\frac{c_1}{2} \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((-1+4t)c_3+4c_2)e^{4t}}{8} - \frac{c_1}{2} \\ \frac{(c_3 t+c_2)e^{4t}}{2} - \frac{c_1}{2} \\ (c_3 t + c_2)\, e^{4t} + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\frac{((-1+4t)c_3+4c_2)\,\mathrm{e}^{4t}}{8}-\frac{c_1}{2},y_2(t)=\frac{(c_3t+c_2)\,\mathrm{e}^{4t}}{2}-\frac{c_1}{2},y_3(t)=(c_3t+c_2)\,\mathrm{e}^{4t}+c_1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
dsolve([diff(y__1(t),t)=0*y__1(t)+2*y__2(t)+1*y__3(t),diff(y__2(t),t)=-4*y__1(t)+6*y__2(t)+1
```

$$y_1(t)=\frac{c_2\mathrm{e}^{4t}}{2}+\frac{c_3\mathrm{e}^{4t}t}{2}-\frac{c_1}{2}$$
$$y_2(t)=\frac{c_2\mathrm{e}^{4t}}{2}+\frac{c_3\mathrm{e}^{4t}t}{2}+\frac{c_3\mathrm{e}^{4t}}{4}-\frac{c_1}{2}$$
$$y_3(t)=c_1+c_2\mathrm{e}^{4t}+c_3\mathrm{e}^{4t}t$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 131

```
DSolve[{y1'[t]==0*y1[t]+2*y2[t]+1*y3[t],y2'[t]==-4*y1[t]+6*y2[t]+1*y3[t],y3'[t]==0*y1[t]+4*y
```

$$\text{y1}(t)\to\frac{1}{4}\left(c_1\left(e^{4t}(2-8t)+2\right)+e^{4t}(8c_2t+c_3)-c_3\right)$$
$$\text{y2}(t)\to\frac{1}{4}\left(-2c_1\left(e^{4t}(4t+1)-1\right)+e^{4t}(c_2(8t+4)+c_3)-c_3\right)$$
$$\text{y3}(t)\to c_1\left(e^{4t}(1-4t)-1\right)+\frac{1}{2}\left(e^{4t}(8c_2t+c_3)+c_3\right)$$

22.9 problem section 10.5, problem 9

Internal problem ID [1612]
Internal file name [`OUTPUT/1613_Sunday_June_05_2022_02_24_34_AM_13780462/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= \frac{y_1(t)}{3} + \frac{y_2(t)}{3} - y_3(t) \\
y_2'(t) &= -\frac{4y_1(t)}{3} - \frac{4y_2(t)}{3} + y_3(t) \\
y_3'(t) &= -\frac{2y_1(t)}{3} + \frac{y_2(t)}{3}
\end{aligned}
$$

22.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}(2t+3)}{6} + \frac{e^t}{2} & \frac{t\,e^{-t}}{3} & -\frac{e^t}{2} + \frac{e^{-t}}{2} \\ \frac{(-2t+3)e^{-t}}{6} - \frac{e^t}{2} & e^{-t}\left(1 - \frac{t}{3}\right) & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^{-t}(2t+3)}{6} - \frac{e^t}{2} & \frac{t\,e^{-t}}{3} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{-t}(2t+3)}{6} + \frac{e^t}{2} & \frac{t\,e^{-t}}{3} & -\frac{e^t}{2} + \frac{e^{-t}}{2} \\ \frac{(-2t+3)e^{-t}}{6} - \frac{e^t}{2} & e^{-t}\left(1 - \frac{t}{3}\right) & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^{-t}(2t+3)}{6} - \frac{e^t}{2} & \frac{t\,e^{-t}}{3} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-t}(2t+3)}{6} + \frac{e^t}{2}\right)c_1 + \frac{t\,e^{-t}c_2}{3} + \left(-\frac{e^t}{2} + \frac{e^{-t}}{2}\right)c_3 \\ \left(\frac{(-2t+3)e^{-t}}{6} - \frac{e^t}{2}\right)c_1 + e^{-t}\left(1 - \frac{t}{3}\right)c_2 + \left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right)c_3 \\ \left(\frac{e^{-t}(2t+3)}{6} - \frac{e^t}{2}\right)c_1 + \frac{t\,e^{-t}c_2}{3} + \left(\frac{e^{-t}}{2} + \frac{e^t}{2}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1(2t+3)+2c_2t+3c_3)e^{-t}}{6} + \frac{e^t(c_1-c_3)}{2} \\ \frac{((-2t+3)c_1+(6-2t)c_2-3c_3)e^{-t}}{6} - \frac{e^t(c_1-c_3)}{2} \\ \frac{(c_1(2t+3)+2c_2t+3c_3)e^{-t}}{6} - \frac{e^t(c_1-c_3)}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{1}{3} - \lambda & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} - \lambda & 1 \\ -\frac{2}{3} & \frac{1}{3} & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{1}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{4}{3} & \frac{1}{3} & -1 & 0 \\ -\frac{4}{3} & -\frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} \frac{4}{3} & \frac{1}{3} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} \frac{4}{3} & \frac{1}{3} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} \frac{4}{3} & \frac{1}{3} & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{7}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{2}{3} & \frac{1}{3} & -1 & 0 \\ -\frac{4}{3} & -\frac{7}{3} & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -\frac{2}{3} & \frac{1}{3} & -1 & 0 \\ 0 & -3 & 3 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -\frac{2}{3} & \frac{1}{3} & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	2	1	Yes	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

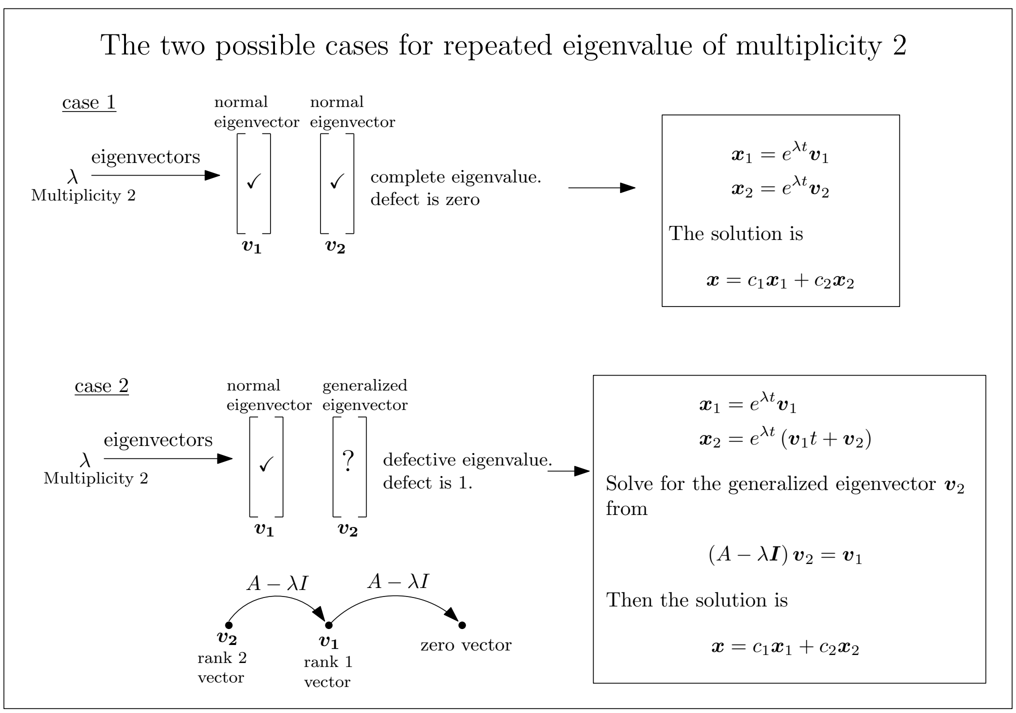


Figure 588: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{1}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\ &= \begin{bmatrix} \mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ -\mathrm{e}^{-t}(t-2) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ \mathrm{e}^{-t}(-t+2) \\ \mathrm{e}^{-t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} ((t+1)\, c_2 + c_1)\, \mathrm{e}^{-t} - c_3 \mathrm{e}^{t} \\ ((-t+2)\, c_2 - c_1)\, \mathrm{e}^{-t} + c_3 \mathrm{e}^{t} \\ ((t+1)\, c_2 + c_1)\, \mathrm{e}^{-t} + c_3 \mathrm{e}^{t} \end{bmatrix}
$$

22.9.3 Maple step by step solution

Let's solve

$$\left[y_1'(t)=\frac{y_1(t)}{3}+\frac{y_2(t)}{3}-y_3(t), y_2'(t)=-\frac{4y_1(t)}{3}-\frac{4y_2(t)}{3}+y_3(t), y_3'(t)=-\frac{2y_1(t)}{3}+\frac{y_2(t)}{3}\right]$$

- Define vector

$$\underline{y}^{\rightarrow}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t)=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}\cdot \underline{y}^{\rightarrow}(t)+\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t)=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}\cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t)=A\cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\underline{y}_1(t) = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\underline{y}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{y}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\underline{y}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{array}\right] - (-1) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} \frac{3}{4} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(t) = \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(t) + c_2 \vec{y}_2(t) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_2(4t+3)+4c_1)\mathrm{e}^{-t}}{4} - c_3 \mathrm{e}^{t} \\ (-c_2 t - c_1)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{t} \\ (c_2 t + c_1)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = \frac{(c_2(4t+3)+4c_1)\mathrm{e}^{-t}}{4} - c_3 \mathrm{e}^{t}, y_2(t) = (-c_2 t - c_1)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{t}, y_3(t) = (c_2 t + c_1)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve([diff(y__1(t),t)=1/3*y__1(t)+1/3*y__2(t)-1*y__3(t),diff(y__2(t),t)=-4/3*y__1(t)-4/3*y
```

$$y_1(t) = -c_1\mathrm{e}^t + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{-t}t$$
$$y_2(t) = c_1\mathrm{e}^t - c_2\mathrm{e}^{-t} - c_3\mathrm{e}^{-t}t + 3c_3\mathrm{e}^{-t}$$
$$y_3(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{-t}t$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 136

```
DSolve[{y1'[t]==1/3*y1[t]+1/3*y2[t]-1*y3[t],y2'[t]==-4/3*y1[t]-4/3*y2[t]+1*y3[t],y3'[t]==-2/
```

$$\mathrm{y1}(t) \to \frac{1}{6}e^{-t}\left(c_1\left(2t + 3e^{2t} + 3\right) + 2c_2t - 3c_3\left(e^{2t} - 1\right)\right)$$
$$\mathrm{y2}(t) \to \frac{1}{6}e^{-t}\left(c_1\left(-2t - 3e^{2t} + 3\right) - 2c_2(t-3) + 3c_3\left(e^{2t} - 1\right)\right)$$
$$\mathrm{y3}(t) \to \frac{1}{6}e^{-t}\left(c_1\left(2t - 3e^{2t} + 3\right) + 2c_2t + 3c_3\left(e^{2t} + 1\right)\right)$$

22.10 problem section 10.5, problem 10

Internal problem ID [1613]
Internal file name [`OUTPUT/1614_Sunday_June_05_2022_02_24_36_AM_49790449/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_1(t) + y_2(t) - y_3(t) \\
y_2'(t) &= -2y_1(t) + 2y_3(t) \\
y_3'(t) &= -y_1(t) + 3y_2(t) - y_3(t)
\end{aligned}
$$

22.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t}(t+1) & t\,e^{-2t} & -t\,e^{-2t} \\ \frac{e^{-2t}}{2} - \frac{e^{2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{2t}}{2} \\ t\,e^{-2t} + \frac{e^{-2t}}{2} - \frac{e^{2t}}{2} & t\,e^{-2t} - \frac{e^{-2t}}{2} + \frac{e^{2t}}{2} & \frac{e^{-2t}}{2} - t\,e^{-2t} + \frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t}(t+1) & t\,e^{-2t} & -t\,e^{-2t} \\ \frac{e^{-2t}}{2} - \frac{e^{2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{2t}}{2} \\ t\,e^{-2t} + \frac{e^{-2t}}{2} - \frac{e^{2t}}{2} & t\,e^{-2t} - \frac{e^{-2t}}{2} + \frac{e^{2t}}{2} & \frac{e^{-2t}}{2} - t\,e^{-2t} + \frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(t+1)\,c_1 + t\,e^{-2t}c_2 - t\,e^{-2t}c_3 \\ \left(\frac{e^{-2t}}{2} - \frac{e^{2t}}{2}\right)c_1 + \left(\frac{e^{-2t}}{2} + \frac{e^{2t}}{2}\right)c_2 + \left(-\frac{e^{-2t}}{2} + \frac{e^{2t}}{2}\right)c_3 \\ \left(t\,e^{-2t} + \frac{e^{-2t}}{2} - \frac{e^{2t}}{2}\right)c_1 + \left(t\,e^{-2t} - \frac{e^{-2t}}{2} + \frac{e^{2t}}{2}\right)c_2 + \left(\frac{e^{-2t}}{2} - t\,e^{-2t} + \frac{e^{2t}}{2}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} ((c_1 + c_2 - c_3)\,t + c_1)\,e^{-2t} \\ \frac{(c_1+c_2-c_3)e^{-2t}}{2} - \frac{e^{2t}(c_1-c_2-c_3)}{2} \\ \frac{(c_1(1+2t)+2\left(t-\frac{1}{2}\right)(c_2-c_3))e^{-2t}}{2} - \frac{e^{2t}(c_1-c_2-c_3)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 & -1 \\ -2 & -\lambda & 2 \\ -1 & 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 2\lambda^2 - 4\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & 2 & 2 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \implies \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \implies \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -1 \\ -2 & -2 & 2 \\ -1 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ -1 & 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & -\frac{8}{3} & \frac{8}{3} & 0 \\ -1 & 3 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & -\frac{8}{3} & \frac{8}{3} & 0 \\ 0 & \frac{8}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & -\frac{8}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 & -1 \\ 0 & -\frac{8}{3} & \frac{8}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	2	1	Yes	$\begin{bmatrix}1\\0\\1\end{bmatrix}$
2	1	1	No	$\begin{bmatrix}0\\1\\1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

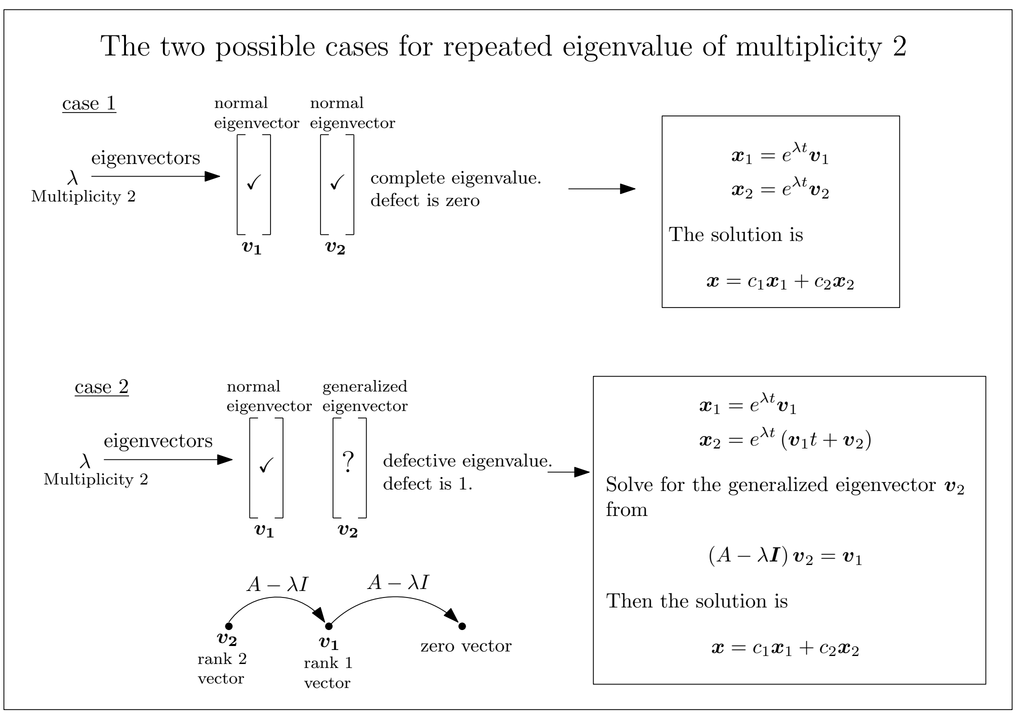


Figure 589: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{-2t} \\ &= \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{-2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-2t}(2t+3)}{2} \\ \frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-2t} \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-2t}\left(t + \frac{3}{2}\right) \\ \frac{\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-2t}\left(c_1 + c_2 t + \frac{3}{2} c_2\right) \\ \frac{c_2 \mathrm{e}^{-2t}}{2} + c_3 \mathrm{e}^{2t} \\ \left((t+1)\, c_2 + c_1\right) \mathrm{e}^{-2t} + c_3 \mathrm{e}^{2t} \end{bmatrix}
$$

22.10.3 Maple step by step solution

Let's solve

$[y_1'(t) = -y_1(t) + y_2(t) - y_3(t), y_2'(t) = -2y_1(t) + 2y_3(t), y_3'(t) = -y_1(t) + 3y_2(t) - y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -2

$$\vec{y}_1(t) = e^{-2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, an

$$\vec{y}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{y}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} - (-2) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -2

$$\underline{y^{\rightarrow}}_2(t) = e^{-2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y^{\rightarrow}}_3 = e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{y^{\rightarrow}} = c_1 \underline{y^{\rightarrow}}_1(t) + c_2 \underline{y^{\rightarrow}}_2(t) + c_3 \underline{y^{\rightarrow}}_3$$

- Substitute solutions into the general solution

$$\underline{y^{\rightarrow}} = c_1 e^{-2t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e^{-2t} c_2 \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}(c_2 t + c_1 + c_2) \\ c_3 e^{2t} \\ e^{-2t}(c_2 t + c_1) + c_3 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ y_1(t) = e^{-2t}(c_2 t + c_1 + c_2), y_2(t) = c_3 e^{2t}, y_3(t) = e^{-2t}(c_2 t + c_1) + c_3 e^{2t} \right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve([diff(y__1(t),t)=-1*y__1(t)+1*y__2(t)-1*y__3(t),diff(y__2(t),t)=-2*y__1(t)+0*y__2(t)+
```

$$y_1(t) = (2c_2t + c_1)\,\mathrm{e}^{-2t}$$
$$y_2(t) = c_2\mathrm{e}^{-2t} + c_3\mathrm{e}^{2t}$$
$$y_3(t) = 2c_2\mathrm{e}^{-2t}t + c_3\mathrm{e}^{2t} + c_1\mathrm{e}^{-2t} - c_2\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 125

```
DSolve[{y1'[t]==-1*y1[t]+1*y2[t]-1*y3[t],y2'[t]==-2*y1[t]+0*y2[t]+2*y3[t],y3'[t]==-1*y1[t]+3
```

$$\mathrm{y1}(t) \to e^{-2t}(c_1(t+1) + (c_2 - c_3)t)$$
$$\mathrm{y2}(t) \to \frac{1}{2}e^{-2t}\left(-\left(c_1\left(e^{4t}-1\right)\right) + c_2\left(e^{4t}+1\right) + c_3\left(e^{4t}-1\right)\right)$$
$$\mathrm{y3}(t) \to \frac{1}{2}e^{-2t}\left(c_1\left(2t - e^{4t} + 1\right) + c_2\left(2t + e^{4t} - 1\right) + c_3\left(-2t + e^{4t} + 1\right)\right)$$

22.11 problem section 10.5, problem 11

Internal problem ID [1614]
Internal file name [`OUTPUT/1615_Sunday_June_05_2022_02_24_38_AM_32416070/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 4y_1(t) - 2y_2(t) - 2y_3(t) \\
y_2'(t) &= -2y_1(t) + 3y_2(t) - y_3(t) \\
y_3'(t) &= 2y_1(t) - y_2(t) + 3y_3(t)
\end{aligned}
$$

22.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{4t} & -e^{4t} + e^{2t} & -e^{4t} + e^{2t} \\ -2t\, e^{4t} & \frac{3\, e^{2t}}{2} + 2t\, e^{4t} - \frac{e^{4t}}{2} & 2t\, e^{4t} - \frac{3\, e^{4t}}{2} + \frac{3\, e^{2t}}{2} \\ 2t\, e^{4t} & -2t\, e^{4t} + \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & -2t\, e^{4t} + \frac{3\, e^{4t}}{2} - \frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{4t} & -e^{4t} + e^{2t} & -e^{4t} + e^{2t} \\ -2t\, e^{4t} & \frac{3\, e^{2t}}{2} + 2t\, e^{4t} - \frac{e^{4t}}{2} & 2t\, e^{4t} - \frac{3\, e^{4t}}{2} + \frac{3\, e^{2t}}{2} \\ 2t\, e^{4t} & -2t\, e^{4t} + \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & -2t\, e^{4t} + \frac{3\, e^{4t}}{2} - \frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t} c_1 + \left(-e^{4t} + e^{2t}\right) c_2 + \left(-e^{4t} + e^{2t}\right) c_3 \\ -2t\, e^{4t} c_1 + \left(\frac{3\, e^{2t}}{2} + 2t\, e^{4t} - \frac{e^{4t}}{2}\right) c_2 + \left(2t\, e^{4t} - \frac{3\, e^{4t}}{2} + \frac{3\, e^{2t}}{2}\right) c_3 \\ 2t\, e^{4t} c_1 + \left(-2t\, e^{4t} + \frac{e^{4t}}{2} - \frac{e^{2t}}{2}\right) c_2 + \left(-2t\, e^{4t} + \frac{3\, e^{4t}}{2} - \frac{e^{2t}}{2}\right) c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(c_1 - c_2 - c_3) + e^{2t}(c_2 + c_3) \\ \frac{((-1+4t)c_2 + (-3+4t)c_3 - 4c_1 t)e^{4t}}{2} + \frac{3\, e^{2t}(c_2+c_3)}{2} \\ \frac{((1-4t)c_2 + (-4t+3)c_3 + 4c_1 t)e^{4t}}{2} - \frac{e^{2t}(c_2+c_3)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\, \vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}4 & -2 & -2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}4-\lambda & -2 & -2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 10\lambda^2 + 32\lambda - 32 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}4 & -2 & -2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2 & -2 & -2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & -1 & -3 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t, v_2 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & -2 & 0 \\ -2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -2 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
4	2	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

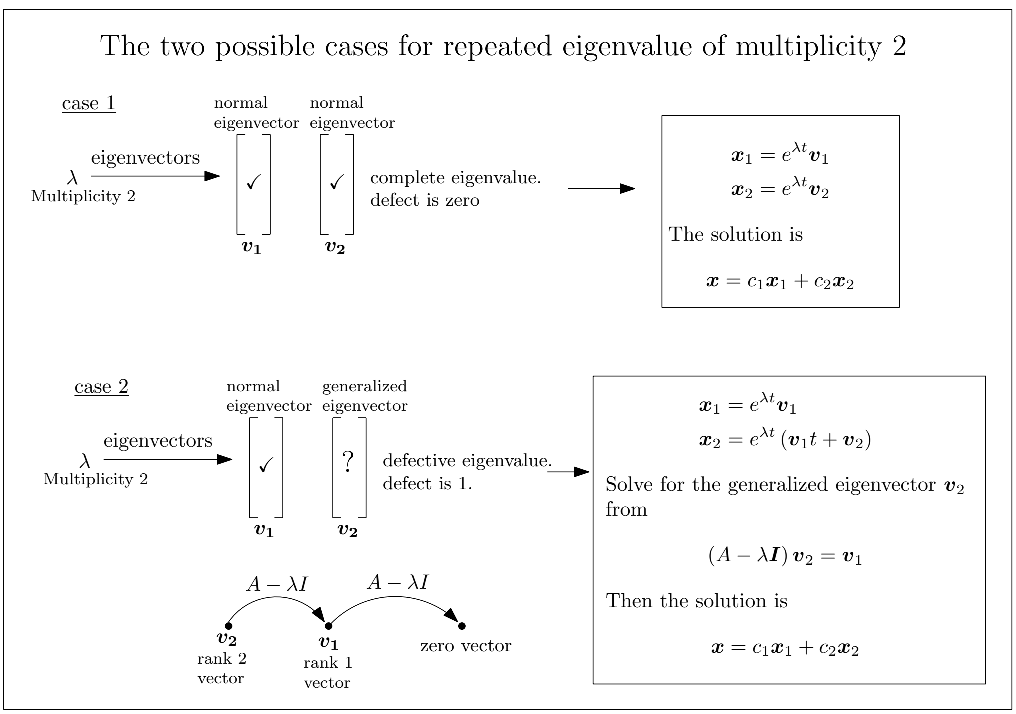


Figure 590: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{4t}}{2} \\ -\mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{4t}}{2} \\ \mathrm{e}^{4t}(-t-1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} -2\,\mathrm{e}^{2t} \\ -3\,\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{c_2 \mathrm{e}^{4t}}{2} - 2c_3 \mathrm{e}^{2t} \\ ((-t-1)\, c_2 - c_1)\, \mathrm{e}^{4t} - 3c_3 \mathrm{e}^{2t} \\ ((t+1)\, c_2 + c_1)\, \mathrm{e}^{4t} + c_3 \mathrm{e}^{2t} \end{bmatrix}
$$

22.11.3 Maple step by step solution

Let's solve

$[y_1'(t) = 4y_1(t) - 2y_2(t) - 2y_3(t), y_2'(t) = -2y_1(t) + 3y_2(t) - y_3(t), y_3'(t) = 2y_1(t) - y_2(t) + 3y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} -2 \\ -3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} -2 \\ -3 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[4, \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 4

$$\overrightarrow{y}_{2}(t) = \mathrm{e}^{4t} \cdot \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

$$\overrightarrow{y}_{3}(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{y}_{3}(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{3}(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 4

$$\underline{\overrightarrow{y}}_3(t) = \mathrm{e}^{4t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$\underline{\overrightarrow{y}} = c_1 \underline{\overrightarrow{y}}_1 + c_2 \underline{\overrightarrow{y}}_2(t) + c_3 \underline{\overrightarrow{y}}_3(t)$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{y}} = c_1 \mathrm{e}^{2t} \cdot \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{4t} \cdot \left(t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \mathrm{e}^{2t} \\ (-c_3 t - c_2)\,\mathrm{e}^{4t} - 3c_1 \mathrm{e}^{2t} \\ (c_3 t + c_2)\,\mathrm{e}^{4t} + c_1 \mathrm{e}^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$\{y_1(t) = -2c_1 \mathrm{e}^{2t}, y_2(t) = (-c_3 t - c_2)\,\mathrm{e}^{4t} - 3c_1 \mathrm{e}^{2t}, y_3(t) = (c_3 t + c_2)\,\mathrm{e}^{4t} + c_1 \mathrm{e}^{2t}\}$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 69

```
dsolve([diff(y__1(t),t)=4*y__1(t)-2*y__2(t)-2*y__3(t),diff(y__2(t),t)=-2*y__1(t)+3*y__2(t)-1
```

$$y_1(t) = c_2 e^{4t} + c_3 e^{2t}$$
$$y_2(t) = -2c_2 e^{4t} t + c_1 e^{4t} + \frac{3c_3 e^{2t}}{2}$$
$$y_3(t) = 2c_2 e^{4t} t - c_1 e^{4t} - \frac{c_3 e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 126

```
DSolve[{y1'[t]==4*y1[t]-2*y2[t]-2*y3[t],y2'[t]==-2*y1[t]+3*y2[t]-1*y3[t],y3'[t]==2*y1[t]-1*y
```

$$\text{y1}(t) \to e^{2t}\left((c_1 - c_2 - c_3)e^{2t} + c_2 + c_3\right)$$
$$\text{y2}(t) \to \frac{1}{2}\left(3(c_2 + c_3)e^{2t} - e^{4t}(4(c_1 - c_2 - c_3)t + c_2 + 3c_3)\right)$$
$$\text{y3}(t) \to -\frac{1}{2}e^{2t}\left(-e^{2t}(4(c_1 - c_2 - c_3)t + c_2 + 3c_3) + c_2 + c_3\right)$$

22.12 problem section 10.5, problem 12

Internal problem ID [1615]
Internal file name [`OUTPUT/1616_Sunday_June_05_2022_02_24_40_AM_88006293/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 6y_1(t) - 5y_2(t) + 3y_3(t)\\
y_2'(t) &= 2y_1(t) - y_2(t) + 3y_3(t)\\
y_3'(t) &= 2y_1(t) + y_2(t) + y_3(t)
\end{aligned}
$$

22.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{4t}(1+2t) & -2t\,e^{4t} - \frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} \\ 2t\,e^{4t} & \frac{e^{-2t}}{2} - 2t\,e^{4t} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} \\ 2t\,e^{4t} & -2t\,e^{4t} + \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{4t}(1+2t) & -2t\,e^{4t} - \frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} \\ 2t\,e^{4t} & \frac{e^{-2t}}{2} - 2t\,e^{4t} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} \\ 2t\,e^{4t} & -2t\,e^{4t} + \frac{e^{4t}}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{4t}(1+2t)\,c_1 + \left(-2t\,e^{4t} - \frac{e^{4t}}{2} + \frac{e^{-2t}}{2}\right) c_2 + \left(\frac{e^{4t}}{2} - \frac{e^{-2t}}{2}\right) c_3 \\ 2t\,e^{4t} c_1 + \left(\frac{e^{-2t}}{2} - 2t\,e^{4t} + \frac{e^{4t}}{2}\right) c_2 + \left(\frac{e^{4t}}{2} - \frac{e^{-2t}}{2}\right) c_3 \\ 2t\,e^{4t} c_1 + \left(-2t\,e^{4t} + \frac{e^{4t}}{2} - \frac{e^{-2t}}{2}\right) c_2 + \left(\frac{e^{-2t}}{2} + \frac{e^{4t}}{2}\right) c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{((-4t-1)c_2+(4t+2)c_1+c_3)e^{4t}}{2} + \frac{e^{-2t}(c_2-c_3)}{2} \\ \frac{((1-4t)c_2+4c_1t+c_3)e^{4t}}{2} + \frac{e^{-2t}(c_2-c_3)}{2} \\ \frac{((1-4t)c_2+4c_1t+c_3)e^{4t}}{2} - \frac{e^{-2t}(c_2-c_3)}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 6-\lambda & -5 & 3 \\ 2 & -1-\lambda & 3 \\ 2 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 32 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -5 & 3 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 8 & -5 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -5 & 3 & 0 \\ 0 & \frac{9}{4} & \frac{9}{4} & 0 \\ 2 & 1 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -5 & 3 & 0 \\ 0 & \frac{9}{4} & \frac{9}{4} & 0 \\ 0 & \frac{9}{4} & \frac{9}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 8 & -5 & 3 & 0 \\ 0 & \frac{9}{4} & \frac{9}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -5 & 3 \\ 0 & \frac{9}{4} & \frac{9}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & 3 \\ 2 & -5 & 3 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -5 & 3 & 0 \\ 2 & -5 & 3 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & -5 & 3 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -5 & 3 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
4	2	1	Yes	$\begin{bmatrix}1\\1\\1\end{bmatrix}$
-2	1	1	No	$\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

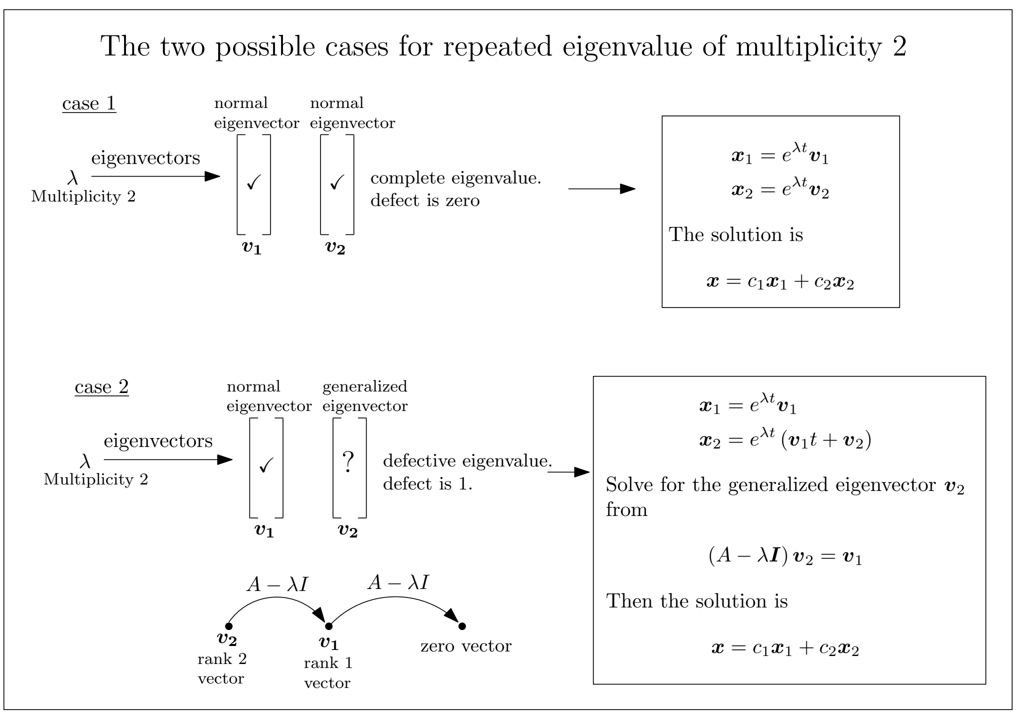


Figure 591: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & 3 \\ 2 & -5 & 3 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\ &= \begin{bmatrix} \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \end{bmatrix} \right) \mathrm{e}^{4t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{4t}(2t+3)}{2} \\ \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}
\end{aligned}
$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\
&= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}\left(t+\frac{3}{2}\right) \\ \mathrm{e}^{4t}(t+1) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((2t+3)c_2+2c_1)\mathrm{e}^{4t}}{2} - c_3 \mathrm{e}^{-2t} \\ ((t+1)\, c_2 + c_1)\, \mathrm{e}^{4t} - c_3 \mathrm{e}^{-2t} \\ ((t+1)\, c_2 + c_1)\, \mathrm{e}^{4t} + c_3 \mathrm{e}^{-2t} \end{bmatrix}
$$

22.12.3 Maple step by step solution

Let's solve

$[y_1'(t) = 6y_1(t) - 5y_2(t) + 3y_3(t), y_2'(t) = 2y_1(t) - y_2(t) + 3y_3(t), y_3'(t) = 2y_1(t) + y_2(t) + y_3(t)]$

- Define vector

$$\underline{\vec{y}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{y}}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 4

$$\underline{\overrightarrow{y}}_2(t) = \mathrm{e}^{4t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and

$$\underline{\overrightarrow{y}}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\overrightarrow{y}}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\underline{\overrightarrow{y}}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4

$$\left(\left[\begin{array}{ccc} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{array}\right]-4\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 4

$$\vec{y}_3(t)=\mathrm{e}^{4t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(t)+c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-2t}\cdot\left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]+c_2\mathrm{e}^{4t}\cdot\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]+c_3\mathrm{e}^{4t}\cdot\left(t\cdot\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 0 \end{array}\right]\right)$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right]=\left[\begin{array}{c} \frac{((1+2t)c_3+2c_2)\mathrm{e}^{4t}}{2}-c_1\mathrm{e}^{-2t} \\ (c_3t+c_2)\,\mathrm{e}^{4t}-c_1\mathrm{e}^{-2t} \\ (c_3t+c_2)\,\mathrm{e}^{4t}+c_1\mathrm{e}^{-2t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\frac{((1+2t)c_3+2c_2)\mathrm{e}^{4t}}{2}-c_1\mathrm{e}^{-2t}, y_2(t)=(c_3t+c_2)\,\mathrm{e}^{4t}-c_1\mathrm{e}^{-2t}, y_3(t)=(c_3t+c_2)\,\mathrm{e}^{4t}+c_1\mathrm{e}^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 86

```
dsolve([diff(y__1(t),t)=6*y__1(t)-5*y__2(t)+3*y__3(t),diff(y__2(t),t)=2*y__1(t)-1*y__2(t)+3*
```

$$y_1(t) = c_1 e^{-2t} + c_2 e^{4t} + c_3 e^{4t} t$$
$$y_2(t) = c_1 e^{-2t} + c_2 e^{4t} + c_3 e^{4t} t - \frac{c_3 e^{4t}}{2}$$
$$y_3(t) = -c_1 e^{-2t} + c_2 e^{4t} + c_3 e^{4t} t - \frac{c_3 e^{4t}}{2}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 127

```
DSolve[{y1'[t]==6*y1[t]-5*y2[t]+3*y3[t],y2'[t]==2*y1[t]-1*y2[t]+3*y3[t],y3'[t]==2*y1[t]+1*y2
```

$$\text{y1}(t) \rightarrow \frac{1}{2} e^{-2t} \left(e^{6t}(c_1(4t+2) - c_2(4t+1) + c_3) + c_2 - c_3\right)$$
$$\text{y2}(t) \rightarrow \frac{1}{2} e^{-2t} \left(e^{6t}(4(c_1 - c_2)t + c_2 + c_3) + c_2 - c_3\right)$$
$$\text{y3}(t) \rightarrow \frac{1}{2} e^{-2t} \left(e^{6t}(4(c_1 - c_2)t + c_2 + c_3) - c_2 + c_3\right)$$

22.13 problem section 10.5, problem 13

Internal problem ID [1616]
Internal file name [`OUTPUT/1617_Sunday_June_05_2022_02_24_42_AM_36547832/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -11y_1(t) + 8y_2(t)\\
y_2'(t) &= -2y_1(t) - 3y_2(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = 6, y_2(0) = 2]
$$

22.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-7t}(1-4t) & 8t\,\mathrm{e}^{-7t} \\ -2t\,\mathrm{e}^{-7t} & \mathrm{e}^{-7t}(4t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{-7t}(1-4t) & 8t\,e^{-7t} \\ -2t\,e^{-7t} & e^{-7t}(4t+1) \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6\,e^{-7t}(1-4t)+16t\,e^{-7t} \\ -12t\,e^{-7t}+2\,e^{-7t}(4t+1) \end{bmatrix} \\ &= \begin{bmatrix} (-8t+6)\,e^{-7t} \\ (-4t+2)\,e^{-7t} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -11-\lambda & 8 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 14\lambda + 49 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -7$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} - (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 8 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -4 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-7	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -7 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

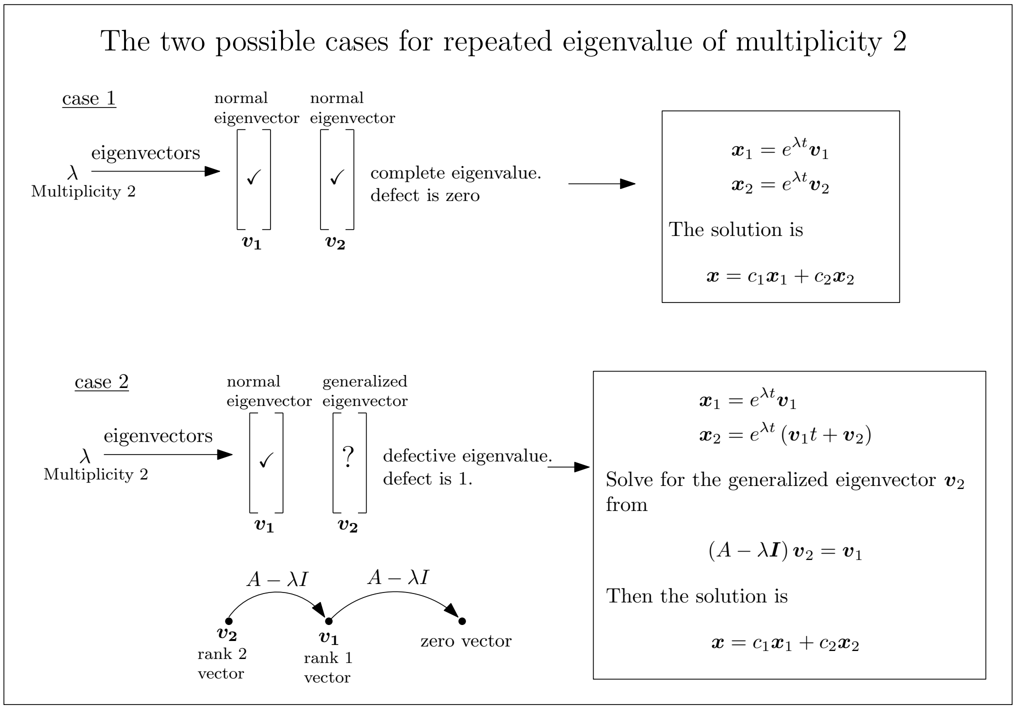


Figure 592: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} - (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -7. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{-7t}\\ &= \begin{bmatrix} 2\,\mathrm{e}^{-7t} \\ \mathrm{e}^{-7t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right) \mathrm{e}^{-7t}\\ &= \begin{bmatrix} \frac{\mathrm{e}^{-7t}(4t+3)}{2} \\ \mathrm{e}^{-7t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-7t} \\ \mathrm{e}^{-7t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-7t}\left(2t + \frac{3}{2}\right) \\ \mathrm{e}^{-7t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-7t}\left(2c_1 + 2c_2 t + \frac{3}{2}c_2\right) \\ \mathrm{e}^{-7t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} y_1(0) = 6 \\ y_2(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + \frac{3c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 6 \\ c_2 = -4 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (-8t + 6)\,\mathrm{e}^{-7t} \\ (-4t + 2)\,\mathrm{e}^{-7t} \end{bmatrix}$$

The following is the phase plot of the system.

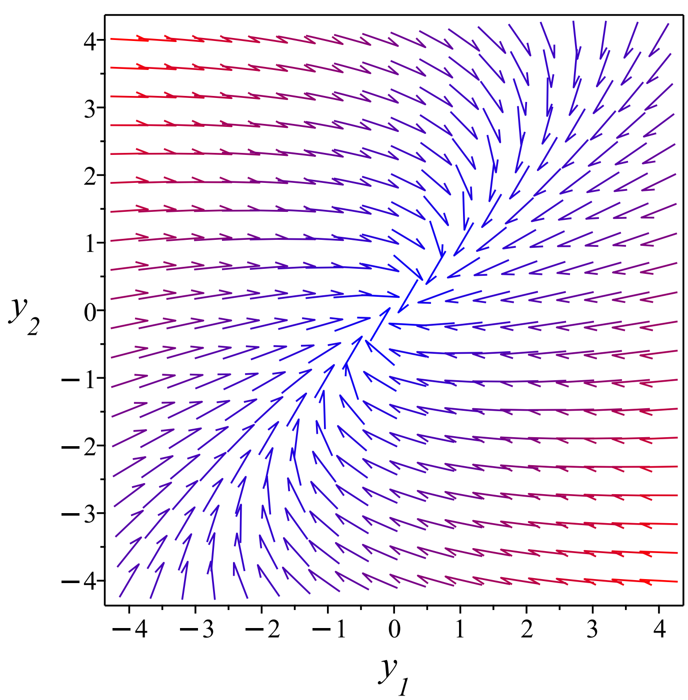
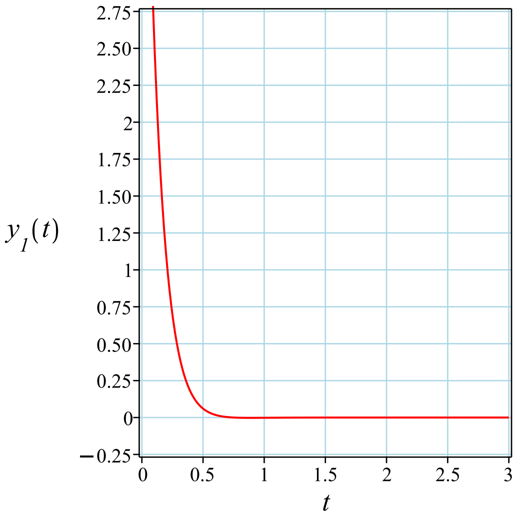
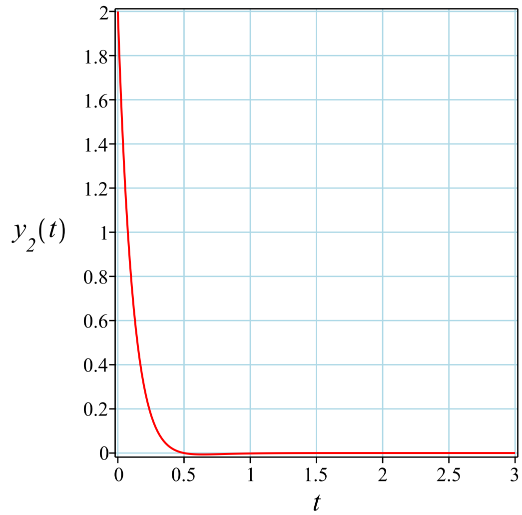


Figure 593: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(y__1(t),t) = -11*y__1(t)+8*y__2(t), diff(y__2(t),t) = -2*y__1(t)-3*y__2(t), y__
```

$$y_1(t) = e^{-7t}(-8t+6)$$
$$y_2(t) = \frac{e^{-7t}(-32t+16)}{8}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{y1'[t]==-11*y1[t]+8*y2[t],y2'[t]==-2*y1[t]-3*y2[t]},{y1[0]==6,y2[0]==2},{y1[t],y2[t]
```

$$\text{y1}(t) \to e^{-7t}(6-8t)$$
$$\text{y2}(t) \to e^{-7t}(2-4t)$$

22.14 problem section 10.5, problem 14

Internal problem ID [1617]

Internal file name [`OUTPUT/1618_Sunday_June_05_2022_02_24_43_AM_42943667/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001

Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555

Problem number: section 10.5, problem 14.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 15y_1(t) - 9y_2(t) \\
y_2'(t) &= 16y_1(t) - 9y_2(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = 5, y_2(0) = 8]
$$

22.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(1+12t) & -9\,\mathrm{e}^{3t}t \\ 16\,\mathrm{e}^{3t}t & \mathrm{e}^{3t}(1-12t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{3t}(1+12t) & -9\,e^{3t}t \\ 16\,e^{3t}t & e^{3t}(1-12t) \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 5\,e^{3t}(1+12t) - 72\,e^{3t}t \\ 80\,e^{3t}t + 8\,e^{3t}(1-12t) \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(-12t+5) \\ (-16t+8)\,e^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 15-\lambda & -9 \\ 16 & -9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 12 & -9 & 0 \\ 16 & -12 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 12 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 12 & -9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{3t}{4}\}$

Hence the solution is

$$\begin{bmatrix}\frac{3t}{4}\\ t\end{bmatrix}=\begin{bmatrix}\frac{3t}{4}\\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{3t}{4}\\ t\end{bmatrix}=t\begin{bmatrix}\frac{3}{4}\\ 1\end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix}\frac{3t}{4}\\ t\end{bmatrix}=\begin{bmatrix}\frac{3}{4}\\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{3t}{4}\\ t\end{bmatrix}=\begin{bmatrix}3\\ 4\end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m>k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix}\frac{3}{4}\\ 1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

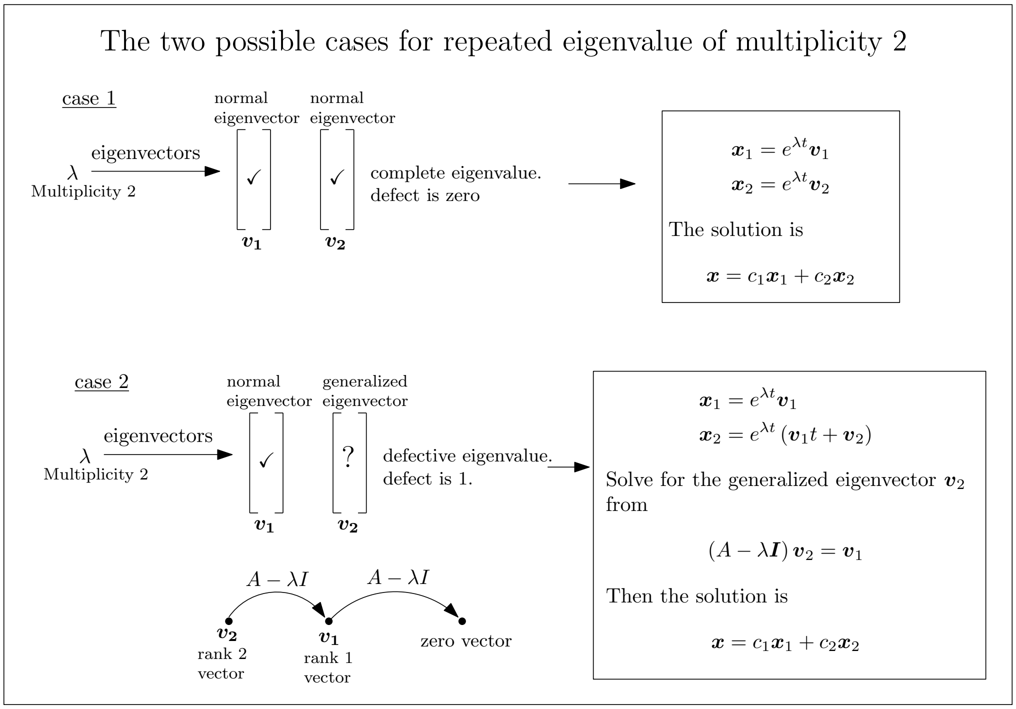


Figure 594: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{5}{4} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \left[\begin{array}{c}\frac{3}{4}\\ 1\end{array}\right] e^{3t}\\ &= \left[\begin{array}{c}\frac{3\,e^{3t}}{4}\\ e^{3t}\end{array}\right]\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}\\ &= \left(\left[\begin{array}{c}\frac{3}{4}\\ 1\end{array}\right] t + \left[\begin{array}{c}1\\ \frac{5}{4}\end{array}\right]\right) e^{3t}\\ &= \left[\begin{array}{c}\frac{e^{3t}(3t+4)}{4}\\ \frac{e^{3t}(4t+5)}{4}\end{array}\right]\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}y_1(t)\\ y_2(t)\end{array}\right] = c_1\left[\begin{array}{c}\frac{3\,e^{3t}}{4}\\ e^{3t}\end{array}\right] + c_2\left[\begin{array}{c}e^{3t}\left(\frac{3t}{4}+1\right)\\ e^{3t}\left(t+\frac{5}{4}\right)\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}y_1(t)\\ y_2(t)\end{array}\right] = \left[\begin{array}{c}\frac{((3t+4)c_2+3c_1)e^{3t}}{4}\\ e^{3t}\left(c_1+c_2t+\frac{5}{4}c_2\right)\end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c}y_1(0) = 5\\ y_2(0) = 8\end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} c_2 + \frac{3c_1}{4} \\ c_1 + \frac{5c_2}{4} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 28 \\ c_2 = -16 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(-48t+20)\,\mathrm{e}^{3t}}{4} \\ (-16t + 8)\,\mathrm{e}^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

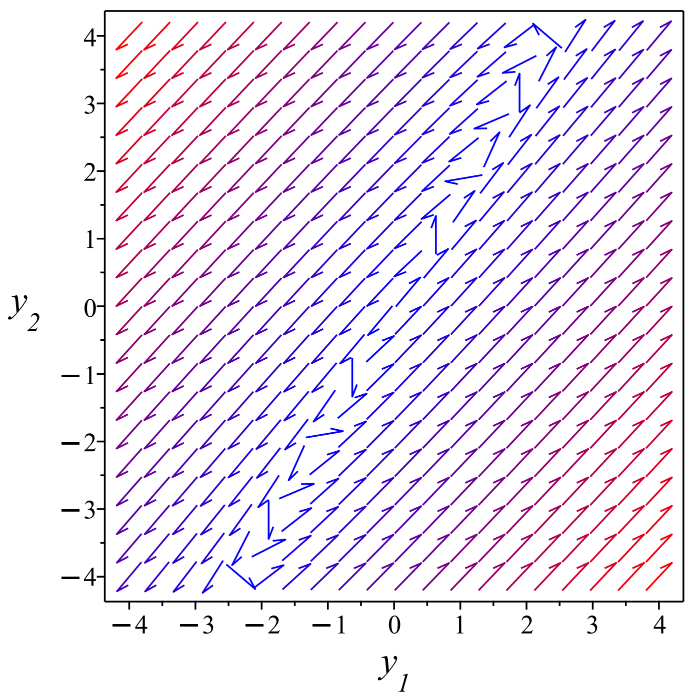
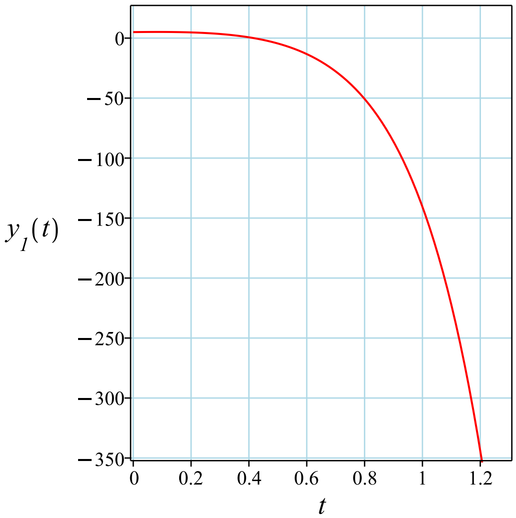
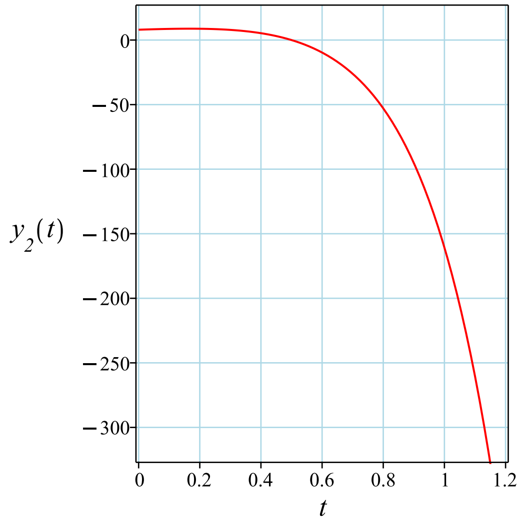


Figure 595: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(y__1(t),t) = 15*y__1(t)-9*y__2(t), diff(y__2(t),t) = 16*y__1(t)-9*y__2(t), y__1
```

$$y_1(t) = \mathrm{e}^{3t}(-12t + 5)$$
$$y_2(t) = \frac{\mathrm{e}^{3t}(-144t + 72)}{9}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 31

```
DSolve[{y1'[t]==15*y1[t]-9*y2[t],y2'[t]==16*y1[t]-9*y2[t]},{y1[0]==5,y2[0]==8},{y1[t],y2[t]}
```

$$\text{y1}(t) \to e^{3t}(5 - 12t)$$
$$\text{y2}(t) \to -8e^{3t}(2t - 1)$$

22.15 problem section 10.5, problem 15

Internal problem ID [1618]
Internal file name [`OUTPUT/1619_Sunday_June_05_2022_02_24_45_AM_66646725/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) - 4y_2(t) \\
y_2'(t) &= y_1(t) - 7y_2(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = 2, y_2(0) = 3]
$$

22.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{-5t}(1+2t) & -4t\,e^{-5t} \\ t\,e^{-5t} & e^{-5t}(1-2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-5t}(1+2t) & -4t\,e^{-5t} \\ t\,e^{-5t} & e^{-5t}(1-2t) \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2\,e^{-5t}(1+2t) - 12t\,e^{-5t} \\ 2t\,e^{-5t} + 3\,e^{-5t}(1-2t) \end{bmatrix} \\
&= \begin{bmatrix} (2-8t)\,e^{-5t} \\ e^{-5t}(-4t+3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & -4 \\ 1 & -7-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$
\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-5	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

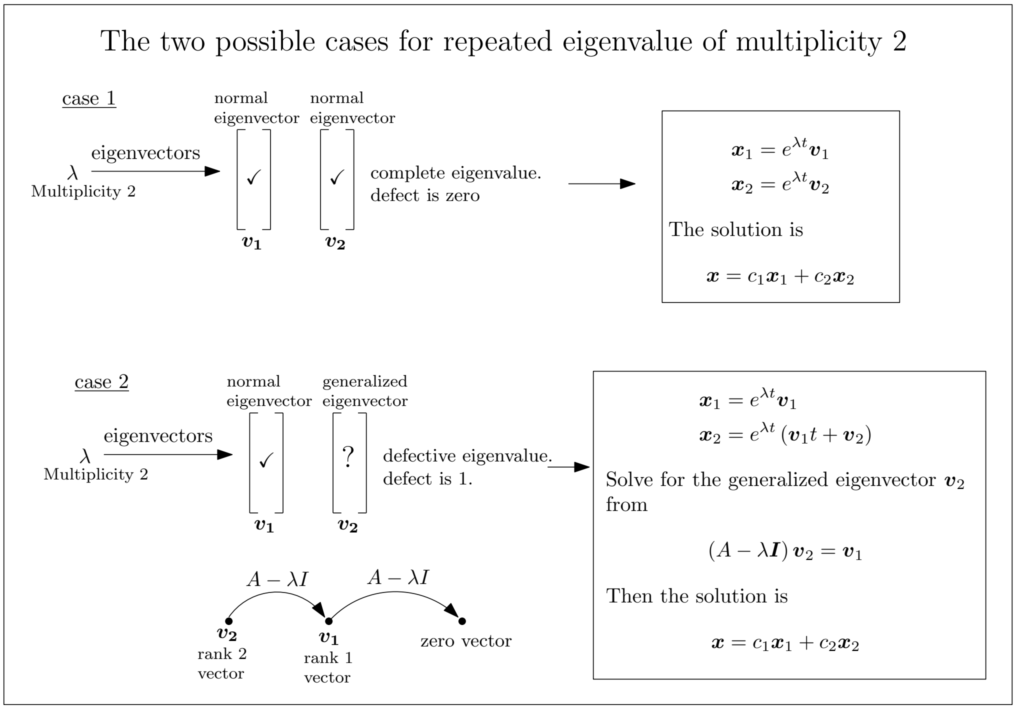


Figure 596: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -5. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathrm{e}^{-5t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-5t} \\
&= \begin{bmatrix} \mathrm{e}^{-5t}(2t+3) \\ \mathrm{e}^{-5t}(t+1) \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-5t}(2t+3) \\ \mathrm{e}^{-5t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ((2t+3)\, c_2 + 2c_1)\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} y_1(0) = 2 \\ y_2(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_2 + 2c_1 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 7 \\ c_2 = -4 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (2 - 8t)\,\mathrm{e}^{-5t} \\ \mathrm{e}^{-5t}(-4t + 3) \end{bmatrix}$$

The following is the phase plot of the system.

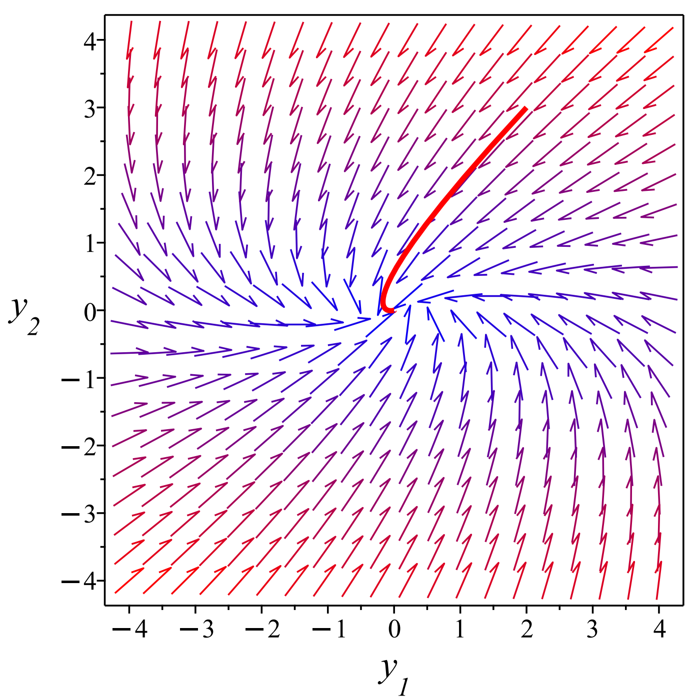
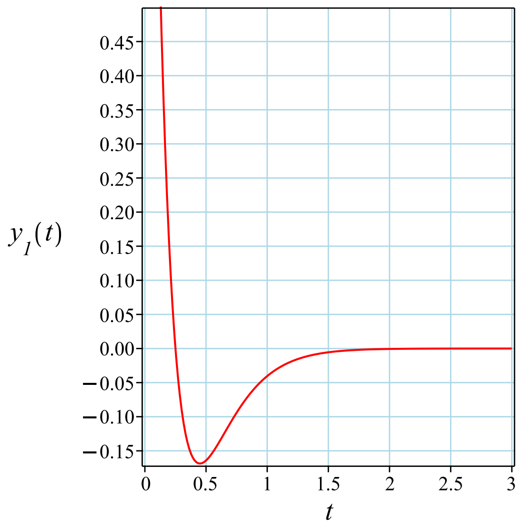
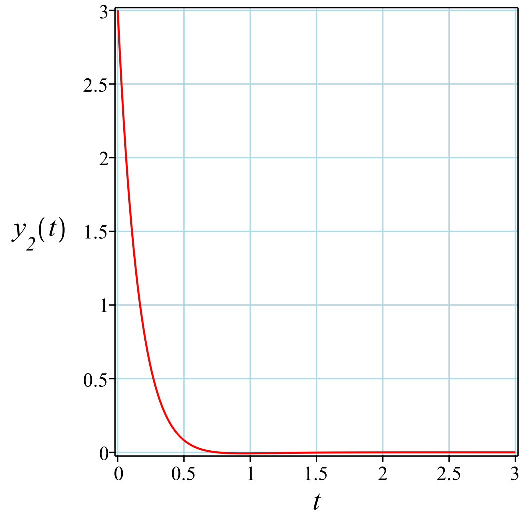


Figure 597: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(y__1(t),t) = -3*y__1(t)-4*y__2(t), diff(y__2(t),t) = y__1(t)-7*y__2(t), y__1(0)
```

$$y_1(t) = e^{-5t}(-8t + 2)$$
$$y_2(t) = \frac{e^{-5t}(-16t + 12)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[{y1'[t]==-3*y1[t]-4*y2[t],y2'[t]==1*y1[t]-7*y2[t]},{y1[0]==2,y2[0]==3},{y1[t],y2[t]},
```

$$\text{y1}(t) \to e^{-5t}(2 - 8t)$$
$$\text{y2}(t) \to e^{-5t}(3 - 4t)$$

22.16 problem section 10.5, problem 16

Internal problem ID [1619]
Internal file name [`OUTPUT/1620_Sunday_June_05_2022_02_24_46_AM_32482012/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -7y_1(t) + 24y_2(t) \\
y_2'(t) &= -6y_1(t) + 17y_2(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = 3, y_2(0) = 1]
$$

22.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{5t}(1-12t) & 24t\,e^{5t} \\ -6t\,e^{5t} & e^{5t}(1+12t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{5t}(1-12t) & 24t\,e^{5t} \\ -6t\,e^{5t} & e^{5t}(1+12t) \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3\,e^{5t}(1-12t) + 24t\,e^{5t} \\ -18t\,e^{5t} + e^{5t}(1+12t) \end{bmatrix} \\
&= \begin{bmatrix} (-12t+3)\,e^{5t} \\ e^{5t}(1-6t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -7-\lambda & 24 \\ -6 & 17-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 24 \\ -6 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -12 & 24 & 0 \\ -6 & 12 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -12 & 24 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -12 & 24 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

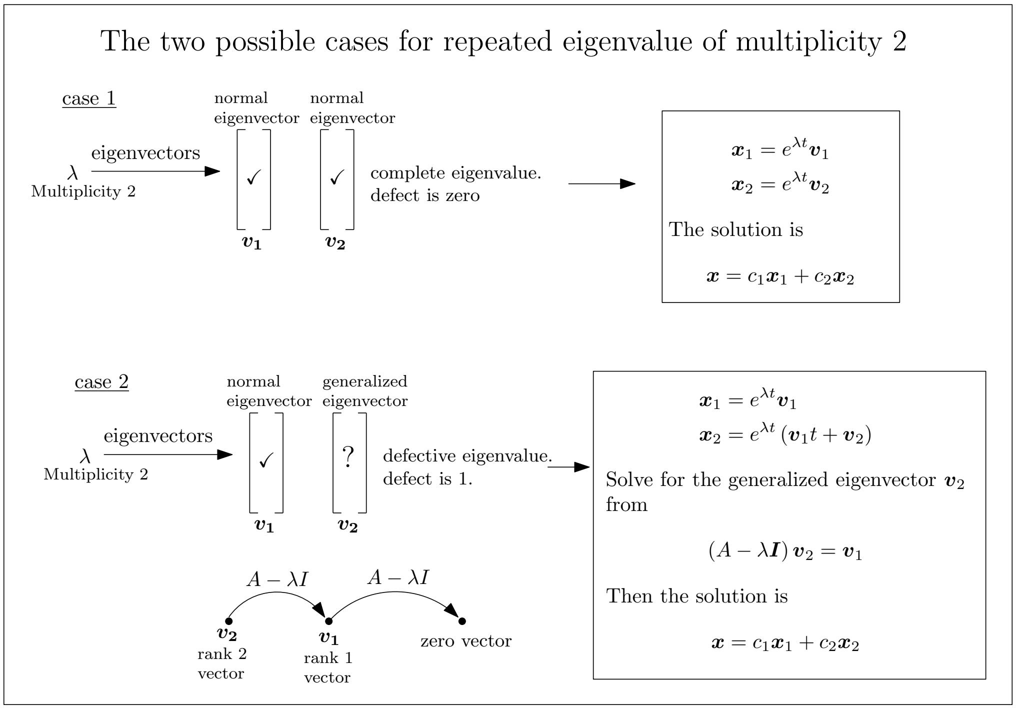


Figure 598: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 24 \\ -6 & 12 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{11}{6} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} \\
&= \begin{bmatrix} 2\,e^{5t} \\ e^{5t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{11}{6} \\ 1 \end{bmatrix} \right) e^{5t} \\
&= \begin{bmatrix} \frac{e^{5t}(12t+11)}{6} \\ e^{5t}(t+1) \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t}\left(2t + \frac{11}{6}\right) \\ e^{5t}(t+1) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{5t}\left(2c_1 + 2c_2 t + \frac{11}{6} c_2\right) \\ e^{5t}(c_2 t + c_1 + c_2) \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} y_1(0) = 3 \\ y_2(0) = 1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + \frac{11c_2}{6} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 7 \\ c_2 = -6 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (-12t + 3)\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t}(1 - 6t) \end{bmatrix}$$

The following is the phase plot of the system.

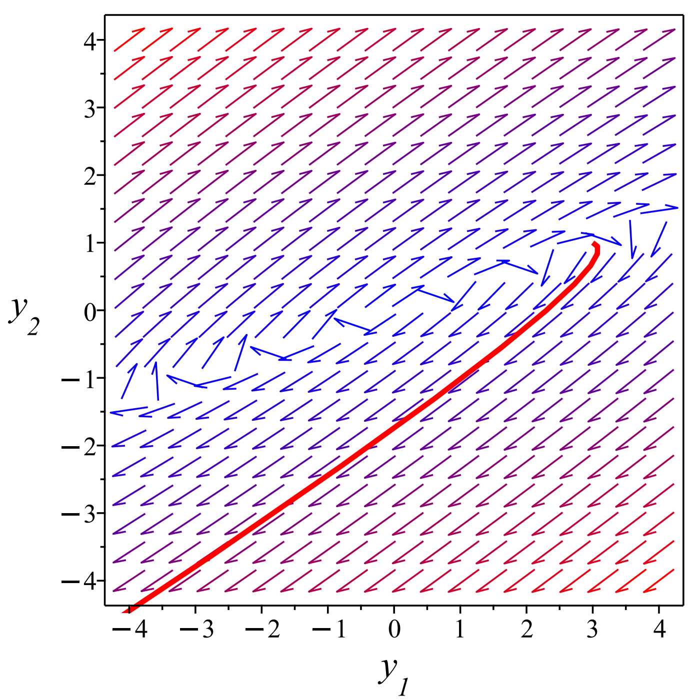
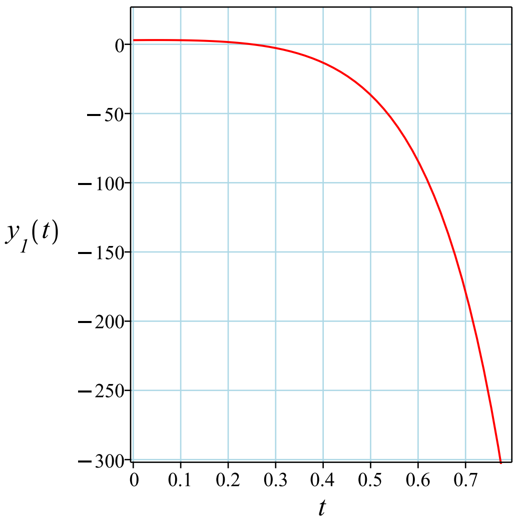
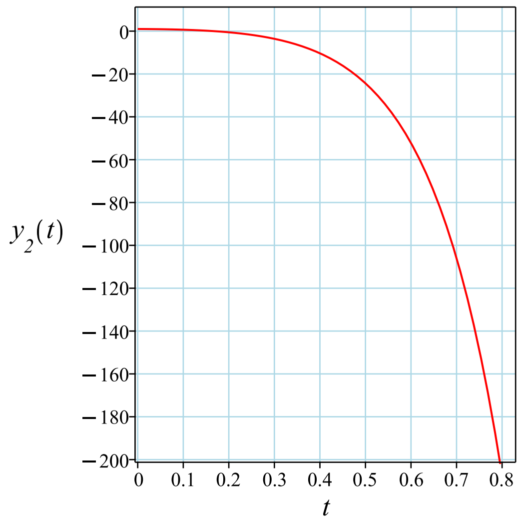


Figure 599: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(y__1(t),t) = -7*y__1(t)+24*y__2(t), diff(y__2(t),t) = -6*y__1(t)+17*y__2(t), y_
```

$$y_1(t) = \mathrm{e}^{5t}(-12t+3)$$
$$y_2(t) = \frac{\mathrm{e}^{5t}(-144t+24)}{24}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 31

```
DSolve[{y1'[t]==-7*y1[t]+24*y2[t],y2'[t]==-6*y1[t]+17*y2[t]},{y1[0]==3,y2[0]==1},{y1[t],y2[t
```

$$\mathrm{y1}(t) \to -3e^{5t}(4t-1)$$
$$\mathrm{y2}(t) \to e^{5t}(1-6t)$$

22.17 problem section 10.5, problem 17

Internal problem ID [1620]
Internal file name [`OUTPUT/1621_Sunday_June_05_2022_02_24_48_AM_26617334/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -7y_1(t) + 3y_2(t) \\
y_2'(t) &= -3y_1(t) - y_2(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = 0, y_2(0) = 2]
$$

22.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-4t}(1-3t) & 3t\,\mathrm{e}^{-4t} \\ -3t\,\mathrm{e}^{-4t} & \mathrm{e}^{-4t}(1+3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-4t}(1-3t) & 3t\,e^{-4t} \\ -3t\,e^{-4t} & e^{-4t}(1+3t) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 6t\,e^{-4t} \\ 2\,e^{-4t}(1+3t) \end{bmatrix} \\
&= \begin{bmatrix} 6t\,e^{-4t} \\ (6t+2)\,e^{-4t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 3 \\ -3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

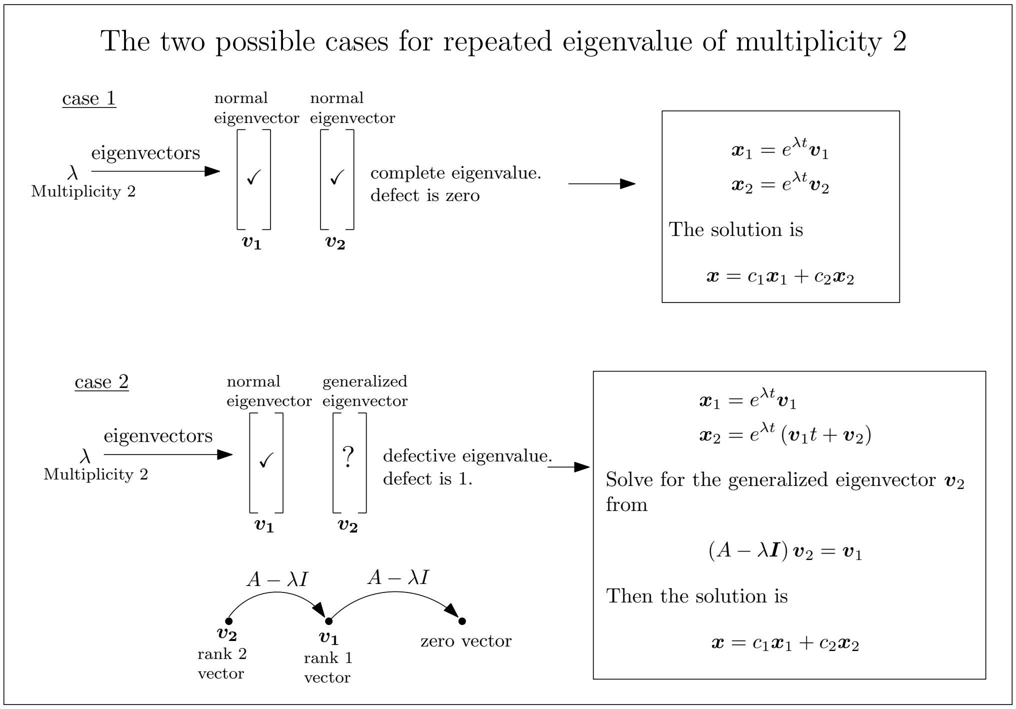


Figure 600: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{-4t} \\ &= \begin{bmatrix} \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right) \mathrm{e}^{-4t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-4t}(3t+2)}{3} \\ \mathrm{e}^{-4t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-4t}\left(t + \frac{2}{3}\right) \\ \mathrm{e}^{-4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-4t}\left(c_1 + c_2 t + \frac{2}{3} c_2\right) \\ \mathrm{e}^{-4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} y_1(0) = 0 \\ y_2(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + \frac{2c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -4 \\ c_2 = 6 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 6t\,\mathrm{e}^{-4t} \\ (6t + 2)\,\mathrm{e}^{-4t} \end{bmatrix}$$

The following is the phase plot of the system.

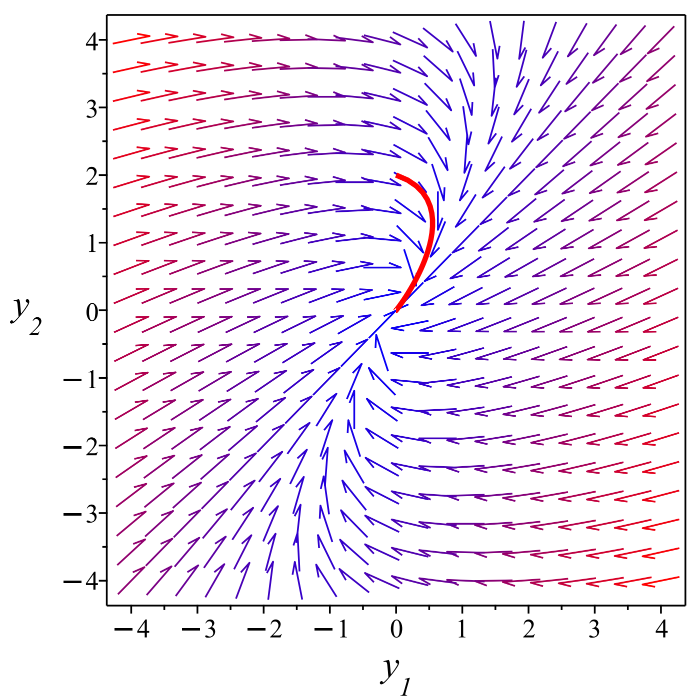
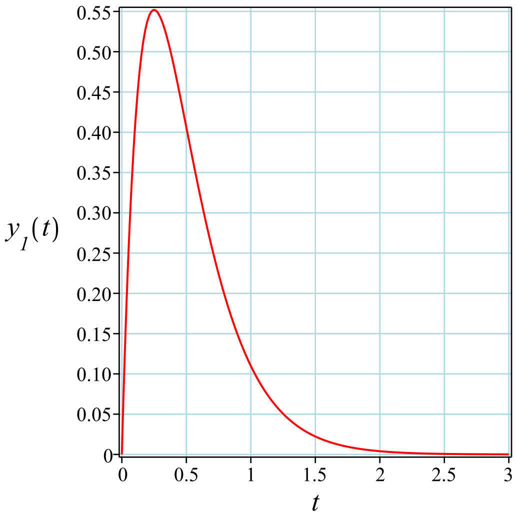
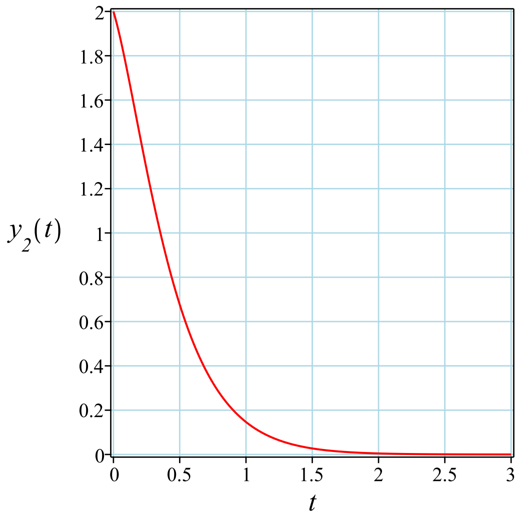


Figure 601: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve([diff(y__1(t),t) = -7*y__1(t)+3*y__2(t), diff(y__2(t),t) = -3*y__1(t)-y__2(t), y__1(0
```

$$y_1(t) = 6\,\mathrm{e}^{-4t}t$$
$$y_2(t) = \frac{\mathrm{e}^{-4t}(18t + 6)}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[{y1'[t]==-7*y1[t]+3*y2[t],y2'[t]==-3*y1[t]-1*y2[t]},{y1[0]==0,y2[0]==2},{y1[t],y2[t]}
```

$$\mathrm{y1}(t) \to 6e^{-4t}t$$
$$\mathrm{y2}(t) \to e^{-4t}(6t + 2)$$

22.18 problem section 10.5, problem 18

Internal problem ID [1621]
Internal file name [`OUTPUT/1622_Sunday_June_05_2022_02_24_50_AM_89919339/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_1(t) + y_2(t) \\
y_2'(t) &= y_1(t) - y_2(t) - 2y_3(t) \\
y_3'(t) &= -y_1(t) - y_2(t) - y_3(t)
\end{aligned}
$$

With initial conditions

$$[y_1(0) = 6, y_2(0) = 5, y_3(0) = -7]$$

22.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(2\,\mathrm{e}^{3t}+3t+7)\,\mathrm{e}^{-2t}}{9} & \frac{(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-3t-1)\,\mathrm{e}^{-2t}}{9} \\ \frac{(4\,\mathrm{e}^{3t}-3t-4)\,\mathrm{e}^{-2t}}{9} & \frac{(4\,\mathrm{e}^{3t}-3t+5)\,\mathrm{e}^{-2t}}{9} & -\frac{2(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} \\ -\frac{(\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t}}{3} & -\frac{(\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t}}{3} & \frac{(\mathrm{e}^{3t}+2)\,\mathrm{e}^{-2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{(2\,\mathrm{e}^{3t}+3t+7)\,\mathrm{e}^{-2t}}{9} & \frac{(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-3t-1)\,\mathrm{e}^{-2t}}{9} \\ \frac{(4\,\mathrm{e}^{3t}-3t-4)\,\mathrm{e}^{-2t}}{9} & \frac{(4\,\mathrm{e}^{3t}-3t+5)\,\mathrm{e}^{-2t}}{9} & -\frac{2(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} \\ -\frac{(\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t}}{3} & -\frac{(\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t}}{3} & \frac{(\mathrm{e}^{3t}+2)\,\mathrm{e}^{-2t}}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(2\,\mathrm{e}^{3t}+3t+7)\,\mathrm{e}^{-2t}}{3} + \frac{5(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} + \frac{14(\mathrm{e}^{3t}-3t-1)\,\mathrm{e}^{-2t}}{9} \\ \frac{2(4\,\mathrm{e}^{3t}-3t-4)\,\mathrm{e}^{-2t}}{3} + \frac{5(4\,\mathrm{e}^{3t}-3t+5)\,\mathrm{e}^{-2t}}{9} + \frac{14(2\,\mathrm{e}^{3t}+3t-2)\,\mathrm{e}^{-2t}}{9} \\ -\frac{11(\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t}}{3} - \frac{7(\mathrm{e}^{3t}+2)\,\mathrm{e}^{-2t}}{3} \end{bmatrix} \\ &= \begin{bmatrix} -(-4\,\mathrm{e}^{3t}+t-2)\,\mathrm{e}^{-2t} \\ (8\,\mathrm{e}^{3t}+t-3)\,\mathrm{e}^{-2t} \\ (-6\,\mathrm{e}^{3t}-1)\,\mathrm{e}^{-2t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & -2 \\ -1 & -1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & -2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -2 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -2 & 0 \\ 0 & -\frac{3}{2} & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}, v_2 = -\frac{4t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	2	1	Yes	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$
1	1	1	No	$\begin{bmatrix}-\frac{2}{3}\\-\frac{4}{3}\\1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

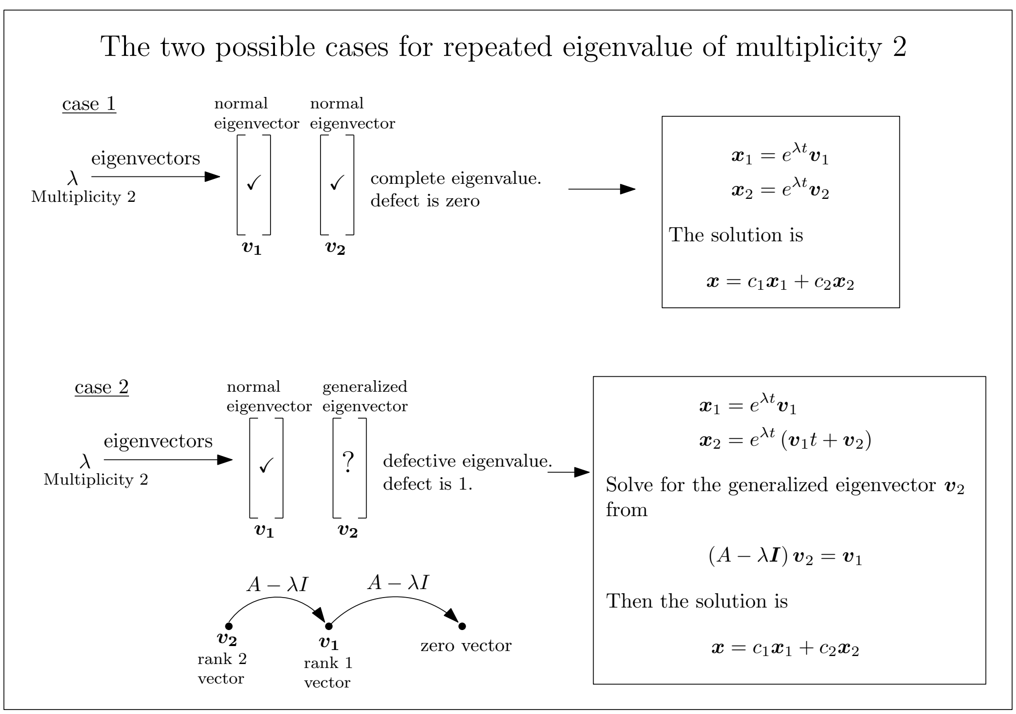


Figure 602: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue -2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-2t} \\ &= \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right] t + \left[\begin{array}{c} -2 \\ 1 \\ -1 \end{array}\right] \right) \mathrm{e}^{-2t} \\
&= \left[\begin{array}{c} -\mathrm{e}^{-2t}(2+t) \\ \mathrm{e}^{-2t}(t+1) \\ -\mathrm{e}^{-2t} \end{array}\right]
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{t} \\
&= \left[\begin{array}{c} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{array}\right] e^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \\ 0 \end{array}\right] + c_2 \left[\begin{array}{c} \mathrm{e}^{-2t}(-t-2) \\ \mathrm{e}^{-2t}(t+1) \\ -\mathrm{e}^{-2t} \end{array}\right] + c_3 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{t}}{3} \\ -\frac{4\,\mathrm{e}^{t}}{3} \\ \mathrm{e}^{t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right] = \left[\begin{array}{c} \frac{(-2c_3\mathrm{e}^{3t} + (-3t-6)c_2 - 3c_1)\mathrm{e}^{-2t}}{3} \\ \frac{(-4c_3\mathrm{e}^{3t} + (3t+3)c_2 + 3c_1)\mathrm{e}^{-2t}}{3} \\ -(-c_3\mathrm{e}^{3t} + c_2)\,\mathrm{e}^{-2t} \end{array}\right]
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\left[\begin{array}{c} y_1(0) = 6 \\ y_2(0) = 5 \\ y_3(0) = -7 \end{array}\right] \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{2c_3}{3} - 2c_2 - c_1 \\ -\frac{4c_3}{3} + c_2 + c_1 \\ c_3 - c_2 \end{bmatrix}$$

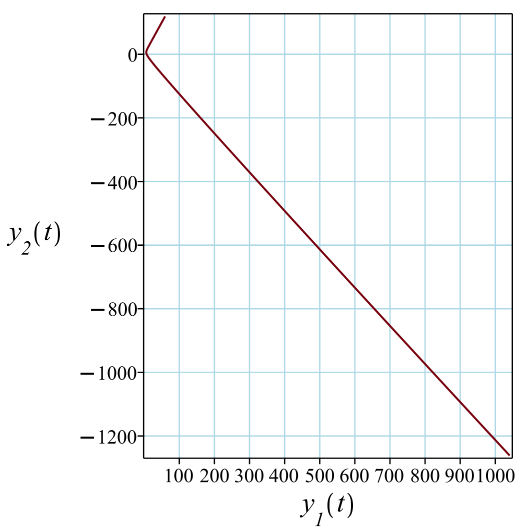
Solving for the constants of integrations gives

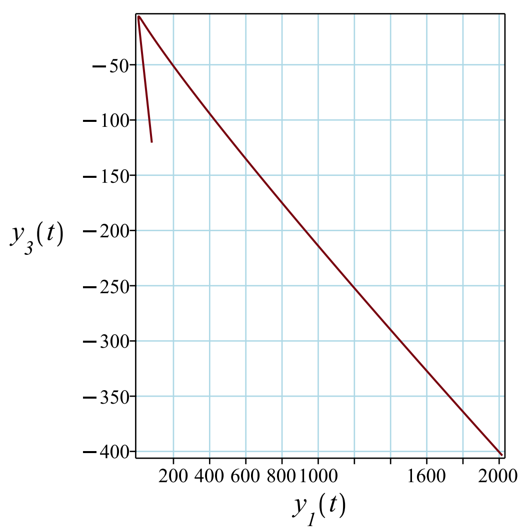
$$\begin{bmatrix} c_1 = -4 \\ c_2 = 1 \\ c_3 = -6 \end{bmatrix}$$

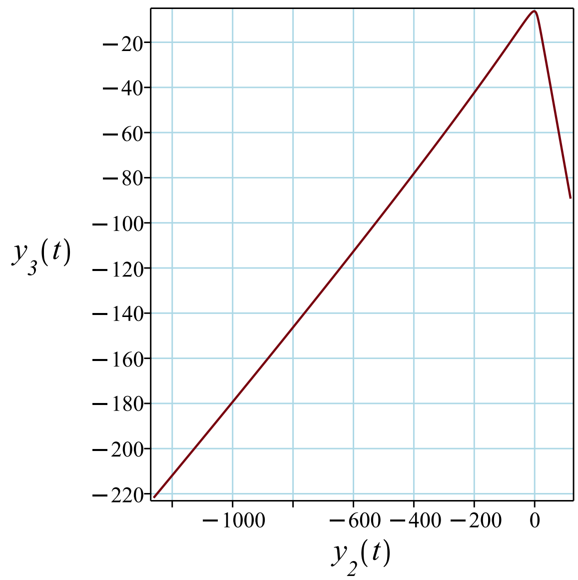
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{(12\,e^{3t}-3t+6)\,e^{-2t}}{3} \\ \frac{(24\,e^{3t}+3t-9)\,e^{-2t}}{3} \\ -(6\,e^{3t}+1)\,e^{-2t} \end{bmatrix}$$

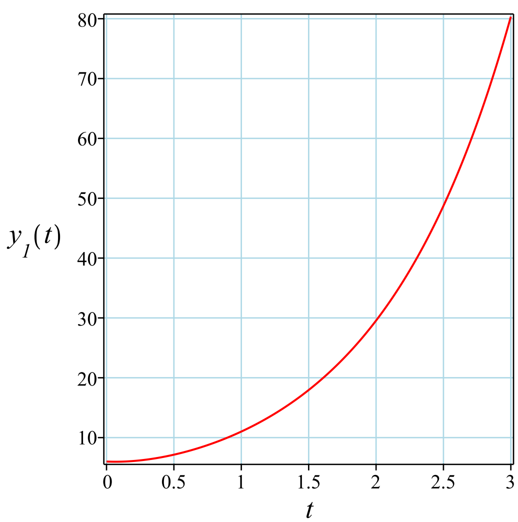
The following are plots of each solution against another.

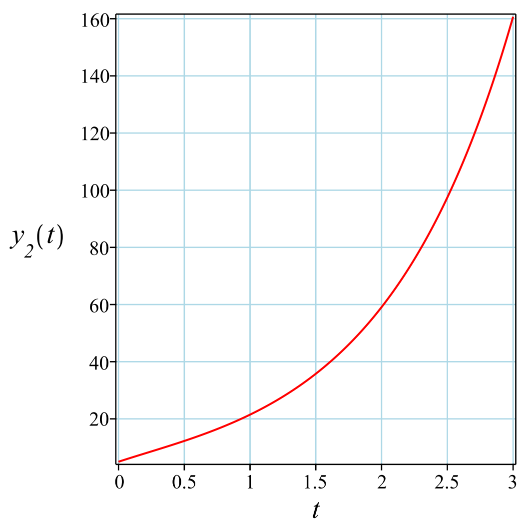


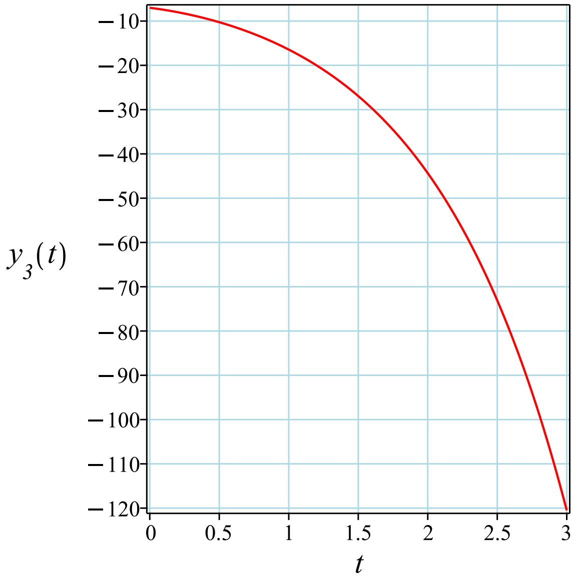




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 57

```
dsolve([diff(y__1(t),t) = -y__1(t)+y__2(t), diff(y__2(t),t) = y__1(t)-y__2(t)-2*y__3(t), dif
```

$$y_1(t) = 4\,\mathrm{e}^{t} + 2\,\mathrm{e}^{-2t} - \mathrm{e}^{-2t}t$$
$$y_2(t) = 8\,\mathrm{e}^{t} - 3\,\mathrm{e}^{-2t} + \mathrm{e}^{-2t}t$$
$$y_3(t) = -6\,\mathrm{e}^{t} - \mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 58

```
DSolve[{y1'[t]==-1*y1[t]+1*y2[t]+0*y3[t],y2'[t]==1*y1[t]-1*y2[t]-2*y3[t],y3'[t]==-1*y1[t]-1*
```

$$\mathrm{y1}(t) \to e^{-2t}\left(-t + 4e^{3t} + 2\right)$$
$$\mathrm{y2}(t) \to e^{-2t}\left(t + 8e^{3t} - 3\right)$$
$$\mathrm{y3}(t) \to -e^{-2t} - 6e^{t}$$

22.19 problem section 10.5, problem 19

Internal problem ID [1622]
Internal file name [`OUTPUT/1623_Sunday_June_05_2022_02_24_51_AM_64767384/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -2y_1(t) + 2y_2(t) + y_3(t)\\
y_2'(t) &= -2y_1(t) + 2y_2(t) + y_3(t)\\
y_3'(t) &= -3y_1(t) + 3y_2(t) + 2y_3(t)
\end{aligned}
$$

With initial conditions

$$[y_1(0) = -6, y_2(0) = -2, y_3(0) = 0]$$

22.19.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{7}{4}-\frac{t}{2}-\frac{3\,\mathrm{e}^{2t}}{4} & \frac{t}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{3}{4} & \frac{\mathrm{e}^{2t}}{2}-\frac{1}{2} \\ -\frac{t}{2}-\frac{3\,\mathrm{e}^{2t}}{4}+\frac{3}{4} & \frac{1}{4}+\frac{t}{2}+\frac{3\,\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{2}-\frac{1}{2} \\ -\frac{3\,\mathrm{e}^{2t}}{2}+\frac{3}{2} & \frac{3\,\mathrm{e}^{2t}}{2}-\frac{3}{2} & \mathrm{e}^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{7}{4}-\frac{t}{2}-\frac{3\,\mathrm{e}^{2t}}{4} & \frac{t}{2}+\frac{3\,\mathrm{e}^{2t}}{4}-\frac{3}{4} & \frac{\mathrm{e}^{2t}}{2}-\frac{1}{2} \\ -\frac{t}{2}-\frac{3\,\mathrm{e}^{2t}}{4}+\frac{3}{4} & \frac{1}{4}+\frac{t}{2}+\frac{3\,\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{2}-\frac{1}{2} \\ -\frac{3\,\mathrm{e}^{2t}}{2}+\frac{3}{2} & \frac{3\,\mathrm{e}^{2t}}{2}-\frac{3}{2} & \mathrm{e}^{2t} \end{bmatrix} \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -9+2t+3\,\mathrm{e}^{2t} \\ 2t+3\,\mathrm{e}^{2t}-5 \\ 6\,\mathrm{e}^{2t}-6 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.19.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 2 & 1 \\ -2 & 2-\lambda & 1 \\ -3 & 3 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ -2 & 2 & 1 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & 1 \\ -2 & 0 & 1 \\ -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ -3 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{4} \implies \left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{2} \implies \left[\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 2 & 1 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{2}, v_2 = \frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	1	Yes	$\begin{bmatrix}1\\1\\0\end{bmatrix}$
2	1	1	No	$\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

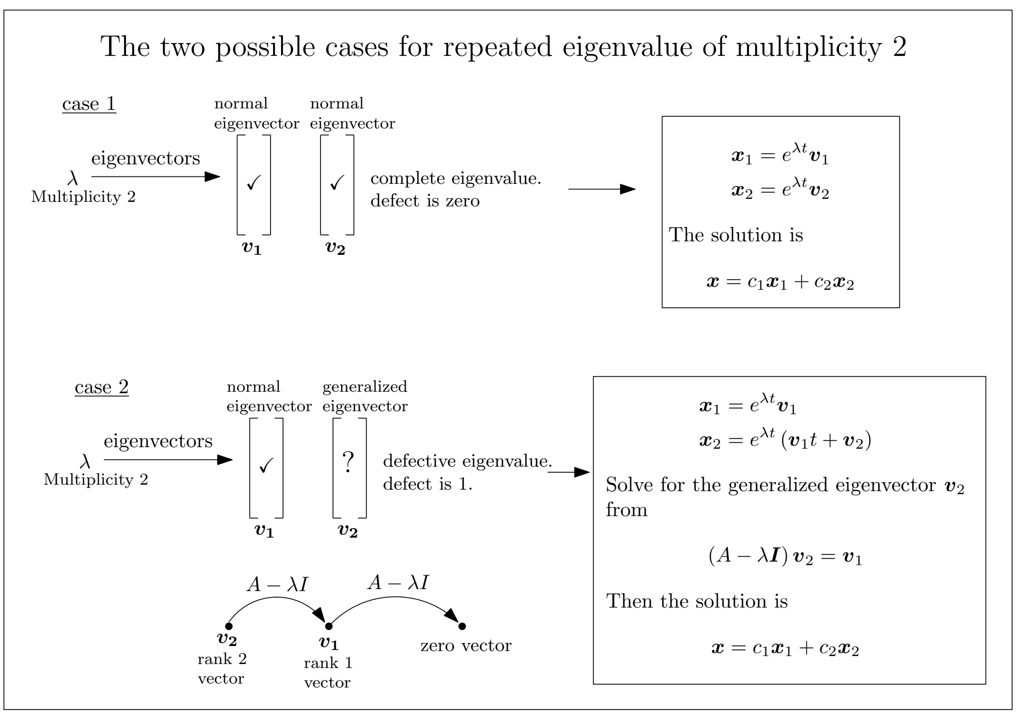


Figure 603: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} 1 \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right) 1 \\
&= \begin{bmatrix} -1+t \\ t+1 \\ -3 \end{bmatrix}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1+t \\ t+1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^{2t}}{2} \\ \frac{e^{2t}}{2} \\ e^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_1 + c_2(-1+t) + \frac{c_3 e^{2t}}{2} \\ c_1 + c_2 t + c_2 + \frac{c_3 e^{2t}}{2} \\ -3c_2 + c_3 e^{2t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} y_1(0) = -6 \\ y_2(0) = -2 \\ y_3(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + \frac{c_3}{2} \\ c_1 + c_2 + \frac{c_3}{2} \\ -3c_2 + c_3 \end{bmatrix}$$

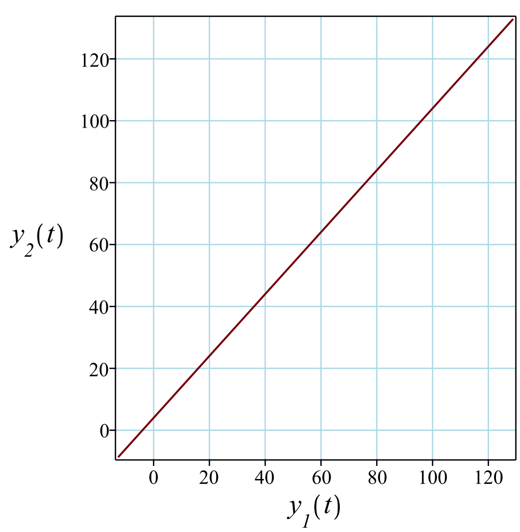
Solving for the constants of integrations gives

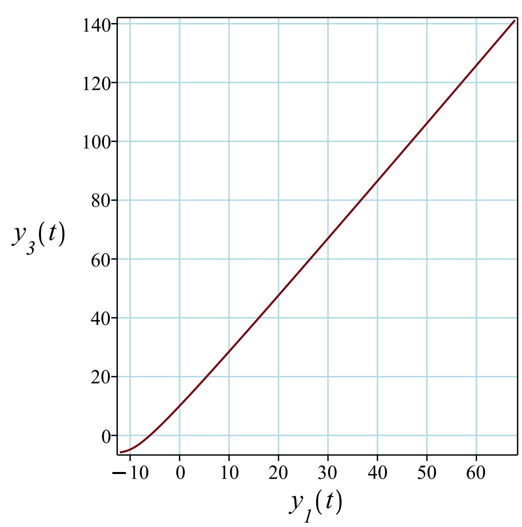
$$\begin{bmatrix} c_1 = -7 \\ c_2 = 2 \\ c_3 = 6 \end{bmatrix}$$

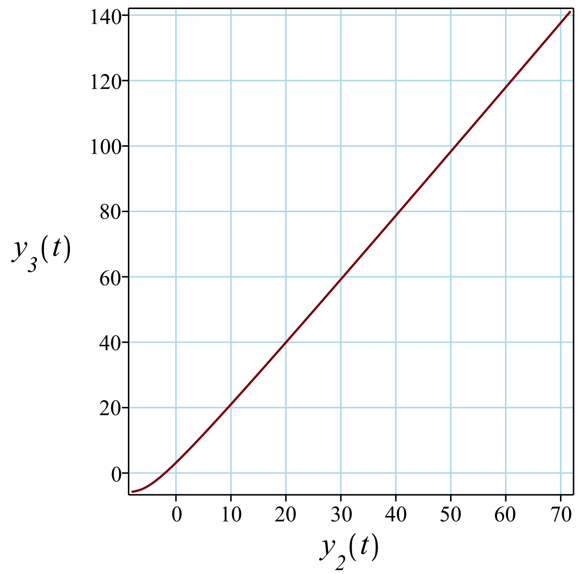
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -9 + 2t + 3\,\mathrm{e}^{2t} \\ 2t + 3\,\mathrm{e}^{2t} - 5 \\ 6\,\mathrm{e}^{2t} - 6 \end{bmatrix}$$

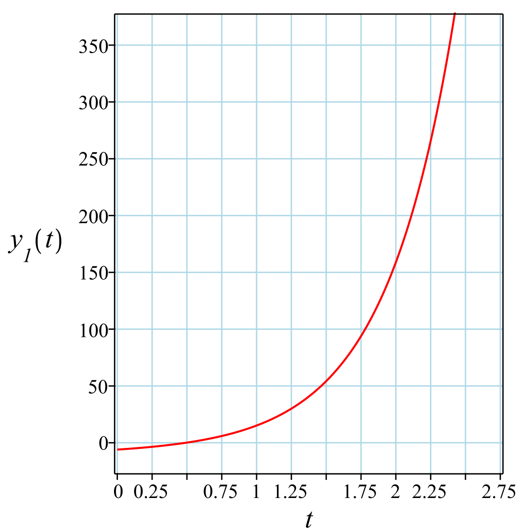
The following are plots of each solution against another.

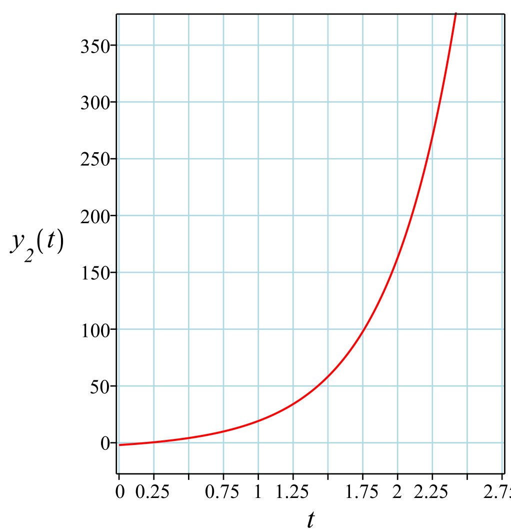


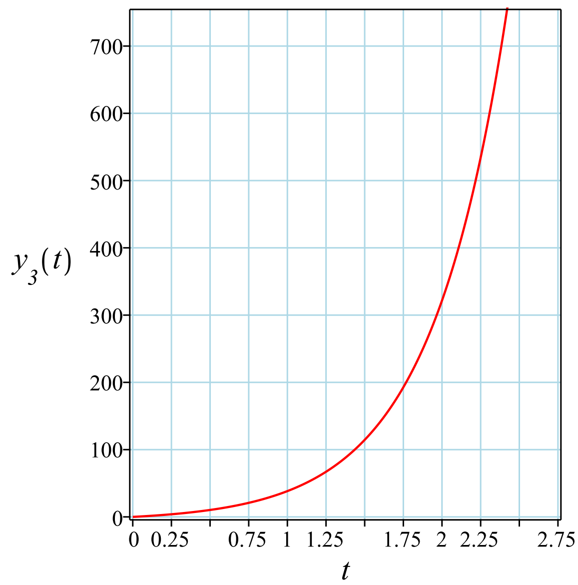




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve([diff(y__1(t),t) = -2*y__1(t)+2*y__2(t)+y__3(t), diff(y__2(t),t) = -2*y__1(t)+2*y__2(
```

$$y_1(t) = -9 + 2t + 3\,\mathrm{e}^{2t}$$
$$y_2(t) = 3\,\mathrm{e}^{2t} - 5 + 2t$$
$$y_3(t) = -6 + 6\,\mathrm{e}^{2t}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 44

```
DSolve[{y1'[t]==-2*y1[t]+2*y2[t]+1*y3[t],y2'[t]==-2*y1[t]+2*y2[t]+1*y3[t],y3'[t]==-3*y1[t]+3
```

$$\mathrm{y1}(t) \to 2t + 3e^{2t} - 9$$
$$\mathrm{y2}(t) \to 2t + 3e^{2t} - 5$$
$$\mathrm{y3}(t) \to 6\left(e^{2t} - 1\right)$$

22.20 problem section 10.5, problem 20

Internal problem ID [1623]
Internal file name [`OUTPUT/1624_Sunday_June_05_2022_02_24_53_AM_17576291/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -7y_1(t) - 4y_2(t) + 4y_3(t) \\
y_2'(t) &= y_1(t) + y_3(t) \\
y_3'(t) &= -9y_1(t) - 5y_2(t) + 6y_3(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = -6, y_2(0) = 9, y_3(0) = -1]
$$

22.20.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(2t) - 3e^{-t}\sin(2t) & -2e^{-t}\sin(2t) & 2e^{-t}\sin(2t) \\ \frac{7e^{-t}\cos(2t)}{4} + \frac{9e^{-t}\sin(2t)}{4} - \frac{7e^{t}}{4} & -\frac{e^{t}}{2} + \frac{3e^{-t}\cos(2t)}{2} + e^{-t}\sin(2t) & -\frac{3e^{-t}\cos(2t)}{2} - e^{-t}\sin(2t) + \frac{3e^{t}}{2} \\ \frac{7e^{-t}\cos(2t)}{4} - \frac{11e^{-t}\sin(2t)}{4} - \frac{7e^{t}}{4} & \frac{e^{-t}\cos(2t)}{2} - 2e^{-t}\sin(2t) - \frac{e^{t}}{2} & \frac{3e^{t}}{2} - \frac{e^{-t}\cos(2t)}{2} + 2e^{-t}\sin(2t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\cos(2t) - 3\sin(2t)) & -2e^{-t}\sin(2t) & 2e^{-t}\sin(2t) \\ \frac{(7\cos(2t)+9\sin(2t))e^{-t}}{4} - \frac{7e^{t}}{4} & \frac{(3\cos(2t)+2\sin(2t))e^{-t}}{2} - \frac{e^{t}}{2} & \frac{(-3\cos(2t)-2\sin(2t))e^{-t}}{2} + \frac{3e^{t}}{2} \\ \frac{(7\cos(2t)-11\sin(2t))e^{-t}}{4} - \frac{7e^{t}}{4} & \frac{(\cos(2t)-4\sin(2t))e^{-t}}{2} - \frac{e^{t}}{2} & \frac{(-\cos(2t)+4\sin(2t))e^{-t}}{2} + \frac{3e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} e^{-t}(\cos(2t) - 3\sin(2t)) & -2e^{-t}\sin(2t) & 2e^{-t}\sin(2t) \\ \frac{(7\cos(2t)+9\sin(2t))e^{-t}}{4} - \frac{7e^{t}}{4} & \frac{(3\cos(2t)+2\sin(2t))e^{-t}}{2} - \frac{e^{t}}{2} & \frac{(-3\cos(2t)-2\sin(2t))e^{-t}}{2} + \frac{3e^{t}}{2} \\ \frac{(7\cos(2t)-11\sin(2t))e^{-t}}{4} - \frac{7e^{t}}{4} & \frac{(\cos(2t)-4\sin(2t))e^{-t}}{2} - \frac{e^{t}}{2} & \frac{(-\cos(2t)+4\sin(2t))e^{-t}}{2} + \frac{3e^{t}}{2} \end{bmatrix} \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -6e^{-t}(\cos(2t) - 3\sin(2t)) - 20e^{-t}\sin(2t) \\ -\frac{3(7\cos(2t)+9\sin(2t))e^{-t}}{2} + \frac{9e^{t}}{2} + \frac{9(3\cos(2t)+2\sin(2t))e^{-t}}{2} - \frac{(-3\cos(2t)-2\sin(2t))e^{-t}}{2} \\ -\frac{3(7\cos(2t)-11\sin(2t))e^{-t}}{2} + \frac{9e^{t}}{2} + \frac{9(\cos(2t)-4\sin(2t))e^{-t}}{2} - \frac{(-\cos(2t)+4\sin(2t))e^{-t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-t}(3\cos(2t) + \sin(2t)) \\ \frac{(9\cos(2t)-7\sin(2t))e^{-t}}{2} + \frac{9e^{t}}{2} \\ \frac{(-11\cos(2t)-7\sin(2t))e^{-t}}{2} + \frac{9e^{t}}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.20.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & -4 & 4 \\ 1 & -\lambda & 1 \\ -9 & -5 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 + 3\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 + 2i \\ \lambda_3 &= -1 - 2i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - 2i$	1	complex eigenvalue
1	1	real eigenvalue
$-1 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -4 & 4 \\ 1 & -1 & 1 \\ -9 & -5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & -4 & 4 & 0 \\ 1 & -1 & 1 & 0 \\ -9 & -5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & 4 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ -9 & -5 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{9R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & 4 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & 4 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & -4 & 4 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} - (-1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 + 2i & -4 & 4 \\ 1 & 1 + 2i & 1 \\ -9 & -5 & 7 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -6 + 2i & -4 & 4 & 0 \\ 1 & 1 + 2i & 1 & 0 \\ -9 & -5 & 7 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{20} + \frac{i}{20}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6+2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}+\frac{9i}{5} & \frac{8}{5}+\frac{i}{5} & 0 \\ -9 & -5 & 7+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{27}{20} - \frac{9i}{20}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6+2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}+\frac{9i}{5} & \frac{8}{5}+\frac{i}{5} & 0 \\ 0 & \frac{2}{5}+\frac{9i}{5} & \frac{8}{5}+\frac{i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -6+2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}+\frac{9i}{5} & \frac{8}{5}+\frac{i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6+2i & -4 & 4 \\ 0 & \frac{2}{5}+\frac{9i}{5} & \frac{8}{5}+\frac{i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{16}{17} - \frac{4i}{17}\right) t, v_2 = -\frac{5}{17}t + \frac{14}{17}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{16}{17} - \frac{4\,\mathrm{I}}{17}\right) t \\ -\frac{5t}{17} + \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{16}{17} - \frac{4i}{17}\right) t \\ -\frac{5}{17}t + \frac{14}{17}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{16}{17} - \frac{4\,\mathrm{I}}{17}\right) t \\ -\frac{5t}{17} + \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{16}{17} - \frac{4i}{17} \\ \frac{-\frac{5}{17}t+\frac{14}{17}it}{t} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{16}{17}-\frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17}+\frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} \frac{16}{17}-\frac{4i}{17} \\ -\frac{5}{17}+\frac{14i}{17} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{16}{17}-\frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17}+\frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} 16-4i \\ -5+14i \\ 17 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3=-1+2i$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & -4 & 4 \\ 1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} - (-1+2i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6-2i & -4 & 4 \\ 1 & 1-2i & 1 \\ -9 & -5 & 7-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -6-2i & -4 & 4 & 0 \\ 1 & 1-2i & 1 & 0 \\ -9 & -5 & 7-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{20}-\frac{i}{20}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6-2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}-\frac{9i}{5} & \frac{8}{5}-\frac{i}{5} & 0 \\ -9 & -5 & 7-2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{27}{20}+\frac{9i}{20}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6-2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}-\frac{9i}{5} & \frac{8}{5}-\frac{i}{5} & 0 \\ 0 & \frac{2}{5}-\frac{9i}{5} & \frac{8}{5}-\frac{i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -6-2i & -4 & 4 & 0 \\ 0 & \frac{2}{5}-\frac{9i}{5} & \frac{8}{5}-\frac{i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6-2i & -4 & 4 \\ 0 & \frac{2}{5}-\frac{9i}{5} & \frac{8}{5}-\frac{i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{16}{17} + \frac{4i}{17}\right)t, v_2 = -\frac{5}{17}t - \frac{14}{17}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{16}{17} + \frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17} - \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{16}{17} + \frac{4i}{17}\right)t \\ -\frac{5}{17}t - \frac{14}{17}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{16}{17} + \frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17} - \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{16}{17} + \frac{4i}{17} \\ \frac{-\frac{5}{17}t - \frac{14}{17}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{16}{17} + \frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17} - \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} \frac{16}{17} + \frac{4i}{17} \\ -\frac{5}{17} - \frac{14i}{17} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{16}{17} + \frac{4\,\mathrm{I}}{17}\right)t \\ -\frac{5t}{17} - \frac{14\,\mathrm{I}\,t}{17} \\ t \end{bmatrix} = \begin{bmatrix} 16+4i \\ -5-14i \\ 17 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
$-1+2i$	1	1	No	$\begin{bmatrix} \frac{16}{17}+\frac{4i}{17} \\ -\frac{5}{17}-\frac{14i}{17} \\ 1 \end{bmatrix}$
$-1-2i$	1	1	No	$\begin{bmatrix} \frac{16}{17}-\frac{4i}{17} \\ -\frac{5}{17}+\frac{14i}{17} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{16}{17} + \frac{4i}{17}\right) e^{(-1+2i)t} \\ \left(-\frac{5}{17} - \frac{14i}{17}\right) e^{(-1+2i)t} \\ e^{(-1+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \left(\frac{16}{17} - \frac{4i}{17}\right) e^{(-1-2i)t} \\ \left(-\frac{5}{17} + \frac{14i}{17}\right) e^{(-1-2i)t} \\ e^{(-1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{16}{17} + \frac{4i}{17}\right) c_2 e^{(-1+2i)t} + \left(\frac{16}{17} - \frac{4i}{17}\right) c_3 e^{(-1-2i)t} \\ c_1 e^t + \left(-\frac{5}{17} - \frac{14i}{17}\right) c_2 e^{(-1+2i)t} + \left(-\frac{5}{17} + \frac{14i}{17}\right) c_3 e^{(-1-2i)t} \\ c_1 e^t + c_2 e^{(-1+2i)t} + c_3 e^{(-1-2i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} y_1(0) = -6 \\ y_2(0) = 9 \\ y_3(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} \left(\frac{16}{17} + \frac{4i}{17}\right) c_2 + \left(\frac{16}{17} - \frac{4i}{17}\right) c_3 \\ \left(-\frac{5}{17} - \frac{14i}{17}\right) c_2 + \left(-\frac{5}{17} + \frac{14i}{17}\right) c_3 + c_1 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

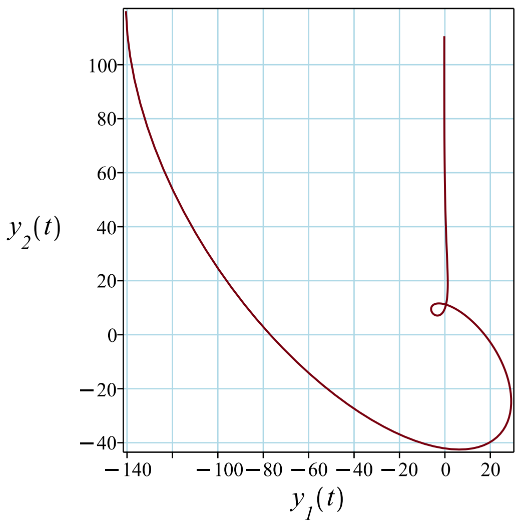
Solving for the constants of integrations gives

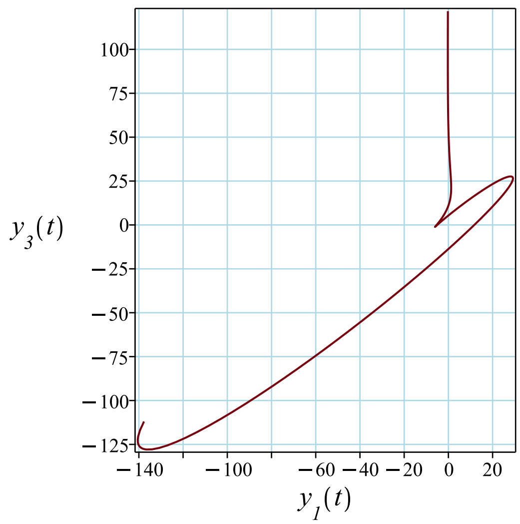
$$\begin{bmatrix} c_1 = \frac{9}{2} \\ c_2 = -\frac{11}{4} + \frac{7i}{4} \\ c_3 = -\frac{11}{4} - \frac{7i}{4} \end{bmatrix}$$

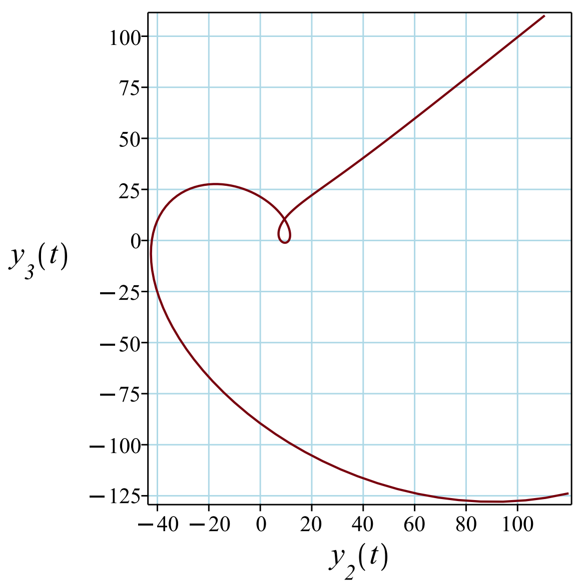
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (-3+i) e^{(-1+2i)t} + (-3-i) e^{(-1-2i)t} \\ \frac{9 e^t}{2} + \left(\frac{9}{4} + \frac{7i}{4}\right) e^{(-1+2i)t} + \left(\frac{9}{4} - \frac{7i}{4}\right) e^{(-1-2i)t} \\ \frac{9 e^t}{2} + \left(-\frac{11}{4} + \frac{7i}{4}\right) e^{(-1+2i)t} + \left(-\frac{11}{4} - \frac{7i}{4}\right) e^{(-1-2i)t} \end{bmatrix}$$

The following are plots of each solution against another.







The following are plots of each solution.

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 80

```
dsolve([diff(y__1(t),t) = -7*y__1(t)-4*y__2(t)+4*y__3(t), diff(y__2(t),t) = y__1(t)+y__3(t),
```

$$y_1(t) = -\frac{4\,e^{-t}\left(\frac{13\sin(2t)}{2}+\frac{39\cos(2t)}{2}\right)}{13}$$
$$y_2(t) = \frac{9\,e^{t}}{2}-\frac{7\,e^{-t}\sin(2t)}{2}+\frac{9\,e^{-t}\cos(2t)}{2}$$
$$y_3(t) = \frac{9\,e^{t}}{2}-\frac{7\,e^{-t}\sin(2t)}{2}-\frac{11\,e^{-t}\cos(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 50

```
DSolve[{y1'[t]==-7*y1[t]-4*y2[t]+4*y3[t],y2'[t]==-1*y1[t]-0*y2[t]+1*y3[t],y3'[t]==-9*y1[t]-5
```

$$\text{y1}(t) \to -2e^{-3t}-4e^{t}$$
$$\text{y2}(t) \to e^{t}(9-4t)$$
$$\text{y3}(t) \to e^{t}(1-4t)-2e^{-3t}$$

22.21 problem section 10.5, problem 21

Internal problem ID [1624]
Internal file name [OUTPUT/1625_Sunday_June_05_2022_02_24_56_AM_17935540/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_1(t) - 4y_2(t) - y_3(t) \\
y_2'(t) &= 3y_1(t) + 6y_2(t) + y_3(t) \\
y_3'(t) &= -3y_1(t) - 2y_2(t) + 3y_3(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = -2, y_2(0) = 1, y_3(0) = 3]
$$

22.21.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t}(1-3t) & -3\,e^{2t}t + \frac{e^{2t}}{2} - \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ 3\,e^{2t}t & \frac{e^{2t}}{2} + 3\,e^{2t}t + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ -3\,e^{2t}t & -3\,e^{2t}t - \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} e^{2t}(1-3t) & -3\,e^{2t}t + \frac{e^{2t}}{2} - \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ 3\,e^{2t}t & \frac{e^{2t}}{2} + 3\,e^{2t}t + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ -3\,e^{2t}t & -3\,e^{2t}t - \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -2\,e^{2t}(1-3t) - 3\,e^{2t}t + 2\,e^{2t} - 2\,e^{4t} \\ -3\,e^{2t}t - e^{2t} + 2\,e^{4t} \\ 3\,e^{2t}t + e^{2t} + 2\,e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} 3\,e^{2t}t - 2\,e^{4t} \\ (-3t-1)\,e^{2t} + 2\,e^{4t} \\ 3\,e^{2t}t + e^{2t} + 2\,e^{4t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.21.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -4 & -1 \\ 3 & 6-\lambda & 1 \\ -3 & -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -1 \\ 3 & 4 & 1 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -4 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -2 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -3 & -4 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -4 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -4 & -1 \\ 3 & 2 & 1 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & -4 & -1 & 0 \\ 3 & 2 & 1 & 0 \\ -3 & -2 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -4 & -1 & 0 \\ 0 & -\frac{2}{5} & \frac{2}{5} & 0 \\ -3 & -2 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -4 & -1 & 0 \\ 0 & -\frac{2}{5} & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & -\frac{2}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -5 & -4 & -1 & 0 \\ 0 & -\frac{2}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -4 & -1 \\ 0 & -\frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	2	1	Yes	$\left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$
4	1	1	No	$\left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

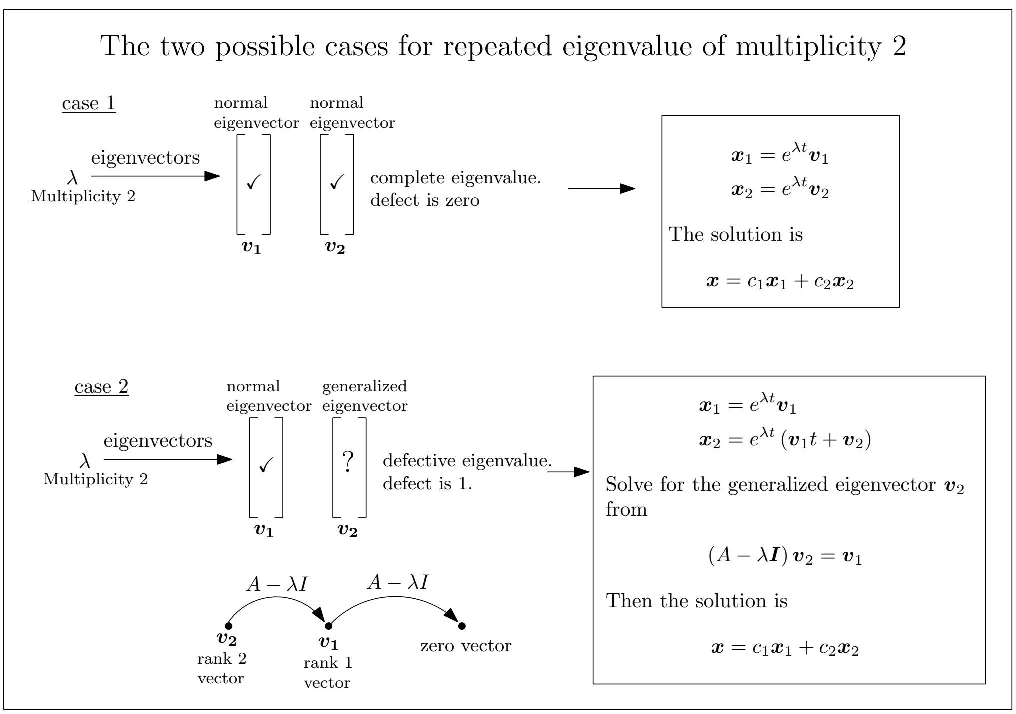


Figure 604: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore

this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -4 & -1 \\ 3 & 4 & 1 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{4}{3} \\ 1 \\ -1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 2. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\ &= \begin{bmatrix} \mathrm{e}^{2t} \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{4}{3} \\ 1 \\ -1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(3t-4)}{3} \\ -\mathrm{e}^{2t}(-1+t) \\ \mathrm{e}^{2t}(-1+t) \end{bmatrix}
\end{aligned}
$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{4t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{4t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{2t}\left(t - \frac{4}{3}\right) \\ \mathrm{e}^{2t}(1-t) \\ \mathrm{e}^{2t}(-1+t) \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((3t-4)c_2 + 3c_1)\mathrm{e}^{2t}}{3} - c_3 \mathrm{e}^{4t} \\ ((1-t)\, c_2 - c_1)\, \mathrm{e}^{2t} + c_3 \mathrm{e}^{4t} \\ ((-1+t)\, c_2 + c_1)\, \mathrm{e}^{2t} + c_3 \mathrm{e}^{4t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} y_1(0) = -2 \\ y_2(0) = 1 \\ y_3(0) = 3 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{4c_2}{3} + c_1 - c_3 \\ c_2 - c_1 + c_3 \\ -c_2 + c_1 + c_3 \end{bmatrix}$$

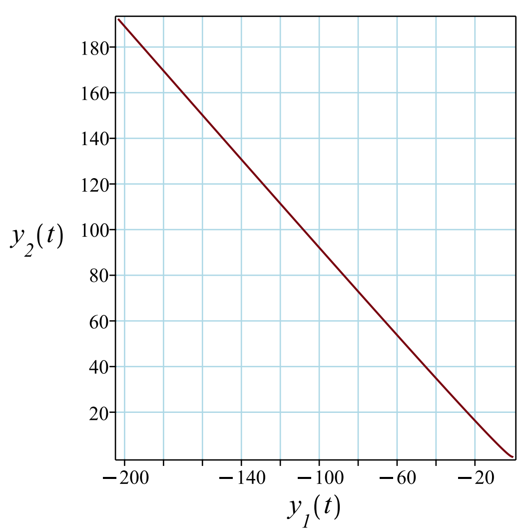
Solving for the constants of integrations gives

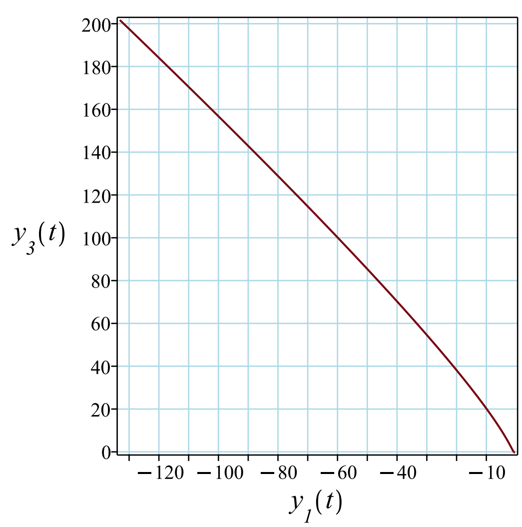
$$\begin{bmatrix} c_1 = 4 \\ c_2 = 3 \\ c_3 = 2 \end{bmatrix}$$

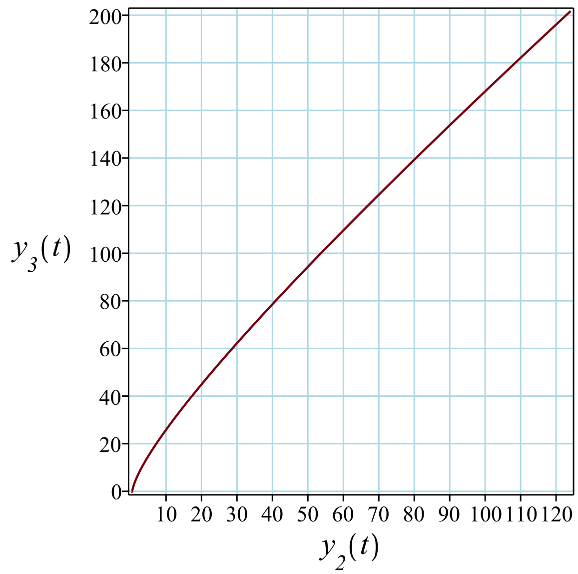
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 3\,e^{2t}t - 2\,e^{4t} \\ (-3t-1)\,e^{2t} + 2\,e^{4t} \\ e^{2t}(1+3t) + 2\,e^{4t} \end{bmatrix}$$

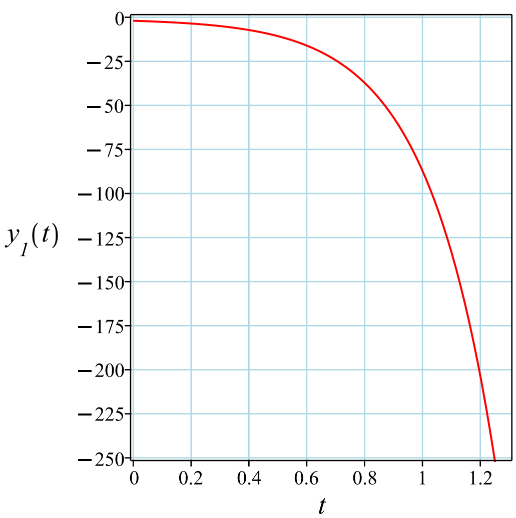
The following are plots of each solution against another.

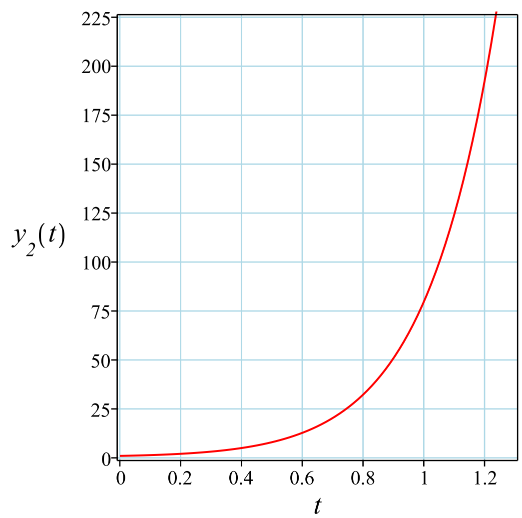


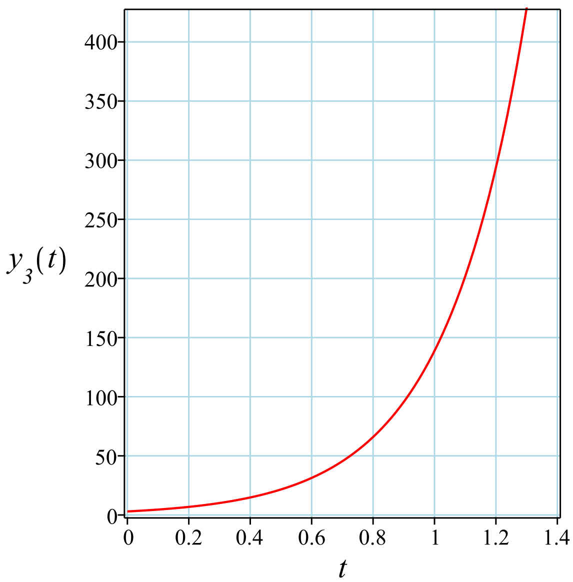




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 63

```
dsolve([diff(y__1(t),t) = -y__1(t)-4*y__2(t)-y__3(t), diff(y__2(t),t) = 3*y__1(t)+6*y__2(t)+
```

$$y_1(t) = -2\,e^{4t} + 3\,e^{2t}t$$
$$y_2(t) = 2\,e^{4t} - 3\,e^{2t}t - e^{2t}$$
$$y_3(t) = 2\,e^{4t} + 3\,e^{2t}t + e^{2t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 63

```
DSolve[{y1'[t]==-1*y1[t]-4*y2[t]-1*y3[t],y2'[t]==3*y1[t]+6*y2[t]+1*y3[t],y3'[t]==-3*y1[t]-2*
```

$$\text{y1}(t) \to 3e^{2t}t - 2e^{4t}$$
$$\text{y2}(t) \to e^{2t}\left(-3t + 2e^{2t} - 1\right)$$
$$\text{y3}(t) \to e^{2t}\left(3t + 2e^{2t} + 1\right)$$

22.22 problem section 10.5, problem 22

Internal problem ID [1625]
Internal file name [`OUTPUT/1626_Sunday_June_05_2022_02_24_58_AM_90731928/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 4y_1(t) - 8y_2(t) - 4y_3(t) \\
y_2'(t) &= -3y_1(t) - y_2(t) - 4y_3(t) \\
y_3'(t) &= y_1(t) - y_2(t) + 9y_3(t)
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = -4, y_2(0) = 1, y_3(0) = -3]
$$

22.22.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(22\,e^{13t}-130\,e^{11t}-35)e^{-4t}}{143} & \frac{2(11\,e^{13t}-65\,e^{11t}+54)e^{-4t}}{143} & -\frac{4(e^{13t}-1)e^{-4t}}{13} \\ -\frac{(22\,e^{13t}+13\,e^{11t}-35)e^{-4t}}{143} & \frac{(22\,e^{13t}+13\,e^{11t}+108)e^{-4t}}{143} & -\frac{4(e^{13t}-1)e^{-4t}}{13} \\ \frac{e^{9t}}{2}-\frac{e^{7t}}{2} & -\frac{e^{9t}}{2}+\frac{e^{7t}}{2} & e^{9t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} -\frac{(22\,e^{13t}-130\,e^{11t}-35)e^{-4t}}{143} & \frac{2(11\,e^{13t}-65\,e^{11t}+54)e^{-4t}}{143} & -\frac{4(e^{13t}-1)e^{-4t}}{13} \\ -\frac{(22\,e^{13t}+13\,e^{11t}-35)e^{-4t}}{143} & \frac{(22\,e^{13t}+13\,e^{11t}+108)e^{-4t}}{143} & -\frac{4(e^{13t}-1)e^{-4t}}{13} \\ \frac{e^{9t}}{2}-\frac{e^{7t}}{2} & -\frac{e^{9t}}{2}+\frac{e^{7t}}{2} & e^{9t} \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4(22\,e^{13t}-130\,e^{11t}-35)e^{-4t}}{143} + \frac{2(11\,e^{13t}-65\,e^{11t}+54)e^{-4t}}{143} + \frac{12(e^{13t}-1)e^{-4t}}{13} \\ \frac{4(22\,e^{13t}+13\,e^{11t}-35)e^{-4t}}{143} + \frac{(22\,e^{13t}+13\,e^{11t}+108)e^{-4t}}{143} + \frac{12(e^{13t}-1)e^{-4t}}{13} \\ -\frac{11\,e^{9t}}{2}+\frac{5\,e^{7t}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2(121\,e^{13t}-325\,e^{11t}-82)e^{-4t}}{143} \\ \frac{(242\,e^{13t}+65\,e^{11t}-164)e^{-4t}}{143} \\ -\frac{11\,e^{9t}}{2}+\frac{5\,e^{7t}}{2} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.22.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -8 & -4 \\ -3 & -1-\lambda & -4 \\ 1 & -1 & 9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 12\lambda^2 - \lambda + 252 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 7$$
$$\lambda_2 = -4$$
$$\lambda_3 = 9$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-4	1	real eigenvalue
7	1	real eigenvalue
9	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -8 & -4 \\ -3 & 3 & -4 \\ 1 & -1 & 13 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 8 & -8 & -4 & 0 \\ -3 & 3 & -4 & 0 \\ 1 & -1 & 13 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & -4 & 0 \\ 0 & 0 & -\frac{11}{2} & 0 \\ 1 & -1 & 13 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & -4 & 0 \\ 0 & 0 & -\frac{11}{2} & 0 \\ 0 & 0 & \frac{27}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{27R_2}{11} \Longrightarrow \left[\begin{array}{ccc|c} 8 & -8 & -4 & 0 \\ 0 & 0 & -\frac{11}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 8 & -8 & -4 \\ 0 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -8 & -4 \\ -3 & -8 & -4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -8 & -4 & 0 \\ -3 & -8 & -4 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -8 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{11}{3} & \frac{2}{3} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -3 & -8 & -4 & 0 \\ 0 & -\frac{11}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -8 & -4 \\ 0 & -\frac{11}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{20t}{11}, v_2 = \frac{2t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{20t}{11} \\ \frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{20t}{11} \\ \frac{2t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{20t}{11} \\ \frac{2t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{20}{11} \\ \frac{2}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{20t}{11} \\ \frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{20}{11} \\ \frac{2}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{20t}{11} \\ \frac{2t}{11} \\ t \end{bmatrix} = \begin{bmatrix} -20 \\ 2 \\ 11 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 9$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -4 \\ 1 & -1 & 9 \end{bmatrix} - (9)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -8 & -4 \\ -3 & -10 & -4 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & -8 & -4 & 0 \\ -3 & -10 & -4 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{5} \implies \left[\begin{array}{ccc|c} -5 & -8 & -4 & 0 \\ 0 & -\frac{26}{5} & -\frac{8}{5} & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{5} \implies \left[\begin{array}{ccc|c} -5 & -8 & -4 & 0 \\ 0 & -\frac{26}{5} & -\frac{8}{5} & 0 \\ 0 & -\frac{13}{5} & -\frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \implies \left[\begin{array}{ccc|c} -5 & -8 & -4 & 0 \\ 0 & -\frac{26}{5} & -\frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -8 & -4 \\ 0 & -\frac{26}{5} & -\frac{8}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{4t}{13}, v_2 = -\frac{4t}{13}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{13} \\ -\frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{13} \\ -\frac{4t}{13} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{13} \\ -\frac{4t}{13} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{13} \\ -\frac{4}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{13} \\ -\frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{13} \\ -\frac{4}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{13} \\ -\frac{4t}{13} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 13 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
7	1	1	No	$\begin{bmatrix} -\frac{20}{11} \\ \frac{2}{11} \\ 1 \end{bmatrix}$
-4	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
9	1	1	No	$\begin{bmatrix} -\frac{4}{13} \\ -\frac{4}{13} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{7t} \\
&= \begin{bmatrix} -\frac{20}{11} \\ \frac{2}{11} \\ 1 \end{bmatrix} e^{7t}
\end{aligned}$$

Since eigenvalue -4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-4t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-4t}
\end{aligned}$$

Since eigenvalue 9 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{9t} \\
&= \begin{bmatrix} -\frac{4}{13} \\ -\frac{4}{13} \\ 1 \end{bmatrix} e^{9t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{20\,\mathrm{e}^{7t}}{11} \\ \frac{2\,\mathrm{e}^{7t}}{11} \\ \mathrm{e}^{7t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{4\,\mathrm{e}^{9t}}{13} \\ -\frac{4\,\mathrm{e}^{9t}}{13} \\ \mathrm{e}^{9t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{(44c_3\mathrm{e}^{13t}+260c_1\mathrm{e}^{11t}-143c_2)\mathrm{e}^{-4t}}{143} \\ -\frac{(44c_3\mathrm{e}^{13t}-26c_1\mathrm{e}^{11t}-143c_2)\mathrm{e}^{-4t}}{143} \\ c_1\mathrm{e}^{7t}+c_3\mathrm{e}^{9t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} y_1(0) = -4 \\ y_2(0) = 1 \\ y_3(0) = -3 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{4c_3}{13} - \frac{20c_1}{11} + c_2 \\ -\frac{4c_3}{13} + \frac{2c_1}{11} + c_2 \\ c_1 + c_3 \end{bmatrix}
$$

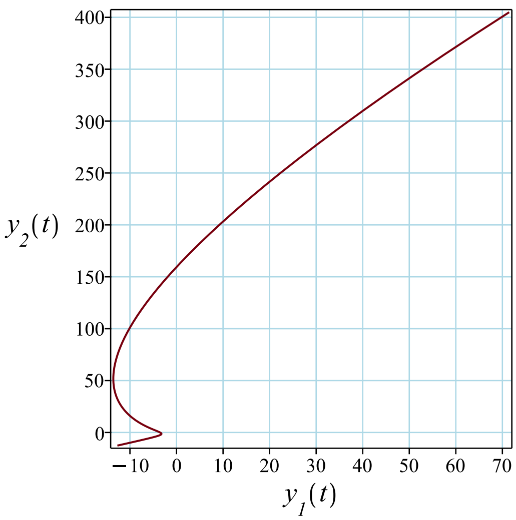
Solving for the constants of integrations gives

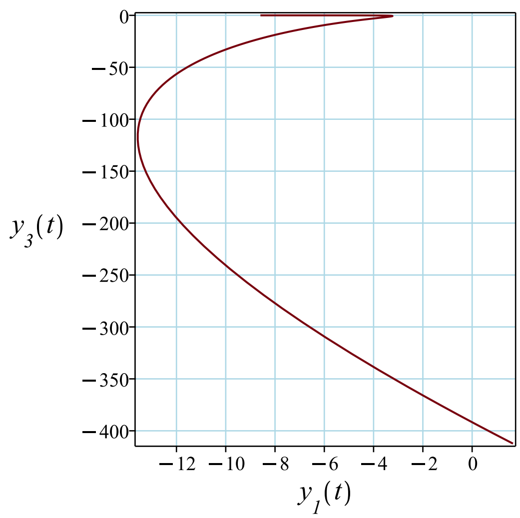
$$
\begin{bmatrix} c_1 = \frac{5}{2} \\ c_2 = -\frac{164}{143} \\ c_3 = -\frac{11}{2} \end{bmatrix}
$$

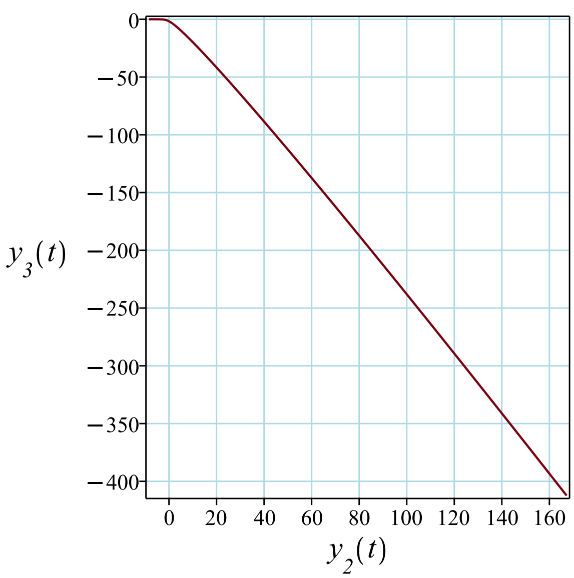
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(-242\,e^{13t}+650\,e^{11t}+164\right)e^{-4t}}{143} \\ -\frac{\left(-242\,e^{13t}-65\,e^{11t}+164\right)e^{-4t}}{143} \\ -\frac{11\,e^{9t}}{2}+\frac{5\,e^{7t}}{2} \end{bmatrix}$$

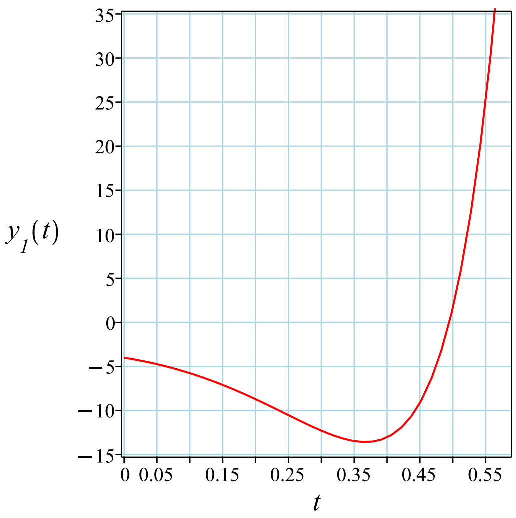
The following are plots of each solution against another.

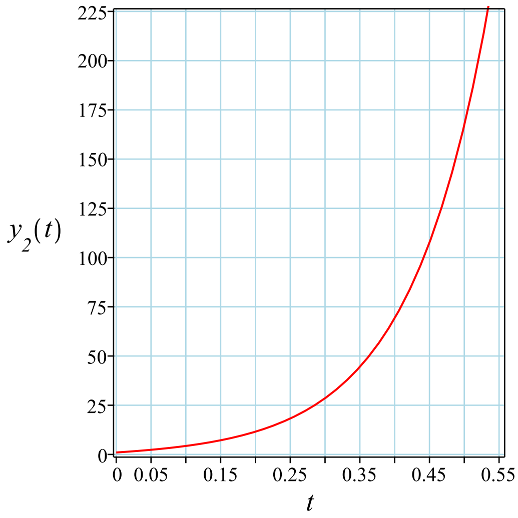


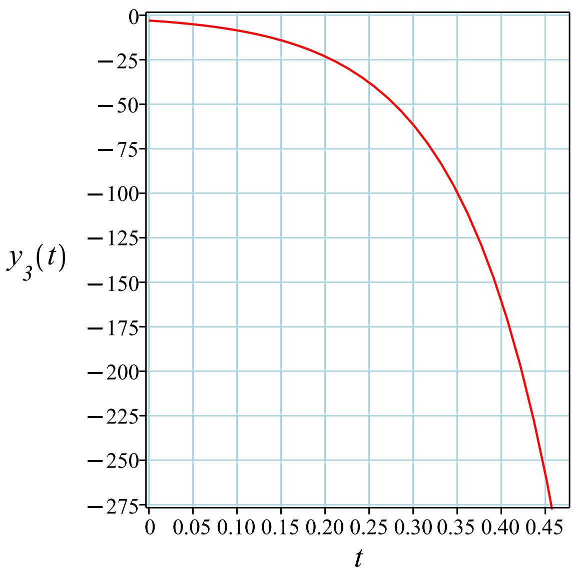




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 62

```
dsolve([diff(y__1(t),t) = 4*y__1(t)-8*y__2(t)-4*y__3(t), diff(y__2(t),t) = -3*y__1(t)-y__2(t
```

$$y_1(t) = -\frac{50\,\mathrm{e}^{7t}}{11} - \frac{164\,\mathrm{e}^{-4t}}{143} + \frac{22\,\mathrm{e}^{9t}}{13}$$
$$y_2(t) = \frac{5\,\mathrm{e}^{7t}}{11} - \frac{164\,\mathrm{e}^{-4t}}{143} + \frac{22\,\mathrm{e}^{9t}}{13}$$
$$y_3(t) = \frac{5\,\mathrm{e}^{7t}}{2} - \frac{11\,\mathrm{e}^{9t}}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 57

```
DSolve[{y1'[t]==4*y1[t]-8*y2[t]-4*y3[t],y2'[t]==-3*y1[t]-1*y2[t]-3*y3[t],y3'[t]==1*y1[t]-1*y
```

$$y1(t) \to e^{8t}(8t-3) - e^{-4t}$$
$$y2(t) \to 2e^{8t} - e^{-4t}$$
$$y3(t) \to -e^{8t}(8t+3)$$

22.23 problem section 10.5, problem 23

Internal problem ID [1626]
Internal file name [`OUTPUT/1627_Sunday_June_05_2022_02_25_01_AM_85775010/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -5y_1(t) - y_2(t) + 11y_3(t) \\
y_2'(t) &= -7y_1(t) + y_2(t) + 13y_3(t) \\
y_3'(t) &= -4y_1(t) + 8y_3(t)
\end{aligned}
$$

With initial conditions

$$[y_1(0) = 0, y_2(0) = 2, y_3(0) = 2]$$

22.23.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{7}{4} - 2t - \frac{3\,e^{4t}}{4} & \frac{e^{4t}}{4} - \frac{1}{4} - 2t & \frac{5\,e^{4t}}{4} - \frac{5}{4} + 6t \\ -t - \frac{3\,e^{4t}}{2} + \frac{3}{2} & \frac{1}{2} + \frac{e^{4t}}{2} - t & \frac{5\,e^{4t}}{2} - \frac{5}{2} + 3t \\ -t - \frac{3\,e^{4t}}{4} + \frac{3}{4} & \frac{e^{4t}}{4} - \frac{1}{4} - t & -\frac{1}{4} + \frac{5\,e^{4t}}{4} + 3t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{7}{4} - 2t - \frac{3\,e^{4t}}{4} & \frac{e^{4t}}{4} - \frac{1}{4} - 2t & \frac{5\,e^{4t}}{4} - \frac{5}{4} + 6t \\ -t - \frac{3\,e^{4t}}{2} + \frac{3}{2} & \frac{1}{2} + \frac{e^{4t}}{2} - t & \frac{5\,e^{4t}}{2} - \frac{5}{2} + 3t \\ -t - \frac{3\,e^{4t}}{4} + \frac{3}{4} & \frac{e^{4t}}{4} - \frac{1}{4} - t & -\frac{1}{4} + \frac{5\,e^{4t}}{4} + 3t \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -3 + 3\,e^{4t} + 8t \\ -4 + 6\,e^{4t} + 4t \\ -1 + 3\,e^{4t} + 4t \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.23.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & -1 & 11 \\ -7 & 1-\lambda & 13 \\ -4 & 0 & 8-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & -1 & 11 & 0 \\ -7 & 1 & 13 & 0 \\ -4 & 0 & 8 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{7R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -1 & 11 & 0 \\ 0 & \frac{12}{5} & -\frac{12}{5} & 0 \\ -4 & 0 & 8 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -1 & 11 & 0 \\ 0 & \frac{12}{5} & -\frac{12}{5} & 0 \\ 0 & \frac{4}{5} & -\frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -5 & -1 & 11 & 0 \\ 0 & \frac{12}{5} & -\frac{12}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -1 & 11 \\ 0 & \frac{12}{5} & -\frac{12}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & -1 & 11 \\ -7 & -3 & 13 \\ -4 & 0 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -9 & -1 & 11 & 0 \\ -7 & -3 & 13 & 0 \\ -4 & 0 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{7R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -9 & -1 & 11 & 0 \\ 0 & -\frac{20}{9} & \frac{40}{9} & 0 \\ -4 & 0 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -9 & -1 & 11 & 0 \\ 0 & -\frac{20}{9} & \frac{40}{9} & 0 \\ 0 & \frac{4}{9} & -\frac{8}{9} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{5} \Longrightarrow \left[\begin{array}{ccc|c} -9 & -1 & 11 & 0 \\ 0 & -\frac{20}{9} & \frac{40}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -9 & -1 & 11 \\ 0 & -\frac{20}{9} & \frac{40}{9} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

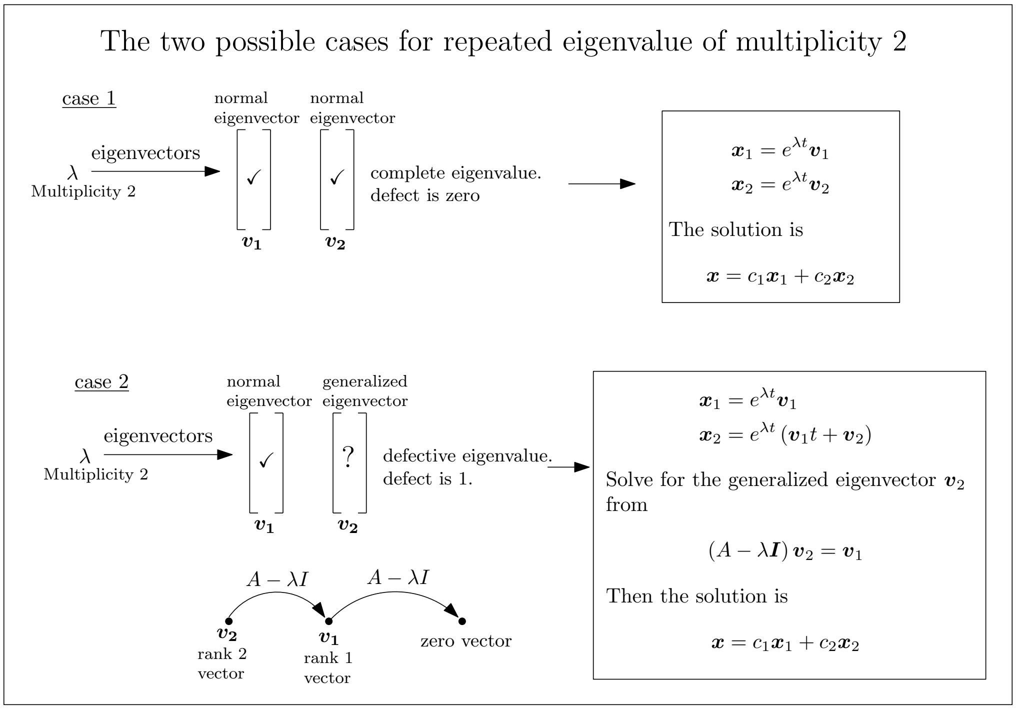


Figure 605: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} 1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{7}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right) 1 \\ &= \begin{bmatrix} 2t + \frac{7}{4} \\ t + \frac{1}{4} \\ t + 1 \end{bmatrix} \end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{4t}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{4t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2t + \frac{7}{4} \\ t + \frac{1}{4} \\ t + 1 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{4t} \\ 2\,\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 2c_1 + 2c_2 t + \frac{7c_2}{4} + c_3 \mathrm{e}^{4t} \\ c_1 + c_2 t + \frac{c_2}{4} + 2c_3 \mathrm{e}^{4t} \\ c_3 \mathrm{e}^{4t} + c_2 t + c_1 + c_2 \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} y_1(0) = 0 \\ y_2(0) = 2 \\ y_3(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + \frac{7c_2}{4} + c_3 \\ c_1 + \frac{c_2}{4} + 2c_3 \\ c_3 + c_1 + c_2 \end{bmatrix}$$

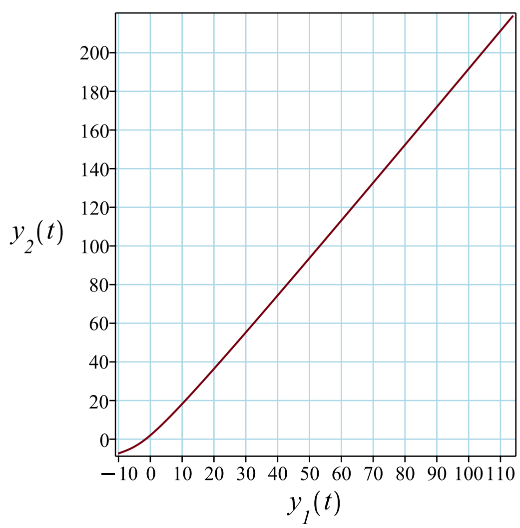
Solving for the constants of integrations gives

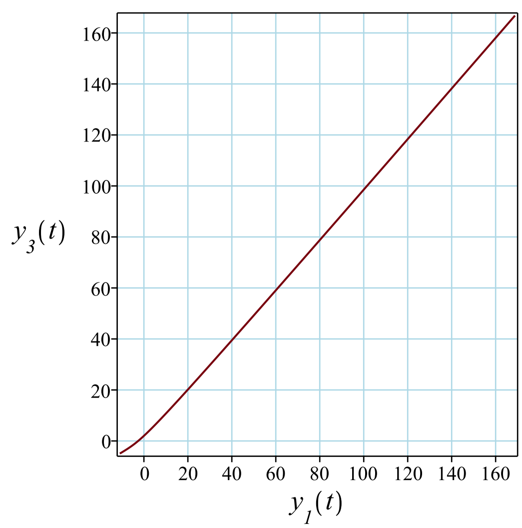
$$\begin{bmatrix} c_1 = -5 \\ c_2 = 4 \\ c_3 = 3 \end{bmatrix}$$

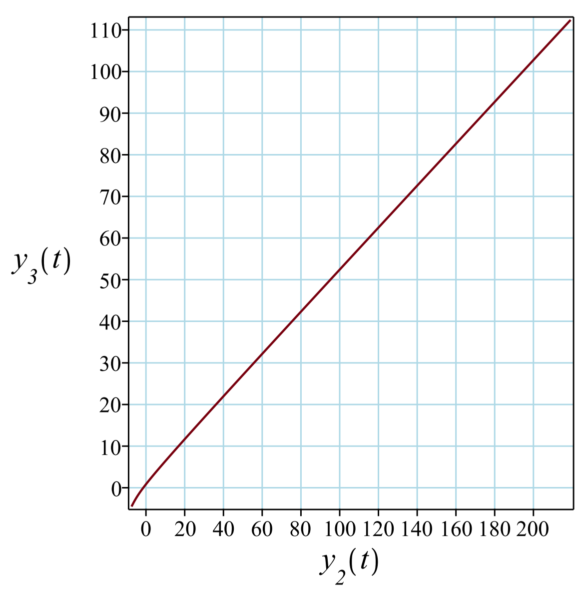
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -3 + 3\,\mathrm{e}^{4t} + 8t \\ -4 + 6\,\mathrm{e}^{4t} + 4t \\ -1 + 3\,\mathrm{e}^{4t} + 4t \end{bmatrix}$$

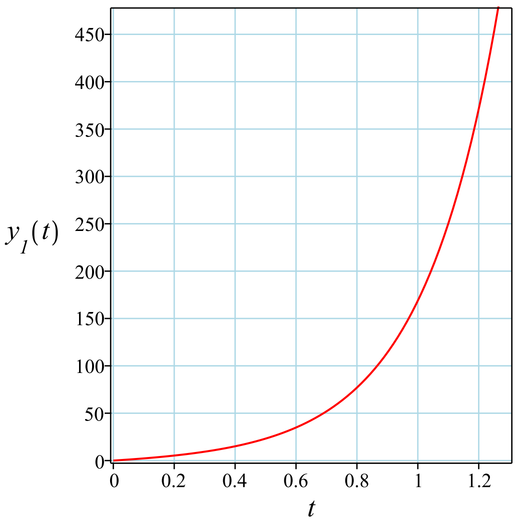
The following are plots of each solution against another.

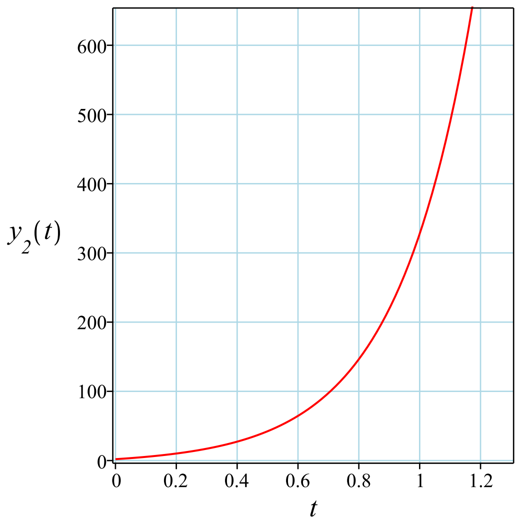


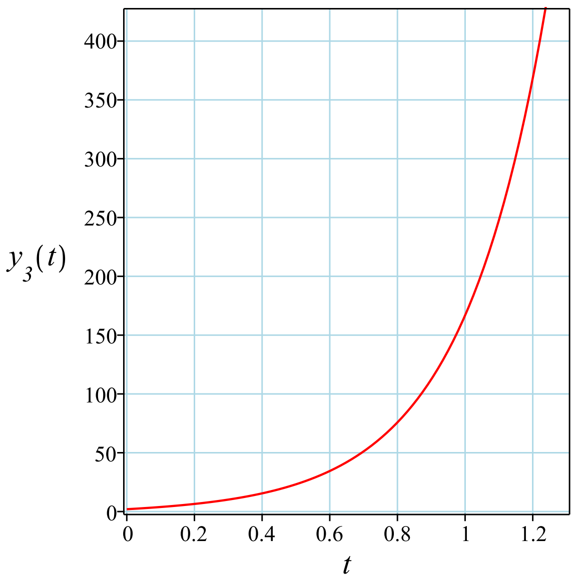




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve([diff(y__1(t),t) = -5*y__1(t)-y__2(t)+11*y__3(t), diff(y__2(t),t) = -7*y__1(t)+y__2(t
```

$$y_1(t) = -3 + 3\,e^{4t} + 8t$$
$$y_2(t) = 6\,e^{4t} - 4 + 4t$$
$$y_3(t) = -1 + 4t + 3\,e^{4t}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 47

```
DSolve[{y1'[t]==-5*y1[t]-1*y2[t]+11*y3[t],y2'[t]==-7*y1[t]+1*y2[t]+13*y3[t],y3'[t]==-4*y1[t]
```

$$y1(t) \to 8t + 3e^{4t} - 3$$
$$y2(t) \to 4t + 6e^{4t} - 4$$
$$y3(t) \to 4t + 3e^{4t} - 1$$

22.24 problem section 10.5, problem 24

Internal problem ID [1627]
Internal file name [`OUTPUT/1628_Sunday_June_05_2022_02_25_03_AM_9188363/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 5y_1(t) - y_2(t) + y_3(t) \\
y_2'(t) &= -y_1(t) + 9y_2(t) - 3y_3(t) \\
y_3'(t) &= -2y_1(t) + 2y_2(t) + 4y_3(t)
\end{aligned}
$$

22.24.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{6t}(1-t) & -t\,\mathrm{e}^{6t} & t\,\mathrm{e}^{6t} \\ \mathrm{e}^{6t}t(2t-1) & \mathrm{e}^{6t}(2t^2+3t+1) & \mathrm{e}^{6t}(-2t^2-3t) \\ 2\,\mathrm{e}^{6t}t(-1+t) & 2\,\mathrm{e}^{6t}t(t+1) & \mathrm{e}^{6t}(-2t^2-2t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{6t}(1-t) & -t\,e^{6t} & t\,e^{6t} \\ e^{6t}t(2t-1) & e^{6t}(2t^2+3t+1) & e^{6t}(-2t^2-3t) \\ 2\,e^{6t}t(-1+t) & 2\,e^{6t}t(t+1) & e^{6t}(-2t^2-2t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(1-t)\,c_1 - t\,e^{6t}c_2 + t\,e^{6t}c_3 \\ e^{6t}t(2t-1)\,c_1 + e^{6t}(2t^2+3t+1)\,c_2 + e^{6t}(-2t^2-3t)\,c_3 \\ 2\,e^{6t}t(-1+t)\,c_1 + 2\,e^{6t}t(t+1)\,c_2 + e^{6t}(-2t^2-2t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -((c_1+c_2-c_3)\,t - c_1)\,e^{6t} \\ 2\left((c_1+c_2-c_3)\,t^2 + \frac{(-c_1+3c_2-3c_3)t}{2} + \frac{c_2}{2}\right)e^{6t} \\ 2\left((c_1+c_2-c_3)\,t^2 + (-c_1+c_2-c_3)\,t + \frac{c_3}{2}\right)e^{6t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.24.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -1 & 1 \\ -1 & 9-\lambda & -3 \\ -2 & 2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 18\lambda^2 + 108\lambda - 216 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & 3 & -3 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 4 & -4 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 6 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

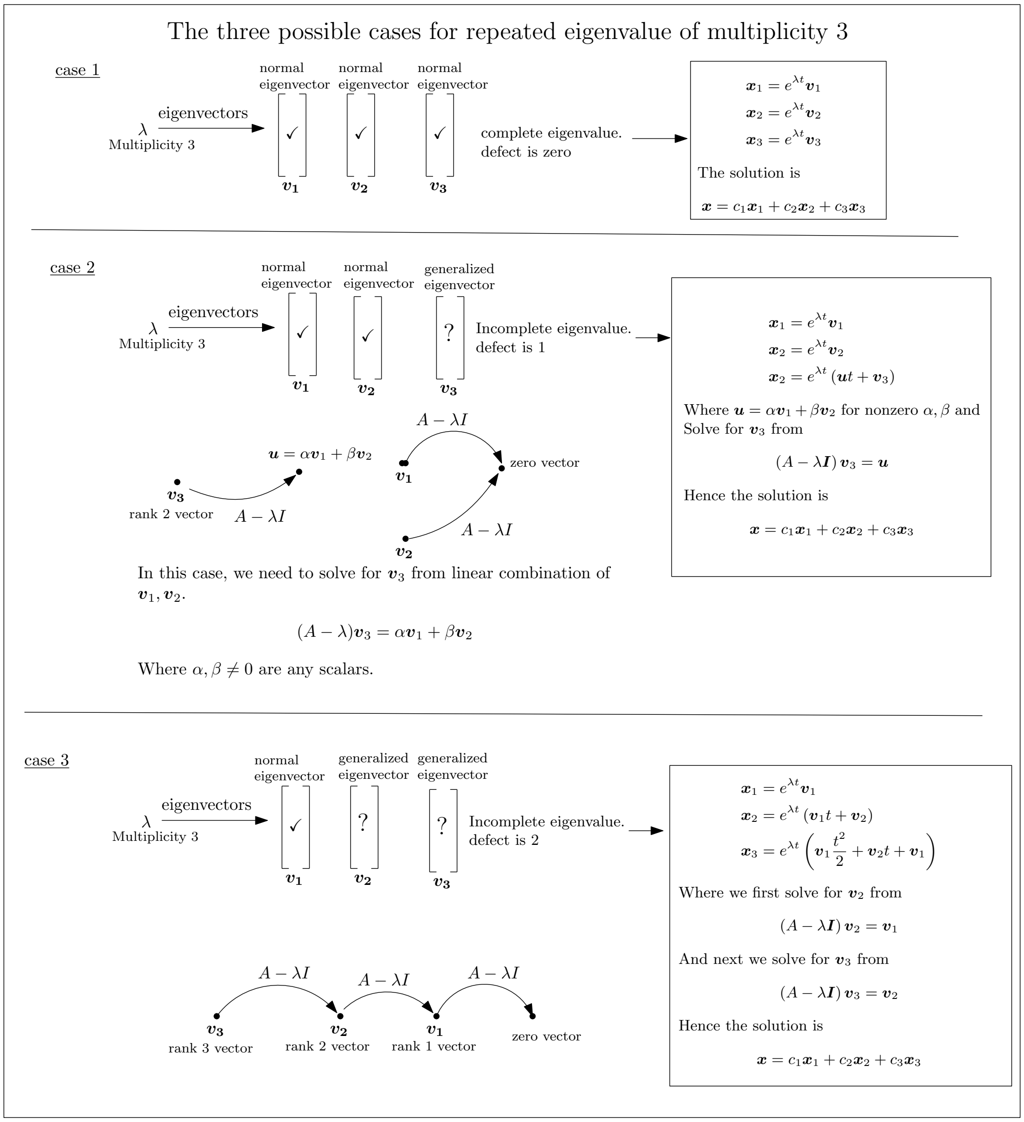


Figure 606: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} -\frac{1}{8} \\ \frac{11}{8} \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 6. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{6t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{6t}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{6t}}{4} \\ \frac{\mathrm{e}^{6t}(4t+5)}{4} \\ \mathrm{e}^{6t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{8} \\ \frac{11}{8} \\ 1 \end{bmatrix}\right) \mathrm{e}^{6t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{6t}(1+2t)}{8} \\ \frac{\mathrm{e}^{6t}(4t^2+10t+11)}{8} \\ \frac{\mathrm{e}^{6t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{6t} \\ e^{6t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{6t}}{4} \\ e^{6t}\left(t+\frac{5}{4}\right) \\ e^{6t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{6t}\left(-\frac{t}{4}-\frac{1}{8}\right) \\ e^{6t}\left(\frac{1}{2}t^2+\frac{5}{4}t+\frac{11}{8}\right) \\ e^{6t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{6t}(2c_3t+2c_2+c_3)}{8} \\ \frac{((4t^2+10t+11)c_3+8c_2t+8c_1+10c_2)e^{6t}}{8} \\ \frac{((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)e^{6t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
dsolve([diff(y__1(t),t)=5*y__1(t)-1*y__2(t)+1*y__3(t),diff(y__2(t),t)=-1*y__1(t)+9*y__2(t)-3
```

$$y_1(t) = e^{6t}(c_3t + c_2)$$
$$y_2(t) = \left(-2c_3t^2 - 4c_2t - 3c_3t + c_1\right)e^{6t}$$
$$y_3(t) = e^{6t}\left(-2c_3t^2 - 4c_2t - 2c_3t + c_1 + c_2 + c_3\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 106

```
DSolve[{y1'[t]==5*y1[t]-1*y2[t]+1*y3[t],y2'[t]==-1*y1[t]+9*y2[t]-3*y3[t],y3'[t]==-2*y1[t]+2*
```

$$\text{y1}(t) \to -e^{6t}(c_1(t-1) + (c_2 - c_3)t)$$
$$\text{y2}(t) \to e^{6t}\left(2(c_1 + c_2 - c_3)t^2 - (c_1 - 3c_2 + 3c_3)t + c_2\right)$$
$$\text{y3}(t) \to e^{6t}\left(2(c_1 + c_2 - c_3)t^2 - 2(c_1 - c_2 + c_3)t + c_3\right)$$

22.25 problem section 10.5, problem 25

Internal problem ID [1628]
Internal file name [OUTPUT/1629_Sunday_June_05_2022_02_25_04_AM_7451132/index.tex]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= y_1(t) + 10y_2(t) - 12y_3(t)\\
y_2'(t) &= 2y_1(t) + 2y_2(t) + 3y_3(t)\\
y_3'(t) &= 2y_1(t) - y_2(t) + 6y_3(t)
\end{aligned}
$$

22.25.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{3t}(1-2t) & e^{3t}(-9t^2+10t) & (9t^2-12t)\,e^{3t} \\ 2\,e^{3t}t & e^{3t}(9t^2-t+1) & (-9t^2+3t)\,e^{3t} \\ 2\,e^{3t}t & e^{3t}t(9t-1) & e^{3t}(-9t^2+3t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(1-2t) & e^{3t}(-9t^2+10t) & (9t^2-12t)\,e^{3t} \\ 2\,e^{3t}t & e^{3t}(9t^2-t+1) & (-9t^2+3t)\,e^{3t} \\ 2\,e^{3t}t & e^{3t}t(9t-1) & e^{3t}(-9t^2+3t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1-2t)\,c_1 + e^{3t}(-9t^2+10t)\,c_2 + (9t^2-12t)\,e^{3t}c_3 \\ 2\,e^{3t}tc_1 + e^{3t}(9t^2-t+1)\,c_2 + (-9t^2+3t)\,e^{3t}c_3 \\ 2\,e^{3t}tc_1 + e^{3t}t(9t-1)\,c_2 + e^{3t}(-9t^2+3t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -9\,e^{3t}\left((c_2-c_3)\,t^2 + \left(\frac{2c_1}{9} - \frac{10c_2}{9} + \frac{4c_3}{3}\right)t - \frac{c_1}{9}\right) \\ 9\left((c_2-c_3)\,t^2 + \left(\frac{2c_1}{9} - \frac{c_2}{9} + \frac{c_3}{3}\right)t + \frac{c_2}{9}\right)e^{3t} \\ 9\,e^{3t}\left((c_2-c_3)\,t^2 + \left(\frac{2c_1}{9} - \frac{c_2}{9} + \frac{c_3}{3}\right)t + \frac{c_3}{9}\right) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.25.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 10 & -12 \\ 2 & 2-\lambda & 3 \\ 2 & -1 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 10 & -12 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 10 & -12 & 0 \\ 2 & -1 & 3 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 10 & -12 & 0 \\ 0 & 9 & -9 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 10 & -12 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & 9 & -9 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 10 & -12 & 0 \\ 0 & 9 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 10 & -12 \\ 0 & 9 & -9 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	3	1	Yes	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

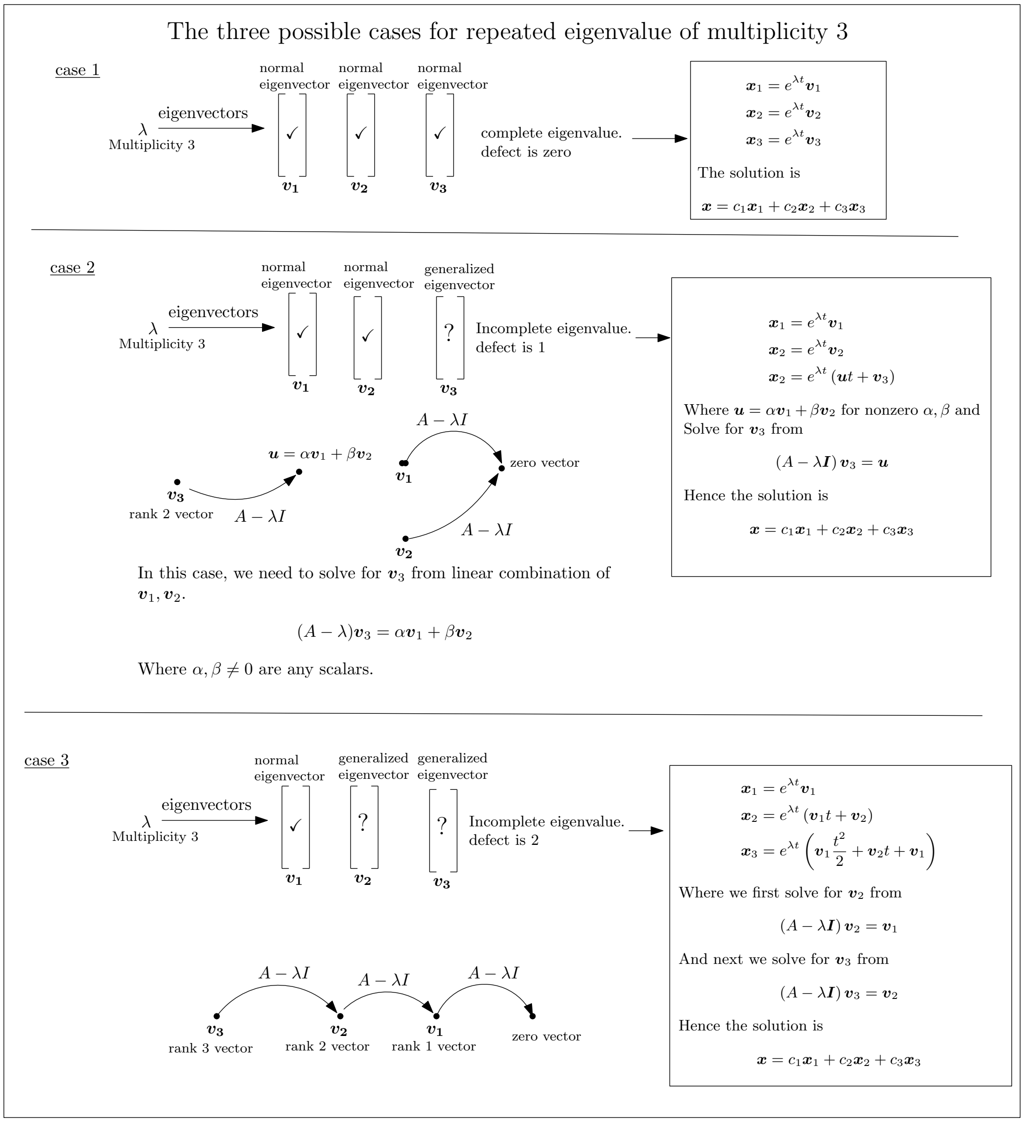


Figure 607: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 10 & -12 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 10 & -12 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} -\frac{17}{36} \\ \frac{19}{18} \\ 1 \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 3. Therefore the three basis

solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\
&= \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{3t}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{3t}(1+2t)}{2} \\ \mathrm{e}^{3t}(t+1) \\ \mathrm{e}^{3t}(t+1) \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{17}{36} \\ \frac{19}{18} \\ 1 \end{bmatrix}\right) \mathrm{e}^{3t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{3t}(18t^2+18t+17)}{36} \\ \frac{\mathrm{e}^{3t}(9t^2+18t+19)}{18} \\ \frac{\mathrm{e}^{3t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t}\left(-t-\frac{1}{2}\right) \\ \mathrm{e}^{3t}(t+1) \\ \mathrm{e}^{3t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{3t}\left(-\frac{1}{2}t^2-\frac{1}{2}t-\frac{17}{36}\right) \\ \mathrm{e}^{3t}\left(\frac{1}{2}t^2+t+\frac{19}{18}\right) \\ \mathrm{e}^{3t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(\left(t^2+t+\frac{17}{18}\right)c_3+2c_2t+2c_1+c_2\right)\mathrm{e}^{3t}}{2} \\ \frac{\mathrm{e}^{3t}\left(\left(t^2+2t+\frac{19}{9}\right)c_3+2c_2t+2c_1+2c_2\right)}{2} \\ \frac{\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)\mathrm{e}^{3t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 86

```
dsolve([diff(y__1(t),t)=1*y__1(t)+10*y__2(t)-12*y__3(t),diff(y__2(t),t)=2*y__1(t)+2*y__2(t)+
```

$$y_1(t) = \mathrm{e}^{3t}\left(c_3t^2 + c_2t + c_1\right)$$
$$y_2(t) = -\frac{\mathrm{e}^{3t}(6c_3t^2 + 6c_2t + 6c_3t + 6c_1 + 3c_2 + 4c_3)}{6}$$
$$y_3(t) = -\frac{\mathrm{e}^{3t}(18c_3t^2 + 18c_2t + 18c_3t + 18c_1 + 9c_2 + 10c_3)}{18}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 118

```
DSolve[{y1'[t]==1*y1[t]+10*y2[t]-12*y3[t],y2'[t]==2*y1[t]+2*y2[t]+3*y3[t],y3'[t]==2*y1[t]-1*
```

$$\mathrm{y1}(t) \to -e^{3t}(c_1(2t-1) + c_2t(9t-10) + 3c_3(4-3t)t)$$
$$\mathrm{y2}(t) \to e^{3t}\left(9(c_2-c_3)t^2 + (2c_1-c_2+3c_3)t + c_2\right)$$
$$\mathrm{y3}(t) \to e^{3t}\left(9(c_2-c_3)t^2 + (2c_1-c_2+3c_3)t + c_3\right)$$

22.26 problem section 10.5, problem 26

Internal problem ID [1629]
Internal file name [`OUTPUT/1630_Sunday_June_05_2022_02_25_06_AM_41661909/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(t) &= -6y_1(t) - 4y_2(t) - 4y_3(t) \\ y_2'(t) &= 2y_1(t) - y_2(t) + y_3(t) \\ y_3'(t) &= 2y_1(t) + 3y_2(t) + y_3(t) \end{aligned}$$

22.26.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t}(1-4t) & -4t\,e^{-2t} & -4t\,e^{-2t} \\ -2\,e^{-2t}t(-1+t) & e^{-2t}(-2t^2+t+1) & e^{-2t}(-2t^2+t) \\ 2\,e^{-2t}t(t+1) & e^{-2t}t(2t+3) & e^{-2t}(2t^2+3t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t}(1-4t) & -4t\,e^{-2t} & -4t\,e^{-2t} \\ -2\,e^{-2t}t(-1+t) & e^{-2t}(-2t^2+t+1) & e^{-2t}(-2t^2+t) \\ 2\,e^{-2t}t(t+1) & e^{-2t}t(2t+3) & e^{-2t}(2t^2+3t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(1-4t)\,c_1 - 4t\,e^{-2t}c_2 - 4t\,e^{-2t}c_3 \\ -2\,e^{-2t}t(-1+t)\,c_1 + e^{-2t}(-2t^2+t+1)\,c_2 + e^{-2t}(-2t^2+t)\,c_3 \\ 2\,e^{-2t}t(t+1)\,c_1 + e^{-2t}t(2t+3)\,c_2 + e^{-2t}(2t^2+3t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -4\left((c_1+c_2+c_3)\,t - \frac{c_1}{4}\right)e^{-2t} \\ -2\left((c_1+c_2+c_3)\,t^2 + \left(-c_1 - \frac{c_2}{2} - \frac{c_3}{2}\right)t - \frac{c_2}{2}\right)e^{-2t} \\ 2\left((c_1+c_2+c_3)\,t^2 + \left(c_1 + \frac{3c_2}{2} + \frac{3c_3}{2}\right)t + \frac{c_3}{2}\right)e^{-2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.26.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-6-\lambda & -4 & -4\\ 2 & -1-\lambda & 1\\ 2 & 3 & 1-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3+6\lambda^2+12\lambda+8=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-2$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-6 & -4 & -4\\ 2 & -1 & 1\\ 2 & 3 & 1\end{bmatrix}-(-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-4 & -4 & -4\\ 2 & 1 & 1\\ 2 & 3 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-4 & -4 & -4 & 0\\ 2 & 1 & 1 & 0\\ 2 & 3 & 3 & 0\end{array}\right]$$

$$R_2=R_2+\frac{R_1}{2}\Longrightarrow\left[\begin{array}{ccc|c}-4 & -4 & -4 & 0\\ 0 & -1 & -1 & 0\\ 2 & 3 & 3 & 0\end{array}\right]$$

$$
R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -4 & -4 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & -4 & -4 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -4 & -4 & -4 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	3	1	Yes	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

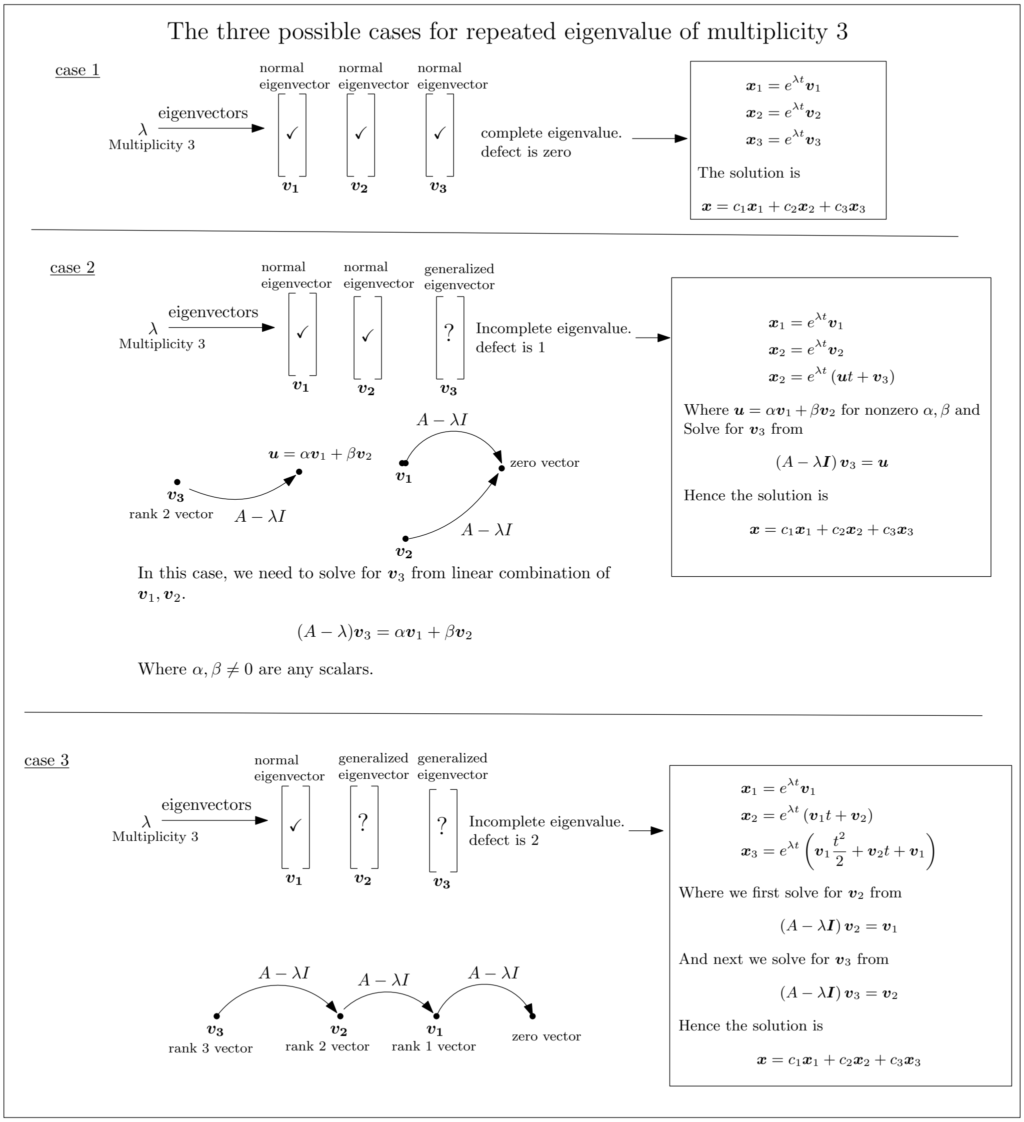


Figure 608: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-6 & -4 & -4\\ 2 & -1 & 1\\ 2 & 3 & 1\end{bmatrix} - (-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & -4 & -4\\ 2 & 1 & 1\\ 2 & 3 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-6 & -4 & -4\\ 2 & -1 & 1\\ 2 & 3 & 1\end{bmatrix} - (-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & -4 & -4\\ 2 & 1 & 1\\ 2 & 3 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}-\frac{1}{4}\\ -\frac{1}{2}\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -2. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mathrm{e}^{-2t}\\ &= \begin{bmatrix} 0 \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\ &= \mathrm{e}^{-2t}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)\\ &= \begin{bmatrix} -\mathrm{e}^{-2t} \\ -t\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}\\ &= \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right) \mathrm{e}^{-2t}\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-2t}(4t+1)}{4} \\ -\frac{\mathrm{e}^{-2t}(t^2+1)}{2} \\ \frac{\mathrm{e}^{-2t}(t^2+2t+2)}{2} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-2t} \\ -t\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-2t}\left(-t-\frac{1}{4}\right) \\ \mathrm{e}^{-2t}\left(-\frac{t^2}{2}-\frac{1}{2}\right) \\ \mathrm{e}^{-2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-2t}\left(-c_2-c_3t-\frac{1}{4}c_3\right) \\ -\frac{\mathrm{e}^{-2t}(c_3t^2+2tc_2+2c_1+c_3)}{2} \\ \frac{((t^2+2t+2)c_3+2tc_2+2c_1+2c_2)\mathrm{e}^{-2t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 73

```
dsolve([diff(y__1(t),t)=-6*y__1(t)-4*y__2(t)-4*y__3(t),diff(y__2(t),t)=2*y__1(t)-1*y__2(t)+1
```

$$y_1(t) = \mathrm{e}^{-2t}(c_3t+c_2)$$
$$y_2(t) = \frac{(2c_3t^2+4c_2t-c_3t+4c_1)\,\mathrm{e}^{-2t}}{4}$$
$$y_3(t) = -\frac{\mathrm{e}^{-2t}(2c_3t^2+4c_2t+3c_3t+4c_1+4c_2+c_3)}{4}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 100

```
DSolve[{y1'[t]==-6*y1[t]-4*y2[t]-4*y3[t],y2'[t]==2*y1[t]-1*y2[t]+1*y3[t],y3'[t]==2*y1[t]+3*y
```

$$\mathrm{y1}(t) \to e^{-2t}(c_1(1-4t)-4(c_2+c_3)t)$$
$$\mathrm{y2}(t) \to e^{-2t}\left(-2(c_1+c_2+c_3)t^2+(2c_1+c_2+c_3)t+c_2\right)$$
$$\mathrm{y3}(t) \to e^{-2t}\left(2(c_1+c_2+c_3)t^2+2c_1t+3(c_2+c_3)t+c_3\right)$$

22.27 problem section 10.5, problem 27

Internal problem ID [1630]
Internal file name [`OUTPUT/1631_Sunday_June_05_2022_02_25_08_AM_57956600/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 2y_2(t) - 2y_3(t)\\
y_2'(t) &= -y_1(t) + 5y_2(t) - 3y_3(t)\\
y_3'(t) &= y_1(t) + y_2(t) + y_3(t)
\end{aligned}
$$

22.27.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} (1-2t)\,\mathrm{e}^{2t} & 2\,\mathrm{e}^{2t}t & -2\,\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}(-2t^2-t) & \mathrm{e}^{2t}(2t^2+3t+1) & \mathrm{e}^{2t}(-2t^2-3t) \\ \mathrm{e}^{2t}(-2t^2+t) & \mathrm{e}^{2t}(2t^2+t) & \mathrm{e}^{2t}(-2t^2-t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} (1-2t)\,\mathrm{e}^{2t} & 2\,\mathrm{e}^{2t}t & -2\,\mathrm{e}^{2t}t \\ \mathrm{e}^{2t}(-2t^2-t) & \mathrm{e}^{2t}(2t^2+3t+1) & \mathrm{e}^{2t}(-2t^2-3t) \\ \mathrm{e}^{2t}(-2t^2+t) & \mathrm{e}^{2t}(2t^2+t) & \mathrm{e}^{2t}(-2t^2-t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (1-2t)\,\mathrm{e}^{2t}c_1 + 2\,\mathrm{e}^{2t}tc_2 - 2\,\mathrm{e}^{2t}tc_3 \\ \mathrm{e}^{2t}(-2t^2-t)\,c_1 + \mathrm{e}^{2t}(2t^2+3t+1)\,c_2 + \mathrm{e}^{2t}(-2t^2-3t)\,c_3 \\ \mathrm{e}^{2t}(-2t^2+t)\,c_1 + \mathrm{e}^{2t}(2t^2+t)\,c_2 + \mathrm{e}^{2t}(-2t^2-t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -2\left((c_1-c_2+c_3)\,t - \frac{c_1}{2}\right)\mathrm{e}^{2t} \\ -2\left((c_1-c_2+c_3)\,t^2 + \frac{(c_1-3c_2+3c_3)t}{2} - \frac{c_2}{2}\right)\mathrm{e}^{2t} \\ -2\,\mathrm{e}^{2t}\left((c_1-c_2+c_3)\,t^2 + \frac{(-c_1-c_2+c_3)t}{2} - \frac{c_3}{2}\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.27.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 & -2 \\ -1 & 5-\lambda & -3 \\ 1 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ -1 & 3 & -3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ -1 & 3 & -3 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$
R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -2 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$
\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
2	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

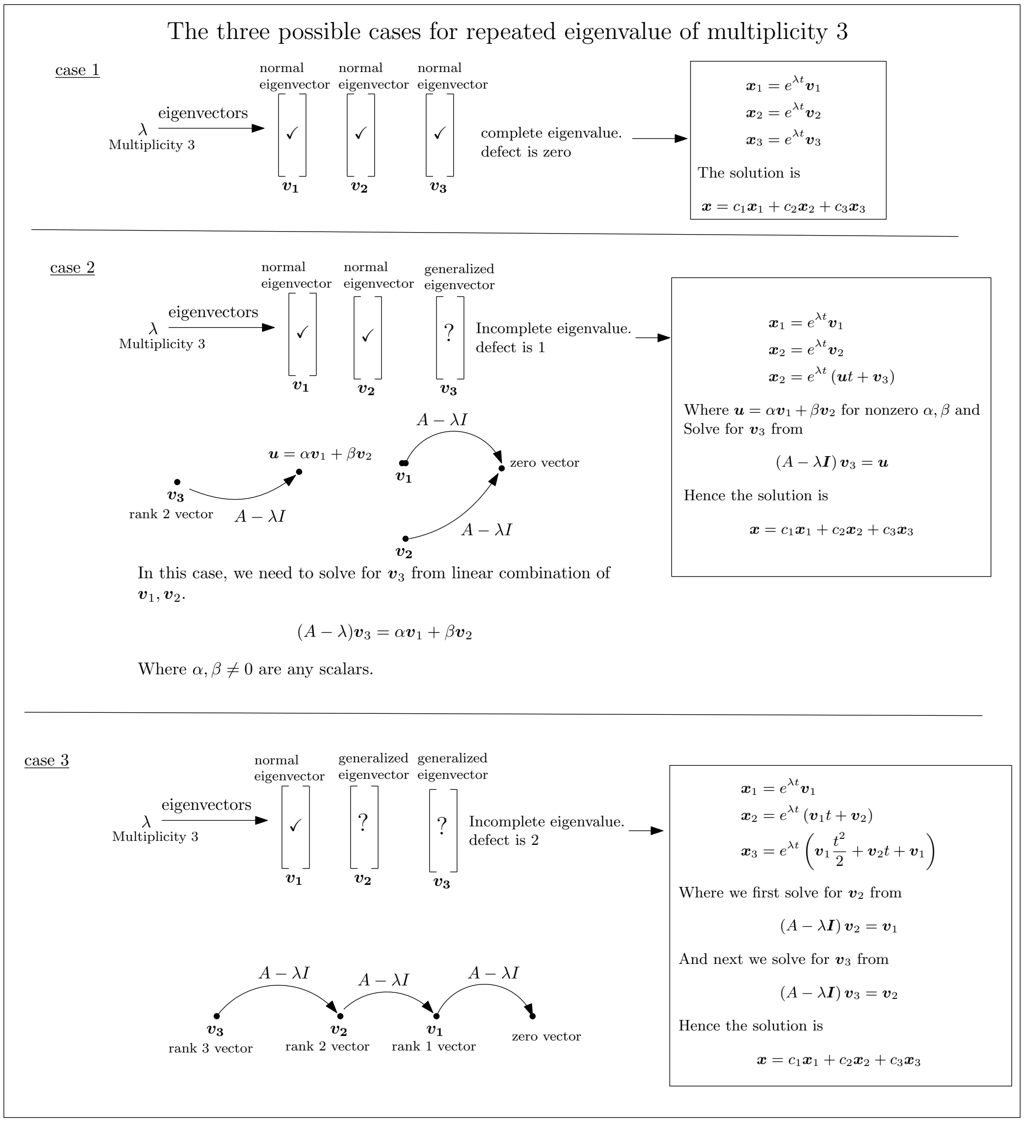


Figure 609: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}0 & 2 & -2\\ -1 & 5 & -3\\ 1 & 1 & 1\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-2 & 2 & -2\\ -1 & 3 & -3\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}0 & 2 & -2\\ -1 & 5 & -3\\ 1 & 1 & 1\end{bmatrix} - (2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ 1\end{bmatrix}$$

$$\begin{bmatrix}-2 & 2 & -2\\ -1 & 3 & -3\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}\frac{3}{8}\\ \frac{13}{8}\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis

solutions associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mathrm{e}^{2t} \\
&= \begin{bmatrix} 0 \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{2t} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}}{2} \\ \frac{\mathrm{e}^{2t}(2t+3)}{2} \\ \mathrm{e}^{2t}(t+1) \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3 \right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{3}{8} \\ \frac{13}{8} \\ 1 \end{bmatrix} \right) \mathrm{e}^{2t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}(4t+3)}{8} \\ \frac{\mathrm{e}^{2t}(4t^2+12t+13)}{8} \\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2} \end{bmatrix}
\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{2t}}{2} \\ e^{2t}\left(t+\frac{3}{2}\right) \\ e^{2t}(t+1) \end{bmatrix} + c_3 \begin{bmatrix} e^{2t}\left(\frac{t}{2}+\frac{3}{8}\right) \\ e^{2t}\left(\frac{1}{2}t^2+\frac{3}{2}t+\frac{13}{8}\right) \\ e^{2t}\left(t+\frac{1}{2}t^2+1\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((4t+3)c_3+4c_2)e^{2t}}{8} \\ \frac{\left(\left(t^2+3t+\frac{13}{4}\right)c_3+2c_2t+2c_1+3c_2\right)e^{2t}}{2} \\ \frac{\left((t^2+2t+2)c_3+2c_2t+2c_1+2c_2\right)e^{2t}}{2} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 74

```
dsolve([diff(y__1(t),t)=0*y__1(t)+2*y__2(t)-2*y__3(t),diff(y__2(t),t)=-1*y__1(t)+5*y__2(t)-3
```

$$y_1(t) = e^{2t}(c_3t + c_2)$$
$$y_2(t) = \frac{(2c_3t^2 + 4c_2t + 3c_3t + 2c_1)e^{2t}}{2}$$
$$y_3(t) = \frac{e^{2t}(2c_3t^2 + 4c_2t + c_3t + 2c_1 - 2c_2 - c_3)}{2}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 108

```
DSolve[{y1'[t]==0*y1[t]+2*y2[t]-2*y3[t],y2'[t]==-1*y1[t]+5*y2[t]-3*y3[t],y3'[t]==1*y1[t]+1*y
```

$$\text{y1}(t) \to -e^{2t}(c_1(2t-1) + 2(c_3 - c_2)t)$$
$$\text{y2}(t) \to e^{2t}\left(-2(c_1 - c_2 + c_3)t^2 - (c_1 - 3c_2 + 3c_3)t + c_2\right)$$
$$\text{y3}(t) \to e^{2t}\left(-2(c_1 - c_2 + c_3)t^2 + (c_1 + c_2 - c_3)t + c_3\right)$$

22.28 problem section 10.5, problem 28

Internal problem ID [1631]
Internal file name [`OUTPUT/1632_Sunday_June_05_2022_02_25_10_AM_77355816/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -2y_1(t) - 12y_2(t) + 10y_3(t) \\
y_2'(t) &= 2y_1(t) - 24y_2(t) + 11y_3(t) \\
y_3'(t) &= 2y_1(t) - 24y_2(t) + 8y_3(t)
\end{aligned}
$$

22.28.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{-6t}(6t^2+4t+1) & (-36t^2-12t)\,e^{-6t} & (24t^2+10t)\,e^{-6t} \\ e^{-6t}(-3t^2+2t) & e^{-6t}(18t^2-18t+1) & e^{-6t}(-12t^2+11t) \\ (-6t^2+2t)\,e^{-6t} & (36t^2-24t)\,e^{-6t} & e^{-6t}(-24t^2+14t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-6t}(6t^2+4t+1) & (-36t^2-12t)\,e^{-6t} & (24t^2+10t)\,e^{-6t} \\ e^{-6t}(-3t^2+2t) & e^{-6t}(18t^2-18t+1) & e^{-6t}(-12t^2+11t) \\ (-6t^2+2t)\,e^{-6t} & (36t^2-24t)\,e^{-6t} & e^{-6t}(-24t^2+14t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-6t}(6t^2+4t+1)\,c_1 + (-36t^2-12t)\,e^{-6t}c_2 + (24t^2+10t)\,e^{-6t}c_3 \\ e^{-6t}(-3t^2+2t)\,c_1 + e^{-6t}(18t^2-18t+1)\,c_2 + e^{-6t}(-12t^2+11t)\,c_3 \\ (-6t^2+2t)\,e^{-6t}c_1 + (36t^2-24t)\,e^{-6t}c_2 + e^{-6t}(-24t^2+14t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} 6\left((c_1-6c_2+4c_3)\,t^2 + \left(\frac{2c_1}{3} - 2c_2 + \frac{5c_3}{3}\right)t + \frac{c_1}{6}\right)e^{-6t} \\ -3\left((c_1-6c_2+4c_3)\,t^2 + \left(-\frac{2c_1}{3} + 6c_2 - \frac{11c_3}{3}\right)t - \frac{c_2}{3}\right)e^{-6t} \\ -6\left((c_1-6c_2+4c_3)\,t^2 + \left(-\frac{c_1}{3} + 4c_2 - \frac{7c_3}{3}\right)t - \frac{c_3}{6}\right)e^{-6t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.28.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -12 & 10 \\ 2 & -24-\lambda & 11 \\ 2 & -24 & 8-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 18\lambda^2 + 108\lambda + 216 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -12 & 10 \\ 2 & -18 & 11 \\ 2 & -24 & 14 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & -12 & 10 & 0 \\ 2 & -18 & 11 & 0 \\ 2 & -24 & 14 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} 4 & -12 & 10 & 0 \\ 0 & -12 & 6 & 0 \\ 2 & -24 & 14 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -12 & 10 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -18 & 9 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -12 & 10 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -12 & 10 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$
\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-6	3	1	Yes	$\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -6 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

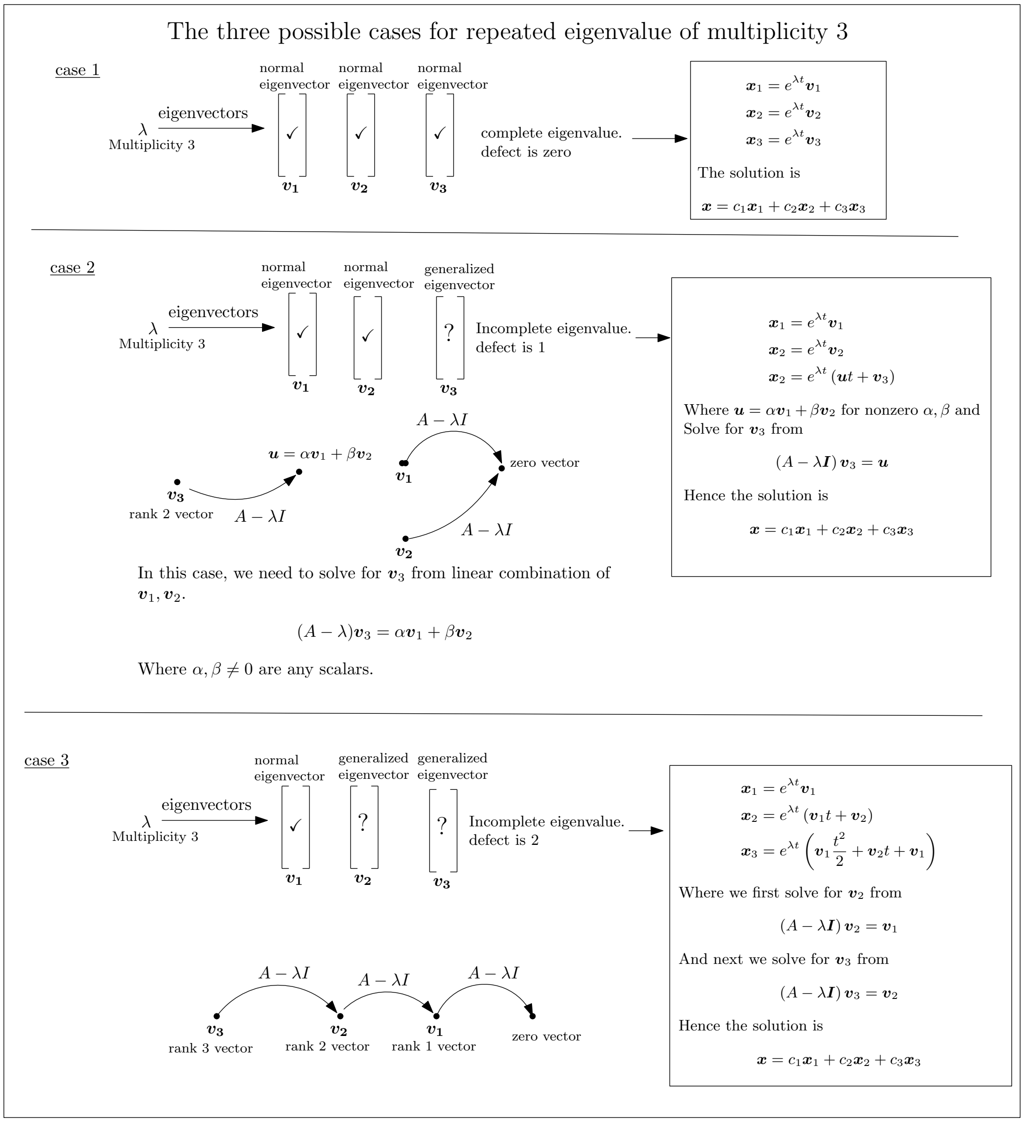


Figure 610: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -12 & 10 \\ 2 & -18 & 11 \\ 2 & -24 & 14 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{8}{3} \\ 1 \\ \frac{13}{6} \end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{8}{3} \\ 1 \\ \frac{13}{6} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -12 & 10 \\ 2 & -18 & 11 \\ 2 & -24 & 14 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{8}{3} \\ 1 \\ \frac{13}{6} \end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix} -\frac{131}{36} \\ 1 \\ \frac{43}{18} \end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -6. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{-6t}\\ &= \begin{bmatrix} -\mathrm{e}^{-6t} \\ \frac{\mathrm{e}^{-6t}}{2} \\ \mathrm{e}^{-6t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\ &= \mathrm{e}^{-6t}\left(\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{8}{3} \\ 1 \\ \frac{13}{6} \end{bmatrix}\right)\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-6t}(3t+8)}{3} \\ \frac{\mathrm{e}^{-6t}(2+t)}{2} \\ \frac{\mathrm{e}^{-6t}(6t+13)}{6} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t}\\ &= \left(\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} -\frac{8}{3} \\ 1 \\ \frac{13}{6} \end{bmatrix} t + \begin{bmatrix} -\frac{131}{36} \\ 1 \\ \frac{43}{18} \end{bmatrix}\right) \mathrm{e}^{-6t}\\ &= \begin{bmatrix} -\frac{\mathrm{e}^{-6t}(18t^2+96t+131)}{36} \\ \frac{\mathrm{e}^{-6t}(2+t)^2}{4} \\ \frac{\mathrm{e}^{-6t}(9t^2+39t+43)}{18} \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{-6t} \\ \frac{e^{-6t}}{2} \\ e^{-6t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-6t}\left(-t-\frac{8}{3}\right) \\ e^{-6t}\left(\frac{t}{2}+1\right) \\ e^{-6t}\left(t+\frac{13}{6}\right) \end{bmatrix} + c_3 \begin{bmatrix} e^{-6t}\left(-\frac{1}{2}t^2-\frac{8}{3}t-\frac{131}{36}\right) \\ e^{-6t}\left(\frac{1}{4}t^2+t+1\right) \\ e^{-6t}\left(\frac{1}{2}t^2+\frac{13}{6}t+\frac{43}{18}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(\left(t^2+\frac{16}{3}t+\frac{131}{18}\right)c_3+2c_2t+2c_1+\frac{16c_2}{3}\right)e^{-6t}}{2} \\ \frac{\left((2+t)^2c_3+2c_2t+2c_1+4c_2\right)e^{-6t}}{4} \\ \frac{\left((9t^2+39t+43)c_3+18c_2t+18c_1+39c_2\right)e^{-6t}}{18} \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 84

```
dsolve([diff(y__1(t),t)=-2*y__1(t)-12*y__2(t)+10*y__3(t),diff(y__2(t),t)=2*y__1(t)-24*y__2(t
```

$$y_1(t) = e^{-6t}\left(c_3t^2+c_2t+c_1\right)$$
$$y_2(t) = -\frac{e^{-6t}(18c_3t^2+18c_2t-24c_3t+18c_1-12c_2+5c_3)}{36}$$
$$y_3(t) = -\frac{e^{-6t}(6c_3t^2+6c_2t-6c_3t+6c_1-3c_2+c_3)}{6}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 131

```
DSolve[{y1'[t]==-2*y1[t]-12*y2[t]+10*y3[t],y2'[t]==2*y1[t]-24*y2[t]+11*y3[t],y3'[t]==2*y1[t]
```

$$\text{y1}(t) \to e^{-6t}\left(c_1\left(6t^2+4t+1\right)+2t(c_3(12t+5)-6c_2(3t+1))\right)$$
$$\text{y2}(t) \to e^{-6t}\left(-3(c_1-6c_2+4c_3)t^2+(2c_1-18c_2+11c_3)t+c_2\right)$$
$$\text{y3}(t) \to e^{-6t}\left(-6(c_1-6c_2+4c_3)t^2+2(c_1-12c_2+7c_3)t+c_3\right)$$

22.29 problem section 10.5, problem 29

Internal problem ID [1632]
Internal file name [`OUTPUT/1633_Sunday_June_05_2022_02_25_12_AM_42281549/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_1(t) - 12y_2(t) + 8y_3(t)\\
y_2'(t) &= y_1(t) - 9y_2(t) + 4y_3(t)\\
y_3'(t) &= y_1(t) - 6y_2(t) + y_3(t)
\end{aligned}
$$

22.29.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(1+2t) & -12t\,\mathrm{e}^{-3t} & 8t\,\mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(1-6t) & 4t\,\mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} & -6t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t}(4t+1) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-3t}(1+2t) & -12t\,e^{-3t} & 8t\,e^{-3t} \\ t\,e^{-3t} & e^{-3t}(1-6t) & 4t\,e^{-3t} \\ t\,e^{-3t} & -6t\,e^{-3t} & e^{-3t}(4t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1+2t)\,c_1 - 12t\,e^{-3t}c_2 + 8t\,e^{-3t}c_3 \\ t\,e^{-3t}c_1 + e^{-3t}(1-6t)\,c_2 + 4t\,e^{-3t}c_3 \\ t\,e^{-3t}c_1 - 6t\,e^{-3t}c_2 + e^{-3t}(4t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} 2\left((c_1 - 6c_2 + 4c_3)\,t + \frac{c_1}{2}\right)e^{-3t} \\ ((c_1 - 6c_2 + 4c_3)\,t + c_2)\,e^{-3t} \\ ((c_1 - 6c_2 + 4c_3)\,t + c_3)\,e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.29.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -12 & 8 \\ 1 & -9-\lambda & 4 \\ 1 & -6 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -12 & 8 \\ 1 & -6 & 4 \\ 1 & -6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -12 & 8 & 0 \\ 1 & -6 & 4 & 0 \\ 1 & -6 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -12 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -6 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -12 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -12 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 6t - 4s\}$

Hence the solution is

$$\begin{bmatrix} 6t - 4s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 6t - 4s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} 6t - 4s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 6t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} 6t - 4s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	3	2	Yes	$\begin{bmatrix} -4 & 6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

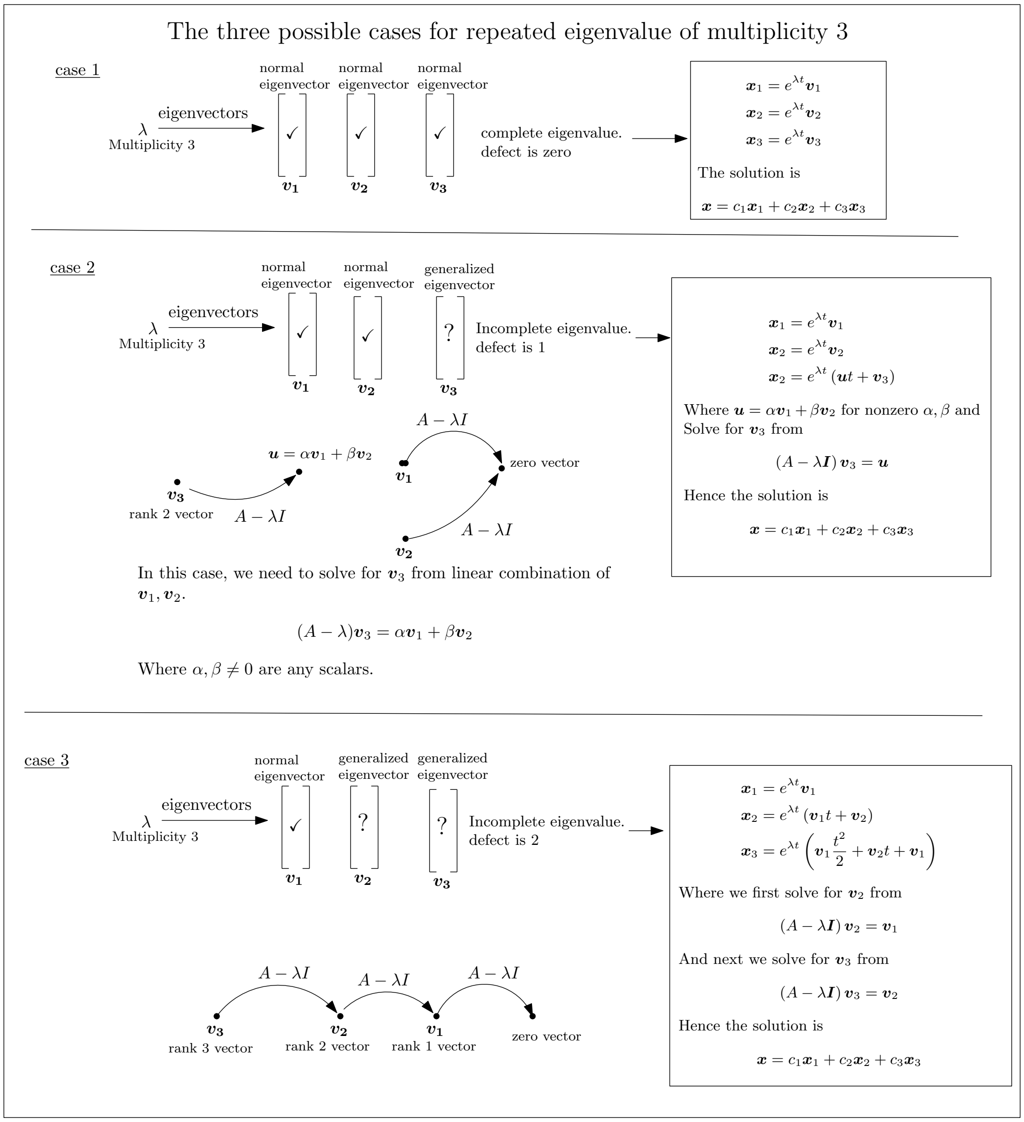


Figure 611: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$
\begin{aligned}
(A-\lambda I)^2 &= \left(\begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} - -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
\begin{aligned}
(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -12 & 8 \\ 1 & -6 & 4 \\ 1 & -6 & 4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2\eta_1 - 12\eta_2 + 8\eta_3 \\ \eta_1 - 6\eta_2 + 4\eta_3 \\ \eta_1 - 6\eta_2 + 4\eta_3 \end{bmatrix} &= \begin{bmatrix} -4\alpha + 6\beta \\ \beta \\ \alpha \end{bmatrix}
\end{aligned}
$$

Expanding the above gives the following equations equations

$$\begin{aligned}2\eta_1 - 12\eta_2 + 8\eta_3 &= -4\alpha + 6\beta \\ \eta_1 - 6\eta_2 + 4\eta_3 &= \beta \\ \eta_1 - 6\eta_2 + 4\eta_3 &= \alpha\end{aligned}$$

solving for α, β from the above gives

$$\begin{aligned}2\eta_1 - 12\eta_2 + 8\eta_3 &= -4\alpha + 6\beta \\ \eta_1 - 6\eta_2 + 4\eta_3 &= \beta\end{aligned}$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$[\eta_1 = 0, \eta_2 = 1, \eta_3 = 0]$$

Hence we found the missing generalized eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Which implies that

$$\begin{aligned}\alpha &= -6 \\ \beta &= -6\end{aligned}$$

Therefore

$$\begin{aligned}\vec{u} &= \alpha\vec{v}_1 + \beta\vec{v}_2 \\ &= -6\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + (-6)\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -12 \\ -6 \\ -6 \end{bmatrix}\end{aligned}$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue -3. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \left[\begin{array}{c}-4\\ 0\\ 1\end{array}\right] \mathrm{e}^{-3t}\\ &= \left[\begin{array}{c}-4\,\mathrm{e}^{-3t}\\ 0\\ \mathrm{e}^{-3t}\end{array}\right]\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t}\\ &= \left[\begin{array}{c}6\\ 1\\ 0\end{array}\right] \mathrm{e}^{-3t}\\ &= \left[\begin{array}{c}6\,\mathrm{e}^{-3t}\\ \mathrm{e}^{-3t}\\ 0\end{array}\right]\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t}\\ &= \left(\left[\begin{array}{c}-12\\ -6\\ -6\end{array}\right] t + \left[\begin{array}{c}0\\ 1\\ 0\end{array}\right]\right) \mathrm{e}^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\left[\begin{array}{c}y_1(t)\\ y_2(t)\\ y_3(t)\end{array}\right] = c_1 \left[\begin{array}{c}-4\,\mathrm{e}^{-3t}\\ 0\\ \mathrm{e}^{-3t}\end{array}\right] + c_2 \left[\begin{array}{c}6\,\mathrm{e}^{-3t}\\ \mathrm{e}^{-3t}\\ 0\end{array}\right] + c_3 \left[\begin{array}{c}-12t\,\mathrm{e}^{-3t}\\ \mathrm{e}^{-3t}(1-6t)\\ -6t\,\mathrm{e}^{-3t}\end{array}\right]$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} (-12tc_3 - 4c_1 + 6c_2)\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t}(-6tc_3 + c_2 + c_3) \\ \mathrm{e}^{-3t}(-6tc_3 + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 53

```
dsolve([diff(y__1(t),t)=-1*y__1(t)-12*y__2(t)+8*y__3(t),diff(y__2(t),t)=1*y__1(t)-9*y__2(t)+
```

$$y_1(t) = \mathrm{e}^{-3t}(c_3t + c_2)$$
$$y_2(t) = \frac{\mathrm{e}^{-3t}(c_3t + 2c_1 + c_2)}{2}$$
$$y_3(t) = \frac{\mathrm{e}^{-3t}(4c_3t + 12c_1 + 4c_2 + c_3)}{8}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 79

```
DSolve[{y1'[t]==-1*y1[t]-12*y2[t]+8*y3[t],y2'[t]==1*y1[t]-9*y2[t]+4*y3[t],y3'[t]==1*y1[t]-6*
```

$$\mathrm{y1}(t) \to e^{-3t}(2c_1t - 12c_2t + 8c_3t + c_1)$$
$$\mathrm{y2}(t) \to e^{-3t}((c_1 - 6c_2 + 4c_3)t + c_2)$$
$$\mathrm{y3}(t) \to e^{-3t}((c_1 - 6c_2 + 4c_3)t + c_3)$$

22.30 problem section 10.5, problem 30

Internal problem ID [1633]
Internal file name [`OUTPUT/1634_Sunday_June_05_2022_02_25_14_AM_43649733/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -4y_1(t) - y_3(t) \\
y_2'(t) &= -y_1(t) - 3y_2(t) - y_3(t) \\
y_3'(t) &= y_1(t) - 2y_3(t)
\end{aligned}
$$

22.30.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-3t}(1-t) & 0 & -t\,\mathrm{e}^{-3t} \\ -t\,\mathrm{e}^{-3t} & \mathrm{e}^{-3t} & -t\,\mathrm{e}^{-3t} \\ t\,\mathrm{e}^{-3t} & 0 & \mathrm{e}^{-3t}(t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-3t}(1-t) & 0 & -t\,e^{-3t} \\ -t\,e^{-3t} & e^{-3t} & -t\,e^{-3t} \\ t\,e^{-3t} & 0 & e^{-3t}(t+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-3t}(1-t)\,c_1 - t\,e^{-3t}c_3 \\ -t\,e^{-3t}c_1 + e^{-3t}c_2 - t\,e^{-3t}c_3 \\ t\,e^{-3t}c_1 + e^{-3t}(t+1)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -(c_1(-1+t) + c_3 t)\,e^{-3t} \\ -((c_1+c_3)\,t - c_2)\,e^{-3t} \\ e^{-3t}(tc_1 + c_3 t + c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.30.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 0 & -1 \\ -1 & -3-\lambda & -1 \\ 1 & 0 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -s\}$

Hence the solution is

$$\begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-3	3	2	Yes	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -3 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

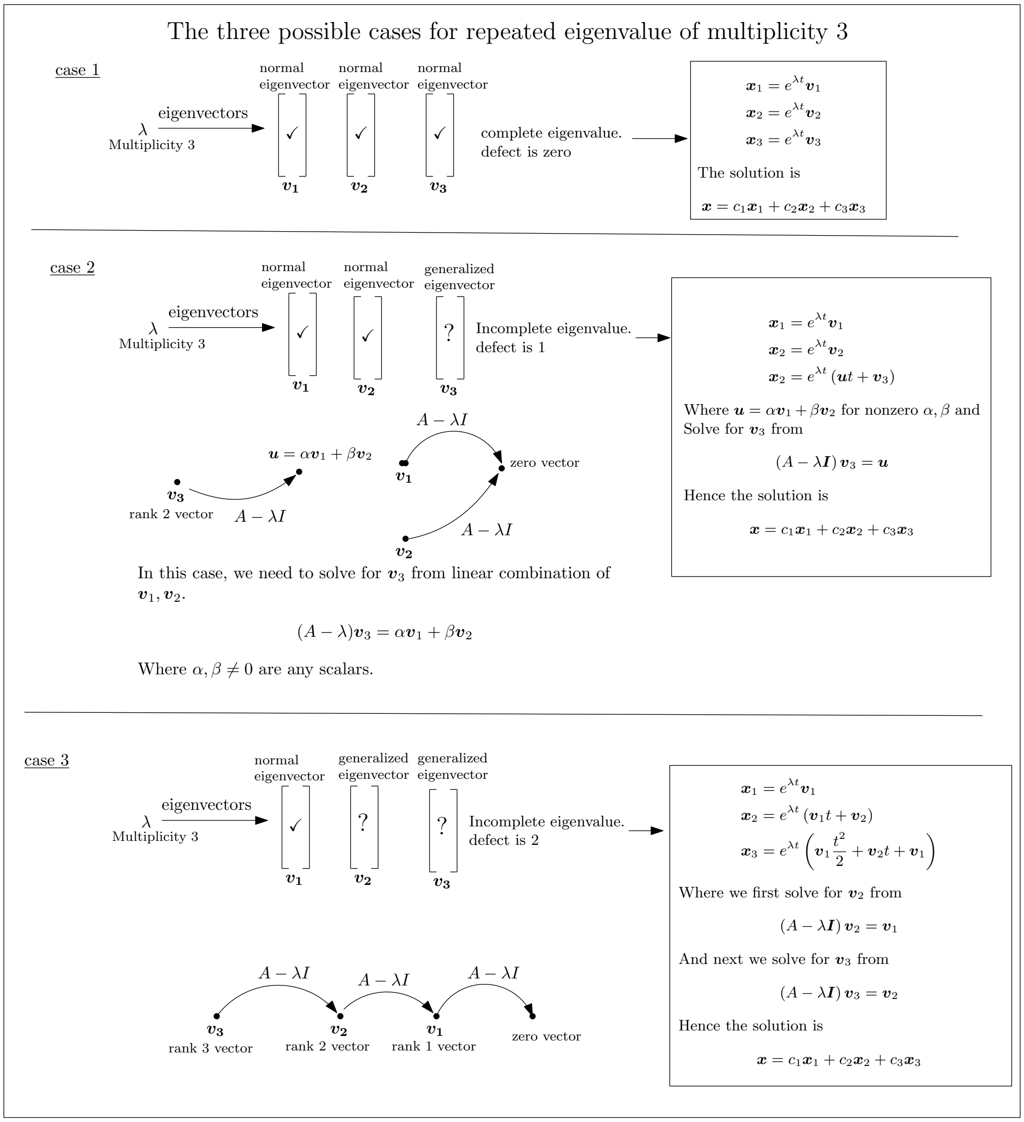


Figure 612: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$
(A-\lambda I)^2 = \left(\begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} - -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2
$$
$$
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha\vec{v}_1 + \beta\vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$
$$
\begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$
$$
\begin{bmatrix} -\eta_1 - \eta_3 \\ -\eta_1 - \eta_3 \\ \eta_1 + \eta_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \\ \alpha \end{bmatrix}
$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
-\eta_1 - \eta_3 &= -\alpha \\
-\eta_1 - \eta_3 &= \beta \\
\eta_1 + \eta_3 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
-\eta_1 - \eta_3 &= -\alpha \\
-\eta_1 - \eta_3 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$[\eta_1 = -1, \eta_3 = 0]$$

Hence we found the missing generalized eigenvector

$$\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Which implies that

$$
\begin{aligned}
\alpha &= -1 \\
\beta &= 1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= -1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue -3. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} -\mathrm{e}^{-3t} \\ 0 \\ \mathrm{e}^{-3t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-3t} \\ &= \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) \mathrm{e}^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-3t} \\ 0 \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-3t}(-1+t) \\ t\,\mathrm{e}^{-3t} \\ -t\,\mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} ((-1+t)\,c_3 - c_1)\,e^{-3t} \\ e^{-3t}(c_3 t + c_2) \\ e^{-3t}(-c_3 t + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve([diff(y__1(t),t)=-4*y__1(t)-0*y__2(t)-1*y__3(t),diff(y__2(t),t)=-1*y__1(t)-3*y__2(t)-
```

$$y_1(t) = e^{-3t}(c_3 t + c_2)$$
$$y_2(t) = e^{-3t}(c_3 t + c_1 + c_2)$$
$$y_3(t) = -e^{-3t}(c_3 t + c_2 + c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 63

```
DSolve[{y1'[t]==-4*y1[t]-0*y2[t]-1*y3[t],y2'[t]==-1*y1[t]-3*y2[t]-1*y3[t],y3'[t]==1*y1[t]-0*
```

$$\text{y1}(t) \to e^{-3t}(c_1(-t) - c_3 t + c_1)$$
$$\text{y2}(t) \to e^{-3t}(c_2 - (c_1 + c_3)t)$$
$$\text{y3}(t) \to e^{-3t}((c_1 + c_3)t + c_3)$$

22.31 problem section 10.5, problem 31

Internal problem ID [1634]
Internal file name [`OUTPUT/1635_Sunday_June_05_2022_02_25_16_AM_27684398/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) - 3y_2(t) + 4y_3(t) \\
y_2'(t) &= 4y_1(t) + 5y_2(t) - 8y_3(t) \\
y_3'(t) &= 2y_1(t) + 3y_2(t) - 5y_3(t)
\end{aligned}
$$

22.31.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{-t}(1-2t) & -3t\,\mathrm{e}^{-t} & 4t\,\mathrm{e}^{-t} \\ 4t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1+6t) & -8t\,\mathrm{e}^{-t} \\ 2t\,\mathrm{e}^{-t} & 3t\,\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-4t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(1-2t) & -3t\,e^{-t} & 4t\,e^{-t} \\ 4t\,e^{-t} & e^{-t}(1+6t) & -8t\,e^{-t} \\ 2t\,e^{-t} & 3t\,e^{-t} & e^{-t}(1-4t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(1-2t)\,c_1 - 3t\,e^{-t}c_2 + 4t\,e^{-t}c_3 \\ 4t\,e^{-t}c_1 + e^{-t}(1+6t)\,c_2 - 8t\,e^{-t}c_3 \\ 2t\,e^{-t}c_1 + 3t\,e^{-t}c_2 + e^{-t}(1-4t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} -2\left(\left(c_1 + \frac{3c_2}{2} - 2c_3\right)t - \frac{c_1}{2}\right)e^{-t} \\ 4\left(\left(c_1 + \frac{3c_2}{2} - 2c_3\right)t + \frac{c_2}{4}\right)e^{-t} \\ 2\left(\left(c_1 + \frac{3c_2}{2} - 2c_3\right)t + \frac{c_3}{2}\right)e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.31.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -3 & 4 \\ 4 & 5-\lambda & -8 \\ 2 & 3 & -5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 & 4 \\ 4 & 6 & -8 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -3 & 4 & 0 \\ 4 & 6 & -8 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} -2 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2} + 2s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} + 2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} + 2s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{3t}{2} + 2s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{3t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{3t}{2} + 2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	3	2	Yes	$\begin{bmatrix} 2 & -\frac{3}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

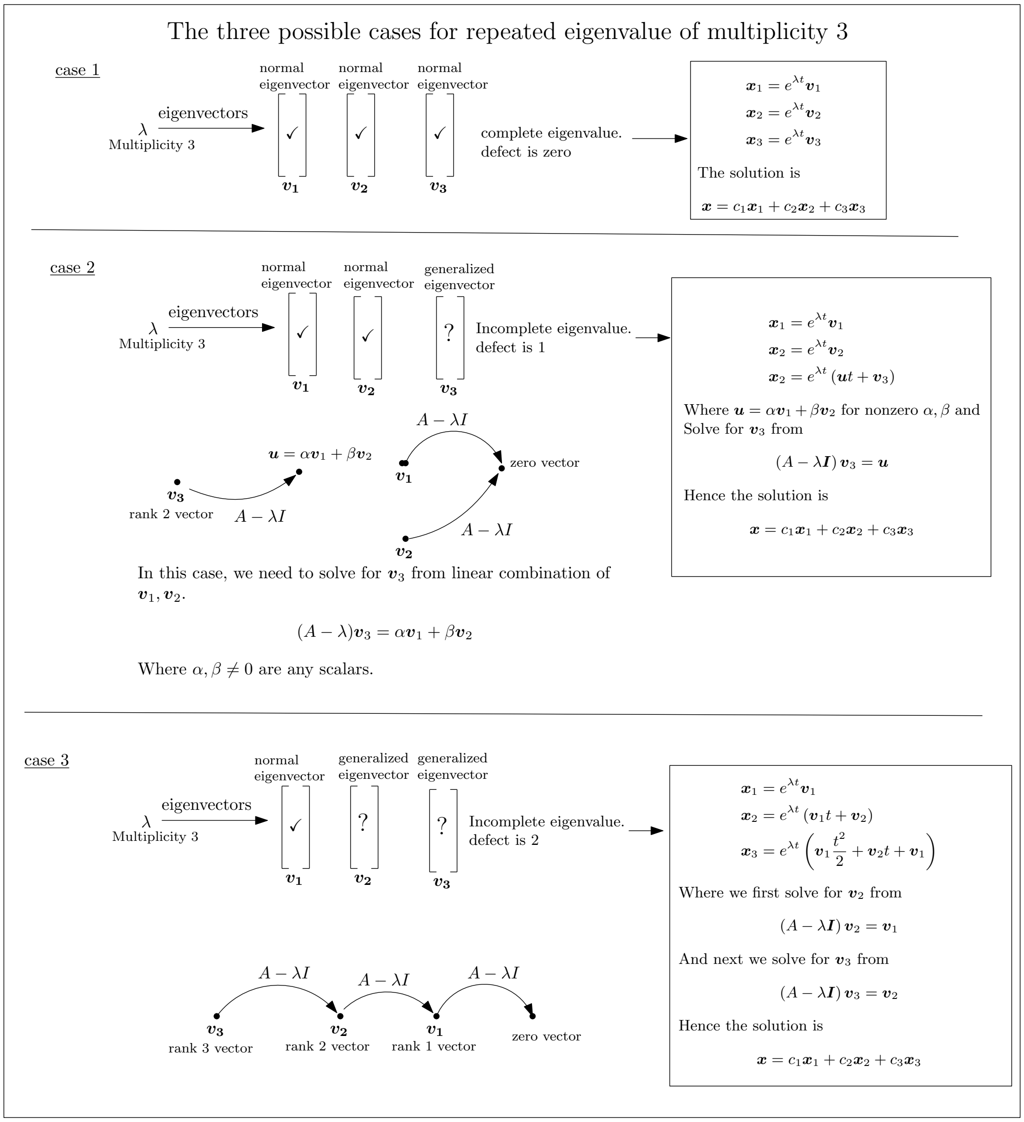


Figure 613: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$
(A-\lambda I)^2 = \left(\begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} - -1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2
$$

$$
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -2 & -3 & 4 \\ 4 & 6 & -8 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -2\eta_1 - 3\eta_2 + 4\eta_3 \\ 4\eta_1 + 6\eta_2 - 8\eta_3 \\ 2\eta_1 + 3\eta_2 - 4\eta_3 \end{bmatrix} = \begin{bmatrix} 2\alpha - \frac{3\beta}{2} \\ \beta \\ \alpha \end{bmatrix}
$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
-2\eta_1 - 3\eta_2 + 4\eta_3 &= 2\alpha - \frac{3\beta}{2} \\
4\eta_1 + 6\eta_2 - 8\eta_3 &= \beta \\
2\eta_1 + 3\eta_2 - 4\eta_3 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
-2\eta_1 - 3\eta_2 + 4\eta_3 &= 2\alpha - \frac{3\beta}{2} \\
4\eta_1 + 6\eta_2 - 8\eta_3 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = 0, \eta_2 = 0, \eta_3 = 1]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= -4 \\
\beta &= -8
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= -4 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + (-8) \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ -8 \\ -4 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue -1. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} -\frac{3\,\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} 4 \\ -8 \\ -4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \mathrm{e}^{-t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 4t\,\mathrm{e}^{-t} \\ -8t\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t}(1-4t) \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} e^{-t}\left(2c_1 - \frac{3c_2}{2} + 4tc_3\right) \\ e^{-t}(-8tc_3 + c_2) \\ e^{-t}(-4tc_3 + c_1 + c_3) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 53

```
dsolve([diff(y__1(t),t)=-3*y__1(t)-3*y__2(t)+4*y__3(t),diff(y__2(t),t)=4*y__1(t)+5*y__2(t)-8
```

$$y_1(t) = e^{-t}(c_3t + c_2)$$
$$y_2(t) = e^{-t}(-2c_3t + c_1 - 2c_2)$$
$$y_3(t) = \frac{e^{-t}(-4c_3t + 3c_1 - 4c_2 + c_3)}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 83

```
DSolve[{y1'[t]==-3*y1[t]-3*y2[t]+4*y3[t],y2'[t]==4*y1[t]+5*y2[t]-8*y3[t],y3'[t]==2*y1[t]+3*y
```

$$\text{y1}(t) \to e^{-t}(-2c_1t - 3c_2t + 4c_3t + c_1)$$
$$\text{y2}(t) \to e^{-t}((4c_1 + 6c_2 - 8c_3)t + c_2)$$
$$\text{y3}(t) \to e^{-t}((2c_1 + 3c_2 - 4c_3)t + c_3)$$

22.32 problem section 10.5, problem 32

Internal problem ID [1635]
Internal file name [`OUTPUT/1636_Sunday_June_05_2022_02_25_17_AM_304506/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.5, constant coefficient homogeneous system II. Page 555
Problem number: section 10.5, problem 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) - y_2(t) \\
y_2'(t) &= y_1(t) - y_2(t) \\
y_3'(t) &= -y_1(t) - y_2(t) - 2y_3(t)
\end{aligned}
$$

22.32.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-2t}(1-t) & -t\,\mathrm{e}^{-2t} & 0 \\ t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}(t+1) & 0 \\ -t\,\mathrm{e}^{-2t} & -t\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t}(1-t) & -t\,e^{-2t} & 0 \\ t\,e^{-2t} & e^{-2t}(t+1) & 0 \\ -t\,e^{-2t} & -t\,e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t}(1-t)\,c_1 - t\,e^{-2t}c_2 \\ t\,e^{-2t}c_1 + e^{-2t}(t+1)\,c_2 \\ -t\,e^{-2t}c_1 - t\,e^{-2t}c_2 + e^{-2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} -(c_1(-1+t) + c_2 t)\,e^{-2t} \\ e^{-2t}(tc_1 + c_2 t + c_2) \\ -((c_1+c_2)\,t - c_3)\,e^{-2t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

22.32.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -1 & 0 \\ 1 & -1-\lambda & 0 \\ -1 & -1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	3	2	Yes	$\begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

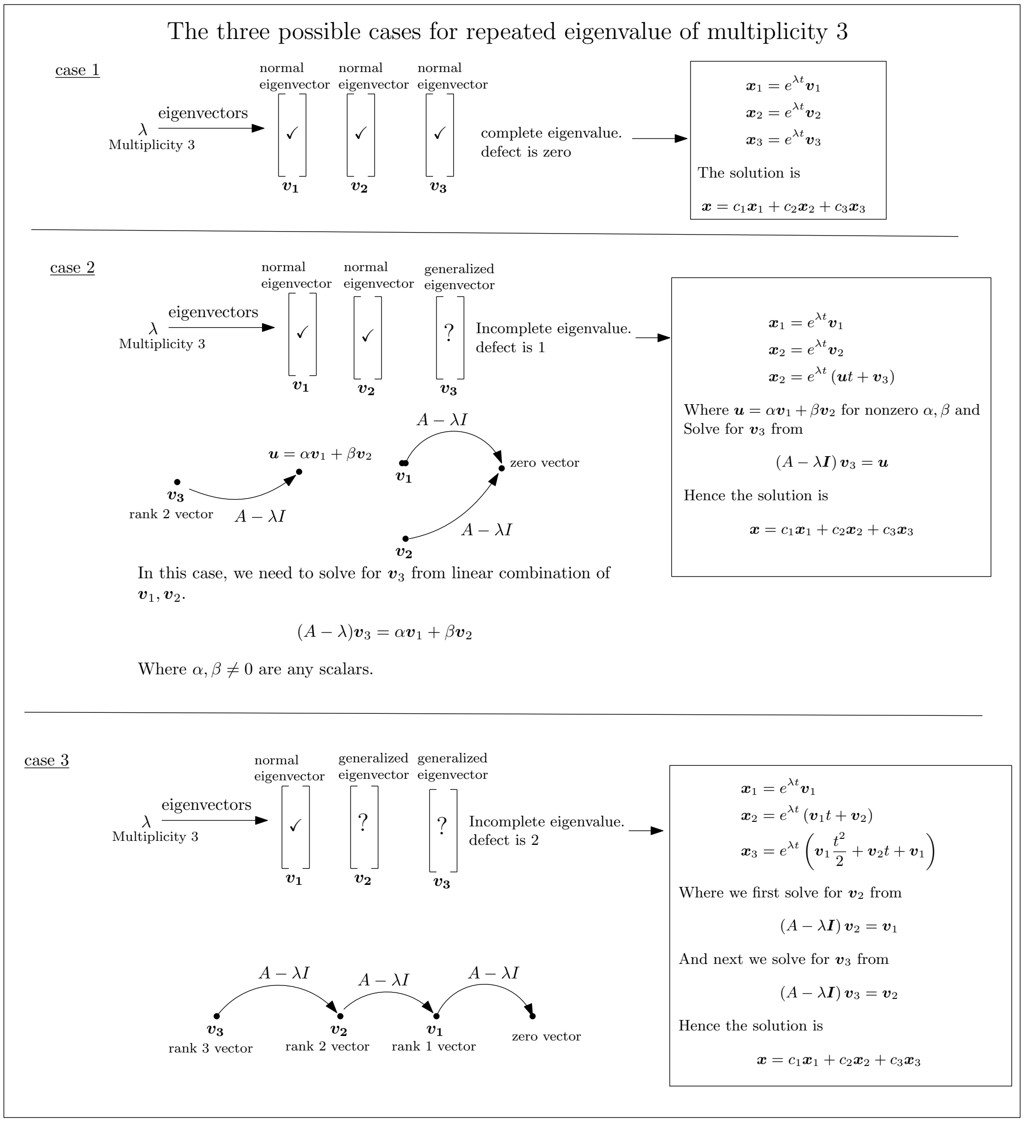


Figure 614: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$.

But

$$(A-\lambda I)^2 = \left(\begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} - -2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$\vec{u} = \alpha\vec{v}_1 + \beta\vec{v}_2$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$(A-\lambda I)\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\eta_1 - \eta_2 \\ \eta_1 + \eta_2 \\ -\eta_1 - \eta_2 \end{bmatrix} = \begin{bmatrix} -\beta \\ \beta \\ \alpha \end{bmatrix}$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
-\eta_1 - \eta_2 &= -\beta \\
\eta_1 + \eta_2 &= \beta \\
-\eta_1 - \eta_2 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
-\eta_1 - \eta_2 &= -\beta \\
\eta_1 + \eta_2 &= \beta
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$
[\eta_1 = -1, \eta_2 = 0]
$$

Hence we found the missing generalized eigenvector

$$
\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
$$

Which implies that

$$
\begin{aligned}
\alpha &= 1 \\
\beta &= -1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\
&= 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue -2. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{-2t}\\ &= \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t}\\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \mathrm{e}^{-2t}\\ &= \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t}\\ &= \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) \mathrm{e}^{-2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-2t}(-1+t) \\ -t\,\mathrm{e}^{-2t} \\ t\,\mathrm{e}^{-2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} ((-1+t)\,c_3 - c_2)\,e^{-2t} \\ e^{-2t}(-c_3 t + c_2) \\ e^{-2t}(c_3 t + c_1) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve([diff(y__1(t),t)=-3*y__1(t)-1*y__2(t)+0*y__3(t),diff(y__2(t),t)=1*y__1(t)-1*y__2(t)+0
```

$$y_1(t) = e^{-2t}(c_3 t + c_2)$$
$$y_2(t) = -e^{-2t}(c_3 t + c_2 + c_3)$$
$$y_3(t) = e^{-2t}(c_3 t + c_1 + c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 63

```
DSolve[{y1'[t]==-3*y1[t]-1*y2[t]+0*y3[t],y2'[t]==1*y1[t]-1*y2[t]+0*y3[t],y3'[t]==-1*y1[t]-1*
```

$$\mathrm{y1}(t) \to e^{-2t}(c_1(-t) - c_2 t + c_1)$$
$$\mathrm{y2}(t) \to e^{-2t}((c_1 + c_2)t + c_2)$$
$$\mathrm{y3}(t) \to e^{-2t}(c_3 - (c_1 + c_2)t)$$

23 Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566

23.1 problem section 10.6, problem 1

Internal problem ID [1636]
Internal file name [`OUTPUT/1637_Sunday_June_05_2022_02_25_19_AM_86847735/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -y_1(t) + 2y_2(t) \\
y_2'(t) &= -5y_1(t) + 5y_2(t)
\end{aligned}
$$

23.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{2t}\cos(t) - 3\,e^{2t}\sin(t) & 2\,e^{2t}\sin(t) \\ -5\,e^{2t}\sin(t) & e^{2t}\cos(t) + 3\,e^{2t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(-3\sin(t) + \cos(t)) & 2\,e^{2t}\sin(t) \\ -5\,e^{2t}\sin(t) & e^{2t}(\cos(t) + 3\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(-3\sin(t)+\cos(t)) & 2\,e^{2t}\sin(t) \\ -5\,e^{2t}\sin(t) & e^{2t}(\cos(t)+3\sin(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(-3\sin(t)+\cos(t))\,c_1 + 2\,e^{2t}\sin(t)\,c_2 \\ -5\,e^{2t}\sin(t)\,c_1 + e^{2t}(\cos(t)+3\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-3c_1+2c_2)\sin(t)+c_1\cos(t))\,e^{2t} \\ e^{2t}(c_2\cos(t)-5c_1\sin(t)+3\sin(t)\,c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 2 \\ -5 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 + i$	1	complex eigenvalue
$2 - i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} - (2 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 + i & 2 \\ -5 & 3 + i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 + i & 2 & 0 \\ -5 & 3 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 + i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 + i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{5} + \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5} + \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3 + i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} - (2+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 - i & 2 \\ -5 & 3 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 - i & 2 & 0 \\ -5 & 3 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 - i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3-i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+i$	1	1	No	$\left[\begin{array}{c}\frac{3}{5}-\frac{i}{5} \\ 1\end{array}\right]$
$2-i$	1	1	No	$\left[\begin{array}{c}\frac{3}{5}+\frac{i}{5} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}y_1(t) \\ y_2(t)\end{array}\right] = c_1\left[\begin{array}{c}\left(\frac{3}{5}-\frac{i}{5}\right)\mathrm{e}^{(2+i)t} \\ \mathrm{e}^{(2+i)t}\end{array}\right] + c_2\left[\begin{array}{c}\left(\frac{3}{5}+\frac{i}{5}\right)\mathrm{e}^{(2-i)t} \\ \mathrm{e}^{(2-i)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c}y_1(t) \\ y_2(t)\end{array}\right] = \left[\begin{array}{c}\left(\frac{3}{5}-\frac{i}{5}\right)c_1\mathrm{e}^{(2+i)t} + \left(\frac{3}{5}+\frac{i}{5}\right)c_2\mathrm{e}^{(2-i)t} \\ c_1\mathrm{e}^{(2+i)t} + c_2\mathrm{e}^{(2-i)t}\end{array}\right]$$

The following is the phase plot of the system.

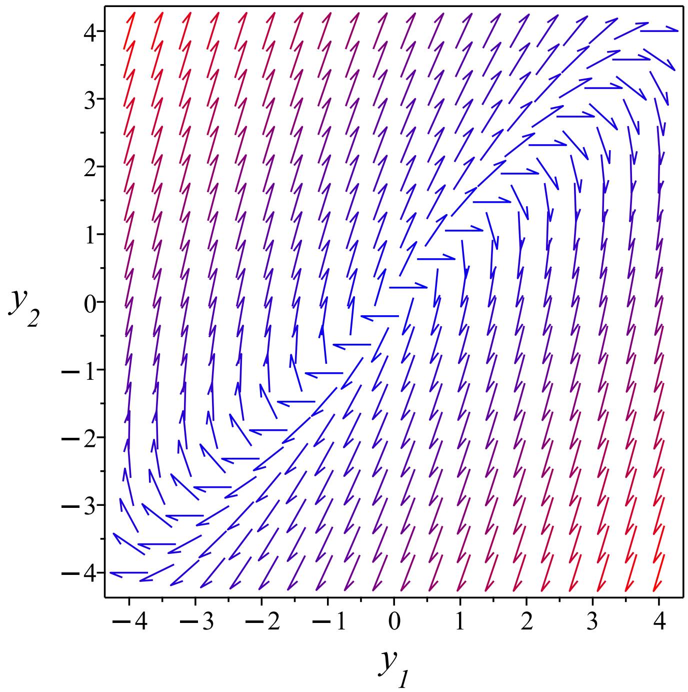


Figure 615: Phase plot

23.1.3 Maple step by step solution

Let's solve

$[y_1'(t) = -y_1(t) + 2y_2(t), y_2'(t) = -5y_1(t) + 5y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix}$$

- Rewrite the system as

  $\underline{y}^{\rightarrow}{}'(t) = A \cdot \underline{y}^{\rightarrow}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[2 - \mathrm{I}, \begin{bmatrix} \frac{3}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[2 + \mathrm{I}, \begin{bmatrix} \frac{3}{5} - \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[2 - \mathrm{I}, \begin{bmatrix} \frac{3}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$
- Solution from eigenpair

  $$\mathrm{e}^{(2-\mathrm{I})t} \cdot \begin{bmatrix} \frac{3}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$
- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{2t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \frac{3}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$
- Simplify expression

  $$\mathrm{e}^{2t} \cdot \begin{bmatrix} \left(\frac{3}{5} + \frac{\mathrm{I}}{5}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$
- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\underline{y}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{3\cos(t)}{5} + \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix}, \underline{y}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{3\sin(t)}{5} + \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}\right]$$
- General solution to the system of ODEs

  $\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1}(t) + c_2 \underline{y}^{\rightarrow}{}_{2}(t)$
- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{3\cos(t)}{5} + \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -\frac{3\sin(t)}{5} + \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3\left(\left(c_1+\frac{c_2}{3}\right)\cos(t)+\frac{\sin(t)(c_1-3c_2)}{3}\right)e^{2t}}{5} \\ e^{2t}(-c_2\sin(t)+c_1\cos(t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ y_1(t) = \frac{3\left(\left(c_1+\frac{c_2}{3}\right)\cos(t)+\frac{\sin(t)(c_1-3c_2)}{3}\right)e^{2t}}{5}, y_2(t) = e^{2t}(-c_2\sin(t)+c_1\cos(t)) \right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve([diff(y__1(t),t)=-1*y__1(t)+2*y__2(t),diff(y__2(t),t)=-5*y__1(t)+5*y__2(t)],singsol=a
```

$$y_1(t) = e^{2t}(c_1\sin(t) + c_2\cos(t))$$
$$y_2(t) = \frac{e^{2t}(3c_1\sin(t) - c_2\sin(t) + c_1\cos(t) + 3c_2\cos(t))}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 55

```
DSolve[{y1'[t]==-1*y1[t]+2*y2[t],y2'[t]==-5*y1[t]+5*y2[t]},{y1[t],y2[t]},t,IncludeSingularSo
```

$$\text{y1}(t) \rightarrow e^{2t}(c_1\cos(t) + (2c_2 - 3c_1)\sin(t))$$
$$\text{y2}(t) \rightarrow e^{2t}(c_2(3\sin(t) + \cos(t)) - 5c_1\sin(t))$$

23.2 problem section 10.6, problem 2

Internal problem ID [1637]
Internal file name [`OUTPUT/1638_Sunday_June_05_2022_02_25_21_AM_82160191/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -11y_1(t) + 4y_2(t) \\
y_2'(t) &= -26y_1(t) + 9y_2(t)
\end{aligned}
$$

23.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{-t}\cos(2t) - 5\,\mathrm{e}^{-t}\sin(2t) & 2\,\mathrm{e}^{-t}\sin(2t) \\ -13\,\mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}\cos(2t) + 5\,\mathrm{e}^{-t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t}(\cos(2t) - 5\sin(2t)) & 2\,\mathrm{e}^{-t}\sin(2t) \\ -13\,\mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}(\cos(2t) + 5\sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(\cos(2t) - 5\sin(2t)) & 2\,e^{-t}\sin(2t) \\ -13\,e^{-t}\sin(2t) & e^{-t}(\cos(2t) + 5\sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(2t) - 5\sin(2t))\,c_1 + 2\,e^{-t}\sin(2t)\,c_2 \\ -13\,e^{-t}\sin(2t)\,c_1 + e^{-t}(\cos(2t) + 5\sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-5c_1 + 2c_2)\sin(2t) + c_1\cos(2t))\,e^{-t} \\ e^{-t}(c_2\cos(2t) - 13c_1\sin(2t) + 5\sin(2t)\,c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -11 - \lambda & 4 \\ -26 & 9 - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1-2i$	1	complex eigenvalue
$-1+2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} - (-1-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -10+2i & 4 \\ -26 & 10+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -10+2i & 4 & 0 \\ -26 & 10+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{5}{2} - \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -10+2i & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -10+2i & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \left(\frac{5}{13} + \frac{i}{13}\right) t\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{13} + \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{13} + \frac{i}{13}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{13} + \frac{I}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{13} + \frac{i}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{13} + \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{13} + \frac{i}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{13} + \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 5 + i \\ 13 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} - (-1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -10 - 2i & 4 \\ -26 & 10 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -10 - 2i & 4 & 0 \\ -26 & 10 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{5}{2} + \frac{i}{2}\right) R_1 \implies \left[\begin{array}{cc|c} -10 - 2i & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -10-2i & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{13} - \frac{i}{13}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{13} - \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{13} - \frac{i}{13}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{13} - \frac{I}{13}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{13} - \frac{i}{13} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{13} - \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{13} - \frac{i}{13} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{13} - \frac{I}{13}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 5-i \\ 13 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$-1+2i$	1	1	No	$\left[\begin{array}{c} \frac{5}{13}-\frac{i}{13} \\ 1 \end{array}\right]$
$-1-2i$	1	1	No	$\left[\begin{array}{c} \frac{5}{13}+\frac{i}{13} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(\frac{5}{13}-\frac{i}{13}\right) e^{(-1+2i)t} \\ e^{(-1+2i)t} \end{array}\right] + c_2 \left[\begin{array}{c} \left(\frac{5}{13}+\frac{i}{13}\right) e^{(-1-2i)t} \\ e^{(-1-2i)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] = \left[\begin{array}{c} \left(\frac{5}{13}-\frac{i}{13}\right) c_1 e^{(-1+2i)t} + \left(\frac{5}{13}+\frac{i}{13}\right) c_2 e^{(-1-2i)t} \\ c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \end{array}\right]$$

The following is the phase plot of the system.

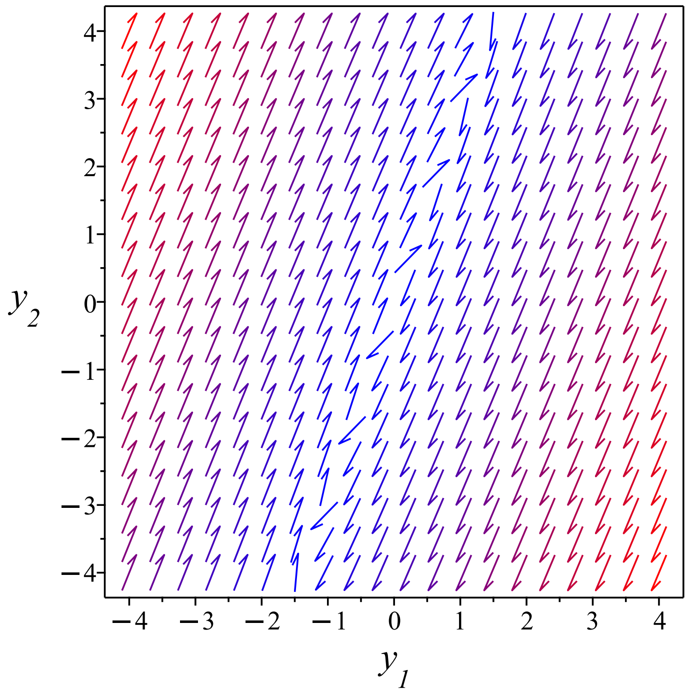


Figure 616: Phase plot

23.2.3 Maple step by step solution

Let's solve

$[y_1'(t) = -11y_1(t) + 4y_2(t), y_2'(t) = -26y_1(t) + 9y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-2\,\mathrm{I}, \begin{bmatrix} \frac{5}{13}+\frac{\mathrm{I}}{13} \\ 1 \end{bmatrix}\right], \left[-1+2\,\mathrm{I}, \begin{bmatrix} \frac{5}{13}-\frac{\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \begin{bmatrix} \frac{5}{13}+\frac{\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{5}{13}+\frac{\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{5}{13}+\frac{\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \begin{bmatrix} \left(\frac{5}{13}+\frac{\mathrm{I}}{13}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{y}^{\rightarrow}{}_{1}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{5\cos(2t)}{13}+\frac{\sin(2t)}{13} \\ \cos(2t) \end{bmatrix}, \underline{y}^{\rightarrow}{}_{2}(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{5\sin(2t)}{13}+\frac{\cos(2t)}{13} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1}(t) + c_2 \underline{y}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 e^{-t} \cdot \begin{bmatrix} \frac{5\cos(2t)}{13} + \frac{\sin(2t)}{13} \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -\frac{5\sin(2t)}{13} + \frac{\cos(2t)}{13} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{5\left(\left(c_1+\frac{c_2}{5}\right)\cos(2t)+\frac{\sin(2t)(c_1-5c_2)}{5}\right)e^{-t}}{13} \\ e^{-t}(-c_2\sin(2t)+c_1\cos(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = \frac{5\left(\left(c_1+\frac{c_2}{5}\right)\cos(2t)+\frac{\sin(2t)(c_1-5c_2)}{5}\right)e^{-t}}{13}, y_2(t) = e^{-t}(-c_2\sin(2t)+c_1\cos(2t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
dsolve([diff(y__1(t),t)=-11*y__1(t)+4*y__2(t),diff(y__2(t),t)=-26*y__1(t)+9*y__2(t)],singsol
```

$$y_1(t) = e^{-t}(c_1\sin(2t)+c_2\cos(2t))$$
$$y_2(t) = \frac{e^{-t}(5c_1\sin(2t)-c_2\sin(2t)+c_1\cos(2t)+5c_2\cos(2t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 64

```
DSolve[{y1'[t]==-11*y1[t]+4*y2[t],y2'[t]==-26*y1[t]+9*y2[t]},{y1[t],y2[t]},t,IncludeSingular
```

$$\text{y1}(t) \to e^{-t}(c_1\cos(2t)+(2c_2-5c_1)\sin(2t))$$
$$\text{y2}(t) \to e^{-t}(c_2\cos(2t)+(5c_2-13c_1)\sin(2t))$$

23.3 problem section 10.6, problem 3

Internal problem ID [1638]
Internal file name [`OUTPUT/1639_Sunday_June_05_2022_02_25_23_AM_44078137/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= y_1(t) + 2y_2(t) \\
y_2'(t) &= -4y_1(t) + 5y_2(t)
\end{aligned}
$$

23.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{3t}\cos(2t) - e^{3t}\sin(2t) & e^{3t}\sin(2t) \\ -2\,e^{3t}\sin(2t) & e^{3t}\cos(2t) + e^{3t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(\cos(2t) - \sin(2t)) & e^{3t}\sin(2t) \\ -2\,e^{3t}\sin(2t) & e^{3t}(\cos(2t) + \sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(\cos(2t) - \sin(2t)) & e^{3t}\sin(2t) \\ -2\,e^{3t}\sin(2t) & e^{3t}(\cos(2t) + \sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(\cos(2t) - \sin(2t))\,c_1 + e^{3t}\sin(2t)\,c_2 \\ -2\,e^{3t}\sin(2t)\,c_1 + e^{3t}(\cos(2t) + \sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_2 - c_1)\sin(2t) + c_1\cos(2t))\,e^{3t} \\ e^{3t}(c_2\cos(2t) - 2c_1\sin(2t) + \sin(2t)\,c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ -4 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 13 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 - 2i$	1	complex eigenvalue
$3 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} - (3 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 + 2i & 2 \\ -4 & 2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 + 2i & 2 & 0 \\ -4 & 2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 + 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 + 2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} - (3 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - 2i & 2 \\ -4 & 2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ -4 & 2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i) R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 - 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3+2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
$3-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(3+2i)t} \\ e^{(3+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(3-2i)t} \\ e^{(3-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 e^{(3+2i)t} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(3-2i)t} \\ c_1 e^{(3+2i)t} + c_2 e^{(3-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

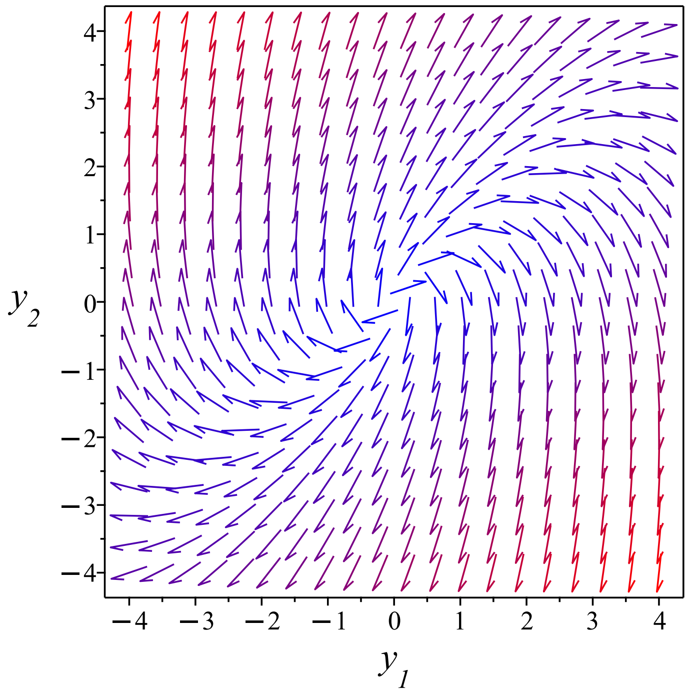


Figure 617: Phase plot

23.3.3 Maple step by step solution

Let's solve

$[y_1'(t) = y_1(t) + 2y_2(t), y_2'(t) = -4y_1(t) + 5y_2(t)]$

- Define vector

$$\underline{y^{\rightarrow}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \cdot \underline{y^{\rightarrow}}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y^{\rightarrow}}'(t) = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \cdot \underline{y^{\rightarrow}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[3+2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{3t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{3t} \cdot \begin{bmatrix} \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_1(t) = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2}+\frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \underline{\vec{y}}_2(t) = \mathrm{e}^{3t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2}-\frac{\sin(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1(t) + c_2 \underline{\vec{y}}_2(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{3t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2} + \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \frac{((c_1+c_2)\cos(2t)+\sin(2t)(c_1-c_2))e^{3t}}{2} \\ e^{3t}(-c_2 \sin(2t) + c_1 \cos(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = \frac{((c_1+c_2)\cos(2t)+\sin(2t)(c_1-c_2))e^{3t}}{2}, y_2(t) = e^{3t}(-c_2 \sin(2t) + c_1 \cos(2t))\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve([diff(y__1(t),t)=1*y__1(t)+2*y__2(t),diff(y__2(t),t)=-4*y__1(t)+5*y__2(t)],singsol=al
```

$$y_1(t) = e^{3t}(c_1 \sin(2t) + c_2 \cos(2t))$$
$$y_2(t) = e^{3t}(c_1 \sin(2t) - c_2 \sin(2t) + c_1 \cos(2t) + c_2 \cos(2t))$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 60

```
DSolve[{y1'[t]==1*y1[t]+2*y2[t],y2'[t]==-4*y1[t]+5*y2[t]},{y1[t],y2[t]},t,IncludeSingularSol
```

$$\text{y1}(t) \to e^{3t}(c_1 \cos(2t) + (c_2 - c_1)\sin(2t))$$
$$\text{y2}(t) \to e^{3t}(c_2 \cos(2t) + (c_2 - 2c_1)\sin(2t))$$

23.4 problem section 10.6, problem 4

Internal problem ID [1639]
Internal file name [`OUTPUT/1640_Sunday_June_05_2022_02_25_25_AM_17172965/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(t) &= 5y_1(t) - 6y_2(t) \\ y_2'(t) &= 3y_1(t) - y_2(t) \end{aligned}$$

23.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{2t}\cos(3t) + e^{2t}\sin(3t) & -2\,e^{2t}\sin(3t) \\ e^{2t}\sin(3t) & e^{2t}\cos(3t) - e^{2t}\sin(3t) \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}(\sin(3t) + \cos(3t)) & -2\,e^{2t}\sin(3t) \\ e^{2t}\sin(3t) & e^{2t}(\cos(3t) - \sin(3t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t}(\sin(3t)+\cos(3t)) & -2\,e^{2t}\sin(3t) \\ e^{2t}\sin(3t) & e^{2t}(\cos(3t)-\sin(3t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(\sin(3t)+\cos(3t))\,c_1 - 2\,e^{2t}\sin(3t)\,c_2 \\ e^{2t}\sin(3t)\,c_1 + e^{2t}(\cos(3t)-\sin(3t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_1-2c_2)\sin(3t)+c_1\cos(3t))\,e^{2t} \\ (\sin(3t)(-c_2+c_1)+c_2\cos(3t))\,e^{2t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -6 \\ 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 13 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2+3i$	1	complex eigenvalue
$2-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} - (2-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3+3i & -6 \\ 3 & -3+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3+3i & -6 & 0 \\ 3 & -3+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3+3i & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+3i & -6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2+3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} - (2+3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-3i & -6 \\ 3 & -3-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3-3i & -6 & 0 \\ 3 & -3-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3-3i & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-3i & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+3i$	1	1	No	$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$
$2-3i$	1	1	No	$\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} (1+i)\,\mathrm{e}^{(2+3i)t} \\ \mathrm{e}^{(2+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} (1-i)\,\mathrm{e}^{(2-3i)t} \\ \mathrm{e}^{(2-3i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} (1+i)\,c_1\mathrm{e}^{(2+3i)t} + (1-i)\,c_2\mathrm{e}^{(2-3i)t} \\ c_1\mathrm{e}^{(2+3i)t} + c_2\mathrm{e}^{(2-3i)t} \end{bmatrix}$$

The following is the phase plot of the system.

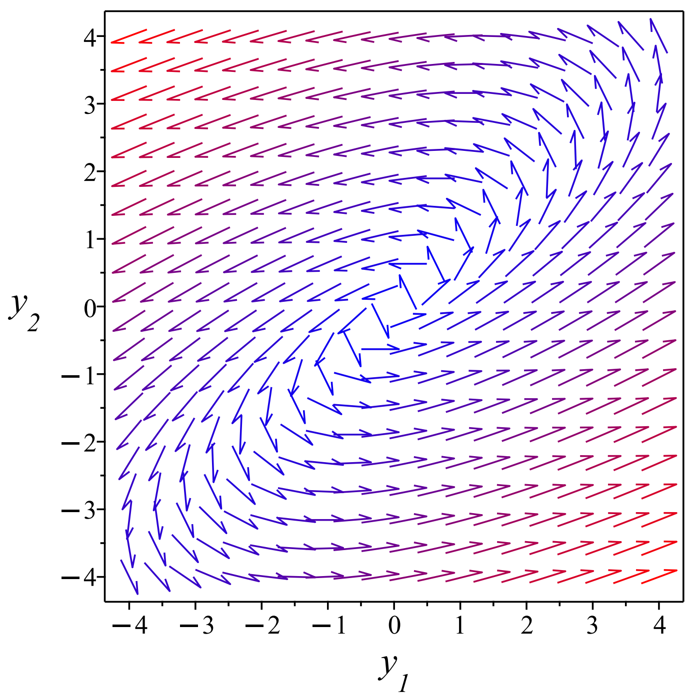


Figure 618: Phase plot

23.4.3 Maple step by step solution

Let's solve

$[y_1'(t) = 5y_1(t) - 6y_2(t)\,, y_2'(t) = 3y_1(t) - y_2(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2-3\,\mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2+3\,\mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-3\,\mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-3\,\mathrm{I})t} \cdot \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2t} \cdot \begin{bmatrix} (1-I)(\cos(3t) - I\sin(3t)) \\ \cos(3t) - I\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{y}^{\rightarrow}{}_{1}(t) = e^{2t} \cdot \begin{bmatrix} \cos(3t) - \sin(3t) \\ \cos(3t) \end{bmatrix}, \underline{y}^{\rightarrow}{}_{2}(t) = e^{2t} \cdot \begin{bmatrix} -\sin(3t) - \cos(3t) \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1}(t) + c_2 \underline{y}^{\rightarrow}{}_{2}(t)$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 e^{2t} \cdot \begin{bmatrix} \cos(3t) - \sin(3t) \\ \cos(3t) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -\sin(3t) - \cos(3t) \\ -\sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}((c_1 - c_2)\cos(3t) - \sin(3t)(c_1 + c_2)) \\ e^{2t}(-c_2\sin(3t) + c_1\cos(3t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = e^{2t}((c_1 - c_2)\cos(3t) - \sin(3t)(c_1 + c_2)), y_2(t) = e^{2t}(-c_2\sin(3t) + c_1\cos(3t))\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
dsolve([diff(y__1(t),t)=5*y__1(t)-6*y__2(t),diff(y__2(t),t)=3*y__1(t)-1*y__2(t)],singsol=all
```

$$y_1(t) = e^{2t}(c_1\sin(3t) + c_2\cos(3t))$$
$$y_2(t) = \frac{e^{2t}(c_1\sin(3t) + c_2\sin(3t) - c_1\cos(3t) + c_2\cos(3t))}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 60

```
DSolve[{y1'[t]==5*y1[t]-6*y2[t],y2'[t]==3*y1[t]-1*y2[t]},{y1[t],y2[t]},t,IncludeSingularSolu
```

$$\text{y1}(t) \to e^{2t}(c_1\cos(3t) + (c_1 - 2c_2)\sin(3t))$$
$$\text{y2}(t) \to e^{2t}(c_2\cos(3t) + (c_1 - c_2)\sin(3t))$$

23.5 problem section 10.6, problem 5

Internal problem ID [1640]
Internal file name [`OUTPUT/1641_Sunday_June_05_2022_02_25_27_AM_40857027/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) - 3y_2(t) + y_3(t)\\
y_2'(t) &= 2y_2(t) + 2y_3(t)\\
y_3'(t) &= 5y_1(t) + y_2(t) + y_3(t)
\end{aligned}
$$

23.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \text{Expression too large to display}\\
&= \text{Expression too large to display}
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \text{Expression too large to display} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \text{Expression too large to display} \\ &= \text{Expression too large to display}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -3 & 1 \\ 0 & 2-\lambda & 2 \\ 5 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 14\lambda + 40 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$$

$$\lambda_2 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eiger
$\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$	1	complex eige
$-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$	1	real eigenval
$\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$	1	complex eige

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} - \left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) [$$

$$\begin{bmatrix} \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1 \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2 \\ 5 & 1 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} \end{bmatrix} [$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} -3+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1 \\ 0 & 2+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2 \\ 5 & 1 & 1+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{604\ldots}\right.} \end{bmatrix}$$

$$R_3 = R_3 - \frac{5R_1}{-3+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -\ldots \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(\ldots\right.}{3\left(540+6\sqrt{\ldots}\right.} \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+36\left(\ldots\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(\ldots\right.} \end{bmatrix}$$

$$R_3 = R_3 - \frac{3\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+36\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)\left(540+6\sqrt{6042}\right)^{\frac{1}{3}} R_2}{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+4\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1 \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-3\sqrt{6042}+24}, v_2 = -\right.$

Hence the solution is

$$\begin{bmatrix} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-3\sqrt{6042}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\right.}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right.} \\ -\frac{6t\left(540+6\sqrt{\right.}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\right.} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-3\sqrt{6042}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{604\right.}}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{604\right.}} \\ -\frac{6\left(540+6\sqrt{\right.}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(\right.} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-3\sqrt{6042}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\cdots\right.}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)\cdots} \\ -\frac{6\left(540+6\sqrt{6\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\cdots\right.} \\ 1 \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-3\sqrt{6042}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\cdots\right.}{5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\left(540+6\sqrt{6042}\right)\cdots} \\ -\frac{6\left(540+6\sqrt{6\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\cdots\right.} \\ 1 \end{array}\right.$$

Considering the eigenvalue $\lambda_2 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{array}\right] - \left(\frac{(540\cdots}{\cdots}\right.\right.$$

$$\left[\begin{array}{cc} -\frac{42+18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\sqrt{20\cdots}\right)^{\frac{1}{3}}-i\left(\left(540+6\sqrt{3}\sqrt{20\cdots}\right.\right.}{6\left(540+6\sqrt{3}\sqrt{20\cdots}\right.} \\ 5 & 1 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\begin{bmatrix} -3 - \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} - \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2} & \\ 0 & 2 - \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} - \frac{7}{\left(540+6\sqrt{6042}\right.} \\ 5 & \end{bmatrix}$$

$$R_3 = R_3 - \frac{5R_1}{-3 - \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} - \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}} \implies \begin{bmatrix} -\frac{42+18\left(540+6\sqrt{3}\right.}{} \\ \\ \\ \end{bmatrix}$$

$$R_3 = R_3 - \frac{6\left(i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3} + \left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}} - 42i\right.}{\left(i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3} + \left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}} - 42i\sqrt{3} + 18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}} + 42\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} -\frac{42+18\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}-i\left(\left(540+6\sqrt{3}\,\sqrt{20\ldots}\right.\right.}{6\left(540+6\sqrt{3}\,\sqrt{20\ldots}\right.} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3t\left(3i\sqrt{2014}-7i\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90i\sqrt{3}+\sqrt{3}\,\sqrt{2014}-8\ldots\right.}{3i\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69i\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+9\ldots} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{3t\left(3\,I\sqrt{2014}-7\,I\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,I\sqrt{3}+\sqrt{3}\,\sqrt{2014}-8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\,\sqrt{201\ldots}\right.\right.}{3\,I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,I\sqrt{2014}+3\sqrt{3}\,\sqrt{2014}+10\left(540+6\sqrt{3}\,\sqrt{2014}\right.\ldots} \\ \frac{12t\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}}{I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-42\,I\sqrt{3}-12\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{3t\left(3\,I\sqrt{2014}-7\,I\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,I\sqrt{3}+\sqrt{3}\,\sqrt{2014}-8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\,\sqrt{201\ldots}\right.\right.}{3\,I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,I\sqrt{2014}+3\sqrt{3}\,\sqrt{2014}+10\left(540+6\sqrt{3}\,\sqrt{2014}\right.\ldots} \\ \frac{12t\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}}{I\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-42\,I\sqrt{3}-12\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\sqrt{3}+\sqrt{3}\sqrt{2014}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\sqrt{3}+\sqrt{3}\sqrt{2014}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_3 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{array}\right] - \left(\frac{(540}{}\right.\right.$$

$$\left[\begin{array}{cc} \frac{-42-18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{201\right.\right.}{6\left(540+6\sqrt{3}\sqrt{201\right.} \\ 5 & 1 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc} -3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2} & & \\ 0 & & 2-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right. } \\ 5 & & \end{array}\right.$$

$$R_3 = R_3 - \frac{5R_1}{-3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}} \implies \left[\begin{array}{c} \frac{-42-18\left(540+6\sqrt{3}\right.}{} \\ \\ \end{array}\right.$$

$$R_3 = R_3 - \frac{6\left(i\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42i\sqrt{3}+7\right.}{\left(i\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42i\sqrt{3}-18\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42\right)\left(-42+1\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cc} \frac{-42-18\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}-\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\,\sqrt{201\right.\right.}{6\left(540+6\sqrt{3}\,\sqrt{201\right.}} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3t\left(3i\sqrt{2014}-7i\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90i\sqrt{3}-\sqrt{3}\,\sqrt{2014}+8\right.}{3i\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69i\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+9} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\sqrt{3}-\sqrt{3}\,\sqrt{2014}+8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\,\sqrt{2014\right.\right.}}{3\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}-3\sqrt{3}\,\sqrt{2014}-10\left(540+6\sqrt{3}\,\sqrt{2014}\right.} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\sqrt{3}-\sqrt{3}\,\sqrt{2014}+8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\,\sqrt{2014\right.\right.}}{3\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}-3\sqrt{3}\,\sqrt{2014}-10\left(540+6\sqrt{3}\,\sqrt{2014}\right.} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\,\sqrt{2014}-7\,\mathrm{I}\,\sqrt{3}\,\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\,\sqrt{3}-\sqrt{3}\,\sqrt{2014}+8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\,\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\,\sqrt{2014}-3\sqrt{3}\,\sqrt{2014}-10\left(540+6\sqrt{3}\,\sqrt{2014}\right)} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\,\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\,\sqrt{2014}-7\,\mathrm{I}\,\sqrt{3}\,\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+90\,\mathrm{I}\,\sqrt{3}-\sqrt{3}\,\sqrt{2014}+8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\,\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\,\sqrt{2014}-3\sqrt{3}\,\sqrt{2014}-10\left(540+6\sqrt{3}\,\sqrt{2014}\right)} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\,\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right.$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defectiv
	algebraic m	geometric k	
$-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$	1	1	No
$\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$	1	1	No
$\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$	1	1	No

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \\
&= \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right] e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right.}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}\left(-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2\,e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.}}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.} \\ \\ \\ \end{bmatrix}$$

Which becomes

Expression too large to display

23.5.3 Maple step by step solution

Let's solve

$[y_1'(t) = -3y_1(t) - 3y_2(t) + y_3(t)\,, y_2'(t) = 2y_2(t) + 2y_3(t)\,, y_3'(t) = 5y_1(t) + y_2(t) + y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}, \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\right.} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right.\right.\right.$$

- Consider eigenpair

$$\left[-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}, \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{1} = \mathrm{e}^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} \cdots\right)} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}, \left[\frac{\cdots}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\cdots\right.}\right.\right.$$

- Solution from eigenpair

$$e^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{I\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}\right)t} \cdot \left[\frac{}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{I\sqrt{3}\left(\cdot\right.}{}\right.}\right.$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \cdot \left(\cos\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}{2}\right)-I\sin\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right.}{3}\right.}{}\right.\right.$$

- Simplify expression

$$\mathrm{e}^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\frac{\left(\cos\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{\cdots}{\left(540\cdots\right.}\cdots\right.}{2}\cdots\right.\right.}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)\cdots}\cdots\right.}{2}\cdots\right.} \cdots \right.$$

$$\frac{2\left(\cos\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\cdots\right.}\cdots\right.}{2}\cdots\right.\right.}{\cdots}$$

$$\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)\cdots}\cdots$$

$$\cos\left(\frac{\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\cdots\right.}{2}\cdots\right.$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_{2}(t) = \mathrm{e}^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \cdot \left[-\frac{9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}\left(4\sqrt{3}\left(540+6\sqrt{6042}\right)^{\frac{5}{3}}\sin\left(\frac{\sqrt{3}\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\cdots\right.}{6\left(540+6\sqrt{6042}\right)\cdots}\cdots\right.\right.}{\cdots}\cdots\right]\right]$$

- General solution to the system of ODEs
  $\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(t) + c_3\overrightarrow{y}_3(t)$
- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \cdot \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{\cdots}\right.}\right.} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right] = \left[\begin{array}{c} \frac{1031\left(\left(\left(\left(\sqrt{3}\,c_2+3c_3\right)\sqrt{2014}+\frac{80442c_3\sqrt{3}}{1031}+\frac{80442c_2}{1031}\right)\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+\frac{\left(12\left(\sqrt{3}\,c_2-3c_3\right)\sqrt{2014}-1423c_3\sqrt{3}+1423\cdots\right.}{1031}\right.\right.}{\left(\mathrm{e}^{\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{2\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}\left(\left(\left(\sqrt{3}\,c_2-3c_3\right)\sqrt{2014}-48c_3\sqrt{3}+48c_2\right)\left(540+6\sqrt{3}\,\sqrt{20\cdots}\right.\right.\right.} \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{ y_1(t) = \frac{1031\left(\left(\left(\left(\sqrt{3}\,c_2+3c_3\right)\sqrt{2014}+\frac{80442c_3\sqrt{3}}{1031}+\frac{80442c_2}{1031}\right)\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+\frac{\left(12\left(\sqrt{3}\,c_2-3c_3\right)\sqrt{2014}-1423c_3\sqrt{3}+1423c_2\right)\left(540\right.}{1031}\right.\right.}{}\right.$$

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 1975

```
dsolve([diff(y__1(t),t)=-3*y__1(t)-3*y__2(t)+1*y__3(t),diff(y__2(t),t)=0*y__1(t)+2*y__2(t)+2
```

Expression too large to display

$$y_2(t) = \mathrm{e}^{\frac{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}}{6}+7\right)t}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}} \sin\left(\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\right)t\sqrt{3}\,36^{\frac{1}{3}}}{36\left(90+\sqrt{6042}\right)^{\frac{1}{3}}}\right)c_2 + \mathrm{e}^{\frac{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}}{6}+7\right)t}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}} \cos\left(\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\right)t\sqrt{3}\,36^{\frac{1}{3}}}{36\left(90+\sqrt{6042}\right)^{\frac{1}{3}}}\right)c_3 + c_1\mathrm{e}^{-\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}$$

$$y_3(t) = \frac{-2c_1\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)\mathrm{e}^{-\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}} + c_2\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)\mathrm{e}^{\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}\,\mathrm{sin}}{}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 187

```
DSolve[{y1'[t]==3*y1[t]-3*y2[t]+1*y3[t],y2'[t]==0*y1[t]+2*y2[t]+2*y3[t],y3'[t]==5*y1[t]+1*y2
```

$$\text{y1}(t) \to \frac{1}{4}e^{-2t}\left((3c_1 - c_2 + c_3)e^{6t}\cos(2t) + (c_1 - 3c_2 - c_3)e^{6t}\sin(2t) + c_1 + c_2 - c_3\right)$$
$$\text{y2}(t) \to \frac{1}{4}e^{-2t}\left(-(c_1 - 3c_2 - c_3)e^{6t}\cos(2t) + (3c_1 - c_2 + c_3)e^{6t}\sin(2t) + c_1 + c_2 - c_3\right)$$
$$\text{y3}(t) \to \frac{1}{2}e^{-2t}\left((c_1 + c_2 + c_3)e^{6t}\cos(2t) + 2(c_1 - c_2)e^{6t}\sin(2t) - c_1 - c_2 + c_3\right)$$

23.6 problem section 10.6, problem 6

Internal problem ID [1641]
Internal file name [`OUTPUT/1642_Sunday_June_05_2022_02_25_36_AM_76040409/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) + 3y_2(t) + y_3(t) \\
y_2'(t) &= y_1(t) - 5y_2(t) - 3y_3(t) \\
y_3'(t) &= -3y_1(t) + 7y_2(t) + 3y_3(t)
\end{aligned}
$$

23.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t} \sin(2t) & \mathrm{e}^{-2t} \cos(2t) + 2\,\mathrm{e}^{-2t} \sin(2t) - \mathrm{e}^{-t} & \mathrm{e}^{-2t} \cos(2t) + \mathrm{e}^{-2t} \sin(2t \\ -\mathrm{e}^{-2t} \cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \cos(2t) - \mathrm{e}^{-2t} \sin(2t) & \mathrm{e}^{-2t} \cos(2t) - \mathrm{e}^{-2t} \sin(2t \\ \mathrm{e}^{-2t} \cos(2t) - \mathrm{e}^{-2t} \sin(2t) - \mathrm{e}^{-t} & -\mathrm{e}^{-2t} \cos(2t) + 3\,\mathrm{e}^{-2t} \sin(2t) + \mathrm{e}^{-t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \sin(2t \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t} \sin(2t) & -\mathrm{e}^{-t} + (\cos(2t) + 2\sin(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) + \sin(2t)) \\ -\mathrm{e}^{-2t} \cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + (2\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t)) \\ -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + (-\cos(2t) + 3\sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \sin(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t} \sin(2t) & -\mathrm{e}^{-t} + (\cos(2t) + 2\sin(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) + \sin(2t \\ -\mathrm{e}^{-2t} \cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + (2\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t \\ -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + (-\cos(2t) + 3\sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \sin(2t) \end{bmatrix}$$

$$= \begin{bmatrix} (\mathrm{e}^{-t} - \mathrm{e}^{-2t} \sin(2t))\, c_1 + (-\mathrm{e}^{-t} + (\cos(2t) + 2\sin(2t))\,\mathrm{e}^{-2t})\, c_2 + (-\mathrm{e}^{-t} + (\cos(2t) + \sin(2t))\,\mathrm{e} \\ (-\mathrm{e}^{-2t} \cos(2t) + \mathrm{e}^{-t})\, c_1 + (-\mathrm{e}^{-t} + (2\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t})\, c_2 + (-\mathrm{e}^{-t} + (\cos(2t) - \sin(2t)) \\ (-\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t})\, c_1 + (\mathrm{e}^{-t} + (-\cos(2t) + 3\sin(2t))\,\mathrm{e}^{-2t})\, c_2 + (\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \sin \end{bmatrix}$$

$$= \begin{bmatrix} ((-c_1 + 2c_2 + c_3)\sin(2t) + \cos(2t)(c_2 + c_3))\,\mathrm{e}^{-2t} + \mathrm{e}^{-t}(c_1 - c_2 - c_3) \\ ((-c_1 + 2c_2 + c_3)\cos(2t) - \sin(2t)(c_2 + c_3))\,\mathrm{e}^{-2t} + \mathrm{e}^{-t}(c_1 - c_2 - c_3) \\ ((-c_1 + 3c_2 + 2c_3)\sin(2t) + \cos(2t)(-c_2 + c_1))\,\mathrm{e}^{-2t} - \mathrm{e}^{-t}(c_1 - c_2 - c_3) \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 3 & 1 \\ 1 & -5-\lambda & -3 \\ -3 & 7 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 + 12\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + 2i$$
$$\lambda_2 = -2 - 2i$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
$-2+2i$	1	complex eigenvalue
$-2-2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & -4 & -3 \\ -3 & 7 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 1 & -4 & -3 & 0 \\ -3 & 7 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ -3 & 7 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 & 1 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-2 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 + 2i & 3 & 1 \\ 1 & -3 + 2i & -3 \\ -3 & 7 & 5 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 + 2i & 3 & 1 & 0 \\ 1 & -3 + 2i & -3 & 0 \\ -3 & 7 & 5 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ -3 & 7 & 5+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{3}{5} - \frac{6i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ 0 & \frac{26}{5} - \frac{18i}{5} & \frac{22}{5} + \frac{4i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{3}{2} + \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+2i & 3 & 1 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t, v_2 = -\frac{1}{2}t - \frac{1}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ -\frac{1}{2}t - \frac{1}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ \frac{-\frac{1}{2}t - \frac{1}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ -1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -2 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-2 + 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 2i & 3 & 1 \\ 1 & -3 - 2i & -3 \\ -3 & 7 & 5 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 1 & -3 - 2i & -3 & 0 \\ -3 & 7 & 5 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ -3 & 7 & 5 - 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{3}{5} + \frac{6i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ 0 & \frac{26}{5} + \frac{18i}{5} & \frac{22}{5} - \frac{4i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{3}{2} - \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1-2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-2i & 3 & 1 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t, v_2 = -\frac{1}{2}t + \frac{1}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ -\frac{1}{2}t + \frac{1}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ \frac{-\frac{1}{2}t + \frac{1}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated

with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-2+2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$-2-2i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} \\ \left(-\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} \\ e^{(-2+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ \left(-\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ e^{(-2-2i)t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 e^{(-2+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 e^{(-2-2i)t} - c_3 e^{-t} \\ \left(-\frac{1}{2}+\frac{i}{2}\right) c_1 e^{(-2+2i)t} + \left(-\frac{1}{2}-\frac{i}{2}\right) c_2 e^{(-2-2i)t} - c_3 e^{-t} \\ c_1 e^{(-2+2i)t} + c_2 e^{(-2-2i)t} + c_3 e^{-t} \end{bmatrix}$$

23.6.3 Maple step by step solution

Let's solve

$[y_1'(t) = -3y_1(t) + 3y_2(t) + y_3(t), y_2'(t) = y_1(t) - 5y_2(t) - 3y_3(t), y_3'(t) = -3y_1(t) + 7y_2(t) + 3y_3($

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(t) = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow}{}'(t) = A \cdot \underline{y}^{\rightarrow}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-2-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-2+2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}_{1} = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-2t}\cdot(\cos(2t)-\mathrm{I}\sin(2t))\cdot\begin{bmatrix}\frac{1}{2}-\frac{\mathrm{I}}{2}\\ -\frac{1}{2}-\frac{\mathrm{I}}{2}\\ 1\end{bmatrix}$$

- Simplify expression

$$e^{-2t}\cdot\begin{bmatrix}\left(\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(2t)-\mathrm{I}\sin(2t))\\ \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(2t)-\mathrm{I}\sin(2t))\\ \cos(2t)-\mathrm{I}\sin(2t)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_{2}(t)=e^{-2t}\cdot\begin{bmatrix}\frac{\cos(2t)}{2}-\frac{\sin(2t)}{2}\\ -\frac{\sin(2t)}{2}-\frac{\cos(2t)}{2}\\ \cos(2t)\end{bmatrix},\underline{\vec{y}}_{3}(t)=e^{-2t}\cdot\begin{bmatrix}-\frac{\sin(2t)}{2}-\frac{\cos(2t)}{2}\\ -\frac{\cos(2t)}{2}+\frac{\sin(2t)}{2}\\ -\sin(2t)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}}=c_1\underline{\vec{y}}_{1}+c_2\underline{\vec{y}}_{2}(t)+c_3\underline{\vec{y}}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}}=c_1e^{-t}\cdot\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}+e^{-2t}c_2\cdot\begin{bmatrix}\frac{\cos(2t)}{2}-\frac{\sin(2t)}{2}\\ -\frac{\sin(2t)}{2}-\frac{\cos(2t)}{2}\\ \cos(2t)\end{bmatrix}+c_3e^{-2t}\cdot\begin{bmatrix}-\frac{\sin(2t)}{2}-\frac{\cos(2t)}{2}\\ -\frac{\cos(2t)}{2}+\frac{\sin(2t)}{2}\\ -\sin(2t)\end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}y_1(t)\\ y_2(t)\\ y_3(t)\end{bmatrix}=\begin{bmatrix}\frac{((c_2-c_3)\cos(2t)-\sin(2t)(c_2+c_3))e^{-2t}}{2}-c_1e^{-t}\\ \frac{((-c_2-c_3)\cos(2t)-\sin(2t)(c_2-c_3))e^{-2t}}{2}-c_1e^{-t}\\ c_1e^{-t}+e^{-2t}c_2\cos(2t)-c_3e^{-2t}\sin(2t)\end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t)=\frac{((c_2-c_3)\cos(2t)-\sin(2t)(c_2+c_3))e^{-2t}}{2}-c_1e^{-t},y_2(t)=\frac{((-c_2-c_3)\cos(2t)-\sin(2t)(c_2-c_3))e^{-2t}}{2}-c_1e^{-t},y_3(t)\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 115

```
dsolve([diff(y__1(t),t)=-3*y__1(t)+3*y__2(t)+1*y__3(t),diff(y__2(t),t)=1*y__1(t)-5*y__2(t)-3
```

$$y_1(t) = e^{-t}c_1 + c_2e^{-2t}\sin(2t) + c_3e^{-2t}\cos(2t)$$
$$y_2(t) = e^{-t}c_1 + c_2e^{-2t}\cos(2t) - c_3e^{-2t}\sin(2t)$$
$$y_3(t) = -e^{-t}c_1 + c_2e^{-2t}\sin(2t) - c_2e^{-2t}\cos(2t) + c_3e^{-2t}\cos(2t) + c_3e^{-2t}\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 158

```
DSolve[{y1'[t]==-3*y1[t]+3*y2[t]+1*y3[t],y2'[t]==1*y1[t]-5*y2[t]-3*y3[t],y3'[t]==-3*y1[t]+7*
```

$$\text{y1}(t) \to e^{-2t}\left((c_1 - c_2 - c_3)e^t + (c_2 + c_3)\cos(2t) + (-c_1 + 2c_2 + c_3)\sin(2t)\right)$$
$$\text{y2}(t) \to e^{-2t}\left((c_1 - c_2 - c_3)e^t + (-c_1 + 2c_2 + c_3)\cos(2t) - (c_2 + c_3)\sin(2t)\right)$$
$$\text{y3}(t) \to e^{-2t}\left((-c_1 + c_2 + c_3)e^t + (c_1 - c_2)\cos(2t) + (-c_1 + 3c_2 + 2c_3)\sin(2t)\right)$$

23.7 problem section 10.6, problem 7

Internal problem ID [1642]
Internal file name [`OUTPUT/1643_Sunday_June_05_2022_02_25_38_AM_33724522/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= 2y_1(t) + y_2(t) - y_3(t) \\
y_2'(t) &= y_2(t) + y_3(t) \\
y_3'(t) &= y_1(t) + y_3(t)
\end{aligned}
$$

23.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^t\cos(t)}{2} + \frac{\sin(t)e^t}{2} & -\frac{e^t\cos(t)}{2} + \frac{\sin(t)e^t}{2} + \frac{e^{2t}}{2} & -\sin(t)e^t \\ -\frac{e^t\cos(t)}{2} - \frac{\sin(t)e^t}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^t\cos(t)}{2} - \frac{\sin(t)e^t}{2} & \sin(t)e^t \\ -\frac{e^t\cos(t)}{2} + \frac{\sin(t)e^t}{2} + \frac{e^{2t}}{2} & -\frac{e^t\cos(t)}{2} - \frac{\sin(t)e^t}{2} + \frac{e^{2t}}{2} & e^t\cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{(\cos(t)+\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2} & -\sin(t)e^t \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\sin(t)+\cos(t))e^t}{2} & \sin(t)e^t \\ \frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & e^t\cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{(\cos(t)+\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2} & -\sin(t)e^t \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\sin(t)+\cos(t))e^t}{2} & \sin(t)e^t \\ \frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & e^t\cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{e^{2t}}{2} + \frac{(\cos(t)+\sin(t))e^t}{2}\right)c_1 + \left(\frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2}\right)c_2 - \sin(t)e^t c_3 \\ \left(\frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2}\right)c_1 + \left(\frac{e^{2t}}{2} + \frac{(-\sin(t)+\cos(t))e^t}{2}\right)c_2 + \sin(t)e^t c_3 \\ \left(\frac{e^{2t}}{2} + \frac{(\sin(t)-\cos(t))e^t}{2}\right)c_1 + \left(\frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2}\right)c_2 + e^t\cos(t)c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(c_1+c_2)e^{2t}}{2} + \frac{((c_1+c_2-2c_3)\sin(t)+\cos(t)(-c_2+c_1))e^t}{2} \\ \frac{(c_1+c_2)e^{2t}}{2} - \frac{((c_1+c_2-2c_3)\sin(t)+\cos(t)(-c_2+c_1))e^t}{2} \\ \frac{(c_1+c_2)e^{2t}}{2} - \frac{((c_1+c_2-2c_3)\cos(t)-\sin(t)(-c_2+c_1))e^t}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 + 6\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 + i \\ \lambda_3 &= 1 - i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
$1 - i$	1	complex eigenvalue
$1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (1 - i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 1 & -1 \\ 0 & i & 1 \\ 1 & 0 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 0 & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 & -1 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (1+i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 & -1 \\ 0 & -i & 1 \\ 1 & 0 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & -i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & -\frac{1}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right)R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 & -1 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$1 + i$	1	1	No	$\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$
$1 - i$	1	1	No	$\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} i e^{(1+i)t} \\ -i e^{(1+i)t} \\ e^{(1+i)t} \end{bmatrix} + c_3 \begin{bmatrix} -i e^{(1-i)t} \\ i e^{(1-i)t} \\ e^{(1-i)t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + i c_2 e^{(1+i)t} - i c_3 e^{(1-i)t} \\ c_1 e^{2t} - i c_2 e^{(1+i)t} + i c_3 e^{(1-i)t} \\ c_1 e^{2t} + c_2 e^{(1+i)t} + c_3 e^{(1-i)t} \end{bmatrix}
$$

23.7.3 Maple step by step solution

Let's solve

$[y_1'(t) = 2y_1(t) + y_2(t) - y_3(t), y_2'(t) = y_2(t) + y_3(t), y_3'(t) = y_1(t) + y_3(t)]$

- Define vector

$$
\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(t) = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$\underline{\vec{y}}'(t) = A \cdot \underline{\vec{y}}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-\mathrm{I})t} \cdot \left[\begin{array}{c} -\mathrm{I} \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\mathrm{I} \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{t} \cdot \left[\begin{array}{c} -\mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_{2}(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -\sin(t) \\ \sin(t) \\ \cos(t) \end{array}\right], \underline{\vec{y}}_{3}(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -\cos(t) \\ \cos(t) \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2}(t) + c_3 \underline{\vec{y}}_{3}(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{2t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{t} \cdot \left[\begin{array}{c} -\sin(t) \\ \sin(t) \\ \cos(t) \end{array}\right] + c_3 \mathrm{e}^{t} \cdot \left[\begin{array}{c} -\cos(t) \\ \cos(t) \\ -\sin(t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} - c_2 e^t \sin(t) - c_3 e^t \cos(t) \\ c_1 e^{2t} + c_2 e^t \sin(t) + c_3 e^t \cos(t) \\ c_1 e^{2t} + c_2 e^t \cos(t) - c_3 e^t \sin(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(t) = c_1 e^{2t} - c_2 e^t \sin(t) - c_3 e^t \cos(t), y_2(t) = c_1 e^{2t} + c_2 e^t \sin(t) + c_3 e^t \cos(t), y_3(t) = c_1 e^{2t} + c_$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 70

```
dsolve([diff(y__1(t),t)=2*y__1(t)+1*y__2(t)-1*y__3(t),diff(y__2(t),t)=0*y__1(t)+1*y__2(t)+1*
```

$$y_1(t) = c_1 e^{2t} + c_2 e^t \cos(t) - c_3 e^t \sin(t)$$
$$y_2(t) = c_1 e^{2t} - c_2 e^t \cos(t) + c_3 e^t \sin(t)$$
$$y_3(t) = c_1 e^{2t} + c_2 e^t \sin(t) + c_3 e^t \cos(t)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 129

```
DSolve[{y1'[t]==2*y1[t]+1*y2[t]-1*y3[t],y2'[t]==0*y1[t]+1*y2[t]+1*y3[t],y3'[t]==1*y1[t]+0*y2
```

$$\text{y1}(t) \to \frac{1}{2} e^t \left(-2c_3 \sin(t) + c_2 \left(e^t + \sin(t) - \cos(t)\right) + c_1 \left(e^t + \sin(t) + \cos(t)\right)\right)$$
$$\text{y2}(t) \to \frac{1}{2} e^t \left((c_1 + c_2) e^t + (c_2 - c_1) \cos(t) - (c_1 + c_2 - 2c_3) \sin(t)\right)$$
$$\text{y3}(t) \to \frac{1}{2} e^t \left((c_1 + c_2) e^t - (c_1 + c_2 - 2c_3) \cos(t) + (c_1 - c_2) \sin(t)\right)$$

23.8 problem section 10.6, problem 8

Internal problem ID [1643]
Internal file name [`OUTPUT/1644_Sunday_June_05_2022_02_25_41_AM_22734179/index.tex`]

Book: Elementary differential equations with boundary value problems. William F. Trench. Brooks/Cole 2001
Section: Chapter 10 Linear system of Differential equations. Section 10.6, constant coefficient homogeneous system III. Page 566
Problem number: section 10.6, problem 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) + y_2(t) - 3y_3(t) \\
y_2'(t) &= 4y_1(t) - y_2(t) + 2y_3(t) \\
y_3'(t) &= 4y_1(t) - 2y_2(t) + 3y_3(t)
\end{aligned}
$$

23.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At}=\begin{bmatrix} e^{-t}\cos(2t)-e^{-t}\sin(2t) & -\frac{e^{-t}\cos(2t)}{2}+\frac{e^t}{2} & \frac{e^{-t}\cos(2t)}{2}-e^{-t}\sin(2t)-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^t}{2}+\frac{e^{-t}\cos(2t)}{2}-\frac{e^{-t}\sin(2t)}{2} & \frac{e^{-t}\cos(2t)}{2}+\frac{3e^{-t}\sin(2t)}{2}-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^{-t}\cos(2t)}{2}-\frac{e^{-t}\sin(2t)}{2}-\frac{e^t}{2} & \frac{e^t}{2}+\frac{e^{-t}\cos(2t)}{2}+\frac{3e^{-t}\sin(2t)}{2} \end{bmatrix}$$

$$=\begin{bmatrix} e^{-t}(\cos(2t)-\sin(2t)) & -\frac{e^{-t}\cos(2t)}{2}+\frac{e^t}{2} & \frac{(\cos(2t)-2\sin(2t))e^{-t}}{2}-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}+\frac{e^t}{2} & \frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}-\frac{e^t}{2} & \frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}+\frac{e^t}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t)=e^{At}\vec{c}$$

$$=\begin{bmatrix} e^{-t}(\cos(2t)-\sin(2t)) & -\frac{e^{-t}\cos(2t)}{2}+\frac{e^t}{2} & \frac{(\cos(2t)-2\sin(2t))e^{-t}}{2}-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}+\frac{e^t}{2} & \frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}-\frac{e^t}{2} \\ 2e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}-\frac{e^t}{2} & \frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}+\frac{e^t}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$=\begin{bmatrix} e^{-t}(\cos(2t)-\sin(2t))\,c_1+\left(-\frac{e^{-t}\cos(2t)}{2}+\frac{e^t}{2}\right)c_2+\left(\frac{(\cos(2t)-2\sin(2t))e^{-t}}{2}-\frac{e^t}{2}\right)c_3 \\ 2e^{-t}\sin(2t)\,c_1+\left(\frac{e^{-t}(\cos(2t)-\sin(2t))}{2}+\frac{e^t}{2}\right)c_2+\left(\frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}-\frac{e^t}{2}\right)c_3 \\ 2e^{-t}\sin(2t)\,c_1+\left(\frac{e^{-t}(\cos(2t)-\sin(2t))}{2}-\frac{e^t}{2}\right)c_2+\left(\frac{(\cos(2t)+3\sin(2t))e^{-t}}{2}+\frac{e^t}{2}\right)c_3 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{((2c_1-c_2+c_3)\cos(2t)-2\sin(2t)(c_1+c_3))e^{-t}}{2}+\frac{(c_2-c_3)e^t}{2} \\ \frac{((4c_1-c_2+3c_3)\sin(2t)+\cos(2t)(c_2+c_3))e^{-t}}{2}+\frac{(c_2-c_3)e^t}{2} \\ \frac{((4c_1-c_2+3c_3)\sin(2t)+\cos(2t)(c_2+c_3))e^{-t}}{2}-\frac{(c_2-c_3)e^t}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

23.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t)=A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix}=\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 1 & -3 \\ 4 & -1-\lambda & 2 \\ 4 & -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 + 3\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1 + 2i$$
$$\lambda_3 = -1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1-2i$	1	complex eigenvalue
1	1	real eigenvalue
$-1+2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & -3 \\ 4 & -2 & 2 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 4 & -2 & 2 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - 2i$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (-1 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 + 2i & 1 & -3 \\ 4 & 2i & 2 \\ 4 & -2 & 4 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 + 2i & 1 & -3 & 0 \\ 4 & 2i & 2 & 0 \\ 4 & -2 & 4 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 4 & -2 & 4+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 0 & -1+i & 1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{5} - \frac{2i}{5}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+2i & 1 & -3 \\ 0 & 1+3i & -1-3i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (-1 + 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 - 2i & 1 & -3 \\ 4 & -2i & 2 \\ 4 & -2 & 4 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 - 2i & 1 & -3 & 0 \\ 4 & -2i & 2 & 0 \\ 4 & -2 & 4 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1 - i) R_1 \implies \left[\begin{array}{ccc|c} -2 - 2i & 1 & -3 & 0 \\ 0 & 1 - 3i & -1 + 3i & 0 \\ 4 & -2 & 4 - 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1 - i) R_1 \implies \left[\begin{array}{ccc|c} -2 - 2i & 1 & -3 & 0 \\ 0 & 1 - 3i & -1 + 3i & 0 \\ 0 & -1 - i & 1 + i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{5} + \frac{2i}{5}\right) R_2 \implies \left[\begin{array}{ccc|c} -2 - 2i & 1 & -3 & 0 \\ 0 & 1 - 3i & -1 + 3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-2i & 1 & -3 \\ 0 & 1-3i & -1+3i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
$-1+2i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$
$-1-2i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^t \\
&= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^t
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^t \\ -\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^t + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-1+2i)t} + \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-1-2i)t} \\ -c_1 e^t + c_2 e^{(-1+2i)t} + c_3 e^{(-1-2i)t} \\ c_1 e^t + c_2 e^{(-1+2i)t} + c_3 e^{(-1-2i)t} \end{bmatrix}$$

23.8.3 Maple step by step solution

Let's solve

$[y_1'(t) = -3y_1(t) + y_2(t) - 3y_3(t), y_2'(t) = 4y_1(t) - y_2(t) + 2y_3(t), y_3'(t) = 4y_1(t) - 2y_2(t) + 3y_3(t)]$

- Define vector

$$\underline{y}^{\rightarrow}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(t) = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(t) = A \cdot \underline{y}^{\rightarrow}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]\right], \left[-1-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \\ 1 \end{array}\right]\right], \left[-1+2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{y}}_{1} = \mathrm{e}^{t} \cdot \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})t} \cdot \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_2(t) = e^{-t} \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ \cos(2t) \\ \cos(2t) \end{bmatrix}, \underline{\vec{y}}_3(t) = e^{-t} \cdot \begin{bmatrix} -\frac{\cos(2t)}{2} + \frac{\sin(2t)}{2} \\ -\sin(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2(t) + c_3 \underline{\vec{y}}_3(t)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^t \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ \cos(2t) \\ \cos(2t) \end{bmatrix} + c_3 e^{-t} \cdot \begin{bmatrix} -\frac{\cos(2t)}{2} + \frac{\sin(2t)}{2} \\ -\sin(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{((-c_2-c_3)\cos(2t)-\sin(2t)(c_2-c_3))e^{-t}}{2} - c_1 e^t \\ -c_1 e^t + c_2 e^{-t}\cos(2t) - c_3 e^{-t}\sin(2t) \\ c_1 e^t + c_2 e^{-t}\cos(2t) - c_3 e^{-t}\sin(2t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(t) = \frac{((-c_2-c_3)\cos(2t)-\sin(2t)(c_2-c_3))e^{-t}}{2} - c_1 e^t, y_2(t) = -c_1 e^t + c_2 e^{-t}\cos(2t) - c_3 e^{-t}\sin(2t), y_3(t) =\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 133

```
dsolve([diff(y__1(t),t)=-3*y__1(t)+1*y__2(t)-3*y__3(t),diff(y__2(t),t)=4*y__1(t)-1*y__2(t)+2
```

$$y_1(t) = c_1 e^t + c_2 e^{-t}\sin(2t) + c_3 e^{-t}\cos(2t)$$
$$y_2(t) = c_1 e^t - c_2 e^{-t}\sin(2t) - c_2 e^{-t}\cos(2t) - c_3 e^{-t}\cos(2t) + c_3 e^{-t}\sin(2t)$$
$$y_3(t) = -c_1 e^t - c_2 e^{-t}\sin(2t) - c_2 e^{-t}\cos(2t) - c_3 e^{-t}\cos(2t) + c_3 e^{-t}\sin(2t)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 163

```
DSolve[{y1'[t]==-3*y1[t]+1*y2[t]-3*y3[t],y2'[t]==4*y1[t]-1*y2[t]+2*y3[t],y3'[t]==4*y1[t]-2*y
```

$$y1(t) \to \frac{1}{2}e^{-t}\left((c_2-c_3)e^{2t}+(2c_1-c_2+c_3)\cos(2t)-2(c_1+c_3)\sin(2t)\right)$$
$$y2(t) \to \frac{1}{2}e^{-t}\left((c_2-c_3)e^{2t}+(c_2+c_3)\cos(2t)+(4c_1-c_2+3c_3)\sin(2t)\right)$$
$$y3(t) \to \frac{1}{2}e^{-t}\left((c_3-c_2)e^{2t}+(c_2+c_3)\cos(2t)+(4c_1-c_2+3c_3)\sin(2t)\right)$$